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Symmetric Nash Equilibrium Arrivals to Queuing System*

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Abstract We consider a game-theoretic setting for the queuing system models where input process of arrivals is strategic. This paper generalizes a methodology for the symmetric Nash equilibrium exploring in queuing system with loss. We assume that the system admits customer requests at the time interval $[0, T]$. Each of customers chooses the moment to send his request into the system maximizing his payoff. Several models of certain systems are presented as examples demonstrating a result of the methodology application.

Keywords: queueing system, strategic customers, optimal arrivals, Kolmogorov backward equations, Nash equilibrium.

1. Introduction

Usually in queuing theory models the input process is determined by given rate of requests. But game-theoretic approach assumes that customers are strategic and form the input process choosing the arrival time moments to maximize their payoff. The paper (Glazer and Hassin, 1983) is the first work where selfish customers choose when to enter the system trying to minimize their waiting time. A single-server model with loss is presented in (Ravner and Haviv, 2014) and a model with several servers and a limited buffer is explored in (Ravner and Haviv, 2015). The common property in such games is the mixed symmetric Nash equilibrium strategy which is a probability distribution of arrival time moments on a working interval of the system. Further researches develop this area adding variations into the system and payoff functions of players. In (Mazalov and Chuiko, 2006) we explore a single-server queuing system without buffer where the players payoff includes a time-sensitivity convenience function. A model where the players take into account not only waiting, but also tardiness costs is considered in (Jain, et al., 2011). In the paper (Haviv, 2013) authors combine the tardiness costs, waiting costs, and restrictions on the opening and closing times. The paper (Breinbjerg et al., 2022) presents a queueing system with the last-come first-served discipline and preemptive-resume.

In the second section we present a generalized game-theoretic model of a queue system with loss. Section 3 describes a common methodology how to explore a symmetric Nash equilibrium determining its structure. Remaining sections contain examples of certain models demonstrating a result of the methodology application.

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2. Generalized Model

We describe a model in a general setting. We consider a system of one or more servers which admits customer requests at a working interval $[0, T]$. It serves a set U of $N+1$ customers which are ready to enter the system from the initial time moment $t = 0$. The system has no queues. At each moment the system can simultaneously perform only limited number of requests $k = 0, 1, ...$, depending on model setting and system state. It may happen that $n > k$ customers arrive to the system at the same time. Then the system chooses with equal probability k of the currently arrived requests for further servicing and remaining $n-k$ customers are lost. Service discipline, rates are determined by model settings.

The request discipline is not defined in the system. It is actually formed by the customers. They use some strategy to choose an instant on a time interval $[0, T]$ to enter the system maximizing some given payoff. The payoff can be a probability to obtain service in the system or some reward depending on service time and/or waiting costs.

The pure strategy of player i is the arrival time t_i of his request in the system. The mixed strategy of player i is the distribution function $F_i(t)$ (having density $f_i(t)$) of the arrival times in the system on the time interval $[0, T]$. Let $F = \{F_i(t), i \in U\}$ be the strategy profile.

All players are identical, independent and demonstrate the selfish behavior without cooperation. As the optimality criterion we choose the symmetric Nash equilibrium. In this case, the strategies of all players coincide, i.e., $F_i(t) = F(t)$ for all i.

The payoff for each player is his expected reward which he obtains sending and performing his request in the system.

Definition 1. A distribution function $F(t)$ of the arrival times t in the system is a symmetric Nash equilibrium if there exists a constant C such that at any time $t \in [0, T]$ the payoff does not exceed C and is equal to C on the support of $F(t)$.

3. Equilibrium Structure

Analysing the structure of the equilibrium we consider any of customers (e.g. first, without loss of generality) who plays against N opponents using the same strategies. The equilibrium distribution of time instants to arrive into the system can possess the following properties which need to be explored.

