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Analysis and Control of Macroeconomic Trends Based on the Leontief Model

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Abstract In the present study, we consolidate the forty-five industrial sectors delineated in the U.S. input-output tables, as disseminated by the OECD, into three overarching sectors: advanced manufacturing, modern services, and a residual category termed 'others.' We adopt gross fixed capital formation as a proxy for the proportion of net profit allocated to investment, positing that the investment requisite for augmenting output is commensurate with the requisite capital intensity. This framework enables us to forecast the trajectory of total output and GDP, taking into account the interplay of multiple determinants. In addressing the inherent linear control dynamics of the input-output model, we apply classical control theory to regulate the advanced manufacturing sector. By deriving control equations that accommodate multifactorial influences, we substantiate the efficacy of this control mechanism through rigorous numerical analysis. Moreover, we reconceptualize the dynamic input-output system as a game-theoretic model characterized by a saddle-point equilibrium. By leveraging the saddle-point equilibrium strategy, we pioneer an innovative approach to resolving the complexities of dynamic input-output analysis. This methodological innovation not only enhances the precision of our predictions but also contributes a novel perspective to the literature on economic modeling and control theory.

Keywords: dynamic input-output model, program control, saddle point equilibrium strategy, differential game.

1. Introduction

The application of static input-output models and the theory of dynamic input-output models are the main focuses of current input-output research. Few researchers have employed dynamic input-output models in conjunction with game theory to investigate the optimal control of an unbalanced economy.

In the era of the digital economy, it is crucial to consider the time component in the input-output table and to be able to control the parameters to achieve the desired goals. In particular, in the case of unbalanced input-output, providing optimal control under the influence of multiple factors can help enterprises to optimize their benefits. Similarly, the government can alter tax policy based on the predicted result of the input-output table to guide resource allocation and encourage the development of each sector.

Leontief (1956) proposed the dynamic inverse model, laying the foundation of dynamic input-output modeling. Miller (2009) included the time factor in the input-output analysis. Smirnov (2021) considered government consumption as the fourth quadrant of the input-output table in the input-output equilibrium equation, and defined total output as the derivative of time. In this paper, based on (Miller, 2009) and (Smirnov, 2021), we remove Taxes fewer subsidies on intermediate and final

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domestic products, which correspond directly to each sector, as the fourth quadrant according to the input-output table published by OECD. The time series is utilized to forecast current economic trends in the United States.

Given the linear control characteristic in the dynamic input-output model, we combine the classical program control theory proposed by Tamasyan (2008) and based on the viewpoint of the advanced manufacturing industry, i.e., the industry that utilizes the emerging technology as the fundamental means, proposed by Singhry (2016), Mourtzis et al. (2018) and Jin et al. (2017), we classify what sectors are included in the advanced manufacturing industry. The equations for the control of multifactor influence are obtained after applying complete control to the advanced manufacturing industry in the U.S. The effectiveness of the program control is verified by numerical computation.

Qu and Huang (1999) described optimal production strategies for consumption tracking in dynamic input-output systems. Kang et al. (1992) discussed the dynamic input-output optimal control model. And Mao (1992) indicated the dynamic input-output optimal control model with constraints. Based on the dynamic input-output model proposed by Leontief, we study the optimal policy design problem of the continuous dynamic input-output model. The dynamic input-output system is abstracted as a saddle-point equilibrium game model in the current paper using ideas from Qu, Kang, and Mao. The saddle-point equilibrium strategy is used to design a new method for solving the dynamic input-output problem, which provides a basis for decision-making by macroeconomic policymakers. The current paper also uses the saddle-point equilibrium strategy to design a new method for solving the dynamic input-output problem, which provides a basis for decision-making by macroeconomic policymakers.

2. Dynamic Input-output Modeling of a Balanced Economy

2.1. Relationship between input-output table

The input-output tables published by the OECD cover 45 economic sectors, and all values are represented in current dollars. For the input-output table, the rows upward represent the destination of the allocated uses of the product of a sector, the sum of which equals total output X_i; the columns upward represent the various inputs to the production of the product, and the sum of the values of these inputs is the total inputs, i.e., X_j. Each element of the matrix $Q = \{x_{ij}\}_{ij=1}^{45}$ represents, from the row-wise view, the amount of i product allocated to the production of j product, and from the column-wise direction, the amount of i product consumed in the production of j product. The row vector E_j represents the value added in each j sector. It contains labor wages W_j , net production taxes ONT_j, and net profits Prh_j. TF denotes Taxes less subsidies on intermediate and final imported products $(TXS-IMP-FNL)$. TD denotes Taxes fewer subsidies on intermediate and final domestic products $(TXS - INT - FNL)$. The column vector Y denotes the final expenditure on output in the industrial sector, which includes Gross Fixed Capital Formation($GFCF$), Changes in inventories(III), Direct purchases abroad by residents (imports), Direct purchases abroad, Exports (cross border), and Imports (cross border).

