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Abstract

The present thesis is devoted to Navier-Stokes equations theory. We describe a way to study local regularity of a weak solution to the classical Navier-Stokes equations. Namely, we examine a certain modification of duality method developed by G. Seregin, which in turn leads to the obtaining the Liouville-type theorem. We show the limits of applicability of the duality method setting the right-hand side of the dual problem to zero and take non-zero initial data. In this case, we prove the results that are exactly the same as those in the paper of M. Schonbeck and G. Seregin.

Contents

1. Introduction

1.1. **Problem statement and main results.** Let X be a Banach space, $p \in [1, +\infty)$ and $t_1 \in \mathbb{R}, t_2 \in \mathbb{R} \cup \{+\infty\}$ such that $t_1 < t_2$. By $L_p(t_1, t_2; X)$ we denote the Banach space of all Banach-valued measurable functions $u : [t_1, t_2] \to X$ such that

$$\|u\|^p_{L_p(t_1,t_2;X)} = \int\limits_{t_1}^{t_2} \|u(t)\|^p_X \, dt < +\infty.$$

Also, denote by $L_\infty(t_1, t_2; X)$ the Banach space of all Banach-valued measurable functions $u : [t_1, t_2] \to X$ such that

$$\|u\|_{L_\infty(t_1,t_2;X)} = \text{ess sup}\{\|u(t)\|_X \mid t \in (t_1, t_2)\} < +\infty.$$

For simplicity of notation we omit the spatial variable of function by matching them as

$$u(\cdot, t) \mapsto u(t) \in L_p(t_1, t_2; X), \quad \text{a.e. in } (t_1, t_2), 1 \leqslant p \leqslant +\infty.$$

Let Ω be a hypercube in $\mathbb{R}^n$, and $|\Omega|$ is the volume of Ω, i.e., its Lebesgue measure. Denote by $BMO(\mathbb{R}^3)$ the Banach space of all locally integrable functions whose mean oscillation is bounded, namely, the following norm if finite:

$$\|f\|_{BMO(\mathbb{R}^3)} = \sup_{\Omega \subset \mathbb{R}^n} \frac{1}{|\Omega|} \int\limits_{\Omega} |f(x) - [f]_\Omega| \, dx < +\infty,$$

where $[f]_\Omega = \frac{1}{|\Omega|} \int_\Omega f(x) \, dx$.

One of the open problems of local regularity of weak solutions to the Navier-Stokes equations is as follows. Consider so-called suitable weak solution $w \in L_\infty(-1, 0; L_2(B_r(0))) \cap L_2(-1, 0; W^1_2(B_r(0)))$ and $g \in L_{\frac{3}{2}}(Q)$ to the classical Navier-Stokes system

$$\partial_t w - \Delta w + (w \cdot \nabla)w + \nabla g = 0, \quad \operatorname{div} w = 0$$

in the unit parabolic space-time ball $Q = [-1, 0] \times B_r(0) \subset \mathbb{R} \times \mathbb{R}^n$. Here, $B_r(x)$ stands for the ball in $\mathbb{R}^n$ of radius r centred at the point $x \in \mathbb{R}^n$. For a definition of suitable weak solutions, we refer to the paper [1]. Let us assume that function w satisfies the additional restriction:

$$|w(x, t)| \leqslant \frac{c}{|x| + \sqrt{-t}}, \qquad \forall (x, t) \in Q, \tag{1.1}$$

for some constant $c > 0$. In that case we say that w has a singularity of type I.

The question is to understand whether or not the origin $z = (x, t) = (0, 0) \in \mathbb{R}^{n+1}$ is a regular point of w, i.e., there exists $\delta > 0$ such that w is essentially bounded in the parabolic ball $Q(\delta) = [-\delta^2, 0] \times B_\delta(0)$.

In this thesis we consider only the case of a three-dimensional space $\mathbb{R}^n$, so $n = 3$. Denote $Q_+ = \mathbb{R}^3 \times (0, +\infty)$. We say that the functions u and p are a mild bounded ancient solution of the backward Navier-Stokes equations, if

$$u \in C^\infty(\bar{Q}_+) \cap L_\infty(Q_+), \quad p \in C^\infty(\bar{Q}_+) \cap L_\infty(0, +\infty; BMO(\mathbb{R}^3)),$$

and these functions u and p obey following equations

$$(\mathcal{NS}) \ : \begin{cases} -\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases} \quad \text{in } Q_+.$$

In [10], it has been shown that if the origin $z = 0$ is a singular point of w, then there exists the non-trivial mild bounded ancient solution u and p such that $|u(0, 0)| = 1$ and

$$|u| \leqslant 1 \text{ in } Q_+. \tag{1.2}$$

Since u is a smooth function let us fix such constant $M > 0$ that

$$\|u\|_{L_\infty(Q_+)} + \|\nabla u\|_{L_\infty(Q_+)} \leqslant M. \tag{1.3}$$

If w has type I singularity satisfying (1.1) then the mild bounded ancient solution u corresponding to w also satisfies the condition (1.4):

$$\exists c_* > 0 : \quad |u(x,t)| \leqslant \frac{c_*}{|x| + \sqrt{t}}, \qquad \forall (x,t) \in Q_+. \tag{1.4}$$

The duality method has been first developed and exploited by G. Seregin in [7]. This method allows to prove Liouville type theorems not only for scalar equations, but also for systems. Soon after that M. Schonbeck and G. Seregin considered the application of this method to the above-mentioned problem, see e.g., [5]. In particular, the following dual problem, namely, the Stokes system with a drift u, has been considered:

$$\partial_t v - \Delta v - (u \cdot \nabla) v + \nabla q = -\operatorname{div} F, \quad \operatorname{div} v = 0, \tag{1.5}$$

in Q_+ and $v(x,0) = 0$ for all $x \in \mathbb{R}^3$. It has been supposed that a tensor-valued field $F \in C_0^\infty(\mathbb{R}^3)$ is smooth and compactly supported in Q_+. In addition, it has been assumed that F is skew symmetric and therefore

$$\operatorname{div} \operatorname{div} F = 0.$$

The following identity takes place:

$$-\int\limits_{Q_+} \nabla u : F\, dx\, dt = \int\limits_{Q_+} u \cdot \operatorname{div} F\, dx\, dt = -\lim_{T \to +\infty} \int\limits_{\mathbb{R}^3} u(x,T) \cdot v(x,T)\, dx. \tag{1.6}$$

If the solution v to the dual problem has a certain decay, the limit on the right hand side vanishes. This means that the skew symmetric part of ∇u vanishes in Q_+. Then one can easily show that u must be a function of time only. Since u is a divergence free field, u is a bounded harmonic function. But also u is a bounded mild ancient solution to the Navier-Stokes equation and thus must be a constant. However, the condition (1.4) means that u is identically equal to zero. Thus, if we had a statement about the decay of v, we would immediately obtain a Liouville-type statement.

Therefore, since we know that $|u(0,0)| = 1$, this finally would prove that $z = 0$ is not a singular point of w and it, in turns, says that the origin is a regular point of w. So, the problem of local regularity of weak solutions to the Navier-Stokes equations stated in the beginning is solved under the additional assumption: u has a singularity of type I.

In this paper, our main goal is to investigate the limits of applicability of the duality method developed by G. Seregin. We examine a certain modification of duality method letting $F = 0$ but taking non-zero initial data, and prove results similar to those of M. Schonbeck and G. Seregin in [5] and [8]. The following theorems are results of the thesis.

Theorem 1.1. Let u and p be a mild bounded ancient solution of the Navier-Stokes equations $\mathcal{NS}$. Let $a \in C_0^\infty(\mathbb{R}^3)$, $\operatorname{div} a = 0$. Then there are unique v and q, such that

$$v \in C^\infty(\bar{Q}_+) \cap L_{2,\infty}(Q_+), \quad \nabla v \in L_2(Q_+),$$

$$q \in C^\infty(\bar{Q}_+) \cap L_{2,\infty}(Q_+),$$

and they are solutions of the Stokes system

$$(\mathcal{S}_u) \ : \ \begin{cases} \partial_t v - \Delta v + (u \cdot \nabla) v + \nabla q = 0 \\ \qquad \operatorname{div} v = 0 \\ \qquad v|_{t=0} = a \end{cases} \qquad \text{in } Q_+.$$

Here $L_{2,\infty}(Q_+)$ denotes the following Banach space $L_\infty(0,+\infty;L_2(\mathbb{R}^3))$ and for given function h on Q_+ and $A\subset\mathbb{R}$, let $h|_A$ denote the restriction of h to a subset $\mathbb{R}^3\times A\subset Q_+$. In case when instead of A we have for $a\in\mathbb{R}$ the following notion $t=a$, we denote $h|_{t=a}=h(\cdot,a)$ and for given function f on $\mathbb{R}$ we denote $f|_{t=a}=f(a)$.

Let X be a Banach space and $\mathcal{I}\subset\mathbb{R}$. Denote by $C(\mathcal{I};X)$ the Banach space of all Banach-valued continuous functions $u:\mathcal{I}\to X$ such that

$$\|u\|_{C(\mathcal{I};X)}=\sup\{\|u(t)\|_X\mid t\in\mathcal{I}\}<+\infty.$$

The most important statement in the case we are considering is the fact that for any $T>0$ the solution v of the Stokes system $\mathcal{S}_u$ belongs to the Banach space $C([0,T];L_1(\mathbb{R}^3))$. That is, for any $t>0$ we have $\|v(t)\|_{L_1(\mathbb{R}^3)}<+\infty$. This statement comes from the following theorem.

Theorem 1.2. Let u and p be a mild bounded ancient solution of the Navier-Stokes equations $\mathcal{NS}$. Let $a\in C_0^\infty(\mathbb{R}^3)$ and $\operatorname{div}a=0$. Then for any $T>0$ there is unique smooth solution v of the Stokes system $\mathcal{S}_u$ such that

$$v\in C([0,T];L_\infty(\mathbb{R}^3)\cap L_1(\mathbb{R}^3)).$$

The duality method in this case gives the following theorem.