The equilibrium distribution density of arrivals is not necessarily strictly positive over the entire interval $[0, T]$. Moreover the distribution can have atoms, i.e. special isolated time instants when there is a positive probability to arrive. E.g. the initial or the last time of the working interval can become atoms. It's necessary to check each time instant which is a candidate to be an atom. The technique is as follows. We consider a certain time instant t in $[0, T]$ and suppose that p is an equilibrium probability for each on N opponents to arrive at this moment. If $p = 0$, nobody of opponents arrive at t, providing the maximum possible payoff for the first customer, arriving at t, then $p = 0$ cannot be a symmetric Nash equilibrium, so $p > 0$. To set that the moment t is isolated we compare the payoff value for the first customer arriving at this moment when opponents arrive with some positive probability p, and his payoff at the moment immediately after or/and before t. If it is better to arrive at t than at time immediately after or/and before t, then there is some neighborhood of t without arrivals. We look for its bounds based on the fact that the equilibrium payoffs at the moment t and at some moment t_e are the same, assuming that there are no arrivals between t and t_e.

Excluding atoms and intervals without arrivals we obtain some area with the strictly positive distribution density of arrival time moments. At each such moment the payoff depends on a state of the system, including the information on count of requests arriving into the system to this moment and about current state of each server in the system (busy or idle). The backward Kolmogorov system is used to determine state probabilities. Note that the process is inhomogeneous in time, since the request rate in the system decreases in jumps as soon as a new request is received from a successive player. In particular, the request rate has the form $\lambda_k(t) = (N-k)\frac{f(t)}{1-F(t)}$ where k is the count of request arriving into the system to the moment t. The payoff expressed through state probabilities must be constant on the area with positive arrival time distribution density, so, its derivative by time must be zero. The corresponding Cauchy problem together with normalization condition $F(T) = 1$ allows to compute the density.

Further sections demonstrate resulting equilibria found for examples of certain systems.

4. A Preemptive System

Consider the following queueing system with one server and preemptive access (Chirkova and Mazalov, 2022). When some customer arrives then he captures server for the exponentially distributed time with parameter μ. But when the next customer arrives in the system the current one must leave the system even it has not finished its service yet. The input process is formed by the strategic customers who try to maximize the probability to be served in the system completely.

We established in (Chirkova and Mazalov, 2022) the following qualitative properties of symmetric Nash equilibria:

- the system possesses a unique symmetric Nash equilibrium;
- at the last time T, there is a strictly positive probability of request arrival in the system;
- on a time interval (t_e, T), no requests arrive in the system;
- on the time interval $[0, t_e]$, there exists a strictly positive density function of request arrivals in the system.

The structure of the Nash equilibrium is determined by the following theorem.

Theorem 1. *The symmetric Nash equilibrium in the $N+1$-person queueing game with preemptive access is described by the distribution function $F(t)$ on the interval $[0, T]$, which has the following properties.*

1. *There is a non-zero probability p of a request entering the system at the instant T.*
2. *At the interval $[t_e, T)$, where*

$$t_e = T - \frac{1}{\mu}\log\frac{p(N+1)(1-(1-p)^N)}{p(N+1)-1+(1-p)^{N+1}},$$

the players do not enter the service system.

3. *If $t_e < 0$ for $p = 1$, then in equilibrium all players send their requests to the system at instant T. Otherwise, $p < 1$, and t_e is greater than 0; in addition, the PDF $f(t)$ on the support $[0, t_e]$ is determined from equations*
$$(F(t) + p)^N (1 - e^{-\mu(T-t)}) +$$
$$N \int_t^{t_e} (1 - e^{-\mu(s-t)})(F(t) + 1 - F(s))^{N-1} dF(s) + F(t)^N e^{-\mu(T-t)} =$$
$$1 - (1 - (1-p)^N) e^{-\mu(T-t_e)}.$$
4. *The probability p of a request entering the system at the instant T is determined from equation $\int_0^{t_e} dF(t) + p = 1$.*
5. *In equilibrium, the probability that a player receives service is equal to $C(T) = \frac{1-(1-p)^{N+1}}{p(N+1)}$.*

5. A Preemptive System with Fixed Reward for Completing Request

Consider the system which is similar to preemptive system presented in the previous section. Each player chooses the time to send his request to the system, providing its complete service or a maximum service time for his requests. The reward for the player is a fixed value $\alpha > 0$ in case of successful completion of his request service. Otherwise he obtains 1 for each unit of a service time which his request get. We don't lose generality because both values can be divided by profit from each service time unit. This value α can be considered to be proportional to a ticket price in a transport interpretation.

In (Chirkova, 2023) we obtain the following.

