ПРИКЛАДНАЯ МАТЕМАТИКА

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BEAM DYNAMICS OPTIMIZATION IN A LINEAR ACCELERATOR

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The article is devoted to the problems of optimization of the charged particles' beam dynamics in accelerators. The increasing requirements to the output parameters of the accelerated particles call for the development of new methods and approaches in the field of beam control for charged particles. The present paper considers and sets out particular tasks of optimization of the longitudinal motion of the charged particles in an RFQ accelerator. The particles' dynamics is considered in the accelerating field of an equivalent travelling wave. As was shown earlier, that approach allows one to consider the longitudinal motion and the transverse motion separately. Besides, certain requirements for transverse motion can be considered in the study of the longitudinal motion, which facilitates further optimization of the transverse dynamics. Particular quality functionals are specified and explained in the article. What distinguishes the present work is that it considers non-smooth functionals in combination with smooth functionals, taking the particles distribution density along the beam of trajectories into consideration. The mathematical model of simultaneous optimization of smooth and non-smooth functionals is considered. The variation of the combined functional is obtained as well as the necessary optimality condition. It should be noted that the considered approach might be applied to the control problems in case of partial information about the initial conditions, i. e. the problems of control of the beam of trajectories of various dynamic systems. Refs 15. Figs 4.

Keywords: control, optimization, minimax, linear accelerator.

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ОПТИМИЗАЦИЯ ДИНАМИКИ ПУЧКА В ЛИНЕЙНОМ УСКОРИТЕЛЕ

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Описываются проблемы оптимизации динамики заряженных частиц в ускорителях. Необходимость разработки новых методов и подходов в задачах управления пучками заряженных частиц вызвана высокими требованиями, предъявляемыми к качеству выходных параметров ускоренных частиц. Ставятся и решаются конкретные задачи оптимизации продольного движения заряженных частиц в ускорителе с пространственно-однородной квадрупольной фокусировкой. Динамика частиц изучается в ускоряющем поле эквивалентной бегущей волны. Как было показано ранее, такой подход позволяет отдельно рассматривать продольное и поперечное движения. При этом при исследовании продольного движения можно учесть некоторые требования к поперечному движению, что облегчает в дальнейшем решение проблемы оптимизации поперечной динамики частиц. Приводятся конкретные функционалы и дается их физический смысл. Особенностью данной статьи является то, что наряду с гладкими функционалами исследуются и негладкие функционалы. При этом учитывается плотность распределения заряженных частиц вдоль пучка траекторий. Рассмотрена математическая модель оптимизации связки гладких и негладких функционалов. Получена вариация построенного функционала и даны необходимые условия оптимальности. Следует отметить, что предложенный подход может быть использован и в задачах управления при неполной информации о начальных данных, т. е. в задачах управления ансамблями траекторий различных динамических систем. Библиогр. 15 назв. Ил. 4.

Ключевые слова: управление, оптимизация, минимакс, линейный ускоритель.

1. Introduction. The problem of charged particles dynamics optimization in accelerating structures is well studied. The approach to that problem presented in the current paper is based on two major developments in this field: simultaneous optimization of a program motion and an ensemble of trajectories and minimax optimization.

Minimization of smooth and non-smooth functionals on the beam of trajectories in various statements was considered in [1–5]. In the works by D. A. Ovsyannikov [6, 7] problems of the charged particles' beam control were studied. As for simultaneous optimization of some program motion and an ensemble of trajectories, we should mention works by D. A. Ovsyannikov and A. D. Ovsyannikov [8, 9]. But those works dealt only with smooth functionals, so once we bring a minimax functional into action, we get something new — an approach based on simultaneous use of integral and minimax functionals for optimization of a program motion and an ensemble of trajectories that was introduced in [10, 11].

In the present paper application of the combined functional for the problem of optimization of the longitudinal motion of the charged particles in a radio-frequency quadrupole (RFQ) accelerator is proposed. The latest version of the combined functional contains a density variable which allows us to take particles distribution density into consideration.