Fig. 1 has the following equilibrium relations.

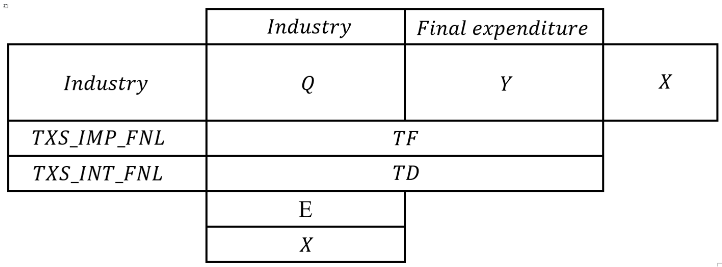

Fig. 1. Schematic diagram of OECD input-output table

$$X_j = \sum_{i=1}^{n} x_{ij} + TF_j + TD_j + E_j = \sum_{j=1}^{n} x_{ij} + Y_i = X_i, \tag{1}$$

where $i, j = 1, \ldots, 45$.

GDP is the sum of the value added E_j and $TXS - INT - FNL$, that is,

$$GDP = \sum_{j=1}^{n} E_j + \sum_{j=1}^{m} TD_j = \sum_{i=1}^{n} Y_i + \sum_{j=m-n}^{m} TF_j + \sum_{j=m-n}^{m} TD_j, \tag{2}$$

where $n = 45, m = 54$.

Similarly, we can obtain:

$$a_{ij} = \frac{x_{ij} - E_j - TD_j}{X_j}, \quad td_j = \frac{TD_j}{X_i}. \tag{3}$$

a_{ij} denotes the quantity consumed by sector i per unit of output in sector j, i.e., the direct consumption coefficient. td_j denotes the sum of taxes fewer subsidies on intermediate and final domestic products in sector j as a share of total output X_i.

$$rw_j = \frac{W_j}{E_j}, \quad rt_j = \frac{ONT_j}{E_j}, \quad ap_j = \sum_{j=1}^{n} a_{ij}. \tag{4}$$

rw_j represents sector j's labor wages as a share of its value added, rt_j represents sector j's taxes on production as a share of its value added, and ap_j is the total share of intermediate consumption in sector j in sector i's annual output X_i.

$$Y_{r_i} = \frac{Y_i}{GDP}, \quad E_j = (1 - ap_j)\, X_j, \quad td = \frac{\sum_{j=m-n}^{m} TD_j}{GDP}. \tag{5}$$

Y_{r_i} denotes the Y_i share of GDP for final consumption in sector i, and td indicates the GDP share of the sum of taxes fewer subsidies on intermediate and final domestic products in sector m corresponding to final consumption to $m - n$.

By combining Eqs. (1)–(5), we can obtain the equation for GDP as affected by each relative variable:

$$GDP = \sum_{j=1}^{n} E_j + \sum_{j=1}^{m} TD_j = (1 - ap_j)\, X + td \cdot GDP. \tag{6}$$

2.2. Dynamic Input-output Modeling

The characteristic properties of the necessary state variables, which provide the basis for simulating the anticipated evolution of the dynamic economy, are listed in the equation above. The economic sector is taken into account while creating the input-output table, and to treat it like a dynamic system, we must first create a set of differential equations that describe the course of economic development.

Definition 1 (Smirnov, 2021). At a constant level of technology, the change in total output X_t over time is defined as its derivative at time t, that is $\dot{X}(t) = \frac{dx(t)}{dt}$, where,$X_t = (X_1(t), \ldots, X_n(t))$ is a vector of accrual-based outputs of the economic sector in terms of inputs at time t. The vector $\dot{X}(t)$ describes the acceleration of production in all sectors of the economy.

