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# Tropical curves and a holomorphic potential for symplectic toric manifolds

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# 1. Abstract

In this work we consider an ellipsoid $E(1, b), b \in \mathbb{Q}$ with standard symplectic structure. There is also a complex structure on $E(1, b)$ such that the resulting complex manifold is holomorphic to $\mathbb{C}^2$. These structures are bound by Kähler potential – a function on the ellipsoid whose hessian is a Kähler form. We try to investigate some properties of the maps which establish this connection.

## 1.1. **Motivation.**

The correspondence between symplectic and complex structures can be viewed as one of the various signs of mirror symmetry – a phenomena which came to mathematics from physics. The mirror symmetry is one of the biggest mysteries and I consider the work on this thesis as the first tiny step towards its understanding.

# 2. Symplectic geometry

**Definition.** A pair $(M, \omega)$ is called symplectic manifold, if $M$ is a smooth manifold and $\omega \in \Omega^2(M)$ is a closed non-degenerate 2-form.

It is immediate corollary from the definition that all symplectic manifolds are orientable and of even dimension. Here are some basic examples:

(1) $(\mathbb{R}^{2n}, \omega_0)$, where $\mathbb{R}^{2n}$ has coordinates $(x_1, y_1, \ldots, x_n, y_n)$ and $\omega_0 = \sum_{k=1}^{n} dx_k \wedge dy_k$
(2) $(T^*X, \omega)$, where $X$ – is any smooth manifold and $\omega = -d\alpha$, where $\alpha$ is a 1-form given by $\alpha_{(x,\xi)} = (d_{(x,\xi)}\pi)^*\xi$ at point $(x, \xi) \in T^*X$.
(3) For $n > 1$ there no symplectic structure on $S^{2n}$ since in this case $H^2(S^{2n}; \mathbb{R}) = 0$.

**Definition.** A map $f: (M, \omega_M) \to (N, \omega_N)$ of symplectic manifolds is called symplectomorphism if it is a diffeomorphism and $f^*\omega_N = \omega_M$.

Darboux theorem says that all symplectic manifolds of the same dimension are locally symplectomorhpic. In other words, symplectic manifolds do not have local invariants (such as curvature in Riemannian geometry). Thus to understand something about these objects we need to study global invariants. Lagrangian submanifolds may play such role.

## 2.1. **Lagrangian submanifolds.**

**Definition.** Let $(M^{2n}, \omega)$ be a $2n$-dimensional symplectic manifold. A submanifold $X$ of $M$ is called Lagrangian if, at each point $p \in X$, $\omega_p|_{T_pX} \equiv 0$ and $\dim T_pX = n$.

Studding Lagrangian submanifolds may help to understand the structure of ambient manifold.

# 3. Almost complex structure

**Definition.** A vector space $V$ is said to have almost complex structure if there is an endomorphism $J: V \to V$ such that $J^2 = -\mathrm{Id}$

Once $V$ has an almost complex structure $J$ we can define the multiplication of vectors by complex numbers as follows:

$$(a + ib)v = av + bJv$$

**Definition.** A manifold $M$ is said to be endowed with almost complex structure if there is a smooth family $\{J_p\}_{p\in M}$ of almost complex structures for each $T_pM$. The pair $(M, J)$ is called almost complex manifold.

### 3.1. Pseudo-holomorphic curves.

**Definition.** Let $j$ be a complex structure on Riemannian surface $\Sigma^2$.
Pseudo-holomorphic curve is a map $u\colon (\Sigma^2, j) \to (M, J)$ such that

$$du \circ j = J \circ du$$

As in symplectic case Lagrangian submanifolds posses some information about ambient symplectic manifold. So psedo-holomorphic curves might be viewed likewise in almost complex manifold.

## 4. Compatible structures. Kähler potential

### 4.1. Compatible almost complex structures.

**Definition.** Let $(V, \Omega)$ be symplectic vector space. A complex structure $J$ on $V$ is called compatible with $\Omega$ (or $\Omega$-compatible) if $G(u, v) := \Omega(u, Jv)$ is a positive defined inner product on $V$.