2. Beam dynamics in a RFQ structure. The longitudinal motion of the charged particles in an RFQ structure is described by the following equation [12]:

$$\frac{d\beta}{d\tau} = \frac{4eUT}{W_0 L} \cos(Kz) \cos(\tilde{\omega}\tau + \phi). \qquad (1)$$

Here $\tau = ct$ is the independent variable (t is time, c is the speed of light), z is the longitudinal coordinate of a particle in the beam, β is the reduced speed of a particle, $\tilde{\omega} = 2\pi\omega/c$, ω is the effective frequency of the accelerating RF field, U is the voltage on the electrodes, T is the acceleration effectiveness, W_0 and e are the rest energy and the charge of the particle, $K = 2\pi/L$, where L is the length of the period, ϕ is the phase of the synchronous particle. We also assume that $L = \beta_s\lambda$, where λ is the wave length of the accelerating field and β_s is the reduced velocity of the synchronous particle.

We will consider the longitudinal motion of the particles in an equivalent travelling wave and take into account only the accelerating half-wave, so that the motion equation (1) for the synchronous particle can be rewritten in the following form [3, 13]:

$$(\Lambda^2)' = 2k\eta\cos\phi. \tag{2}$$

The equation in deviations from the synchronous particle will be [14]

$$\psi'' + 2\frac{\Lambda'}{\Lambda}\psi' + \frac{\Lambda''}{\Lambda}\psi - \frac{\eta}{\Lambda^2}(\cos\phi - \cos(\phi+\psi)) = 0. \tag{3}$$

Here $\psi = K(z_s - z)$, z_s is the coordinate of the synchronous particle, $\Lambda = \beta_s/\beta_0$ (β_0 is the initial reduced velocity of the synchronous particle), $\eta = \frac{UT}{(UT)_{\max}}$, $k = \Omega/\tilde{\omega}$, $s = \Omega\tau \in [0, T_s]$ is the new independent variable, Ω is defined by the expression below:

$$\Omega^2 = \frac{4\pi e(UT)_{\max}}{W_0 L_0^2},$$

where $L_0 = \beta_0\lambda$.

In equations (2) and (3) the derivatives are taken with respect to the new independent variable s.

3. Numerical simulation. The mathematical model of the charged particles' beam dynamics in an RFQ accelerator was implemented in the BDO-RFQ software developed at the Faculty of Applied Mathematics and Control Processes of Saint Petersburg State University.

The target parameters for optimization are:

— synchronous particle output energy equal to 5 MeV;

— minimum deviation of the beam from the synchronous particle in energy at the output of the structure;

— the value of the defocusing factor — an important parameter of the accelerating process, should be less than 1 along the process.

The limitation on the value of the defocusing factor in this case is of special importance due to the separate modeling and optimization of the longitudinal and transverse motions of the particles. Some of the results, based on the mathematical model described in the previous section, are shown in the following pictures.

Figure 1 represents the dynamics of the synchronous particle (its energy) and the value of the defocusing factor. With the chosen initial conditions the defocusing factor is well below 1, which is good and it should remain this way, while the energy of the synchronous particle is somewhat less than the target 5 MeV.

Beam reduced energy depicted in Figure 2 doesn't reach 5 MeV and also shows some spread at the end of the accelerating structure.

Phase deviations oscillations shown in Figure 3 decrease well into the last third of the process, which is well but can be further improved.

Figure 4 shows the phase portrait at the output of the process, each dot corresponding to a certain particle in the beam. Decreasing the energy spread in these coordinates would mean squeezing the picture vertically.

