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O. I. Drivotin

COVARIANT DESCRIPTION OF PHASE SPACE DISTRIBUTIONS

St. Petersburg State University, 7–9, Universitetskaya nab.,
St. Petersburg, 199034, Russian Federation

The concept of phase space for particles moving in the 4-dimensional space time is formulated. Definition of particle distribution density as differential form is given. The degree of the distribution density form may be different in various cases. The Liouville and the Vlasov equations are written in tensor form with use of such tensor operations as the Lie dragging and the Lie derivative. The presented approach is valid in both non-relativistic and relativistic cases. It should be emphasized that this approach does not include the concepts of phase volume and distribution function. The covariant approach allows using arbitrary systems of coordinates for description of the particle distribution. In some cases, making use of special coordinates grants the possibility to construct analytical solutions. Besides, such an approach is convenient for description of degenerate distributions, for example, of the Kapchinsky–Vladimirsky distribution, which is well-known in the theory of charged particle beams. It can be also applied for description of particle distributions in curved space time. Refs 25.

Keywords: Liouville equation, Vlasov equation, phase space, phase density, particle distribution density, self-consistent distribution, degenerate distribution.

О. И. Дривотин

КОВАРИАНТНОЕ ОПИСАНИЕ РАСПРЕДЕЛЕНИЙ В ФАЗОВОМ ПРОСТРАНСТВЕ

Санкт-Петербургский государственный университет, Российская Федерация,
199034, Санкт-Петербург, Университетская наб., 7–9

Сформулирована концепция фазового пространства для частиц, движущихся в 4-мерном пространстве-времени. Дано определение плотности распределения частиц как дифференциальной формы. Уравнения Лиувилля и Власова записаны в тензорном виде с использованием таких тензорных операций как перенос Ли и производная Ли. Представленный подход применим как в нерелятивистском, так и в релятивистском случаях. Следует подчеркнуть, что данный подход не содержит понятий фазового объема и функции распределения. Ковариантный подход позволяет использовать любые системы координат для описания распределений частиц. В некоторых случаях использование специальных координат дает возможность строить аналитические решения. Кроме того, такой подход удобен для описания вырожденных распределений, например распределения Капчинского–Владимирского, хорошо известного в теории пучков заряженных частиц. Он также может быть применен для описания распределений частиц в искривленном пространстве-времени. Библиогр. 25 назв.

Ключевые слова: уравнение Лиувилля, уравнение Власова, фазовое пространство, фазовая плотность, плотность распределения частиц, самосогласованное распределение, вырожденное распределение.

1. Introduction. Covariant approach is commonly in use in general relativity. But it can be also applied in classical theory, where processes in space and time are considered. Covariance means using of objects that can be defined without introducing coordinates, for

Drivotin Oleg Igorevich — doctor of physical and mathematical sciences, professor; o.drivotin@spbu.ru

Дривотин Олег Игоревич — доктор физико-математических наук, профессор; o.drivotin@spbu.ru

example, tensors. Covariant tensor equality is not associated with any coordinate chart. Nevertheless, if coordinates are specified, a tensor equality can be regarded as a set of equalities of corresponding tensor components.

Customary forms of the Liouville and the Vlasov equations contain partial derivatives of a distribution function with respect to space coordinates and components of the momentum vector [1, 2]. Therefore, they cannot be regarded as covariant, because differentiating with respect to vector components is not covariant operation. Covariant approach allows making use such operation as gradient of function, which components are derivatives with respect to coordinates in some space. But in general case, components of the momentum vector cannot be taken as global coordinates in a phase space, as it is explained in section 2.

In all of these works, the concept of distribution function was employed. The definition of a distribution function requires making use the concept of phase volume. As distinct from the works cited above, presented here approach does not include the concepts of distribution function and phase volume. The particle distribution density is introduced by a natural way as a differential form of some degree. The main ideas of the present work were previously formulated in works [3, 4].

Kinetic equations in covariant form were also written down in some works (see, for example, [5–7]). But forms of top degree only [5, 7] and particle distribution function [6, 7] were employed there.

The presented here approach is valid both for nonrelativistic and relativistic cases equally. The difference between these cases relates only to the form of dynamics equations. In classical problems covariant approach has also great importance. It allows making use of various coordinates in the phase space. It is required, for example, in problems concerned with self-consistent particle distributions for a charged particle beam. Besides, this approach allows to consider degenerate distributions that are described by forms of lower degree.

In fact, such approach was used in numerous works devoted to self-consistent distributions for a cylindrical beam and for a beam with varying cross-section radius ("breathing" beam) [8–15]. In these works, the space of motion integrals was introduced, and particle density was defined in this space. Under some conditions, the phase density can be expressed through the introduced density. It was convenient to consider the phase density as a differential form, and the motion integrals as phase coordinates. It gave possibility to construct new solutions of the Vlasov equation. For example, they can be obtained as a linear combinations of degenerate distributions, or as solutions of some integral equation [9–15].

