Санкт-Петербургский государственный университет

***КОХОВИЧ Дарья Игоревна***

**Выпускная квалификационная работа**

# *On upward straight-line embeddings of graphs*

Уровень образования: Магистратура
Направление 01.04.01 «Математика»
Основная образовательная программа ВМ.5832.2021 «Современная математика»

Научный руководитель:
Факультет математики
и компьютерных наук
Профессор
Кандидат ф.-м. наук
Охотин Александр Сергеевич

Рецензент:
Доцент
Утрехтский университет
Мчедлидзе Тамара

Санкт-Петербург
2023

# 1 Basic Definitions

A **planar graph** is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints. Such a drawing is called a **planar embedding**. A planar embedding divides the plane into a set of regions called **faces**. Face incident to vertices $u_1, \ldots, u_m$, where $m \geqslant 3$, we denote by $\langle u_1, \ldots, u_m \rangle$. The face with unbounded area is an **outer face**. Other faces are **inner**. An edge is **outer** if it belongs to the outer face, and it is **inner** otherwise.

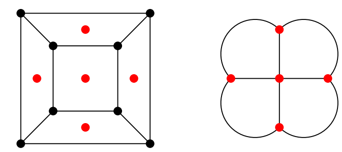

Figure 1: Example of a planar graph (left) and a weak dual graph (right).

An **outerplanar** graph is a graph that admits an **outerplanar drawing**, i.e., a planar drawing in which all vertices are on the outer face. The **weak dual** $G^*$ of a planar graph $G$ is the graph having a node for each inner face of $G$, and an edge between two nodes if and only if the two corresponding faces share an edge. For an outerplanar graph $G$, its weak dual $G^*$ is a tree. **Outerpath** is a outerplanar graph, whose weak dual graph $G^*$ is a path.

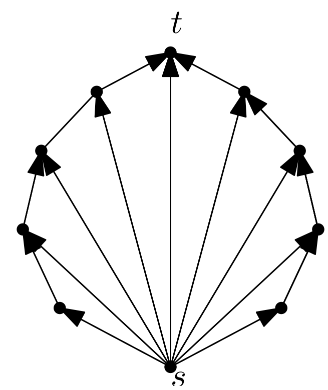


Figure 2: Example of a *st*-fan.

A **directed** graph $G = (V, E)$, or a digraph, is a graph whose edges have an orientation. We assume each edge $e = (u, v)$ of $G$ to be oriented from $u$ to $v$, and hence denote $u$ and $v$ as the **tail** and **head** of $e$, respectively. A vertex $u$ of $G$ is a **source** (resp. a **sink**) if it is the tail (resp. the head) of all its incident edges. A **directed cycle** is a cycle in directed graph in which each edge is traversed in the same direction. A **directed acyclic graph** (DAG) is a digraph that contains no directed cycle. An ***st*-DAG** is a DAG with a single source $s$ and a single sink $t$.

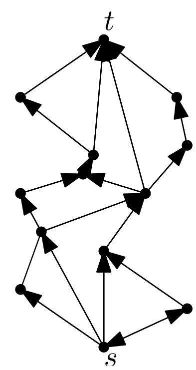


Figure 3: Example of an *st*-outerplanar graph.

An outerplanar graph is **internally triangulated** if it is biconnected and all inner faces are cycles of length 3. A **fan** is an internally triangulated outerpath whose inner edges all share an end-vertex. An ***st*-outerplanar graph** is an *st*-DAG whose underlying undirected graph is a outerplanar graph. An *st*-outerplanar graph is **one-sided** if the edge $(s, t)$ is an outer edge. An ***st*-fan** is an *st*-DAG whose underlying undirected graph is a fan and whose inner edges have $s$ as an end-vertex. An ***st*-outerpath** is an *st*-DAG whose underlying undirected graph is an outerpath.

A **topological book embedding** (TBE) of $G$ is a planar drawing such that all vertices of $G$ are represented as points of a horizontal line $l$, called the **spine**. All vertices of $G$ are embedded on the spine in some order $v_1, v_2, \ldots, v_n$, hence we can use the notation $v_i < v_j$, if $i < j$ or we can say that $v_i$ is lower than $v_j$. Each of the two half-planes defined by $l$ is a **page**. Each edge in a TBE is either in the left page, or completely in the right page, or it can be on both pages, in which case it crosses the spine. We assume that in a topological book embedding every edge is drawn as a sequence of one or more circular arcs, in particular semi-circles, such that no two consecutive arcs are in the same page.

