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Densities of lattices of translates

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Abstract

Let K be a convex body in $\mathbb{R}^n$. Let $d_{n,n-1}(K)$ be the smallest possible density of a non-separable lattice of translates of K. In this paper we prove the estimate $d_{2,1}(K) \le \frac{\pi\sqrt{3}}{8}$ for $K \subset \mathbb{R}^2$, with equality if and only if K is an ellipse, which was conjectured by E. Makai. Also we prove the estimate $d_{3,2}(K) \le \frac{\pi}{4\sqrt{3}}$ for $K \subset \mathbb{R}^3$ using projection bodies.

In the last section we show an easy way to improve E.H. Smith's packing density bound in $\mathbb{R}^3$ from 0.53835... to 0.54755... .

Organisation of the paper

The paper is organised in the following way. In the first section, we recall the main definitions; in the second section, we review the literature and formulate our results; in the third section we list some properties of the support function. The remaining sections are devoted to proving the results. The fourth and fifth sections contain the proof of the Makai conjecture. The sixth section refers to its three-dimensional case. The last section refers to the Ulam's conjecture and improving E.H. Smith's estimate.

Contents

1 Preliminaries

A set $K \subset \mathbb{R}^n$ is a convex body if it is convex, compact, and its interior $\mathrm{int}K$ is non-empty. We denote the volume of K by $|K|$. The difference body of K is defined as $K - K := \{x - y | x, y \in K\}$. For any $\lambda \in \mathbb{R}$ we define $\lambda K := \{\lambda x | x \in K\}$. K is centrally symmetric if $K = -K$.

Notation 1.1. *Let $K, L \subset \mathbb{R}^2$ be two planar convex bodies. We denote their mixed volume by (K, L).*

Definition 1.1. *If $0 \in \mathrm{int}K$ then the polar body of K is*

$$K^\circ := \{x \in \mathbb{R}^n | \forall y \in K \ \langle y, x\rangle \leq 1\}.$$

Definition 1.2. *If $0 \in \mathrm{int}K$ and K is centrally symmetric then the Mahler volume of K is $|K||K^\circ|$.*

Proposition 1.1 (Blaschke-Santaló inequality)**.** *For a centrally symmetric convex body $K \subset \mathbb{R}^n$ we have*

$$|K||K^\circ| \leq |B|^2, \text{ where } B \text{ is the unit ball in } \mathbb{R}^n.$$

Definition 1.3. *For a convex body K define the projection body of K by its support function:*

$$h_{\Pi K}(u) = |Pr_{u^\perp}K| \text{ for all } u \in S^{n-1}.$$

It is well-known that the definition is correct and it defines a convex body ΠK.

Proposition 1.2 (Petty's inequality)**.** *Let B be the unit ball in $\mathbb{R}^n$. Then for a convex body K we have:*

$$|(\Pi K)^\circ||K|^{n-1} \leq |(\Pi B)^\circ||B|^{n-1}.$$

We will also need some definitions from the geometry of numbers.

Definition 1.4. *$\Lambda \subset \mathbb{R}^n$ is a lattice if $\Lambda = A\mathbb{Z}^n$ for some $A \in GL(n, \mathbb{R})$. We also define $d(\Lambda) := |detA|$.*

Definition 1.5. *A lattice Λ is called K-admissible if $\Lambda \cap \mathrm{int}K = \{0\}$.*

Definition 1.6. *For a convex body K its critical determinant is defined as*

$$\Delta(K) := \min\{d(\Lambda) | \Lambda \text{ is } K\text{-admissible}\}$$

Definition 1.7. *A lattice Λ is called K-critical if Λ is K-admissible and $d(\Lambda) = \Delta(K)$.*

Definition 1.8. *Let Λ be a lattice, then a lattice of translates of K is defined as*

$$\Lambda + K = \{x + y | x \in \Lambda, y \in K\}.$$

Definition 1.9. *Let $\Lambda + K$ be a lattice of translates of K. Then its density is defined as*

$$D(\Lambda, K) := \frac{|K|}{d(\Lambda)}.$$

Definition 1.10. *We denote by $\delta_L(K)$ the density of the densest lattice packing in $\mathbb{R}^n$ by translates of K. It is easy to see that for centrally symmetric K we have:*

$$\delta_L(K) = \frac{|K|}{2^n \Delta(K)}.$$

Definition 1.11. *A lattice of translates of K is non-separable if each affine $(n-1)$-subspace in $\mathbb{R}^n$ meets $x + K$ for some $x \in \Lambda$.*