At the same level of technology and initial output X_0, the relative increase in output $\frac{\triangle X}{X_0}$ requires a proportional increase in the comparative gross fixed asset formation $\frac{\triangle GFCF}{GFCF_0}$ and the change in inventories $\frac{\triangle III}{III_0}$, where $GFCF_0$ and III_0 are the initial gross fixed asset formation and the initial inventory change, respectively. The amount of investment Cp required to increase output is proportional to the requisite acceleration $\dot{X}_i(t)$. As a result, when combined with the definition 1, the ratio that defines the amount of investment in each sector of the economy will take the following form:

$$Cp_i(t) = Fe_i \cdot \dot{X}_i(t), \tag{7}$$

where the Fe_i value represents the capital density of each sector of the economy. It is the coefficient of proportionality between the growth of output $\dot{X}_i(t)$ and the amount of investment $Cp_i(t)$ required to ensure it.

Fe_i indicates the output per unit time $\frac{\Delta X_i}{\Delta t} = \frac{X_i(T_2)-X_i(T_1)}{T_2-T_1}$. Consequently, Fe_i leads to an acceleration in the production of goods and services. Its value to each manufacturer is determined by the ratio, i.e:

$$Fe_i = \frac{Cp_i(t)}{\frac{\Delta X_i}{\Delta_t}} = \frac{Cp_i(t)\,(T_2 - T_1)}{X_i\,(T_2) - X_i\,(T_1)} = \frac{Cp_i(t)\,(T_2 - T_1)}{X_i\,(T_1)\left(\frac{X_i(T_2)}{X_i(T_1)} - 1\right)}. \tag{8}$$

For the input-output tables published by the OECD, capital intensity can be written as:

$$Fe_i(t) = \frac{Cp_i(t)}{X_i(t+1) - X_i(t)}. \tag{9}$$

$$Fe_{n+1}(t) = \frac{Cp(t)}{GDP(t+1) - GDP(t)}. \tag{10}$$

In an input-output model, the construction of differential equations is related to the sources of investment. In general, the generators of investment include $GFCF$, III, government expenditures imports, and exports. However, for investment, we usually consider only two factors, $GFCF$ and III, because they are sources of investment based on internal demand.

Based on the formula (7), we regard the investment as part of the net profit, which is written as:

$$Cp_i = rn_j \cdot Prh_j, \quad j = 1, \ldots, n, \tag{11}$$

where rn_j denotes the share of net profit used for investment.

The formula (11) can be written as follows:

$$Cp_i = GFCF_j + III_j, \tag{12}$$

$$rn = \frac{Cp}{GDP}, \tag{13}$$

where rn denotes the volume of investment Cp as a share of GDP.

Based on (3), (4), (5), (7), we can obtain the set of dynamic differential equations describing the economic development of each sector considered.

$$\dot{X} = \frac{rn_j \cdot (1 - ap_j - td_j)\,(1 - rw_j - rt_j)}{Fe_i} \left(\sum_{j=1}^{n} a_{ij} \cdot X_j + Y_{r_j} \cdot GDP \right), \tag{14}$$

$$G\dot{D}P = \frac{rn}{Fe_{n+1}} \left(\sum_{j=1}^{n} (1 - ap_j)\, X_j + td \cdot GDP \right). \tag{15}$$

In vector form, the system (14),(15) is shown below:

$$\dot{\mathbf{X}} = \mathbf{DX}, \quad \mathbf{D} = \mathbf{M}\widetilde{\mathbf{Q}} \tag{16}$$

where,

$$\widetilde{\mathbf{Q}} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} & Yr_1 \\ x_{21} & x_{22} & \cdots & x_{2n} & Yr_2 \\ \vdots & I & \ddots & \vdots & \vdots \\ x_{n1} & x_{n1} & \cdots & x_{nn} & Yr_n \\ 1 - ap_1 & 1 - ap_2 & \cdots & 1 - ap_n & td \end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} \frac{\alpha_1}{Fe_1} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\alpha_2}{Fe_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\alpha_n}{Fe_n} & 0 \\ 0 & 0 & \cdots & 0 & \frac{td}{Fe_{n+1}} \end{bmatrix},$$

$\alpha_j = rn_j\,(1 - rw_j - rt_j)\,(1 - td_j - ap_j),\ j = 1, \cdots, n.$

Considering the direct consumption coefficient ap_j can better estimate the production cost and resource utilization, considering the wage rate rw_j can assist in analyzing the relationship between production cost and output, and considering the net production tax rt_j can more accurately calculate the production cost and profit, the model can help us to more accurately predict the total output as well as the GDP by considering all of these factors, which is helpful to the government and the enterprises to formulate the economic policy and strategic planning.

