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Optimal control of the Navier — Stokes system with a space variable in a network-like domain

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The research of the problem of optimal control of the Navier — Stokes evolutionary differential system, considered in Sobolev spaces, the elements of which are functions with carriers in an n-dimensional network-like domain, is presented. Such domain consists of a finite number of subdomains, mutually adjacent to certain parts of the surfaces of their boundaries according to the graph type. For functions that are elements of these spaces, the conditions for the existence of traces on the surfaces of the joining are presented and the conditions of adjacency subdomains to which these functions satisfy are described. In applied questions of the analysis of the processes of transport of continuous media, the conditions of adjacency describe the regularities of the flow of fluid through the boundaries of the adjacent domains. The paper presents the results of following main research questions: 1) weak solvability of the initial boundary value problem for the Navier — Stokes system and obtaining the conditions for the existence of a weak solution to this problem; 2) the formation and solution of optimal control problems of various types of Navier — Stokes system. The fundamental approach to the analysis of the weak solvability of the initial boundary value problem is its reduction to the differential-difference problem (semi-digitization of the original system by a time variable) and subsequent use of a priori estimates for weak solutions of the obtained boundary value problems. The obtained a priori estimates are used to prove the theorem of the existence of a weak solution of the original differential system and indicate the way of the actual construction of this solution. A universal approach to solving the problems of optimal distributed and starting control of the Navier — Stokes evolutionary system is presented. The latter essentially expands the possibilities of analyzing non-stationary network-like processes of applied hydrodynamics (for example, processes of transporting various types of liquids through network or main line pipelines) and optimal control of these processes.

Keywords: differential-difference system, evolutionary Navier — Stokes system, network-like domain, solvability, optimal control.

1. Introduction. The paper be considered the question of the existence of a weak solution and the associated problems of optimal control of the Navier — Stokes evolutionary system, the spatial variable of which belongs to a network-like domain. The structure of such a domain is similar to the geometry of a connected graph (see [1–4] and the bibliography there). The work is a continuation of the studies presented in [5, 6], a essential difference from which was the use of a priori estimates of the differential-difference system to analyze the weak solvability of the Navier — Stokes differential system using a priori estimates. Namely, according to the obtained weak solutions of the differential-difference system, piecewise-constant approximations on the time variable are constructed, which form a weakly compact sequence of approximations of the solutions of the Navier — Stokes system. The obtained results are the basis for the analysis of the problems of optimal distributed and starting control of the Navier — Stokes evolutionary system, which have interesting analogies with the applied problems of optimization of multiphase hydrodynamic flows and composite polimers [7, 8].

2. Notations and concepts. Consider a bounded network-like domain $\Im \in \mathbb{R}^n$ ($n \geqslant 2$), consisting of N subdomains $\Im_l$ ($l \in I_N = \{1, 2, ..., N\}$), united in a certain way to each other by means of M, $1 \leqslant M \leqslant N-1$, nodal places ω_j ($j \in I_M = \{1, 2, ..., M\}$): $\Im = \hat{\Im} \bigcup \hat{\omega}$, here $\hat{\Im} = \bigcup\limits_{l=1}^{N} \Im_l$, $\hat{\omega} = \bigcup\limits_{j=1}^{M} \omega_j$, where $\Im_l \bigcap \Im_{l'} = \emptyset$ ($l \neq l'$), $\omega_j \bigcap \omega_{j'} = \emptyset$ ($j \neq j'$), $\Im_l \bigcap \omega_j = \emptyset$ [5]. In these nodal places $\Im_l$ have common boundaries, which are the surfaces of the joining. If the index $j \in I_M$ is fixed, then the nodal place is a totality of subdomains. Such a set consists of subdomain $\Im_{l_j}$ and subdomains $\Im_{l'_\iota}$, $l'_\iota \in I_M(j) \subset I_M$, $\iota = \overline{1, m_j}$, from where follows the existence of the surface of the adjoining S_j ($meas\, S_j > 0$) of subdomain $\Im_{l_j}$ to the subdomains its respective subsurfaces $S_{j,\iota}$ ($meas\, S_{j\,\iota} > 0$), $\iota = \overline{1, m_j} : S_j = \bigcup\limits_{\iota=1}^{m_j} S_{j\,\iota}$. In this case, S_j is part of the boundary $\partial\Im_{l_j}$, and $S_{j,\iota}$ ($\iota = \overline{1, m_j}$) are the corresponding parts of the boundaries $\partial\Im_{l'_\iota}$. The latter means that ω_j it is defined by the adjacency surface S_j, and the subsurfaces $S_{j\,\iota}$ are the surfaces of the joining to $\Im_{l'_\iota}$, $l'_\iota \in I_M(j)$, $\iota = \overline{1, m_j}$. The boundary $\partial\Im$ of a domain $\Im$ is defined by the ratio $\partial\Im = \bigcup\limits_{l=1}^{N} \partial\Im_l \setminus \bigcup\limits_{j=1}^{M} S_j$. It is assumed that the surfaces S_j ($j \in I_M$) are smooth, subregions $\Im_l$ ($l \in I_N$) are star-shaped relative to the ball fixed at each l ($l \in I_N$).

It should be noted that the structure of the network-like domain $\Im$ is similar to the structure of the graph-tree [1, 2], we also note that any connected subdomain of the domain $\Im$ also has a network-like structure.

Next, the initial boundary value problem for the evolutionary transfer equation is considered, which is a mathematical model of the transportation of a viscous fluid through pipeline networks.

3. Navier — Stokes evolutionary system. For functions $Y(x,t) = \{y_1(x,t), y_2(x,t), ..., y_n(x,t)\}$, $x, t \in \Im_T = \Im \times (0, T)$ ($x = \{x_1, x_2, ..., x_n\}$, $T < \infty$), consider the system

$$\frac{\partial Y}{\partial t} - \nu \Delta Y + \sum_{i=1}^{n} Y_i \frac{\partial Y}{\partial x_i} + \operatorname{grad} p = f, \tag{1}$$

$$\operatorname{div} Y = 0 \left(\sum_{i=1}^{n} \frac{\partial Y}{\partial x_i} = 0 \right), \tag{2}$$

however, for $Y(x,t)$ in the nodal places ω_j ($j \in I_M$) there are conditions (conditions of adjacency the subdomain $\Im_{l_j}$ to $\Im_{l'_\iota}$, $l'_\iota \in I_M$, $\iota = \overline{1, m_j}$)

$$Y(x,t)|_{x \in S_{j\,\iota} \subset \partial\Im_{l_j}} = Y(x,t)|_{x \in S_{j\,\iota} \subset \partial\Im_{l'_\iota}}, \quad \iota = \overline{1, m_j}, \tag{3}$$

$$\int\limits_{S_j} \frac{\partial Y(x,t)}{\partial n_j} ds + \sum_{\iota=1}^{m_j} \int\limits_{S_{j\,\iota}} \frac{\partial Y(x,t)}{\partial n_{j\,\iota}} ds = 0, \tag{4}$$

on surfaces S_j, $S_{j\iota}$ $(\iota = \overline{1, m_j})$ at $t \in (0, T)$, where n_j and $n_{j\iota}$ are the external normals to S_j and $S_{j\iota}$, respectively, $\iota = \overline{1, m_j}$, $j = \overline{1, M}$. The relationships

$$Y(x,t)|_{t=0} = Y_0(x),\ x \in \Im, \tag{5}$$

$$Y(x,t)|_{x\in\partial\Im} = 0, \tag{6}$$

describe the initial and boundary conditions. The set of relations (1)–(6) are the initial marginal boundary value problem (differential system (1)–(6)) for functions $Y(x,t)$, $p(x,t)$ in a closed domain $\overline{\Im}_T$ $(\overline{\Im}_T = (\Im \bigcup \partial\Im) \times [0, T])$.