Notice that, by using semi-circles, two arcs in the same page cross each other only if their end-points alternate along the spine (fig. 4).

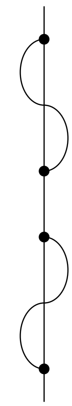

Figure 4: $S$-edges.

A **monotone** topological book embedding is a topological book embedding such that every edge crosses the spine at most once. An edge that crosses the spine once is called an $S$**-edge** and each of the two arcs composing it are called **sub-edges**. A non-$S$-edge $(x, y)$ is called **free**, if there are no edges that **cover** the edge $(x, y)$, i.e. there are no vertices $u, v$, such that $u \leq x < y \leq v$ and edges $(u, v)$ and $(x, y)$ are on the same page (fig. 5). Let $\gamma$ be a monotone topological book embedding. TBE $\gamma$ is **nice** if there is no pair of $S$-edges whose sub-edges cover in the right page.

## 2 *Motivation and related work*

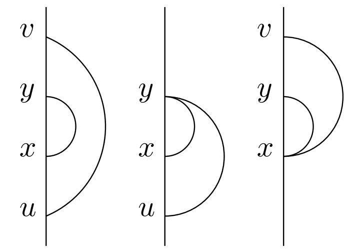


Figure 5: Forbidden cases for free edges.

**Book embedding**. An **upward** $k$**-page book embedding** (for short, kUBE) $\langle \pi, \sigma \rangle$ of a directed graph $G = (V, E)$ consists of a vertex ordering $\pi = \pi(V)$ and of an edge assignment $\sigma = \sigma(E)$ to one of $k$ sets, called **pages**, so that for any two edges $(a, b)$ and $(c, d)$ in the same page, we have neither $\pi(a) < \pi(c) < \pi(b) < \pi(d)$ nor $\pi(c) < \pi(a) < \pi(d) < \pi(b)$. Similar to TBE canonical drawing $\Gamma(\pi, \sigma)$ of $G$ is a drawing where the $k$ pages correspond to $k$ half-planes sharing a vertical line, called the spine. The page number of a digraph $G$ (also called book thickness) is the minimum number $k$ such that $G$ admits an upward $k$-page book embedding.

Results on book embeddings of undirected graphs are described in the paper (5) by Bernhart and Kainen. They prove that only outerplanar graphs have page number one, and graphs with page number two are sub-Hamiltonian graphs. This result implies the NP-completeness of deciding whether a graph admits a 2-page book embedding.

As for undirected graphs, there are many papers devoted to the study of upper and lower bounds on the page number of directed graphs. Heath et al. (16) show that directed trees and unicyclic digraphs have page number one and two, respectively. Bhore (7) shows that every $st$-outerpath have 4-page book embedding and every outerpath have a 16-page book embedding. Alzohairi and Rival (1), and later Di Giacomo et al. (14) with an improved linear-time construction, show that series-parallel digraphs have page number two. Mchedlidze and Symvonis (19) generalize this result and prove that N-free upward planar digraphs, which contain series-parallel digraphs, also have page number two.

About the lower bounds, Nowakowski and Parker (22) give an example of a planar $st$-graph that requires three pages for an upward book embedding. Pupyrev (23) computes book embeddings of all maximal planar graphs of size $n \leqslant 18$ and found no instance that requires four pages. However Bekos (4) and Yannakakis (24) prove that there are class of planar graphs with book number four. Later Jungeblut (17) finds slightly more accurate estimates: every upward planar graph $G$ on $n$ vertices has page number at most $O(n^{2/3}log^{2/3}(n))$ and there is an upward planar graph with page number at least 5. Overall, the question, whether the page number of upward planar digraphs is bounded, remains open.

Binucci and others (9) prove that testing if graph has kUBE is NP-complete if $k \geqslant 3$. In paper (9), a polynomial algorithm checks whether the $st$-outerplanar graph has 2UBE. This algorithm is given for $st$-outerpath, whose faces have a special structure. The running time of this algorithm depends on the branchwidth of the graph.

**Planar straight-line embedding**. Consider an arbitrary planar directed graph $G$ and some set of points $S$ on the plane such that the number of points coincides with the number of vertices in the graph. Embed all the vertices into points and draw straight-line edges. If the planarity condition is satisfied and the fact that for each directed edge from the point $(x_1, y_1)$ to the point $(x_2, y_2)$ it is true that $y_1 < y_2$, we say that the graph $G$ has an **upward planar straight-line embedding (UPSE)** into a point set $S$. A set of points is in **general position** if no three points lie on the same line. A point set is in **convex position** no point is in the convex hull of the others. We call a **one-sided convex point set** any convex point set in which the lowest point and the highest point on the convex hull are neighbors.