Definition 1.12. *For a convex body K its *critical determinant is defined as*

$$\Delta^*(K) := \max\{d(\Lambda) | \Lambda + K \text{ is non-separable}\}$$

Definition 1.13. *A lattice Λ is called K-*critical if $\Lambda + K$ is non-separable and $d(\Lambda) = \Delta^*(K)$.*

Definition 1.14. $d_{n,n-1}(K) = \min\{d(\Lambda, K) | \Lambda + K \text{ is non-separable}\} = \frac{|K|}{\Delta^*(K)}$

Definition 1.15. *Lattice width $\omega_K : \mathbb{Z}^n \setminus 0 \to \mathbb{R}$ is defined as*

$$\omega_K(u) = \max_{x,y \in K} \langle u, x - y \rangle.$$

2 Introduction

The Reinhardt Conjecture is an open long-standing problem about finding a centrally symmetric body $K \subset \mathbb{R}^2$ with the smallest possible value of $\delta_L(K)$. Reinhardt conjectured that the unique solution up to an affine transformation is the smoothed octagon (an octagon rounded at corners by arcs of hyperbolas) and the conjectured minimum of $\delta_L(K)$ is $\frac{8-\sqrt{32}-\ln 2}{\sqrt{8}-1} \approx 0.902414$, while the best known estimation is $\delta_L(K) \geq 0.8926...$ by Tammela [6]. Comprehensive information about this conjecture can be found in [5].

It is well known that the dual problem to the Reinhardt conjecture is the question about upper bounds for $d_{2,1}(K)$. This problem was considered by Endre Makai Jr. in [1], [2]. He conjectured that $d_{2,1}(K) \leq \frac{\pi\sqrt{3}}{8}$ with equality only for ellipses. In his works Endre Makai Jr. obtained the following dual property and nearly accurate estimate.

Example 2.1. *For the unit ball B in $\mathbb{R}^2$ we have $d_{2,1}(B) = \frac{\pi\sqrt{3}}{8}$ and $\delta_L(B) = \frac{\pi}{2\sqrt{3}} \approx 0.90689$.*

Example 2.2. *For a triangle T we have $d_{2,1}(T) = \frac{3}{8}$. Let $T = (0,0)(1,1/2)(1/2,1)$. Then $\mathbb{Z}^2$ is the *critical lattice.*

The following property was introduced by Ende Makai to show the connection between non-separable lattices concept and the Mahler volume:

Proposition 2.1 (Duality condition, [2]). *For a convex body $K \subset \mathbb{R}^n$ we have:*

$$d_{n,n-1}(K) = \frac{|K||((K-K)/2)^\circ|}{4^n \delta_L(((K-K)/2)^\circ)}.$$

Proposition 2.2. *[1] For a convex body $K \subset \mathbb{R}^2$ we have $d_{2,1}(K) \leq 0.6910\ldots$*

Conjecture 2.1 (Endre Makai Jr., [1]). $\max\{d_{2,1}(K) | K \subset \mathbb{R}^2$ *is a convex body*$\} = \frac{\pi\sqrt{3}}{8} = 0.68017\ldots$*, with equality possible only for ellipses.*

From Proposition 2.1 we get the idea of using estimation of $\delta_L(K)$ to get upper bounds for $d_{2,1}(K)$. But unfortunately, accurate estimates in $\mathbb{R}^2$ cannot be obtained in that way and the main reason is that the optimal body for δ_L optimisation is not an ellipse. Nevertheless, the estimate obtained in [1] is very close to the conjectured value.

This paper will provide a proof of Conjecture 2.1.

Theorem 2.1.

$$\max\{d_{2,1}(K) | K \subset \mathbb{R}^2 \text{ is a convex body}\} = \frac{\pi\sqrt{3}}{8} = 0.68017\ldots.$$

In addition, if $d_{2,1}(K) = \frac{\pi\sqrt{3}}{8}$, then K is an ellipse.

Corollary 2.1. *For each centrally symmetric convex body $K \subset \mathbb{R}^2$ we have*

$$\delta_L(K) \geq \frac{1}{2\pi\sqrt{3}}|K||K^\circ|,$$

with equality if and only if K is an ellipse.