In the mathematical description of the processes of transportation of viscous fluids, $\Im$ it belongs to $\mathbb{R}^3$ and models the network (or main) hydraulic system, which is the carrier of the hydraulic flow. The function $Y(x,t)$ describes the quantitative characteristics of the velocities of the hydraulic flow, $\sum\limits_{i=1}^{n} Y_i \frac{\partial Y}{\partial x_i}$ are the convective change of the velocity vector. The ratios (1), (2) and (3), (4) define the Navier—Stokes system, which simulates the flow of a viscous fluid (viscosity is equal ν) through the hydraulic system and forms the law of fluid flow at the branch places of the hydraulic system, (5) and (6) are initial and boundary conditions, respectively, $p(x,t)$ is pressure in the hydraulic system.

Remark 1. You can use other conditions of adjacency (3), (4) depending on the goals pursued of an applied nature. It is necessary that the requirement of solvability of the problem (1)–(6) be satisfied (see work [5]).

To obtain the conditions of solvability of the differential system (1)–(6), a differential-difference system of the form

$$\begin{gathered} \tfrac{1}{\tau}[Y(k) - Y(k-1)] - \nu\Delta Y(k) + \sum_{i=1}^{n} Y_i(k)\tfrac{\partial Y(k)}{\partial x_i} = f_\tau(k) - \operatorname{grad} p(k), \\ \operatorname{div} Y(k) = 0,\ \ k = 1,2,...,K,\ \ y(0) = Y_0(x), \end{gathered} \tag{7}$$

$$Y(k)|_{x\in\partial\Im} = 0,\ \ k = 1,2,...,K, \tag{8}$$

is used, where $\tau = T/K$ is step partition of segment $[0,T]$ by points $k\tau$ $(k = 1,2,...,K-1)$; $Y(k) := Y(x;k)$; $Y(k)_t := \frac{1}{\tau}[Y(k) - Y(k-1)]$; $f_\tau(k) := f_\tau(x;k) = \frac{1}{\tau}\int\limits_{(k-1)}^{k\tau} f(x,t)dt$; $p_\tau(k) := p_\tau(x;k) = \frac{1}{\tau}\int\limits_{(k-1)}^{k\tau} p(x,t)dt$ $(k = 1,2,...,K)$.

Let's introduce the necessary spaces, using the classical lebesgue and Sobolev spaces. Denote through $L_2(\Im)^n$ space, the elements of which are real Lebesgue measurable vector-functions $u(x) = \{u_1(x,t), u_2(x,t), ..., u_n(x,t)\}$, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. The ratios $(u,v) = \int\limits_{\Im} u(x)v(x)dx$ and $\|u\| = \sqrt{(u,u)}$ (here $\int\limits_{\Im} \varphi(x)dx = \sum\limits_{l=1}^{N}\int\limits_{\Im_l} \varphi(x)dx$) in $L_2(\Im)^n$ define the scalar product and the norm, respectively. Let $D(\Im)^n$ is set of infinitely differentiable finite functions $\varphi(x)$, for which $\operatorname{div}\varphi = 0$: $\mathfrak{D}(\Im)^n = \{\varphi : \varphi \in D(\Im)^n,\ \operatorname{div}\varphi = 0\}$. The closure $\mathfrak{D}(\Im)^n$ in $L_2(\Im)^n$ defines space $\mathcal{H}(\Im)$, the elements of space $\mathcal{H}^1(\Im)$ are functions $\varphi(x) \in \mathcal{H}(\Im)$ with a generalized derivative $\frac{\partial\varphi}{\partial x} \in L_2(\Im)^n$. The scalar product in $\mathcal{H}^1(\Im)$ is defined by the formula $(\varphi,\psi)_1 = (\varphi,\psi) + (\frac{\partial\varphi}{\partial x}, \frac{\partial\psi}{\partial x})$, $\|\varphi\|_1 = \Big(\|\varphi\|^2 + \|\frac{\partial\varphi}{\partial x}\|^2\Big)^{1/2}$. For the differential-difference system (7), (8) we introduce the state space $V_0^1(\Im)$ as a closure in $\mathcal{H}^1(\Im)$ the set of functions $\varphi \in \mathfrak{D}(\Im)^n$ that satisfy the relations $\int\limits_{S_j} \frac{\partial\varphi(x)}{\partial n_j}ds + \sum\limits_{\iota=1}^{m_j}\int\limits_{S_{j\iota}} \frac{\partial\varphi(x)}{\partial n_{j\iota}}ds = 0$.

We will first analyze two differential forms:

$$\rho(u,v)=\sum_{i,j=1}^{n}\int\limits_{\Im}\frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}dx,\quad \varrho(u,v,\omega)=\sum_{i,k=1}^{n}\int\limits_{\Im}u_k\frac{\partial v_i}{\partial x_k}\omega_i dx$$

for which the integrals $\int\limits_{\Im}\frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}dx$ and $\int\limits_{\Im}u_k\frac{\partial v_i}{\partial x_k}\omega_i dx$ converge (here $u(x)=\{u_1(x), u_2(x),...,u_n(x)\}$, $u(x)=\{u_1(x),u_2(x),...,u_n(x)\}$, $u(x)=\{u_1(x),u_2(x),...,u_n(x)\}$) (see also [9, pp. 79–81]).