Note that

$$J \text{ is } \Omega\text{-compatible} \Leftrightarrow \begin{cases} \Omega(Ju, Jv) = \Omega(u, v) \\ \Omega(u, Ju) > 0, \forall u \neq 0 \end{cases}$$

It can be shown that for any symplectic vector space $(V, J)$ there exists $\Omega$-compatible complex structure $J$ on $V$.

**Definition.** Let $(M, \omega)$ be a symplectic manifold. An almost complex structure $J$ on $M$ is called compatible with $\omega$ (or $\omega$-compatible) if $g(u, v) := \omega(u, Ju)$ is a Riemannian metric. The triple $(\omega, g, J)$ is called compatible triple.

It also can be shown that any symplectic manifold has compatible almost complex structures.

### 4.2. Kähler manifolds and Kähler potential.

**Definition.** A Kähler manifold is a symplectic manifol $(M, \omega)$ equipped with an integrable compatible almost complex structure. The symplectic form $\omega$ is then called a Kähler form.

Here integrability property of an almost complex structure means that some tensor $\mathcal{N}$ of type (2, 0) vanishes. It can be shown that in this case our manifold is complex.

**Definition.** Let $M$ be a complex manifold of compldex dimension $n$. A function $\rho \in C^\infty(M; \mathbb{R})$ is strictly plurisubharmonic if, on each local complex chart $(\mathcal{U}, z_1, \dots, z_n)$, the matrix $\left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p)\right)$ is positive-defined for all $p \in \mathcal{U}$.

**Proposition 1.** *Let $M$ be a complex manifold and let $\rho \in C^\infty(M; \mathbb{R})$ be strictly plurisubharmonic function. Then*

$$\omega = \frac{i}{2}\partial\bar{\partial}\rho$$

*is Kähler form.*

In this case $\rho$ is said to be Kähler potential.

**Example.** Let $M = \mathbb{C}^n \cong \mathbb{R}^{2n}$ with coordinates $(z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n)$. Let

$$\rho(z_1, \dots, z_n) = \sum_{j=1}^{n} |z_j|^2 = \sum_{j=1}^{n} z_j \bar{z}_j$$

Then

$$\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} = \delta_{jk} \Leftrightarrow \left( \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) \right) = \mathrm{Id}$$

i.e. $\rho$ is strictly plurisubharmonic. The corresponding Kähler form

$$\omega = \frac{i}{2}\partial\bar{\partial}\rho = \frac{i}{2}\sum_{j,k} \delta_{jk} dz_j \wedge d\bar{z}_k = \sum_{j=1}^{n} dx_j \wedge dy_j$$

is the standard form.

Kähler potential is the main object under investigation in this work.

## 5. Symplectic toric varieties

### 5.1. **Moment map.**

**Definition.** Let $(M, \omega)$ be a symplectic manifold. We say that there is a symplectic action of Lie group $G$ on $(M, \omega)$ if there exists a homomorhphism

$$\psi \colon G \to \mathrm{Symp}(M, \omega) \subset \mathrm{Diff}(M)$$

**Definition.** Let $\mathfrak{g}$ be a Lie algebra of Lie group $G$. The action $\psi$ is a hamiltonian action on symplectic manifold $(M^{2n}, \omega)$ if there exists a map

$$\mu \colon M \to \mathfrak{g}^*$$

such that

(1) For each $X \in \mathfrak{g}$ let
- $\mu^X \colon M \to \mathbb{R}$, $\mu^X(p) := \langle \mu(p), X \rangle$
- $X^\#$ be a vector field on $M$ generated by $\{\exp tX | t \in \mathbb{R}\} \subset G$

Then

$$d\mu^X = \iota_{X^\#}\omega$$

(2) For all $g \in G$

$$\mu \circ \psi_g = \mathrm{Ad}_g^* \circ \mu$$

The map $\mu$ is called moment map for a hamiltonian $G$-space $(M, \omega, G, \mu)$.

We will be interested only in the case when $G = \mathbb{T}^n$, i.e. our Lie group is a torus of half dimension of symplectic manifold. Lie algebra of $\mathbb{T}^n$ is $\mathbb{R}^n$ with trivial Lie bracket which makes the definition of the moment map much simpler.