The Ulam's packing conjecture states that for a centrally symmetric convex body $K \subset \mathbb{R}^3$ we have $\delta_L(K) \geq \frac{\pi}{\sqrt{18}}$. If this statement holds, then by using duality we could obtain the sharp estimate $d_{3,2}(K) \leq \frac{\pi}{6\sqrt{2}}$, with equality only for ellipsoids. Nevertheless, the best known estimate for a packing constant in $\mathbb{R}^3$ is $\delta_L(K) \geq 0.53835\ldots$ by E.H. Smith [7]. Therefore we can use duality to get the estimate:

$$d_{3,2}(K) = \frac{|K||K^\circ|}{64\delta_L(K^\circ)} \leq \frac{(\frac{4\pi}{3})^2}{64 \cdot 0.53835\ldots} = 0.509251\ldots$$

This paper provides an idea of using projection bodies to improve $d_{3,2}$ estimates without using known bounds for δ_L. We prove the following theorem.

Theorem 2.2.

$$\max\{d_{3,2}(K) | K \subset \mathbb{R}^3 \text{ is a convex body}\} \leq \frac{\pi}{4\sqrt{3}} = 0.453449\ldots.$$

Example 2.3. *For the unit ball $B \subset \mathbb{R}^3$ we have $d_{3,2}(B) = \frac{\pi}{6\sqrt{2}} = 0.37024\ldots.$*

In the last section we prove the following theorem.

Theorem 2.3. *For a centrally symmetric convex body $K \subset \mathbb{R}^3$ we have*

$$\delta_L(K) \geq 0.54755\ldots$$

3 Calculations

Notation 3.1. *In this section we assume that $K \subset \mathbb{R}^2$ is strongly convex and has C^∞ boundary.*

Notation 3.2. *The support function of K $h : S^1 \to \mathbb{R}$ is defined as $h(\theta) = \max_{x \in K} \langle (\cos\theta, \sin\theta), x \rangle$.*

Notation 3.3. *Let $a = (a_1, a_2), b = (b_1, b_2)$. Then we denote*

$$det(a, b) := det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

The following statements are well-known.

Proposition 3.1. *Parameterization in terms of the support function: consider*

$$\gamma(x) = \begin{pmatrix} cos(x) & -sin(x) \\ sin(x) & cos(x) \end{pmatrix} \begin{pmatrix} h(x) \\ h'(x) \end{pmatrix}.$$

Then

- $\gamma(x) \in K$
- $\langle (\cos x, \sin x), \gamma(x) \rangle = \max_{y \in K} \langle (\cos x, \sin x), y \rangle$
- $\gamma'(x) = (h(x) + h''(x))(-sin(x), cos(x))$
- $det(\gamma(x), \gamma'(x)) = h(x)(h(x) + h''(x))$
- $det(\gamma'(x), \gamma''(x)) = (h(x) + h''(x))^2$

Proposition 3.2. *We list some properties of the support function:*

- $h + h'' \geq 0$
- $|K| = \frac{1}{2} \int_0^{2\pi} h^2 + hh'' = \frac{1}{2} \int_0^{2\pi} h^2 - h'^2$
- $|K^\circ| = \frac{1}{2} \int_0^{2\pi} \frac{1}{h^2}$

Proposition 3.3. *Let $\Gamma(t) = \gamma(t) + l(t)\frac{\gamma'(t)}{|\gamma'(t)|}$, where $\gamma : \mathbb{R} \to \mathbb{R}^2$ and $l : \mathbb{R} \to \mathbb{R}$ are smooth periodic functions with period 2π. We also assume that $|\gamma'| > 0$. Then*

$$\frac{1}{2} \int_0^{2\pi} det(\Gamma, \Gamma') - \frac{1}{2} \int_0^{2\pi} det(\gamma, \gamma') = \frac{1}{2} \int_0^{2\pi} \frac{l^2}{|\gamma'|^2} det(\gamma', \gamma'').$$

Proof.

