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Dependent Retailers' Demand in Game Theoretic Model of Supply Chain

Suriya Sh. Kumacheva and Victor V. Zakharov

St.Petersburg State University,
7/9, Universitetskaya nab., St. Petersburg, 199034, Russia
E-mail: s.kumacheva@spbu.ru, v.zaharov@spbu.ru

Abstract Supply chain management is one of the intensively developing areas of applied research. One of the main tools for studying the problems of this area is game theory. This study is based on a two-level supply chain model mathematically described using a hierarchical Stackelberg game. The top player in the hierarchy is the manufacturer and the bottom players are two retailers interacting according to the Cournot game scheme. Unlike previous models, this one assumes that their demands are dependent and jointly distributed. Next, the focus shifts to the study of the interaction pattern of retailers when trading substitute goods. A special case of joint distribution of demand is considered.

Keywords: two-echelon supply chain, Stackelberg game, Cournot game, dependent demand, joint distribution, substitutable demand, Nash equilibrium.

1. Introduction

Recently, new economic conditions associated with geopolitical and macroeconomic changes have been adjusting patterns of interaction and scenarios in various areas of economic activity. One of the areas affected by such changes is the risk of non-delivery and failure to fulfill other contractual obligations in business and trade. The need to manage these risks is driving the search for new solutions related to production and distribution, purchasing, and supply chains.

Distribution of some goods to the final customers is one of the most important problems of modern business, trade, and economy. Therefore, supply chain management (SCM) is aimed at optimizing the entire process from source to consumer. To improve the economic efficiency of each supply chain (Hennetx and Ardax, 2008), it is necessary to take into account such factors as cost reduction and satisfaction of demand for final products, i.e. synchronization of supply with demand.

The practical value and relevance of problems related to supply chain management are reflected in such works as (Bonci et al., 2017, Kherbach and Mocan, 2016, Kumara et al., 2017).

Game-theoretic approach to modeling various dynamic processes is one of the most popular settings in modern analysis in economics, sociology, and management. Few recent decades, SCM has been very intensively developed, and mathematical game theory has become an important tool for analyzing supply chains with multiple agents, often having conflicting interests (Cachon and Netessine, 2004, Sharma et al., 2019).

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To illustrate the game-theoretic approach to supply chain analysis, many models consider a simple system with one supplier and two retailers. For example, the model presented in (Dai et al., 2005) considers the capacity constraint of the manufacturer, the possibility of stockouts, and the reallocation of customers between retailers.

Among the recent works devoted to mathematical models of SCM, we should mention such works as (Kuchesfehani et al., 2022) and (Yao et al., 2022). In (Kuchesfehani et al., 2022), a closed supply chain consisting of one manufacturer and one retailer is modeled as a stochastic dynamic game. By analyzing two scenarios of player interaction, the authors search for an equilibrium in the game under study and give a parametric analysis of the solution found. (Yao et al., 2022) considers a supply chain in which demand depends on the environmental reputation of the producer. The supply chain under study also consists of one manufacturer and one retailer, and a constructive analysis of the impact of consumer sensitivity to the environmental reputation of the manufacturer on the strategies and outcomes of the supply chain participants and the consumer is presented.

The idea of analyzing a supply chain consisting of one manufacturer and two retailers as a basic model was explored by (Dai et al., 2005, Hennetx and Ardax, 2008, Yao et al., 2008) and further developed by (Ghiami and Williams, 2015). The problem of competition between two retailers (Yao et al., 2008) necessitates the need for coordination and revenue sharing between retailers. In this case, the retailers are considered as the two adversaries in the competition model, and the manufacturer is seen as the leader playing a Stackelberg game with them both. For models with this structure, it was shown in (Dai et al., 2005) that the equilibrium between retailers exists and is the unique Nash equilibrium.

The (Ghiami and Williams, 2015) continues to investigate the coordination mechanism for a supply chain of the same structure. One of the factors affecting this coordination is demand, which may be different for each retailer. Starting from the earliest game-theoretic models, such as (Parlar, 1988), it is common to consider demand as a random variable. In the model (Hennetx and Ardax, 2008), retailers were represented by stohastic demand and the supplier by random lead time. In turn, this led to the need to apply queueing theory along with a game-theoretic approach.

