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Irrational-Behavior-Proof Conditions for the Group Pursuit Game

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1 Abstract

This paper focuses on a group pursuit game problem involving a pursuer P and multiple evaders $E_i\,(i = 1, \ldots, 4)$. In a model highly disadvantageous to evaders, an additional velocity α^+ is introduced for pursuer. The pursuer possesses a velocity α^- for capturing evaders moving in prescribed directions and a velocity α^+ for capturing evaders deviating from the prescribed direction. However, α^+ can only be utilized once throughout the entire game.

The game model is described in the form of differential equations, with strategies and payoff functions defined for both the pursuer and evaders. We assume the pursuer employs a discriminatory strategy, while the evaders' movement directions are highly disadvantageous to themselves. In the nonzero-sum game, Nash equilibrium of the game is found, and conditions for the effectiveness of the pursuers' punishment strategy are proved, as presented in the paper. In the matrix game, strategies of the evaders under rational and irrational behavior are separately studied. Through simulation, payoff matrices for evaders are obtained under scenarios of both equal and unequal velocity. It is revealed that for the evader group, even under irrational behavior, better payoffs can be achieved than under rational behavior. This situation may exist but might not necessarily materialize, as it also depends on the strategy of the pursuer.

2 Introduction

Game theory is a research domain in mathematics and economics, representing a new branch of modern mathematics and an important discipline in operations research. Key concepts in game theory include strategies, payoffs, game forms, and Nash equilibrium. It explores competition, cooperation, and conflict among participants, analyzing their behavior and potential outcomes. Through game theory, we gain a better understanding of market competition, strategic decision-making, resource allocation, and the optimal decision-making of decision-makers under mutual influence, providing a powerful framework for analyzing decisions and strategic interactions among rational entities.

The origin of game theory can be traced back to the early 20th century. John von Neumann, often referred to as "the father of computer science," collaborated with economist Oskar Morgenstern to publish "Theory of Games and Economic Behavior" in 1944. This seminal work laid the foundation for game theory and marked its true inception. They applied game theory to the study of economic and social issues, discussing many important topics including multiplayer games. In 1950, John Nash proposed the theory of Nash equilibrium in his doctoral dissertation, laying the cornerstone of modern non-cooperative game theory. The Nash equilibrium theory describes the optimal strategy combination of participants in the game, where each participant's strategy is the best response to the strategies of others. This theory has had profound impacts on game theory and other fields, earning Nash the Nobel Prize in Economics in 1994.

Over time, game theory has evolved into a multidisciplinary field spanning economics, biology, computer science, political science and so on. It plays a crucial role in addressing diverse decision-making problems, strategic planning, and behavioral pattern analysis. As such, it has become a vital tool for understanding human behavior and societal interactions.

At its core, game theory investigates how individuals or entities make choices in situations where the outcome of each decision depends not only on their own actions but also on the actions of others. This framework enables researchers to model and analyze various scenarios, from competitive markets and negotiations to conflicts and cooperation.

Over the years, game theory has branched out into several subfields, each focusing on different aspects of strategic decision-making. Classical game theory, as a foundational branch, studies the behavior of rational players with complete information in competitive environments. In contrast, cooperative game theory examines how groups form alliances to achieve common interests. Non-cooperative game theory analyzes situations where players act independently to maximize their own interests, often resulting in strategic outcomes like Nash equilibrium. Evolutionary game theory extends this framework to dynamic systems, exploring how strategies evolve over time in interacting populations, revealing the emergence of cooperation and social norms. Computational game theory, as an amalgamation of game theory and computer science, applies numerical methods and computational techniques to tackle large-scale and intricate games, while multi-agent systems explore how agents interact, coordinate, and make decisions to achieve individual or collective objectives in various fields, including artificial intelligence, economics, and social sciences.

In modern society, people are often faced with various complex decision-making problems involving competition and cooperation among multiple stakeholders. Game theory, as a powerful analytical tool, provides an important theoretical framework and methodology for understanding and solving these decision-making problems. Among the various models in game theory, pursuit games stand out as a typical scenario that attracts considerable attention.

Pursuit games refer to a form of game played between pursuers and evaders, the pursuers are focused on capturing the evaders swiftly, whereas the evaders are determined to evade capture. This gaming scenario is commonly observed in various real-life situations, such as law enforcement operations, security defense, and predator-prey dynamics in natural ecosystems. In pursuit games, the strategic choices and behavioral decisions made by the participants directly impact the final outcomes and payoff distributions.

Of particular note, one-versus-many pursuit games represent a special scenario within pursuit dynamics, where a single pursuer needs to simultaneously pursue multiple evaders. In contrast to traditional pursuit games, one-versus-many pursuit games present greater challenges and complexity, as pursuer must effectively allocate resources and devise strategies to address the diverse behaviors and dynamic changes of multiple evaders.

In this paper, we will focus on exploring one-versus-many pursuit games and utilize the methods and theories of game theory to model, analyze, and solve them. Through the study of one-versus-many pursuit games, we aim to gain a deeper understanding of the competition and cooperation relationships among the participants, explore optimal decision-making strategies, and provide theoretical guidance and practical insights for decision-making problems in real-world applications.

However, unlike in the past, in this paper, the pursuer has two velocities. One is the usual velocity used during the pursuit, denoted as α^{-}, and the other is an acceleration of the pursuer, denoted as α^+, which is greater than α^-. When using velocity α^+, the pursuer can catch the evader more quickly, but it can only be used once. Therefore, this paper aims to explore how the strategy selection for the evader changes when the pursuer has two velocities.

This article is based on the model in paper [1, 2], and explores the impact of the punishment strategy at a velocity different from that in the original model. The conditions for effective punishment strategies for the pursuer were identified under both scenarios where evaders have identical and differing velocities. When punishment strategies prove ineffective, a payment matrix is derived through matrix game simulation, revealing the existence of a strategy that allows evaders to prolong their survival time even under irrational behavior.

Through research, i hope to contribute new perspectives and methodologies to address complex decision-making problems in real-life situations, thereby making a contribution to social and economic development.

3 Related works

In this section, the literature on group pursuit game is briefly introduced.

In the early 1960s, Isaacs [3] conducted in-depth research on dynamic game theory, including aspects of pursuit games. His research introduced the concept of zero-sum differential pursuit games, emphasizing optimal strategies and equilibrium concepts in dynamic games. Tarashnina [4], through studying nonzero-sum simple pursuit games in group chasing problems, revealed the phenomenon of the existence of infinitely many Nash equilibria. Specifically, by threatening to change the pursuit order, pursuers can compel evaders to adopt highly unfavorable behaviors. This has played an indispensable role in understanding the strategic dynamics and equilibrium states in group pursuit games, providing important insights for the research in this paper. Pankratova et al. [1] analyzed the dynamic equilibrium between a pursuer and m independently acting evaders in a nonzero-sum simple pursuit game, particularly focusing on how to find a time-optimal equilibrium state between pursuers and evaders, especially when evaders are discriminated against and pursuers have punishment strategy. In subsequent research, Pankratova et al. [2] focused on group pursuit games on a bounded velocity plane and described and solved this problem through two different formalized methods: non-cooperative and cooperative. The concepts of Nash equilibrium and core were introduced by them, along with the proof that the core is non-empty in all initial positions.

Furthermore, Fang et al. [5] conducted a study on the pursuit game in scenarios involving different maximum velocities among a single evader and several pursuers. Wang et al. [6] mainly investigated the case where evaders with higher velocities have an advantage in pursuit games, as well as the case where there is individual maximum volecity heterogeneity among a group of heterogeneous pursuers. Lin et al. [7] mainly studied the linear quadratic differential games between N pursuers and a single evader, particularly considering observation restrictions and asymmetric information structures. Tarashnina [8] focused on a traveling salesman problem in a dynamic environment, modeled as a nonzero-sum pursuit problem, where the salesman and customers chase each other. They ultimately found the solution to this problem, namely the Nash equilibrium, and illustrated their research results through some examples. Petrov [9] studied the problem of a group of pursuers chasing a group of evaders in Euclidean space, where all participants have equal opportunities. Assuming that evaders remain within a convex set during the game, the goal of the pursuers is to capture at least q evaders, with each evader being captured by at least r different pursuers, and capture times do not overlap.

4 Model of pursuit game

In this paper, we study a pursuit game in which there are five players including one pursuer P captures 4 evaders E_1,E_2,E_3,E_4 - the evaders move at a constant velocity in the coordinate system, while the pursuers have two velocities. Denote by $N = \{P, E_1, ..., E_4\}$ the set of players. And pursuer and evaders all have possibility of changing their movement direction at each time instant. This is a game of perfect information, at each time instant $t > 0$, the pursuer and every evader, possess knowledge of the time t, as well as their own positions and all others. Additionally, we assume that at each instant t the pursuer P knows the direction that is also the velocity-vectors chosen by the evaders E_i at that time. This is a discriminatory strategies used by pursuer against evaders.

At time instant $t = 0$, both the pursuer and the evaders begin to move from their initial positions:

$$z_p^0 = \left(x_p^0, y_p^0\right), z_i^0 = \left(x_i^0, y_i^0\right), i = 1, \ldots, 4 \tag{4.1}$$

And we denote by $P^t = z_p^t = \left(x_p^t, y_p^t\right)$ and $E_i^t = z_i^t = (x_i^t, y_i^t)$ the current positions of pursuer P and evader E_i at time instant t, respectively.

Let α^- and α^+ are the velocities of P, but α^+ can only be used once, also let β_i is the velocity of E_i $(i = 1, \ldots, 4)$. Because it is necessary to ensure that the pursuer catch all the evaders, we suppose that $\alpha^+ > \alpha^- > \max_{i=1,\ldots,4} \beta_i$, here α^-, α^+, β_i are constant.

The player's movement can be characterized by the subsequent set of differential equations[10]:

$$\begin{aligned} \dot{z}_p &= v_p, v_p \in V_p, \\ \dot{z}_i &= v_i, v_i \in V_i, i = 1, \ldots, 4 \end{aligned} \tag{4.2}$$

with initial conditions

$$z_p(0) = z_p^0, z_i(0) = z_i^0, i = 1, \ldots, 4 \tag{4.3}$$

where $z_p^t,\ z_1^t,\ z_2^t,\ z_3^t,\ z_4^t \in R^2$. And the set of control variables has the following forms [11], as shown in the figure 4.1.

$$\begin{aligned} V_p^- &= \left\{ v_p^- = \left(v_p^{-1}, v_p^{-2}\right) : \left(v_p^{-1}\right)^2 + \left(v_p^{-2}\right)^2 \leq \left(\alpha^-\right)^2 \right\} \\ V_p^+ &= \left\{ v_p^+ = \left(v_p^{+1}, v_p^{+2}\right) : \left(v_p^{+1}\right)^2 + \left(v_p^{+2}\right)^2 \leq \left(\alpha^+\right)^2 \right\} \\ V_i &= \left\{ v_i = \left(v_i^1, v_i^2\right) : \left(v_i^1\right)^2 + \left(v_i^2\right)^2 \leq \beta_i^2 \right\}, i = 1, \ldots, 4 \end{aligned} \tag{4.4}$$

and $V_p^-,\ V_p^+ \in V_p$.