2.3. Input-output Analysis in the United States

For the U.S. input-output tables 2000-2018, the current paper utilized R Programming Language and Python to calculate capital intensity and control for errors between total output and actual total output, as detailed in (Dan, 2023).

We aggregated 45 sectors into three major categories, namely advanced manufacturing, abbreviated as AMI (comprising 12 sectors), modern services, abbreviated as MSI (comprising 12 sectors), and other industries (23 sectors). Based on Eqs. (14), (15), we used linear regression to predict capital intensity under inflation, denoted as Fe_i, Fe_{n+1}. Additionally, we employed time series analysis to forecast GDP and the total output of the three aggregated sectors from 2019 to 2024.

According to Fig. 2, the year 2007 required the highest capital expenditure per unit of total output for both advanced manufacturing and modern services, while the years 2001 and 2008 saw the lowest capital expenditure per unit of total output for these sectors. The minimal capital required for advanced manufacturing in 2001 reflects the sector's sluggish development during that period. The financial crisis of 2007-2008 is also one of the key reasons for the notably low capital expenditure in modern services in 2008. From the projections, considering inflation, the capital intensity for advanced manufacturing is consistently higher than that for modern services and other industries.

According to Fig. 3, from the historical data, the unit capital expenditure required to increase the unit of GDP was the highest in 2007, and the unit capital expenditure required to increase the unit of GDP was the lowest in 2008. For the 2019-2024 forecast, the lowest unit capital expenditure is required to increase unit GDP in 2022.

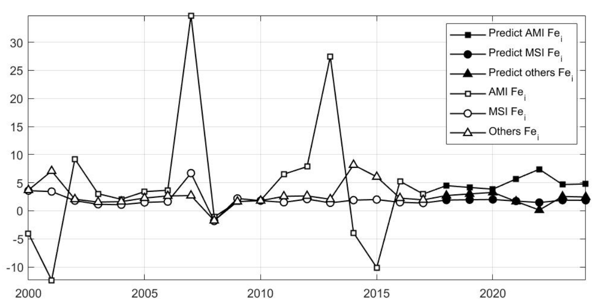

Fig. 2. Predicted values of capital density Fe_i under real GDP

According to Fig. 4, nominal GDP is consistent with forecast GDP from 2000 to 2018. At this stage, the official U.S. GDP was released only until 2021. For our forecasts of 215,906,93 , 224, 878, 63, 233,095,49 and 240, 621, 01 million for the years 2019, 2020, 2021, 2022, respectively, the variances are: 0.97%, 6.34%, 0.04% and 1.84%. The projected nominal GDP for 2023 and 2024 will be \$ 247, 513, 36 and \$ 253, 825, 81 million respectively.

According to Fig. 5, historical data from 2000 to 2018 shows that the actual values are consistent with the predicted values. Overall, the total output of the modern services industry has always been higher than that of advanced Manufacturing. Notably, the year 2008 serves as an inflection point for both sectors, with the financial

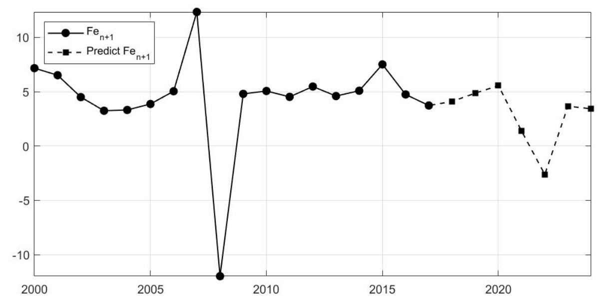

Fig. 3. Predicted values of capital density Fe_{n+1} under real GDP

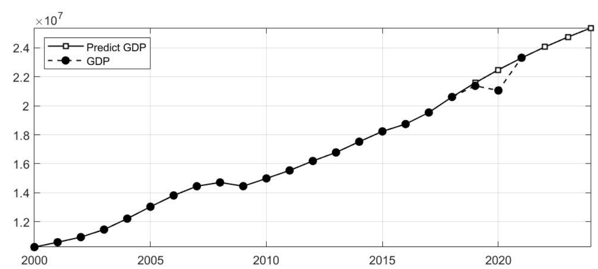

Fig. 4. Nominal GDP and predicted GDP

crisis being an evident and significant cause. In the forecast for 2019-2024, the total output for modern services industry is predicted to rise, while the total output for advanced manufacturing industry is expected to decline. This trend further confirms that, in developed countries, the modern services industry contributes more significantly to the economy.