The paper (Yuqing et al., 2015) considers a two-echelon supply chain with one manufacturer and two retailers. The interaction between the manufacturer and retailers is represented as a hierarchical scheme similar to the Stackelberg game. At the same time, the retailers interact according to the Cournot game. The objective of each player is to choose a positive value of the order quantity to maximize his/her own expected profit.

In contrast to the mentioned paper and earlier models with a similar problem formulation, the present study rejects the assumption of retailers' demand independence. In the paper (Kumacheva and Zakharov, 2022) a constructive method of finding the Nash equilibrium in the game was proposed for the case when the demand for both retailers can be represented as continuous variables with joint distribution. In this paper, we use the above result to further analyze the behavior of producers and obtain the equilibrium in a general game-theoretic model.

Under the conditions of the new economic reality, the issues related to substitute goods become especially relevant. In this regard, a special case of substitute demand is studied in detail.

Mathematical modeling of interactions between retailers selling substitute goods has been previously studied in many works, such as (Parlar, 1988) etc. Developing the direction outlined in these works, in the presented paper we study a model similar in formulation to the (Kumacheva and Zakharov, 2022) model, shifting the focus from a two-echelon supply chain to the study of the interaction between two lower-level players. Due to the fact that they trade a substitute good, the dependence of their demand functions here can be considered as linear one. The application of the probabilistic properties studied for the two-echelon model in (Kumacheva and Zakharov, 2022) is illustrated here using a joint normal distribution.

The paper has the following structure. Section 1 presents an overview of existing SCM problems, mainly focusing on one manufacturer and two retailers models with random demand. Section 2 formulates the model under study in comparison to the previous model discussed in the section 1. Section 3 is devoted to obtaining the Nash equilibrium conditions in the developed game for the case of dependent demand. Section 4 investigates the transformation of the Cornout game under the assumption that retailers sell substitute goods. In section 5, a special case of normally distributed demand is considered. The section 6 summarizes the results of the study.

2. Model Formulation

2.1. Two-echelon Supply Chain Model

First consider a two-echelon supply chain model involving one manufacturer and two retailers. The manufacturer produces a product, which is then sold to two retailers. The cost of producing a unit of this product is c and the wholesale price is w. The market price of the product is p, which is fixed and common to both retailers.

Let δ_i be the random demand for retailer i's product, $i = \overline{1,2}$. Similarly, δ_j is the random demand for retailer's product j, $j = 3 - i$.

Unlike (Yuqing et al., 2015), where retailers' demands are treated as mutually independent variables, in the current study δ_i and δ_j are continuous variables, having a joint cumulative distribution function $F_{\delta_i,\delta_j}(x, y)$ (which is zero when at least one demand δ_i, $i = \overline{1,2}$, is negative) and a joint probability density function $f_{\delta_i,\delta_j}(x, y)$. In (Kumacheva and Zakharov, 2022) a similar model built under the same assumption was investigated.

If i-th retailer's local demand exceeds its order quantity q_i, then the unsatisfied demand $(\delta_i - q_i)$ is transferred to retailer j with probability β_i, and retailer i loses sales, the unit cost of which is b_i. If retailer j also does not have enough order quantity to satisfy demand shifted from retailer i, then the manufacturer loses b_m, which is the cost of brand loyalty.

The model under study can be considered as a hierarchical game with rational players. The goal of each player is to choose a positive value of the order quantity to maximize his/her own expected profit.

As a high-level player in this hierarchy, the manufacturer chooses a strategy that involves setting a wholesale price w for the product. Retailers choose order quantities at the price set by the manufacturer. Thus, the manufacturer can be considered as the leader of the game, and the behavior of retailers is shaped as the best response to the leader's strategy. Therefore, the interaction between the manufacturer and retailers is a Stackelberg game (Pechersky and Belyaeva, 2001, Tirole, 1988). In our paper, we assume that the wholesale price w is fixed, and focus only on searching the Nash equilibrium of the Cournot game between retailers. In general, the strategy profile can be described as a system of variables (w, q_i, q_j). But in the scope of this study, the producer's control instrument is the wholesale price w and the retailers' control instruments are q_i and q_j. Therefore, for further analysis, we consider the players' profits only as functions of their own strategies.