Lemma 1. *The form $\rho(u,v)$ is continuous by u, v on $V_0^1(\Im)\times V_0^1(\Im)$, the form $\varrho(u,v,\omega)$ is continuous by u, v, ω on $L_4(\Im)^n\times V_0^1(\Im)\times L_4(\Im)^n$.*

P r o o f. For $\frac{\partial u_j}{\partial x_i}$ and $\frac{\partial v_j}{\partial x_i}$ of form $\rho(u,v)$ we get

$$|\int\limits_{\Im}\frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}dx|\leqslant\sqrt{\int\limits_{\Im}(\frac{\partial u_j}{\partial x_i})^2dx}\sqrt{\int\limits_{\Im}(\frac{\partial v_j}{\partial x_i})^2dx}\leqslant\|u_j\|_1\,\|v_j\|_1 \tag{9}$$

(where the Cauchy—Bunyakovskii inequality is used). Similar actions for $u_k\omega_i$ and $\frac{\partial v_i}{\partial x_k}$, and then for u_k^2 and ω_i^2 of form $\sigma(u,v,\omega)$, reduce to the following inequalities:

$$\begin{gathered}|\int\limits_{\Im}u_k\frac{\partial v_i}{\partial x_k}\omega_i dx|\leqslant\sqrt{\int_{\Im}(u_k\omega_i)^2dx}\sqrt{\int\limits_{\Im}(\frac{\partial v_j}{\partial x_i})^2dx}\leqslant\\ \leqslant\sqrt[4]{\int\limits_{\Im}u_k^4dx}\sqrt{\int\limits_{\Im}(\frac{\partial v_j}{\partial x_i})^2dx}\sqrt[4]{\int\limits_{\Im}\omega_i^4dx}\leqslant\|u_k\|_{L_4(\Im)}\,\|v_j\|_1\,\|\omega_i\|_{L_4(\Im)}.\end{gathered} \tag{10}$$

From inequality (9) follows continuity $\rho(u,v)$ on $V_0^1(\Im)\times V_0^1(\Im)$, continuity $\varrho(u,v,\omega)$ on $(V_0^1(\Im)\cap L_4(\Im)^n)\times V_0^1(\Im)\times L_4(\Im)^n$ follows from (10).

Lemma 2. *Let u, ω are arbitrary elements of space $V_0^1(\Im)$, then*: 1) $\varrho(u,u,\omega)=-\varrho(u,\omega,u)$; 2) $\varrho(u,\omega,\omega)=0$; 3) $\varrho(\omega,\omega,\omega)=0$.

P r o o f. The first statement follows from the sum $\sum_{i,k=1}^{n}\int\limits_{\Im}u_k\frac{\partial v_i}{\partial x_k}\omega_i dx$, when integrating in parts all its integrals, the following statements follow from the first.

Lemma 3. *From the weak convergence of sequences $\{u_m\}_{m\geqslant1}$, $\{v_m\}_{m\geqslant1}$ in $L_2(\Im)^n$ to the elements u and v follows the convergence of the sequence $\{u_mv_m\}_{m\geqslant1}$ in norm $L_2(\Im)^n$ to the element uv.*

P r o o f. Let's show convergence $\int\limits_{\Im_T}u_mv_m\zeta dxdt\underset{m\to\infty}{\to}\int\limits_{\Im_T}uv\zeta dxdt$ on any element of $\zeta(x)\in V_0^1(\Im)$. Since the sequences $\{u_m\}_{m\geqslant1}$, $\{v_m\}_{m\geqslant1}$ converge weakly, the elements u_m, v_m are bounded in total and $\|v_m\|+\|v\|\leqslant c$, $\|u_m\|+\|u\|\leqslant c$. The sequence $\{v_m\zeta\}_{m\geqslant1}$ converges strongly in $L_2(\Im)^n$ to the element $v\zeta$. Indeed, with arbitrarily small given $\varepsilon>0$, let's take as $\zeta(x)$ a function $\frac{\varepsilon}{\|\zeta\|}\zeta(x)$, then

$$\|v_m\zeta-v\zeta\|_{L_2(\Im)^n}\leqslant\|v_m-v\|_{L_2(\Im)^n}\|\zeta\|_{L_2(\Im)^n}\leqslant\varepsilon(\|v_m\|_{L_2(\Im)^n}+\|v\|_{L_2(\Im)^n})\leqslant\varepsilon c,$$

what means convergence $\{v_m\zeta\}_{m\geqslant1}$ to $v\zeta$ in $L_2(\Im)^n$. This and the estimates presented below

$$\begin{gathered}|\int\limits_{\Im}u_mv_m\zeta dx-\int\limits_{\Im}uv\zeta dx|=\int\limits_{\Im}|(u_mv_m-uv)\zeta|dx\leqslant\\ \leqslant\int\limits_{\Im}(\|u_m\|+\|v_m\zeta-v\zeta\|+\|v\|\|u_m\zeta-u\zeta\|)\,dx\end{gathered}$$

lead to the completion of the proof of the lemma statement.

The following approach for analyzing the weak solvability of the system (1)–(6) is based on the construction of a priori estimates of the solutions of the differential-difference system (7), (8) and use of the Galerkin method, which assume look for functions $Y(k) \in V_0^1(\Im)$, $k = 1, 2, ..., K$, in the form of expansions on a special basis of space $V_0^1(\Im)$ — system of generalized eigenfunctions of the operator $\Delta Y = \sum\limits_{i=1}^{n} \frac{\partial^2 Y}{\partial x_i^2}$. Such a system forms the basis in the spaces $V_0^1(\Im)$ and $L_2(\Im)^n$ (proof similar to the one given in the work [10]).

Let us turn to the issue of constructing a priori estimates of the weak solution of the system (7), (8). Let the input data $Y_0(x)$, $f(x,t)$ of systems (1)–(6) satisfy the conditions $Y_0(x) \in V_0^1(\Im)$, $f(x,t) \in L_{2,1}(\Im_T)^n$ (the elements of space $L_{2,1}(\Im_T)^n$ belong to $L_1(\Im_T)^n$ and have a finite norm $\|u\|_{2,1} = \int\limits_0^T (\int\limits_{\Im} u(x,t)^2 dx)^{1/2} dt$). This means that for the system (7), (8) the input data $Y_0(x) \in V_0^1(\Im)$, $f_\tau(k) \in L_2(\Im)^n$.

Definition 1. *The set of functions* $\{Y(k) \in V_0^1(\Im), k = 1, 2, ..., K\}$, *where for each* k $(k = 1, 2, ..., K)$ *the function* $Y(k)$ *satisfies the relation*

$$\begin{gathered} (Y(k)_t, \eta) + \nu\rho(Y(k), \eta) + \varrho(Y(k), Y(k), \eta) = (f_\tau(k), \eta), \\ Y(0) = Y_0(x), \end{gathered} \tag{11}$$

under any function $\eta(x) \in V_0^1(\Im)$, *is called the weak solution of the differential-difference system* (7), (8).

Since the set of generalized eigenfunctions $\{U_i(x)\}_{i \geqslant 1}$ is the basis in space $V_0^1(\Im)$, then for approximations $Y_m(k) = \sum\limits_{i=1}^{m} g_{i,m}^k U_i(x)$ of the functions $Y(k)$, $k = 1, 2, ..., K$, consider the system

$$\begin{gathered} (Y_m(k)_t, U_i) + \nu\rho(Y_m(k), U_i) + \varrho(Y_m(k), Y_m(k), U_i) = (f_\tau(k), U_i), \\ i = 1, 2, ..., m, \quad k = 1, 2, ..., K, \end{gathered} \tag{12}$$

$$Y_m(0) = Y_{0\,m}(x), \tag{13}$$

where $Y_{0\,m}(x) = \sum\limits_{i=1}^{m} g_{im}^0 U_i(x)$ (g_{im}^0 — const), $Y_{0\,m}(x) \to Y_0(x)$ in norm $\mathcal{H}^1(\Im)$.