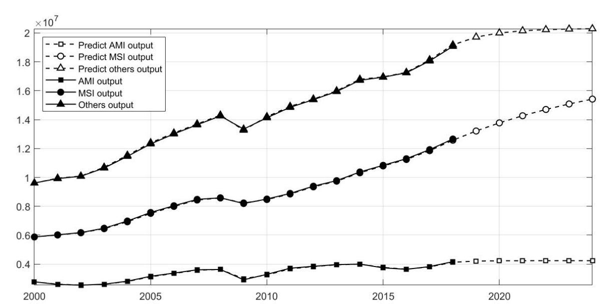

Fig. 5. Total outputs and predicted total inputs of the 3 sectors

3. Equilibrium Economic Program Control

The control system:

$$\dot{X}(t) = \mathbf{D}X(t) + \mathbf{Q}u + f(t), \tag{17}$$

where
$\mathbf{X} = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ - state vector, T- transposition,
$\mathbf{u} = (u_1, \dots, u_r)^T \in \mathbb{R}^r$- control vector,
$\mathbf{P}(t)$, $\mathbf{Q}(t)$ are $(n \times n)$ and $(n \times r)$ matrices with continuous components,
$\mathbf{f}(t)$ - n-dimensional continuous vector function - a disturbance.

This is a particular case of continuous one when $\mathbf{D}$ = constant, $\mathbf{Q}$ = constant.

The general solution of the Cauchy form of system (17) is:

$$x(t, 0, x_0) = Y(t)\left[x_0 + \int_0^t Y^{-1}(\tau)(Q(\tau)u(\tau) + f(\tau))d\tau\right]. \tag{18}$$

Let two points x_0, x_1 and interval $t \in [0, T]$ are given. It is necessary to find an admissible control $\mathbf{u}(t)$ such that

$$\mathbf{x}(T, \mathbf{x}_0, \mathbf{u}(\cdot)) = \mathbf{x}_1. \tag{19}$$

The pair $\mathbf{x}_0$, $\mathbf{x}_1$ is said to be controllable on interval $t \in [0, T]$ if there exists an admissible control that is a solution of equation (18).

3.1. Program control algorithm

① Check the complete controllable of the system.
Stationary systems: the Kalman criterion. rang $\left[Q, DQ, \dots, D^{n-1}Q\right] = n$.
Nonstationary systems: Sufficient conditions for complete controllability.
$rangS\,(t_*) = n$.
② Calculate the fundamental matrix $\mathbf{Y}(t)$ of the homogeneous system $\dot{\mathbf{x}} = \mathbf{D}(t)\mathbf{x}$.
③ Construct the matrix $\mathbf{B}(t) = \mathbf{Y}^{-1}(t)\mathbf{Q}(t)$.
④ Calculate the Gramian $\mathbf{A}(T) = \int_0^T \mathbf{B}(\tau)\mathbf{B}^T(\tau)d\tau$.
⑤ Calculate the vector $\eta = \mathbf{Y}^{-1}(T)\mathbf{x}_1 - \mathbf{x}_0 - \int_0^T \mathbf{Y}^{-1}(\tau)\mathbf{f}(\tau)d\tau$.
⑥ Solve the system $\mathbf{A}(T)\mathbf{c} = \eta$.
⑦ Solve the integral equation $\int_0^T \mathbf{B}(\tau)\mathbf{v}(\tau)d\tau = \mathbf{0}$
or take $\mathbf{v}(t) \equiv \mathbf{0}$.
⑧ Form the program control $\mathbf{u}(t) = \mathbf{B}^T(t)\mathbf{c} + \mathbf{v}(t)$.

3.2. The United States input-output program control

Informed by the computational analysis presented in Section 2.3, we have acquired data ($n = 4$) for the United States during 2017-2018. We set control for the advanced manufacturing industry to obtain the following non-homogeneous equation.

$$\dot{X} = \begin{bmatrix} 0.0195 & 0.0013 & 0.0031 & 0.0047 \\ 0.0081 & 0.0155 & 0.0098 & 0.0151 \\ 0.0240 & 0.0067 & 0.0114 & 0.0336 \\ 0.0375 & 0.0379 & 0.0315 & 0.0011 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u,$$

$$x_0 = \begin{bmatrix} 3823518.3 \\ 11922242.2 \\ 18063252.8 \\ 19541523.1 \end{bmatrix}, x_1 = \begin{bmatrix} 4164850 \\ 12656899.5 \\ 19074639.5 \\ 20611103.4 \end{bmatrix}$$