$$\int_0^{2\pi} det(\Gamma,\Gamma') = \int_0^{2\pi} det(\gamma(t)+l(t)\frac{\gamma'(t)}{|\gamma'(t)|}, \gamma'(t)+\left(l(t)\frac{\gamma'(t)}{|\gamma'(t)|}\right)')$$

$$= \int_0^{2\pi} det(\gamma,\gamma') + \int_0^{2\pi} det(\gamma, \left(l(t)\frac{\gamma'(t)}{|\gamma'(t)|}\right)') + \int_0^{2\pi} det(l(t)\frac{\gamma'(t)}{|\gamma'(t)|}, \left(l(t)\frac{\gamma'(t)}{|\gamma'(t)|}\right)')$$

$$= \int_0^{2\pi} det(\gamma,\gamma') - \int_0^{2\pi} det(\gamma', l(t)\frac{\gamma'(t)}{|\gamma'(t)|}) + \int_0^{2\pi} \frac{l^2}{|\gamma'|^2} det(\gamma',\gamma'')$$

$$= \int_0^{2\pi} det(\gamma,\gamma') + \int_0^{2\pi} \frac{l^2}{|\gamma'|^2} det(\gamma',\gamma'')$$

□

4 Proof of the Theorem 2.1 (inequality part)

Lemma 4.1. *[2]*

$$\max\{d_{2,1}(K) | K \subset \mathbb{R}^2 \textit{ is a convex body}\} = \max\{d_{2,1}(K) | K \subset \mathbb{R}^2 \textit{is a centrally symmetric convex body}\}$$

Therefore, we can assume that K is centrally-symmetric.

Notation 4.1. *A triangle is a central triangle if its barycenter is in the origin.*

Lemma 4.2. *Let K be a centrally symmetric convex body and suppose that there exists a central triangle T° whose vertices belong to ∂K°, such that $|K||T^\circ| \leq \frac{3\sqrt{3}}{4}\pi$. Then $d_{2,1}(K) \leq \frac{\sqrt{3}\pi}{8}$.*

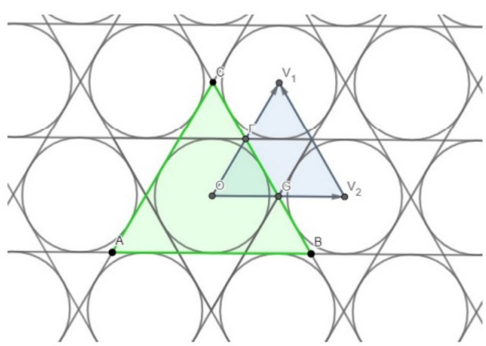

Figure 1: Non-separable lattice for a convex body K.

Proof. $(T^\circ)^\circ = T = ABC$ is a central triangle circumscribed to K. $|T||T^\circ| = \frac{27}{4}$, hence $\frac{|K|}{|T|} \leq \frac{\pi}{3\sqrt{3}}$. Further, we can generate a non-separable lattice Λ with $d(\Lambda) = \frac{8}{9}|T|$ by $V_1 = \frac{2}{3}(C-A)$ and $V_2 = \frac{2}{3}(B-A)$ as in Figure 1.