Thus, in such game the manufacturer's total expected profit is

$$\Pi_m(w) = (w-c)\sum_{i=1}^{2} q_i + b_m E\left[\left(\sum_{i=1}^{2}(\delta_i - q_i)\right)^+\right]. \tag{1}$$

The first summand in the equation (1) represents the profit that manufacturer receives from sales. The second summand is related to expected loss of brand loyalty cost. The plus sign at the last bracket here and hereafter denotes those values which are included in the general expression only if the expression in brackets takes positive values.

In turn, the retailer relationship can be constructed as a Cournot duopoly (Pechersky and Belyaeva, 2001, Tirole, 1988). The total demand of retailer i, $i = \overline{1,2}$, consists of local demand δ_i and demand switched from another retailer j, $j = 3 - i$, in the absence of sufficient stock. Thus, the expected profit of retailer i can be expressed as follows:

$$\Pi_i(q_i) = pE\left(\min\{\delta_i + \beta_j(\delta_j - q_j)^+, q_i\}\right) - b_i E\,(\delta_i - q_i)^+ - wq_i. \tag{2}$$

3. The Case of Dependent Demands: Equilibrium Conditions

To obtain the equilibrium conditions for the dependent demand case, we first need to analyze the expected profit of the retailer (3).

The first term is the market price p multiplied by the expected value of the expression $\min\{\delta_i + \beta_j(\delta_j - q_j)^+, q_i\}$. Let us first represent this expression in another form. Two cases are possible:

1. If $\delta_i + \beta_j(\delta_j - q_j)^+ < q_i$ then $\min\{\delta_i + \beta_j(\delta_j - q_j)^+, q_i\} = \delta_i + \beta_j(\delta_j - q_j)^+$ and therefore, the specified expected value is $E(\delta_i) + \beta_j E(\delta_j - q_j)^+$ (even when δ_i and δ_j are dependent).
2. If $\delta_i + \beta_j(\delta_j - q_j)^+ \geq q_i$ then $E\left(\min\{\delta_i + \beta_j(\delta_j - q_j)^+, q_i\}\right) = q_i$ since q_i is determined by a real constant.

Finally, the expected value of the first summand of the retailer i's expected profit (3) can be calculated according to the formula:

$$\begin{gathered} E\left(\min\{\delta_i + \beta_j(\delta_j - q_j)^+, q_i\}\right) = \\ = Pr\{\delta_i + \beta_j(\delta_j - q_j)^+ < q_i\}\left(E(\delta_i) + \beta_j E(\delta_j - q_j)^+\right) + \\ + Pr\{\delta_i + \beta_j(\delta_j - q_j)^+ \geq q_i\} q_i = \\ = Pr\{\delta_i + \beta_j(\delta_j - q_j)^+ < q_i\}\left(E(\delta_i) + \beta_j E(\delta_j - q_j)^+ - q_i\right) + q_i. \end{gathered} \tag{3}$$

Now let us analyze the value $Pr\{\delta_i + \beta_j(\delta_j - q_j)^+ < q_i\}$, or, which is equivalent under the conditions of the problem formulated above, the value $Pr\{0 \leq \delta_i +$

$\beta_j(\delta_j - q_j)^+ < q_i\}$. Due to the non-negativity of both summands in probability, this expression can be written as $F_\xi(q_i)$, where $\xi = \delta_i + \beta_j(\delta_j - q_j)^+$, and $F_\xi(x)$ is a cumulative distribution function of the variable ξ for its fixed value $x = q_i$. This form is the convolution of the probability distributions in the case where the variables δ_i and $(\delta_j - q_j)^+$ are independent. But we consider another case, assuming joint probability of δ_i and δ_j.

To obtain the main result, we can apply the auxiliary lemma proved in (Kumacheva and Zakharov, 2022). Let's formulate it here.

