# ПРОЦЕССЫ УПРАВЛЕНИЯ

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## A LYAPUNOV MATRIX BASED STABILITY CRITERION FOR A CLASS OF TIME-DELAY SYSTEMS*

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This paper is devoted to the stability analysis of linear time-invariant systems with multiple delays. First, we recover some basic elements of our research. Namely, we introduce the complete type functionals, the delay Lyapunov matrix, and a space of special functions that allow to present a family of necessary stability conditions. Then, we prove a sufficient stability condition (instability condition) in terms of a quadratic Lyapunov—Krasovskii functional. Summarizing these results, we finally obtain an exponential stability criterion for a class of linear time-delay systems. The criterion requires only a finite number of mathematical operations to be tested and depends uniquely on the delay Lyapunov matrix. Refs 15.

*Keywords*: time-delay system, Lyapunov matrix, stability criterion.

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## КРИТЕРИЙ УСТОЙЧИВОСТИ ДЛЯ ОДНОГО КЛАССА СИСТЕМ С ЗАПАЗДЫВАНИЕМ, ОСНОВАННЫЙ НА МАТРИЦЕ ЛЯПУНОВА*

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Статья посвящена анализу устойчивости линейных стационарных систем с несколькими запаздываниями. Сначала мы вводим известные элементы, на которых базируется наше исследование, а именно: квадратичные функционалы Ляпунова—Красовского полного типа, матрицу Ляпунова для систем с запаздыванием и пространство некоторых функций, которые позволяют получить семейство необходимых условий устойчивости. Эти функции строятся на основе значений фундаментальной матрицы на некотором конечном отрезке. Затем доказываем достаточное условие устойчивости (условие неустойчивости), выраженное через специальный квадратичный функционал Ляпунова—Красовского: показываем, что для неустойчивой системы найдется точка из некоторого компактного бесконечномерного множества, значение функционала в которой отрицательно. Суммируя данные результаты, в итоге получаем критерий экспоненциальной устойчивости для некоторого класса линейных систем с запаздыванием. Проверка критерия требует лишь конечного числа математических операций, а проверяемые условия зависят только от матрицы Ляпунова для систем с запаздыванием. Библиогр. 15 назв.

*Ключевые слова*: система с запаздыванием, матрица Ляпунова, критерий устойчивости.

**1. Introduction.** The importance of the Lyapunov—Krasovskii functionals [1] in the stability analysis of time-delay systems is well known. The main ideas in the construction of this class of functionals lies in the early contributions by [2–4]. A decisive impulse was given in the past decade in [5], where the so-called Lyapunov—Krasovskii functionals of complete type were introduced. The main characteristics of this class of functionals is that they have a quadratic lower bound, and they are determined by the delay Lyapunov matrix, solution of the dynamic, symmetry and algebraic properties. For a deeper study on functionals of complete type and the delay Lyapunov matrix, the reader is referred to the book [6].

The availability of the analogous of the Lyapunov matrix for time-delay systems has allowed the extension of well-known results for delay free systems to the time-delay case (see, Section 2.12 in [6]). One of the most interesting problems is the extension of the stability criterion for linear systems, which depends on the positivity of the Lyapunov matrix $V$, solution of the Lyapunov equation $A^TV + VA = -W$. The extension of this result to the time-delay case has been object of study in numerous contributions in the last years. See, for instance, the necessary stability conditions depending on the delay Lyapunov matrix introduced in [7] for systems with pointwise delays, in [8] for distributed delays and in [9] for the neutral-type case, also the stability criteria in [10, 11], for retarded and neutral type scalar equations, respectively.