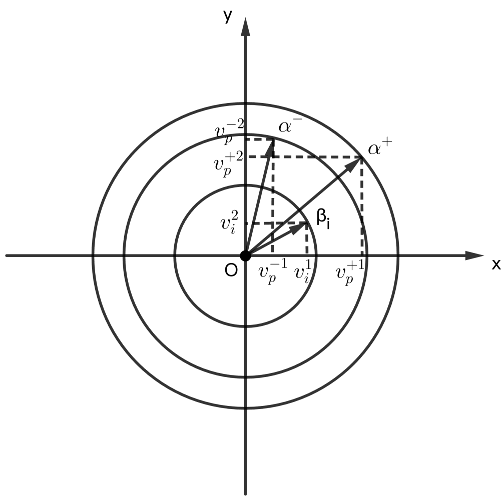

Figure 4.1: Velocity vector

As we all know, strategies are typically defined as the set of actions or choices available to each player when making decisions. Therefore, let's now define the behaviors of the pursuer P and evaders E_i throughout the entire game process, i.e., their strategies.

Definition 4.1. *The strategy of evader E_i includes time instant and its own and pursuer's position at time instant t which is a function $s_{E_i}(t, z_p^t, z_1^t, \ldots, z_4^t)$ in $S_{E_i} = \left\{s_{E_i(\cdot)}\right\}$. And evaders use open-loop strategies. Denote by S_{E_i} the set of admissible strategies of player E_i, $i = 1, \ldots, 4$.*

Definition 4.2. *Let $\pi\left(z_p^0, z_1^0, \ldots, z_4^0, s_{E_1}, \ldots, s_{E_4}\right)$ be a pursuit order chosen by the pursuer at the initial instant $t = 0$ for some fixed strategy profile of the evaders and $\pi \in \Pi$, Π is the set of all possible order. Obviously, depending on the pursuit order, the pursuer P can choose different velocity to catch evader E_i, resulting in the pursuer using different strategy s_p. Let $\pi = \{1, ..., 4\}$ be a pursuit order chosen by pursuer P. Then we can denote by $v_p = v_i(\pi)$ the velocity of pursuer P, $v_p = \left\{v_i(\pi) : v_i(\pi) \in \{\alpha^-, \alpha^+\}, \sum_{i=1}^4 v_i(\pi) = 3\alpha^- + \alpha^+\right\} \in V_p$.*

Definition 4.3. *There are more elements in a pursuer strategy than an evader strategy, the pursuer's strategy not only includes the time, its position at time instant t, but also the velocity-vector of the pursuer and evader and pursuit order chosen by the pursuer at the initial instant. A strategy of pursuer P is a function $s_p\left(t, z_p^t, z_1^t, \ldots, z_4^t, v_i(\pi), v_{E_1}^t, \ldots, v_{E_4}^t\right)$.*

Since evader E_i may deviate from the direction dictated by the pursuer at initial time during the game, there are many ways to solve this problem, the most common way is to define a punishment strategy to get Nash equilibrium. So, defining a punishment strategy $u_p^\pi = \langle \pi, s_p, p \rangle$, where $s_p\left(t, z_p^t, z_1^t, \ldots, z_4^t, v_i(\pi), v_{E_1}^t, \ldots, v_{E_4}^t\right)$ is a pursuit strategy of P and $p =$

$(t, s_{E_1}, \ldots, s_{E_4})$ is an element of punishment which includes the changed order of the evader deviating from the original direction at any time instant t. Then we say that the triple $u_p^\pi = \langle \pi, s_p\ , p \rangle$ is a strategy of pursuer P and also refer it as the punishment strategy of P, denote by $U_p = \left\{ u_p^\pi \right\}_{\pi \in \Pi}$ the strategy of pursuer P.

In the game, we assume that the pursuer P aims to capture all evaders with the minimal payoff, while each evader E_i wants to survive as long as possible to maximize its own payoff, so they only consider themselves without caring about others in the nonzero-sum pursuit game. But in matrix games, we consider the group of evaders as a whole.

The pursuit game in this article is played as follows, and the process is shown in the figure 4.2: at the beginning of the game, the pursuer P will assign the evader E_i move with a certain behavior. The pursuer calculates the minimal payoff to capture the evader who uses the prescribed behavior as the pursuit order. This means selecting the order with the minimal payoff from 4! pursuit orders as the initial pursuit order. If there are no deviations from the evaders, the pursuer P will capture the evaders in the initial pursuit order. If any evader E_i deviates from the direction prescribed by the pursuer at initial time, pursuer P will change the pursuit order and immediately pursuit the evader E_i to punish it, then calculates the minimum payoff of chasing the remaining evaders to determine the order of subsequent captures. Repeat this process until all evaders are captured, then the game ends. We define the coincidence of position of pursuer P and evaders E_i as evaders E_i being caught.

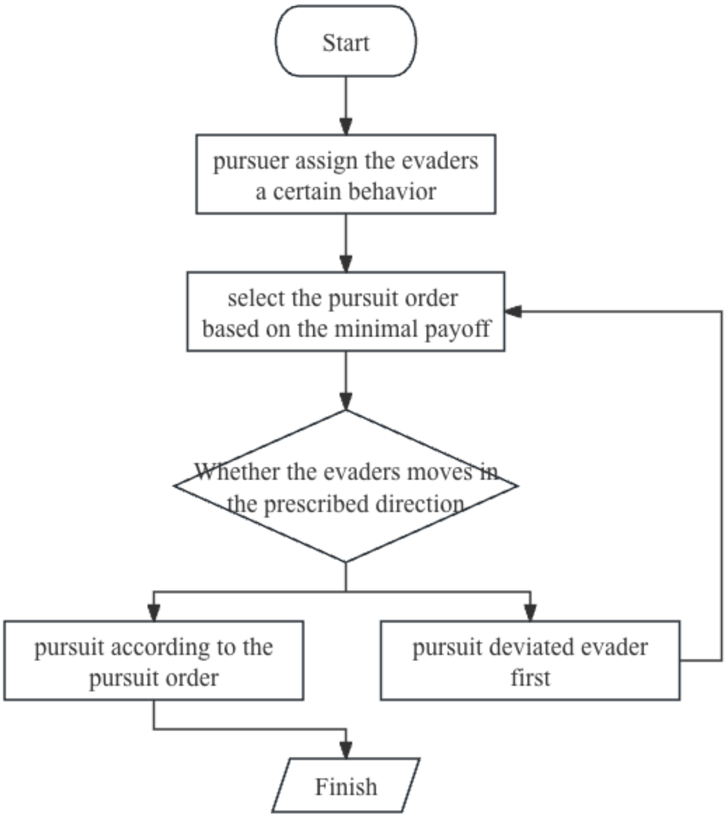

Figure 4.2: Game process

During the game process, to simplify the difficulty of the game, we stipulate that only one evader can deviate from the direction before the moment when the pursuer pursuit any evader, we do not consider the scenarios where the group of evaders deviate. The velocity of the pursuer P is greater than any evader E_i, which means that pursuer P can catch all evaders E_i within a finite time.

Definition 4.4. *Denote the payoff function of evader E_i and pursuer P by K_{E_i} and K_p, respectively [12].*

$$K_{E_i}\left(s_p, s_{E_1}, \ldots, s_{E_4}\right) = \sum_{k \leq i} T_k^\pi \tag{4.5}$$

where $T_k^\pi = \frac{\left|N^{k-1} E_k^{T_{k-1}}\right|}{v_i(\pi)-\beta_i}$ is the time it takes for the pursuer to capture the evader $E_k\,(k = 1, \ldots, 4)$ according to the pursuit order $\pi \in \Pi$. Here, i

denotes the sequence number of the evader E_i in the pursuit order $\pi = \{1, ..., 4\}$*, while* k *represents the number of evaders captured before* E_i*.* $\left|N^{k-1}E_k^{T_{k-1}}\right|$ *is the Euclidean distance between point* N^{k-1} *and* $E_k^{T_{k-1}}$*.* $v_i(\pi) - \beta_i$ *is the difference between the velocity used by pursuer* P *to capture the evaders and the velocity of the evader. And at initial time, we set* $N^0 = P^0$*.*

Definition 4.5. *In order to maximize its payoff, the pursuer strives to pursuit all the evaders as soon as possible, we define the payoff function of pursuer* P *as the negative value of the payoff function of the last caught evader* E_i *[12].*

$$K_P(s_p, s_{E_1}, \ldots, s_{E_4}) = -T^\pi \tag{4.6}$$

where $T^\pi = \sum_{k=1}^{4} T_k^\pi$ *is the total pursuit time, and* π *the pursuit order.*

5 Nonzero-sum pursuit game

Definition 5.1. *Define the nonzero-sum pursuit game in normal form as the following [13]:*

$$\Gamma\left(z_p^0, z_1^0, \ldots, z_4^0\right) = \left\langle N\ , \{S_i\}_{i\in N}\ , \{K_i\}_{i\in N}\right\rangle \tag{5.1}$$

where $N = \{P, E_1, ..., E_4\}$ is the set of players, S_i is the set of admissible strategies of player i and K_i is a payoff function of player $i\,(i \in N)$ defined by (4.5) and (4.6). The outcome of each game is influenced by the selection of the players' initial positions.

In this game, there exists a Nash equilibrium where the pursuer selects one evader for pursuit, while the other evaders move in the prescribed direction. If anyone deviates from this direction, the pursuer immediately changes the pursuit order to punish them.

There are two types of behaviors for the evaders:

1. behavior $u_i^{j'}$ requires moving towards the current capture point $N^{j'}$ along the line connecting the positions of the pursuer P and the evader;

2. behavior u_i^j requires moving towards the capture point of the currently pursued evader $E_i^{j'}$, $j > j'$, namely, towards the current capture point $N^{j'}$, where $N^{j'} = P^{T_{j'}}$, $j' \in \{1, ..., 4\}$.

Suppose the pursuit order $\pi = \{1, 2, 3, 4\}$ and the pursuit process without any evader deviating from the prescribed direction is illustrated in Figure 5.1.

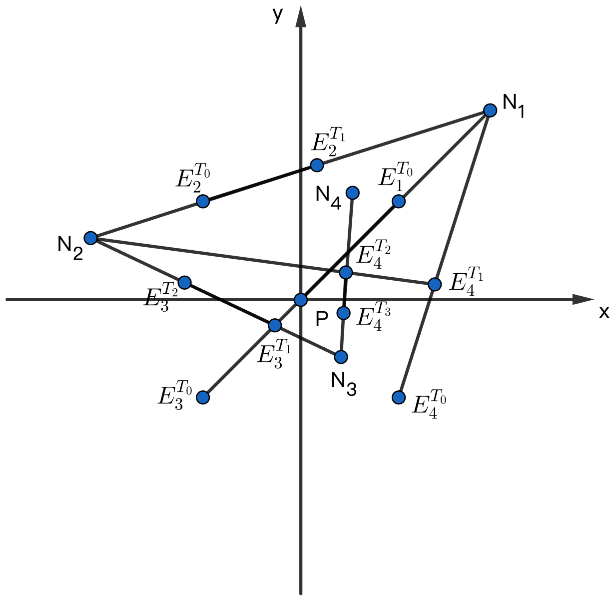

Figure 5.1: Capture process

We divide the set of evaders $E_i = \{E_1, ..., E_4\}$ into two types, the first is the evaders that have not been caught, denote as $S = \left\{E_i^j\right\}_{j>j'}$. Here E_i^j represents the type of the j_{th} evader in the pursuit order, among those who have not been caught yet. The second is the evaders who have been caught, denote as $M = \left\{E_i^{j'}\right\}$, and $E_i^{j'}$ is the evader currently being pursued.

And suppose that $T^0 = 0, N^0 = P^0$.

At initial time $t = 0$, pursuer P will calculate the time it takes to capture the evader in all possible orders and select the one with the minimal payoff as the original capture order. We denote this order as the optimal order $\pi^* = \{E_1, ..., E_4\}$, and its corresponding strategy as $u_p^{\pi^*}$. If there is evader deviating from the original direction during the game, for the convenience of calculation, we can renumber the evader according to

$\pi^* = \{E_1, ..., E_4\}$. Hence, we have

$$T^{\pi^*} = \min_{\pi \in \Pi} T^{\pi} \tag{5.2}$$

We need to now determine the circumstances in which the pursuer's total time to capture the evader in the original pursuit order using α^- exceeds the total time for the pursuer to pursue all evaders once using α^+ in the presence of deviations from the prescribed direction.