To verify that these distributions are solutions of the Vlasov equation, we can take into account that the phase density conserves along the characteristic lines of this equation. To provide rigorous proof, we need an equation where one can substitute these distribution. One of the purposes of this article is to write down such equation. Therefore, formalization of the approach applied in works [8–15] is offered in this article.

The article is organized as follows. In section 2 the concept of phase space is introduced. Section 3 is devoted to a definition of phase density. Section 4 deals with the Liouville and the Vlasov equations, which are written in covariant form. The difference between these two equations consists only in a way of computation of the force acting on the particles. In the Liouville equation the force is external one, while in the Vlasov equation it should be computed using a particle distribution the equation is written for. In section 5 the case of electric charged matter is considered. Relation between the phase

density and the 4-dimensional current density entering the Maxwell equations is obtained. As the conception of the four-dimensional current density is applicable both in relativistic as in nonrelativistic cases, this relation is valid in both cases. In section 6 two known examples are considered to show how new approach works. They are the Brillouin flow [16] and the Kapchinsky–Vladimirsky distribution [17]. Section 7 contains short discussion of the results.

2. Phase space. Physical processes occur in the 4-dimensional spacetime. Description of motion of a particle requires specifying not only its position in spacetime, but also its velocity. To identify state of a particle with account of its velocity we should define a phase space. When a phase space is defined, state of a particle can be regarded as a point in it.

Consider an open domain D in the spacetime. Assume that there exists some foliation formed by disjoint spacelike surfaces filling D. Let's parameterize these surfaces with a continuous parameter t.

Assume also that there exists a system of diffeomorphic mappings of these surfaces to some selected surface. Call this selected surface the configuration space and denote it by C. The assumption introduced above means that the domain D can be represented as $T \times C$, where T denotes an interval where the parameter t varies, $t \in T$. Such approach is widely used in general relativity, and it is called $3 + 1$ splitting of the spacetime [18–20].

When time passes, particles move from one surface to another. One can describe dynamics of particles as dynamics of their images in the configuration space. Call the tangent bundle [21] of the configuration space the phase space, and denote it by M, $\dim M = 6$. Points in M will be denoted by q, $q \in M$.

Such phase space is senseless from the physical point of view, but it can be used. In order to construct a meaningful phase space, let's apply the fundamental physical concept of reference frame. Following to works [22, 23], let us introduce a congruence of observers in D. Call one of them the main observer and call the others the local observers. Assume that the main observer can measure the time in D. Assume that at any instant of time measured by the main observer one can find a surface such that all events of it are simultaneous from the point of view of all observers of the congruence. Call such surface the layer of simultaneous events. It is easy to see that the layers of simultaneous events form a foliation in D.

Assume that there exists a system of diffeomorphic mappings of these layers to one selected layer such that for any event P its image is the intersection of the selected layer with worldline of an observer from the congruence that passes through P. The selected layer will be also called the configuration space, as previously.

The congruence of the observers and the system of the introduced mapping form a reference frame. It can be shown that coordinates of image of an event in the configuration space can be regarded as spatial coordinates of the event and time measured by the main observer is temporal coordinate of the event [22, 23]. As previously, call the tangent bundle of the configuration space the phase space. Note that this definition is valid both in classical and relativistic cases.

Phase space associated with a reference frame is preferable, because all values entering equations describing physical processes have physical sense. For example, in temporal-spatial coordinates associated with a reference frame components of the electromagnetic field tensor are components of the electric field and components of the magnetic field. Further we shall consider only such phase space.

According to definition of the tangent bundle [21], for every $x \in C$ there exists a

neighborhood U and a homeomorphic mapping $\Phi : p^{-1}(U) \mapsto U \times T_xU$ such that $p_1\Phi = p$. Here T_xU denotes the tangent space at the point x, which is fiber of the bundle, p and p_1 denote projections of M and $U \times T_xU$ on U correspondingly. It is clear that if one take three spatial coordinates x^1, x^2, x^3, then corresponding components of velocities can be taken as coordinates in T_xU, and these 6 numbers can be taken as coordinates locally in $p^{-1}(U)$.

Global coordinates in the phase space can be introduced as follows. Consider a subregion in the phase space corresponding to some subregion in the configuration space, which will be denoted also by C, instead of the whole phase space. It allows taking into account cases when spatial coordinates of some coordinate chart are defined not in all configuration space.

Assume that C is simply connected. Specify coordinates in C, and assume that components of the metric tensor in these coordinates are continuously differentiable in C.

Take some point $x_0 \in C$. Consider the boundary problem

$$\frac{\partial v^i}{\partial x^j} + \Gamma^i_{jk} v^k = 0, \quad v(x_0) = v_0, \tag{1}$$

where v_0 is some vector, $v_0 \in T_{x_0}C$, and Γ^i_{jk} denotes the Christoffel symbol of the second kind. Here and further we use the Einstein summation convention according to which summation is meant over all allowed values of repeated upper and lower indices.