Theorem 1. *Let* $Y_0(x) \in V_0^1(\Im)$, $f_\tau(k) \in L_2(\Im)^n$ $(k = 1, 2, ..., K)$. *For functions* $Y_m(k)$, $k = 1, 2, ..., K$, *valid estimates*

$$\|Y_m(k)\| \leqslant \|Y_m(0)\| + 2\|f_\tau(k)\|_{2,1}^{'}, \quad k = 1, 2, ..., K,$$

$$\|Y_m(k)\|^2 + 2\tau\nu \sum_{k'=1}^{k} \|\tfrac{\partial Y_m(k')}{\partial x}\|^2 \leqslant C\left(\|Y_0\|^2 + (\|f_\tau(k)\|_{2,1}^{'})^2\right), \quad k = 1, 2, ..., K,$$

where the constant C *is independent of* τ; $\|f_\tau(k)\|_{2,1}^{'} = \tau \sum\limits_{k'=1}^{k} \|f_\tau(k')\|$.

P r o o f. From the relation $Y(k-1) = Y(k) - \tau Y(k)_t$ follows the obvious relation $2\tau(Y(k), Y(k)_t) = Y^2(k) + \tau^2 Y(k)_t^2 - Y^2(k-1)$. The ratio (11), (12) multiplied by $2\tau g_{i,m}^k$ and summed by i from 1 to m, we get

$$\begin{gathered} Y_m^2(k) - Y_m^2(k-1) + \tau^2 Y_m^2(k)_t + 2\tau\nu\varrho(Y_m(k), Y_m(k)) = 2\tau(f_\tau(k), Y_m(k)), \\ k = 1, 2, ..., K, \end{gathered}$$

($\rho(Y_m(k), Y_m(k), Y_m(k)) = 0$ by virtue of statement 3 of Lemma 2), which follows

$$\|Y_m(k)\|^2 - \|Y_m(k-1)\|^2 + \tau^2 \|Y_m(k)_t\|^2 + 2\tau\nu \|\tfrac{\partial Y_m(k)}{\partial x}\|^2 \leqslant 2\tau \|f_\tau(k)\| \|Y_m(k)\|, \quad k = 1, 2, ..., K, \tag{14}$$

and further

$$\|Y_m(k)\|^2 - \|Y_m(k-1)\|^2 \leqslant 2\tau \|f_\tau(k)\| \|Y_m(k)\|, \quad k = 1, 2, ..., K. \tag{15}$$

For inequality (15), consider two cases:

• $\|Y_m(k)\| + \|Y_m(k-1)\| > 0$. Since $\frac{\|Y_m(k)\|}{\|Y_m(k)\| + \|Y_m(k-1)\|} \leqslant 1$, then, dividing (15) by $\|Y_m(k)\| + \|Y_m(k-1)\|$, we get

$$\|Y_m(k)\| - \|Y_m(k-1)\| \leqslant 2\tau \|f_\tau(k)\|, \quad k = 1, 2, ..., K; \tag{16}$$

• $\|Y_m(k)\| + \|Y_m(k-1)\| = 0$. From (15) follows $0 \leqslant 2\tau \|f_\tau(k)\| \, \|Y_m(k)\|$ and $\|Y_m(k)\|^2 - \|Y_m(k-1)\|^2 \leqslant 2\tau \|f_\tau(k)\| \, \|Y_m(k)\|, \ k = 1, 2, ..., K$, and again we get (16).

If we sum (16) by k' from 1 to k, we get the first estimate in the statements of the theorem:

$$\|Y_m(k)\| \leqslant \|Y_m(0)\| + 2 \sum_{k'=1}^{k} \tau \|f_\tau(k')\| = \|Y_0\| + 2\|f_\tau(k)\|'_{2,1}, \quad k = 1, 2, ..., K, \tag{17}$$

$\|f_\tau(k)\|'_{2,1} = \sum_{k'=1}^{k} \tau \|f_\tau(k')\|$ ($\|\cdot\|'_{2,1}$ is analogue of the norm $\|\cdot\|_{2,1}$).

If we sum (14) by k' from 1 to k, using inequalities (17), we get a second estimate in the statements of the theorem:

$$\begin{aligned} &\|Y_m(k)\|^2 + 2\tau\nu \sum_{k'=1}^{k} \|\tfrac{\partial Y_m(k')}{\partial x}\|^2 \leqslant \\ &\leqslant \|Y_m(k)\|^2 + \tau^2 \sum_{k'=1}^{k} \|Y_m(k)_t\|^2 + 2\tau\nu \sum_{k'=1}^{k} \|\tfrac{\partial Y_m(k')}{\partial x}\|^2 \leqslant \\ &\leqslant C\left(\|Y_0\|^2 + (\|f_\tau(k)\|'_{2,1})^2\right), \quad k = 1, 2, ..., K, \end{aligned} \tag{18}$$

where the constant C does not depend on τ.

Consequence. The obtained a priori estimates make it possible to show a weak solvability of the differential-difference system (7), (8) (and hence (12), (13)), which is established similarly to the reasoning given in the work.

Remark 2. It is easy to show that from the estimates (17), (18) follows the continuous dependence of the weak solution $\{Y(k) \in V_0^1(\Im), \ k = 1, 2, ..., K\}$ of the differential-difference system (7), (8) on the input data $Y_0(x)$, $f_\tau(k)$.

To analyze the differential system (1)–(6), we will introduce the necessary spaces. Denote through $W^{1,0}(\Im_T)$ the space with elements $u(x,t)$, the generalized derivatives $\frac{\partial u(x,t)}{\partial x}$ of which belong to $L_2(\Im_T)^n$, $\|u\|_{W^{1,0}(\Im_T)} = \left(\|u\|^2 + \|\frac{\partial u}{\partial x}\|^2\right)^{1/2}$, and let $W^1(\Im_T)$ is the space with elements $u(x,t) \in L_2(\Im_T)^n$, for which $\frac{\partial u(x,t)}{\partial x}$, $\frac{\partial u(x,t)}{\partial t}$ belong to $L_2(\Im_T)^n$, $\|u\|_{W^1(\Im_T)} = \left(\|u\|^2 + \|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial u}{\partial x}\|^2\right)^{1/2}$. For the elements of space $W^1(\Im_T)$ the following properties are valid [11, p. 32]: elements are continuous according to t in norm $L_2(\Im)^n$ and traces of elements on sections $\Im_T$ by planes $t = t_0$ $(t_0 \in [0,T])$ belong to $L_2(\Im)^n$.