For the group of evader this is true if the following inequality holds:

$$\begin{aligned} &\frac{\left|N^0 E_1^{T_0}\right|}{v_i(\pi) - \beta_1} + \frac{\left|N^1 E_2^{T_1}\right|}{v_i(\pi) - \beta_2} + \frac{\left|N^2 E_3^{T_2}\right|}{v_i(\pi) - \beta_3} + \frac{\left|N^3 E_4^{T_3}\right|}{v_i(\pi) - \beta_4} > \\ &\frac{\left|N^0 E_1'^{T_0}\right|}{v_i(\pi) - \beta_1'} + \frac{\left|N^1 E_2'^{T_1}\right|}{v_i(\pi) - \beta_2'} + \frac{\left|N^2 E_3'^{T_2}\right|}{v_i(\pi) - \beta_3'} + \frac{\left|N^3 E_4'^{T_3}\right|}{v_i(\pi) - \beta_4'} \end{aligned} \tag{5.3}$$

where E_i' refers to the evader E_i in the reordering when any evader deviates from the prescribed direction, while β_i' refers to the order of the i_{th} evader in this sequence.

These two formulas (5.2) and (5.3) provide existence of the given Nash equilibrium.

Therefore, the effectiveness of the pursuer's punishment strategy for evaders with the same and different velocities can be determined by the following two theorems.

Theorem 5.1. *For evader with the same velocity, if an evader deviates from the prescribed direction at any time instant $t > 0$, use α^+ to capture the i_{th} evader in new pursuit order. The pursuer's punishment strategy is effective if*

$$\frac{\sum_{k=1}^{4}\left|N^{k-1}E_k^{T_{k-1}}\right| - \sum_{k \neq i}\left|N^{k-1}E_k'^{T_{k-1}}\right|}{\left|N^{k-1}E_k'^{T_{k-1}}\right|_{k=i}} > \frac{1-l}{m-l}, k = 1, 2, 3, 4 \tag{5.4}$$

Proof. Let $\alpha^+ = m\alpha^-$, where $m > 1$; $\beta_i = l_n\alpha^-$, where $l_n < 1$, $n = 1,\cdots,4$, and m, l_n are constant.

Here $l_1 = l_2 = l_3 = l_4 = l$.

For evaders with the same velocity, if an evader deviates from the prescribed direction at any time instant $t > 0$, use α^+ to capture the first evader in new pursuit order:

$$\frac{\left|N^0E_1^{T_0}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^1E_2^{T_1}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^2E_3^{T_2}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^3E_4^{T_3}\right|}{\alpha^- - l\alpha^-} >$$
$$\frac{\left|N^0E_1^{\prime T_0}\right|}{m\alpha^- - l\alpha^-} + \frac{\left|N^1E_2^{\prime T_1}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^2E_3^{\prime T_2}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^3E_4^{\prime T_3}\right|}{\alpha^- - l\alpha^-}$$

$$\frac{\left|N^0E_1^{T_0}\right| + \left|N^1E_2^{T_1}\right| + \left|N^2E_3^{T_2}\right| + \left|N^3E_4^{T_3}\right| - \left|N^1E_2^{\prime T_1}\right| - \left|N^2E_3^{\prime T_2}\right| - \left|N^3E_4^{\prime T_3}\right|}{\left|N^0E_1^{\prime T_0}\right|}$$

$$> \frac{1-l}{m-l}$$

For evaders with the same velocity, if an evader deviates from the prescribed direction at any time instant $t > 0$, use α^+ to capture the second evader in new pursuit order:

$$\frac{\left|N^0E_1^{T_0}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^1E_2^{T_1}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^2E_3^{T_2}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^3E_4^{T_3}\right|}{\alpha^- - l\alpha^-} >$$
$$\frac{\left|N^0E_1^{\prime T_0}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^1E_2^{\prime T_1}\right|}{m\alpha^- - l\alpha^-} + \frac{\left|N^2E_3^{\prime T_2}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^3E_4^{\prime T_3}\right|}{\alpha^- - l\alpha^-}$$

$$\frac{\left|N^0E_1^{T_0}\right| + \left|N^1E_2^{T_1}\right| + \left|N^2E_3^{T_2}\right| + \left|N^3E_4^{T_3}\right| - \left|N^0E_1^{\prime T_0}\right| - \left|N^2E_3^{\prime T_2}\right| - \left|N^3E_4^{\prime T_3}\right|}{\left|N^1E_2^{\prime T_1}\right|}$$

$$> \frac{1-l}{m-l}$$

For evaders with the same velocity, if an evader deviates from the prescribed direction at any time instant $t > 0$, use α^+ to capture the third evader in new pursuit order:

$$\frac{\left|N^0E_1^{T_0}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^1E_2^{T_1}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^2E_3^{T_2}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^3E_4^{T_3}\right|}{\alpha^- - l\alpha^-} >$$
$$\frac{\left|N^0E_1^{\prime T_0}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^1E_2^{\prime T_1}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^2E_3^{\prime T_2}\right|}{m\alpha^- - l\alpha^-} + \frac{\left|N^3E_4^{\prime T_3}\right|}{\alpha^- - l\alpha^-}$$

$$\frac{\left|N^0E_1^{T_0}\right| + \left|N^1E_2^{T_1}\right| + \left|N^2E_3^{T_2}\right| + \left|N^3E_4^{T_3}\right| - \left|N^0E_1^{\prime T_0}\right| - \left|N^1E_2^{\prime T_1}\right| - \left|N^3E_4^{\prime T_3}\right|}{\left|N^2E_3^{\prime T_2}\right|}$$

$$> \frac{1-l}{m-l}$$

For evaders with the same velocity, if an evader deviates from the prescribed direction at any time instant $t > 0$, use α^+ to capture the fourth evader in new pursuit order:

$$\frac{\left|N^0E_1^{T_0}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^1E_2^{T_1}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^2E_3^{T_2}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^3E_4^{T_3}\right|}{\alpha^- - l\alpha^-} >$$
$$\frac{\left|N^0E_1^{\prime T_0}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^1E_2^{\prime T_1}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^2E_3^{\prime T_2}\right|}{\alpha^- - l\alpha^-} + \frac{\left|N^3E_4^{\prime T_3}\right|}{m\alpha^- - l\alpha^-}$$

$$\frac{\left|N^0E_1^{T_0}\right| + \left|N^1E_2^{T_1}\right| + \left|N^2E_3^{T_2}\right| + \left|N^3E_4^{T_3}\right| - \left|N^0E_1^{\prime T_0}\right| - \left|N^1E_2^{\prime T_1}\right| - \left|N^2E_3^{\prime T_2}\right|}{\left|N^3E_4^{\prime T_3}\right|}$$

$$> \frac{1-l}{m-l}$$

□

Theorem 5.2. *For evader with the different velocity, if an evader deviates from the prescribed direction at any time instant $t > 0$, use α^+ to capture the i_{th} evader in new pursuit order. The pursuer's punishment strategy is effective if*

$$\frac{\sum_{k=1}^{4}(\left|N^{k-1}E_k^{T_{k-1}}\right|\prod_{n\neq k}(1-l_n))}{\prod_{n=1}^{4}(1-l_n)} >$$

$$\frac{\left|N^{k-1}E_k^{\prime T_{k-1}}\right|_{k=i}}{(m-l_n')_{n=i}} + \frac{\sum_{k\neq i}(\left|N^{k-1}E_k^{T_{k-1}}\right|\prod_{n\neq i,k}(1-l_n))}{\prod_{n\neq k}(1-l_n')}, k = 1,2,3,4 \tag{5.5}$$

Proof. Let $\alpha^+ = m\alpha^-$, where $m > 1$; $\beta_i = l_n\alpha^-$, where $l_n < 1$, $n = 1,\cdots,4$, and m, l are constant.

For evaders with the different velocity, if an evader deviates from the prescribed direction at any time instant $t > 0$, use α^+ to capture the first evader in new pursuit order:

$$\frac{\left|N^0E_1^{T_0}\right|}{\alpha^- - l_1\alpha^-} + \frac{\left|N^1E_2^{T_1}\right|}{\alpha^- - l_2\alpha^-} + \frac{\left|N^2E_3^{T_2}\right|}{\alpha^- - l_3\alpha^-} + \frac{\left|N^3E_4^{T_3}\right|}{\alpha^- - l_4\alpha^-} >$$
$$\frac{\left|N^0E_1^{\prime T_0}\right|}{m\alpha^- - l_1'\alpha^-} + \frac{\left|N^1E_2^{\prime T_1}\right|}{\alpha^- - l_2'\alpha^-} + \frac{\left|N^2E_3^{\prime T_2}\right|}{\alpha^- - l_3'\alpha^-} + \frac{\left|N^3E_4^{\prime T_3}\right|}{\alpha^- - l_4'\alpha^-}$$

$$\frac{\left|N^0E_1^{T_0}\right|}{1-l_1} + \frac{\left|N^1E_2^{T_1}\right|}{1-l_2} + \frac{\left|N^2E_3^{T_2}\right|}{1-l_3} + \frac{\left|N^3E_4^{T_3}\right|}{1-l_4} > \frac{\left|N^0E_1^{\prime T_0}\right|}{m-l_1'} +$$

$$\frac{\left|N^1E_2^{\prime T_1}\right|(1-l_3')(1-l_4') + \left|N^2E_3^{\prime T_2}\right|(1-l_2')(1-l_4') + \left|N^3E_4^{\prime T_3}\right|(1-l_2')(1-l_3')}{(1-l_2')(1-l_3')(1-l_4')}$$

For evaders with the different velocity, if an evader deviates from the prescribed direction at any time instant $t > 0$, use α^+ to capture the second evader in new pursuit order:

$$\frac{\left|N^0 E_1^{T_0}\right|}{\alpha^- - l_1\alpha^-} + \frac{\left|N^1 E_2^{T_1}\right|}{\alpha^- - l_2\alpha^-} + \frac{\left|N^2 E_3^{T_2}\right|}{\alpha^- - l_3\alpha^-} + \frac{\left|N^3 E_4^{T_3}\right|}{\alpha^- - l_4\alpha^-} > \frac{\left|N^0 E_1^{\prime T_0}\right|}{\alpha^- - l_1'\alpha^-} + \frac{\left|N^1 E_2^{\prime T_1}\right|}{m\alpha^- - l_2'\alpha^-} + \frac{\left|N^2 E_3^{\prime T_2}\right|}{\alpha^- - l_3'\alpha^-} + \frac{\left|N^3 E_4^{\prime T_3}\right|}{\alpha^- - l_4'\alpha^-}$$

$$\frac{\left|N^0 E_1^{T_0}\right|}{1 - l_1} + \frac{\left|N^1 E_2^{T_1}\right|}{1 - l_2} + \frac{\left|N^2 E_3^{T_2}\right|}{1 - l_3} + \frac{\left|N^3 E_4^{T_3}\right|}{1 - l_4} > \frac{\left|N^1 E_2^{\prime T_1}\right|}{m - l_2'} + \frac{\left|N^0 E_1^{\prime T_0}\right|(1-l_3')(1-l_4') + \left|N^2 E_3^{\prime T_2}\right|(1-l_1')(1-l_4') + \left|N^3 E_4^{\prime T_3}\right|(1-l_1')(1-l_3')}{(1-l_1')(1-l_3')(1-l_4')}$$