The equation of parallel transport of a vector v along a line $x = x(\lambda)$ can be written in the form

$$\frac{dv^i}{d\lambda} + \Gamma^i_{jk} v^k \frac{dx^j}{d\lambda} = \left(\frac{\partial v^i}{\partial x^j} + \Gamma^i_{jk} v^k \right) \frac{dx^j}{d\lambda} = 0.$$

Therefore, the problem (1) describes a vector field such that any vector of this field at a point x can be regarded as a result of parallel transport of the vector v_0 along any line connecting points x and x_0.

This boundary problem for the system of linear differential equations has unique solution for a simply connected region. Indeed, solution of the problem (1) consists in finding of 3-dimensional surfaces in the phase space which tangent vectors are annihilated by forms $\omega_{(i)} = dv^i + \Gamma^i_{jk} v^k \, dx^j$, $i = \overline{1, n}$. It is sufficient for solvability that all $d\omega_{(i)} = 0$. Differentiating $\omega_{(i)}$, we get $d\omega_{(i)} = \Gamma^i_{jk} dv^k \wedge dx^j = 0$, as Christoffel symbols are permutation symmetric.

Therefore, for each vector $v \in T_xC$ we can find parallel vector $P_{x_0}v \in T_{x_0}C$ at the point x_0. Mapping $v \mapsto P_{x_0}v$ is continuously differentiable, as it is set by the Green functions of the problem (1). Then three coordinates in the configuration space and three components of $P_{x_0}v$ in these coordinates can be taken as coordinates of the point of the phase space specified by x and v. Besides, continuous differentiability of the mapping ensures that transition functions between various coordinates are differentiable. It means that phase space M is differentiable manifold.

If particles always lie on the same surface in the phase space, or distribution density does not depend on some coordinate, then the phase space M can be taken as corresponding subspace of the initial phase space. In this case dimension of the phase space under consideration is less then dimension of the initial phase space, and coordinates are not necessarily spatial coordinates and velocity coordinates and can represent a mixture of both kinds.

3. Phase density. In this section we shall concern various types of distributions, all of which can be described on the base of a common approach.

As a simplest case, consider continuous media that occupies an open set in the phase space. Within the framework of this model, particle number in an open subregion G, $G \subset M$, is not necessarily integer. Call the differential form $n(t, q)$ of degree $m = \dim M$ such that integration of the form over each open set G gives particle number in G the particle distribution density in the phase space, or the phase density:

$$\int\limits_G n = N_G.$$

The boundaries of G and the form n are assumed sufficiently smooth for integration being possible.

Consider also another case when particle are distributed on an oriented surface S in the phase space that can move, $\dim S = p$, $0 < p < m$. Call the differential form $n(t, q)$ of degree p defined on the surface S such that for any open set G, $G \subset M$,

$$\int\limits_{G \cap S} n = N_G$$

the particle distribution density for this case. This form depends on orientation of the surface, which is defined by an ordered set of $m - p$ vectors. A change of the orientation can result in change of sign of the form components [24]. Assume that form n and the surface S are also sufficiently smooth for integration being possible.

At last, consider the case of ensemble of pointlike particles. Define the scalar function

$$\delta_{q'}(q) = \begin{cases} 1, & q = q', \\ 0, & q \neq q'. \end{cases} \tag{2}$$

If q' depends on t, then this function is also function of t. All functions which values are nonzero only in finite set of points can be represented as linear combination of the functions of form (2). Restrict ourselves only to combinations with all coefficients equal to 1:

$$n(t, q) = \sum_{i=1}^{N} \delta_{q_{(i)}}(q), \qquad q_{(i)} \neq q_{(j)}, \quad \text{if} \quad i \neq j. \tag{3}$$

In this class of functions, define an operation of taking sum of function values in all points $q_{(i)}$, where the function value is nonzero:

$$\sum_{q \in G} n(t, q) \equiv \sum_{i:\, q_{(i)} \in G} n(t, q_{(i)}). \tag{4}$$

Operation defined by equation (4) is analogous to integration of the form of higher degree over G. A scalar function can be regarded as the differential form of degree 0. Therefore, equation (4) set a rule of integration of a form of degree 0 over open set G. As previously, call function of form (3) the phase density for system of pointlike particles if

$$\sum_{q \in G} n(t, q) = N_G.$$

It is easy to understand that the phase density is given by equality (3), where $q_{(i)}$ are positions of the particles in the phase space, $i = \overline{1, N}$, N is the total number of particles in the ensemble.