The Atiyah-Guillemin-Sternberg theorem tells that for $G = \mathbb{T}^m$ the image $\mu(M)$ of the moment map is a convex polytope which is called moment polytope.

Even apart from this observation we know that the case $G = \mathbb{T}^n$ is particularly interesting since toric topology arises frequently in mathematics. This leads us to the following definition.

**Definition.** A $2n$-dimensional symplectic toric manifold is a compact connected symplectic manifold $(M^{2n}, \omega)$ equipped with an effective hamiltonian action of an $n$-torus $\mathbb{T}^n$ and with a corresponding moment map $\mu : M \to \mathbb{R}^n$.

It turns out that for the symplectic toric manifolds the moment polytope is of very special type.

5.2. **Delzant polytopes.**

**Definition.** A Delzant polytope $P \subset \mathbb{R}^n$ is a convex polytope satisfying:

- it is simple, i.e. there are $n$ edges meeting at each vertex;
- it is rational, i.e., the edges meeting at the vertex $p$ are rational in the sense that each edge is of the form $p + tu_i$ , $t \geq 0$, where $u_i \in \mathbb{Z}^n$;
- t is smooth, i.e., for each vertex, the corresponding $u_1, \ldots, u_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

There is remarkable theorem of Delzant.

**Theorem 1** (Delzant[3])**.** *Symplectic toric manifolds are classified by Delzant polytopes. More specifically, there is the following one-to-one correspondence*

$$\{\textit{Symplectic toric manifolds}\} \longleftrightarrow \{\textit{Delzant polytopes}\}$$
$$(M^{2n}, \omega, \mathbb{T}^n, \mu) \longmapsto \mu(M)$$

## 6. Kähler potential for toric varieties

It turns out that symplectic toric varieties apart from symplectic structure also posses compatible complex structure and then they are Kähler manifolds. Moreover, the Kähler potential can be written explicitly in terms of Delzant polytope corresponding to this manifold. Let us make it more precise.

6.1. **Guillemin's theorem.** A Delzant polytope $P$ can be described by a set of inequalities of the form $\langle x, v_r \rangle \geq \lambda_r, r = 1, \ldots, d$, where $d$ is the number of faces of Delzant polytope $P$, each $v_r$ being a primitive element of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ and inward-pointing normal to the $r$-th $(n-1)$-dimensional face of $P$. Consider the affine functions $\ell_r \colon \mathbb{R}^n \to \mathbb{R}, r = 1, ..., d$, defined by

$$\ell_r(x) = \langle x, v_r \rangle - \lambda_r$$

Then $x \in \mathring{P}$ if and only if $\ell_r(x) > 0$ for all $r$ and hence the function

$$g_P(x) = \frac{1}{2}\sum_{j=1}^{d} \ell_r(x)\log(\ell_r(x)) \quad (\heartsuit)$$

is smooth on $\mathring{P}$.

**Theorem 2** (Guillemin[2])**.** *The "canonical" compatible complex structure on toric symplectic manifold* $(M^{2n}, \omega)$ *is given in symplectic coordinates* $(x, y)$ *of* $\mathring{M} \cong \mathring{P} \times \mathbb{T}^n$ *by*

$$J_P = \begin{pmatrix} 0 & -Hess(g_P)^{-1} \\ Hess(g_P) & 0 \end{pmatrix}$$

## 7. Problem statement

In this work we consider the ellipsoid

$$E(1, b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + \frac{|z_2|^2}{b^2} < \frac{1}{\pi} \right\}$$

with standard symplectic structure given by $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ and standard torus action:

$$(t_1, t_2) \cdot (z_1, z_2) = (t_1 z_1, t_2 z_2)$$

In this case $P$ is just right triangle with legs 1 and $b \in \mathbb{Q}$. While symplectic structure is standard the complex structure is tricky and it can given in various ways by Kähler potential.