For evaders with the different velocity, if an evader deviates from the prescribed direction at any time instant $t > 0$, use α^+ to capture the third evader in new pursuit order:

$$\frac{\left|N^0 E_1^{T_0}\right|}{\alpha^- - l_1\alpha^-} + \frac{\left|N^1 E_2^{T_1}\right|}{\alpha^- - l_2\alpha^-} + \frac{\left|N^2 E_3^{T_2}\right|}{\alpha^- - l_3\alpha^-} + \frac{\left|N^3 E_4^{T_3}\right|}{\alpha^- - l_4\alpha^-} > \frac{\left|N^0 E_1^{\prime T_0}\right|}{\alpha^- - l_1'\alpha^-} + \frac{\left|N^1 E_2^{\prime T_1}\right|}{\alpha^- - l_2'\alpha^-} + \frac{\left|N^2 E_3^{\prime T_2}\right|}{m\alpha^- - l_3'\alpha^-} + \frac{\left|N^3 E_4^{\prime T_3}\right|}{\alpha^- - l_4'\alpha^-}$$

$$\frac{\left|N^0 E_1^{T_0}\right|}{1 - l_1} + \frac{\left|N^1 E_2^{T_1}\right|}{1 - l_2} + \frac{\left|N^2 E_3^{T_2}\right|}{1 - l_3} + \frac{\left|N^3 E_4^{T_3}\right|}{1 - l_4} > \frac{\left|N^2 E_3^{\prime T_2}\right|}{m - l_3'} + \frac{\left|N^0 E_1^{\prime T_0}\right|(1-l_2')(1-l_4') + \left|N^1 E_2^{\prime T_1}\right|(1-l_1')(1-l_4') + \left|N^3 E_4^{\prime T_3}\right|(1-l_1')(1-l_2')}{(1-l_1')(1-l_2')(1-l_4')}$$

For evaders with the different velocity, if an evader deviates from the prescribed direction at any time instant $t > 0$, use α^+ to capture the fourth evader in new pursuit order:

$$\frac{\left|N^0E_1^{T_0}\right|}{\alpha^- - l_1\alpha^-} + \frac{\left|N^1E_2^{T_1}\right|}{\alpha^- - l_2\alpha^-} + \frac{\left|N^2E_3^{T_2}\right|}{\alpha^- - l_3\alpha^-} + \frac{\left|N^3E_4^{T_3}\right|}{\alpha^- - l_4\alpha^-} >$$
$$\frac{\left|N^0E_1'^{T_0}\right|}{\alpha^- - l_1'\alpha^-} + \frac{\left|N^1E_2'^{T_1}\right|}{\alpha^- - l_2'\alpha^-} + \frac{\left|N^2E_3'^{T_2}\right|}{\alpha^- - l_3'\alpha^-} + \frac{\left|N^3E_4'^{T_3}\right|}{m\alpha^- - l_4'\alpha^-}$$

$$\frac{\left|N^0E_1^{T_0}\right|}{1-l_1} + \frac{\left|N^1E_2^{T_1}\right|}{1-l_2} + \frac{\left|N^2E_3^{T_2}\right|}{1-l_3} + \frac{\left|N^3E_4^{T_3}\right|}{1-l_4} > \frac{\left|N^3E_4'^{T_3}\right|}{m-l_4'} +$$
$$\frac{\left|N^0E_1'^{T_0}\right|(1-l_2')(1-l_3') + \left|N^1E_2'^{T_1}\right|(1-l_1')(1-l_3') + \left|N^2E_3'^{T_2}\right|(1-l_1')(1-l_2')}{(1-l_1')(1-l_2')(1-l_3')}$$

□

Now let us consider Figure 5.1: The x-axis extends along the horizontal direction and y-axis is going up. Denote by θ^i the angle between x-axis and the straight line N^iN^{i+1}. Introduce the following notations: $E_i^0 = \left(x_i^0, y_i^0\right)$, $N^i = (x_{N^i}, x_{N^i})$ for all $i \in 1, \ldots, 4$, and $E_i^{T_k} = \left(x_i^{T_k}, y_i^{T_k}\right)$ for all $i > k$, $i \in 1, \ldots, 4$, $k \in 1, \ldots, 4$, $T_0 = 0$. The coordinate $E_i^{T_k}$ is a position of evader E_i at the moment T_k, where $T_k\,(k = 1, \ldots, 4)$ are the capture moments of the previously caught evaders.

Because there are four evaders, we have 4 capture points $N^1, \ldots, N^4$ that correspond to the capture moments $T_1, \ldots, T_4$.

According to Figure 5.1, based on the initial known coordinates of the pursuer and evaders, the coordinates of capture points N^i and the coordinates of the evaders at time T_k can be determined using geometric knowledge.

The coordinates of N^i are

$$x_{N^i} = x_{N^{i-1}} - v_i(\pi)T_i\cos\theta^{i-1}, y_{N^i} = y_{N^{i-1}} - v_i(\pi)T_i sin\theta^{i-1}$$

where

$$\cos\theta^{i-1} = \frac{x_{N^{i-1}} - x_i^{T_{i-1}}}{\sqrt{\left(x_{N^{i-1}} - x_i^{T_{i-1}}\right)^2 + \left(y_{N^{i-1}} - y_i^{T_{i-1}}\right)^2}}$$

$$\sin\theta^{i-1} = \frac{y_{N^{i-1}} - y_i^{T_{i-1}}}{\sqrt{\left(x_{N^{i-1}} - x_i^{T_{i-1}}\right)^2 + \left(y_{N^{i-1}} - y_i^{T_{i-1}}\right)^2}}$$

The coordinates of $E_i^{T_k}$ are

$$x_i^{T_k} = x_i^{T_{k-1}} - \beta_i T_k cos\theta^i, y_i^{T_k} = x_i^{T_{k-1}} - \beta_i T_k sin\theta^i$$

where

$$\cos\theta^{i} = \frac{x_{N^{k}} - x_i^{T_{k-1}}}{\sqrt{\left(x_{N^{k}} - x_i^{T_{k-1}}\right)^2 + \left(y_{N^{k-1}} - y_i^{T_{k-1}}\right)^2}}$$

$$\sin\theta^{i} = \frac{y_{N^{k}} - y_i^{T_{k-1}}}{\sqrt{\left(x_{N^{k}} - x_i^{T_{k-1}}\right)^2 + \left(y_{N^{k-1}} - y_i^{T_{k-1}}\right)^2}}$$

Now, we obtain the conditions that geometrically prove the existence of Nash equilibrium.

6 Matrix game

In modern society, people often face various complex decision-making problems involving competition and cooperation among multiple stakeholders. Matrix games provide a clear framework for describing decision-makers' choices and payoff situations. By simplifying decision problems into a matrix form, we can intuitively observe the impact of different participants' choices on each other and the potential competition or cooperation among them. Additionally, matrix games serve as an effective decision support tool, helping us formulate optimal decision strategies. By analyzing the solutions of matrix games, we can identify the optimal strategy choices for participants, thus gaining an advantage in competition or achieving win-win outcomes in cooperation.

In the described pursuit game problem, this chapter focuses on the overall survival time of the evaders, which involves complex competition and cooperation between them and the pursuer. The pursuer can employ different velocity strategy combinations to chase the evaders, while the evaders attempt to prolong their escape time through various strategies. By representing the choices and payoffs of both pursuer and evaders as a payoff matrix, we utilize the theory and methods of matrix games to analyze the optimal solutions under different strategies, thereby identifying the best behavior choices for both pursuer and evaders.

Definition 6.1. *The system*

$$\Gamma^M \left(z_p^0, z_1^0, \ldots, z_4^0\right) = \langle X\ , Y\ , K \rangle \tag{6.1}$$

where X and Y are called the strategies of evader and pursuer, respectively, and the function K is the payoff of evader defined by (4.5), is called a two-

person zero-sum game in normal form [13].

Definition 6.2. *Two-person zero-sum game in which both players have finite sets of strategies are called matrix game [13].*

Definition 6.3. *The strategy of the evader is defined as the combination of whether to deviate from the prescribed direction, i.e.*

$$X = \{(oooo), (dooo), (odoo), (oodo), (oood)\}$$

where o represents movement in the prescribed direction, d represents deviation from the prescribed direction.

The pursuer's strategy is defined as a combination of different velocities for the four evaders, i.e.

$$Y = \{(\alpha^-, \alpha^-, \alpha^-, \alpha^-), (\alpha^+, \alpha^-, \alpha^-, \alpha^-), (\alpha^-, \alpha^+, \alpha^-, \alpha^-), \\ (\alpha^-, \alpha^-, \alpha^+, \alpha^-), (\alpha^-, \alpha^-, \alpha^-, \alpha^+)\}$$

Now, we know that the evaders have a total of five strategies, but since the evader can choose to escape at different time instants T_0, T_1, T_2, T_3, it means before E_1,E_2,E_3,E_4 are captured respectively. At T_0, all five strategies can be chosen. At T_1, E_1 is caught, so only E_2, E_3, and E_4 can choose to change direction, resulting in four strategies, and so on. Then total number of strategies for the evaders is 14. Let us order the strategy set X of the evader, i.e. set up a direct correspondence between the sets $I = \{1, 2, ..., 14\}$ and X. Similarly, pursuer has 5 strategies, it is possible to set up a direct correspondence between the sets $G = \{1, 2, ..., 5\}$ and Y. The game Γ^M is then fully defined by specifying the matrix $A = \{a_{ij}\}$, where $a_{ij} = K(x_i, y_i)$, $(i, j) \in I \times G$, $(x_i, y_i) \in X \times Y$, $i \in I$, $j \in G$ (whence comes the name of the game - the matrix game).

$$a_{ij} = K(x_i, y_i) = \begin{cases} \underset{\pi\in\Pi}{min} \sum_{i=1}^{4} \frac{\left|N^{i-1}(\pi)E_i^{T_{i-1}}(\pi)\right|}{v_i(\pi)-\beta_i(\pi)}, \; no\ evader\ deviates \\ \underset{\pi\in\Pi}{min} \sum_{i=1}^{4} \frac{\left|N^{i-1}\left(\pi_T^i\right)E_i^{T_{i-1}}\left(\pi_T^i\right)\right|}{v_i\left(\pi_T^i\right)-\beta_i\left(\pi_T^i\right)}, \; any\ evader\ deviates \end{cases}$$

where π is represents the pursuit order chosen by the pursuer when there are no deviations from the prescribed direction by the evaders. π_T^i represents the pursuer recalculating the pursuit order when E_i deviates from the prescribed direction at time instant T.

Pursuer P has two velocities during the pursuit. Our aim is to find out in which case pursuer P will use strategy s_p with α^+ to capture the evader better and how the evader's optimal strategy changes when the pursuer changes his strategy. When an evader deviates from the original direction, pursuer P will recalculate the payoff at this time, different evader changes its direction at different times will eventually lead to a change in payoff, but pursuer P can choose to use α^+ to capture the evader who changes direction to maximize its total payoff because it is the negative of the payoff for the group of evaders. So the payoff function of group of evaders is as follows:

$$\sum_{i=1}^{4} K_{E_i} = \underset{\pi\,\in\,\Pi}{min} \sum_{i=1}^{4} \frac{\left|N^{i-1}\left(\pi_T^i\right) E_i^{T_{i-1}}\left(\pi_T^i\right)\right|}{v_i\left(\pi_T^i\right) - \beta_i\left(\pi_T^i\right)}$$

Suppose the original pursuit order is $\pi = \{E_1, E_2, E_3, E_4\}$.