These three cases can be combined as follows. Denote by G_0 a region in the phase space enclosing all open sets G which can be considered. Consider a linear space $\mathcal{F}$ of some integrable test functions defined on the set G_0. Define functional $< n, f >$ as

$$\int\limits_{G_0} n(t,q)f(q), \qquad \int\limits_{G_0 \cap S} n(t,q)f(q), \qquad \sum_{q \in G_0} n(t,q)f(q)$$

in the first, the second, and the third cases correspondingly, $f \in \mathcal{F}$. Then definition of the phase density n can be written in the form

$$< n, \chi_G >= N_G,$$

where χ_G is the characteristic function of the set G, $G \subset G_0$:

$$\chi_G(q) = \begin{cases} 1, & q \in G, \\ 0, & q \notin G. \end{cases}$$

4. The Liouville and the Vlasov Equations. At each instant of time $t \in T$, particle dynamics equations define a vector field $f(t,q)$ in the phase space. Assume that for each $t \in T$, $q \in M$ there exists a unique integral line passing through point q. For example, if components of $f(t,q)$ in Cartesian coordinates are continuously differentiable with respect to coordinates and the time, it will be so. The time can be taken as a parameter for the integral lines.

We shall use operation of the Lie dragging [24] to describe how the phase density changes when the time passes. Lie dragging of a point q is the point gotten by displacement of q along the integral curve passing through q by the parameter increment δt. Denote it by $F_{f,\delta t}q$. By virtue of uniqueness of the integral curve passing through a point, this mapping is reversible.

The mapping $F_{f,\delta t}q$ induces the following coordinate transformation. For each point q, let's take coordinates of its preimage. Such transformation can be regarded as shift of the system of coordinates. Denote these coordinates as $q^1_{(f,\delta t)}, \ldots, q^6_{(f,\delta t)}$.

Let some tensor field T be defined in the phase space. Define tensor at the point $F_{f,\delta t}q$ as follows: its components in coordinates $q^i_{(f,\delta t)}$, are equal to components of T at the point q in the initial coordinates. Such tensor is called the Lie dragging of the tensor T along vector field f by the parameter increment δt. Denote this tensor by $F_{f,\delta t}q$.

How do phase density changes when particles move? It is easy to understand that degree of the differential form describing particle distribution does not change, because we can take dragged vectors of basis at the initial point q as the basis vectors at the dragged point $F_{f,\delta t}q$. Assume also that the particles do not appear and disappear. Therefore, integrals of the phase density over any set should be the same as integral over dragged set. For example, for a form n of top degree m

$$\int\limits_{F_{f,\delta t}G} n(t+\delta t, q) = \int\limits_{G} n(t).$$

Here $F_{f,\delta t}G$ denotes an image of the set G, that is set all points of which are images of points of G. Introducing coordinates $q^i_{(f,\delta t)}$ we see that $F_{f,\delta t}G$ looks in these coordinates as D in coordinates q^i. Therefore, to ensure equality of the integrals for any region it is necessary and sufficient that the following equation will be satisfied

$$n(t+\delta t, F_{f\,\delta t}q) = F_{f,\,\delta t}n(t,q). \tag{5}$$

It is easy to see that analogous reasons take place when degree of form n is less than m. Therefore, equation (5) is valid in these cases also.

Vector field f depends on the force acting on a particle. Let us call equation (5) the covariant form of the Liouville equation if the force is purely external. Let us call equation (5) the covariant form of the Vlasov equation if the force is determined also by a self field, by which we mean field produced by the particle ensemble. In the first case, the equation (5) can be regarded as a differential equation. Though it does not contain derivatives, it describes such object as differential form. In the second case, the equation (5) can be regarded as an integro-differential equation, because usually it is possible to represent the self field in an integral form with use of the Green functions. We assume here that structure of separate particles can be ignored, and one should take into account only phase density while computing the force acting from the self field. Account of a particle structure gives an additional term in the Vlasov equation which is called the collision integral. This term is sufficient only at high densities, and it will be omitted from further consideration here.

Consider the case when the particle distribution is described by a top degree form. How does this form change at some point q of the phase space depending on the time? Assume that its only component $\tilde{n}$ is continuously differentiable with respect to phase coordinates and time.

Let at some instance of time t the phase density at a point q is equal to $n(t,q)$. At the instance $t+\delta t$ it will be equal to $n(t+\delta t,q) = F_{f\,\delta t}n(t, F_{f\,-\delta t}q)$, as the phase density changes according to equation (5). Introduce the derivative of a differential form with respect to the parameter t as a form which components are derivatives of corresponding components with respect to t. Then we obtain the Liouville and the Vlasov equation in the form

$$\frac{\partial n}{\partial t} = \lim_{\delta t\to 0}\frac{n(t+\delta t,q)-n(t,q)}{\delta t} = -\mathcal{L}_f n(t,q). \tag{6}$$

Here $\mathcal{L}_f n(t,q)$ denotes the Lie derivative of the phase density along the vector field f, which can be defined as follows. The Lie derivative of a tensor field T along a vector field f is

$$\mathcal{L}_f T = \lim_{\delta t\to 0}\frac{T - F_{f,\,\delta t}T}{\delta t}.$$

The equation (6) for density form of top degree was considered also in works [5, 7]. For this case, it is easy to understand that the solution of (6) also satisfies to the equation (5) [5].