In the previous section we have seen the explicit formula for Kähler potential for toric symplectic varieties. Let us note that the function

$$\begin{aligned} \Phi \colon \mathring{P} &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \nabla_x g_P \end{aligned}$$

establishes a diffeomorphism between $\mathring{P}$ and $\mathbb{R}^2$. Moreover, regarded as a diffeomorphism between $\mathring{P} \times \mathbb{T}^2$ with symplectic structure and $\mathbb{R}^2 \times \mathbb{T}^2$ with complex structure it respects both of them. It easy to see that the matrix $d\Phi$ is self-adjoint.

**Problem.** Given a diffeomorhism $\Phi$ between symplectic domain and complex domain such that it is equivariant with respect to torus action. Then show that $d\Phi$ is self-adjoint.

It turned out that the solution of the problem is written in the work of Miguel Abreu ([2]). Namely, he proves the following theorem.

**Theorem 3.** *Let* $(M_P, \omega_P, \mu_P)$ *be the toric symplectic manifold associated to a Delzant polytope* $P \subset \mathbb{R}^n$*, and* $J$ *any compatible toric complex structure. Then* $J$ *is determined by a potential* $g \in C^\infty(\mathring{P})$ *of the form*

$$g = g_P + h$$

*where* $g_P$ *is given by* $(\heartsuit)$*,* $h$ *is smooth on the whole* $P$*, and the matrix* $G = Hess_x(g)$ *is positive definite on* $\mathring{P}$*.*

*Proof.* • At first let us understand that $(M_P, J_P, \mu_P)$ is equivariantly biholomorphic to $(M_P, J, \mu_P)$ for any $\omega_P$-compatible almost complex structure $J$.

For every point $p \in M_P$ there is an open invariant affine neighbourhood $U_p$ which is equvariantly biholomorphic to $\mathbb{C}^n/2\pi i\mathbb{Z}^k$, where $p$ belongs to $k$-face for $0 \leq k \leq n$. These neigbourhoods exist due to the fact that there are $2n$ holomorphic vector fields $(\eta_1, \ldots, \eta_n, J_P\eta_1, \ldots, J_P\eta_n)$ on $(M_P, J_P)$ and $(\eta_1, \ldots, \eta_n, J\eta_1, \ldots, J\eta_n)$ on $(M_P, J)$, where $\eta_k$ are induced by the Hamiltonian torus action. Integrability of $J_P$ and $J$ and $\mathbb{T}^n$-invarince implies that these vector fields commute with each other. The $\omega_P$-compatibility condition gives their linear independence on the interior of $M_P$.

Then for inner points there is a equivariant biholomorphism to $\mathbb{C}^n/2\pi i\mathbb{Z}^n$. The case when $p$ belongs to some $k$-face is completely determined by the combinatorics of $P$.

Once we understood how the neighbourhood for every point looks like we can patch them together. The patching is also determined by combinatorial structure of $P$. The gluing between the neighbourhood of $p$ laying in $k$-face for which the corresponding inward pointing normal vectors form a part of a standard basis and the neighbourhood of inner point is given by

$$\begin{aligned} \mathbb{C}^n/2\pi i\mathbb{Z}^n &\longrightarrow \mathbb{C}^n/2\pi i\mathbb{Z}^k \\ (z_1, \ldots, z_n) &\longmapsto (z_1, \ldots, z_k, e^{z_{k+1}}, \ldots, e^{z_n}) \end{aligned}$$

• Now let $\phi\colon (M_P, J_P) \to (M_P, J)$ be such equivariant biholomorhpism. From the construction above it is clear that $\phi$ can be chosen to be the identity in cohomology. Thus $\omega_J = \phi_J^*\omega_P$ and $[\omega_P] = [\omega_J] \in H^2(M_P)$. Then the basics of Hodge theory says that there exists $\mathbb{T}^n$-invariant function $f_J \in C^\infty(M_P)$ such that

$$\omega_J = \omega_P + 2i\partial\bar{\partial} f_J,$$

where the $\partial$- and $\bar{\partial}$-operators are defined with respect to $J_P$.