If the E_2 changes direction at $T = 0$, we will have a new order π_0^2 and $\pi_0^2(1) = E_2$, $\pi_0^2(2) = E_1, E_3, E_4$, $\pi_0^2(3) = \pi_0^2(2) - E_i$, $i \in 1, 3, 4$, $\pi_0^2(4) = \pi_0^2(3) - E_i$, $i \in 1, 3, 4$. We suppose the new pursuit order $\pi_0^2 = \{2, 1, 4, 3\}$

and the capture process is illustrated in Figure 6.1.

$$\sum_{i=1}^{4} K_{E_i} = \frac{min}{\pi \in \Pi} \sum_{i=1}^{4} \frac{\left| N^{i-1}\left(\pi_0^2\right) E_i^{T_{i-1}}\left(\pi_0^2\right) \right|}{\upsilon_i\left(\pi_0^2\right) - \beta_i\left(\pi_0^2\right)}$$

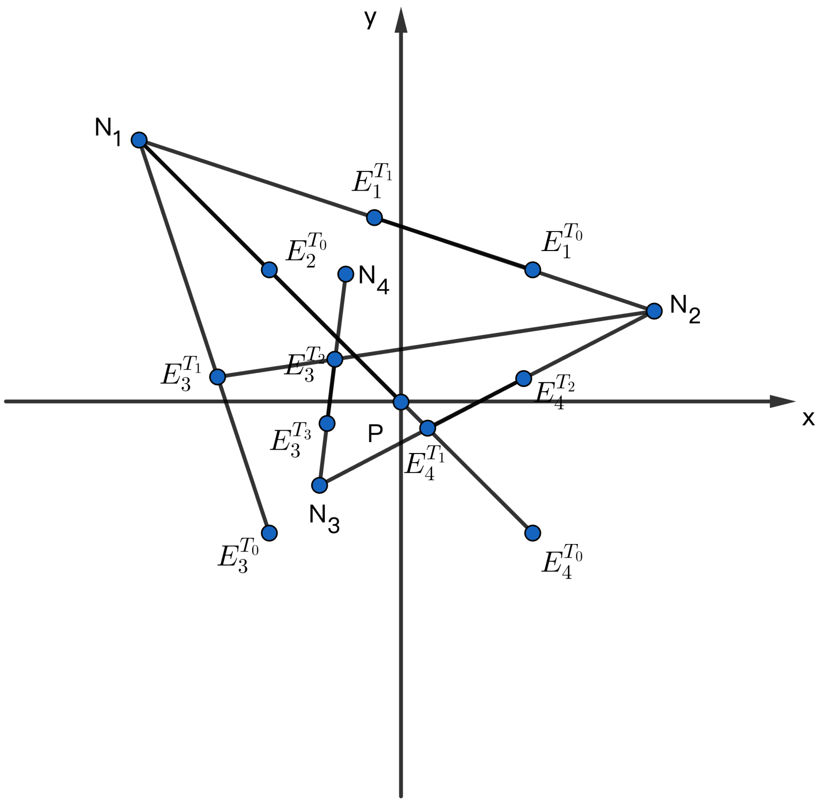

Figure 6.1: Capture process of E_2 deviates from the prescribed direction at $T = 0$

If the E_3 changes direction at $T = 0$, we will have a new order π_0^3 and $\pi_0^3(1) = E_3$, $\pi_0^3(2) = E_1, E_2, E_4$, $\pi_0^3(3) = \pi_0^3(2) - E_i$, $i \in 1, 2, 4$, $\pi_0^3(4) = \pi_0^3(3) - E_i$, $i \in 1, 2, 4$. We suppose the new pursuit order $\pi_0^3 = \{3, 4, 1, 2\}$ and the capture process is illustrated in Figure 6.2.

$$\sum_{i=1}^{4} K_{E_i} = \frac{min}{\pi \in \Pi} \sum_{i=1}^{4} \frac{\left| N^{i-1}\left(\pi_0^3\right) E_i^{T_{i-1}}\left(\pi_0^3\right) \right|}{\upsilon_i\left(\pi_0^3\right) - \beta_i\left(\pi_0^3\right)}$$

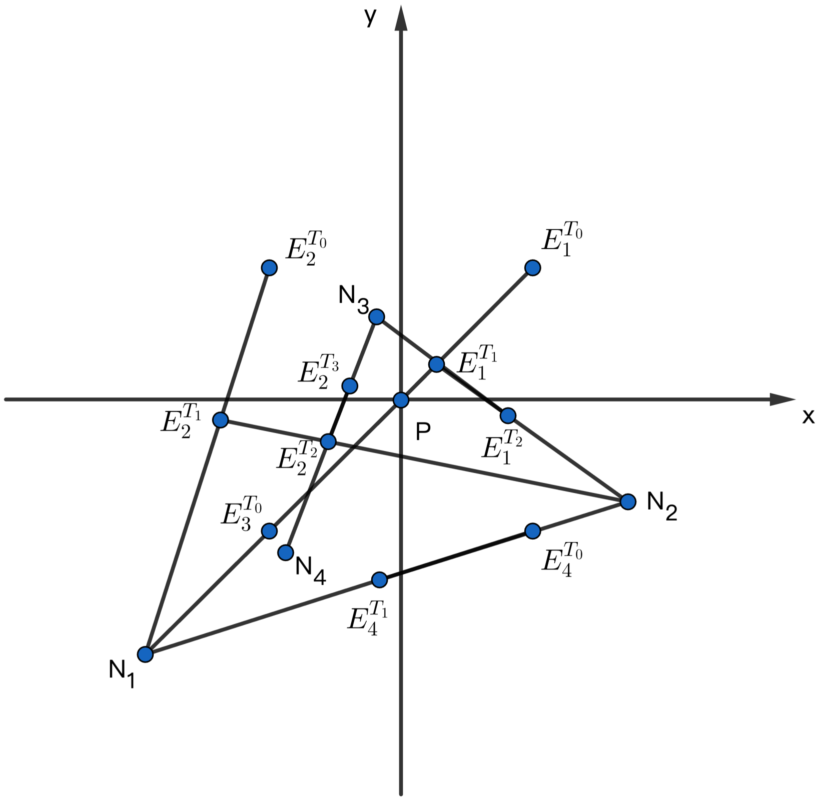

Figure 6.2: Capture process of E_3 deviates from the prescribed direction at $T = 0$

If the E_4 changes direction at $T = 0$, we will have a new order π_0^4 and $\pi_0^4(1) = E_4$, $\pi_0^4(2) = E_1, E_2, E_3$, $\pi_0^4(3) = \pi_0^4(2) - E_i$, $i \in 1, 2, 3$, $\pi_0^4(4) = \pi_0^4(3) - E_i$, $i \in 1, 2, 3$. We suppose the new pursuit order $\pi_0^4 = \{4, 1, 2, 3\}$ and the capture process is illustrated in Figure 6.3.

$$\sum_{i=1}^{4} K_{E_i} = \underset{\pi \in \Pi}{min} \sum_{i=1}^{4} \frac{\left| N^{i-1}\left(\pi_0^4\right) E_i^{T_{i-1}}\left(\pi_0^4\right) \right|}{v_i\left(\pi_0^4\right) - \beta_i\left(\pi_0^4\right)}$$

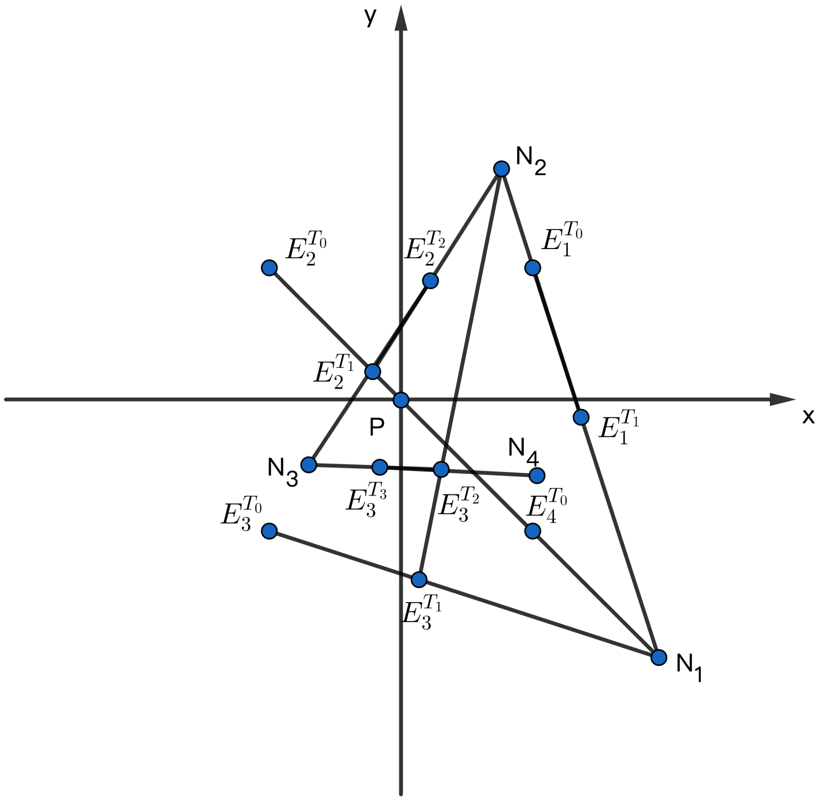

Figure 6.3: Capture process of E_4 deviates from the prescribed direction at $T = 0$

If the E_3 changes direction at $T = 1$, we will have a new order π_1^3 and $\pi_1^3(1) = E_1$, $\pi_1^3(2) = E_3$, $\pi_1^3(3) = E_2, E_4$, $\pi_1^3(4) = \pi_1^3(3) - E_i$, $i \in 2, 4$. We suppose the new pursuit order $\pi_1^3 = \{1, 3, 2, 4\}$ and the capture process is illustrated in Figure 6.4.

$$\sum_{i=1}^{4} K_{E_i} = \underset{\pi \in \Pi}{min} \sum_{i=1}^{4} \frac{\left| N^{i-1}\left(\pi_1^3\right) E_i^{T_{i-1}}\left(\pi_1^3\right) \right|}{v_i\left(\pi_1^3\right) - \beta_i\left(\pi_1^3\right)}$$

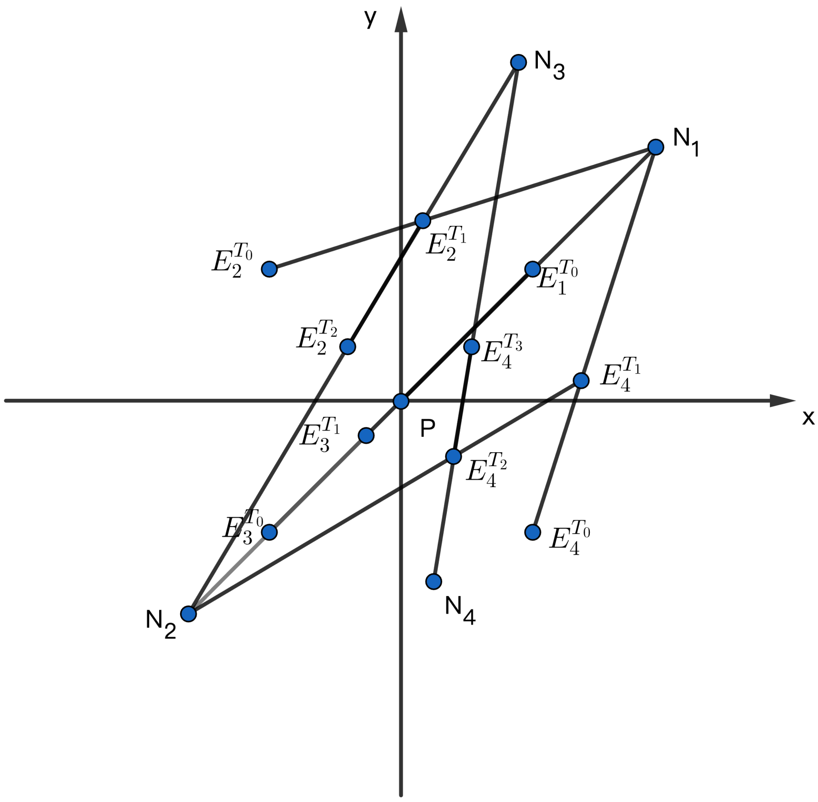

Figure 6.4: Capture process of E_3 deviates from the prescribed direction at $T = 1$