Theorem 1.3. Let u and p be a mild bounded ancient solution of the Navier-Stokes equations $\mathcal{NS}$ and the singularity of type I takes place. Let $a\in C_0^\infty(\mathbb{R}^3)$, $\operatorname{div}a=0$, and let v be a solution of the Stokes system $\mathcal{S}_u$. Then for any $T>0$ we have

$$\int\limits_{\mathbb{R}^3}u(\cdot,0)\cdot a\,dx=\int\limits_{\mathbb{R}^3}u(\cdot,T)\cdot v(\cdot,T)\,dx. \tag{1.7}$$

Theorem 1.4. Let v be a solution of the Stokes system $\mathcal{S}_u$ and u and p be a mild bounded ancient solution of the Navier-Stokes equations $\mathcal{NS}$ and for $c_*>0$ the singularity of type I takes place. Then there is a constant $\varepsilon_0>0$ such that if $c_*<\varepsilon_0$, the following is valid:

$$\int\limits_{\mathbb{R}^3}u(\cdot,t)\cdot v(\cdot,t)\,dx\to0\text{ as }t\to+\infty.$$

Our main result is the following Liouville-type theorem.

Theorem 1.5. Let u and p be a mild bounded ancient solution of the Navier-Stokes equations $\mathcal{NS}$ and for $c_*>0$ the singularity of type I takes place. Then there is a constant $\varepsilon_0>0$ such that if $c_*<\varepsilon_0$, the following is valid:

$$u(x,0)=0,\qquad\forall x\in\mathbb{R}^3.$$

The last Theorem contradicts to the fact that $|u(0,0)|=1$. So, it is a proof of the statement that the origin $z=0$ is a regular point of the weak solution w. Thus, in this scenario we solve the problem of local regularity of weak solutions to the Navier-Stokes equations stated above.

In fact, using the last Theorem it is possible to obtain a more precise mode of the Liouville-type theorem, i.e., there is a constant $\varepsilon_0>0$ such that if $c_*<\varepsilon_0$, the following is valid:

$$u(x,t)=0,\qquad\forall(x,t)\in Q_+.$$

In 2002, L. Escuriazo, G. Seregin, V. Šverak [2] proved backward uniqueness under certain assumptions, namely, that $u\equiv0$ on a half-space $\mathbb{R}^n_+\times[0,1]$, with $\mathbb{R}_+=\{x\in\mathbb{R}^n:x_n>0\}$, see Chapter 5 for details. This is a rather complicated statement, which implies, in particular, $u\equiv0$ on Q_+. We discuss this further in Section 8.

1.2. **Known result.** In 2018, M. Schonbek and G. Seregin [5] proved that if we have the decay assumption (1.4) on the drift u of the Stokes system (1.5), then for any integer $m \geqslant 0$ the decay estimates for the solution v are valid:

$$\|v(\cdot,t)\|_{L_1(\mathbb{R}^3)} \leqslant c(m,c_*,F)\frac{\sqrt[4]{t^3}}{\ln^m(t+e)}, \qquad t \geqslant 0,$$

and

$$\|v(\cdot,t)\|_{L_2(\mathbb{R}^3)} \leqslant \frac{c(m,c_*,F)}{\ln^m(t+e)}, \qquad t \geqslant 0.$$

Since for the mild bounded ancient solution u of the Navier-Stokes equations the singularity of type I holds, one can easily get that

$$\|u(\cdot,t)\|_{L_\infty(\mathbb{R}^3)} \leqslant \frac{c_*}{\sqrt{t}}, \qquad \forall t > 0.$$

Therefore, for instance, it is possible to obtain the following estimate:

$$\left|\int\limits_{\mathbb{R}^3} u(x,t)\cdot v(x,t)\,dx\right| \leqslant \|u(\cdot,t)\|_{L_\infty(\mathbb{R}^3)}\,\|v(\cdot,t)\|_{L_1(\mathbb{R}^3)} \leqslant c_F\cdot t^{\frac{1}{4}}, \tag{1.8}$$

where $c_F = c_*\cdot c(0,c_*,F)$. Unfortunately, the above estimate does not allow us to conclude the Liouville-type theorem, since the function $t^{1/4}$ is an increasing function. Moreover, no matter how we try to estimate the integral from above, the available estimates for the function v always yield an estimate by the function $t^{1/4}$ as above.

However, M. Schonbek and G. Seregin obtained a result directly asserting the vanishing of the right hand side limit in the (1.6) for sufficiently small constant $c_* > 0$, namely, there is $\varepsilon > 0$ such that if $c_* < \varepsilon$, then

$$\int\limits_{\mathbb{R}^3} u(x,t)\cdot v(x,t)\,dx \to 0 \text{ as } t \to +\infty. \tag{1.9}$$

1.3. **Plan of the proof.** In this section, we describe the structure of the present thesis. In Section 2, we prove an auxiliary statement that we use next. In Section 3 we start with preliminaries and show the existence of the smooth mild solution of the Stokes system $\mathcal{S}_u$ with drift u using classical technique, consequently, obtaining the Theorem 1.1.

Then, in Section 4 we prove some auxiliary estimates on the L_p norm of u and v. This is the most crucial estimates in the thesis, that shows that our results are similar to those obtained by M. Schonbeck and G. Seregin. For the Liouville-type theorem the most important section is Section 5, since we prove the Theorem 1.2, namely, the mild solution v is a smooth function with finite $C([0,T];L_1(\mathbb{R}^3))$ norm.

The Theorem 1.3 in Section 6 shows that the duality method in the case we are considering works in a generally similar way.

In particular, let us to show here how it works in case of smooth u and v with compact support in Q_T. Note that, for the sake of economy of space, we will sometimes omit the time variable t of functions. Thus, for instance, the notation $u(t)$ will be replaced by u. Also, for positive T denote $Q_T = \mathbb{R}^3\times(0,T)$. Clearly, integration by parts gives us

$$\int\limits_{Q_T}(u\,\partial_t v + v\,\partial_t u)\,dx\,dt = \int\limits_{Q_T}(u(\Delta v - (u\cdot\nabla)v - \nabla q) + v(-\Delta u + (u\cdot\nabla)u + \nabla p))\,dx\,dt =$$

$$= \int\limits_{Q_T}(-\nabla u\cdot\nabla v - u\cdot(u\cdot\nabla)v + q\cdot\operatorname{div} u + \nabla v\cdot\nabla u + v\cdot(u\cdot\nabla)u - p\cdot\operatorname{div} v)\,dx\,dt = 0,$$

since,

$$\begin{aligned}\int_{\mathbb{R}^3} u\cdot(u\cdot\nabla)v\,dx &= \int_{\mathbb{R}^3} u_iu_jv_{i,j}\,dx\\ &= -\int_{\mathbb{R}^3} v_i(u_iu_j)_{,j}\,dx\\ &= -\int_{\mathbb{R}^3} v_iu_{i,j}u_j\,dx - \int_{\mathbb{R}^3} v_iu_iu_{j,j}\,dx\\ &= -\frac{1}{2}\int_{\mathbb{R}^3} v\cdot\nabla|u|^2\,dx - \int_{\mathbb{R}^3} v_iu_i\,\mathrm{div}\,u\,dx\\ &= \frac{1}{2}\int_{\mathbb{R}^3} \mathrm{div}\,v\cdot|u|^2\,dx \;=\; 0\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3} v\cdot(u\cdot\nabla)u\,dx &= \int_{\mathbb{R}^3} v_iu_ju_{i,j}\,dx\\ &= -\int_{\mathbb{R}^3} u_i(v_iu_j)_{,j}\,dx\\ &= -\int_{\mathbb{R}^3} u_iv_{i,j}u_j\,dx - \int_{\mathbb{R}^3} u_iv_iu_{j,j}\,dx\\ &= -\int_{\mathbb{R}^3} u\cdot(u\cdot\nabla)v\,dx - \int_{\mathbb{R}^3} u_iv_i\,\mathrm{div}\,u\,dx \;=\; 0.\end{aligned}$$

On the other hand we get

$$\int_{Q_T} (u\,\partial_t v + v\,\partial_t u)\;dx\,dt = \int_{\mathbb{R}^3}\int_0^T \partial_t(uv)\,dt\,dx = \int_{\mathbb{R}^3} \left(u(\cdot,T)\cdot v(\cdot,T) - u(\cdot,0)\cdot v(\cdot,0)\right)\,dx.$$

Hence we obtain:

$$\int_{\mathbb{R}^3} u(\cdot,0)\cdot a\,dx = \int_{\mathbb{R}^3} u(\cdot,T)\cdot v(\cdot,T)\,dx.$$

In the end, Section 7 shows that the result of M. Schonbeck and G. Seregin shown in Subsection 1.2 holds, namely, the Theorem 1.4. Finally, in Section 8 we prove the Liouville-type Theorem 1.5, which is the main result of the present thesis, and discuss the precise mode of Liouville-type theorem.

2. Integral identities

In this subsection we prove some auxiliary lemmas on the convective terms. Note that for any vectors $a, b \in \mathbb{R}^n$ we denote $a \otimes b$ as $n \times n$ matrix such that $(a \otimes b)_{ij} = a_i b_j$. Also, the symbol : denotes the scalar product of matrices, that is, for any matrices $A, B \in \mathbb{R}^{n\times n}$ we have $A : B = A_{ij} B_{ij}$.