□

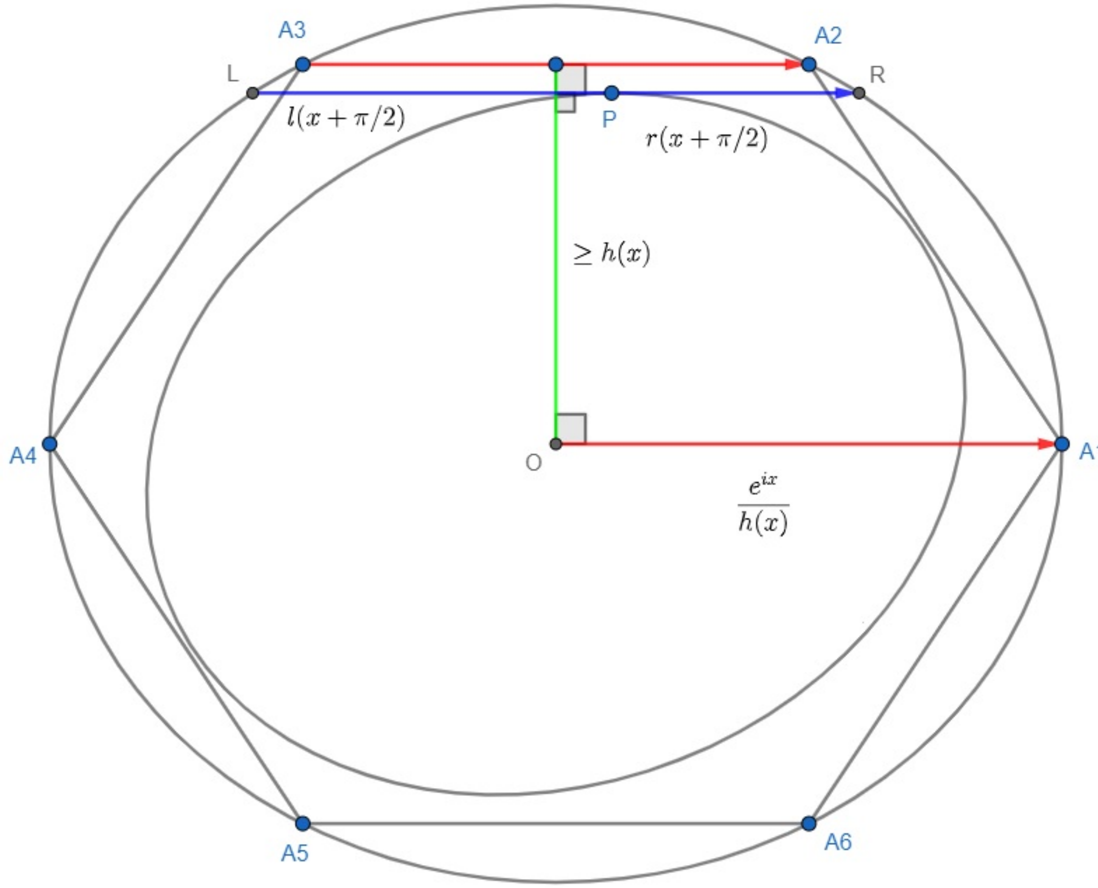

Figure 2: The outer body is K°, the inner body is iK; the point P is $\gamma(x+\pi/2)$, LR is the tangent line to iK at the point P, hence we have $|LP| = l(x+\pi/2)$, $|PR| = r(x+\pi/2)$.

Lemma 4.3. *Assume that K is centrally symmetric, strictly convex and C^∞ boundary and let the minimal area of a central triangle inscribed in K° be equal to $\frac{3}{2}$. Then we have $\frac{3}{4}|K^\circ| \geq |K|$.*

Proof. For each central triangle T inscribed in K° consider an inscribed affine regular hexagon $conv(T \cup -T)$ and let $A_1..A_6$ be one of them. We put $A_1 = \frac{e^{ix}}{h(x)} = A_2 - A_3$.

From the minimum area condition we have $|A_2A_3O| \geq \frac{1}{2}$, hence the distance $d(O, A_2A_3) \geq \frac{1}{|A_2A_3|} = h(x)$. Further, let A be the intersection of all inscribed in K° affine regular hexagons. Then we have $iK \subset A \subset K^\circ$, where iK is $\frac{\pi}{2}$-rotated K.

Let γ parametrize iK in terms of the support function and let $L(x)$ be the tangent line to iK at the point $\gamma(x)$. Further, define $l(x)$ and $r(x)$ as lengths of left and right parts of $|L(x) \cap K^\circ|$ with respect to $\gamma(x)$. Since $iK \subset A$ and since K° is centrally symmetric we have $r(x) + l(x) \geq \frac{1}{h(x-\pi/2)}$. Then using Proposition 3.3 we get:

$$|K^\circ| - |K| = |K^\circ| - |iK| = \frac{1}{2}\int_0^{2\pi} l^2 = \frac{1}{2}\int_0^{2\pi} r^2.$$

Therefore:

$$2(|K^\circ|-|K|)=\frac{1}{2}\int_0^{2\pi}(l^2+r^2)\geq\frac{1}{4}\int_0^{2\pi}(l+r)^2\geq\frac{1}{4}\int_0^{2\pi}\frac{1}{h(x-\pi/2)^2}=\frac{1}{2}|K^\circ|.$$

Thus we have $\frac{3}{4}|K^\circ|\geq|K|$. ☐

Lemma 4.4. *Let $K\subset\mathbb{R}^2$ be a centrally symmetric convex body and let the minimal area of a central triangle inscribed in K° be equal to $\frac{3}{2}$. Then we have $|K|\leq\frac{\sqrt{3}\pi}{2}$, with equality if and only if K is an ellipse.*