Components of the Lie derivative of a differential form T of degree p can be determined from the equalities

$$(\mathcal{L}_f T)_{i_1\ldots i_p} = \frac{\partial T_{i_1\ldots i_p}}{\partial q^k}f^k + \frac{\partial f^j}{\partial q^{i_1}}\cdot T_{j\,i_2\ldots i_p} + \ldots \frac{\partial f^j}{\partial q^{i_p}}\cdot T_{i_1\ldots i_{p-1}\,j}. \tag{7}$$

Summation is meant over possible values of repeated indices.

When the phase space is associated with a reference frame, the dynamics equations can be written in the form [24]

$$\frac{dx^i}{dt} = v^i, \quad m\sum_{j=1}^{3} g_{ij}(\frac{d}{dt}(\gamma v))^j = Q_i, \quad i=1,2,3, \tag{8}$$

where g_{ij} and $(d/dt(\gamma v))^j$ are spatial components of the metric tensor and of covariant derivative of vector γv correspondingly, γ is reduce energy of a particle, Q denotes 3-dimensional force vector acting on a particle.

For example, if particles move in the electromagnetic field, then

$$Q_i = e\left(E_i + \sum_{j=1}^{3} v^j B_{ij}\right), \tag{9}$$

where e and m are electric charge and mass of a particle, E is the electric field intensity, and B is the magnetic flux density, $B_{ij} = \mu_0 g^{-1/2} g_{il} g_{jm} \varepsilon^{klm} H_k$ [24], g is the determinant of the spatial part of the metric tensor, ε^{klm} are the 3-dimensional Levi–Civita symbols, H is the magnetic field intensity, μ_0 is the magnetic constant.

As an example, consider ensemble of nonrelativistic particles in the flat spacetime which dynamics is described by equation (8) with force term (9). Take Cartesian coordinates in the configuration space and Cartesian components of velocities as coordinates in the phase space. According to (9) components of the force acting on a particle does not depend on corresponding components of velocities. Then in the right hand side of equality (7) one should take into account only the first term, and equation (6) takes the form

$$\frac{\partial \tilde{n}}{\partial t} + \sum_{i=1}^{3} v^i \frac{\partial \tilde{n}}{\partial x^i} + \sum_{i=1}^{3} \frac{e}{m}(E_i + \sum_{i=1}^{3} B_{ij} v^j)\frac{\partial \tilde{n}}{\partial v^i} = 0.$$

This form of the Vlasov equation is widely used in applied problems, for example, in the theory of charged particle beams.

The simplest particular case is an ensemble consisting of one particle. In this case, the particle density is $\delta_{q_{(1)}(t)}(q)$. This scalar function is equal to 1 only at the point where the particle is located at the instant of time t, and is equal to 0 at other points. It is easy to understand that this density satisfies to the equation (5).

Density for a system of N particles taken in the form (3) also satisfies to the equation (5), which can be regarded as the Liouville equation if particle interaction is neglected and the Vlasov equation otherwise.

5. Computation of self electromagnetic field. The electromagnetic field can be described by the tensor of electromagnetic field F. When the first coordinate is temporal, for example, $x^0 = ct$, and the other three are spatial, components of the tensor are

$$\|F_{ik}\| = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_{12} & -B_{13} \\ -E_2/c & B_{12} & 0 & -B_{23} \\ -E_3/c & B_{13} & B_{23} & 0 \end{pmatrix}.$$

For simplicity, consider the case when the source of the electromagnetic field is electrically charged matter describing by the differential form J of the third degree in the 4-dimensional spacetime, the integral of which over smooth oriented 3-dimensional surface S gives the quantity of charge passing through S in the direction of its orientation:

$$\int_S J = Q_S.$$

This form is known as the current density form [24]. In the coordinates associated with a reference frame, spatial component J_{123} is the charge density, and the other components are

equal to components of the current density form of the second degree in the 3-dimensional configuration space with an accuracy up to the sign.

Then the Maxwell equations can be written in the form [24]

$$dF = 0, \quad d\,\overline{{}^*}F = J/(\varepsilon_0 c),$$

where $\overline{{}^*}$ denotes sequential application of the Hodge operator denoted as * and lowering of indices with the 4-dimensional metric tensor:

$$(\overline{{}^*}F)_{ik} = g_{il}g_{km}({}^*F)^{lm}, \qquad ({}^*F)^{lm} = |g|^{-1/2}\varepsilon^{ijlm}F_{ij},$$

g is determinant of the metric tensor, ε^{ijlm} is the 4-dimensional Levi–Civita symbol, ε_0 is electric constant.

In what follows, we shall concern ourselves how to express the current density J through the phase density n. For simplicity, assume that the phase density is also described by the form of top degree 6.