A first criterion depending uniquely on the delay Lyapunov matrix systems with multiple pointwise delays was presented in [12] (see also [13]). However, the sufficiency part is only theoretical, as one has to make an infinite number of mathematical operations in order to check the stability condition (in this case, adopting the terminology used in [14], we say that the stability criterion is *infinite*). A first attempt to overcome this problem was recently reported in [14]. There, a new stability criterion depending on the delay Lyapunov matrix and the fundamental matrix of the system is introduced. The main distinctive feature of this criterion is that it is *finite*, i. e. a finite number of mathematical operations is required to test the stability condition. Nonetheless, the introduction of the fundamental matrix demands a greater computational effort in the test of the conditions.

Inspired by the results in [14], we present a new finite necessary and sufficient stability conditions for a class of retarded type systems depending uniquely on the delay Lyapunov matrix. The new criterion is constrained to systems with sufficiently small parameter values and delays. However, despite such restriction, it differs from the previously mentioned results in two aspects: (i) in contrast with [12], the stability criterion presented in this contribution is finite and (ii) unlike [14], it does not require the computation of the fundamental matrix, as it only depends on the delay Lyapunov matrix.

The rest of the contribution is organized as follows. The Lyapunov—Krasovskii framework and some basic definitions are introduced in the next section. In Section 3 the problem formulation is presented. Some auxiliary results concerning instability conditions are provided in Section 4. In Section 5 the main result of the contribution is presented: necessary and sufficient stability condition for a class of retarded type systems. Finally we conclude with some remarks and future work.

Throughout the paper, $\mathbb{PC}\left([-H,0],\mathbf{R}^n\right)$ and $\mathbb{C}^{(1)}\left([-H,0],\mathbf{R}^n\right)$ denotes the space of piecewise continuous and continuously differentiable vector functions defined on the interval $[-H,0]$, respectively. The maximum and minimum eigenvalues of a matrix $Q$ are represented by $\lambda_{\max}(Q)$ and $\lambda_{\min}(Q)$, respectively. The notation $Q>0$ means that the matrix $Q$ is positive definite and $\|\cdot\|$ stands for the standard Euclidian vector norm. The square block matrix with $i$-th row and $j$-th column element $Q_{ij}$ is represented by $[Q_{ij}]_{i,j=1}^{r}$.

**2. Preliminary results.** Consider the retarded type system

$$\dot{x}(t)=\sum_{i=0}^{m}A_i x(t-h_i), \tag{1}$$

where $0=h_0<h_1<\ldots<h_m=H$ and $A_i\in\mathbf{R}^{n\times n}$. The initial function $\varphi$ belongs to the space $\mathbb{PC}\left([-H,0],\mathbf{R}^n\right)$. The restriction of the solution $x(t,\varphi)$, $t\geqslant -H$, of system (1) (such that $x(\theta,\varphi)=\varphi(\theta)$, $\theta\in[-H,0]$) on the interval $[t-H,t]$, $t\geqslant 0$, is represented by

$$x_t(\varphi):\theta\to x(t+\theta,\varphi),\ \ \theta\in[-H,0].$$

We equip the set of piecewise continuous initial functions with the norm

$$\|\varphi\|_H=\sup_{\theta\in[-H,0]}\|\varphi(\theta)\|.$$

**Definition 1.** System (1) is exponentially stable if there exist constants $\mu>0$ and $\sigma>0$ such that

$$\|x(t,\varphi)\|\leqslant \mu e^{-\sigma t}\|\varphi\|_H,\quad t\geqslant 0.$$

The fundamental matrix of system (1), denoted by $K$, satisfies the equation

$$\dot{K}(t)=\sum_{i=0}^{m}A_iK(t-h_i),\quad t>0,$$

with the initial conditions $K(0)=I$ and $K(\theta)=0$ for $\theta<0$.

Let us introduce the delay Lyapunov matrix.

**Definition 2.** The delay Lyapunov matrix $U(\tau)$, $\tau\in[-H,H]$, associated with a symmetric matrix $W$ is a continuous matrix function that satisfies the following equations:

$$U'(\tau)=\sum_{i=0}^{m}U(\tau-h_i)A_i,\ \ \tau\in(0,H),$$

$$U^T(\tau)=U(-\tau),\ \ \tau\in[-H,H],$$

$$U'(+0)-U'(-0)=-W,$$

called the dynamic, symmetry and algebraic properties, respectively. Here $U'(+0)$ and $U'(-0)$ denote the corresponding one-sided limits.