If the E_4 changes direction at $T = 1$, we will have a new order π_1^4 and $\pi_1^4(1) = E_1$, $\pi_1^4(2) = E_4$, $\pi_1^4(3) = E_2, E_3$, $\pi_1^4(4) = \pi_1^4(3) - E_i$, $i \in 2, 3$. We suppose the new pursuit order $\pi_1^4 = \{1, 4, 3, 2\}$ and the capture process is illustrated in Figure 6.5.

$$\sum_{i=1}^{4} K_{E_i} = \begin{matrix} min \\ \pi \in \Pi \end{matrix} \sum_{i=1}^{4} \frac{\left| N^{i-1}\left(\pi_1^4\right) E_i^{T_{i-1}}\left(\pi_1^4\right) \right|}{v_i\left(\pi_1^4\right) - \beta_i\left(\pi_1^4\right)}$$

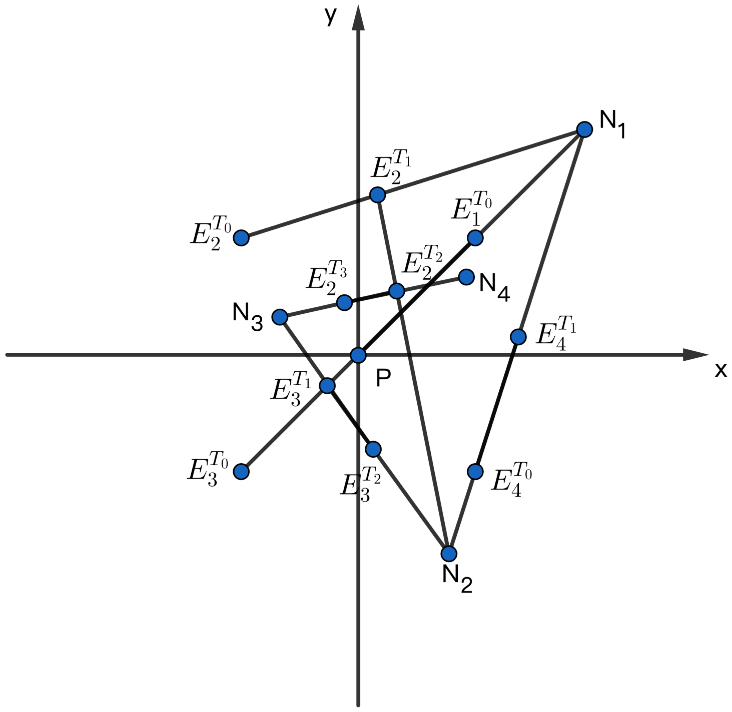

Figure 6.5: Capture process of E_4 deviates from the prescribed direction at $T = 1$

If the E_4 changes direction at $T = 2$, we will have a new order π_2^4 and $\pi_2^4(1) = E_1$, $\pi_2^4(2) = E_2$, $\pi_2^4(3) = E_4$, $\pi_2^4(4) = E_3$. We suppose the new pursuit order $\pi_2^4 = \{1, 2, 4, 3\}$ and the capture process is illustrated in Figure 6.6.

$$\sum_{i=1}^{4} K_{E_i} = \min_{\pi \in \Pi} \sum_{i=1}^{4} \frac{\left| N^{i-1}\left(\pi_2^4\right) E_i^{T_{i-1}}\left(\pi_2^4\right) \right|}{v_i\left(\pi_2^4\right) - \beta_i\left(\pi_2^4\right)}$$

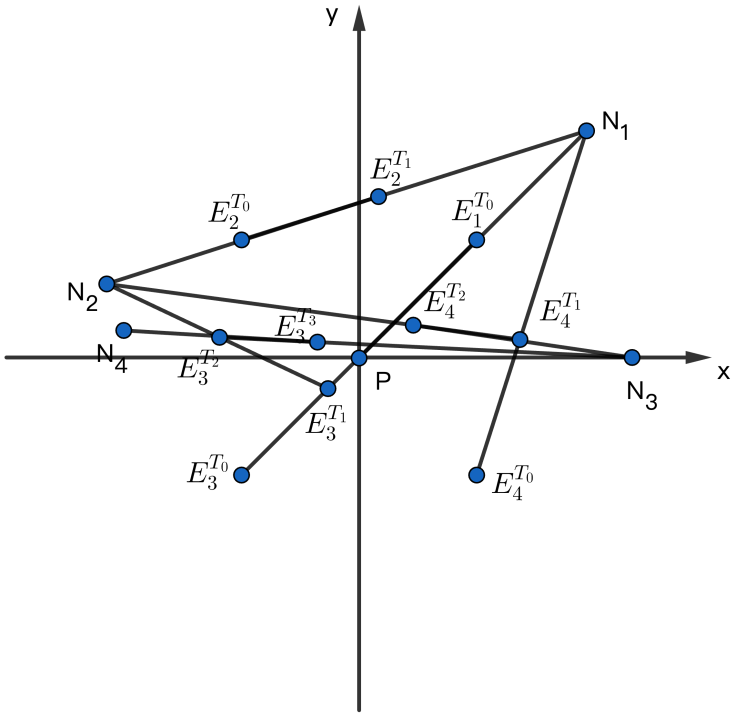

Figure 6.6: Capture process of E_4 deviates from the prescribed direction at $T = 2$

6.1 Rational behavior for evaders

It is supposed that the behavior of both evaders and pursuer are rational, i.e. in order to achieve the highest possible payoff, assuming that the opponent is acting in the best (for himself) possible way. Let's say the evader selects strategy x_i. In the worst-case scenario, the evader will secure a payoff of $min_{y_i} K(x_i, y_i)$. Consequently, the evader can consistently ensure a payoff of $max_{x_i} min_{y_i} K(x_i, y_i)$.

Let's consider some examples. The evaders with the same and different velocities are located at a symmetrical position in the coordinate axis and observe the survival time of the evaders under rational behavior.

6.2 Simulation

First, suppose the four evaders are symmetrically positioned in the four quadrants of the coordinate axis, and they all have the same velocity. The pursuer continues to increase velocity α^+ while the pursuers' velocity α^- remains constant for simulation.

In the strategy of the evaders, * represents the order in which E_i has been captured.

Example 6.1. *Let* $E_1 = (1,1),\ E_2 = (-1,1),\ E_3 = (-1,-1),\ E_4 = (1,-1),\ \alpha^- = 2,\ \alpha^+ = 10,\ \beta_i = 1.$

Table 6.1: Payoff matrix1

		$(\alpha^-\alpha^-\alpha^-\alpha^-)$	$(\alpha^+\alpha^-\alpha^-\alpha^-)$	$(\alpha^-\alpha^+\alpha^-\alpha^-)$	$(\alpha^-\alpha^-\alpha^+\alpha^-)$	$(\alpha^-\alpha^-\alpha^-\alpha^+)$
T_0	oooo	4.53	3.18	3.64	4.39	3.45
	dooo	4.53	3.18	3.64	4.39	3.45
	odoo	4.53	3.18	3.64	4.39	3.45
	oodo	4.53	3.18	3.64	4.39	3.45
	oood	4.53	3.18	3.64	4.39	3.45
T_1	*ooo	4.53	3.18	3.64	4.39	3.45
	*doo	4.53	3.18	3.64	4.39	3.45
	*odo	5.76	5.04	4.05	4.79	5.60
	*ood	4.53	3.18	3.64	4.39	3.45
T_2	**oo	4.53	3.18	3.64	4.39	3.45
	**do	4.53	3.18	3.64	4.39	3.45
	**od	5.45	5.95	4.10	4.50	4.37
T_3	***o	4.53	3.18	3.64	4.39	3.45
	***d	4.53	3.18	3.64	4.39	3.45

Example 6.2. *Let* $E_1 = (1,1)$, $E_2 = (-1,1)$, $E_3 = (-1,-1)$, $E_4 = (1,-1)$, $\alpha^- = 2$, $\alpha^+ = 50$, $\beta_i = 1$.

Table 6.2: Payoff matrix2

		$(\alpha^-\alpha^-\alpha^-\alpha^-)$	$(\alpha^+\alpha^-\alpha^-\alpha^-)$	$(\alpha^-\alpha^+\alpha^-\alpha^-)$	$(\alpha^-\alpha^-\alpha^+\alpha^-)$	$(\alpha^-\alpha^-\alpha^-\alpha^+)$
T_0	oooo	4.53	3.06	3.89	4.38	3.34
	dooo	4.53	3.06	3.89	4.38	3.34
	odoo	4.53	3.06	3.89	4.38	3.34
	oodo	4.53	3.06	3.89	4.38	3.34
	oood	4.53	3.06	3.89	4.38	3.34
T_1	*ooo	4.53	3.06	3.89	4.38	3.34
	*doo	4.53	3.06	3.89	4.38	3.34
	*odo	5.76	5.03	4.06	4.75	3.59
	*ood	4.53	3.06	3.89	4.38	3.34
T_2	**oo	4.53	3.06	3.89	4.38	3.34
	**do	4.53	3.06	3.89	4.38	3.34
	**od	5.44	6.12	4.08	4.40	4.26
T_3	***o	4.53	3.06	3.89	4.38	3.34
	***d	4.53	3.06	3.89	4.38	3.34

Example 6.3. *Let* $E_1 = (1,1)$, $E_2 = (-1,1)$, $E_3 = (-1,-1)$, $E_4 = (1,-1)$, $\alpha^- = 2$, $\alpha^+ = 100$, $\beta_i = 1$.

Table 6.3: Payoff matrix3

		$(\alpha^-\alpha^-\alpha^-\alpha^-)$	$(\alpha^+\alpha^-\alpha^-\alpha^-)$	$(\alpha^-\alpha^+\alpha^-\alpha^-)$	$(\alpha^-\alpha^-\alpha^+\alpha^-)$	$(\alpha^-\alpha^-\alpha^-\alpha^+)$
T_0	oooo	4.53	3.05	3.92	4.38	3.33
	dooo	4.53	3.05	3.92	4.38	3.33
	odoo	4.53	3.05	3.92	4.38	3.33
	oodo	4.53	3.05	3.92	4.38	3.33
	oood	4.53	3.05	3.92	4.38	3.33
T_1	*ooo	4.53	3.05	3.92	4.38	3.33
	*doo	4.53	3.05	3.92	4.38	3.33
	*odo	5.76	5.03	4.07	4.75	5.58
	*ood	4.53	3.05	3.92	4.38	3.33
T_2	**oo	4.53	3.05	3.92	4.38	3.33
	**do	4.53	3.05	3.92	4.38	3.33
	**od	5.45	6.15	4.08	4.39	4.25
T_3	***o	4.53	3.05	3.92	4.38	3.33
	***d	4.53	3.05	3.92	4.38	3.33

In the above three examples, based on the simulation, it is determined that the initial pursuit order for pursuer is all $\pi = \{1,2,3,4\}$, and from the matrix, the optimal strategy for the evader is all that evader E_4 deviates from the prescribed direction at time instant T_2 and the new pursuit order is $\pi_2^4 = \{1, 2, 4, 3\}$. This is the strategy that evaders would choose under rational behavior and the correspond value in the payoff matrix is what evaders can get with rational behavior. At this time, the pursuers' strategy is denoted as $(\alpha^-\alpha^+\alpha^-\alpha^-)$.