Theorem 2. *The symmetric Nash equilibrium in the $N+1$-person queueing game with preemptive access and fixed reward for completing request is described by the distribution function $F(t)$ on the interval $[0, T]$, which has the following properties.*

1. *There is a strictly positive probability p for a request to arrive into the system at the instant T.*
2. *There is the interval $[t_e, T)$, where t_e is determined by*
$$\frac{1-(1-p)^{N+1}}{p(N+1)} = 1 - (1-(1-p)^N)(1 - \frac{1}{\alpha}(T - t_e)) e^{-\mu(T-t_e)}, \quad (1)$$
when the requests do not enter the service system.
3. *If the solution of equation (1) is negative for $p = 1$, then requests arrive into the system at instant T. Otherwise, $p < 1$, and t_e is greater than 0; in addition, the density function $f(t)$ on the support $[0, t_e]$ is determined from equation*
$$(F(t) + p)^N (\alpha - e^{-\mu(T-t)}(\alpha - (T-t))) + F(t)^N e^{-\mu(T-t)}(\alpha - (T-t)) +$$
$$N \int_t^{t_e} (\alpha - e^{-\mu(s-t)}(\alpha - (s-t)))(F(t) + 1 - F(s))^{N-1} dF(s) =$$
$$\alpha - (1 - (1-p)^N)(\alpha - (T - t_e)) e^{-\mu(T-t_e)}.$$
4. *The probability p to arrive into the system at the instant T is determined from equation*
$$\int_0^{t_e} dF(t) + p = 1.$$

5. *In equilibrium, the expected reward which a player receives sending and performing his request in the system is equal to* $S(T) = \alpha \frac{1-(1-p)^{N+1}}{\mu p(N+1)}$.

6. A Random-Access Two-Server System

Consider a two-server system. Each arriving request is redirected by a manager to one of two servers, each of which may simultaneously perform only one request. It may happen that several customers send their requests to the system at the same time. Then the system may redirect two or more requests to the same server. If the server is currently busy, then all requests arriving in it at this time are lost. Otherwise, the server chooses one of the currently arrived requests for further service by a uniform random draw. The service times of the requests are independent random variables obeying the exponential distribution with rates μ_1 and μ_2 for the first and second servers, respectively. Customers choose arrival time moments to maximize the probability of service for their requests.

We analyse two request redistribution models. In the first model the system has no information about the server states, and each request can be redirected to a busy server even given a free server. In the second model, the access to servers is random only if both are free.

For both models we established the following qualitative properties of symmetric Nash equilibria:

- the system possesses a unique symmetric Nash equilibrium;
- at the zero time, there is a strictly positive probability of request arrival in the system;
- on a time interval $(0, t_e)$, no requests arrive in the system;
- on the time interval $[t_e, T]$, there exists a strictly positive density function of request arrivals in the system.

6.1. A Pure Random Access

This system is investigated in (Chirkova, 2017) in details. A user request that arrives into the system is redirected with probability r to the first server and probability $\overline{r} = 1 - r$ to the second.

Theorem 3. *Any symmetric Nash equilibrium distribution of the arrival times in the two-server pure random-access system with loss described by the distribution function $F(t)$ on the interval $[0, T]$ has the following properties.*

1. *There exists a strictly positive probability $p_e = F(0) > 0$ of request arrival in the system at the zero time.*
2. *On the interval $(0, t_e)$, where t_e satisfies the equation*

$$\frac{2 - (1 - p_e r)^{N+1} - (1 - p_e \overline{r})^{N+1}}{p_e(N + 1)} = 1 - r\left(1 - (1 - p_e r)^N\right) e^{-\mu_1 t_e} - \overline{r}\left(1 - (1 - p_e \overline{r})^N\right) e^{-\mu_2 t_e}. \quad (2)$$

the players send their requests to the system with zero probability.