To assess the system's complete controllability, we implement program control targeting a specific coordinate pair $\{x_0, x_1\}$ within the interval $[0, 1]$. Here, x_0 and

x_1 represent the total output of the advanced manufacturing industry, the modern service industry, others, as well as the GDP for the years 2017 and 2018, respectively.

$$D = \begin{bmatrix} 0.0195 & 0.0013 & 0.0031 & 0.0047 \\ 0.0081 & 0.0155 & 0.0098 & 0.0151 \\ 0.0240 & 0.0067 & 0.0114 & 0.0336 \\ 0.0375 & 0.0379 & 0.0315 & 0.0011 \end{bmatrix}, \; Q = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; f(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$T = 1. \; x_0 = \begin{bmatrix} 3823518.3 \\ 11922242.2 \\ 18063252.8 \\ 19541523.1 \end{bmatrix}, \; x_1 = \begin{bmatrix} 4164850 \\ 12656899.5 \\ 19074639.5 \\ 20611103.4 \end{bmatrix}.$$

Initially, the system's complete controllability is verified using the Kalman criterion for stationary systems, which necessitates a non-zero determinant of the controllability matrix ($det \; D \neq 0$).

$$Rank(Q, DQ, D^2Q, D^3Q) = \begin{bmatrix} 1 & 0.0195 & 0.00064143 & 0.00002 \\ 0 & 0.0081 & 0.00108495 & 0.00006 \\ 0 & 0.024 & 0.00108495 & 0.00010 \\ 0 & 0.0375 & 0.00182829 & 0.00013 \end{bmatrix} = 4.$$

Therefore, it is completely controllable.

Due to the intricate nature of the computational formulae involved, intermediate steps are omitted for brevity. Consequently, the control input $u(t)$ is presented directly. For practical purposes, the constant e is approximated by 2.7183, facilitating a numerical solution to the control problem.

$u(t) = [[1.6118 \times 10^{18} \times ((-0.7006 \times 2.7183^{0.003t} + 0.0113 \times 2.7183^{0.015t} + 0.3318 \times 2.7183^{0.059t} + 0.3574/2.718^{0.03t}) \times (-0.2937 \times 2.718^{0.003t}) + \cdots + (0.6445/2.7183^{0.03t}) \times (0.4216 \times 2.718^{0.003t} + 0.0018 \times 2.7183^{0.015t} + 0.3076 \times 2.7183^{0.059t} + 0.269/2.718^{0.03t}) \times (\cdots + 0.218 \times 2.7183^{0.015t} + 0.217 \times 2.7183^{0.059t} + 0.076/2.7183^{0.03t})))]].$

To evaluate the effectiveness of the proposed control strategy, Table 1 delineates the discrepancies between the aggregated outputs of the advanced manufacturing industry, modern service industry, and other sectors, as influenced by the control variables, in comparison to the actual GDP of the United States for the year 2018.

Figure 6 reveals significant amplitude of fluctuations in the trajectory of key economic indicators under the proposed control scheme. The control mechanism we have constructed here is merely the initial step towards achieving the ultimate objective. In future work, we will consider a discrete form of control functions in order to smooth out fluctuations in the program mode.

Table 1. The error rate between the control total output and the actual total output

Year	2018 actual total output	2018 controlled total output	Error rate
AMI	4164850	4165722.87	0.02%
MSI	12656899.5	12667558.54	0.08%
Others	19074639.5	19085804.38	0.05%
GDP	20611103.4	20620479.56	0.04%

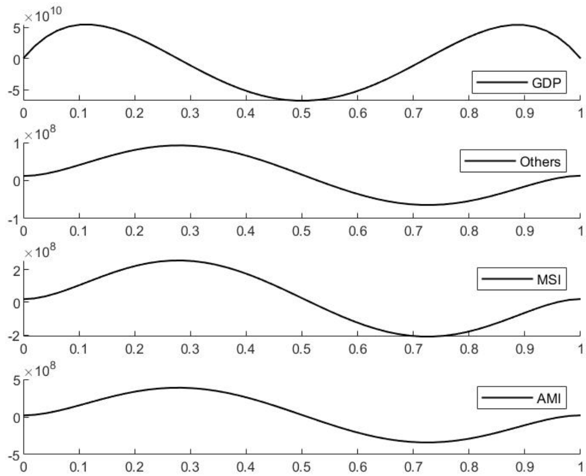

Fig. 6. The control process of total output and GDP among 3 sectors in 2017-2018

4. Input-output Model of Saddle Point Equilibrium Strategy

Drawing upon the Leontief input-output relational framework, we can delineate a general dynamic input-output model. The input-output processes of individual sectors are conceptualized as dynamic economic activities, which determine that the output levels of each sector, as well as the final consumption products, are functions of time, denoted by t. This temporal dimension introduces a dynamic perspective to the traditionally static input-output analysis, allowing for a more nuanced understanding of the economic interdependencies and their evolution over time.

