ПРОЦЕССЫ УПРАВЛЕНИЯ

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THE HYPODIFFERENTIAL DESCENT METHOD IN THE PROBLEM OF CONSTRUCTING AN OPTIMAL CONTROL*

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This paper considers the problem of optimal control of an object, whose motion is described by a system of ordinary differential equations. The original problem is reduced to the problem of unconstrained minimization of a nonsmooth functional. For this, the necessary minimum conditions in terms of subdifferential and hypodifferential are determined. A class of problems, for which these conditions are also sufficient, is distinguished. On the basis of these conditions, the subdifferential descent method and the hypodifferential descent method are applied to the considered problem. The application of the methods is illustrated by numerical examples. Refs 16. Tables 4.

Keywords: nonsmooth functional, variational problem, program control, hypodifferential descent method.

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МЕТОД ГИПОДИФФЕРЕНЦИАЛЬНОГО СПУСКА В ЗАДАЧЕ ПОСТРОЕНИЯ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ

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В статье рассматривается задача оптимального управления объектом, движение которого описывается системой обыкновенных дифференциальных уравнений. Исходная задача сводится к задаче безусловной минимизации некоторого негладкого функционала. Для него найдены необходимые условия минимума в терминах субдифференциала и гиподифференциала. Выделен класс задач, для которых эти условия оказываются и достаточными. На основании данных условий к изучаемой задаче применяются метод субдифференциального спуска и метод гиподифференциального спуска. Приложение методов иллюстрируется на численных примерах. Библиогр. 16 назв. Табл. 4.

Ключевые слова: негладкий функционал; вариационная задача, оптимальное управление, метод гиподифференциального спуска.

Introduction. The technique of exact penalty functions was firstly used in the optimal control problems in [1, 2]. The general idea of such an approach is reduction of

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the original problem with restrictions to the unconstrained minimization of a nonsmooth functional. For this problem one should use nonsmooth optimization methods. The subdifferential descent method and the hypodifferential descent method belong to this class of methods.

The methods used in the paper may be refered to the direct methods of optimal control problems, since the optimization problem in functional space is being solved without necessity for integration of the system, which describes the controlled object. Among the vast arsenal of optimal control problem solving methods an approach based on the variations of the minimized functional is similar with the considered in the article method (see [3–6]).

In this paper the integral restriction on control is considered. Optimal control problems with such constraints were studied in some works, for example [7–9].

Approach used in the article is especially appropriate when it is important to take into account precisely the limitation on the final position of the object and the restriction in the form of differential equalities. It is therefore of interest in the spread of the use of exact penalties over optimal control problems with state constraints, the exact adherence of which is principal in many practical problems.

Statement of the problem. Let us consider a system of ordinary differential equations in normal form

$$\dot{x}(t) = f(x, u, t), \quad t \in [0, T]. \tag{1}$$

It is required to find such a control $u^* \in P_m[0, T]$, satisfying an integral restriction

$$\int_0^T \big(u(t), u(t)\big) dt \leqslant 1, \tag{2}$$

which brings system (1) from the given initial position

$$x(0) = x_0 \tag{3}$$

to the given final state

$$x(T) = x_T \tag{4}$$

and minimizes the integral functional

$$I(x, u) = \int_0^T f_0(x, \dot{x}, u, t) dt. \tag{5}$$

Suppose that there exists an optimal control u^*. In system (1) $T > 0$ is a given moment of time, $f(x, u, t)$ is a real n-dimensional vector-function, $x(t)$ is an n-dimensional vector-function of the phase coordinates, which is supposed to be continuous with partially continuous in the interval $[0, T]$ derivative, $u(t)$ is an m-dimensional vector-function of control, which is supposed to be partially continuous in $[0, T]$. We consider $f(x, u, t)$ to be continuously differentiable in x and u and continuous in all three of its arguments.

If $t_0 \in [0, T)$ is a discontinuity point of the vector-function $u(t)$, then we put

$$u(t_0) = \lim_{t \downarrow t_0} u(t). \tag{6}$$

At the point T we assume that

$$u(T) = \lim_{t \uparrow T} u(t). \tag{7}$$

We consider that $\dot{x}(t_0)$ is a right-handed derivative of the vector-function x at the point t_0, $\dot{x}(T)$ is a left-handed derivative of the vector-function x at the point T.

In functional (5) $f_0(x, \dot{x}, u, t)$ is a real scalar function, which is supposed to be continuously differentiable in x, $\dot{x}$ and u and continuous in all four of its arguments.

Reduction to the variational problem. Put $z(t) = \dot{x}(t)$, $z \in P_n[0, T]$. Then from (3) we get $x(t) = x_0 + \int_0^t z(\tau)d\tau$. With regard to the vector-function $z(t)$ we make a suggestion, analogous to (6), (7). We have

$$f(x, u, t) = f\left(x_0 + \int_0^t z(\tau)d\tau, u, t\right),$$

$$f_0(x, z, u, t) = f\left(x_0 + \int_0^t z(\tau)d\tau, z, u, t\right).$$

Let us introduce the functional

$$F_\lambda(z, u) = I(z, u) + \lambda\left[\varphi(z, u) + \sum_{i=1}^n \psi_i(z) + \max\left\{0, \int_0^T (u(t), u(t))\, dt - 1\right\}\right], \qquad (8)$$

where

$$\varphi(z, u) = \sqrt{\int_0^T \big(z(t) - f\big(x, u, t\big), z(t) - f\big(x, u, t\big)\big) dt},$$

$$\psi_i(z) = \left|\overline{\psi}_i(z)\right|,\ \overline{\psi}_i(z) = x_{0i} + \int_0^T z_i(t)dt - x_{Ti},\ i = \overline{1, n},$$

and x_{0i} is an i-th component of the vector x_0, x_{Ti} is an i-th component of the vector x_T, $i = \overline{1, n}$, $\lambda > 0$ is some constant.

Denote

$$\Phi(z, u) = \varphi(z, u) + \sum_{i=1}^n \psi_i(z) + \max\left\{0, \int_0^T \big(u(t), u(t)\big)dt - 1\right\}. \qquad (9)$$

It is not difficult to see that functional (9) is nonnegative for all $z \in P_n[0, T]$ and for all $u \in P_m[0, T]$ and vanishes at a point $[\overline{z}, \overline{u}] \in P_n[0, T] \times P_m[0, T]$ if and only if the vector-function $\overline{u}(t)$ satisfies constraint (2), and the vector-function $\overline{x}(t) = x_0 + \int_0^t \overline{z}(\tau)d\tau$ satisfies system (1) at $u(t) = \overline{u}(t)$ and constraints (3), (4).

Let us introduce the sets

$$\Omega = \big\{[z, u] \in P_n[0, T] \times P_m[0, T] \;\big|\; \Phi(z, u) = 0\big\},$$

$$\Omega_\delta = \big\{[z,u] \in P_n[0,T] \times P_m[0,T] \;\big|\; \Phi(z,u) < \delta\big\},$$

here $\delta > 0$ is some number. Then

$$\Omega_\delta \setminus \Omega = \big\{[z,u] \in P_n[0,T] \times P_m[0,T] \;\big|\; 0 < \Phi(z,u) < \delta\big\}.$$

Using the same technique as in [1, 10], it can be shown that the following theorem takes place.

Theorem 1. *Suppose there exists such a positive number $\lambda_0 < \infty$ that $\forall \lambda > \lambda_0$ there exists a point $[z(\lambda), u(\lambda)] \in P_n[0,T] \times P_m[0,T]$, for which $F_\lambda\big(z(\lambda), u(\lambda)\big) = \inf\limits_{[z,u]} F_\lambda(z,u)$. Let the functional $I(z,u)$ be Lipschitz on the set $\Omega_\delta \setminus \Omega$. Then functional* (8) *will be an exact penalty function.*

Thus, under the assumptions of Theorem 1 there exists such a number $0 < \lambda^* < \infty$ that $\forall \lambda > \lambda_*$ the initial problem of minimization of functional (5) on the set Ω is equivalent to the problem of minimization of functional (8) on the whole space. Further we suppose that the number λ in functional (8) is fixed and the condition $\lambda > \lambda^*$ holds.