Lemma 1. *The probability $Pr\{0 \le \delta_i + \beta_j(\delta_j - q_j)^+ < q_i\}$ depends only on the joint distribution of demands δ_i and δ_j, $i \neq j$, and the values of the order quantities q_i and q_j in the following form:*

$$\begin{aligned} & Pr\{0 \le \delta_i + \beta_j(\delta_j - q_j)^+ < q_i\} = \\ & = \int_{-\infty}^{+\infty} \int_{0}^{\alpha_1 q_i} f_{\delta_i,\delta_j}(x,y)dxdy + \int_{q_j}^{\frac{1-\alpha_1}{\beta_j} q_i + q_j} \int_{-\infty}^{+\infty} f_{\delta_i,\delta_j}(x,y)dxdy - \\ & -F_{\delta_i,\delta_j}(\alpha_1 q_i, \tfrac{1-\alpha_1}{\beta_j} q_i + q_j) - F_{\delta_i,\delta_j}(0, q_j) + \\ & +F_{\delta_i,\delta_j}(\alpha_1 q_i, q_j) + F_{\delta_i,\delta_j}(0, \tfrac{1-\alpha_1}{\beta_j} q_i + q_j), \end{aligned} \tag{4}$$

where α_1 is a real constant, $\alpha_1 \in (0, 1)$.

The proof of the above lemma can be found in the Appendix.

Using the properties of the joint distribution of (Borovkov, 2009, Dekking et al., 2005), we can find the following expected values:

$$E(\delta_i) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x f_{\delta_i,\delta_j}(x,y)dxdy$$

and

$$\begin{aligned} E(\delta_j - q_j)^+ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (y - q_j)^+ f_{\delta_i,\delta_j}(x,y)dxdy = \\ &= \int_{-\infty}^{+\infty} \int_{q_j}^{+\infty} y f_{\delta_i,\delta_j}(x,y)dxdy - q_j Pr\{\delta_j > q_j\}. \end{aligned}$$

Thus, knowing the retailers' joint random demand distribution δ_i and δ_j, we can obtain the first term (3) of the expected profit of retailer i (3).

Similar reasoning can be applied to the analysis of the second summand in equation (3):

$$\begin{aligned} b_i E\left(\delta_i - q_i\right)^+ &= b_i \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - q_i)^+ f_{\delta_i,\delta_j}(x,y)dxdy = \\ &= b_i \left(\int_{q_i}^{+\infty} \int_{-\infty}^{+\infty} x f_{\delta_i,\delta_j}(x,y)dxdy - q_i Pr\{\delta_i > q_i\} \right) \end{aligned} \tag{5}$$

Now we can use equations (4) and (5) to calculate the first derivative of the i-th retailer's expected profit (3) and obtain the extremum conditions:

$$\frac{\partial \Pi_i}{\partial q_i} = p(1 - \Psi_i) + b_i Pr\{\delta_i > q_i\} - w = 0, \tag{6}$$

where Ψ_i is defined as a right-hand side of the equation (4).

The similar extension can be obtained for the expected profit of retailer j:

$$\frac{\partial \Pi_j}{\partial q_j} = p(1 - \Psi_j) + b_j Pr\{\delta_j > q_j\} - w = 0, \tag{7}$$

where Ψ_j can be defined as

$$\begin{aligned} \Psi_j = \int\limits_0^{\alpha_2 q_j} \int\limits_{-\infty}^{+\infty} f_{\delta_i,\delta_j}(x,y)dxdy + \int\limits_{-\infty}^{+\infty} \int\limits_{q_i}^{\frac{1-\alpha_2}{\beta_i} q_j + q_i} f_{\delta_i,\delta_j}(x,y)dxdy - \\ -F_{\delta_i,\delta_j}(\tfrac{1-\alpha_2}{\beta_i} q_j + q_i, \alpha_2 q_j) - F_{\delta_i,\delta_j}(q_i, 0) + \\ +F_{\delta_i,\delta_j}(q_i, \alpha_2 q_j) + F_{\delta_i,\delta_j}(\tfrac{1-\alpha_2}{\beta_i} q_j + q_i, 0), \end{aligned} \tag{8}$$

where α_2 is a real constant $\alpha_2 \in (0, 1)$.