Lemma 2.1. Let $v \in W^1_2(\mathbb{R}^3)$ and $u \in C^\infty(\mathbb{R}^3) \cap L_\infty(\mathbb{R}^3)$ such that $\operatorname{div} u = 0$. Then

$$\int\limits_{\mathbb{R}^3} v \otimes u : \nabla v \, dx = 0.$$

Proof. Take $\zeta = \xi\left(\frac{x}{R}\right)$, where $\xi \in C^\infty_0(\mathbb{R}^3)$ is such that $\operatorname{supp} \xi \subset B_{2R}(0)$, $0 \leqslant \xi(x) \leqslant 1$ for all $x \in \mathbb{R}^3$, $\xi = 1$ in $B_R(0)$, and, moreover,

$$|\nabla \zeta(x)| \leqslant \frac{C}{R} \qquad \forall x \in \mathbb{R}^3.$$

Consider the following integral and then integrating by parts gives us

$$\begin{aligned}
\int \zeta \, v \otimes u : \nabla v \, dx &= \int \zeta v_i u_j v_{i,j} \, dx \\
&= -\int (\zeta v_i u_j)_{,j} v_i \, dx \\
&= -\int \zeta_{,j} v_i u_j v_i \, dx - \int \zeta v_{i,j} u_j v_i \, dx \\
&\quad -\int \zeta v_i^2 \operatorname{div} u \, dx \\
&= -\int \zeta_{,j} \, v_i^2 u_j \, dx - \frac{1}{2} \int \zeta u \cdot \nabla |v|^2 \, dx \\
&= -\int \nabla \zeta \cdot u (v \cdot v) \, dx + \frac{1}{2} \int \zeta \, |v|^2 \operatorname{div} u \, dx \\
&\quad + \frac{1}{2} \int \nabla \zeta \cdot u \, |v|^2 \, dx.
\end{aligned}$$

Hence, we get the identity

$$\int \zeta \, v \otimes u : \nabla v \, dx = -\frac{1}{2} \int \nabla \zeta \cdot u \, |v|^2 \, dx. \tag{2.1}$$

By Holder inequality we obtain that

$$\left| \int\limits_{\mathbb{R}^3} \nabla \zeta \cdot u \, |v|^2 \, dx \right| \leqslant \frac{C}{R} \|u\|_{L_\infty(\mathbb{R}^3)} \cdot \|v\|^2_{L_2(\mathbb{R}^3)}.$$

Whence, taking the limit $R \to +\infty$ in the identity (2.1) we get the required identity, since since $\zeta(x) = \xi\left(\frac{x}{R}\right) \to \xi(0) = 1$ as $R \to +\infty$. $\square$

3. The Stokes problem with a drift

3.1. **Existence of unique weak solution.** Let H denote the closure of all smooth free-divergent functions with compact support with respect to the $L_2(\mathbb{R}^3)$ norm, so

$$H = \mathrm{clos}_{L_2(\mathbb{R}^3)}\{f \in C_0^\infty(\mathbb{R}^3) \mid \mathrm{div}\, f = 0\}.$$

Moreover, let H^1 denote the following Banach space:

$$H^1 = \mathrm{clos}_{W_2^1(\mathbb{R}^3)}\{f \in C_0^\infty(\mathbb{R}^3) \mid \mathrm{div}\, f = 0\}.$$

Let the bilinear form

$$[f, g] = \int\limits_{\mathbb{R}^3} \nabla f : \nabla g \, dx, \qquad f, g \in C_0^\infty(\mathbb{R}^3)$$

be a scalar product on the Banach space $\dot{H}^1$, which is the closure of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|f\|_{\dot{H}^1} = \sqrt{[f, f]}$, so in other words,

$$\dot{H}^1 = \mathrm{clos}_{\|\cdot\|_{\dot{H}^1}}\{f \in C_0^\infty(\mathbb{R}^3) \mid \mathrm{div}\, f = 0\}.$$

The following theorem is the key in proving the existence of the solution of the Stokes system $\mathcal{S}_u$, and its proof basically repeats the technique of O. A. Ladyzhenskaya in book [4] (see Chapter 4 and Chapter 6). We give a detailed proof of the following Theorem in view of the specificity of our problem.

Theorem 3.1. Given $T > 0$. Let u and p be a mild bounded ancient solution of the Navier-Stokes equations $\mathcal{NS}$. Let $a \in C_0^\infty(\mathbb{R}^3)$, $\mathrm{div}\, a = 0$. Then there is unique v, such that

$$v \in L_{2,\infty}(Q_T) \cap L_2(0, T; H^1), \quad v(0) = a,$$

and this function satisfies

$$\int\limits_{Q_T} \Big(-v \cdot \partial_t \eta + \nabla v : \nabla \eta - v \otimes u : \nabla \eta\Big)\, dx\, dt = \int\limits_{\mathbb{R}^3} a \cdot \eta(\cdot, 0)\, dx, \tag{3.1}$$

for any $\eta \in C_0^\infty(\mathbb{R}^3 \times [0, T))$ such that $\mathrm{div}\, \eta = 0$. Moreover, the following estimate is met:

$$\|v\|_{L_{2,\infty}(Q_T)} + \|\nabla v\|_{L_2(Q_T)} \leqslant 2\, \|a\|_{L_2(\mathbb{R}^3)}\,.$$

Proof. Let $\{\varphi_k\}_{k\in\mathbb{N}} \subset H^1$ be a sequence such that it is an orthonormal basis in H and its linear span is a complete set in $\dot{H}^1$, i.e.,

$$\mathrm{clos}_{\dot{H}^1} \mathrm{span}\, \{\varphi_k\}_{k\in\mathbb{N}} = \dot{H}^1.$$

For simplicity we denote the scalar product in $L_2(\mathbb{R}^3)$ as follows:

$$(f_1, f_2) = \int f_1 \cdot f_2 \, dx.$$

We claim that for any $N \in \mathbb{N}$ there are unique set of functions $\{C_k^N\}_{k=1}^N \subset W_2^1(0, T)$ such that

$$v^N(x, t) = \sum_{k=1}^N C_k^N(t) \varphi_k(x), \qquad v^N \in W_2^{1,1}(Q_T),$$

and, moreover, for any $k = 1, \ldots, N$ we have the following identity

$$(\partial_t v^N(t), \varphi_k) + (\nabla v^N(t), \nabla\varphi_k) - (v^N(t) \otimes u(t), \nabla\varphi_k) = 0, \tag{3.2}$$

and the initial condition $v^N(x,0) = a^N(x)$, a.e. in $\mathbb{R}^3$ is valid, where a^N is the partial sum of the Fourier series of a, namely,

$$a^N(x) = \sum_{k=1}^{N} (a, \varphi_k)\varphi_k(x).$$

Indeed, the last statement is valid, since these functions $\{C_k^N\}_{k=1}^N$ are the solutions of the following linear ordinary differential equations:

$$\begin{cases} \dfrac{d}{dt} C_k^N(t) = \displaystyle\sum_{j=1}^{N} (\varphi_j \otimes u(t), \nabla\varphi_k) C_j^N(t) - \lambda_k C_k^N(t) \\ \quad C_k^N(0) = (a, \varphi_k) \end{cases} \qquad k = 1, \ldots, N.$$

Multiplying each of the equations (3.2) by the corresponding function C_k^N and summing them we get

$$(\partial_t v^N(t), v^N(t)) + \left\|\nabla v^N(t)\right\|^2_{L_2(\mathbb{R}^3)} - (v^N(t) \otimes u(t), \nabla v^N(t)) = 0. \tag{3.3}$$

The last term equals zero by Lemma 2.1.

Whence, by the Strong Continuity Theorem and definition of $\dot{H}^1$ we have

$$\frac{1}{2}\frac{d}{dt}\left\|v^N(t)\right\|^2_{L_2(\mathbb{R}^3)} + \left\|v^N(t)\right\|^2_{\dot{H}^1} = 0. \tag{3.4}$$

Clearly, $\left\|v^N(t)\right\|^2_{\dot{H}^1} \geqslant 0$, so one can omit this term, and integration of the identity above gives us that for any $t > 0$ we get

$$\left\|v^N(t)\right\|^2_{L_2(\mathbb{R}^3)} \leqslant \left\|a^N\right\|^2_{L_2(\mathbb{R}^3)} \leqslant \|a\|^2_{L_2(\mathbb{R}^3)},$$

and then

$$\left\|v^N\right\|_{L_{2,\infty}(Q_T)} = \text{ess sup}\left\{\left\|v^N(t)\right\|_{L_2(\mathbb{R}^3)} \mid 0 < t < T\right\} \leqslant \|a\|_{L_2(\mathbb{R}^3)}.$$

Moreover, from (3.4) one can derive

$$\begin{aligned} \left\|v^N\right\|^2_{L_2(0,T;\dot{H}^1)} &= \int_0^T \left\|v^N(t)\right\|^2_{\dot{H}^1} dt \\ &= \frac{1}{2}\left(\left\|a^N\right\|^2_{L_2(\mathbb{R}^3)} - \left\|v^N(T)\right\|^2_{L_2(\mathbb{R}^3)}\right) \\ &\leqslant \frac{1}{2}\|a\|^2_{L_2(\mathbb{R}^3)}. \end{aligned}$$

Therefore, clearly,

$$\left\|v^N\right\|_{L_{2,\infty}(Q_T)} + \left\|v^N\right\|_{L_2(0,T;\dot{H}^1)} \leqslant 2\|a\|_{L_2(\mathbb{R}^3)}.$$

Hence, there is $v \in L_{2,\infty}(Q_T) \cap L_2(0,T;\dot{H}^1)$ such that for the sequence $\{v^N\} \subset W_2^{1,1}(Q_T)$ we have the following weak and weak-$*$ convergences correspondingly:

$$v^N \xrightarrow{w} v \text{ in } L_2(0,T;\dot{H}^1), \quad v^N \xrightarrow{w*} v \text{ in } L_{2,\infty}(Q_T),$$

from which we obtain the energy inequality:

$$\|v\|_{L_{2,\infty}(Q_T)} + \|\nabla v\|_{L_2(Q_T)} = \|v\|_{L_{2,\infty}(Q_T)} + \|v\|_{L_2(0,T;\dot{H}^1)} \leqslant 2\|a\|_{L_2(\mathbb{R}^3)}.$$