Proof. Consider the approximation of K by smooth and strongly convex centrally symmetric bodies $K_\epsilon\to K$ and let the minimal area of a central triangle inscribed in K_ϵ be equal to S_ϵ. Then it is obvious that $S_\epsilon\to\frac{3}{2}$. Further, $\sqrt{\frac{2}{3}S_\epsilon}K_\epsilon$ satisfies the condition of Lemma 4.3, hence we have $\frac{3}{4}|(\sqrt{\frac{2}{3}S_\epsilon}K_\epsilon)^\circ|\geq|\sqrt{\frac{2}{3}S_\epsilon}K_\epsilon|$. Since the volume is continuous, we have $\frac{3}{4}|K^\circ|\geq|K|$. Finally, using the Blaschke-Santaló inequality we get $\frac{3}{4}\pi^2\geq\frac{3}{4}|K^\circ||K|\geq|K|^2$, with equality only for an ellipse. ☐

Proof of Theorem 2.1 (inequality part). Obviously follows from Lemma 4.1, Lemma 4.2 and Lemma 4.4. ☐

5 The equality case

The following Lemmas are well-known.

Lemma 5.1. *[4] $K+\mathbb{Z}^n$ is non-separable if and only if $\omega_K(u)\geq 1$ for all $u\in\mathbb{Z}^n\setminus 0$.*

Lemma 5.2. *Let K be a centrally-symmetric convex body. Then $\mathbb{Z}^n+K$ is non-separable if and only if $\mathbb{Z}^n$ is $\frac{1}{2}K^\circ$-admissible.*

Proof. Obviously follows from Lemma 5.1. ☐

Lemma 5.3. *$\mathbb{Z}^n$ is *critical for K if and only if $\mathbb{Z}^n$ is critical for $\frac{1}{2}K^\circ$.*

Proof. Let $\Lambda=A\mathbb{Z}^2$ be a $\frac{1}{2}K^\circ$-admissible lattice with $d(\Lambda)<1$, that is, $|detA|<1$. Therefore $\mathbb{Z}^2$ is $\frac{1}{2}A^{-1}K^\circ$-admissible, hence we have that $(A^*)^{-1}\mathbb{Z}^2+K$ is non-separable, but $det(A^*)^{-1}>1$, a contradiction.

Proof in the opposite direction is similar. ☐

Lemma 5.4. *If $K\neq -K$, than $d_{n,n-1}(K)<d_{n,n-1}(K-K)$.*

Proof. From The Brunn–Minkowski inequality we have the estimation $|K|<|(K-K)/2|$. Also it is obvious that $d_{2,1}(K-K)=d_{2,1}((K-K)/2)$ and that $\omega_K\equiv\omega_{(K-K)/2}$. Therefore from Lemma 5.1 it follows that for any lattice Λ the lattice of translates $K+\Lambda$ is non-separable if and only if $(K-K)/2+\Lambda$ is non-separable. ☐

Lemma 5.5. *[3] Let Λ be K-critical for a centrally-symmetric $K \subset \mathbb{R}^2$, and let C be the boundary of K. Then one can find three pairs of points $\pm p_1, \pm p_2, \pm p_3$ of the lattice on C. Moreover these three points can be chosen such that $p_1 + p_2 = p_3$ and any two vectors among p_1, p_2, p_3 form a basis of Λ.*

Conversely, if p_1, p_2, p_3 satisfying $p_1 + p_2 = p_3$ are on C, then the lattice generated by p_1 and p_2 is K-admissible

Proof of Theorem 2.1 (the equality case). Suppose that $d_{2,1}(K) = \frac{\pi\sqrt{3}}{8}$. It follows from Lemma 5.4 that K has to be a centrally symmetric body. Let $\mathbb{Z}^2$ be a critical lattice for K°, so $|K| = \frac{\sqrt{3}\pi}{2}$. Then, from Lemma 5.5 we conclude that the minimal area of a central triangle inscribed in K° is equal to $\frac{3}{2}$. Hence there is an equality in Lemma 4.4. Thus K is an ellipse. □

6 Proof of the Theorem 2.2

Proof of Theorem 2.2. Since $d_{3,2}(K) \le d_{3,2}(K-K)$, it suffices to consider the case of a centrally symmetric K. We also assume that K is strongly convex and has C^∞ boundary.

Let us construct a lattice packing for K°. For $h \in S^2$ let the critical lattice for $K^\circ \cap h^\perp$ corresponds to an affine regular hexagon $A_1..A_6$ inscribed in $K^\circ \cap h^\perp$. The horizontal part of of the lattice to be constructed will be generated by the vectors $2A_1$ and $2A_2$. Denote

$$\Lambda_{h^\perp} := \{2A_1 m + 2A_2 n | n, m \in \mathbb{Z}\}.$$

It is easy to see that for any $w \in \Lambda_{h^\perp}$ we have $\operatorname{int}(K^\circ) \cap \operatorname{int}(K^\circ + w) = \emptyset$. Further, define the third generating vector $v := h \cdot 2\max_{x \in K^\circ}\langle x, h\rangle$. Thus we constructed the lattice $\Lambda := \{vn + \Lambda_{h^\perp} | n \in \mathbb{Z}\}$ and $K^\circ + \Lambda$ is obviously a lattice packing of K°.