Denote by $\Omega_1(\Im_T) \subset W^{1,0}(\Im_T)$, $\Omega_2(\Im_T) \subset W^1(\Im_T)$ sets whose elements with fixed $t \in (0,T)$ belong to the space $V_0^1(\Im)$. The closures $\Omega_1(\Im_T)$ and $\Omega_2(\Im_T)$ in spaces $W^{1,0}(\Im_T)$ and $W^1(\Im_T)$ denote by $W_0^{1,0}(\Im_T)$ and $W_0^1(\Im_T)$, respectively [3]. Clearly, $u(x,t)|_{\partial\Im} = 0$ for $u(x,t) \in W_0^{1,0}(\Im_T)$ or $u(x,t) \in W_0^1(\Im_T)$. The space $W_0^{1,0}(\Im_T)$ is the state space $Y(x,t)$ of the Navier—Stokes system, $W_0^1(\Im_T)$ is helper space. As above $Y_0(x) \in V_0^1(\Im)$, $f(x,t) \in L_2(\Im)^n$, the scalar function $p(x,t)$, characterizing quantitative changes of pressure, belong to the class $C(\Im_T)$.

Definition 2. *A pair of functions*

$$\Big\{Y(x,t),\, p(x,t): \quad Y(x,t) \in W_0^{1,0}(\Im_T),\, p(x,t) \in C(\Im_T)\Big\}$$

is called a weak solution of a differential system (1)–(6), *if the function* $Y(x,t)$ *satisfies the integral identity*

$$\begin{gathered} -\int\limits_{\Im_T} Y(x,\tau)\tfrac{\partial\eta(x,\tau)}{\partial\tau}dxd\tau + \nu\int\limits_0^T \rho(Y,\eta)d\tau + \int\limits_0^T \varrho(Y,Y,\eta)d\tau = \\ = \int\limits_{\Im} Y_0(x)\eta(x,0)dx + \int\limits_{\Im_T} f(x,\tau)\eta(x,\tau)dxd\tau \end{gathered} \tag{19}$$

for any $\eta(x,t) \in W_0^1(\Im_T)$ *and* $\eta(x,T) = 0$.

Remark 3. By virtue of Definition 2 for a function $p(x,t)$ it is necessary that the relation $(\operatorname{grad} p(x,t),\, \eta(x,\tau)) = 0$ at any $\eta(x,t)$ from $W_0^1(\Im_T)$. The latter is possible, for example, when $p(x,t)$ it belongs to the class $C(\Im_T)$. Note also that in many application problems of continuum transport, the function $p(x,t)$ refers to the input data, therefore its existence not depend on the existence of the function $Y(x,t) \in W_0^{1,0}(\Im_T)$.

Further, using the obtained a priori estimates (17), (18) (statements of Theorem 1), consider the issue of weak solvability of the differential system (1)–(6) [6] (see also [12, p. 191]).

Theorem 2. *Let the conditions* $Y_0(x) \in V_0^1(\Im)$, $f(x,t) \in L_{2,1}(\Im_T)^n$, *then the initial boundary value problem* (1)–(6) *is weakly solvable.*

P r o o f. Let's denote through $Y_K(x,t)$ piecewise constant interpolations by t: $Y_K(x,t) = Y(k)$, $t \in ((k-1)\tau, k\tau]$, $k = 1,2,...,K$, $Y_K(x,0) = Y_0(x)$. Here we proceed from the existence of the solution $\{Y(k) \in V_0^1(\Im),\, k = 1,2,...,K\}$ (consequence of Theorem 1). It is clear that $u_K(x,t) \in W_0^{1,0}(\Im_T)$ and satisfies the a priori estimates (17) and (18), then for $u_K(x,t)$ a fair estimate

$$\|Y_K\| + \|\tfrac{\partial Y_K}{\partial x}\| \leqslant C^* \tag{20}$$

with an independent of τ the constant $C^* > 0$. A similar representation is set for the function $f_K(x,t)$: $f_K(x,t) = f(x;k)$, $t \in ((k-1)\tau, k\tau]$, $k = 1,2,...,K$. With an unlimited increase K to infinity, we get a sequence $\{Y_K(x,t)\}$, from which, given (20), we select the subsequence $\{\widetilde{Y}_K(x,t)\}$, converging to $Y(x,t) \in W_0^{1,0}(\Im_T)$. Let's assume that $Y(x,t)$ is weak solution of the system (1)–(6). To do this, we will show what $Y(x,t)$ satisfies the identity (19) with arbitrary $\eta(x,t) \in C^1(\Im_{T+\tau})^n$, satisfying the conditions (3), (4) under any $t \in (0,T)$ and for which $\eta|_{\partial\Im_T} = 0$, $\eta|_{t\in[T,T+\tau]} = 0$. By $\eta(x,t)$ are defined $\eta(k)$: $\eta(k) = \eta(x,k\tau)$, $k = 1,2,...,K$, at the same time $\eta(k)_{t'} = \frac{1}{\tau}[\eta(k+1) - \eta(k)]$ (here $\eta(k)_{t'}$, $\eta(k)_t$ are right and left approximations $\frac{\partial\eta}{\partial t}$ at the point $t = k\tau$, respectively). By functions $\eta(k)$ piecewise continuous approximations $\eta_K(x,t)$, $\frac{\partial\eta_K(x,t)}{\partial x}$, $\frac{\partial\eta_K(x,t)}{\partial t}$ are formed for functions $\eta(x,t)$, $\frac{\partial\eta(x,t)}{\partial x}$, $\frac{\partial\eta(x,t)}{\partial t}$ by analogy with $Y_K(x,t)$, moreover $\eta_K(x,t)$, $\frac{\partial\eta_K(x,t)}{\partial x}$, $\frac{\partial\eta_K(x,t)}{\partial t}$ uniformly converge on $\overline{\Im}_T$ to $\eta(x,t)$, $\frac{\partial\eta(x,t)}{\partial x}$, $\frac{\partial\eta(x,t)}{\partial t}$ at $K\to\infty$, respectively; $\eta_K(x,t)=0$, $t\in[T,T+\tau]$.

The identity (19) will be summed up by k from 1 to N, replaced $\eta(x)$ by $\tau\eta(x)$:

$$
\begin{gathered}
-\tau\sum_{k=1}^{N}\int_{\Im}Y(k)\eta(k)_t dxdt-\int_{\Im}Y_0\eta(1)dx+\nu\sum_{k=1}^{N}\tau\rho(Y(k),\eta(k))+\\
+\sum_{k=1}^{N}\tau\varrho(Y(k),Y(k),\eta(k))=\sum_{k=1}^{N}\tau\int_{\Im}f_\tau(k)\eta(k)
\end{gathered}
\tag{21}
$$

(here $\tau\sum_{k=1}^{N}Y(k)_t\eta(k)=-\tau\sum_{k=1}^{N}Y(k)\eta(k)_t-Y(0)\eta(1),\ \eta(N)=\eta(N+1)=0$). The relation (21), given the representation $\eta_K(x,t)$, takes the form

$$
\begin{gathered}
-\int_{\Im_T}Y_K(x,t)\eta_K(x,t)_t dxdt-\int_{\Im}Y_0(x,t)\eta_K(x,\tau)dx+\nu\int_0^T\rho(Y_K,\eta_K)dt+\\
+\int_0^T\varrho(Y_K,Y_K,\eta_K)dt=\int_{\Im_T}f_K(x,t)\eta_K(x,t)dxdt.
\end{gathered}
\tag{22}
$$

Passing on to the limit in (22) by subsequence $\{\widetilde{Y}_K(x,t)\}$ ($\eta_K(x,t)$ replaced by $\widetilde{\eta}_K(x,t)$, which corresponds to $\{\widetilde{Y}_K(x,t)\}$) taking into account Lemma 3, we obtain the identity (19) for $Y(x,t)$. This proves the weak solvability of the initial boundary value problem (differential system) (1)–(6). It should also be noted, by virtue of comment 3, the continuous dependence $Y(x,t)$ on the input data $Y_0(x)$, $f_\tau(k)$. The theorem is proven.