**UPSE problem**.
**Input:** a planar DAG $G$, a point set $S$ on the plane in general position.
**Output:** is there a planar straight-line embedding of the graph $G$ on point set $S$.

Cabello (11) prove that the UPSE problem is $NP$-complete even when $G$ is a 2-connected and 2-outerplanar. And Arseneva et al. (3) prove that the UPSE problem, where $G$ is an directed tree with a fixed point, is $NP$-complete.

However, if we look at the straight-line embedding problem (UPSE for undirected graphs), we know that any outerplanar graph can be embedded in any set of points $P$ in general position. For example, Bose (10) describes an algorithm that runs in $O(n \log^3(n))$, where $n$ is the number of points $P$.

According to (8), there are necessary and sufficient conditions for graphs that can always be embedded in an one-sided convex set. Therefore, interest fell on solving UPSE problem for $st$-outerplanar graphs in a one-sided convex point set with a condition that the embedding can be with one bend. The solution of this task for an one-sided convex point set turned out to be simpler and is formulated in Theorem 1.

**UPSE with 1 bend problem**.
**Input:** a planar DAG $G$, a point set $S$ on the plane in general position.
**Output:** is there a planar embedding of the graph $G$ on point set $S$ such that every directed edge $e \in E(G)$ from point $(x_1, y_1)$ to point $(x_2, y_2)$ can have at most one bend at a point $(x_3, y_3)$ where $y_1 < y_3 < y_2$.

**Theorem 1.** *Every st-outerplanar graph G admits a 1-bend point-set embedding on every one-sided convex point set S.*

The proof consists of the following two theorems. The first part was proved jointly with colleagues. The idea of the main proof is that, first, it is necessary to make a transition from the graph to its TBE, and then, according to the resulting embedding, embed the graph on a set of points. Theorem 2 shows that if a graph (not necessarily $st$-outerplanar) has a nice TBE, then we can embed the graph with the necessary conditions. The main proof of this paper is formulated in Theorem 3 and proved in Section 5.

**Theorem 2.** *Let $G = (V, E)$ be a planar graph. If G admits a nice topological book embedding $\gamma$, then G adimts a 1-bend point-set embedding on every one-sided convex point set.*

**Theorem 3.** *Every st-outerplanar graph G has a nice monotone topological book embedding.*

## 3 Preliminaries

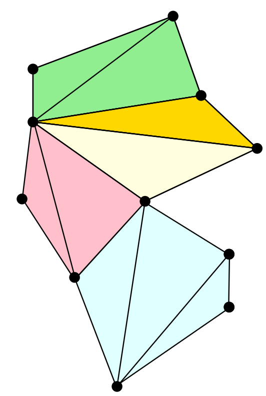

Figure 6: Example of $st$-fan decomposition.

Let $G$ be an internally-triangulated $st$-outerpath. Let $f_1, \ldots, f_h$ be the ordered list of faces forming the path $G^*$. An $st$**-fan decomposition** of an outerpath $G$ is a sequence of $s_i t_i$-fans $F_i \subset G$, with $i = 1, 2, \ldots, k$, such that:

1. $F_i$ is incrementally maximal, i.e. let $F_i = \cup_{j=i_1}^{i_s} f_j$, then:
   - or $F_i \cup f_{i_s+1}$ is not an $s_i t_i$-fan;
   - or $i_s = h$;
2. For any $1 \le i < j \le k$, $F_i$ and $F_j$ share a single edge if $j = i + 1$, which we denote by $e_i$, while they do not share any edge otherwise;
3. $s_1 = s$;
4. the tail of $e_i$ is $s_{i+1}$;
5. edge $e_i \neq (s_i, t_i)$;
6. $\cup_{i=1}^{k} F_i = G$.

**Fact 1.** *In any fan $F_i$, the edge $e_i$ is an outer edge of $F_i$ and has tail $t_i$. And the edge $e_{i-1}$ is an outer edge of $F_i$ and has head $s_i$.*

The **extreme faces** of an *st*-outerpath $G$ are the two faces that correspond to the vertices of $G^*$ having degree one. An *st*-outerpath $G$ is **primary** if and only if the path forming $G^*$ has one extreme face incident to $s$ and one extreme face incident to $t$. In the proof we utilize the following results.