Lemma 1. *If system* (1) *is linear in the phase variables x and in control u, and the functional $I(z,u)$ is convex, then the functional $F_\lambda(z,u)$ is convex.*

P r o o f. Let us present functional (8) in the form

$$F_\lambda(z,u) = I(z,u) + \lambda\varphi(z,u) + \lambda F_1(z) + \lambda F_2(u),$$

where $I(z,u)$, $\varphi(z,u)$, $F_1(z)$, $F_2(u)$ are the corresponding summands from the right-hand side of (8). The functionals $F_1(z)$ and $F_2(u)$ are convex as maximum of convex functionals. The functional $I(z,u)$ is convex by the lemma assumption. Let us show the convexity of the functional $\varphi(z,u)$ in the case of the linearity of system (1).

Let system (1) be of the form

$$\dot{x} = A(t)x + B(t)u + c(t),$$

where $A(t)$ is an $n \times n$-matrix; $B(t)$ is an $n \times m$-matrix; $c(t)$ is an n-dimensional vector-function. Suppose $A(t)$, $B(t)$, $c(t)$ be real and continuous in $[0,T]$. Let $z_1, z_2 \in P_n[0,T]$, $u_1, u_2 \in P_m[0,T]$, $\alpha \in (0,1)$. Denote $\overline{\varphi}(z,u,t) = z(t) - f(z,u,t)$. We have

$$\varphi^2\big(\alpha(z_1,u_1) + (1-\alpha)(z_2,u_2)\big) = \big|\big|\alpha z_1(t) + (1-\alpha) z_2(t) \; -$$

$$- \; A(t)\big[x_0 + \int_0^t \big(\alpha z_1(\tau) + (1-\alpha) z_2(\tau)\big) d\tau\big] - B(t)\big[\alpha u_1(t) + (1-\alpha) u_2(t)\big] - c(t)\big|\big|^2 =$$

$$= \big|\big|\alpha\overline{\varphi}(z_1,u_1) + (1-\alpha)\overline{\varphi}(z_2,u_2)\big|\big|^2 = \alpha^2 \int_0^T \big(\overline{\varphi}(z_1,u_1,t), \overline{\varphi}(z_1,u_1,t)\big) dt \; + \quad (10)$$

$$+ \; 2\alpha(1-\alpha) \int_0^T \big(\overline{\varphi}(z_1,u_1,t), \overline{\varphi}(z_2,u_2,t)\big) dt + (1-\alpha)^2 \int_0^T \big(\overline{\varphi}(z_2,u_2,t), \overline{\varphi}(z_2,u_2,t)\big) dt,$$

$$\big(\alpha\varphi(z_1,u_1) + (1-\alpha)\varphi(z_2,u_2)\big)^2 = \alpha^2 \int_0^T \big(\overline{\varphi}(z_1,u_1,t), \overline{\varphi}(z_1,u_1,t)\big) dt \; +$$

$$+ 2\alpha(1-\alpha)\sqrt{\int\limits_0^T \big(\overline{\varphi}(z_1,u_1,t),\overline{\varphi}(z_1,u_1,t)\big)dt\int\limits_0^T \big(\overline{\varphi}(z_2,u_2,t),\overline{\varphi}(z_2,u_2,t)\big)dt} \;+$$

$$+ (1-\alpha)^2\int\limits_0^T \big(\overline{\varphi}(z_2,u_2,t),\overline{\varphi}(z_2,u_2,t)\big)dt. \qquad (11)$$

Using Hölder's inequality, for all z_1, z_2, u_1, u_2 one gets

$$\int\limits_0^T \big(\overline{\varphi}(z_1,u_1,t),\overline{\varphi}(z_2,u_2,t)\big)dt \leqslant$$

$$\leqslant \sqrt{\int\limits_0^T \big(\overline{\varphi}(z_1,u_1,t),\overline{\varphi}(z_1,u_1,t)\big)dt}\sqrt{\int\limits_0^T \big(\overline{\varphi}(z_2,u_2,t),\overline{\varphi}(z_2,u_2,t)\big)dt},$$

hence from (10) and (11) we obtain

$$\varphi^2\big(\alpha(z_1,u_1)+(1-\alpha)(z_2,u_2)\big) \leqslant \big(\alpha\varphi(z_1,u_1)+(1-\alpha)\varphi(z_2,u_2)\big)^2. \qquad (12)$$

Since $\varphi\big(\alpha(z_1,u_1)+(1-\alpha)(z_2,u_2)\big) \geqslant 0$, $\alpha\varphi(z_1,u_1)+(1-\alpha)\varphi(z_2,u_2) \geqslant 0$, then from inequality (12) $\forall\ z_1$, z_2, u_1, u_2 and $\alpha \in (0,1)$ follows:

$$\varphi\big(\alpha(z_1,u_1)+(1-\alpha)(z_2,u_2)\big) \leqslant \alpha\varphi(z_1,u_1)+(1-\alpha)\varphi(z_2,u_2),$$

that proves the convexity of the functional $\varphi(z,u)$ in the case of the original system linearity.

Now note that the functional $F_\lambda(z,u)$ is convex (in the case of the initial system linearity) as a sum of convex functionals.

Lemma 1 is proved.

Necessary minimum conditions. Let us introduce the sets

$$\Omega_1 = \left\{ z \in P_n[0,T] \;\Big|\; x_0 + \int\limits_0^T z(t)dt = x_T \right\},$$

$$\Omega_2 = \left\{ u \in P_m[0,T] \;\Big|\; \int\limits_0^T \big(u(t),u(t)\big)dt \leqslant 1 \right\},$$

$$\Omega_3 = \big\{ [z,u] \in P_n[0,T]\times P_m[0,T] \;\big|\; \varphi(z,u)=0 \big\}$$

and the following index sets:

$$I_0 = \{ i=\overline{1,n} \mid \overline{\psi}_i(z)=0 \},$$

$$I_- = \{ i=\overline{1,n} \mid \overline{\psi}_i(z)<0 \},$$

$$I_+ = \{ i=\overline{1,n} \mid \overline{\psi}_i(z)>0 \}.$$

Let us also introduce the control sets

$$U_0 = \left\{ u \in P_m[0,T] \;\Big|\; \int\limits_0^T \big(u(t), u(t)\big)dt - 1 = 0 \right\},$$

$$U_- = \left\{ u \in P_m[0,T] \;\Big|\; \int\limits_0^T \big(u(t), u(t)\big)dt - 1 < 0 \right\},$$

$$U_+ = \left\{ u \in P_m[0,T] \;\Big|\; \int\limits_0^T \big(u(t), u(t)\big)dt - 1 > 0 \right\}.$$

Using the same technique as in [1, 10], it is easy to see, that the following two theorems take place.