The existence and uniqueness of matrix $U$ depends on the Lyapunov condition. This result is recalled in the next theorem (see [6]).

**Theorem 1.** *The delay Lyapunov matrix U associated with a matrix W exists and is unique if and only if the Lyapunov condition holds, i. e. if the spectrum of system* (1)

$$\Lambda = \left\{ s \,\middle|\, \det\left( sI - \sum_{i=0}^{m} A_i e^{-sh_i} \right) = 0 \right\}$$

*does not contain a root $\widehat{s}$ such that $-\widehat{s}$ also belongs to the spectrum.*

In [5] a functional, such that

$$\frac{d}{dt} v_0(x_t(\varphi)) = -x^T(t,\varphi) W x(t,\varphi), \quad W > 0, \quad \varphi \in \mathbb{PC}([-H,0],\mathbf{R}^n), \tag{2}$$

is introduced and is determined by the delay Lyapunov matrix as follows:

$$v_0(\varphi) = \varphi^T(0)U(0)\varphi(0) + 2\varphi^T(0)\sum_{i=1}^{m}\int_{-h_i}^{0} U(-h_i-\theta)A_i\varphi(\theta)d\theta +$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-h_i}^{0}\int_{-h_j}^{0} \varphi^T(\theta_1)A_i^T U(\theta_1-\theta_2+h_i-h_j)A_j\varphi(\theta_2)d\theta_2 d\theta_1,$$

$$\varphi \in \mathbb{PC}\left([-H,0],\mathbf{R}^n\right).$$

A functional based on $v_0$ that plays a key role for obtaining necessary and sufficient stability condition depending on the delay Lyapunov matrix is presented next [7, 12]:

$$v_1(\varphi) = v_0(\varphi) + \int_{-H}^{0} \varphi^T(\theta)W\varphi(\theta)d\theta, \quad \varphi \in \mathbb{PC}\left([-H,0],\mathbf{R}^n\right). \tag{3}$$

Its derivative along the solutions of system (1) satisfies

$$\frac{d}{dt} v(x_t(\varphi)) = -x^T(t-H,\varphi) W x(t-H,\varphi), \quad \varphi \in \mathbb{PC}([-H,0],\mathbf{R}^n).$$

We introduce now the following bilinear functional:

$$z(\varphi_1,\varphi_2) = \varphi_1^T(0)U(0)\varphi_2(0) +$$

$$+ \varphi_1^T(0)\sum_{i=1}^{m}\int_{-h_i}^{0} U(-h_i-\theta)A_i\varphi_2(\theta)d\theta + \sum_{i=1}^{m}\int_{-h_i}^{0} \varphi_1^T(\theta)A_i^T U(h_i+\theta)d\theta\varphi_2(0) +$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-h_i}^{0}\int_{-h_j}^{0} \varphi_1^T(\theta_1)A_i^T U(\theta_1-\theta_2+h_i-h_j)A_j\varphi_2(\theta_2)d\theta_2 d\theta_1 +$$

$$+ \int_{-H}^{0} \varphi_1^T(\theta)W\varphi_2(\theta)d\theta, \quad \varphi_1,\varphi_2 \in \mathbb{PC}\left([-H,0],\mathbf{R}^n\right).$$

Notice that this bilinear functional is related to the functional $v_1$ by the equality $v_1(\varphi) = z(\varphi, \varphi)$. In the next lemma (see [14]) we recall some results concerning the quadratic upper bounds of the functional $v_1$ and the bilinear functional $z$.