Since embedding $L_\infty(0,T;L_2(\mathbb{R}^3)) \hookrightarrow L_2(Q_T)$ is continuous, the latter implies

$$v \in L_{2,\infty}(Q_T) \cap L_2(0,T;H^1).$$

Furthermore, by multiplying for each k identities (3.2) by arbitrary functions ξ_k that belong to the space

$$\{h \in W_2^1(0,T) : h|_{t=T} = 0\},$$

and, consequently, integrating new obtained identities, we get

$$\int_0^T \big((\nabla v^N(t), \nabla\varphi_k) - (v^N(t) \otimes u(t), \nabla\varphi_k)\big)\, \xi_k(t)\, dt = -\int_0^T (\partial_t v^N(t), \varphi_k)\xi_k(t)\, dt.$$

Integration by parts of the right-hand side of the identity above gives

$$\begin{aligned} \int_0^T (\partial_t v^N(t), \varphi_k)\xi_k(t)\, dt &= \int_0^T \xi_k(t)\, \frac{d}{dt}(v^N(t), \varphi_k)\, dt \\ &= (v^N(T), \varphi_k)\, \xi_k(T) - (a^N, \varphi_k)\, \xi_k(0) \\ &- \int_0^T (v^N(t), \varphi_k)\, \xi_k'(t)\, dt, \end{aligned}$$

whence we obtain

$$\int_0^T \big((\nabla v^N(t), \nabla\varphi_k) - (v^N(t) \otimes u(t), \nabla\varphi_k)\big)\, \xi_k(t)\, dt - \int_0^T (v^N(t), \varphi_k)\xi_k'(t)\, dt = (a^N, \varphi_k)\xi_k(0).$$

This yields that for any $\eta^m(x,t) = \sum\limits_{k=1}^{m} \xi_k(t)\varphi_k(x)$, where $\xi_k \in \{h \in W_2^1(0,T) : h|_{t=T} = 0\}$ for each $m \in \mathbb{N}$ and $k = 1, \dots, m$, we have

$$\int_{Q_T} \Big(-v^N \cdot \partial_t\eta^m + \nabla v^N : \nabla\eta^m - v^N \otimes u : \nabla\eta^m\Big)\, dx\, dt = \int_{\mathbb{R}^3} a^N \cdot \eta^m(\cdot, 0)\, dx.$$

Obviously, the sequence $\{\eta^m(x,t)\}_{m=1}^{+\infty}$ is dense in

$$\mathrm{clos}_{W_2^{1,1}(Q_T)} \big\{\eta \in C_0^\infty(\mathbb{R}^3 \times [0,T)) : \operatorname{div} \eta = 0\big\}$$

with respect to the $W_2^{1,1}(Q_T)$ norm. From the continuity of the trace operator, we obtain $\eta^m(0) \to \eta(0)$ as $m \to +\infty$. Then we go to the limit in the equation above, when N tends to $+\infty$, then when m tends to $+\infty$, and, finally, get

$$\int_{Q_T} \Big(-v \cdot \partial_t\eta + \nabla v : \nabla\eta - v \otimes u : \nabla\eta\Big)\, dx\, dt = \int_{\mathbb{R}^3} a \cdot \eta(\cdot, 0)\, dx. \tag{3.5}$$

To sum up, energy inequality entails the uniqueness of the function v. Suppose there are two different weak solutions v_1 and v_2, then

$$\|v_1 - v_2\|_{L_{2,\infty}(Q_T)} + \|\nabla(v_1 - v_2)\|_{L_2(Q_T)} \leqslant 0,$$

which, in turn, means that $v_1 - v_2 = 0$ identically in Q_T. $\square$

3.2. **Smoothness of solutions to the Stokes problem.** Assume that F belongs to some functional class in which the singular integral is bounded. The pressure field p_F, which is associated with F, is defined by

$$p_F(x,t) = -\frac{1}{3}\operatorname{trace} F(x,t) + \lim_{\delta\to 0} \int\limits_{\mathbb{R}^3\setminus B_\delta(x)} \nabla^2\mathcal{E}(x-y) : F(y,t)\,dy,$$

where

$$\mathcal{E}(x) = \frac{1}{4\pi|x|}, \qquad x\in\mathbb{R}^3.$$

Note that for each $s\in(1,+\infty)$ and for any $t\in(0,+\infty)$ the following estimate holds:

$$\|p_F(\cdot,t)\|_{L_s(\mathbb{R}^3)} \leqslant c_s \|F(\cdot,t)\|_{L_s(\mathbb{R}^3)}.$$

Note that the function v, that is the solution of the integral identity (3.1) obtained in Theorem 3.1 is, in fact, the weak solution of the following Cauchy problem:

$$(\mathcal{S}_u)\ :\ \begin{cases} \partial_t v - \Delta v + (u\cdot\nabla)v + \nabla q = 0 \\ \operatorname{div} v = 0 \\ v|_{t=0} = a \end{cases}$$

in the sense of distributions in Q_T. Similarly to the Proposition 1.1 from M. Schonbeck and G. Seregin's paper (see proof in Appendix A in [5]) we can get the following result.

Proposition 3.2. There exists a unique solution v to Stokes system $\mathcal{S}_u$ with properties:

$$\partial_t^k\,\nabla^\ell v \in L_2(Q_+)$$

for integer $k,\ell\geqslant 0$ except $k+l=0$,

$$\partial_t^k\,\nabla^{\ell+1} q \in L_2(Q_+)$$

for integer $k,\ell\geqslant 0$, and, moreover,

$$v\in L_{2,\infty}(Q_+), \quad q\in L_{2,\infty}(Q_+).$$

Combining the Proposition 3.2 above and estimates of solutions to the Stokes problem with lower order terms on Hölder spaces that were obtained by V. A. Solonnikov in [12] (see Theorem 9.1), we get the further smoothness of v in a standard way by taking derivatives of a solution with respect to the spatial and time variable, and thus establish that

$$v\in C^\infty(\bar{Q}_+).$$

Also, for any $T>0$ by Sobolev Embedding Theorem for each integer $d>\frac{n}{2}=\frac{3}{2}$, we have that the following embedding is continuous:

$$W_2^d(Q_T)\hookrightarrow C(\bar{Q}_T).$$

This implies that for certain constant $c>0$ we obtain

$$\|v\|_{L_\infty(Q_T)} \leqslant c\,\|v\|_{W_2^d(Q_T)}.$$

Hence, we get that the smooth solution v of the Stokes problem $\mathcal{S}_u$ belongs to $L_\infty(Q_T)$ for any $T>0$, and there is a unique smooth pressure field $q\in L_{2,\infty}(Q_+)$, which is associated with $v\otimes u$, and the equations of the Stokes system $\mathcal{S}_u$ are identities in the pointwise sense in Q_+. Also, since singular integral above is an operator from $L_\infty(\mathbb{R}^3)$ onto $BMO(\mathbb{R}^3)$, then the pressure field

$$q\in L_\infty(0,+\infty;BMO(\mathbb{R}^3))$$

and $\|q\|_{L_{2,\infty}(Q_+)} \leqslant c_u\,\|a\|_{L_2(\mathbb{R}^3)}$ for some constant $c_u>0$ depending only on u.

3.3. **Solution of the Stokes problem as a mild solution.** Define the function

$$\Gamma(x,t)=\frac{1}{(4\pi t)^{\frac{3}{2}}}\,e^{-\frac{|x|^2}{4t}},\qquad (x,t)\in Q_+.$$

Note that Γ is a fundamental solution of the heat equation and for any $t>0$ we have

$$\|\Gamma(\cdot,t)\|_{L_1(\mathbb{R}^3)}=1.$$

The Ozin's tensor is a fundamental solution to the Stokes problem, for example, see formulas (39) and (40) in [12]. The Ozin's tensor (K_{ij}) is defined with the help of the standard heat kernel in the following way:

$$K_{ij}(x,t)=\Gamma(x,t)\delta_{ij}+\frac{\partial^2\Phi}{\partial x_i\partial x_j}(x,t),$$

where

$$\Delta\Phi(x,t)=\Gamma(x,t),\qquad \Phi(x,t)=\int\limits_{\mathbb{R}^3}\mathcal{E}(x-y)\Gamma(y,t)\,dy.$$

The tensor $K=(K_{ijk})$ is obtained from Ozin's tensor by

$$K_{ijk}(x,t)=\frac{\partial K_{ij}}{\partial x_k}(x,t).$$

Theorem 3.3. Let u and p be a mild bounded ancient solution of the Navier-Stokes equations $\mathcal{NS}$. Let $a\in C_0^\infty(\mathbb{R}^3)$ and $\operatorname{div}a=0$. Then the smooth solution v of the Stokes system $\mathcal{S}_u$ is a mild solution, that is, for any $i=1,\dots,3$ and $t>0$ we have

$$v_i(x,t)=\int\limits_{\mathbb{R}^3}\Gamma(x-y,t)a_i(y)\,dy-\int\limits_0^t\int\limits_{\mathbb{R}^3}K_{ijk}(x-y,t-\tau)v_j(y,\tau)u_k(y,t)\,dy\,dt. \tag{3.6}$$

The proof of this Theorem is written in [9], see formulas (1.4)-(1.7). In short, by previous Subsection 3.2, we have that the smooth function

$$v\in L_\infty(Q_T),\qquad \forall T>0,$$

is the solution to the Cauchy problem for the Stokes system

$$(\mathcal{S}_u)\ :\ \begin{cases}\partial_t v-\Delta v+\nabla q=\operatorname{div}F\\ \qquad \operatorname{div}v=0\\ \qquad v|_{t=0}=a\end{cases}\qquad \text{in } Q_T,$$

where $F=-v\otimes u$. Therefore, the solution of the Cauchy problem for the Stokes system $\mathcal{S}_u$ is given with the help of the Ozin's tensor by formula (3.6). Indeed, since the solution of the Cauchy problem for the heat equation is represented by the fundamental solution Γ, one can get that

$$\int\limits_{\mathbb{R}^3}K_{ij}(x-y,t-\tau)\frac{\partial F_{jk}}{\partial y_k}(y,\tau)\,dy=-\int\limits_{\mathbb{R}^3}K_{ijk}(x-y,t-\tau)F_{jk}(y,\tau)\,dy.$$

From which we obtain the required Theorem.