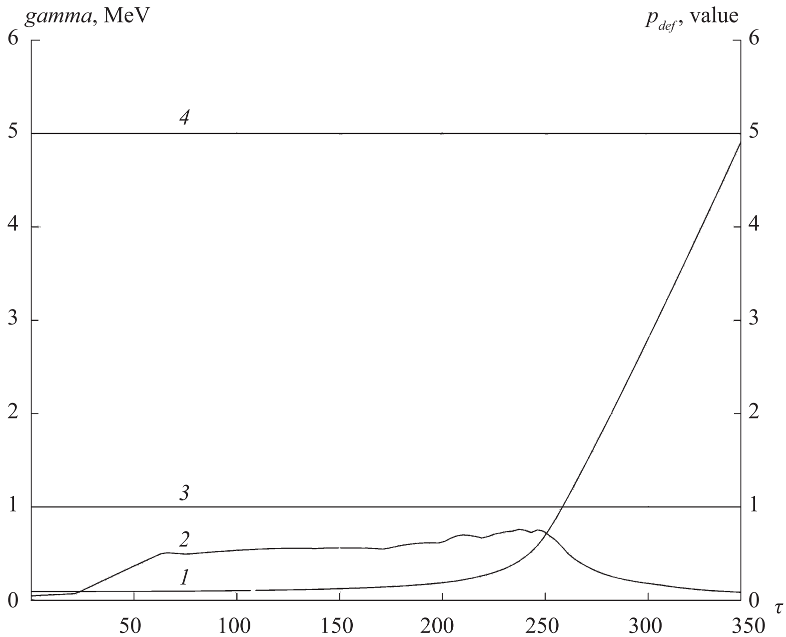

Figure 1. Synchronous particle reduced energy and defocusing factor
1 — reduced energy of the synchronous particle; *2* — actual value of the defocusing factor; *3* — limitation of the defocusing factor; *4* — target energy of the synchronous particle at the output of the accelerating structure.