**Lemma 1.** *There exists a positive number $\beta$ such that*

$$v_1(\varphi) \leqslant \beta \|\varphi\|_H^2, \quad \varphi \in \mathbb{PC}\left([-H, 0], \mathbf{R}^n\right),$$

$$|z(\varphi_1, \varphi_2)| \leqslant \beta \|\varphi_1\|_H \|\varphi_2\|_H, \quad \varphi_1, \varphi_2 \in \mathbb{PC}\left([-H, 0], \mathbf{R}^n\right).$$

**3. Problem statement.** Consider the following function that depends on the fundamental matrix:

$$\psi_r(\theta) = \sum_{i=1}^{r} K(\tau_i + \theta)\gamma_i, \quad \theta \in [-H, 0], \tag{4}$$

where $\gamma_i \in \mathbf{R}^n$, $\tau_i \in [0, H]$, $i = \overline{1, r}$.

Based on new properties that connect the fundamental matrix $K$ with the delay Lyapunov matrix $U$, and the introduction of the functional $z(\cdot, \cdot)$, the next equality has been provided in [7]:

$$v_1(\psi_r) = \gamma^T \left[U(\tau_j - \tau_i)\right]_{i,j=1}^{r} \gamma \tag{5}$$

with $\gamma = \begin{pmatrix} \gamma_1^T & \dots & \gamma_r^T \end{pmatrix}^T$.

Take equidistant points from the segment $[0, H]$:

$$\tau_i = \frac{i-1}{r-1} H, \quad i = \overline{1, r}.$$

For these points introduce block-matrix

$$\mathcal{K}_r = \left[U(\tau_j - \tau_i)\right]_{i,j=1}^{r} = \left[U\left(\frac{j-i}{r-1} H\right)\right]_{i,j=1}^{r}$$

for $r \geqslant 2$ and separately $\mathcal{K}_1 = U(0)$. To make the notation more clear, we present three particular cases:

$$\mathcal{K}_2 = \begin{pmatrix} U(0) & U(H) \\ * & U(0) \end{pmatrix},$$

$$\mathcal{K}_3 = \begin{pmatrix} U(0) & U(H/2) & U(H) \\ * & U(0) & U(H/2) \\ * & * & U(0) \end{pmatrix},$$

$$\mathcal{K}_4 = \begin{pmatrix} U(0) & U(H/3) & U(2H/3) & U(H) \\ * & U(0) & U(H/3) & U(2H/3) \\ * & * & U(0) & U(H/3) \\ * & * & * & U(0) \end{pmatrix}.$$

Stars here denote the blocks, which are obvious, as we deal with the symmetric matrices.

In [12] an infinite stability criterion for system (1) is introduced.

**Theorem 2.** *Assume that the Lyapunov condition holds. System* (1) *is exponentially stable if and only if for every natural number $r$, the following holds*:

$$\mathcal{K}_r > 0. \tag{6}$$

The result is based on the fact that one can make an arbitrarily close approximation of any continuous function by the function $\psi_r$ given by (4) for sufficiently large $r$. This criterion can be successfully used as a necessary stability condition to discard unstable systems, but checking of the stability requires an infinite number of mathematical operations, which leads us to formulate the problem addressed in this contribution.

**Problem 1.** *Find an estimate of the number $r$ for which condition* (6) *of Theorem 2 is necessary and sufficient for the stability of system* (1).

**4. Auxiliary results.** Consider the compact set [14]

$$\mathcal{S} = \left\{ \varphi \in \mathbb{C}^{(1)}\left([-H,0],\mathbf{R}^n\right) \quad \|\varphi\|_H = \|\varphi(0)\| = 1, \quad \|\varphi'(\theta)\| \leqslant M, \quad \theta \in [-H,0] \right\},$$

where $M = \sum\limits_{i=0}^{m} \|A_i\|$. We introduce some instrumental results that allow us to obtain the main contribution of the paper. The first one concerns the approximation of a function $\varphi$ of the compact set $\mathcal{S}$ by the function $\psi_r$ given by (4). This approximation has been introduced in [14] and is included here for the sake of completeness. The second one is inspired by [15] and corresponds to an instability condition of system (1) based on the functional $v_1$.