Theorem 2. *If $[z,u] \notin \Omega_3$, then the functional $F_\lambda(z,u)$ is subdifferentiable, and its subdifferential at the point $[z,u]$ is expressed by the formula*

$$\begin{aligned}\partial F_\lambda(z,u) = \Big\{ \Big[\int\limits_t^T \frac{\partial f_0}{\partial x} d\tau + \frac{\partial f_0}{\partial z} + \lambda\big[w(t) - \int\limits_t^T \Big(\frac{\partial f}{\partial x}\Big)' w(\tau)d\tau + \sum_{i\in I_0} \omega_i e_i + \sum_{j=1}^n \mu_j e_j\big], \\ \frac{\partial f_0}{\partial u} + \lambda\big[- \Big(\frac{\partial f}{\partial u}\Big)' w(t) + 2\nu u(t)\big]\Big] \;\Big|\; \omega_i \in [-1,1],\ i \in I_0, \\ \mu_j = 0,\ j \in I_0,\ \mu_j = 1,\ j \in I_+,\ \mu_j = -1,\ j \in I_-, \\ \nu \in [0,1],\ u \in U_0,\ \nu = 1,\ u \in U_+,\ \nu = 0,\ u \in U_-, \\ w(t) = \frac{z(t) - f(x,u,t)}{\varphi(z,u)} \Big\}.\end{aligned} \tag{13}$$

Theorem 3. *If $[z,u] \in \Omega_3$, then the functional $F_\lambda(z,u)$ is subdifferentiable, and its subdifferential at the point $[z,u]$ is expressed by the formula*

$$\begin{aligned}\partial F_\lambda(z,u) = \Big\{ \Big[\int\limits_t^T \frac{\partial f_0}{\partial x} d\tau + \frac{\partial f_0}{\partial z} + \lambda\big[v(t) - \int\limits_t^T \Big(\frac{\partial f}{\partial x}\Big)' v(\tau)d\tau + \sum_{i\in I_0} \omega_i e_i + \sum_{j=1}^n \mu_j e_j\big], \\ \frac{\partial f_0}{\partial u} + \lambda\big[- \Big(\frac{\partial f}{\partial u}\Big)' v(t) + 2\nu u(t)\big]\Big] \;\Big|\; v \in P_n[0,T],\ ||v|| \leqslant 1 \Big\}.\end{aligned} \tag{14}$$

In (14) $\omega_i \in [-1,1],\ i \in I_0,\ \mu_j,\ j = \overline{1,n},\ \nu$ *are defined by* (13).

Corollary 1. *If $[z,u] \in \Omega_3$, $z \in \Omega_1$, $u \in \Omega_2$, then the functional $F_\lambda(z,u)$ is subdifferentiable, and its subdifferential at the point $[z,u]$ is expressed by the formula*

$$\begin{aligned}\partial F_\lambda(z,u) = \Big\{ \Big[\int\limits_t^T \frac{\partial f_0}{\partial x} d\tau + \frac{\partial f_0}{\partial z} + \lambda\big[v(t) - \int\limits_t^T \Big(\frac{\partial f}{\partial x}\Big)' v(\tau)d\tau + \sum_{i\in I_0} \omega_i e_i\big], \\ \frac{\partial f_0}{\partial u} + \lambda\big[- \Big(\frac{\partial f}{\partial u}\Big)' v(t) + 2\nu u(t)\big]\Big] \;\Big|\; \omega_i \in [-1,1],\ i = \overline{1,n}, \\ \nu \in [0,1],\ u \in U_0,\ \nu = 0,\ u \in U_-,\ v \in P_n[0,T],\ ||v|| \leqslant 1 \Big\}.\end{aligned} \tag{15}$$

It is known [11] that necessary and in the case of the convexity also sufficient condition for the minimum of functional (8) at the point $[z^*, u^*]$ in terms of subdifferential is the condition

$$0_{n+m} \in \partial F_\lambda(z^*, u^*),$$

where 0_{n+m} is a zero element of the space $P_n[0,T] \times P_m[0,T]$. Hereof and in view of Lemma 1 we conclude that the following theorem takes place.

Theorem 4. *For the control* $u^* \in \Omega_2$ *to bring system* (1) *from initial position* (3) *to final state* (4) *and to minimize functional* (5)*, it is necessary, and in the case of the linearity of system* (1) *and the convexity of functional* (5) *also sufficient that*

$$0_{n+m} \in \partial F_\lambda(z^*, u^*), \tag{16}$$

where the expression for the subdifferential $\partial F_\lambda(z,u)$ *is given by* (15).

The subdifferential descent method. Let us find the smallest by norm subgradient $h = h(t,z,u) \in \partial F_\lambda(z,u)$ at the point $[z,u]$, i. e. solve the problem $\min\limits_{h \in \partial F_\lambda(z,u)} ||h||^2$.

Fix a point $[z,u]$ and consider two cases.

A. Let $\varphi(z,u) > 0$. In this case

$$\min_{h \in \partial F_\lambda(z,u)} ||h||^2 := \min_{\omega_i,\ i \in I_0,\ \nu} \left[\int_0^T \Big(s_1(t) + \lambda \sum_{i \in I_0} \omega_i e_i\Big)^2 dt + \int_0^T \big(s_2(t) + 2\lambda\nu u(t)\big)^2 dt \right], \tag{17}$$

where

$$s_1(t) = \overline{s}_1(t) + \lambda \sum_{j=1}^n \mu_j e_j,$$

$$\overline{s}_1(t) = \int_t^T \frac{\partial f_0}{\partial x} d\tau + \frac{\partial f_0}{\partial z} + \lambda \Big[w(t) - \int_t^T \Big(\frac{\partial f}{\partial x}\Big)' w(\tau) d\tau\Big],$$

$$s_2(t) = \frac{\partial f_0}{\partial u} - \lambda \Big(\frac{\partial f}{\partial u}\Big)' w(t),$$

and numbers ω_i, $i \in I_0$, μ_j, $j = \overline{1,n}$, ν and the vector-function $w(t)$ are defined by (13).

Problem (17) is a problem of quadratic programming with linear constraints and can be solved using one of the known methods [12]. Denote ω_i^*, $i \in I_0$, ν^* its solution. Then the vector-function

$$G(t,z,u) := h^* = \Big[s_1(t) + \lambda \sum_{i \in I_0} \omega_i^* e_i, s_2(t) + 2\lambda\nu^* u(t)\Big] \tag{18}$$

is the smallest by norm subgradient of the functional F_λ at a point $[z,u]$ in this case (if $\varphi(z,u) > 0$). If $||G|| > 0$, then the vector-function $-G(t,z,u)/||G||$ is the subdifferential descent direction of the functional F_λ at the point $[z,u]$.

B. Let $\varphi(z,u) = 0$. In this case

$$\min_{h \in \partial F_\lambda(z,u)} ||h||^2 := \min \Big[||h_1||^2 + ||h_2||^2 \Big] = \min_{\omega_i,\ i \in I_0,\ \nu,\ v} \Big[\int_0^T \Big\{ \int_t^T \frac{\partial f_0}{\partial x} d\tau + \frac{\partial f_0}{\partial z} +$$

$$+\lambda\Big[v(t)-\int_t^T\Big(\frac{\partial f}{\partial x}\Big)'v(\tau)d\tau+\sum_{i\in I_0}\omega_i e_i+\sum_{j=1}^{n}\mu_j e_j\Big]\Big\}^2 dt+ \qquad (19)$$

$$+\int_0^T\Big\{\frac{\partial f_0}{\partial u}+\lambda\Big[-\Big(\frac{\partial f}{\partial u}\Big)'v(t)+2\nu u(t)\Big]\Big\}^2 dt\Bigg],$$

where $h_1=h_1(t,z,u)$, $h_2=h_2(t,z,u)$, and numbers ω_i, $i\in I_0$, μ_j, $j=\overline{1,n}$, ν and the vector-function $v(t)$ are defined by (14).

Construct the functional

$$H_\mu(v,\omega,\overline{\nu})=||h||^2+\mu\big[\max\{0,||v||^2-1\}+\max\{0,\overline{\nu}^2-1\}+\sum_{i\in I_0}\max\{0,\omega_i^2-1\}\big], \quad (20)$$

here $\overline{\nu}=2\nu-1$, and the vector $\omega\in R^{|I_0|}$ consists of the components ω_i, $i\in I_0$.