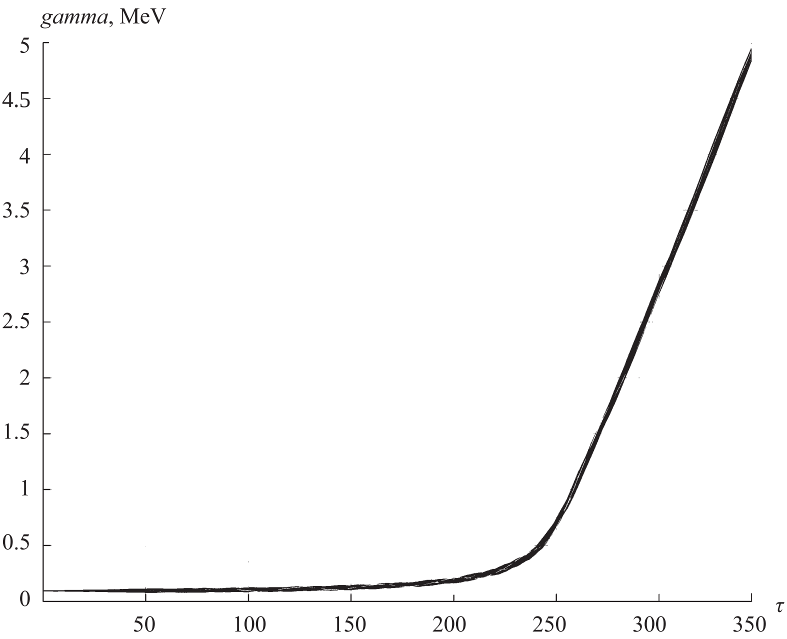

Figure 2. Beam reduced energy

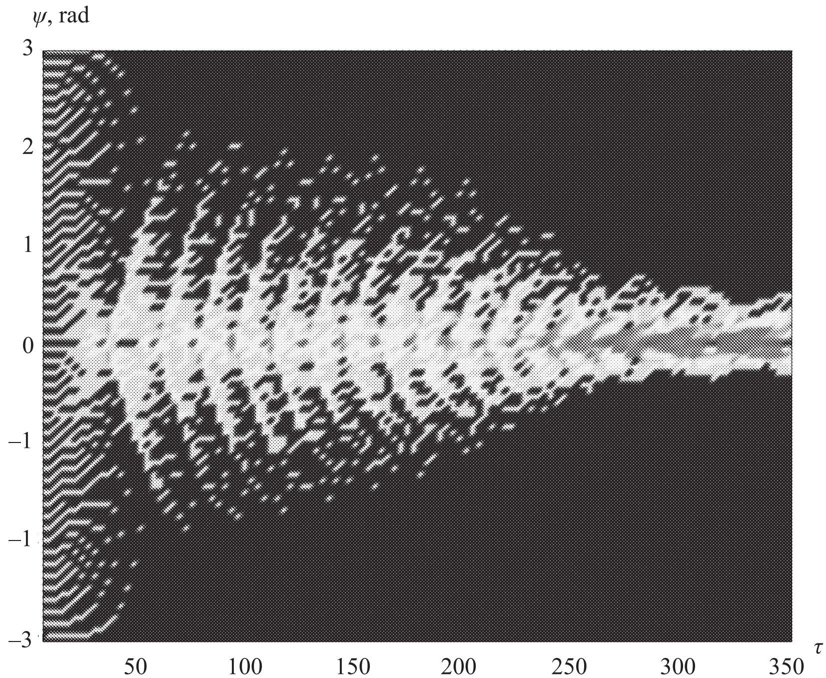

Figure 3. Phase deviations of the beam

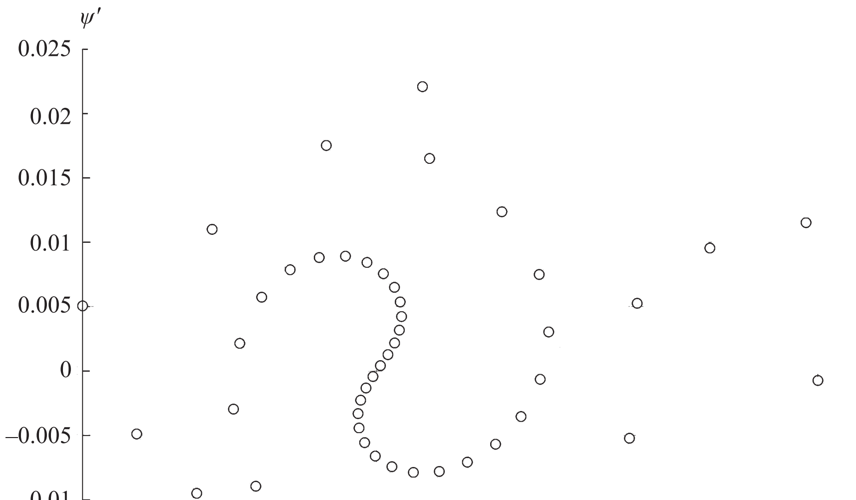

Figure 4. Phase portrait at the end of accelerating structure

4. Statement of the optimization problem. Once the optimization objectives are formulated we can move on to the statement of the optimization problem and write down the quality functionals.

Let us introduce a smooth functional that evaluates the deviation of the kinetic energy of the synchronous particle from the target value at the end of the accelerating structure and also considers the defocusing factor restrictions for $s \in [0, T_s]$

$$J_1(u) = \int\limits_0^T h(p_{def}, a_d)\, ds + (\Lambda^2(T_s) - a_E)^2. \tag{4}$$

Here a_d, a_E are some fixed values, defocusing factor p_{def} is defined as follows [14]:

$$p_{def} = \frac{2k^2|\sin\phi|}{\Lambda^2}.$$

Penalty function $h(p, a)$ is defined by expression

$$h(p,a) = \begin{cases} (p-a)^2, & p > a, \\ 0, & p \leqslant a. \end{cases}$$

Let us also introduce a minimax functional with the particles distribution density variable $\rho = \rho(s, \psi, \psi')$ on the set of terminal positions of the system (3)

$$J_2(u) = \max_{(\psi_{T_s}, \psi'_{T_s}) \in Y} p_w^2 \rho(T_s, \psi_{T_s}, \psi'_{T_s}). \tag{5}$$

Parameter $p_w = (W_k - W_k^s)/W_k^s$ relates to the deviations of the energies of the particles in the beam from the energy of the synchronous particle, which in terms of Λ and ψ can be written

$$p_w = (p_\beta + 1)^2 - 1, \quad p_\beta = -k\left(\psi' + \psi\frac{\Lambda'}{\Lambda}\right).$$

Functional (5) is a minimax functional that allows to include the most deviating particles into the optimization considering the particles distribution density at the output of the accelerating structure.

5. Mathematical optimization. The problem stated in the previous sections can be generalized in the following form.

Let us consider systems of differential equations

$$\frac{dx}{dt} = f(t, x, u), \qquad x(0) = x_0, \tag{6}$$

$$\frac{dy}{dt} = F(t, x, y, u), \qquad y(0) = y_0 \in M_0, \tag{7}$$

where $t \in [0, T]$ is independent variable instead of s; x is n-dimensional phase-vector; $u = u(t)$ is r-dimensional piecewise continuous control vector-function from a class D that takes value in a compact set U; y is n-dimensional phase-vector; $f(t, x, u)$, $F(t, x, y, u)$ is n-dimensional reasonably smooth vector-functions; M_0 is a compact set.