***4.1. Approximation of the set $\mathcal{S}$.*** Consider an arbitrary initial function $\varphi$ from the set $\mathcal{S}$ and the function $\psi_r$ of the form (4), which satisfies the following equalities:

$$\psi_r(-\tau_i) = \varphi(-\tau_i), \quad i = \overline{1,r},$$

with $\tau_i = (i-1)H/(r-1)$, $i = \overline{1,r}$. Such function always exists [14], and vectors $\gamma_i$, $i = \overline{1,r}$, can be computed iteratively. The next lemma (see [14]) gives an estimate of the approximation error between $\varphi \in \mathcal{S}$ and $\psi_r$, denoted by $R_r = \varphi - \psi_r$.

**Lemma 2.** *For every $\varphi \in \mathcal{S}$,*

$$\|R_r\|_H = \|\varphi - \psi_r\|_H \leqslant \varepsilon_r,$$

*where*

$$\varepsilon_r = \frac{(M+L)e^{LH}}{1/\delta_r + L}.$$

*Here $\delta_r = \dfrac{H}{r-1}$ and $L$ is the Lipschitz constant for the fundamental matrix on $[0,H]$, i. e. it is such that $\|K'(t)\| \leqslant L$ on $[0,H]$ almost everywhere.*

***4.2. Instability condition.*** The next lemma is useful for proving the main result of this section.

**Lemma 3.** *Let $P$ and $Q$ be real matrices. If $\det(P+iQ) = 0$, then there exist two vectors $C_1$ and $C_2$ such that*

1) $(P+iQ)(C_1+iC_2) = 0$;
2) $\|C_1\| = 1$;
3) $\|C_2\| \leqslant 1$;
4) $C_1^T C_2 = 0$. *Here $i$ is the imaginary unit.*

P r o o f. Since $\det(P+iQ) = 0$, there exists a complex vector $\xi_1 + i\xi_2 \neq 0$, such that

$$(P+iQ)(\xi_1 + i\xi_2) = 0.$$

Let us introduce now the following vectors:

$$\widehat{C}_1 = \xi_1 + b\xi_2,$$

$$\widehat{C}_2 = -b\xi_1 + \xi_2,$$

where $b$ is a real number, and observe that

$$(P + iQ)\left(\widehat{C}_1 + i\widehat{C}_2\right) = 0.$$

Consider the product

$$\widehat{C}_1^T \widehat{C}_2 = (1 - b^2)\xi_1^T \xi_2 - b\left(\|\xi_1\|^2 - \|\xi_2\|^2\right).$$

We need to choose $b$ to guarantee that $\widehat{C}_1^T \widehat{C}_2 = 0$. If $\xi_1^T \xi_2 = 0$ one can take $b = 0$, otherwise, one can take any real solution of the quadratic equation, which always has two:

$$b^2 + b\frac{\|\xi_1\|^2 - \|\xi_2\|^2}{\xi_1^T \xi_2} - 1 = 0.$$

Now, at least one of the vectors $\widehat{C}_1$ and $\widehat{C}_2$ is nonzero and the desired vectors $C_1$ and $C_2$ can be constructed as follows: if $\|\widehat{C}_1\| \geqslant \|\widehat{C}_2\|$, we have

$$C_1 = \frac{\widehat{C}_1}{\|\widehat{C}_1\|}, \quad C_2 = \frac{\widehat{C}_2}{\|\widehat{C}_1\|},$$

and if $\|\widehat{C}_1\| < \|\widehat{C}_2\|$,

$$C_1 = \frac{\widehat{C}_2}{\|\widehat{C}_2\|}, \quad C_2 = -\frac{\widehat{C}_1}{\|\widehat{C}_2\|}.$$

□

We provide an instability condition for system (1) depending on the functional $v_1$. The result is based on [15], where the condition is obtained for the functional $v_0$.