As the value of the pursuer's velocity α^+ gradually increases in simulation experiments, it is observed that with the increasing ratio of α^+ to β_i, the survival time of the evader remains almost unchanged, and it does not alter the optimal strategy for the evader.

When other conditions remain unchanged, the velocity of the pursuer α^+ remains unchanged and α^- is continuously increased for simulation.

Example 6.4. *Let* $E_1 = (1, 1)$, $E_2 = (-1, 1)$, $E_3 = (-1, -1)$, $E_4 = (1, -1)$, $\alpha^- = 10$, $\alpha^+ = 100$, $\beta_i = 1$.

Table 6.4: Payoff matrix4

		$(\alpha^-\alpha^-\alpha^-\alpha^-)$	$(\alpha^+\alpha^-\alpha^-\alpha^-)$	$(\alpha^-\alpha^+\alpha^-\alpha^-)$	$(\alpha^-\alpha^-\alpha^+\alpha^-)$	$(\alpha^-\alpha^-\alpha^-\alpha^+)$
T_0	oooo	0.73	0.62	0.57	0.58	0.58
	dooo	0.73	0.62	0.57	0.58	0.58
	odoo	0.73	0.62	0.57	0.58	0.58
	oodo	0.73	0.62	0.57	0.58	0.58
	oood	0.73	0.62	0.57	0.58	0.58
T_1	*ooo	0.73	0.62	0.57	0.58	0.58
	*doo	0.73	0.62	0.57	0.58	0.58
	*odo	0.91	0.80	0.68	0.74	0.70
	*ood	0.73	0.62	0.57	0.58	0.58
T_2	**oo	0.73	0.62	0.57	0.58	0.58
	**do	0.73	0.62	0.57	0.58	0.58
	**od	0.85	0.74	0.67	0.59	0.68
T_3	***o	0.73	0.62	0.57	0.58	0.58
	***d	0.73	0.62	0.57	0.58	0.58

Example 6.5. *Let* $E_1 = (1, 1)$, $E_2 = (-1, 1)$, $E_3 = (-1, -1)$, $E_4 = (1, -1)$, $\alpha^- = 50$, $\alpha^+ = 100$, $\beta_i = 1$.

Table 6.5: Payoff matrix5

		$(\alpha^-\alpha^-\alpha^-\alpha^-)$	$(\alpha^+\alpha^-\alpha^-\alpha^-)$	$(\alpha^-\alpha^+\alpha^-\alpha^-)$	$(\alpha^-\alpha^-\alpha^+\alpha^-)$	$(\alpha^-\alpha^-\alpha^-\alpha^+)$
T_0	oooo	0.15	0.13	0.13	0.13	0.13
	dooo	0.15	0.13	0.13	0.13	0.13
	odoo	0.15	0.13	0.13	0.13	0.13
	oodo	0.15	0.13	0.13	0.13	0.13
	oood	0.15	0.13	0.13	0.13	0.13
T_1	*ooo	0.15	0.13	0.13	0.13	0.13
	*doo	0.15	0.13	0.13	0.13	0.13
	*odo	0.18	0.17	0.15	0.16	0.16
	*ood	0.15	0.13	0.13	0.13	0.13
T_2	**oo	0.15	0.13	0.13	0.13	0.13
	**do	0.15	0.13	0.13	0.13	0.13
	**od	0.17	0.15	0.15	0.14	0.15
T_3	***o	0.15	0.13	0.13	0.13	0.13
	***d	0.15	0.13	0.13	0.13	0.13

In the example 6.4 - 6.5, based on the simulation, it is determined that the initial pursuit order for pursuer is all $\pi = \{1, 2, 3, 4\}$, and from the matrix, the optimal strategy for the evader is all that evader E_3 deviates from the prescribed direction at time instant T_1 and the new pursuit order is $\pi_1^3 = \{1, 3, 2, 4\}$. This is the strategy that evaders would choose under rational behavior and the correspond value in the payoff matrix is what evaders can get with rational behavior. At this time, the pursuers' strategy is denoted as $(\alpha^-\alpha^+\alpha^-\alpha^-)$.

Through simulation experiments, also compare with example 6.3, it is found that as the value of pursuer velocity α^- gradually increases, the relative change in the evader's survival time becomes significant with the increasing ratio of α^- to β_i. Differently from the previous experiment, the optimal strategy for the evaders have changed.

Then, similarly assume that the four evaders are symmetrically positioned in the four quadrants of the coordinate axis, but their velocities are not the same. Simulation proceeds by continually increasing α^- while keeping the pursuer velocity α^+ constant.

Example 6.6. *Let* $E_1 = (1,1)$, $E_2 = (-1,1)$, $E_3 = (-1,-1)$, $E_4 = (1,-1)$, $\alpha^- = 2$, $\alpha^+ = 10$, $\beta_1 = 0.3$, $\beta_2 = 0.5$, $\beta_3 = 0.7$, $\beta_4 = 0.9$.

Table 6.6: Payoff matrix6

		$(\alpha^-\alpha^-\alpha^-\alpha^-)$	$(\alpha^+\alpha^-\alpha^-\alpha^-)$	$(\alpha^-\alpha^+\alpha^-\alpha^-)$	$(\alpha^-\alpha^-\alpha^+\alpha^-)$	$(\alpha^-\alpha^-\alpha^-\alpha^+)$
T_0	oooo	3.03	2.67	2.90	2.93	2.71
	dooo	3.03	2.67	2.90	2.93	2.71
	odoo	3.20	2.74	2.86	3.19	2.66
	oodo	2.85	2.79	3.43	3.62	3.33
	oood	3.89	2.85	3.42	3.67	3.43
T_1	*ooo	3.03	2.67	2.90	2.93	2.71
	*doo	3.03	2.67	2.90	2.93	2.71
	*odo	4.45	4.29	3.96	3.65	3.92
	*ood	3.43	3.51	3.00	3.21	2.89
T_2	**oo	3.03	2.67	2.90	2.93	2.71
	**do	3.03	2.67	2.90	2.93	2.71
	**od	4.28	4.46	4.09	2.96	3.50
T_3	***o	3.03	2.67	2.90	2.93	2.71
	***d	3.03	2.67	2.90	2.93	2.71

Example 6.7. *Let* $E_1 = (1,1)$, $E_2 = (-1,1)$, $E_3 = (-1,-1)$, $E_4 = (1,-1)$, $\alpha^- = 2$, $\alpha^+ = 50$, $\beta_1 = 0.3$, $\beta_2 = 0.5$, $\beta_3 = 0.7$, $\beta_4 = 0.9$.

Table 6.7: Payoff matrix7

		$(\alpha^-\alpha^-\alpha^-\alpha^-)$	$(\alpha^+\alpha^-\alpha^-\alpha^-)$	$(\alpha^-\alpha^+\alpha^-\alpha^-)$	$(\alpha^-\alpha^-\alpha^+\alpha^-)$	$(\alpha^-\alpha^-\alpha^-\alpha^+)$
T_0	oooo	3.03	2.62	2.91	2.93	2.68
	dooo	3.03	2.62	2.91	2.93	2.68
	odoo	3.20	2.70	2.89	3.19	2.59
	oodo	3.85	2.67	3.38	3.59	3.27
	oood	3.89	2.76	3.36	3.64	3.37
T_1	*ooo	3.03	2.62	2.91	2.93	2.68
	*doo	3.03	2.62	2.91	2.93	2.68
	*odo	4.45	4.28	3.96	3.59	3.84
	*ood	3.43	3.53	3.03	3.19	2.81
T_2	**oo	3.03	2.62	2.91	2.93	2.68
	**do	3.03	2.62	2.91	2.93	2.68
	**od	4.28	4.52	4.08	2.80	3.50
T_3	***o	3.03	2.62	2.91	2.93	2.68
	***d	3.03	2.62	2.91	2.93	2.68

Example 6.8. *Let* $E_1 = (1,1)$, $E_2 = (-1,1)$, $E_3 = (-1,-1)$, $E_4 = (1,-1)$, $\alpha^- = 2$, $\alpha^+ = 100$, $\beta_1 = 0.3$, $\beta_2 = 0.5$, $\beta_3 = 0.7$, $\beta_4 = 0.9$.

Table 6.8: Payoff matrix8

		$(\alpha^-\alpha^-\alpha^-\alpha^-)$	$(\alpha^+\alpha^-\alpha^-\alpha^-)$	$(\alpha^-\alpha^+\alpha^-\alpha^-)$	$(\alpha^-\alpha^-\alpha^+\alpha^-)$	$(\alpha^-\alpha^-\alpha^-\alpha^+)$
T_0	oooo	3.03	2.61	2.92	2.93	2.67
	dooo	3.03	2.61	2.92	2.93	2.67
	odoo	3.20	2.70	2.90	3.19	2.58
	oodo	3.85	2.66	3.38	3.58	3.27
	oood	3.89	2.75	3.35	3.64	3.37
T_1	*ooo	3.03	2.61	2.92	2.93	2.67
	*doo	3.03	2.61	2.92	2.93	2.67
	*odo	4.45	4.28	3.96	3.58	3.83
	*ood	3.43	3.53	3.03	3.19	2.80
T_2	**oo	3.03	2.61	2.92	2.93	2.67
	**do	3.03	2.61	2.92	2.93	2.67
	**od	4.28	4.52	4.08	2.80	3.38
T_3	***o	3.03	2.61	2.92	2.93	2.67
	***d	3.03	2.61	2.92	2.93	2.67

In the example 6.6 - 6.8, based on the simulation, it is determined that the initial pursuit order for pursuer is all $\pi = \{1, 2, 3, 4\}$, and from the matrix, the optimal strategy for the evader is all that evader E_3 deviates from the prescribed direction at time instant T_1, and the new pursuit order is $\pi_1^3 = \{1, 3, 4, 2\}$. This is the strategy that evaders would choose under rational behavior and the correspond value in the payoff matrix is what evaders can get with rational behavior. At this time, the pursuers' strategy is denoted as $(\alpha^-\alpha^-\alpha^+\alpha^-)$.

Finally, when the velocity of the evaders is different, keep the velocity of the pursuer α^+ unchanged and continue to increase α^- for simulation.

Example 6.9. *Let* $E_1 = (1,1)$, $E_2 = (-1,1)$, $E_3 = (-1,-1)$, $E_4 = (1,-1)$, $\alpha^- = 10$, $\alpha^+ = 100$, $\beta_1 = 0.3$, $\beta_2 = 0.5$, $\beta_3 = 0.7$, $\beta_4 = 0.9$.

Table 6.9: Payoff matrix9

		$(\alpha^-\alpha^-\alpha^-\alpha^-)$	$(\alpha^+\alpha^-\alpha^-\alpha^-)$	$(\alpha^-\alpha^+\alpha^-\alpha^-)$	$(\alpha^-\alpha^-\alpha^+\alpha^-)$	$(\alpha^-\alpha^-\alpha^-\alpha^+)$
T_0	oooo	0.72	0.61	0.56	0.57	0.56
	dooo	0.72	0.61	0.56	0.57	0.56
	odoo	0.73	0.61	0.57	0.57	0.56
	oodo	0.74	0.61	0.57	0.58	0.58
	oood	0.75	0.61	0.57	0.58	0.58
T_1	*ooo	0.72	0.61	0.56	0.57	0.56
	*doo	0.72	0.61	0.56	0.57	0.56
	*odo	0.90	0.79	0.67	0.72	0.67
	*ood	0.73	0.62	0.56	0.57	0.56
T_2	**oo	0.72	0.61	0.56	0.57	0.56
	**do	0.72	0.61	0.56	0.57	0.56
	**od	0.83	0.72	0.66	0.57	0.65
T_3	***o	0.72	0.61	0.56	0.57	0.56
	***d	0.72	0.61	0.56	0.57	0.56

Example 6.10. *Let* $E_1 = (1,1)$, $E_2 = (-1,1)$, $E_3 = (-1,-1)$, $E_4 = (1,-1)$, $\alpha^- = 50$, $\alpha^+ = 100$, $\beta_1 = 0.3$, $\beta_2 = 0.5$, $\beta_3 = 0.7$, $\beta_4 = 0.9$.