Under some natural assumptions it can be shown, that the functional H_μ is an exact penalty function, then one may use any method (for example, the subdifferential descent method) for the unconstrained minimization of functional (20) to find v^*, ω^*, ν^*.

R e m a r k 1. The subdifferential $\partial F_\lambda(z,u)$ is a convex compact set, therefore necessary minimum condition of the functional $H_\mu(v,\omega,\overline{\nu})$ will be also sufficient.

Denote v^*, ω^*, ν^* the solution of problem (19). Then the vector-function

$$G(t,z,u):=h^*=\Bigg[\int_t^T\frac{\partial f_0}{\partial x}d\tau+\frac{\partial f_0}{\partial z}+\lambda\Big[v^*(t)-\int_t^T\Big(\frac{\partial f}{\partial x}\Big)'v^*(\tau)d\tau+\sum_{i\in I_0}\omega_i^* e_i+\sum_{j=1}^{n}\mu_j e_j\Big],$$

$$\frac{\partial f_0}{\partial u}+\lambda\Big[-\Big(\frac{\partial f}{\partial u}\Big)'v^*(t)+2\nu^* u(t)\Big]\Bigg] \quad (21)$$

is the smallest by norm subgradient of the functional F_λ at the point $[z,u]$ in this case (if $\varphi(z,u)=0$). If $||G||>0$, then the vector-function $-G(t,z,u)/||G||$ is the subdifferential descent direction of the functional F_λ at a point $[z,u]$.

Thus, in items A and B the problem of finding subdifferential descent direction of the functional F_λ at a point $[z,u]$ has been solved. In case of $\varphi(z,u)>0$ (item A) this problem is solved relatively easily, as it is a problem of quadratic programming with linear constraints. In case of $\varphi(z,u)=0$ (item B) besides unknown values ω, ν one must also find the vector-function $v(t)$. It is a more complicated problem, which can be solved with numerical methods, for example, with subdifferential descent method, as it was noted in item B.

Now we can describe the subdifferential descent method for finding stationary points of the functional $F_\lambda(z,u)$. Choose an arbitrary point $[z_1,u_1]\in P_n[0,T]\times P_m[0,T]$ and assume that the point $[z_k,u_k]\in P_n[0,T]\times P_m[0,T]$ is already found. If minimum condition (16) holds, then the point $[z_k,u_k]$ is the stationary point of the functional $F_\lambda(z,u)$ and the process terminates. Otherwise put

$$[z_{k+1},u_{k+1}]=[z_k,u_k]-\alpha_k G_k,$$

where the vector-function $G_k=G(t,z_k,u_k)$ is the smallest by norm subgradient of the functional F_λ at the point $[z_k,u_k]$. The value for the functional G_k is given either by formula (18) if $\varphi(z_k, u_k) > 0$, or by formula (21) if $\varphi(z_k, u_k) = 0$. The value α_k is the solution of the following one-dimensional minimization problem

$$\min_{\alpha \geqslant 0} F_\lambda([z_k, u_k] - \alpha G_k) = F_\lambda([z_k, u_k] - \alpha_k G_k).$$

Then $F_\lambda(z_{k+1}, u_{k+1}) \leqslant F_\lambda(z_k, u_k)$. If the sequence $\{[z_k, u_k]\}$ is finite, then its last point is the stationary point of the functional $F_\lambda(z, u)$ by construction. If the sequence $\{[z_k, u_k]\}$ is infinite, then the described process may not lead to the stationary point of the functional $F_\lambda(z, u)$, because the subdifferential mapping $\partial F_\lambda(z, u)$ is not continuous in Hausdorff metric.

The hypodifferential descent method. Using formulas of codifferential calculus [11], it can be shown that the following two theorems take place.

Theorem 5. *If $[z, u] \notin \Omega_3$, then the functional $F_\lambda(z, u)$ is hypodifferentiable, and its hypodifferential at a point $[z, u]$ is expressed by the formula*

$$dF_\lambda(z, u) = \big[0, \overline{s}_1(t), s_2(t)\big] +$$

$$+ \lambda \sum_{i=1}^{n} \mathrm{co}\big\{\big[\overline{\psi}_i(z) - \psi_i(z), e_i, 0_m\big], \big[-\overline{\psi}_i(z) - \psi_i(z), -e_i, 0_m\big]\big\} +$$

$$+ \lambda \mathrm{co}\big\{\big[\int_0^T \big(u(t), u(t)\big)dt - 1 - \max\{0, ||u||^2 - 1\}, 0_n, 2u(t)\big], \big[-\max\{0, ||u||^2 - 1\}, 0_n, 0_m\big]\big\},$$

where

$$s_1(t) = \overline{s}_1(t) + \lambda \sum_{j=1}^{n} \mu_j e_j,$$

$$\overline{s}_1(t) = \int_t^T \frac{\partial f_0}{\partial x} d\tau + \frac{\partial f_0}{\partial z} + \lambda\big[w(t) - \int_t^T \Big(\frac{\partial f}{\partial x}\Big)' w(\tau) d\tau\big],$$

$$s_2(t) = \frac{\partial f_0}{\partial u} - \lambda \Big(\frac{\partial f}{\partial u}\Big)' w(t),$$

$$w(t) = \frac{z(t) - f(x, u, t)}{\varphi(z, u)},$$

$$\mu_j = 0,\ j \in I_0,\ \mu_j = 1,\ j \in I_+,\ \mu_j = -1,\ j \in I_-.$$

Theorem 6. *If $[z, u] \in \Omega_3$, then the functional $F_\lambda(z, u)$ is hypodifferentiable, and its hypodifferential at a point $[z, u]$ is expressed by the formula*

$$dF_\lambda(z, u) = \Big\{\Big[\lambda\big[\int_0^T \big(z(t) - f(x, u, t)\big)' v(t) dt - \varphi(z, u)\big], \int_t^T \frac{\partial f_0}{\partial x} d\tau + \frac{\partial f_0}{\partial z} +$$

$$+ \lambda\big[v(t) - \int_t^T \Big(\frac{\partial f}{\partial x}\Big)' v(\tau) d\tau\big], \frac{\partial f_0}{\partial u} - \lambda \Big(\frac{\partial f}{\partial u}\Big)' v(t)\Big] +$$

$$+ \lambda \sum_{i=1}^{n} \text{co}\{ [\overline{\psi}_i(z) - \psi_i(z), e_i, 0_m], [- \overline{\psi}_i(z) - \psi_i(z), -e_i, 0_m] \} + \tag{22}$$

$$+ \lambda \text{co}\{ [\int_0^T \big(u(t), u(t)\big) dt - 1 - \max\{0, ||u||^2 - 1\}, 0_n, 2u(t)],$$

$$[- \max\{0, ||u||^2 - 1\}, 0_n, 0_m] \} \;\Big|\; v \in P_n[0,T],\; ||v|| \leqslant 1 \Big\}.$$

It is known [11] that necessary and in the case of the convexity also sufficient condition for the minimum of functional (8) at the point $[z^*, u^*]$ in terms of hypodifferential is the condition

$$0_{n+m+1} \in dF_\lambda(z^*, u^*),$$

where 0_{n+m+1} is a zero element of the space $P_n[0,T] \times P_m[0,T] \times R$. Hereof and in view of Lemma 1 we conclude that the following theorem takes place.

Theorem 7. *For the control $u^* \in \Omega_2$ to bring system* (1) *from initial position* (3) *to final state* (4) *and to minimize functional* (5)*, it is necessary, and in the case of the linearity of system* (1) *and the convexity of functional* (5) *also sufficient that*

$$0_{n+m+1} \in dF_\lambda(z^*, u^*), \tag{23}$$

where the expression for the hypodifferential $dF_\lambda(z,u)$ is given by (22)*.*

Let us find the smallest by norm hypogradient $g = g(t,z,u) \in dF_\lambda(z,u)$ at the point $[z,u]$, i. e. solve the problem $\min\limits_{g \in dF_\lambda(z,u)} ||g||^2$.