Explicit calculation and Theorem 2 gives

$$\omega_J = \sum_{j,k,l=1}^{n} \frac{\partial}{\partial x_j}\left((g_P)^{kl}\frac{\partial(f_P + f_J)}{\partial x_l}\right) dx_j \wedge dy_k =: 2i\partial\bar{\partial} f_{sum} \quad (\diamondsuit)$$

where $g_P$ is given by $(\heartsuit)$ and $f_{sum} = f_J + f_P$ and $f_P$ is Legendre transform of $g_P$:

$$f_P(x) = \frac{1}{2}\sum_{r=1}^{d} (\lambda_r \log \ell_r(x) + \langle x, v_r\rangle)$$

where $v_r$ and $\lambda_r$ are defined in 6.1.

• From $(\diamondsuit)$ it is easy to understand that the transformation

$$\tilde{x} = \tilde{\phi}(x) = x + G_P^{-1} \cdot \frac{\partial f_J}{\partial x}$$

gives a change of coordinates $\tilde{\phi}\colon P \to P$ corresponding to $\phi\colon M_P \to M_P$. The compatibility condition says that $\omega_J(\cdot, J_P\cdot)$ is a Riemannian metric which implies that $(d\tilde{\phi})G_P^{-1}$ is symmetric and positive defined on the interior of $P$ and $\det(d\tilde{\phi}) > 0$ on $P$. It is now immediate that for $p$ belonging to the $r$-th $(n-1)$-face of $P$

$$\langle d_p\tilde{\phi}(v_r), v_r\rangle > 0 \quad (\clubsuit)$$

• Legendre duality between the potentials corresponding to symplectic and complex structures can be written as

$$f(\tilde{x}) + g(\tilde{x}) = \sum_{k=1}^{n} \tilde{x}_k \frac{\partial g}{\partial \tilde{x}_k}(\tilde{x})$$

Knowing that $f(\tilde{x}) = f_{sum}(x)$ and $J = \tilde{\phi}_* J_P$ we can rewrite it as

$$g(\tilde{x}) = \sum_{k=1}^{n} \tilde{x}_k \left(\frac{\partial g_P}{\partial x_k} \circ \tilde{\phi}^{-1}\right)(\tilde{x}) - (f_P \circ \phi^{-1})(\tilde{x}) - (f_J \circ \phi^{-1})(\tilde{x}) \quad (\spadesuit)$$

• The formula above will help us to understand that $h = g - g_P$ is smooth on the whole $P$ which is equivalent to the smoothness of $h \circ \tilde{\phi}$. It is not hard to compute that

$$h(\tilde{\phi}(x)) = \frac{1}{2}\sum_{r=1}^{d} \left( \langle \tilde{\phi}(x) - x, v_r\rangle + \ell_r(\tilde{\phi}(x)) \log\left(\frac{\ell_r(x)}{\ell_r(\tilde{\phi}(x))}\right)\right) - f_J(x)$$

So we only need to prove that $\frac{\ell_r(x)}{\ell_r(\tilde{\phi}(x))}$ are positive and smooth on the whole $P$. This property is evident for the interior of $P$. If $p$ belongs to the $r$-th $(n-1)$-face then by $(\clubsuit)$ we have

$$d_p(\ell_r \circ \tilde{\phi})(v_r) = d_p\ell_r(d_p\tilde{\phi}(v_r)) = \langle d_p\tilde{\phi}(v_r), v_r\rangle > 0$$

The map $\tilde{\phi}$ preserves the faces of $P$ and then $\ell_r(\tilde{\phi})(x) = \ell_r(x) \cdot s_r(x)$, where $s_r$ is some smooth function. Simple computation shows that $s_r$ is positive and then

$$\frac{\ell_r(x)}{\ell_r(\tilde{\phi}(x))} = \frac{1}{s_r(x)}$$

smooth and positive as desired. □

7.1. **Further plan.** Since for a given toric symplectic manifold we know everything about its compatible complex structures, we can study subobjects of these to structures: Lagrangian submanifolds and pseudo-holomorphic curves.

The idea is the following. In Delzant polytope we consider some set (namely a tropical curve) and then we want to lift in our manifold in two ways: one way to a Lagrangian submanifold and the other to a pseudo-holomorhic curve.

There is a hope that it will help us to develop the new method for studding toric symplectic manifolds.

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