3. *If for $p_e = 1$, the equation (2) gives $t_e > T$, then the equilibrium strategy is the pure strategy in which all players send their requests to the system at the zero time.*

4. Otherwise, if $p_e < 1$, then on the interval $[t_e, T]$, there exists a continuous positive density function $f(t)$ of the arrival times in the system that is defined by the Cauchy problem

$$\frac{f(t)}{1-F(t)} = \frac{r\mu_1\left(\sum\limits_{i=1}^{N} p_{10i}(t) + \sum\limits_{i=2}^{N} p_{11i}(t)\right) + \overline{r}\mu_2\left(\sum\limits_{i=1}^{N} p_{01i}(t) + \sum\limits_{i=2}^{N} p_{11i}(t)\right)}{\sum\limits_{i=0}^{N-1}(N-i)(r^2(p_{00i}(t)+p_{01i}(t)) + \overline{r}^2(p_{00i}(t)+p_{10i}(t))}$$

where

$$\begin{aligned}
&p'_{000}(t) = -\lambda_0(t)p_{000}(t),\\
&p'_{101}(t) = r\lambda_0(t)p_{000}(t) - (\lambda_1(t)+\mu_1)p_{101}(t),\\
&p'_{011}(t) = \overline{r}\lambda_0(t)p_{000}(t) - (\lambda_1(t)+\mu_2)p_{011}(t),\\
&p'_{00i}(t) = -\lambda_i(t)p_{00i}(t) + \mu_1 p_{10i}(t) + \mu_2 p_{01i}(t),\\
&p'_{10i}(t) = r\lambda_{i-1}(t)(p_{00i-1}(t)+p_{10i-1}(t)) - (\lambda_i(t)+\mu_1)p_{10i}(t) + \mu_2 p_{11i}(t),\\
&p'_{01i}(t) = \overline{r}\lambda_{i-1}(t)(p_{00i-1}(t)+p_{01i-1}(t)) - (\lambda_i(t)+\mu_2)p_{01i}(t) + \mu_1 p_{11i}(t),\\
&p'_{11i}(t) = \lambda_{i-1}(t)(rp_{01i-1}(t) + \overline{r}p_{10i-1}(t) + p_{11i-1}(t)) - (\mu_1+\mu_2+\lambda_i(t))p_{11i}(t),\\
&\qquad i = 2,\dots,N.
\end{aligned}$$

with initial conditions

$$\begin{aligned}
&p_{00i}(t_e) = C_N^i p_e^i(1-p_e)^{N-i}\left[\mathbb{1}_{i=0} + \mathbb{1}_{i>0}\left(r^i(1-e^{-\mu_1 t_e}) + \overline{r}^i(1-e^{-\mu_2 t_e})\right)\right.\\
&\qquad \left. + \mathbb{1}_{i>1}(1-r^i-\overline{r}^i)(1-e^{-\mu_1 t_e})(1-e^{-\mu_2 t_e})\right],\\
&\qquad i = 0,\dots,N,\\
&p_{10i}(t_e) = C_N^i p_e^i(1-p_e)^{N-i}\left[r^i e^{-\mu_1 t_e} + \mathbb{1}_{i>1}(1-r^i-\overline{r}^i)e^{-\mu_1 t_e}(1-e^{-\mu_2 t_e})\right],\\
&\qquad i = 1,\dots,N,\\
&p_{01i}(t_e) = C_N^i p_e^i(1-p_e)^{N-i}\left[\overline{r}^i e^{-\mu_2 t_e} + \mathbb{1}_{i>1}(1-r^i-\overline{r}^i)(1-e^{-\mu_1 t_e})e^{-\mu_2 t_e}\right],\\
&\qquad i = 1,\dots,N,\\
&p_{11i}(t_e) = C_N^i p_e^i(1-p_e)^{N-i}(1-r^i-\overline{r}^i)e^{-\mu_1 t_e}e^{-\mu_2 t_e},\\
&\qquad i = 2,\dots,N.
\end{aligned}$$

and normalization condition

$$p_e + \int\limits_{t_e}^{T} f(t)dt = 1.$$

5. The value $C(p_e) = \frac{2-(1-p_e r)^{N+1} - (1-p_e\overline{r})^{N+1}}{p_e(N+1)}$ gives the probability of service on the whole strategy carrier.

6.2. A Rational Random Access

The system with rational random access is presented in (Chirkova, 2020). It operates in the following way. If both servers are currently free, then a user request that arrives in the system is redirected with probability r to the first server and probability $\overline{r} = 1 - r$ to the second. If only one server is free, then the request is redirected to it. If both servers are busy, then the request is rejected.