Fix a point $[z,u]$ and consider two cases.

A. Let $\varphi(z,u) > 0$. In this case

$$\min_{g \in dF_\lambda(z,u)} ||g||^2 = \min_{\beta_i \in [0,1],\; i=\overline{1,n+1}} \Big|\Big| [0, \overline{s}_1(t), s_2(t)] +$$

$$+ \lambda \sum_{i=1}^{n} \{ \beta_i [\overline{\psi}_i(z) - \psi_i(z), e_i, 0_m] + (1-\beta_i) [- \overline{\psi}_i(z) - \psi_i(z), -e_i, 0_m] \} +$$

$$+ \lambda \beta_{n+1} \left[\int_0^T \big(u(t), u(t)\big) dt - 1 - \max\{0, ||u||^2 - 1\}, 0_n, 2u(t) \right] + \tag{24}$$

$$+ \lambda (1 - \beta_{n+1}) [- \max\{0, ||u||^2 - 1\}, 0_n, 0_m] \Big|\Big|^2.$$

Problem (24) is a problem of quadratic programming with linear constraints and can be solved using one of the known methods [12]. Denote its solution β_i^*, $i = \overline{1, n+1}$. Let $g = [g_1, g_2]$, where the vector-function g_2 consists of the last $n+m$ components of g. Then the vector-function

$$G(t,z,u) := g_2^* = [\overline{s}_1(t), s_2(t)] + \lambda \sum_{i=1}^{n} \{ \beta_i^* [e_i, 0_m] + (1-\beta_i^*) [- e_i, 0_m] \} +$$

$$+ \lambda \beta_{n+1}^* [0_n, 2u(t)] + \lambda (1 - \beta_{n+1}^*) [0_n, 0_m] \tag{25}$$

consists of the last $n+m$ componets of the smallest by norm hypogradient of the functional F_λ at the point $[z,u]$ in this case (if $\varphi(z,u) > 0$). If $||G|| > 0$, then the vector-function

$-G(t,z,u)/||G||$ is the hypogradient descent direction of the functional F_λ at the point $[z,u]$.

B. Let $\varphi(z,u)=0$. In this case

$$\min_{g\in dF_\lambda(z,u)}||g||^2=\min_{\beta_i\in[0,1],\ i=\overline{1,n+1},\ v}\Big|\Big|\Big[\lambda\Big[\int_0^T\big(z(t)-f(x,u,t)\big)'v(t)dt-\varphi(z,u)\Big],$$

$$\int_t^T\frac{\partial f_0}{\partial x}d\tau+\frac{\partial f_0}{\partial z}+\lambda\Big[v(t)-\int_t^T\Big(\frac{\partial f}{\partial x}\Big)'v(\tau)d\tau\Big],\frac{\partial f_0}{\partial u}-\lambda\Big(\frac{\partial f}{\partial u}\Big)'v(t)\Big]+$$

$$+\lambda\sum_{i=1}^n\big\{\beta_i\big[\overline{\psi}_i(z)-\psi_i(z),e_i,0_m\big]+(1-\beta_i)\big[-\overline{\psi}_i(z)-\psi_i(z),-e_i,0_m\big]\big\}+$$

$$+\lambda\beta_{n+1}\Big[\int_0^T\big(u(t),u(t)\big)dt-1-\max\{0,||u||^2-1\},0_n,2u(t)\Big]+$$

$$+\lambda(1-\beta_{n+1})\big[-\max\{0,||u||^2-1\},0_n,0_m\big]\Big|\Big|^2=$$

$$=\min_{\beta_i\in[0,1],\ i=\overline{1,n+1},\ v}\Big|\Big|\Big[\lambda\Big[\int_0^T\big(z(t)-f(x,u,t)\big)'v(t)dt-\varphi(z,u)\Big],$$

$$\int_t^T\frac{\partial f_0}{\partial x}d\tau+\frac{\partial f_0}{\partial z}+\lambda\Big[v(t)-\int_t^T\Big(\frac{\partial f}{\partial x}\Big)'v(\tau)d\tau\Big],\frac{\partial f_0}{\partial u}-\lambda\Big(\frac{\partial f}{\partial u}\Big)'v(t)\Big]+$$

$$+\lambda\sum_{i=1}^n\big\{\beta_i\big[2\overline{\psi}_i(z),2e_i,0_m\big]+\big[-\overline{\psi}_i(z)-\psi_i(z),-e_i,0_m\big]\big\}+$$

$$+\lambda\beta_{n+1}\Big[\int_0^T\big(u(t),u(t)\big)dt-1,0_n,2u(t)\Big]+\lambda\big[-\max\{0,||u||^2-1\},0_n,0_m\big]\Big|\Big|^2.$$

This expression can be rewritten as follows:

$$\min_{g\in dF_\lambda(z,u)}||g||^2:=\min\Big[||g_1||^2+||g_2||^2+||g_3||^2\Big]=$$

$$=\min_{\overline{\beta}_i\in[-1,1],\ i=\overline{1,n+1},\ v}\Bigg[\Big\{\lambda\Big[\int_0^T\big(z(t)-f(x,u,t)\big)'v(t)dt-\varphi(z,u)\Big]+\lambda\sum_{i=1}^n\overline{\psi}_i(z)\big(\overline{\beta}_i+1\big)-$$

$$-\lambda\sum_{i=1}^n\big(\overline{\psi}_i(z)+\psi_i(z)\big)+\frac{\lambda}{2}\left(\int_0^T\big(u(t),u(t)\big)dt-1\right)\big(\overline{\beta}_{n+1}+1\big)-\lambda\max\{0,||u||^2-1\}\Big\}^2+$$

$$+\int_0^T\left\{\int_t^T\frac{\partial f_0}{\partial x}d\tau+\frac{\partial f_0}{\partial z}+\lambda\Big[v(t)-\int_t^T\Big(\frac{\partial f}{\partial x}\Big)'v(\tau)d\tau+\sum_{i=1}^n\overline{\beta}_ie_i\Big]\right\}^2dt+$$

$$+ \int_0^T \Big\{ \frac{\partial f_0}{\partial u} - \lambda \Big(\frac{\partial f}{\partial u} \Big)' v(t) + \lambda \overline{\beta}_{n+1} u(t) + \lambda u(t) \Big\}^2 dt \Big], \quad (26)$$

where $g_1 = g_1(t, z, u)$, $g_2 = g_2(t, z, u)$, $g_3 = g_3(t, z, u)$, $\overline{\beta}_i = 2\beta_i - 1$, $i = \overline{1, n+1}$, and the vector-function $v(t)$ is defined in (22).

Let the vector $\overline{\beta} \in R^{n+1}$ consist of the components $\overline{\beta}_i$, $i = \overline{1, n+1}$. Write the functional

$$H_\mu(v, \overline{\beta}) = ||g||^2 + \mu \big[\max\{0, ||v||^2 - 1\} + \sum_{i=1}^{n+1} \max\{0, \overline{\beta}_i^2 - 1\} \big]. \quad (27)$$

Denote

$$\Psi(v, \overline{\beta}) = \mu \big[\max\{0, ||v||^2 - 1\} + \sum_{i=1}^{n+1} \max\{0, \overline{\beta}_i^2 - 1\} \big].$$

Introduce the sets

$$\overline{\Omega} = \big\{ [v, \overline{\beta}] \in P_n[0,T] \times R^{n+1} \bigm| \Psi(v, \overline{\beta}) = 0 \big\},$$

$$\overline{\Omega}_\delta = \big\{ [v, \overline{\beta}] \in P_n[0,T] \times R^{n+1} \bigm| \Psi(v, \overline{\beta}) < \delta \big\}.$$

Then

$$\overline{\Omega}_\delta \setminus \overline{\Omega} = \big\{ [v, \overline{\beta}] \in P_n[0,T] \times R^{n+1} \bigm| 0 < \Psi(v, \overline{\beta}) < \delta \big\}.$$

Also introduce the following sets

$$B_{i0} = \big\{ \overline{\beta}_i \in R \bigm| \overline{\beta}_i^2 - 1 = 0 \big\},$$

$$B_{i-} = \big\{ \overline{\beta}_i \in R \bigm| \overline{\beta}_i^2 - 1 < 0 \big\},$$

$$B_{i+} = \big\{ \overline{\beta}_i \in R \bigm| \overline{\beta}_i^2 - 1 > 0 \big\},$$

where $i = \overline{1, n+1}$.