Let $\frac{|K^\circ|}{\delta_L(K^\circ)} = 8\frac{\sqrt{3}\pi}{2}$. Then $d(\Lambda) \ge 8\frac{\sqrt{3}\pi}{2}$, thus we have

$$\frac{1}{8}d(\Lambda) = \frac{1}{4}d(\Lambda_{h^\perp})\frac{1}{2}|v| = \Delta(K^\circ \cap h^\perp)d_h \ge \frac{\sqrt{3}\pi}{2}, \text{ where } d_h := \frac{1}{2}|v|.$$

It is well-known that for $h \in S^2$ we have $K^\circ \cap h^\perp = (Pr_{h^\perp}K)^\circ$. Therefore by Lemma 4.4 we get $\Delta(K^\circ \cap h^\perp)|Pr_{h^\perp}K| \le \frac{\sqrt{3}\pi}{2}$, so $d_h \ge |Pr_{h^\perp}K|$. Therefore $\Pi K \subset K^\circ$ and $K \subset \Pi^\circ K$. Then by using Petty's inequality we obtain the estimate:

$$|K|^3 \le |\Pi^\circ K||K|^2 \le \left(\frac{4}{3}\right)^3.$$

Thus

$$d_{3,2}(K) = \frac{|K||K^\circ|}{64\delta_L(K^\circ)} \le \frac{\frac{4}{3}\frac{8\sqrt{3}\pi}{2}}{64} = \frac{\pi}{4\sqrt{3}}.$$

□

7 Proof of the Theorem 2.3

Lemma 7.1. *Let $H \subset \mathbb{R}^2$ be a convex centrally symmetric hexagon or a parallelogram. Then for a centrally symmetric convex body $C \subset \mathbb{R}^2$ we have:*

$$(C, H) \geq \frac{1}{\sqrt{\delta_L(C)}}\sqrt{|C|}\sqrt{|H|}.$$

Proof. Let $H = A_1..A_6$ and let $H' = A'_1..A'_6$ be the centrally symmetric hexagon circumscribing C, which satisfies the condition: $A'_iA'_{i+1} \parallel A_iA_{i+1}$ for $i = 1, 2, 3$. Then it is easy to observe that $(C, H) = (H', H)$. Further, since $\delta_L(H') = 1$ and $C \subset H'$ we have $\frac{|C|}{|H'|} \leq \delta_L(C)$. Therefore

$$(C, H) = (H', H) \geq \sqrt{|H'|}\sqrt{|H|} \geq \frac{1}{\sqrt{\delta_L(C)}}\sqrt{|C|}\sqrt{|H|}.$$

□

Proof of Theorem 2.3. We use the notation and the construction from [7]. Let d be a chord in K of maximum length. $C := K \cap d^{\perp}$. Let $\Lambda \subset \mathbb{R}^2$ be a lattice such that $C + \Lambda$ is the densest lattice packing of translates of C, $det\Lambda = |P|$, $\frac{|C|}{|P|} = \delta_L(C)$. H_K is a centrally symmetric hexagon with $|H_K| \geq \frac{|P|}{4}$. We may assume that $|H_K| = \frac{|P|}{4}$. h is the distance between two layers in the lattice packing of translates of K. Then as shown in [7] we have:

$$|K| \geq \frac{|H_K|(d-h)}{3} + \frac{h}{3}(|C| + (C, H_K) + |H_K|).$$

Therefore since $\delta_L(K) \geq \frac{|K|}{h|P|}$ and $\delta_L(C) \geq 0.89265...$ we have:

$$\delta_L(K) \geq \frac{1}{12}\frac{d}{h} + \frac{1}{3}\left(\delta_L(C) + \frac{1}{2}\sqrt{\delta_L(C)}\frac{(C, H_K)}{\sqrt{|C|}\sqrt{|H_K|}}\right) \geq \frac{1}{12} + \frac{1}{3}\left(0.89265... + \frac{1}{2}\right) = 0.54755...$$

□

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