4. Upper bounds

In 1964, the following estimates have been obtained by V. A. Solonnikov, see [11]: for any integer $\ell, m \geqslant 0$ there is a constant $C_{\ell,m} > 0$ such that for any $(x,t) \in Q_+$ we have

$$|\partial_t^\ell \nabla^m K_{ij}(x,t)| \leqslant \frac{C_{\ell,m}}{(|x|^2+t)^{\frac{3}{2}+\frac{m}{2}+\ell}},$$

and, as a consequence,

$$|\partial_t^\ell \nabla^m K_{ijk}(x,t)| \leqslant \frac{C_{\ell,m}}{(|x|^2+t)^{2+\frac{m}{2}+\ell}}.$$

So, obviously, for $\kappa = C_{0,0}$ we have

$$|K_{ijk}(x,t)| \leqslant \frac{\kappa}{(|x|^2+t)^2}, \qquad \forall (x,t) \in Q_+.$$

Lemma 4.1. For any $p \geqslant 1$ there is a constant $k_p > 0$ depending only on p such that for any $t > 0$ we have

$$\|K(\cdot,t)\|_{L_p(\mathbb{R}^3)} \leqslant k_p \, t^{\frac{3-4p}{2p}}.$$

Proof. Note that for any $t > 0$ we have

$$|K_{ijk}(x,t)|^p \leqslant \frac{\kappa^p}{(|x|^2+t)^{2p}} \leqslant \frac{\kappa^p}{|x|^{4p}+t^{2p}}, \qquad x \in \mathbb{R}^3.$$

Make the following substitution of variables: $x = y\sqrt{t}$. Then the Jacobian

$$J_{x\to y}(y) = \det\begin{vmatrix} \sqrt{t} & 0 & 0 \\ 0 & \sqrt{t} & 0 \\ 0 & 0 & \sqrt{t} \end{vmatrix} = t^{\frac{3}{2}},$$

Therefore

$$\begin{aligned} \|K(\cdot,t)\|^p_{L_p(\mathbb{R}^3)} &= \int\limits_{\mathbb{R}^3} |K_{ijk}(x,t)|^p \, dx \\ &\leqslant \kappa^p \int\limits_{\mathbb{R}^3} \frac{dx}{|x|^{4p}+t^{2p}} \\ &= \kappa^p \int\limits_{\mathbb{R}^3} \frac{t^{\frac{3}{2}} \, dy}{t^{2p}(y^{4p}+1)} \\ &= \kappa^p \, t^{\frac{3}{2}-2p} \int\limits_{\mathbb{R}^3} \frac{dy}{y^{4p}+1} \\ &\leqslant (k_p)^p \, t^{\frac{3-4p}{2}}, \end{aligned}$$

where $k_p = \kappa \left(\int_{\mathbb{R}^3} \frac{dy}{y^{4p}+1} \right)^{1/p} < +\infty$. Obviously, the required result follows from the upper bound above. $\square$

As a consequence one can show the following fact.

Lemma 4.2. Given $T > 0$. The tensor K belongs to the space $L_1(Q_T)$ and, moreover, there is a constant $c_0 > 0$ such that

$$\|K\|_{L_1(Q_T)} \leqslant c_0 \sqrt{T}.$$

Proof. By Lemma 4.1 above we get

$$\begin{aligned}\|K\|_{L_1(Q_T)} &= \int_0^T \|K(\cdot,t)\|_{L_1(\mathbb{R}^3)}\,dt \\ &\leqslant k_1 \int_0^T t^{-1/2}\,dt = c_0\sqrt{T},\end{aligned}$$

where $c_0 = 2\,k_1$ is an absolute constant. □

Furthermore, acting analogically, one can find the the upper bound on the norm of the mild bounded ancient solution u of the Navier-Stokes equations $\mathcal{NS}$ that satisfies the condition (1.4). In that case we also shall say that u has a singularity of type I, that is, for certain $c_* > 0$ we have the following estimate:

$$|u(x,t)| \leqslant \frac{c_*}{|x|+\sqrt{t}}, \qquad (x,t) \in Q_+.$$

The following statements will be proved under the assumption that the mild bounded ancient solution u of the Navier-Stokes equations $\mathcal{NS}$ has a type I singularity.

Lemma 4.3. Let $p > 3$. Then for any $t > 0$ the function $u(\cdot,t)$ belongs to the space $L_p(\mathbb{R}^3)$ and there is a constant $C_p > 0$ such that

$$\|u(t)\|_{L_p(\mathbb{R}^3)} \leqslant C_p\, t^{\frac{3-p}{2p}}, \qquad t > 0,$$

where

$$C_p = \left(c_* \int_{\mathbb{R}^3} \frac{dy}{(|y|+1)^p} \right)^{\frac{1}{p}} < +\infty.$$

Moreover, for any $t > 0$

$$\|u(t)\|_{L_\infty(\mathbb{R}^3)} \leqslant c_*\, t^{-1/2}.$$

Proof. Clearly, substituting as before $x = y\sqrt{t}$ we get

$$\begin{aligned}\|u(t)\|^p_{L_p(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |u(x,t)|^p\,dx \\ &\leqslant c_* \int_{\mathbb{R}^3} \frac{dx}{(|x|+\sqrt{t})^p} \\ &= c_*\, t^{\frac{3}{2}-\frac{p}{2}} \int_{\mathbb{R}^3} \frac{dy}{(|y|+1)^p} \\ &= (C_p)^p\, t^{\frac{3-p}{2}},\end{aligned}$$

and here, similarly, the integral $\int_{\mathbb{R}^3} \frac{dy}{(|y|+1)^p} < +\infty$. Remark that if $p = 3$, then $C_p = +\infty$, since the integral $\int_{\mathbb{R}^3} (|y|+1)^{-3}\,dy$ diverges.

It remains to show that $\|u(t)\|_{L_\infty(\mathbb{R}^3)} \leqslant c_*\, t^{-1/2}$. The singularity of type I obviously entails the latter. □

Proposition 4.4. Let $\frac{6}{5} < p < 2$ and $a \in C_0^\infty(\mathbb{R}^3)$, $\operatorname{div} a = 0$. Then for any $t > 0$ the strong solution v of the Stokes system $\mathcal{S}_u$ is such that $v(\cdot, t)$ belongs to the space $L_p(\mathbb{R}^3)$, and, moreover, there is a constant $\tilde{C}_p > 0$ depending only on p such that

$$\|v(t)\|_{L_p(\mathbb{R}^3)} \leqslant \|a\|_{L_p(\mathbb{R}^3)} + \tilde{C}_p\, t^{\frac{6-3p}{4p}}.$$

Proof. Consider the function

$$\tilde{v}(x,t) = -\int\limits_0^t \int\limits_{\mathbb{R}^3} K(x-y, t-\tau) : [v \otimes u](y,\tau)\, dy\, d\tau,$$

so by Theorem 3.3 we have that

$$v(x,t) = \tilde{v}(x,t) + \int\limits_{\mathbb{R}^3} \Gamma(x-y,t) a(y)\, dy.$$

Then by Holder inequality for $p > 1$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, Fubini's theorem, Lemma 4.1 and Lemma 4.2 we obtain

$$\begin{aligned}
\|\tilde{v}(t)\|^p_{L_p(\mathbb{R}^3)} &= \int\limits_{\mathbb{R}^3} \left| \int\limits_0^t \int\limits_{\mathbb{R}^3} K(x-y,t-\tau) : [v\otimes u](y,\tau)\, dy\, d\tau \right|^p dx \\
&\leqslant \int\limits_{\mathbb{R}^3} \left| \int\limits_0^t \int\limits_{\mathbb{R}^3} |K(x-y,t-\tau)|^{\frac{1}{p'}+\frac{1}{p}} \cdot |[v\otimes u](y,\tau)|\, dy\, d\tau \right|^p dx \\
&\leqslant \int\limits_{\mathbb{R}^3} \left| \left(\int\limits_0^t \int\limits_{\mathbb{R}^3} |K(x-y,t-\tau)|^{\frac{p'}{p'}}\, dy\, d\tau \right)^{\frac{1}{p'}} \cdot \right. \\
&\quad \cdot \left. \left(\int\limits_0^t \int\limits_{\mathbb{R}^3} |K(x-y,t-\tau)|^{\frac{p}{p}} \cdot |[v\otimes u](y,\tau)|^p\, dy\, d\tau \right)^{\frac{1}{p}} \right|^p dx \\
&= \|K\|^{\frac{p}{p'}}_{L_1(Q_t)} \int\limits_{\mathbb{R}^3} \int\limits_0^t \int\limits_{\mathbb{R}^3} |K(x-y,t-\tau)| \cdot |[v\otimes u](y,\tau)|^p\, dy\, d\tau\, dx \\
&= \|K\|^{p-1}_{L_1(Q_t)} \int\limits_0^t \int\limits_{\mathbb{R}^3} |K(x',t-\tau)|\, dx' \int\limits_{\mathbb{R}^3} |[v\otimes u](y,\tau)|^p\, dy\, d\tau \\
&\leqslant c_0\, k_1\, t^{\frac{p-1}{2}} \cdot \int\limits_0^t (t-\tau)^{-1/2} \int\limits_{\mathbb{R}^3} |[v\otimes u](y,\tau)|^p\, dy\, d\tau.
\end{aligned}$$