Following the proof scheme of the statement in (Yuqing et al., 2015) for the game with independent demands, it is easy to show that the non-negative solution (q_i^*, q_j^*) of the system (6) – (7) forms a Nash equilibrium in the considered game with joint distribution of stochastic demand. Thus, the following theorem holds.

Theorem 1. *The non-negative solution of the system (6) – (7) forms a unique Nash Equilibrium between two retailers in two-echelon supply chain model with market search behavior and dependent retailers' demands.*

Thus, we see that the equilibrium expected manufacturer's profit $\Pi_m(w)$ can be determined from (1) for any solution of the system (6) and (7), which can be obtained in general if the joint distribution $F_{\delta_i,\delta_j}(x, y)$ is a known law.

4. Low Level Cournot Game: the Case of Substitute Goods

Now consider the case where retailers sell substitute products. Under this assumption, we have to abandon the situation when they are buyers of the product of one manufacturer. Moreover, since products and manufacturers are different, wholesale prices w_i and w_j are also different. For the purposes of this study we will not take into account the influence of the manufacturer's price factor and suppose that they are approximately equal $w_i \approx w_j$ and, therefore, can be further replaced by a common value, which we will put equal to w.

From a mathematical point of view, our new assumption means that we artificially narrow down the original model described in the previous section and study only the interaction between two retailers. Therefore, we have to postpone the study of the Stackelberg hierarchical game and focus on the Cournot game of two low-level players.

Let δ_i and δ_j be the demands for substitute goods. In real trade practice, we can interpret such a game as a situation when two retailers trade products with similar demands (e.g., coffee and tea, milk and its substitutes, oranges and grapefruits, etc.). Following (Parlar, 1988, Martagan, 2010), we hypothesize that in this case retailers' demands have the following dependence:

$$\delta_i = \xi \delta_j + \eta, \tag{9}$$

where ξ and η are the real coefficients to obtain a linear relationship between δ_i and δ_j demand.

Thus, with (9), an event $\{0 \leq \delta_i + \beta_j(\delta_j - q_j)^+ < q_i\}$ is equivalent to simultaneous occurrence of events $0 \leq \delta_i < \alpha_1 q_i$ and $q_j \leq \delta_j < \frac{1-\alpha_1}{\beta_j} q_i + q_j$ (where α_1, similarly to Lemma 1, is an arbitrary real constant, $\alpha_1 \in (0;1)$), that gives us the system of these conditions.

Taking into account this system and (9), we obtain that

$$Pr\{0 \leq \delta_i + \beta_j(\delta_j - q_j)^+ < q_i\} = Pr\{A \leq \delta_j < B\},$$

where

$$A = \max\{-\frac{\eta}{\xi}; q_j\} \tag{10}$$

and

$$B = \min\{\alpha_1 q_i; \quad \frac{\alpha_1 q_i - \eta}{\xi}\}. \tag{11}$$

Finally, we can see that

$$Pr\{A \leq \delta_j < B\} = \int_A^B f_{\delta_j}(y)dy = \int_A^B \int_{-\infty}^{+\infty} f_{\delta_i,\delta_j}(x,y)dxdy, \tag{12}$$

where A and B can be defined from (10) and (11).

Using the inverse of (9), we can obtain a similar result for demand δ_i:

$$Pr\{C \leq \delta_i < D\} = \int_C^D f_{\delta_i}(x)dx = \int_{-\infty}^{+\infty} \int_C^D f_{\delta_i,\delta_j}(x,y)dxdy, \tag{13}$$

where C and D are

$$C = \max\{\eta; q_i\} \tag{14}$$

and

$$D = \min\{\frac{1-\alpha_2}{\beta_i} q_j + q_i; \quad \eta + \xi \alpha_2 q_j\} \tag{15}$$

correspondingly, and α_2 is a real constant, $\alpha_2 \in (0;1)$.

5. Special Case: Normally Distributed Demand

In (Kumacheva and Zakharov, 2022) it was discussed that, given a known law of joint distribution of retailers' demand, the structure of the Nash equilibrium can be substantially simplified. Obtained for the Cournot game, when substituted into the expression of the manufacturer's payoff (1), it gives the general equilibrium of the Stackelberg game.