Lemma 2. *Suppose there exists such a positive number $\mu_0 < \infty$ that $\forall \mu > \mu_0$ there exists a point $[v(\mu), \overline{\beta}(\mu)] \in P_n[0,T] \times R^{n+1}$, for which $H_\mu\big(v(\mu), \overline{\beta}(\mu)\big) = \inf\limits_{[v, \overline{\beta}]} H_\mu(v, \overline{\beta})$. Let the functional $g(v, \overline{\beta})$ be Lipschitz on the set $\overline{\Omega}_\delta \setminus \overline{\Omega}$. Then functional* (27) *will be an exact penalty function.*

Thus, under the assumptions of Lemma 2 there exists such a number $0 < \mu^* < \infty$ that $\forall \mu > \mu_*$ problem (26) is equivalent to the problem of minimization of functional (27) on the whole space. Further we suppose that the number μ in functional (27) is fixed and the condition $\mu > \mu^*$ holds.

Lemma 3. *Functional* (27) *is hypodifferentiable, and its hypodifferential at a point $[v, \overline{\beta}]$ is expressed by the formula*

$$dH_\mu(v, \overline{\beta}) = \big[0, g_v, g_{\overline{\beta}_1}, \ldots, g_{\overline{\beta}_{n+1}}\big] +$$

$$+ \mu \Big[\mathrm{co}\big\{ \big[||v||^2 - 1 - \max\{0, ||v||^2 - 1\}, 2v(t), 0_{n+1} \big], \big[-\max\{0, ||v||^2 - 1\}, 0_n, 0_{n+1} \big] \big\} +$$

$$+ \mathrm{co}\big\{ \big[\overline{\beta}_1^2 - 1 - \max\{0, \overline{\beta}_1^2 - 1\}, 0_n, 2\overline{\beta}_1, 0_n \big], \big[-\max\{0, \overline{\beta}_1^2 - 1\}, 0_n, 0_{n+1} \big] \big\} + \cdots + \quad (28)$$

$$+\operatorname{co}\big\{\big[\overline{\beta}_{n+1}^2-1-\max\{0,\overline{\beta}_{n+1}^2-1\},0_n,0_n,2\overline{\beta}_{n+1}\big],\big[-\max\{0,\overline{\beta}_{n+1}^2-1\},0_n,0_{n+1}\big]\big\}\Big].$$

Calculate the following vector-functions in formula (28):

$$g_v=g_{1v}+g_{2v}+g_{3v},$$

where

$$g_{1v}=2\lambda^2\Bigg\{\int\limits_0^T\big(z(t)-f(x,u,t)\big)'v(t)dt-\varphi(z,u)+\sum_{i=1}^n\overline{\beta}_i\overline{\psi}_i(z)+\sum_{i=1}^n\overline{\psi}_i(z)+$$

$$+\sum_{i=1}^n\big(-\overline{\psi}_i(z)-\psi_i(z)\big)+\frac{1}{2}\left(\int\limits_0^T\big(u(t),u(t)\big)dt-1\right)(\overline{\beta}_{n+1}+1)-$$

$$-\max\{0,||u||^2-1\}\Bigg\}\big(z(t)-f(x,u,t)\big),$$

$$g_{2v}=2\lambda\Bigg\{\lambda v(t)-\lambda\int\limits_t^T\Big(\frac{\partial f}{\partial x}\Big)'v(\tau)d\tau-\lambda\frac{\partial f}{\partial x}\int\limits_0^t v(\tau)d\tau+\lambda\frac{\partial f}{\partial x}\int\limits_0^t\int\limits_\tau^T\Big(\frac{\partial f}{\partial x}\Big)'v(\xi)d\xi d\tau\ +$$

$$+\int\limits_t^T\frac{\partial f_0}{\partial x}d\tau+\frac{\partial f_0}{\partial z}+\lambda\sum_{i=1}^n\overline{\beta}_ie_i-\frac{\partial f}{\partial x}\int\limits_0^t\left[\int\limits_\tau^T\frac{\partial f_0}{\partial x}d\xi+\frac{\partial f_0}{\partial z}\right]d\tau-\lambda t\frac{\partial f}{\partial x}\sum_{i=1}^n\overline{\beta}_ie_i\Bigg\},$$

$$g_{3v}=-2\lambda\frac{\partial f}{\partial u}\Big(\frac{\partial f_0}{\partial u}+\lambda\big[-\Big(\frac{\partial f}{\partial u}\Big)'v(t)+\overline{\beta}_{n+1}u(t)+u(t)\big]\Big),$$

$$g_{\overline{\beta}_i}=g_{1\overline{\beta}_i}+g_{2\overline{\beta}_i},\quad i=\overline{1,n},$$

where

$$g_{1\overline{\beta}_i}=2\lambda^2\Bigg\{\int\limits_0^T\big(z(t)-f(x,u,t)\big)'v(t)dt-\varphi(z,u)+\sum_{i=1}^n\overline{\beta}_i\overline{\psi}_i(z)+\sum_{i=1}^n\overline{\psi}_i(z)\ +$$

$$+\sum_{i=1}^n\big(-\overline{\psi}_i(z)-\psi_i(z)\big)+\frac{1}{2}\left[\int\limits_0^T\big(u(t),u(t)\big)dt-1\right](\overline{\beta}_{n+1}+1)-\max\{0,||u||^2-1\}\Bigg\}\overline{\psi}_i(z),$$

$$g_{2\overline{\beta}_i}=2\lambda\int\limits_0^T\left\{\int\limits_t^T\frac{\partial f_0}{\partial x}d\tau+\frac{\partial f_0}{\partial z}+\lambda\left[v(t)-\int\limits_t^T\Big(\frac{\partial f}{\partial x}\Big)'v(\tau)d\tau\right]+\lambda\sum_{i=1}^n\overline{\beta}_ie_i\right\}'e_idt,$$

$$g_{\overline{\beta}_{n+1}}=2\lambda\int\limits_0^T\Big(\frac{\partial f_0}{\partial u}+\lambda\big[-\Big(\frac{\partial f}{\partial u}\Big)'v(t)+\overline{\beta}_{n+1}u(t)+u(t)\big]\Big)'u(t)dt.$$

R e m a r k 2. The hypodifferential $dF_\lambda(z,u)$ is a convex compact set, therefore necessary minimum condition of the functional $H_\mu(v,\overline{\beta})$ will be also sufficient.

Lemma 4. *For the point* $[v^*,\overline{\beta}^*] \in P_n[0,T]\times R^{n+1}$ *to minimize functional* (27)*, it is necessary and sufficient that*

$$0_{n+n+2} \in dH_\mu(v^*,\overline{\beta}^*), \tag{29}$$

where the expression for the hypodifferential $dH_\mu(v,\overline{\beta})$ *is given by* (28).