Then by Holder inequality for parameter $\frac{2}{p} > 1$, since

$$\frac{1}{\frac{2}{p}} + \frac{1}{\frac{2}{2-p}} = \frac{p}{2} + \frac{2-p}{2} = 1,$$

by Lemma 4.3 we get

$$\begin{aligned}\int\limits_{\mathbb{R}^3} |[v\otimes u](y,\tau)|^p\,dy &\leqslant \Bigg(\int\limits_{\mathbb{R}^3} |v(y,\tau)|^2\,dy\Bigg)^{\frac{p}{2}}\cdot\Bigg(\int\limits_{\mathbb{R}^3} |u(y,\tau)|^{\frac{2p}{2-p}}\,dy\Bigg)^{\frac{2-p}{2}}\\ &= \|u(\tau)\|^p_{L_{\frac{2p}{2-p}}(\mathbb{R}^3)}\,\|v(\tau)\|^p_{L_2(\mathbb{R}^3)}\\ &\leqslant c(p)\,\tau^{(3-\frac{2p}{2-p})\frac{2-p}{4}}\,\|v\|^p_{L_{2,\infty}(Q_+)}\\ &= c(p)\,\tau^{\frac{6-5p}{4}}\,\|v\|^p_{L_{2,\infty}(Q_+)},\end{aligned}$$

for $\frac{2p}{2-p} > 3$, where $c(p) = \Big(C_{\frac{2p}{2-p}}\Big)^p$. Hence, here we have $\frac{6}{5} < p < 2$.

Thus, we obtain

$$\begin{aligned}\|\tilde v(t)\|^p_{L_p(\mathbb{R}^3)} &\leqslant c_0\,k_1\,t^{\frac{p-1}{2}}\int\limits_0^t (t-\tau)^{-1/2}\int\limits_{\mathbb{R}^3}|[v\otimes u](y,\tau)|^p\,dy\,d\tau\\ &\leqslant \tilde c(p)\,t^{\frac{p-1}{2}}\int\limits_0^t (t-\tau)^{-1/2}\tau^{\frac{6-5p}{4}}\,d\tau,\end{aligned}$$

where $\tilde c(p) = c_0\,k_1\,c(p)\,\|v\|^p_{L_{2,\infty}(Q_+)}$. Now, consider the integral above and calculate it:

$$\begin{aligned}\int\limits_0^t (t-\tau)^{-1/2}\tau^{\frac{6-5p}{4}}\,d\tau &= \int\limits_0^t (t-\tau)^{\frac{3-4p}{2}}\tau^{\frac{6-5p}{4}}\,d\tau\\ &= t^{1-\frac{1}{2}+\frac{6-5p}{4}}\int\limits_0^t\Big(1-\frac{\tau}{t}\Big)^{-1/2}\Big(\frac{\tau}{t}\Big)^{\frac{6-5p}{4}}\,d\Big(\frac{\tau}{t}\Big)\\ &= t^{\frac{8-5p}{4}}\int\limits_0^1 (\tau')^{\frac{6-5p}{4}}\,(1-\tau')^{-1/2}\,d\tau'\\ &= t^{\frac{8-5p}{4}}\,\mathrm{B}\left(\frac{10-5p}{4},\frac{1}{2}\right).\end{aligned}$$

Here $B(a,b) = \int_0^1 z^{a-1}(1-z)^{b-1}\,dz$ is Beta function for two real arguments $a, b > 0$. In our case it means that all calculations above are correct if and only if we choose such parameters p that $p < \frac{10}{5} = 2$. Hence, clearly, we get

$$\|\tilde v(t)\|^p_{L_p(\mathbb{R}^3)} \leqslant \tilde c(p)\,t^{\frac{p-1}{2}}\,t^{\frac{8-5p}{4}}\,\mathrm{B}\left(\frac{10-5p}{4},\frac{1}{2}\right).$$

Whence we obtain the required inequality:

$$\|v(t)\|_{L_p(\mathbb{R}^3)} \leqslant \|a\|_{L_p(\mathbb{R}^3)} + \tilde C_p\,t^{\frac{6-3p}{4p}},$$

since for any $t > 0$ we have $\|\Gamma(\cdot,t)\|_{L_1(\mathbb{R}^3)} = 1$, where

$$\tilde C_p = \|v\|_{L_{2,\infty}(Q_+)}\,C_{\frac{2p}{2-p}}\left(c_0\,k_1\cdot\mathrm{B}\left(\frac{10-5p}{4},\frac{1}{2}\right)\right)^{1/p} < +\infty$$

and the parameter p is such that $\frac{6}{5} < p < 2$. □

5. Pikard's method

5.1. **Pikard's method for the solutions of the Cauchy problem.**

Proposition 5.1. Given $T > 0$. Let u and p be a bounded ancient solution of the Navier-Stokes equations. Suppose that

$$c_0\, M\, \sqrt{T} \leqslant \delta < 1.$$

Then for any $a \in L_1(\mathbb{R}^3)$, $\operatorname{div} a = 0$, the mild solution v of the problem $\mathcal{S}_u$ belongs to the Banach space $C([0,T]; L_1(\mathbb{R}^3))$.

Proof. Let $t \in [0,T]$. Define the Pikard's iterations on the interval $[0,T]$ in the following way:

$$v^{k+1}(x,t) = \int\limits_{\mathbb{R}^3} \Gamma(x-y,t)a(y)\,dy - \int\limits_0^t \int\limits_{\mathbb{R}^3} K(x-y,t-\tau) : [v^k \otimes u](y,\tau)\,dy\,d\tau$$

and $v^0(x,t) = a(x)$. Then, applying Fubini's theorem and Holder inequality we get

$$\begin{aligned}
\|v^{k+1}(t) - v^k(t)\|_{L_1(\mathbb{R}^3)} &\leqslant \int\limits_0^t \int\limits_{\mathbb{R}^3} \int\limits_{\mathbb{R}^3} |K(x-y,t-\tau)| \cdot |[(v^k - v^{k-1}) \otimes u](y,\tau)|\,dy\,dx\,d\tau \\
&= \int\limits_0^t \int\limits_{\mathbb{R}^3} \big|K(x',t-\tau)\big|\;dx' \int\limits_{\mathbb{R}^3} |[(v^k - v^{k-1}) \otimes u](y,\tau)|\,dy\,d\tau \\
&\leqslant \|K\|_{L_1(Q_t)} \sup_{\tau \in [0,t]} \int\limits_{\mathbb{R}^3} |[(v^k - v^{k-1}) \otimes u](y,\tau)|\,dy \\
&\leqslant \|K\|_{L_1(Q_t)}\; \|u\|_{L_\infty(Q_t)} \sup_{\tau \in [0,t]} \|v^k(\tau) - v^{k-1}(\tau)\|_{L_1(\mathbb{R}^3)},
\end{aligned}$$

therefore by Lemma 4.2 and taking supremum for $t \in [0,T]$ it means that we have

$$\|v^{k+1} - v^k\|_{C([0,T];L_1(\mathbb{R}^3))} \leqslant c_0 \sqrt{T}\; \|u\|_{L_\infty(Q_T)}\; \|v^k - v^{k-1}\|_{C([0,T];L_1(\mathbb{R}^3))}.$$

Since $\|u\|_{L_\infty(Q_T)} \leqslant M$ and $c_0\, M\, T^{1/2} \leqslant \delta$, we get

$$\|v^{k+1} - v^k\|_{C([0,T];L_1(\mathbb{R}^3))} \leqslant \delta\, \|v^k - v^{k-1}\|_{C([0,T];L_1(\mathbb{R}^3))}.$$

Hence, repeating this estimate iteratively, we obtain

$$\|v^{k+1} - v^k\|_{C([0,T];L_1(\mathbb{R}^3))} \leqslant \delta^k\, \|v^1 - a\|_{C([0,T];L_1(\mathbb{R}^3))}.$$

Thus, the sequence $\{v^k\}_{k\in\mathbb{N}}$ is a fundamental Cauchy sequence in $C([0,T]; L_1(\mathbb{R}^3))$ and in that Banach space

$$v_k \to v \in C([0,T]; L_1(\mathbb{R}^3)) \text{ as } k \to +\infty,$$

where the limit v is such that

$$v(x,t) = \int\limits_{\mathbb{R}^3} \Gamma(x-y,t)a(y)\,dy - \int\limits_0^t \int\limits_{\mathbb{R}^3} K(x-y,t-\tau) : [v \otimes u](y,\tau)\,dy\,d\tau.$$

□

5.2. **Extension of the solutions obtained by Pikard's method.**

Theorem 5.2. Let u and p be a bounded ancient solution of the Navier-Stokes equations. Let $a \in C_0^\infty(\mathbb{R}^3)$ and $\operatorname{div} a = 0$. Then for any $T > 0$ there is unique smooth solution v of the Stokes system $\mathcal{S}_u$ such that

$$v \in C([0,T]; L_\infty(\mathbb{R}^3) \cap L_1(\mathbb{R}^3)).$$

Moreover, this solution v is a classical solution, smooth solution, mild solution, and the energy estimate holds:

$$\|v\|_{C([0,T];L_2(\mathbb{R}^3))} + \|\nabla v\|_{L_2(Q_T)} \leqslant 2\,\|a\|_{L_2(\mathbb{R}^3)}\,.$$

Proof. Fix the following constant

$$T_* = \left(\frac{\delta}{c_0 M}\right)^2.$$

Here T_* is the length of the time interval obtained by the Pikard's method in Proposition 5.1. Remark that T_* does not depend on the initial data a. We know that on $[0, T_*]$ there is a unique mild solution

$$v \in C([0,T_*]; L_1(\mathbb{R}^3)).$$

Consider the function

$$a^1 = v(\cdot, T_*) \in L_1(\mathbb{R}^3)$$

as new initial data. Then on the interval $[T_*, 2T_*]$ there also exists a unique mild solution

$$v \in C([T_*, 2T_*]; L_1(\mathbb{R}^3)).$$

We also call it as v since it is an extension of the $v|_{[0,T_*]}$ on the $[T_*, 2T_*]$. Then again, consider the function

$$a^2 = v(\cdot, 2T_*) \in L_1(\mathbb{R}^3)$$

as new initial data and extend this solution on the interval $[2T_*, 3T_*]$ and etc.