**Lemma 4.** *Assume that system* (1) *is such that*

$$\frac{\lambda_{\min}(W)}{\lambda_{\max}(W)} > 4MH, \tag{7}$$

*where the matrix $W$ is the one to which the delay Lyapunov matrix is associated. If system* (1) *is unstable, there exists $\varphi \in \mathcal{S}$, such that*

$$v_1(\varphi) \leqslant -\alpha_1,$$

*here* $\alpha_1 = \dfrac{\lambda_{\min}(W)}{4M} - \lambda_{\max}(W)H > 0.$

P r o o f. As system (1) is unstable, there exist an eigenvalue $\lambda = \alpha + i\beta$ with $\alpha > 0$, and two vectors $C_1$, $C_2 \in \mathbf{R}^n$ that satisfy the conditions of Lemma 3 such that the following expression is a solution of system (1):

$$x(t, \varphi) = e^{\alpha t}\phi(t), \quad \phi(t) = \cos\beta t C_1 - \sin\beta t C_2, \quad t \in (-\infty, \infty),$$

which corresponds to the initial function

$$\varphi(\theta) = x(\theta, \varphi), \quad \theta \in [-H, 0].$$

We prove first that $\varphi \in \mathcal{S}$. By Lemma 3, it is easy to see that $\|\varphi(0)\| = 1$ and

$$\max_{t\in\mathbf{R}} \|\phi(t)\| = 1.$$

Indeed, $\|\phi(t)\|^2 = \|C_1\|^2 + (\|C_2\|^2 - \|C_1\|^2)\sin^2\beta t \leqslant 1$.

Thus,

$$\|x(\theta)\| = e^{\alpha\theta}\,\|\phi(\theta)\| \leqslant 1, \quad \theta \leqslant 0.$$

Finally, since $x(t,\varphi)$ is a solution of system (1), we get

$$\|\varphi'(t)\| = \|\dot{x}(t)\| \leqslant \sum_{i=0}^{m} \|A_i\|\|x(t-h_i)\| \leqslant M\|\varphi(0)\| = M, \quad t \in [-H, 0].$$

We focus next on the functional $v_0$. Let us set $T = 2\pi/\beta$ for $\beta \neq 0$ and $T = 1$ for $\beta = 0$. It is easy to see that $T$ is a period of $\phi(t)$. By equation (2),

$$\frac{d}{dt} v_0(x_t(\varphi)) = -x^T(t,\varphi)Wx(t,\varphi), \quad t \geqslant 0,$$

which implies that

$$v_0(\varphi) = v_0(x_T) + \int\limits_0^T x^T(t,\varphi)Wx(t,\varphi)dt.$$

As $x(T+\theta) = e^{\alpha T}\varphi(\theta)$,

$$v_0(x_T(\varphi)) = e^{2\alpha T} v_0(\varphi),$$

hence,

$$v_0(\varphi) = -\frac{1}{e^{2\alpha T} - 1}\int\limits_0^T x^T(t,\varphi)Wx(t,\varphi)dt \leqslant -\frac{\lambda_{\min}(W)}{(e^{2\alpha T} - 1)}\int\limits_0^T \|x(t)\|^2 dt.$$

By Lemma 3,

$$\int\limits_0^T \|x(t)\|^2 dt = \left(\int\limits_0^T e^{2\alpha t}\left(\cos^2(\beta t)\|C_1\|^2 + \sin^2(\beta t)\|C_2\|^2\right) dt\right) \geqslant \int\limits_0^T e^{2\alpha t}\cos^2(\beta t)dt.$$