The new variables x and y refer to Λ and (ψ, ψ') used in the previous sections, control vector-function u represents the acceleration intensity and the phase of the synchronous particle $(\eta(s), \phi(s))$, so that equations (6), (7) represent equations (2), (3) rewritten in the normal form using new notations.

The solution of sub-system (6) is called program motion and the trajectories of system (7) are called disturbed motions or the ensemble of trajectories.

Let us also introduce the equation, describing the dynamics of the particles distribution density $\rho = \rho(t, y(t))$ on the trajectories of sub-system (7)

$$\frac{d\rho}{dt} = -\rho \cdot \mathrm{div}_y F(t, x, y, u), \quad \rho(0) = \rho_0(y_0). \tag{8}$$

By analogy with functional (4), on the solution of system (6) we will introduce an integral functional $I_1(u)$

$$I_1(u) = \int_0^T \varphi_1(x(t, x_0, u))dt + g(x(T)). \tag{9}$$

And on the trajectories of system (7) we introduce a generalized minimax functional (5), that takes particles distribution density into consideration

$$I_2(u) = \max_{y_T \in Y} \varphi_2(y_T, \rho(y_T)), \tag{10}$$

where Y is the set of terminal positions of the sub-system (7), defined by the following expression:

$$Y = \{y(T, x_0, y_0, u) \mid u \in D, x(0) = x_0, y(0) = y_0 \in M_0\}.$$

Functions φ_1, φ_2, g in the expressions for the functionals (9) and (10) are non-negative smooth functions.

In this paper we consider a combination of $I_1(u)$ and $I_2(u)$

$$I(u) = I_1(u) + I_2(u). \tag{11}$$

The combined functional (11) allows us to simultaneously optimize program motion and the ensemble of trajectories, take particles distribution density into consideration and not only evaluate the dynamic process in general, but also include the “worst”, particles into the optimization process.

6. Variation of the functional. Let us write down the variations equations corresponding to systems (6)–(8) [15]

$$\begin{gathered}
\frac{d\delta x}{dt} = \frac{\partial f}{\partial x}\delta x + \Delta_u f, \quad \delta x(0) = 0; \\
\frac{d\delta y}{dt} = \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \Delta_u F, \quad \delta y(0) = 0; \\
\frac{d\delta\rho}{dt} = -\delta\rho \cdot \mathrm{div}_y F - \rho\frac{d(\mathrm{div}_y \delta y)}{dt}, \quad \delta\rho(0) = 0.
\end{gathered} \tag{12}$$

Also let us introduce the variation equation for $\mathrm{div}_y \delta y$

$$\frac{d(\mathrm{div}_y \delta y)}{dt} = \frac{\partial(\mathrm{div}_y F)}{\partial x}\delta x + \frac{\partial(\mathrm{div}_y F)}{\partial y}\delta y + \Delta_u \mathrm{div}_y F, \quad \mathrm{div}_y \delta y(0) = 0. \tag{13}$$

Here and further operator Δ_u of some function f is defined the following way:

$$\Delta_u f\left(t,x,u\right) = f(t,x,u+\Delta u) - f\left(t,x,u\right).$$

The variation of the functional represented by a smooth function is

$$\delta I_1 = \int\limits_0^T \frac{\partial \varphi_1}{\partial x}\delta x dt + \frac{\partial g(x(T))}{\partial x}\delta x(T). \tag{14}$$

Variation of the functional $I_2(u)$ considering the particles distribution density is [7]

$$\delta I_2 = \max_{y_0 \in R_T(u)} \left[\frac{\partial \varphi_2}{\partial y}\delta y(T) + \frac{\partial \varphi_2}{\partial \rho}\delta \rho(T)\right], \tag{15}$$

where $R_T(u)$ is a set defined by expression

$$R_T(u) = \{\bar{y}_0 : \bar{y}_0 \in M_0, \varphi_2(y(T,x_0,\bar{y}_0,u),\rho) = \max_{y_0 \in M_0} \varphi_2(y(T,x_0,y_0,u),\rho)\}.$$