4. The problem of optimal control. For the Navier—Stokes system two types of optical control problems — distributed and start control are considered, which are most meeting in applications and do not reduce the community of analysis. Everywhere below, the control is indicated by the symbol υ, the state of $Y(x,t)$ of the Navier—Stokes system is indicated by the $Y(x,t;\upsilon)$. In the case of distributed control $\upsilon(x,t)$, the distributed effect operator (this operator determines the density of external forces) is present on the right side of equation (1) of the Navier—Stokes system:

$$
\frac{\partial Y}{\partial t}-\nu\Delta Y+\sum_{i=1}^{n}Y_i\frac{\partial Y}{\partial x_i}+\operatorname{grad}p=f+B\upsilon,
\tag{23}
$$

in the case of start control $\upsilon(x)$, the external effect is realized out by means of the initial state of the Navier—Stokes system and determines the initial condition (5):

$$
Y(x,t)|_{t=0}=\upsilon(x),\ x\in\Im.
\tag{24}
$$

Thus, for the Navier—Stokes system, the initial boundary value problem (23), (2)–(6) determines the problem of optimal distributed control, the initial boundary value problem (1)–(4), (24), (6) determines the problem of optimal start control. At the same time, in both cases, the state of the Navier—Stokes system is monitored both at the domain $\Im_T$ (distributed observation), and $\Im$ at $t_0\in(0,T)$ (observation at a fixed point in time) or at $t=T$ (final observation). Other types of observations are also possible, for example, at the boundary $\partial\Im_T$ or part of it (boundary observation). The physical task is to bring the dynamic characteristics of the viscous fluid (velocity, convective component values) to preassigned levels at an interval $(0,T)$ or by a point in time $t=t_0\in(0,T]$.

The definition of a weak solution of the system (23), (2)–(6) or (1)–(4), (24), (6) exactly repeats the Definition 2 with the only difference that for the system (23), (2)–(6) the right part (19) changes to

$$\int_{\Im} Y_0(x)\eta(x,0)dx + \int_{\Im_T} (f(x,\tau) + B\upsilon(x,t))\eta(x,\tau)dxd\tau,$$

and for the system (1)–(4), (24) the right part (19) changes to

$$\int_{\Im} \upsilon(x)\eta(x,0)dx + \int_{\Im_T} f(x,\tau)\eta(x,\tau)dxd\tau.$$

According to what has been said $Y(x,t)$ replaced on $Y(\upsilon) := Y(x,t;\upsilon)$ or on $Y_t(\upsilon) := Y(x,t;\upsilon)$ ($Y_T(\upsilon) := Y(x,T;\upsilon)$), the latter for the convenience of presenting the results.

To assess the state $Y(x,t;\upsilon)$ of the Navier—Stokes system, we introduce the space of controls U and linear continuous observation operators $C_q : L_2(\Im_T)^n \to H$ $(q = 1,2)$, where H is the observation space: $H = L_2(\Im_T)^n$ in the case of distributed observation and in the case of final observation $H = L_2(\Im)^n$. In addition $C_1Y(\upsilon) = DY_T(\upsilon)$, here operator $D : H \to H$ is the linear bounded operator, $C_2\widetilde{Y}(\upsilon) = \widetilde{Y}_T(\upsilon)$, $\widetilde{Y}_t(\upsilon) := \widetilde{Y}(x,t;\upsilon) := \sum\limits_{i=1}^{n} Y_i(x,t;\upsilon)\frac{\partial Y(x,t;\upsilon)}{\partial x_i}$.

Let's set the functional $J(\upsilon)$ on a closed convex set $U_\partial \subset U$:

$$J(\upsilon) = \|C_1Y(\upsilon) - \Phi\|_H^2 + \|C_2\widetilde{Y}(\upsilon) - \Psi\|_H^2 = J_1(\upsilon) + J_2(\upsilon), \tag{25}$$

where $J_1(\upsilon) = \|DY_T(\upsilon) - \Phi\|_H^2$, $J_2(\upsilon) = \|\widetilde{Y}_T(\upsilon) - \Psi\|_H^2$ and Φ, Ψ are preassigned functions: in the case of distributed observation $\Phi := \Phi(x,t), \Psi := \Psi(x,t) \in H = L_2(\Im_T)^n$, for the case of final observation $\Phi := \Phi(x), \Psi := \Psi(x) \in H = L_2(\Im)^n$. In applications, the functional $J_1(\upsilon)$ establishes the difference between the characteristics of the velocity vector of the hydraulic flow from the defined Φ, the functional $J_2(\upsilon)$ characterizes the difference between the convective change in the velocity vector and the defined Ψ.

Definition 3. *The problem of optimal distributed or start control of the Navier—Stokes system* (23), (2)–(5) *or* (1)–(4), (24) *is to find* $\inf\limits_{\upsilon \in U_\partial} J(\upsilon)$. *Optimal control* $u \in U_\partial$ *of the system will be called the minimizing element of the functional* $J(\upsilon)$: $J(u) = \inf\limits_{\upsilon \in U_\partial} J(\upsilon)$.

In the future, we will assume the existence of optimal control of the Navier—Stokes system. We will prove the auxiliary statements beforehand.

Lemma 4. *Let* $u, \upsilon \in U_\partial$ *and* $\theta \in (0,1)$. *The following relations are valid*:

$$Y_t(u)'(\upsilon - u) = Y_t(\upsilon) - Y_t(u), \quad \widetilde{Y}_t(u)'(\upsilon - u) = \widetilde{Y}_t(\upsilon) - \widetilde{Y}_t(u), \tag{26}$$

here the symbol "′" denotes Frechet derivative by control υ *of function* $Y_t(\upsilon)$ *and* $\widetilde{Y}_t(\upsilon)$.