Let us find the smallest by norm hypogradient $\overline{g}=\overline{g}(t,v,\overline{\beta})\in dH_\mu(v,\overline{\beta})$ at the point $[v,\overline{\beta}]$, i. e. solve the problem

$$\min_{\overline{g}\in dH_\mu(v,\overline{\beta})} ||\overline{g}||^2 = \min_{\gamma_i\in[0,1],\ i=\overline{1,n+2}} \Big|\Big| \big[0, g_v, g_{\overline{\beta}_1},\dots,g_{\overline{\beta}_{n+1}}\big] +$$

$$+\mu\Big[\gamma_1\big[||v||^2-1-\max\{0,||v||^2-1\},2v(t),0_{n+1}\big]+(1-\gamma_1)\big[-\max\{0,||v||^2-1\},0_n,0_{n+1}\big]+$$

$$+\gamma_2\big[\overline{\beta}_1^2-1-\max\{0,\overline{\beta}_1^2-1\},0_n,2\overline{\beta}_1,0_n\big]+(1-\gamma_2)\big[-\max\{0,\overline{\beta}_1^2-1\},0_n,0_{n+1}\big]+\dots+ \tag{30}$$

$$+\gamma_{n+2}\big[\overline{\beta}_{n+1}^2-1-\max\{0,\overline{\beta}_{n+1}^2-1\},0_n,0_n,2\overline{\beta}_{n+1}\big]+$$

$$+(1-\gamma_{n+2})\big[-\max\{0,\overline{\beta}_{n+1}^2-1\},0_n,0_{n+1}\big]\Big]\Big|\Big|^2.$$

Problem (30) is a problem of quadratic programming with linear constraints and can be solved using one of the known methods [12]. Denote its solution γ_i^*, $i=\overline{1,n+2}$. Let $\overline{g}=[\overline{g}_1,\overline{g}_2]$, where the vector-function $\overline{g}_2$ consists of the last $n+n+1$ components of $\overline{g}$. Then the vector-function

$$\overline{G}(t,v,\overline{\beta}) := \overline{g}_2^* = \big[g_v, g_{\overline{\beta}_1},\dots,g_{\overline{\beta}_{n+1}}\big] +$$

$$+\mu\Big[\gamma_1^*\big[2v(t),0_{n+1}\big]+(1-\gamma_1^*)\big[0_n,0_{n+1}\big]+\gamma_2^*\big[0_n,2\overline{\beta}_1,0_n\big]+(1-\gamma_2^*)\big[0_n,0_{n+1}\big]+\dots+$$

$$+\gamma_{n+2}^*\big[0_n,0_n,2\overline{\beta}_{n+1}\big]+(1-\gamma_{n+2}^*)\big[0_n,0_{n+1}\big]\Big]$$

consists of the last $n+n+1$ components of the smallest by norm hypogradient of the functional H_μ at the point $[v,\overline{\beta}]$. If $||\overline{G}||>0$, then the vector-function $-\overline{G}(t,v,\overline{\beta})/||\overline{G}||$ is the hypogradient descent direction of the functional H_μ at the point $[v,\overline{\beta}]$.

Let us describe the following hypodifferential descent method for finding minimum points of the functional $H_\mu(v,\overline{\beta})$. Choose an arbitrary point $[v_1,\overline{\beta}_1]\in P_n[0,T]\times R^{n+1}$ and assume that the point $[v_k,\overline{\beta}_k]\in P_n[0,T]\times R^{n+1}$ is already found. If minimum condition (29) holds, then the point $[v_k,\overline{\beta}_k]$ is the minimum point of the functional $H_\mu(v,\overline{\beta})$ and the process terminates. Otherwise put

$$[v_{k+1},\overline{\beta}_{k+1}] = [v_k,\overline{\beta}_k]-\alpha_k\overline{G}_k,$$

where the vector-function $\overline{G}_k=\overline{G}(t,v_k,\overline{\beta}_k)$ consists of the last $n+n+1$ components of the smallest by norm hypogradient of the functional H_μ at the point $[v_k,\overline{\beta}_k]$ and the value α_k is the solution of the following one-dimensional minimization problem:

$$\min_{\alpha\geqslant 0} H_\mu([v_k,\overline{\beta}_k]-\alpha\overline{G}_k) = H_\mu([v_k,\overline{\beta}_k]-\alpha_k\overline{G}_k). \tag{31}$$

Then $H_\mu(v_{k+1}, \overline{\beta}_{k+1}) \leqslant H_\mu(v_k, \overline{\beta}_k)$. If the sequence $\{[v_k, \overline{\beta}_k]\}$ is infinite, then it can be shown that the hypodifferential descent method converges in the following sense:

$$||\overline{g}(v_k, \overline{\beta}_k)|| \to 0 \text{ if } k \to \infty.$$

If the sequence $\{[v_k, \overline{\beta}_k]\}$ is finite, then its last point is the minimum point of the functional $H_\mu(v, \overline{\beta})$ by construction.

Denote v^*, β^* the solution of problem (26). Let $g = [g_1, g_2]$, where the vector-function g_2 consists of the last $n+m$ components of g. Then the vector-function

$$G(t,z,u) := g_2^* = \left[\int_t^T \frac{\partial f_0}{\partial x} d\tau + \frac{\partial f_0}{\partial z} + \lambda\Big[v^*(t) - \int_t^T \Big(\frac{\partial f}{\partial x}\Big)' v^*(\tau) d\tau\Big],\right.$$
$$\left.\frac{\partial f_0}{\partial u} - \lambda\Big(\frac{\partial f}{\partial u}\Big)' v^*(t)\right] + \lambda \sum_{i=1}^{n} \big\{\beta_i^* \big[e_i, 0_m\big] + (1-\beta_i^*)\big[-e_i, 0_m\big]\big\} + \tag{32}$$
$$+ \lambda \beta_{n+1}^* \big[0_n, 2u(t)\big] + \lambda(1-\beta_{n+1}^*)\big[0_n, 0_m\big]$$

consists of the last $n+m$ components of the smallest by norm hypogradient of the functional F_λ at the point $[z,u]$ in this case (if $\varphi(z,u) = 0$). If $||G|| > 0$, then the vector-function $-G(t,z,u)/||G||$ is the hypogradient descent direction of the functional F_λ at the point $[z,u]$.

Thus, in the points A and B the problem of finding the hypogradient descent direction of the functional F_λ at the point $[z,u]$ was solved. In the case $\varphi(z,u) > 0$ (point A) this problem is sufficiently easy, as it is a problem of quadratic programming with linear constraints. In the case $\varphi(z,u) = 0$ (point B) besides the unknown values β_i, $i = \overline{1, n+1}$, one also has to find the vector-function $v(t)$. This is a more difficult problem, which may be solved with numerical methods, for example, with the hypodifferential descent method as it has been described in the point B.

R e m a r k 3. Note that due to functional H_μ structure problem (31) of finding the descent step can be solved analytically. Moreover, problem (30) of finding the descent direction can be solved in finite number of iterations using quadratic programming methods.

Now we can describe the hypodifferential descent method for finding stationary points of the functional $F_\lambda(z,u)$. Choose an arbitrary point $[z_1, u_1] \in P_n[0,T] \times P_m[0,T]$ and assume that the point $[z_k, u_k] \in P_n[0,T] \times P_m[0,T]$ is already found. If minimum condition (23) holds, then the point $[z_k, u_k]$ is the stationary point of the functional $F_\lambda(z,u)$ and the process terminates. Otherwise put

$$[z_{k+1}, u_{k+1}] = [z_k, u_k] - \alpha_k G_k,$$

where the vector-function $G_k = G(t, z_k, u_k)$ consists of the last $n+m$ components of the smallest by norm hypogradient of the functional F_λ at the point $[z_k, u_k]$. The value for the functional G_k is given either by formula (25) if $\varphi(z_k, u_k) > 0$, or by formula (32) if $\varphi(z_k, u_k) = 0$. The value α_k is the solution of the following one-dimensional minimization problem

$$\min_{\alpha \geqslant 0} F_\lambda([z_k, u_k] - \alpha G_k) = F_\lambda([z_k, u_k] - \alpha_k G_k).$$

Then $F_\lambda(z_{k+1}, u_{k+1}) \leqslant F_\lambda(z_k, u_k)$. If the sequence $\{[z_k, u_k]\}$ is infinite, then it can be shown that the hypodifferential descent method converges in the sense

$$||g(z_k, u_k)|| \to 0 \text{ if } k \to \infty.$$

If the sequence $\{[z_k, u_k]\}$ is finite, then its last point is the stationary point of the functional $F_\lambda(z, u)$ by construction.