Consider a simple connected subregion C of the configuration space. According to the previous, in the phase space, which is tangent bundle of C, one can introduce six coordinates, three of them being coordinates in C. It means that they are spatial coordinates. Another three coordinates can be expressed through these spatial coordinates and components of velocity at this point, and, conversely, velocity at some point can be expressed through these six coordinates.

Denote the spatial coordinates by x^i, $x^i = q^i$, $i = 1, 2, 3$. Assume that the particles occupy some region in the phase space. Denote the section of this region at the point $\{x^1, x^2, x^3\}$ by $\Omega(x^1, x^2, x^3)$. Let us call $\Omega(x^1, x^2, x^3)$ the set of admissible values of the phase coordinates q^4, q^5, q^6 at the point of C with coordinates x^1, x^2, x^3. A value of particle velocity is admissible only if it corresponds to some point from $\Omega(x^1, x^2, x^3)$.

Integrating the current density form over $\Omega(x^1, x^2, x^3)$ we obtain the density in the configuration space

$$J_{123}(x) = \int\limits_{\Omega(x^1,x^2,x^3)} n_{123456}(t, q)\, dq^4 \wedge dq^5 \wedge dq^6.$$

In order to find another components of the current density, for example, J_{012}, we take a point x in the spacetime, and consider a small 3-dimensional cell spanned by the edge vectors $\delta x^0 e_{(0)}$, $\delta x^1 e_{(1)}$, $\delta x^2 e_{(2)}$. Here $e_{(i)}$ denotes basic vector of the coordinate basis. The cell contains points with coordinates $x^0 + \alpha\delta x^0$, $x^1 + \beta\delta x^1$, $x^2 + \gamma\delta x^2$, $\alpha, \beta, \gamma \in [1, 0]$, and is a part of some 3-dimensional surface in the spacetime. Let the orientation of that surface is specified by the basic vector $e_{(3)}$. As configuration space is associated with a reference frame, x^0 is temporal coordinate. For definiteness, let $x^0 = t$. Denote this cell by C_{012}. Its smallness means that the dependence of the phase density and of the current density on x^0, x^1, x^2 can be neglected within the cell with an accuracy up to terms of higher order of smallness.

According to definition of the current density, the quantity of charge passing through the cell C_{012} in the direction $\delta x_{(3)}$ is equal to

$$\int\limits_{C_{012}} J = -J(\delta x^0 e_{(0)}, \delta x^1 e_{(1)}, \delta x^2 e_{(2)}) = -J_{012}\delta x^0 \delta x^1 \delta x^2.$$

The minus sign appears here in accordance with the integration rule [24] because orientation of the cell is specified by the vector $e_{(3)}$, and the set of the vectors $\{e_{(3)}, e_{(0)}, e_{(1)}, e_{(2)}\}$ is negatively oriented.

Let us express this quantity through the phase density. For given δt, particles passing through the cell C_{012} have the coordinate x^3 differing from the coordinate x^3 of the cell less then $(dx^3/dt)\delta t$. Consider the 4-dimensional cell spanned by the edges vectors $\delta x^0 e_{(0)}$, $\delta x^1 e_{(1)}$, $\delta x^2 e_{(2)}, (dx^3/dt)\delta t\, e_{(3)}$. Denote it by C_{0123}. The cell C_{012} is one of its faces. Not all of particles passing through C_{0123} have world lines that cross the 3-dimensional cell C_{012}, because their velocity have nonvanishing components dx^0/dt, dx^1/dt, and dx^2/dt, which can be regarded as longitudinal relative to the cell C_{012}. Therefore, their world lines may cross other faces of the cell C_{0123}, which can be considered as flank faces. But under assumption that the phase density varies slowly, for every particle leaving the 4-dimensional cell through flank boundary, there exists a particle entering the 4-dimensional cell through opposite flank boundary with the same velocity. Therefore number of particles crossing the face C_{012} is equal to number of particles in the cell C_{012}.

In order to calculate their number, consider number of particle in the 6-dimensional cell in the phase space with edge vectors $\delta x^1 e_{(1)}$, $\delta x^2 e_{(2)}$, $(dx^3/dt)\delta t\, e_{(3)}$, $\delta q_{(4)}$, $\delta q_{(5)}$, $\delta q_{(6)}$:

$$n_{123456}(t,q)dx^1\wedge dx^2\wedge dx^3\wedge dq^4\wedge dq^5\wedge dq^6(\delta x^1 e_{(1)}, \delta x^2 e_{(2)}, (dx^3/dt)\,\delta t e_{(3)}, \delta q_{(4)}, \delta q_{(5)}, \delta q_{(6)}) =$$

$$= n_{123456}(t,q)\delta x^0\delta x^1\delta x^2 v^3 dq^4\wedge dq^5\wedge dq^6(\delta q_{(4)}, \delta q_{(5)}, \delta q_{(6)}),$$

here $\delta q_{(4)}$, $\delta q_{(5)}$, $\delta q_{(6)}$ are some linearly independent vectors with vanishing spatial components. Integrating over all admissible values of the phase coordinates q^4, q^5, q^6, we get the equality

$$J_{012} = -\int\limits_{\Omega(x^1,x^2,x^3)} n_{123456}(t,q)v^3(t,q)\, dq^4\wedge dq^5\wedge dq^6.$$