But, in the previous section, we abandoned the setting of the general Stackelberg game, cutting the game to study the low-level game between retailers. Under this assumption, we can obtain an equilibrium in the Cournot game for a fixed distribution law.

To illustrate this approach let's consider a particular case and assume that demand δ_i is distributed normally with parameter μ_i and σ_i, similarly, δ_j has normal distribution with mean μ_j and σ_j:

$$\delta_i \sim N(\mu_i, \sigma_i^2), \quad \delta_j \sim N(\mu_j, \sigma_j^2).$$

According to the assumption of dependence between δ_i and δ_j, their joint distribution is represented by the density function (Borovkov, 2009)

$$f_{\delta_i,\delta_j}(x,y) =$$

$$= \frac{1}{2\pi\sigma_i\sigma_j\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_i)^2}{\sigma_i^2} - 2\rho\frac{(x-\mu_i)(y-\mu_j)}{\sigma_i\sigma_j} + \frac{(y-\mu_j)^2}{\sigma_j^2}\right]},$$

where $\sigma_{ij} = cov(X,Y)$, $\rho = \frac{\sigma_{ij}}{\sigma_i\sigma_j}$.

Thus, (3) becomes

$$E\left(\min\{\delta_i + \beta_j(\delta_j - q_j)^+, q_i\}\right) =$$

$$= \left(\Phi\left(\tfrac{B-\mu_j}{\sigma_j}\right) - \Phi\left(\tfrac{A-\mu_j}{\sigma_j}\right)\right)(\mu_i - q_i + \beta_j(\mu_j - q_j)) + q_i, \tag{16}$$

where $\Phi(z)$ is Laplace integral function, A and B are the values from (10) and (11).

Then the extremum condition for i-th retailer (6) becomes

$$\frac{\partial \Pi_i}{\partial q_i} = p\left(1 - \Phi\left(\frac{B-\mu_j}{\sigma_j}\right) + \Phi\left(\frac{A-\mu_j}{\sigma_j}\right)\right) + b_i Pr\{\delta_i > q_i\} - w = 0,$$

and, with normality of demand distribution:

$$\frac{\partial \Pi_i}{\partial q_i} = p\left(1 - \Phi\left(\frac{B-\mu_j}{\sigma_j}\right) + \Phi\left(\frac{A-\mu_j}{\sigma_j}\right)\right) + b_i\left(\frac{1}{2} - \Phi\left(\frac{q_i-\mu_i}{\sigma_i}\right)\right) - w = 0. \tag{17}$$

Similarly, the extremum condition for j-th retailer is

$$\frac{\partial \Pi_j}{\partial q_j} = p\left(1 - \Phi\left(\frac{D-\mu_i}{\sigma_i}\right) + \Phi\left(\frac{C-\mu_i}{\sigma_i}\right)\right) + b_j\left(\frac{1}{2} - \Phi\left(\frac{q_j-\mu_j}{\sigma_j}\right)\right) - w = 0. \tag{18}$$

and can be obtained in the same way, using the transformation inverse to (9) and (13).

Following (Yuqing et al., 2015) and (Kumacheva and Zakharov, 2022), in the current research Theorem 1 formulates that in the class of similar problems a unique Nash Equilibrium can be found as a solution of equations (6) – (7). For the special case under study we can formulate the following result.

Proposition 1. *The non-negative solution (q_i^*, q_j^*) of the system (17) – (18) forms a unique Nash Equilibrium between two retailers with normally distributed substitute demand. This solution depends only on the values of the integral Laplace function determined by the relations between the parameters forming quantities (10), (11), (14) and (15).*

6. Conclusions

The presented work is a logical continuation of the study (Kumacheva and Zakharov, 2022) related to a two-echelon supply chain with market search behavior and dependent retailers' demand. In the mentioned paper, the model was investigated under the assumption that retailers' demand has a joint distribution. In this framework, a lemma was proved, according to which the probability $Pr\{0 \leq \delta_i + \beta_j(\delta_j - q_j)^+ < q_i\}$ depends only on the joint distribution of demands δ_i and δ_j, $i \neq j$, and the values of the order quantities q_i and q_j.