Table 6.10: Payoff matrix10

		$(\alpha^-\alpha^-\alpha^-\alpha^-)$	$(\alpha^+\alpha^-\alpha^-\alpha^-)$	$(\alpha^-\alpha^+\alpha^-\alpha^-)$	$(\alpha^-\alpha^-\alpha^+\alpha^-)$	$(\alpha^-\alpha^-\alpha^-\alpha^+)$
T_0	oooo	0.15	0.13	0.13	0.13	0.13
	dooo	0.15	0.13	0.13	0.13	0.13
	odoo	0.15	0.13	0.13	0.13	0.13
	oodo	0.15	0.13	0.13	0.13	0.13
	oood	0.15	0.13	0.13	0.13	0.13
T_1	*ooo	0.15	0.13	0.13	0.13	0.13
	*doo	0.15	0.13	0.13	0.13	0.13
	*odo	0.18	0.17	0.15	0.16	0.16
	*ood	0.15	0.13	0.13	0.13	0.13
T_2	**oo	0.15	0.13	0.13	0.13	0.13
	**do	0.15	0.13	0.13	0.13	0.13
	**od	0.16	0.15	0.15	0.14	0.14
T_3	***o	0.15	0.13	0.13	0.13	0.13
	***d	0.15	0.13	0.13	0.13	0.13

In the example 6.9 - 6.10, based on the simulation, it is determined that the initial pursuit order for pursuer is all $\pi = \{1, 2, 3, 4\}$, and from the matrix, the optimal strategy for the evader is all that evader E_3 deviates from the original direction at time instant T_1 and the new pursuit order $\pi_1^3 = \{1, 3, 4, 2\}$. This is the strategy that evaders would choose under rational behavior and the correspond value in the payoff matrix is what evaders can get with rational behavior. At this time, the pursuers' strategy is denoted as $(\alpha^- \alpha^+ \alpha^- \alpha^-)$.

In Example 6.6 - 6.10., the conclusions drawn for the evaders are the same as when their velocities are identical.

6.3 Irrational behavior for evaders

Under rational behavior, according to the maximin principle, the evader can get their optimal strategy. Based on simulation results, the evader's payoff obtained through the optimal strategy may sometimes exceed that obtained by moving in the prescribed direction, and sometimes it may not.

According to Theorem 1,2, when

$$\frac{\sum_{k=1}^{4}\left|N^{k-1}E_k^{T_{k-1}}\right| - \sum_{k\neq i}\left|N^{k-1}E_k^{\prime T_{k-1}}\right|}{\left|N^{k-1}E_k^{\prime T_{k-1}}\right|_{k=i}} \leq \frac{1-l}{m-l}, k = 1, 2, 3, 4$$

$$\frac{\sum_{k=1}^{4}\left(\left|N^{k-1}E_k^{T_{k-1}}\right| \prod_{n\neq k}(1-l_n)\right)}{\prod_{n=1}^{4}(1-l_n)} \leq$$

$$\frac{\left|N^{k-1}E_k^{\prime T_{k-1}}\right|_{k=i}}{(m-l'_n)_{n=i}} + \frac{\sum_{k\neq i}\left(\left|N^{k-1}E_k^{T_{k-1}}\right| \prod_{n\neq i,k}(1-l_n)\right)}{\prod_{n\neq k}(1-l'_n)}, k = 1, 2, 3, 4$$

this implies that the pursuer's punishment strategy is ineffective.

In this case, effective punishment cannot be applied when the evader deviates from the prescribed direction. Therefore, when the ratios among the positions of the pursuer, evaders, capture points, and three types of velocities satisfy the inequalities described above, the evaders can choose to change direction in search of a greater payoff compared to when moving in the prescribed direction. From the payoff matrix, it can be observed that regardless of the ratio between α^- and β_i, and between α^+ and β_i, how their ratio changes, there will always exist a strategy combination that allows evaders to achieve a greater payoff. Similarly, there will always exist a strategy combination that enables the pursuer to effectively punish the evader deviating from the prescribed direction of movement.

Suppose evaders are irrational, then they can choose whether to deviate from the prescribed direction. In this case, they may or may not receive a greater payoff because this also relates to the strategy of the pursuer.

7 Conclusion

In the master's thesis, we delve into the pursuit game problem between one pursuer and four evaders in game theory, exploring variations in strategies within nonzero-sum games and matrix games. Introducing the pursuer α^+, with a velocity that exacerbates the survival conditions for evaders in a game model significantly unfavorable to them.

In nonzero-sum game, we establish the Nash equilibrium, in which the pursuer chooses a target to pursue, prompting the other evaders to move in prescribed directions. If any evader deviates from the prescribed direction, the pursuer adjusts the pursuit order to capture the deviant evader first as a form of punishment. We derive geometric formulas for calculating the coordinates of capture points and evaders at different time instant, and identify effective punishment strategy conditions for the pursuer.

In matrix game, we analyze the strategy variations under rational and irrational behavior of evaders. Keeping α^+ and α^- unchanged respectively while increasing the other velocity for experiments, interestingly, we observe that despite the higher velocity of the newly introduced velocity α^+, the overall survival time of the evaders remains nearly unchanged, unlike the scenario when studying the survival time of individual evaders. Additionally, Under irrational behavior, there exists at least one strategy whereby the evaders' survival time surpasses that achieved by adhering to the prescribed direction of movement.

Through the lens of game theory, we aim to gain deeper insights into strategic behaviors and the dynamics of complex systems. Through theoretical modeling and empirical analysis, i hope to contribute to a broader understanding of strategic interactions and their impacts on reality.

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Appendix

```
import math
import numpy as np
import itertools


def input_coordinates(obj):
    while(True):
        try:
            a=[float(i) for i in input('please enter'
            +obj+'coordinate').split(',')]
            if(len(a)==2):
                return a
        except:
            pass
def meet_time(a1,a2,v1,v2):
    return math.dist(a1,a2)/(v1-v2)
def meet_time1(a1,a2,v_gap):
    return math.dist(a1,a2)/(np.abs(v_gap))
def calsincos(n1,n2):
    long=math.dist(n2,n1)
    sinx=(n2[1]-n1[1])/long
    cosx=(n2[0]-n1[0])/long
    return [cosx,sinx]
def runPos(a,z,catch_time,v):
    return [a[index]+item*v*catch_time
    for index,item in enumerate(calsincos(a,z))]
```

```
def time_one_catch(coordinate_P ,coordinates ,v,v_E):
    times=[meet_time1(coordinate_P ,item ,v-v_E[index])
    for index,item in enumerate(coordinates)]
    catch_time=min(times)
    catch_index=times.index(catch_time)
    catch_point=runPos(coordinate_P ,
    coordinates[catch_index],catch_time ,v)
    catch_coordinates=[catch_point
    if(index==catch_index)
    else runPos(coordinates[index],
    catch_point ,catch_time ,v_E[index])
    for index,item in enumerate(coordinates)]
    return catch_time ,catch_index ,catch_coordinates
def time_one_catch_addindex(coordinate_P ,coordinates ,
    v,v_E,catch_index=-1):
    times=[meet_time1(coordinate_P ,item ,v-v_E[index])
    for index,item in enumerate(coordinates)]
    if(catch_index==-1):
        catch_time=min(times)
        catch_index=times.index(catch_time)
    else:
        catch_time=times[catch_index]
    catch_point=runPos(coordinate_P ,
    coordinates[catch_index],catch_time ,v)
    catch_coordinates=[catch_point
    if(index==catch_index)
```

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    else runPos(coordinates[index],catch_point,
    catch_time,v_E[index])
    for index,item in enumerate(coordinates)]
    return catch_time,catch_index,catch_coordinates
def cal_onecatch_time(catch_seq,vlist,
coordinate_P,coordinates,v_E):
    catch_idx=[item-1 for item in catch_seq]
    t=0
    for index,idx in enumerate(catch_idx):
        re=time_one_catch_addindex(coordinate_P,
        coordinates,vlist[index],v_E,idx)
        t=t+re[0]
        coordinate_P=re[2][idx]
        coordinates=re[2]
    return t
def updatelist(flist,uplist):
    a=list(flist)
    b=list(uplist)
    [b.remove(item) for item in a]
    return a+b
def updatelist_fromindex(flist,uplist):
    a=list(flist)
    b=list(uplist)
    c=[b[index] for index in a]
    [b.remove(item) for item in c]
    return c+b
```

```
def list10(L):
    re=[L]+[updatelist([item],L)
    for item in [1,2,3,4]]+[updatelist(item,L)
    for item in [[1,2],[1,3],[1,4]]]+[updatelist(item,L)
    for item in [[1,2,3],[1,2,4]]]
    return re
def list9(L,vs,coordinate_P,coordinates,v_E):
    re=[L]+[[L[0]]]+[[L[1]]]+[[L[2]]]
    +[[L[3]]]+[[L[0],L[1]]]+[[L[0],
    L[2]]]+[[L[0],L[3]]]+[[L[0],L[1],L[3],L[2]]]
    posibles=[[re_item+list(item)
    for item in itertools.permutations([itemi
    for itemi in range(1,5)
    if(itemi not in  re_item)])]
    for re_item in re]
    time_posibles=[[cal_onecatch_time(item,vs,
    coordinate_P,coordinates,v_E)
    for item in posible] for posible in posibles]
    mintime_posibles_idxes=[item.index(min(item))
    for item in time_posibles]
    return [item2[mintime_posibles_idxes[index2]]
    for index2,item2 in enumerate(posibles)]
def calnashi(A):
    return max([min(item) for item in A])
def calnashi2(A):
    A2=[[item[i] for index,item in enumerate(As[0])]
```

```
    for i in range(len(A[0]))]
    return max([min(item) for item in A2])

vstates=[[0,0,0,0],[1,0,0,0],[0,1,0,0],
[0,0,1,0],[0,0,0,1]]
catch_seqs=[list(item)
for item in itertools.permutations([1,2,3,4])]
coordinate_P=[0,0]
coordinate_E1=[1,1]
coordinate_E2=[-1,1]
coordinate_E3=[-1,-1]
coordinate_E4=[1,-1]
vs=[2,50]
v_E=[0.3,0.5,0.7,0.9]

vs_states=[[vs[item2] for item2 in item]
for item in vstates]
[coordinate_E1,coordinate_E2,coordinate_E3,
coordinate_E4,coordinate_E5]
coordinates=[coordinate_E1,coordinate_E2,
coordinate_E3,coordinate_E4]
catch_seqs_9=[list9(item,vs_states[0],
coordinate_P,coordinates,v_E)
for item in catch_seqs]

A=[[cal_onecatch_time(item2,vs_states_item,
```

```
coordinate_P ,coordinates ,v_E)
for item2 in [list(item) for item in catch_seqs]]
for vs_states_item in vs_states]
tv=A[0]
print('The 24 minimum times are',min(tv))
print('shortest time path is',catch_seqs
[tv.index(min(tv))])
As=[[[cal_onecatch_time(item2,vs_states_item,
coordinate_P ,coordinates ,v_E)
for item2 in [list(item) for item in catch_seqs_item]]
for vs_states_item in vs_states]
for catch_seqs_item in catch_seqs_9]
As
```