$$X(t) = AX(t) + B\dot{X}(t) + Y(t), \tag{20}$$

where, $t \in [t_*, t^*]$(product planning period), n dimension vectors $X(t)$ and $Y(t)$ respectively is the output vector and the final consumption product vector (excluding the investment part). The n time-invariant matrix A is the direct consumption coefficient matrix, whose elements satisfy $\sum a_{ij} < 1$ and $0 \leqslant a_{ij} < 1, i, j = 1, 2, \cdots, n$. $\dot{X}(t)$ is the rate of change in investments in each sector. The n time-invariant matrix B is the investment coefficient matrix.

Building on the foundational work of Mao, 1992, our study takes a holistic approach to addressing the stochastic uncertainties inherent in the national economy. We delve into the formulation of a continuous dynamic input-output model that operates under the most adverse conditions of ambiguity. By incorporating concepts from game theory, we frame the model as a linear quadratic differential game, where the control strategy of the first player, denoted by $\dot{X}(t)$, is juxtaposed against the uncertain factor-represented by the stochastic variable $z(t)$ — which serves as the control strategy for the second player.

Our analysis is geared towards identifying the optimal control strategy for the dynamic input-output system. This is achieved by rigorously deriving the saddle point equilibrium strategy, which encapsulates the most favorable course of action for both players under the given conditions of uncertainty. The resolution of this equilibrium strategy provides valuable insights into the optimization of the dynamic input-output system in the face of stochastic disturbances.

4.1. The game model of dynamic input-output system

Owing to the intricate nature of real economic activities, formula (20) is insufficient to accurately capture genuine economic laws. Consequently, we propose an amendment to the model by introducing a random factor, denoted as variable $z(t)$, to account for uncertain factors in social and economic activities. This results in the formulation of a new model represented by formula (21).

$$X(t) = AX(t) + B\dot{X}(t) + Y(t) + z(t). \tag{21}$$

The investment growth rates $\dot{X}(t)$ of each sector determine the changes in their output capabilities, and by adjusting them, the final consumption product vector $Y(t)$ can be controlled. Therefore, the continuous dynamic input-output model (21) can be transformed into the following state-space representation.

$$\begin{cases} \dot{X}(t) = u(t) \\ Y(t) = (I - A)X(t) - Bu(t) - z(t) \end{cases} \tag{22}$$

Among them, $t \in [t_*, t^*]$. I is the identity matrix of order n. $u(t) \in R^n$ is the decision-making control variable of player 1, $z(t) \in R^n$ is the "natural" decision-making control variable of player 2, output level vector $X(t)$ is the state variable of the system, and final consumption product vector $Y(t)$ is the output variable of the system.

We use the n dimensional column vector $G(t)$ to represent the social demand vector for products, assuming it is a known continuous function vector. When the national economy is in a dynamic equilibrium, the social demand vector $G(t)$ is equal to the system output vector $Y(t)$. However, it is difficult to eliminate the supply-demand imbalance. At this time, on the one hand, we hope to make the performance index $J(u, z)$ obtain a minimum value by adjusting the control variable $u(t)$. On the other hand, it is hoped that under the worst interference of the random factor $z(t)$, the performance index $J(u, z)$ takes the maximum value, where,

$$J(u, z) = \frac{1}{2} \int_{t_*}^{t^*} [Y(t) - G(t)]^{\mathrm{T}} P [Y(t) - G(t)] + u^{\mathrm{T}}(t) R u(t) dt, \tag{23}$$

where, P and R are positive definite matrices of order n respectively.

Their practical significance is to distinguish between the difference between the number of final consumer products provided by each sector and the social demand, and the difference in the primary and secondary degrees required by the changes in the output capacity of each sector. At this point, a complete dynamic input-output system game model is formed. In the following chapters, we will use the method of solving the saddle point equilibrium strategy to solve this problem.