Numerical examples. Let us consider some examples of the application of the hypodifferential descent method.

E x a m p l e 1. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= u_1, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= u_2 - 9.8 \end{aligned}$$

with boundary conditions

$$x(0) = [-1, 0, 0, 0], \quad x(1) = [0, 0, 0, 0].$$

It is required to minimize the functional

$$I = \int_0^1 u_1^2(t) + u_2^2(t) \; dt.$$

For this problem the analytical solution is known [13], which is as follows:

$$u_1^*(t) = -12t + 6,$$

$$u_2^*(t) = 9.8,$$

$$z_1^*(t) = -6t^2 + 6t,$$

$$z_2^*(t) = -12t + 6,$$

$$z_3^*(t) = 0,$$

$$z_4^*(t) = 0,$$

$$I(z^*, u^*) = 108.04.$$

Table 1 presents the hypodifferential descent method results. Here we put $u = [0, 1]$, $z(t) = [1, 0, 0, 0]$ as initial approximation, then $x(t) = [-1 + t, 0, 0, 0]$. Table 1 shows that on the 30-th iteration error does not exceed the value 3×10^{-3}.

Table 1. **Example 1**

k	$I(z_k, u_k)$	$\Phi(z_k, u_k)$	$\|\|u^* - u_k\|\|$	$\|\|z^* - z_k\|\|$	$\|\|G(z_k, u_k)\|\|$
1		1.06044	3.47062	3.21367	197.96324
2		0.94422	3.20293	3.22259	707.22868
10		0.34105	1.15682	1.38112	848.13142
20		0.20739	0.72749	0.69893	256.2921
30	108.0425		0.05774	0.02886	0.425

E x a m p l e 2. Let us consider another example. Let the following system be given

$$\dot{x}_1 = x_2 + u_1,$$
$$\dot{x}_2 = u_2$$

with boundary conditions

$$x(0) = [2, 0.5], \quad x(1) = [x_1(1), 0]$$

and the restriction on the control

$$\int_0^1 u_1^2(t) + u_2^2(t)\ dt \leqslant 1.$$

It is required to minimize the functional

$$I = \int_0^1 z_1(t)\ dt.$$

For this problem the analytical solution is also known [7], which is as

$$u_1^*(t) = -\sqrt{\frac{9}{13}},$$

$$u_2^*(t) = \sqrt{\frac{9}{13}}t - \frac{1}{2}\sqrt{\frac{9}{13}} - \frac{1}{2},$$

$$z_1^*(t) = \frac{1}{2}\sqrt{\frac{9}{13}}t^2 - \frac{1}{2}\left(\sqrt{\frac{9}{13}} + 1\right)t + \frac{1}{2} - \sqrt{\frac{9}{13}},$$

$$z_2^*(t) = \sqrt{\frac{9}{13}}t - \frac{1}{2}\sqrt{\frac{9}{13}} - \frac{1}{2},$$

$$I(z^*, u^*) = \frac{1}{4}(1 - \sqrt{13}).$$

Table 2 presents the hypodifferential descent method results. Here we put $u = [0, 0]$, $z(t) = [0, 0]$ as initial approximation, then $x(t) = [2, 0.5]$. Table 2 shows that on the 7-th iteration error does not exceed the value 5×10^{-3}.

Table 2. **Example 2**

k	$I(z_k, u_k)$	$\Phi(z_k, u_k)$	$\|\|u^* - u_k\|\|$	$\|\|z^* - z_k\|\|$	$\|\|G(z_k, u_k)\|\|$
1		1.0	1.00004	0.86826	188.77058
2		0.51873	0.91483	0.90879	76.71471
5		0.00243	0.79148	0.85081	112.2858
6	−0.61768		0.23167	0.23273	0.70711
7	−0.6464		0.08873	0.1132	0.21357

E x a m p l e 3. Let the following system be given

$$\dot{x}_1 = u,$$
$$\dot{x}_2 = x_1^2$$

with boundary conditions

$$x(0) = [0.25, 0], \quad x(1) = [0.25, x_2(1)]$$

and the restriction on the control

$$\int_0^1 u^2(t)\ dt \leqslant 1.$$

It is required to minimize the functional

$$I = \int_0^1 z_2(t)\ dt.$$

This example was considered in the paper [14] with the heavier restriction on the control $|u(t)| \leqslant 1$, $t \in [0, 1]$, where one may also find the optimal value of the functional

$$I(z^*, u^*) = \frac{1}{96}.$$

Table 3 presents the hypodifferential descent method results. Here we put $u = 10t-5$, $z(t) = [10t-5, (0.25+5t^2-5t)^2]$ as initial approximation, then $x(t) = [0.25+5t^2-5t, 5t^5 - 12.5t^4 + 9.1(6)t^3 - 1.25t^2 + 0.0625t]$. Table 3 shows that on the 8-th iteration error does not exceed the value 5×10^{-3}, however, due to the considered weaker restriction on the control and the nonlinearity of the system we can not guarantee that the obtained value is a global minimum in this problem.

Table 3. **Example 3**

k	$I(z_k, u_k)$	$\Phi(z_k, u_k)$	$\|\|G(z_k, u_k)\|\|$
1		8.3333	486.44
2		0.43953	102.93801
5		0.10272	130.33683
7		0.00025	99.303
8	0.01579		0.1127

E x a m p l e 4. Let us consider one more example. There is a system given

$$\dot{x}_1 = \cos(x_3),$$
$$\dot{x}_2 = \sin(x_3),$$
$$\dot{x}_3 = u$$

with boundary conditions

$$x(0) = [0, 0, 0], \quad x(1) = [3.85, 2.85, x_3(1)]$$

and the restriction on the control

$$\int_0^{5.1228} u^2(t)\ dt \leqslant 1.2807.$$

It is required to minimize the functional

$$I = \int_0^{5.1228} z_3(t)\ dt.$$

This example with other boundary conditions was considered in the papers [15, 16].

Table 4 presents the hypodifferential descent method results. Here we put $u = 0.5$, $z(t) = [0.5, 0.5, 0.5]$ as initial approximation, then $x(t) = [0.5t, 0.5t, 0.5t]$. Analogous to the previous example due to the nonlinearity of the system we can not guarantee that the obtained value is a global minimum in this problem.

Table 4. **Example 4**

k	$I(z_k, u_k)$	$\Phi(z_k, u_k)$	$\|\|G(z_k, u_k)\|\|$
1		328.4571	373.594
2		232.7861	350.5031
10		27.879	81.23427
15		7.18531	48.2351
20	-0.06627		50.3464
25		0.42832	22.2662
30	-0.157194		0.21303
35	-0.19294		0.0573

Conclusion. The considered problem of constructing an optimal control in the form of Lagrange with integral restriction on control reduces to the variational problem of minimizing a nonsmooth functional on the whole space. For this functional the subdifferential and the hypodifferential are obtained, the necessary minimum conditions are found, which are also sufficient in a partial case. The methods of the subdifferential descent and the hypodifferential descent are applied to the problem. The results are illustrated with numerical examples.

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