P r o o f. Let's reason for the case of distributed control $\upsilon(x,t)$, similar reasoning is true in the case of start control. Proceeding from the integral identity of the definition of the weak solution of the system (23), (2)–(6) and taking into account the relations

$$\rho(Y_t(\upsilon),\eta) - \rho(Y_t(u),\eta) = \sum_{i,j=1}^{n} \int_{\Im} \frac{\partial Y_t(\upsilon)_j}{\partial x_i}\frac{\partial \eta_j}{\partial x_i}dx - \sum_{i,j=1}^{n} \int_{\Im} \frac{\partial Y_t(u)_j}{\partial x_i}\frac{\partial \eta_j}{\partial x_i}dx = \\ = \rho(Y_t(\upsilon) - Y_t(u),\eta),$$

$$\varrho(Y_t(\upsilon),Y_t(\upsilon),\eta) - \varrho(Y_t(u),Y_t(u),\eta) = \sum_{j,k=1}^{n} \int_{\Im} Y_t(\upsilon)_k \frac{\partial Y_t(\upsilon)_i}{\partial x_k}\eta_i dx - \sum_{j,k=1}^{n} \int_{\Im} Y_t(u)_k \frac{\partial Y_t(u)_i}{\partial x_k}\eta_i dx = \\ = \widetilde{\varrho}(\widetilde{Y}(\upsilon) - \widetilde{Y}(u),\eta)$$

(here $\widetilde{\varrho}(\widetilde{Y}(\omega),\eta) = \sum\limits_{j,k=1}^{n} \int\limits_{\Im} Y_t(\omega)_k \frac{\partial Y_t(\omega)_i}{\partial x_k}\eta_i dx, \ \ \omega = \upsilon$ or $\omega = u$) for any $u, \upsilon \in U_\partial$ and arbitrary function $\eta(x,t) \in W_0^1(\Im_T)$, we come to integral identities for arbitrary $\eta(x,t) \in W_0^1(\Im_T)$:

$$-\int\limits_{\Im_T} (Y_t(\upsilon) - Y_t(u)) \frac{\partial \eta(x,t)}{\partial t} dxdt + \nu \int\limits_0^T \rho(Y_t(\upsilon) - Y_t(u), \eta)dt + \int\limits_0^T \widetilde{\varrho}(\widetilde{Y}(\upsilon) - \widetilde{Y}(u), \eta)dt = \\ = \int\limits_0^T (B(\upsilon - u), \eta)dt,$$

$$-\int\limits_{\Im_T} (Y_t(u+\theta(\upsilon-u)) - Y_t(u)) \frac{\partial \eta(x,t)}{\partial t} dxdt + \nu \int\limits_0^T \rho(Y_t(u+\theta(\upsilon-u)) - Y_t(u), \eta)dt + \\ + \int\limits_0^T \widetilde{\varrho}(\widetilde{Y}(u+\theta(\upsilon-u)) - \widetilde{Y}(u), \eta)dt = \theta \int\limits_0^T (B(\upsilon - u), \eta)dt.$$

If there are limits $\lim\limits_{\theta\to 0} \frac{Y_t(u+\theta(\upsilon-u)) - Y_t(u)}{\theta}$, $\lim\limits_{\theta\to 0} \frac{\widetilde{Y}(u+\theta(\upsilon-u)) - \widetilde{Y}(u)}{\theta}$, then there are derivatives of Frechet $Y_t(u)'$, $\widetilde{Y}(u)'$, respectively. After dividing the second integral identity by θ and calculating the limit at $\theta \to 0$, we obtain the relations (26) after comparing the left parts of these identities. The lemma is proven.

Lemma 5. *Let $u \in U_\partial$ be the minimizing element of the functional $J(\upsilon)$ then*

$$J(u)'(\upsilon - u) \geqslant 0 \tag{27}$$

for any $\upsilon \in U_\partial$.

P r o o f. Since $u \in U_\partial$ it is a minimizing element of the functional $J(\upsilon)$, then $J(u) = \inf\limits_{\upsilon\in U_\partial} J(\upsilon)$. With any $\upsilon \in U_\partial$ and $\theta \in (0,1)$ element $(1-\theta)u + \theta\upsilon \in U_\partial$ due to the convexity of U_∂, and therefore,

$$J(u) \leqslant J((1-\theta)u + \theta\upsilon) = J(u + \theta(\upsilon - u))$$

and then

$$\frac{J(u+\theta(\upsilon-u)) - J(u)}{\theta} \geqslant 0.$$

When $\theta \to 0$ we get the inequality (27), the lemma is proven.

Next, consider the problems of optimal control of the Navier — Stokes system (1), (2).

Distributed control. The Navier — Stokes system witch distributed control is of the form (23), (2)–(4). Its state $\{Y(\upsilon)(x,t), p(\upsilon)(x,t)\}$ is determined by a weak solution of the initial boundary value problem (23), (2)–(6), for which the integral identity takes the form

$$\begin{gathered} -\int\limits_{\Im_T} Y(x,t;\upsilon) \frac{\partial \eta(x,t)}{\partial t} dxdt + \nu \int\limits_0^T \rho(Y,\eta)dt + \int\limits_0^T \varrho(Y,Y,\eta)dt = \\ = \int\limits_{\Im} Y_0(x)\eta(x,0)dx + \int\limits_{\Im_T} (f(x,t) + B\upsilon(x,t))\eta(x,t)dxdt. \end{gathered} \tag{28}$$

Minimizing functional $J(\upsilon)$ with distributed observation operators C_1, C_2 ($H = L_2(\Im_T)$) has the form (25), control $\upsilon \in U = L_2(\Im_T)$.

Theorem 3. *Let $u(x,t)$ is the optimal control of the system* (23), (2)–(4), *then the state function $Y(u)(x,t)$ of this system satisfies the identity* (28) *under $\upsilon(x,t) = u(x,t)$ for any element $\eta(x,t) \in W_0^1(\Im_T)$, $\eta(x,T) = 0$, and the inequality*

$$(DY_t(u) - \Phi,\, D\,(Y_t(\upsilon) - Y_t(u)))_H + (\widetilde{Y}_t(u) - \Psi,\, \widetilde{Y}_t(\upsilon) - \widetilde{Y}_t(u))_H \geqslant 0 \tag{29}$$

for any $\upsilon(x,t) \in U_\partial$; function $p(x,t)$ is arbitrary element of class $C(\Im_T)$.

P r o o f. Since $u(x,t)$ is the optimal control, it is a minimizing element of the functional $J(\upsilon)$: $J(u) = \inf\limits_{\upsilon \in U_\partial} J(\upsilon)$. By virtue of the statement of Lemma 5, inequality (27) is true. Based on (25), $J(\upsilon)$ it is presented in the following form $\left(\widetilde{Y}(x,t;\upsilon) = \sum\limits_{i=1}^{n} Y_i(x,t;\upsilon)\frac{\partial Y(x,t;\upsilon)}{\partial x_i}\right)$:

$$\begin{gathered} J(\upsilon) = (DY(x,t;\upsilon) - \Phi(x,t),\, DY(x,t;\upsilon) - \Phi(x,t))_H + \\ + (\widetilde{Y}(x,t;\upsilon) - \Psi(x,t),\, \widetilde{Y}(x,t;\upsilon) - \Psi(x,t))_H \end{gathered}$$

and, given the inequalities (26) of Lemma 4, we come to inequality

$$\begin{gathered} \tfrac{1}{2}J(u)'(\upsilon - u) = (DY_t(u) - \Phi,\, D(Y_t(\upsilon) - Y_t(u)))_H + \\ + (\widetilde{Y}_t(u) - \Psi,\, \widetilde{Y}_t(\upsilon) - \widetilde{Y}_t(u))_H \geqslant 0 \end{gathered}$$

and get inequality (29). The relation (28) (at $\upsilon(x,t) = u(x,t)$) for any $\eta(x,t) \in W_0^1(\Im_T)$ $(\eta(x,T) = 0)$ is the integral identity of the initial boundary value problem (23), (2)–(5). By virtue of Remark 4, the function $p(x,t)$ is arbitrary element from $C(\Im_T)$. The theorem is proven.