4.2. Optimal control construction

The above model is a normal linear quadratic differential game model. Next, we use the result of Deissenberg to solve it, and construct the Hamilton function of the system (23) as follows:

$$H(X, u, Z, \theta, t) = \frac{1}{2} [Y(t) - G(t)]^{\mathrm{T}} P [Y(t) - G(t)] + \frac{1}{2} u^{\mathrm{T}} R u + \theta^{\mathrm{T}} u. \tag{24}$$

Then,

$$\frac{\partial H}{\partial u} = \left(R + B^{\mathrm{T}}PB\right)u - B^{\mathrm{T}}P(I-A)X + \lambda + B^{\mathrm{T}}PG(t) + B^{\mathrm{T}}Pz,$$
$$\frac{\partial H}{\partial z} = Pz - P(I-A)X + PG(t) + PBu,$$
$$\frac{\partial H}{\partial X} = (I-A)^{\mathrm{T}}P(I-A)x - (I-A)^{\mathrm{T}}PBu - (I-A)^{\mathrm{T}}Pz - (I-A)^{\mathrm{T}}PG(t).$$

According to the minimum principle, the optimal control of the system (23) $u^*(t)$, $z^*(t)$ and the optimal trajectory $X^*(t)$ and the corresponding co-state variable $\theta^*(t)$ satisfy:

$$\begin{aligned}
&\dot{X} = u, X(t_0) = X_0,\\
&\dot{\theta} = -(I-A)^{\mathrm{T}}P(I-A)X + (I-A)^{\mathrm{T}}PBu + (I-A)^{\mathrm{T}}Pz + (I-A)^{\mathrm{T}}PG(t),\\
&\theta(t^*) = 0,\\
&u = -R^{-1}\theta,\\
&z = (I-A)X - BR^{-1}\theta - G(t).
\end{aligned} \tag{25}$$

Let:

$$\theta(t) = S(t)X(t) + v(t). \tag{26}$$

Then after proper calculation, $S(t)$ and $v(t)$ respectively satisfy the following matrix Riccati differential equation:

$$\begin{aligned}
&\dot{S}(t) + (I-A)^{\mathrm{T}}PBR^{-1}S(t) + S(t)R^{-1}B^{\mathrm{T}}P^{\mathrm{T}}(I-A) - S(t)R^{-1}S(t) = 0,\\
&S(t^*) = 0.\\
&\dot{v}(t) + (I-A)^{\mathrm{T}}PBR^{-1}r(t) + v(t)R^{-1}B^{\mathrm{T}}P^{\mathrm{T}}(I-A) - S(t)R^{-1}v(t) = 0,\\
&v(t^*) = 0.
\end{aligned} \tag{27}$$

Substituting equation (26) into equation (25), the optimal control law of the system is:

$$\begin{aligned}
u^*(t) &= -R^{-1}S(t)X(t) - R^{-1}v(t),\\
z^*(t) &= \left[I - BR^{-1}S(t)\right]X(t) - BR^{-1}v(t) - G(t),
\end{aligned} \tag{28}$$

where, $S(t)$ and $v(t)$ are uniquely determined by equation (27).

Hence, this section uses the saddle point equilibrium theory in the differential game to study the multi-sector dynamic input-output problem in macroeconomic decision-making and designs the realization method of the control strategy.

5. Conclusion

The present study commences with the formulation of a dynamic input-output model, building upon the framework established by the OECD. Using the United States as a case study, the model, in conjunction with the least squares method, is employed to forecast capital intensity amidst inflationary conditions. For the U.S.

input-output tables, time series analysis is utilized to project the trajectory of the maximum aggregate total output as well as the GDP for the period spanning 2019 to 2024. Our findings indicate that the contribution of modern services to the U.S. GDP surpasses that of advanced manufacturing. In light of this, we recommend that the United States should relax trade restrictions and proactively engage in international economic and technological collaborations to foster robust economic growth.

Furthermore, the study integrates classical program control theory with a linear non chi-square differential equation that encompasses advanced manufacturing, modern services, other sectors, and GDP. By applying control to the aggregated advanced manufacturing sector in the United States for the years 2017 to 2018, we derive equations that account for the control of multiple influencing factors and validate the effectiveness of program control through numerical analysis.

As a final point, we explore the optimal strategy design for the continuous variant based on the dynamic input-output model proposed by Leontief. The dynamic input-output system is conceptualized as a saddle-point equilibrium game model, and the saddle-point equilibrium strategy is adopted to devise an innovative approach for resolving dynamic input-output challenges.

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