Analogously, we have

$$J_{013} = \int\limits_{\Omega(x^1,x^2,x^3)} n(t,q)_{123456}v^2(t,q)\, dq^4\wedge dq^5\wedge dq^6,$$

$$J_{023} = -\int\limits_{\Omega(x^1,x^2,x^3)} n(t,q)_{123456}v^1(t,q)\, dq^4\wedge dq^5\wedge dq^6.$$

6. Particle distributions for charged particle beam. Finding analytical solutions of the Vlasov equation is very complicated problem. It arises in charged particle beam physics for high current beam. The solutions of the Vlasov equation for high current beam are often called self-consistent distributions, because particles move in the field that is produced by them. A lot of papers are devoted to self-consistent distributions.

We give here two examples of stationary particle distributions in the phase space for charged particle beam, the Brillouin flow [16], and the Kapchinsky–Vladimirsky distribution [17]. They are degenerate distributions, because dimension of support of them is less then dimension of the phase space. The stationarity means that the phase density does not depend on the time.

In both cases, consider nonrelativistic uniformly charged cylindrical beam in uniform longitudinal magnetic field $H = (0, 0, H_z)$. Self field can be found from the Poisson equation

$$\Delta\varphi = -e\varrho_0/\varepsilon_0,$$

where ϱ_0 is spatial density of the particles inside the beam cross-section.

Assume that all particles have the same longitudinal velocity. Transverse particle motion is described by equations (8) and (9) where $E_r = e\varrho_0 r/(2\varepsilon_0)$, $E_\varphi = E_z = 0$, $B_{r\varphi} = \mu_0 r H_z$, $B_{rz} = B_{\varphi z} = 0$. Integrating these equations, we get following integrals of transverse motion [8–15]:

$$M = r^2(\dot\varphi + \omega_0), \tag{10}$$

$$H = \dot r^2 + \omega^2 r^2 + M^2/r^2. \tag{11}$$

Here r, φ, z are cylindrical coordinates, overdot denotes differentiation with respect to t, $\omega_0 = eB_z/(2m)$, $\omega^2 = \omega_0^2 - e\varrho_0/(m\varepsilon_0) = \text{const}$, e, m are charge and mass of the particle, $B_z = B_{r\varphi}/r = \text{const}$.

The Brillouin flow is trivial example, and can be described without concerning of the phase density. But we apply the approach developed here to show how it works. In the Brillouin flow all particles rotate around the beam axis with the same angular velocity. It is easy to find this velocity and the particle spatial density: $\dot\varphi = -\omega_0$, $\varrho = m\omega_0^2\varepsilon_0/e$. That means that for all particles $M = 0$, and $H = 0$.

In order to introduce the phase space, consider a transverse slice of the beam moving along the axis: $z \in (z_0 + v^z t, z_0 + \delta z + v^z t)$, δz being assumed small enough to consider that all particles of the slice have the same value of the coordinate z. As assumed previously, all particles of the slice have also the same longitudinal component of the velocity. Therefore, the phase space for the particles of this slice is 4-dimensional in the general case. But in the case of the Brillouin flow, all the particles of the slice lie on the surface $M = 0$, and $H = 0$. Let us take this surface as the phase space. This phase space is 2-dimensional. Spatial coordinates r and φ can be taken as coordinates also in the phase space, $r < R$, where R is the beam radius. Then the Vlasov equation (6) takes the form

$$\frac{\partial n_{r\varphi}}{\partial t} + \dot r \frac{\partial n_{r\varphi}}{\partial r} + \dot\varphi \frac{\partial n_{r\varphi}}{\partial \varphi} = 0.$$

This equation is satisfied, because all terms are equal to 0.

Consider the Kapchinsky–Vladimirsky distribution. It is also called the microcanonical distribution, because all the particles have the same value of the energy of the transverse motion $H = H_0$. For every admissible value of M there exists a set of particle trajectories, for which r and φ change in accordance with equations (10), (11). Rotation of some trajectory by an arbitrary angle around the beam axis gives another trajectory with the same M. Assume that the particles are evenly distributed on all the trajectories with the same M differing by the rotation angle.