For the finite number steps of these algorithm we extend solution v on the interval $[0, T]$ for any $T > 0$, and, finally, it proves that the mild solution v of the problem $\mathcal{S}_u$ belongs to the Banach space $C([0,T]; L_1(\mathbb{R}^3))$. Taking into account results of the Subsection 3.2 we obtain the required Theorem. □

Note that in the proof we do not use the type I singularity of u.

6. Duality method

Theorem 6.1. Let u and p be a mild bounded ancient solution of the Navier-Stokes equations $\mathcal{NS}$ and the singularity of type I takes place. Let $a \in C_0^\infty(\mathbb{R}^3)$, $\operatorname{div} a = 0$, and let v be a solution of the Stokes system $\mathcal{S}_u$. Then for any $T \geqslant 0$ we have

$$\int\limits_{\mathbb{R}^3} u(\cdot,0)\cdot a\,dx = \int\limits_{\mathbb{R}^3} u(\cdot,T)\cdot v(\cdot,T)\,dx. \tag{6.1}$$

Proof. Pick $\zeta \in C_0^\infty(\mathbb{R}^3)$ and multiply equations $\mathcal{S}_u$ by ζu in the sense of the scalar product in $L_2(\mathbb{R}^3)$. So, since $\operatorname{div} u = 0$, for any $t > 0$ we get

$$(\zeta\partial_t v, u) + (\zeta\nabla v, \nabla u) + (\nabla v, u\,\nabla\zeta) + ((u\cdot\nabla)v, \zeta u) - (q, u\cdot\nabla\zeta) = 0.$$

On the other hand, multiplying $\mathcal{NS}$ equations by ζv, since $\operatorname{div} v = 0$, we get

$$-(\zeta\partial_t u, v) + (\zeta\nabla u, \nabla v) + (\nabla u, v\,\nabla\zeta) + ((u\cdot\nabla)u, \zeta v) - (p, v\cdot\nabla\zeta) = 0.$$

Now we subtract from the first equation the second equation, then according to the rules for differentiating an integral that depends on a parameter we obtain:

$$\frac{d}{dt}(\zeta u(t), v(t)) + (u_j v_{j,k} - v_j u_{j,k}, \zeta_{,k}) + (\zeta u_j, v_{k,j}u_k - u_{k,j}v_k) - (qu_j - pv_j, \zeta_{,j}) = 0. \tag{6.2}$$

Clearly, since we have

$$(u\cdot v, \Delta\zeta) = -(u_j v_{j,k} + v_j u_{j,k}, \zeta_{,k})$$

we get that

$$(u_j v_{j,k} - v_j u_{j,k}, \zeta_{,k}) = -(u\cdot v, \Delta\zeta) - 2(u_{j,k}v_j, \zeta_{,k}).$$

Also, since we know that $\operatorname{div} u = 0$, we get

$$\begin{aligned}(\zeta u_j, v_{k,j}u_k + u_{k,j}v_k) &= (\zeta u_j u_k, v_{k,j}) + (\zeta u_j v_k, u_{k,j})\\ &= -(\zeta_{,j}u_j u_k, v_k) - (\zeta\operatorname{div} u, u\cdot v)\\ &\quad -(\zeta u_j u_{k,j}, v_k) + (\zeta u_j v_k, u_{k,j})\\ &= -(\nabla\zeta\cdot u, v\cdot u)\end{aligned}$$

Thus, equation (6.2) turns into

$$\frac{d}{dt}(\zeta u(t), v(t)) = (u\cdot v, \Delta\zeta) + 2(u_{j,k}v_j, \zeta_{,k}) + (\nabla\zeta\cdot u, v\cdot u) + (qu - pv, \nabla\zeta).$$

Integrate the latter by $t \in (0,T)$, hence we get the crucial identity:

$$\int\limits_{\mathbb{R}^3} \zeta\,u(T)\cdot v(T)\,dx - \int\limits_{\mathbb{R}^3} \zeta\,u(0)\cdot v(0)\,dx =$$

$$= \int\limits_{Q_T} (2v\,\nabla u\nabla\zeta + u\cdot v\,\Delta\zeta + \nabla\zeta\cdot u(u\cdot v) + (qu - pv)\cdot\nabla\zeta)\,dx\,dt. \tag{6.3}$$

Take $\zeta = \zeta_R$, where $\zeta_R(x) = \xi\left(\frac{x}{R}\right)$, and $\xi \in C_0^\infty(\mathbb{R}^3)$ is such that $\operatorname{supp}\xi \subset B_{2R}(0)$, $0 \leqslant \xi(x) \leqslant 1$ for all $x \in \mathbb{R}^3$, $\xi = 1$ in $B_R(0)$, and, moreover,

$$|\nabla\zeta_R(x)| \leqslant \frac{C}{R}, \qquad |\nabla^2\zeta_R(x)| \leqslant \frac{C}{R^2}, \qquad \forall x \in \mathbb{R}^3.$$

Then by Holder inequality and by condition (1.3) we obtain

$$\begin{aligned}
\int\limits_{Q_T} |v\,\nabla u \nabla \zeta_R|\,dx\,dt \quad &\leqslant \quad \frac{C}{R}\int\limits_0^T \|v(t)\,\nabla u(t)\|_{L_1(\mathbb{R}^3)}\,dt \\
&\leqslant \quad \frac{C}{R}\int\limits_0^T \|v(t)\|_{L_1(\mathbb{R}^3)}\,\|\nabla u(t)\|_{L_\infty(\mathbb{R}^3)}\,dt \\
&\leqslant \quad \frac{C}{R}\,\|v\|_{C([0,T];L_1(\mathbb{R}^3))}\int\limits_0^T \|\nabla u(t)\|_{L_\infty(\mathbb{R}^3)}\,dt \\
&\leqslant \quad \frac{CMT}{R}\,\|v\|_{C([0,T];L_1(\mathbb{R}^3))}.
\end{aligned}$$

Analogically we get

$$\int\limits_{Q_T} |u\cdot v\,\Delta\zeta_R|\,dx\,dt \leqslant \frac{CMT}{R^2}\,\|v\|_{C([0,T];L_1(\mathbb{R}^3))}$$

and

$$\int\limits_{Q_T} |\nabla\zeta_R\cdot u(u\cdot v)|\,dx\,dt \leqslant \frac{CM^2T}{R}\,\|v\|_{C([0,T];L_1(\mathbb{R}^3))}.$$

Note that conditions (1.2) and (1.4) imply that there is $M_* > 0$ such that

$$|u(x,t)| \leqslant M_*\,\min\left\{1,\,\frac{1}{|x|}\right\}, \qquad x\in\mathbb{R}^3, t\in[0,1].$$

So, Lemma 4.3 and the latter yield that $u \in L_\infty(0,T;L_s(\mathbb{R}^3)) = L_{s,\infty}(Q_T)$ for any $s > 3$. Therefore we get that the pressure field p associated with $u\otimes u$ belongs to the space $L_{2,\infty}(Q_T)$, since by Holder inequality for any $t > 0$ we have $\|p(t)\|^2_{L_2(\mathbb{R}^3)} \leqslant c\,\|u(t)\|^4_{L_4(\mathbb{R}^3)}$. Hence, we obtain

$$\begin{aligned}
\int\limits_{Q_T} |pv\cdot\nabla\zeta_R|\,dx\,dt \quad &\leqslant \quad \frac{C}{R}\int\limits_0^T \|p(t)\|_{L_2(\mathbb{R}^3)}\,\|v(t)\|_{L_2(\mathbb{R}^3)}\,dt \\
&\leqslant \quad \frac{CT}{R}\,\|p\|_{L_{2,\infty}(Q_T)}\,\|v\|_{L_{2,\infty}(Q_T)}.
\end{aligned}$$

Since $u \in L_{5,\infty}(Q_T)$ and $v \in L_{2,\infty}(Q_T)$, then $v\otimes u \in L_{\frac{10}{7},\infty}(Q_T)$, because

$$\|[v\otimes u](t)\|_{L_{\frac{10}{7}}(\mathbb{R}^3)} \leqslant \|u^{10/7}(t)\|_{L_{\frac{7}{2}}(\mathbb{R}^3)}\,\|v^{10/7}(t)\|_{L_{\frac{7}{5}}(\mathbb{R}^3)} = \|u(t)\|_{L_5(\mathbb{R}^3)}\,\|v(t)\|_{L_2(\mathbb{R}^3)}.$$

Then, obviously, the pressure field q associated with $v\otimes u$ belongs to the space $L_{\frac{10}{7},\infty}(Q_T)$, because we have

$$\|q(t)\|_{L_{\frac{10}{7}}(\mathbb{R}^3)} \leqslant c\,\|[v\otimes u](t)\|_{L_{\frac{10}{7}}(\mathbb{R}^3)}.$$

Hence, since $u \in L_{\frac{10}{3},\infty}(Q_T)$, by Holder inequality with parameters $\frac{10}{7}$ and $\frac{10}{3}$ we analogically obtain

$$\int\limits_{Q_T} |qu\cdot\nabla\zeta_R|\,dx\,dt \leqslant \frac{CT}{R}\,\|q\|_{L_{\frac{10}{7},\infty}(Q_T)}\,\|u\|_{L_{\frac{10}{3},\infty}(Q_T)}$$