The variation of the functional (11) is

$$\delta I = \delta I_1 + \delta I_2. \tag{16}$$

Using (12)–(15) let us choose auxiliary vector-functions ψ, λ and scalar function χ so that

$$\begin{gathered}
\psi^{*'} + \psi^* \frac{\partial f}{\partial x} = \frac{\partial \varphi_1}{\partial x} - \lambda^* \frac{\partial F}{\partial x} + \chi\rho \frac{\partial(\mathrm{div}_y F)}{\partial x}, \quad \psi^*(T) = -\frac{\partial g(x(T))}{\partial x};\\
\lambda^{*'} + \lambda^* \frac{\partial F}{\partial y} = \chi\rho \frac{\partial(\mathrm{div}_y F)}{\partial y}, \quad \lambda^*(T) = -\frac{\partial \varphi_2(y_T,\rho_T)}{\partial y};\\
\chi' = \chi \mathrm{div}_y F, \quad \chi(T) = -\frac{\partial \varphi_2(y_T,\rho_T)}{\partial \rho}.
\end{gathered} \tag{17}$$

Here and further symbol * stands for the operation of transposition of a vector or matrix.

The variation of the functional (16) using expressions (17) can be written as follows:

$$\delta I(u) = \max_{y_0 \in R_T(u)} - \int\limits_0^T (\psi^* \Delta_u f + \lambda^* \Delta_u F - \chi\rho\Delta_u \mathrm{div}_y F) dt. \tag{18}$$

Expression (18) can be used for the directional methods of optimization.

7. Optimality condition. Let us introduce Hamilton's function

$$H(t,x,y,\rho,\psi,\lambda,\chi,u) = \psi^* f(t,x,u) + \lambda^* F(t,x,y,u) - \chi\rho \mathrm{div}_y F. \tag{19}$$

Using (19) we can rewrite the expression for the variation (18), so that

$$\delta I(u) = \max_{y_0 \in R_T(u)} - \int\limits_0^T \big(H(t,x,y,\rho,\psi,\lambda,\chi,\tilde{u}) - H(t,x,y,\rho,\psi,\lambda,\chi,u)\big) dt,$$

where $\tilde{u}(t) = u(t) + \Delta u \in U$.

Optimal control $u^0 = u^0(t)$, optimal trajectories $x_t^0 = x^0(t)$, $y_t^0 = y^0(t)$ and distribution density on the optimal trajectories $\rho_t^0 = \rho^0(t, y_t^0)$ comprise the so-called optimal process.

Theorem. *If $u^0 = u^0(t)$ is the optimal control, then for all $t \in [0, T]$ except for the discontinuity points of the control function we have*

$$\min_{u \in U} \max_{y_0 \in R_T(u^0)} (H(t, x_t^0, y_t^0, \rho_t^0, \psi_t^0, \lambda_t^0, \chi_t^0, u) - H(t, x_t^0, y_t^0, \rho_t^0, \psi_t^0, \lambda_t^0, \chi_t^0, u^0)) = 0,$$

where $\psi_t^0, \lambda_t^0, \chi_t^0$ are found from equations (17) *alongside the optimal process.*

8. Conclusion. The study of the combination of smooth and non-smooth functionals considering the distribution density of the particles leads us to the conclusion that in the problem of simultaneous optimization of the program motion and disturbed motions not only $y(t)$ depends on the program motion due to the setting of the problem (6), but also $x(t)$ turns out to be affected by the dynamics of the ensemble of the trajectories in the optimization process as can be seen from equations for the auxiliary functions (17).

Simultaneous use of smooth and non-smooth functionals in the problem of optimal control allows to perform optimization not only for the averaged values, but also considering the most deviating particles. The obtained expression for the variation of the functional can be used for directional methods of minimization in various applications, in this particular paper the application for the mathematical model of the charged particle beam dynamics in RFQ structures was considered.

The next steps in the development of this approach in applications to accelerating structures include considering the interaction of the charged particles and constructing more complex functionals considering wider range of characteristics of the dynamics of a charged particle beam.

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