It should be noted that the control effect on the Navier—Stokes system (1)–(4) in a finite number of fixed points of the domain $\Im$ (point control) is a variant of distributed control [2]. For example, such points x_j may belong to the surfaces S_j of the nodal sites ω_j, $j = 1, 2, ..., M$. Then in the ratio (1) $f(x,t) = \sum\limits_{j=1}^{M} \upsilon_j(t) \otimes \delta(x - x_j)$, $\upsilon(t) = \{\upsilon_1(t), \upsilon_2(t), ..., \upsilon_M(t)\} \in L_2(0,T)^M$. In this case, the state $Y(\upsilon) := Y(x,t;\upsilon)$ of the system (1)–(4) is an element $W_0^1(\Im_T)$ and for $Y(x,t;\upsilon)$ fair identity

$$\begin{gathered} -\int\limits_{\Im_T} Y(x,t;\upsilon)\frac{\partial\eta(x,t)}{\partial t}dxdt + \nu\int\limits_0^T \rho(Y,\eta)dt + \int\limits_0^T \varrho(Y,Y,\eta)dt = \\ = \int\limits_{\Im} Y_0(x)\eta(x,0)dx + \int\limits_{\Im_T} f(x,t)\eta(x,t)dxdt + \sum_{j=1}^{m}\int\limits_0^T \upsilon_j(t)\eta(x_j,t)dt \end{gathered}$$

for any $\eta(x,t) \in W_0^1(\Im_T)$, $\eta(x,T) = 0$. Further reasoning is similar to the above.

Starting control. Consider the problem of optimal starting control of the Navier—Stokes system (1)–(4) with a control effect $\upsilon(x) \in U = L_2(\Im)^n$, that determines the initial condition (24). The pair $\{Y(\upsilon)(x,t), p(x,t)\}$ is a weak solution of the initial boundary value problem (1)–(4), (24), (6), $p(x,t) \in C(\Im_T)$, the function $Y(\upsilon)(x,t)$ satisfies the integral identity

$$\begin{gathered} -\int\limits_{\Im_T} Y(x,t;\upsilon)\frac{\partial\eta(x,t)}{\partial t}dxdt + \nu\int\limits_0^T \rho(Y,\eta)dt + \int\limits_0^T \varrho(Y,Y,\eta)dt = \\ = \int\limits_{\Im} \upsilon(x)\eta(x,0)dx + \int\limits_{\Im_T} f(x,t)\eta(x,t)dxdt \end{gathered} \tag{30}$$

for any $\eta(x,t) \in W_0^1(\Im_T)$, $\eta(x,T) = 0$, and $p(x,t)$ is an arbitrary element of $C(\Im_T)$. Minimizing functional $J(\upsilon)$ has the form (25), $\upsilon(x) \in U = H = L_2(\Im)^n$.

Theorem 4. *Let u is the optimal control of the system* (1)–(4), *then the state* $Y(u)(x,t)$ *of this system satisfies the identity* (30), *where* $\upsilon(x)$ *replaced by* $u(x)$, *and the inequality*

$$(DY_t(u) - \Phi,\, D\,(Y_t(\upsilon) - Y_t(u)))_H + (\widetilde{Y}_t(u) - \Psi,\, \widetilde{Y}_t(\upsilon) - \widetilde{Y}_t(u))_H \geqslant 0$$

for any $\upsilon(x) \in U_\partial = U$. *The function* $p(u)(x,t)$, *like above, belongs to space* $C(\Im_T)$.

The proof of the statement of the theorem repeats verbatim the proof of Theorem 3, since the representation of the minimizing functional $J(\upsilon)$ does not change.

Remark 4. Statements of Theorems 3 and 4 are necessary conditions for the existence of optimal distributed and starting controls. For a linearized Navier — Stokes system

$$\frac{\partial Y}{\partial t} - \nu \Delta Y + \operatorname{grad} p = f$$

the necessary and sufficient conditions for optimal control can be established using a conjugate system to this one.

5. Conclusion. The study of the problem of optimal distributed and starting control of the Navier — Stokes evolutionary differential system is considered in the Sobolev spaces of functions with carriers in the network-like region of the n-dimensional Euclidean space ($n \geqslant 2$). The paper presents the results of two main areas of research: obtaining conditions of weak solvability of the initial boundary problem for the Navier — Stokes system; the formation and solution of optimal control problems of the Navier — Stokes system. When analyzing the weak solvability of the initial boundary-boundary problem, it is reduced to the differential-difference system and the construction of a priori estimates for weak solutions of this system is carried out. Based on the Galerkin method with a special basis, an algorithm is formed for the actual construction of a weak solution to the initial boundary problem for the Navier — Stokes system. The obtained results are effectively used in the analysis of optimal control problems not only for network hydrodynamic processes, but also for the analysis of inverse problems of mathematical physics, the problems of determining the minimax of controlled systems, stability and stabilization of mechanical systems [13–17].

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Оптимальное управление системой Навье — Стокса с пространственной переменной в сетеподобной области

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Проведено исследование задачи оптимального управления эволюционной дифференциальной системой Навье — Стокса, рассматриваемой в пространствах Соболева, элементы которых — это функции с носителями в n-мерной сетеподобной области. Такая область состоит из конечного числа подобластей, взаимно примыкающих определенными частями поверхностей своих границ по типу графа. Для функций, являющихся элементами указанных пространств, представлены условия существования следов на поверхностях примыкания и рассмотрены условия примыкания подобластей, которым эти функции удовлетворяют. В прикладных вопросах анализа процессов переноса сплошных сред условия примыкания описывают закономерности протекания потоков жидкостей через границы примыкающих подобластей. Приведены результаты двух основных вопросов исследования: слабая разрешимость начально-краевой задачи для системы Навье — Стокса и получение условий существования слабого решения этой задачи; формирование и решение задач оптимального управления разного типа системой Навье — Стокса. Основополагающим подходом анализа слабой разрешимости начально-краевой задачи является редукция ее к дифференциально-разностной (полудискретизация исходной системы по временной переменной) и последующее использование априорных оценок для слабых решений полученных краевых задач. Такие оценки используются для доказательства теоремы существования слабого решения исходной дифференциальной системы и указывают путь фактического построения этого решения. Представлен универсальный подход к решению задач оптимального распределенного и стартового управления эволюционной системой Навье — Стокса. Последнее существенно расширяет возможности анализа нестационарных сетеподобных процессов прикладной гидродинамики (например, процессов транспортировки разного типа жидкостей по сетевым или магистральным трубопроводам) и оптимального управления этими процессами.

Ключевые слова: дифференциально-разностная система, эволюционная система Навье — Стокса, сетеподобная область, разрешимость, оптимальное управление.

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