Thus, taking the limit $R \to +\infty$ in the identity (6.3) we finally obtain

$$\int_{\mathbb{R}^3} u(0) \cdot a \, dx = \int_{\mathbb{R}^3} u(T) \cdot v(T) \, dx,$$

since $\zeta_R(x) = \xi\left(\frac{x}{R}\right) \to \xi(0) = 1$ as $R \to +\infty$. □

7. Decreasing in time for sufficiently small singularity

Theorem 7.1. Let v be a solution of the Stokes system $\mathcal{S}_u$ and u and p be a mild bounded ancient solution of the Navier-Stokes equations $\mathcal{NS}$ and for $c_* > 0$ the singularity of type I takes place. There is a constant $\varepsilon_0 > 0$ such that if $c_* < \varepsilon_0$, then the following is valid:

$$\lim_{t \to +\infty} \int_{\mathbb{R}^3} u(\cdot, t) \cdot v(\cdot, t) \, dx = 0.$$

Proof. Consider the function

$$\tilde{v}(x,t) = -\int_0^t \int_{\mathbb{R}^3} K(x-y, t-\tau) : [v \otimes u](y,\tau) \, dy \, d\tau,$$

so by Theorem 3.3 we have that

$$v(x,t) = \tilde{v}(x,t) + \int_{\mathbb{R}^3} \Gamma(x-y,t) a(y) \, dy.$$

Therefore, by Fubini's theorem, Lemma 4.1 and Lemma 4.3 we obtain

$$\begin{aligned}
\|\tilde{v}(t)\|_{L_1(\mathbb{R}^3)} &\leqslant \int_0^t \int_{\mathbb{R}^3} |K(x', t-\tau)| \, dx' \int_{\mathbb{R}^3} |[v \otimes u](y,\tau)| \, dy \, d\tau \\
&\leqslant c_0 \, k_1 \int_0^t (t-\tau)^{-1/2} \int_{\mathbb{R}^3} |[v \otimes u](y,\tau)| \, dy \, d\tau \\
&\leqslant c_0 \, k_1 \int_0^t (t-\tau)^{-1/2} \, \|v(\tau)\|_{L_1(\mathbb{R}^3)} \, \|u(\tau)\|_{L_\infty(\mathbb{R}^3)} \, d\tau \\
&\leqslant c_0 \, k_1 \, c_* \int_0^t (\tau(t-\tau))^{-1/2} \, \|v(\tau)\|_{L_1(\mathbb{R}^3)} \, d\tau \\
&\leqslant c_0 \, k_1 \, c_* \, \pi \, \|v\|_{C([0,t];L_1(\mathbb{R}^3))} \, .
\end{aligned}$$

Therefore, since $\int_0^t (\tau(t-\tau))^{-1/2} \, d\tau = \pi$ for any $t > 0$, we get for $c = c_0 \, k_1 \, \pi$ that

$$\|v\|_{C([0,T];L_1(\mathbb{R}^3))} \leqslant c \, c_* \, \|v\|_{C([0,T];L_1(\mathbb{R}^3))} + \|a\|_{L_1(\mathbb{R}^3)} \, .$$

Further, if $1 - c \, c_* > 0$ then this entails that for any $0 < t < T$ we have

$$\|v(t)\|_{L_1(\mathbb{R}^3)} \leqslant \frac{1}{1 - c \, c_*} \, \|a\|_{L_1(\mathbb{R}^3)} = c_a. \tag{7.1}$$

Finally, by Holder inequality and also Lemma 4.3 we get

$$\left|\int\limits_{\mathbb{R}^3} u(t)\cdot v(t)\,dx\right| \leqslant \|u(t)\|_{L_\infty(\mathbb{R}^3)}\,\|v(t)\|_{L_1(\mathbb{R}^3)} \leqslant \frac{c_*\,c_a}{\sqrt{t}} \to 0$$

as $t\to+\infty$. And, furthermore, since $1-c\,c_*>0$, there is an upper bound on c_*, namely,

$$c_* < \frac{1}{c} = \frac{1}{\pi\,c_0\,k_1} = \varepsilon_0.$$

□

Note that if we have $1-c\,c_*<0$, then upper bound (7.1) turns into

$$\|v(t)\|_{L_1(\mathbb{R}^3)} \geqslant 0 > \frac{1}{1-c\,c_*}\,\|a\|_{L_1(\mathbb{R}^3)}$$

and it is absolutely unhelpful. Therefore, the condition that $c_*<\varepsilon_0$ is vital in this proof.

8. Proof of the main result. Concluding remarks

8.1. **Proof of the Liouville-type theorem.** Recall that we want to prove that for mild bounded ancient solution u and p of the Navier-Stokes equations $\mathcal{NS}$ such that for $c_*>0$ the singularity of type I takes place there is a constant $\varepsilon_0>0$ such that if $c_*<\varepsilon_0$, then

$$u(x,0)=0, \qquad \forall x\in\mathbb{R}^3.$$

Proof. By Theorem 6.1 and Theorem 7 we obtain

$$\int\limits_{\mathbb{R}^3} u(0)\cdot a\,dx = \lim_{t\to+\infty}\int\limits_{\mathbb{R}^3} u(t)\cdot v(t)\,dx = 0$$

for any $a\in C_0^\infty(\mathbb{R}^3)$ such that $\operatorname{div} a=0$. This yields that Helmholtz–Leray decomposition implies $u(0)\in L_2(\mathbb{R}^3)$ and that there is $\psi\in W_2^1(\mathbb{R}^3)$ such that $u(0)=\nabla\psi$ almost everywhere in $\mathbb{R}^3$. Since $\operatorname{div} u=0$, therefore $u(0)\in H$, and, also, by Helmholtz–Leray decomposition we have

$$\int\limits_{\mathbb{R}^3} u(0)\cdot\eta\,dx = 0, \qquad \forall\eta\in H.$$

Thereby, by Dubois-Reymond Lemma we immediately get that $u(0)=0$ in $\mathbb{R}^3$ and the Theorem is proved. □

Note that to prove the regularity of the weak solution w in the origin $z=0$ it is sufficient to know that

$$u(x,0)=0, \quad \forall x\in\mathbb{R}^3,$$

since these fact already contradicts to the statement that $|u(0,0)|=1$.

However, our goal is to show that the Liouville-type theorem holds. Since we are interested only in the possibility of obtaining a Liouville-type theorem by applying the duality method to our special case, therefore, as we mentioned in Subsection 1.1, using backward uniqueness theorem proved in the paper of L. Escuriazo, G. Seregin, V. Šverak (see [2]) one can prove that

$$u(x,t)\equiv 0, \quad \forall (x,t)\in Q_+.$$

More about Liouville-type theorems could be found in the G. Seregin's book (see e.g., Chapter 6.4 and Appendix A.3 in [6]) and the paper of G. Koch, N. Nadirashvili, G. Seregin and V. Šverak, [3].

8.2. **Concluding remarks.** In this section, we describe how the Theorem 7.1 could be proved for any constant $c_* > 0$, that is, without assuming that $c_* < \varepsilon_0$ for some $\varepsilon_0 > 0$. Unfortunately, we show that this theorem can not be proved under such assumptions. What is interesting here is that we come to exactly the same estimate as M. Schonbeck and G. Seregin came to in paper [5]. Moreover, we come to some conclusions about the duality method.

Let $p, q \geqslant 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Applying Holder inequality by Proposition 4.4 and Lemma 4.3 for $p \in \left(\frac{6}{5}, 2\right)$ and $q > 3$ we get

$$\begin{aligned} \left| \int\limits_{\mathbb{R}^3} u(t) \cdot v(t)\, dx \right| &\leqslant \|u(t)\|_{L_q(\mathbb{R}^3)} \|v(t)\|_{L_p(\mathbb{R}^3)} \\ &\leqslant C_q\, t^{\frac{3-q}{2q}} \left(\|a\|_{L_p(\mathbb{R}^3)} + \tilde{C}_p\, t^{\frac{6-3p}{4p}} \right) \\ &= C_q \|a\|_{L_p(\mathbb{R}^3)}\, t^{\frac{3-q}{2q}} + C_q \tilde{C}_p\, t^{\frac{3-q}{2q} + \frac{6-3p}{4p}}. \end{aligned}$$

Since $q = \frac{p}{p-1}$, we have $q \in (3, 6)$ and therefore $p \in (\frac{6}{5}, \frac{3}{2})$. Obviously,

$$\frac{3-q}{2q} < 0,$$

which entails that $t^{\frac{3-q}{2q}} \to 0$ as t tends to $+\infty$. However, regarding the second summand term index in the estimate we obtained above, we have

$$\frac{3-q}{2q} + \frac{6-3p}{4p} = \frac{3 - \frac{p}{p-1}}{\frac{2p}{p-1}} + \frac{6-3p}{4p} = \frac{2(3p-3-p) + 6 - 3p}{4p} = \frac{1}{4}.$$

Hence, while t tends to $+\infty$ we obtain

$$\left| \int\limits_{\mathbb{R}^3} u(t) \cdot v(t)\, dx \right| \to C_q \tilde{C}_p\, t^{\frac{1}{4}} \to +\infty.$$

These result is similar to one obtained by M. Schonbeck and G. Seregin, namely, the estimate (1.8) (see Subsection 1.2).

This allows us to conclude that the duality method is applicable under different assumptions and is indeed a method rather than a special case of solving the problem of local regularity of a weak solution of the Navier-Stokes equations. As we mentioned in Section 1.1, our main goal is to investigate the limits of applicability of the duality method developed by G. Seregin. We examined a modification fo duality method expecting that we could obtain results similar to those obtained by M. Schonbeck and G. Seregin in [5]. Since all of our results, both positive and negative, are exactly the same as those of M. Schonbeck and G. Seregin, we finally conclude that the duality method requires the search for further applications in the theory of the Navier-Stokes equations.

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