Theorem 4. *Any symmetric Nash equilibrium distribution of the arrival times in the two-server rational random-access system with loss described by the distribution function $F(t)$ on the interval $[0,T]$ has the following properties.*

1. *There exists a strictly positive probability $p_e = F(0) > 0$ of request arrival in the system at the zero time.*
2. *On the interval $(0, t_e)$, where*

$$t_e = \left(\ln \frac{1-(1-p_e)^N - Np_e(1-p_e)^{N-1}}{1 - \frac{2(1-(1-p_e)^{N+1})}{p_e(N+1)} + (1-p_e)^N} \right) / (\mu_1 + \mu_2), \tag{3}$$

the players send their requests to the system with zero probability.

3. *If for $p_e = 1$, the expression (2) gives $t_e > T$, then the equilibrium strategy is the pure strategy in which all players send their requests to the system at the zero time.*
4. *Otherwise, if $p_e < 1$, then on the interval $[t_e, T]$, there exists a continuous positive density function $f(t)$ of the arrival times in the system that is defined by the Cauchy problem*

$$\frac{f(t)}{1-F(t)} = \frac{(\mu_1+\mu_2)(1-C(p_e))}{\sum\limits_{i=0}^{N-1}(N-i)(p_{01i}(t)+p_{10i}(t))},$$

where

$$\begin{aligned}
&p'_{000}(t) = -\lambda_0(t)p_{000}(t),\\
&p'_{101}(t) = r\lambda_0(t)p_{000}(t) - (\lambda_1(t)+\mu_1)p_{101}(t),\\
&p'_{011}(t) = \overline{r}\lambda_0(t)p_{000}(t) - (\lambda_1(t)+\mu_2)p_{011}(t),\\
&p'_{00i}(t) = -\lambda_i(t)p_{00i}(t) + \mu_1 p_{10i}(t) + \mu_2 p_{01i}(t),\\
&p'_{10i}(t) = r\lambda_{i-1}(t)p_{00i-1}(t) - (\lambda_i(t)+\mu_1)p_{10i}(t) + \mu_2 p_{11i}(t),\\
&p'_{01i}(t) = \overline{r}\lambda_{i-1}(t)p_{00i-1}(t) - (\lambda_i(t)+\mu_2)p_{01i}(t) + \mu_1 p_{11i}(t),\\
&p'_{11i}(t) = \lambda_{i-1}(t)(p_{01i-1}(t)+p_{10i-1}(t)+p_{11i-1}(t)) - (\lambda_i(t)+\mu_1+\mu_2)p_{11i}(t),\\
&\qquad i = 2,\dots,N.
\end{aligned}$$

with initial conditions

$$\begin{aligned}
&p_{00i}(t_e) = C_N^i p_e^i (1-p_e)^{N-i}\left[\mathbb{1}_{i=0} + \mathbb{1}_{i=1}\left(r(1-e^{-\mu_1 t_e}) + \overline{r}(1-e^{-\mu_2 t_e})\right)\right.\\
&\qquad \left. + \mathbb{1}_{i>1}(1-e^{-\mu_1 t_e})(1-e^{-\mu_2 t_e})\right],\\
&\qquad i = 0,\dots,N;\\
&p_{10i}(t_e) = C_N^i p_e^i (1-p_e)^{N-i}\left[\mathbb{1}_{i=1} r e^{-\mu_1 t_e} + \mathbb{1}_{i>1} e^{-\mu_1 t_e}(1-e^{-\mu_2 t_e})\right],\\
&\qquad i = 1,\dots,N;\\
&p_{01i}(t_e) = C_N^i p_e^i (1-p_e)^{N-i}\left[\mathbb{1}_{i=1} \overline{r} e^{-\mu_2 t_e} + \mathbb{1}_{i>1}(1-e^{-\mu_1 t_e}) e^{-\mu_2 t_e}\right],\\
&\qquad i = 1,\dots,N;\\
&p_{11i}(t_e) = C_N^i p_e^i (1-p_e)^{N-i} \mathbb{1}_{i>1} e^{-\mu_1 t_e} e^{-\mu_2 t_e},\\
&\qquad i = 2,\dots,N.
\end{aligned}$$

and normalization condition

$$p_e + \int\limits_{t_e}^{T} f(t)dt = 1.$$

5. *The value* $C(p_e) = 2\frac{1-(1-p_e)^{N+1}}{p_e(N+1)} - (1-p_e)^N$ *gives the probability of service on the whole strategy carrier.*

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