Solving the right hand side integral, we obtain

$$\int\limits_0^T e^{2\alpha t}\cos^2(\beta t)dt = \frac{1}{4\alpha}(e^{2\alpha T} - 1) + \frac{1}{4(\alpha^2+\beta^2)}\left(e^{2\alpha T}(\alpha\cos(2\beta T) + \beta\sin(2\beta T)) - \alpha\right) =$$

$$= \frac{(2\alpha^2+\beta^2)\left(e^{2\alpha T} - 1\right)}{4\alpha(\alpha^2+\beta^2)} \geqslant \frac{e^{2\alpha T} - 1}{4\alpha}.$$

Taking into account that the spectral abscissa can be estimated by $M = \sum_{i=0}^m \|A_i\|$, we arrive at

$$v_0(\varphi) \leqslant -\frac{\lambda_{\min}(W)}{4\alpha} \leqslant -\frac{\lambda_{\min}(W)}{4M}.$$

Finally, from expression (3) for the functional $v_1$, it follows that

$$v_1(\varphi) \leqslant -\frac{\lambda_{\min}(W)}{4M} + \lambda_{\max}(W)H = -\alpha_1.$$

□

**5. Necessary and sufficient conditions.** We provide the main result of the contribution, namely, finite necessary and sufficient stability conditions for the system of the form (1) that satisfies inequality (7).

**Theorem 3.** *Assume that the Lyapunov condition holds and system* (1) *satisfies* (7). *System* (1) *is exponentially stable if and only if*

$$\mathcal{K}_r > 0$$

*for*

$$r = 1 + H\left(e^{LH}(M+L)\left(\alpha + \sqrt{\alpha(\alpha+1)}\right) - L\right), \tag{8}$$

*where* $\alpha = \beta/\alpha_1$.

P r o o f. The necessity directly follows from Theorem 2. In order to prove the sufficiency, we assume by contradiction that system (1) is unstable but the Lyapunov condition and the conditions of the theorem hold. Consider $\varphi \in \mathcal{S}$ and notice that

$$v_1(\varphi) = v_1(R_r + \psi_r) = v_1(\psi_r) + 2z(\varphi, R_r) - v_1(R_r).$$

By Lemmas 1 and 4,

$$v_1(\psi_r) \leqslant -\alpha_1 + 2\beta\|R_r\|_H + \beta\|R_r\|_H^2.$$

By using Lemma 2 and considering $r$ given by (8), we have that

$$\|R_r\|_H \leqslant \frac{(M+L)e^{LH}}{1/\delta_r + L} \leqslant \frac{\alpha_1}{\sqrt{\beta(\beta+\alpha_1)} + \beta},$$

which implies that

$$-\alpha_1 + 2\beta\|R_r\|_H + \beta\|R_r\|_H^2 \leqslant 0.$$

Finally, from the previous inequality and equation (5), we get

$$v_1(\psi_r) = \gamma^T \mathcal{K}_r \gamma \leqslant 0.$$

The obtained contradiction finishes the proof. □

**Remark.** *The number $r$ given by* (8) *has the same form as the estimate obtained in* [14]. *The only difference lies in the computation of $\alpha$ as in this case $\alpha_1$ does not come from the quadratic lower bound of the functional $v_1$, but from the instability result given in Lemma 4.*

Although Theorem 3 partially solves Problem 1 by giving a finite stability criterion, it only works for a constrained class of systems of the form (1). However, in contrast with the finite stability criterion introduced in [14], here one does not need the computation of the fundamental matrix, which increases the computational effort and seems not to be a trivial task in the non-commensurate delays case. Obviously, the constraints introduced by inequality (7) can be relaxed if one considers a positive definite matrix $W$ such that $\lambda_{\min}(W) = \lambda_{\max}(W)$, or uses less conservative estimate on $\alpha_1$ in the proof of Lemma 4.

**6. Conclusions.** A new stability criterion for a class of time delay systems is introduced in the framework of the Lyapunov—Krasovskii functionals. The main characteristic of this criterion is that it depends exclusively on the delay Lyapunov matrix and only requires a finite number of mathematical operations to be checked. Future research directions include the generalization to systems without restrictions in the parameter values and the extension to the neutral-type case.

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