This paper presents a further application of this result based on the study of demand for substitute goods. Under the assumption of linear dependence of demand, a parametric analysis of equilibrium in the Cournot game is carried out. The special case of normally distributed demand is also studied. The results obtained earlier are applied to the problem with a refined form of joint distribution.

Taking into account the results presented in earlier papers, (Yuqing et al., 2015) and (Kumacheva and Zakharov, 2022), this study derives the Nash Equilibrium in the Cournot game between retailers for the case under study.

It should be noted that there are many prospects within the framework of the designed model and the results obtained for it. The following stand out among them.

First, we should consider cases of other joint distributions besides the normal law. Second, it would be interesting to study other types of mutually dependent demand from an economic point of view. It is also necessary to return to the original setting of the problem and obtain a general equilibrium for the manufacturer and retailers in Stackelberg hierarchical game.

To summarize, only a phase of this study has been completed. The relevance of the research problems and the mathematical formulation of the problem allow us to conduct further research, analyzing in detail all stages of the solution and special cases.

7. Appendix

Here the proof of the auxiliary lemma 1 can be found.

Proof. First, we denote the right side of the equality (4) as Φ. Then, let's consider two double inequalities:

$$0 \leq \delta_i < \alpha_1 q_i \tag{19}$$

and

$$q_i \leq \delta_j < \frac{1-\alpha_1}{\beta_j} q_i + q_j. \tag{20}$$

Their union is equivalent to $0 \leq \beta_j(\delta_j - q_j)^+ < q_i$.

Now we can represent the expression $Pr\{0 \leq \delta_i + \beta_j(\delta_j - q_j)^+ < q_i\}$ as the probability of union (Borovkov, 2009) of the events (19) and (20). Therefore:

$$Pr\{0 \leq \delta_i + \beta_j(\delta_j - q_j)^+ < q_i\} = Pr\{0 \leq \delta_i < \alpha_1 q_i\} +$$
$$+ Pr\{q_i \leq \delta_j < \tfrac{1-\alpha_1}{\beta_j} + q_j\} - Pr\{0 \leq \delta_i < \alpha_1 q_i,\ q_j \leq \delta_j < \tfrac{1-\alpha_1}{\beta_j} + q_j\}.$$

Further our reasoning will refer to the properties of joint probability distribution $F(\delta_i, \delta_j)$ (Borovkov, 2009, Dekking et al., 2005).

The first summand $Pr\{0 \le \delta_i < \alpha_1 q_i\} = F_{\delta_i}(\alpha_1 q_i) - F_{\delta_i}(0)$ can be obtained applying our knowledge of the density of joint distribution:

$$Pr\{0 \le \delta_i < \alpha_1 q_i\} = \int\limits_{-\infty}^{+\infty} \int\limits_{0}^{\alpha_1 q_i} f_{\delta_i,\delta_j}(x,y)dxdy.$$

Similarly, the second summand can be obtained:

$$Pr\{q_j \le \delta_j < \frac{1-\alpha_1}{\beta_j} q_i + q_j\} = \int\limits_{q_j}^{\frac{1-\alpha_1}{\beta_j} q_i + q_j} \int\limits_{-\infty}^{+\infty} f_{\delta_i,\delta_j}(x,y)dxdy.$$

Finally the third summand is

$$\begin{gathered} Pr\{0 \le \delta_i < \alpha_1 q_i,\ q_j \le \delta_j < \tfrac{1-\alpha_1}{\beta_j} q_i + q_j\} = \\ = F_{\delta_i,\delta_j}(\alpha_1 q_i, \tfrac{1-\alpha_1}{\beta_j} q_i + q_j) + F_{\delta_i,\delta_j}(0, q_j) - F_{\delta_i,\delta_j}(\alpha_1 q_i, q_j) - \\ - F_{\delta_i,\delta_j}(0, \tfrac{1-\alpha_1}{\beta_j} q_i + q_j) \end{gathered}$$

with negative sign.

The Lemma is proved. □

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