Denote the phase of the particle on the trajectory by θ, and take M, θ and φ as coordinates in the phase space, which is the surface $H = H_0$. Then the Vlasov equation takes the form

$$\frac{\partial n_{\varphi\theta M}}{\partial t} + \dot\varphi \frac{\partial n_{\varphi\theta M}}{\partial \varphi} + \dot\theta \frac{\partial n_{\varphi\theta M}}{\partial \theta} + \dot M \frac{\partial n_{\varphi\theta M}}{\partial M} = 0.$$

The first term is equal to 0 according to the stationarity of the distribution. The second term is equal to 0 according to the above assumption about uniformity of azimuthal distribution. The forth term is equal to 0 as M conserves. Therefore, we obtain from the Vlasov equation that the particles should be evenly distributed on phases θ of the trajectories, as $\dot\theta \neq 0$.

The Vlasov equation is satisfied, and the problem seems to be solved, but it is more complicated because the solution is obtained under assumption that beam is uniformly charged. Let us show that the spatial density of the beam is uniform if the particles are uniformly distributed on M on a segment $(-M_0, M_0)$.

First of all, note that beam boundary $r = R$ can be reached only by particles with $M = 0$. As $\dot{r} = 0$ at $r = R$, we have that $H_0 = \omega^2 R^2$.

It can be seen from (11) that radial velocity changes as

$$\dot{r} = \pm\sqrt{H - \omega^2 r^2 - \frac{M^2}{r^2}}$$

along a particle trajectory. It is easy to get that maximal value of $|M|$ is reached when the particle moves along circular trajectory: $\dot{r} = 0$. For circular trajectory $r = \sqrt{|M|/\omega}$, and therefore $H = 2\omega|M|$. Taking into account that $H = \omega^2 R^2$, we get that maximal value of $|M|$ is $M_0 = \omega R^2/2$.

Let us introduce other coordinates in the phase space: x, y, M, where x and y are Cartesian coordinates in the configuration space. Components n_{xyM} and $n_{\phi\theta M}$ are related by the tensor component transformation rule, which in this case can be written as

$$n_{xyM} = 2n_{\varphi\theta M} \cdot \det\left|\begin{pmatrix} \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \end{pmatrix}\right| = \frac{2n_{\varphi\theta M}}{r|\dot{r}|}.$$

Here we take into account that each trajectory passes twice through every point in the phase space that lies on it, when the particle approaches to the axis, and when the particle moves away from the axis. These cases differ only by the sign of the radial component of velocity. The multiplier 2 is written in compliance with this reason.

Then

$$\varrho = \int\limits_{-M_0}^{M^0} n_{xyM}\, dM = \frac{2n_{\varphi\theta M}}{r} \int\limits_{-M_0}^{M_0} \frac{dM}{(H_0 - M^2/r^2 - \omega^2 r^2)^{1/2}} = 2\pi n_{\varphi\theta M} = \text{const}.$$

Thus, we have proved that spatial density is uniform inside the beam cross-section. That means that uniform phase density defined on the surface $H = H_0$, $M \in (-M_0, M_0)$ corresponds to the Kapchinsky–Vladimirsky distribution.

7. Conclusion. The covariant theory of the phase space distributions for particles moving in the spacetime is presented. If there are introduced coordinates in the spacetime, it is possible to consider a phase space. It turns out that a phase space is not necessarily associated with a reference frame. But from physical point of view, it is preferable to use phase space associated with a reference frame. As it is shown in the works [22, 23] the reference frame can be defined by the same manner both in relativistic and nonrelativistic cases. Therefore, the concept of phase space associated with a reference frame is the same for both cases. The differences between both cases are related only to the way of specifying of the metric tensor and to the form of motion equations.

The approach presented here does not use the concept of phase volume, as compared to the common approach. If one introduces the phase volume, then particle distribution function $\overline{n}$ can be defined as multiplier in the equality $n = \overline{n}\Omega_P$ [2, 7], where n is the phase density form of top degree and Ω_P is the phase volume form (the Liouville 6-form in [7]).

Firstly, that approach requires rigorous definition of the phase volume form. In

work [7] the Liouville 6-form is defined in the 7-dimensional space, and reduction to the 6-dimensional space is carried out only for the case of the flat Minkowski spacetime. In the present approach the phase space is 6-dimensional, and the 7-dimensional space is required only for computation of self force using the Maxwell equations that contain 4-dimensional electric current density.

Secondly, the definition of the phase density as the distribution function can be used only in the case of nondegenerate distribution, when particles are distributed in some open subdomain of the phase space. By this reason, such approach faces difficulties for degenerate distributions, when particles are distributed on some surface in the phase space.

As it is shown by the examples, the approach presented here is simple in use, and plainer, because it does not contain unnecessary conception of the phase volume.

The theory presented here is rigorous basis for consideration of particle distributions for a charged particle beam, particularly, of degenerate distributions. Degenerate particle distributions are often used as model distributions, for example, for numerical solutions of the optimization problems for charged particle accelerators [25].

The theory can be also used in other problems concerned with self-consistent field of moving particles. For example, the problem of self-gravitating star matter is of interest in cosmology. Another problem, where this theory can be applied, is magnetized matter, for example, gas of neutrons.

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