**Lemma 1** (7). *Every primary internally-triangulated st-outerpath G admits an st-fan decomposition.*

**Lemma 2.** *Every internally-triangulated st-outerplanar graph G can be decomposed into a primary st-outerpath $G^o$ and multiple single source and single sink one-sided outerplanar graphs.*

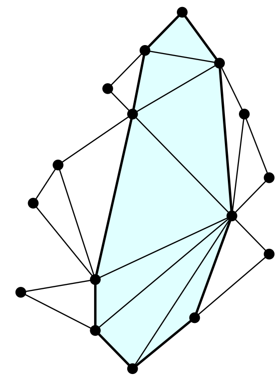

Figure 7: Primary *st*-outerpath from *st*-outerplanar graph.

*Proof.* Consider the weak dual graph $G^*$ of the graph $G$. Let $f_s$ and $f_t$ be some faces incident to $s$ and $t$ respectively. Since $G$ is an outerplanar graph, $G^*$ is a tree. Then, there exists only one path $P$ in $G^*$ from $f_s$ to $f_t$. The graph $G^o = \cup_{f \in P} f$ is a primary *st*-outerpath by construction.

**Fact 2.** *Every acyclic directed graph has at least one source and at least one sink.*

To prove the fact, it suffices to consider the vertices that must be embedded in the highest point and the lowest point of the set. These vertices are, respectively, a sink and a source.

Each outer edge $(u, v)$ of $G^o$ potentially split graph $G$ into two parts: an outerplanar graph consisting $G^o$ and outerplanar graph $G'$. By Fact 2 graph $G'$ has at least one sink and at least one source. Consider set of vertices of $G'$ expect $\{u, v\}$. We call this set $V'$. Note that when the graph $G$ is divided, all edges incident to the vertices $V'$ are preserved. And since the graph $G$ has only one source and one sink, different from the vertices from the set $V'$, none of the vertices from $V'$ is a source and a sink of the graph $G'$. Hence, the candidates for the source and the sink in the graph $G'$ are only the vertices $u$ and $v$. There is also an edge between the vertices $u$ and $v$, which is outer in the graph $G'$. Then $G'$ is a single source and single sink one-sided-outerplanar graph with source $u$ and sink $v$. □

Such one-sided-outerplanar subgraphs we call **appendages**. The edge $(x, y)$ we call **attachment edge**. Di Battista and Tamassia (6) prove that every upward planar graph $G$ can be augmented, by only introducing edges, to an upward planar triangulation. Thus, we can assume that $G$ is internally triangulated.

Consider some outerplanar graph $G$. Assume that by Lemma 2 we split $G$ into primary $st$-outerpath $G^o$ and multiple appendages. In the proof of Lemma 2, we took arbitrary faces that incident to $s$ and $t$. It follows from this that different primary $st$-outerpaths can be cut from the same graph $G$.

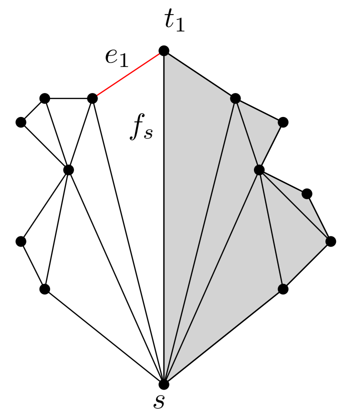

Figure 8: Building of one-sided $F_1$.

Consider $st$-fan decomposition $F_1, \ldots, F_k$ of the graph $G^o$. Consider fan $F_1$. This fan can potentially have a different number of vertices, relative to the edge $(s_1, t_1)$. When constructing $G^o$, we choose a face $f_s$ containing the edges $(s_1, t_1)$ and $e_1$. Then the edge $(s_1, t_1)$ for the fan $F_1$ is outer. Then, for simplicity, we can assume that $F_1$ is one-sided (fig. 8). Similarly, for the fan $F_k$, we can choose face incident to $t$ so that $F_k$ is one-sided.

We also often use the following lemma for embedding of appendages and following fact for proof of Theorem 3.

**Fact 3.** *Edge $e_i$ for each $i$ is not an attachment edge.*

**Lemma 3** (7)**.** *Every one-sided-outerplanar graph can be embedded in one page.*

## 4 *Nice topological book embedding*

This section is devoted to the proof of Theorem 2. Consider a graph $G = (V, E)$, where $|V| = n$. Consider a nice monotone topological book embedding $\gamma$ of graph $G$. Consider also some one-sided convex set of points $S$ ($|S| = n$).

Recall that $\gamma$ can have $S$-edges that intersect the spine at most once. Let get dummy vertices of the graph, which correspond to the intersection of the $S$-edge and the spine. Let the set of basic vertices and dummy vertices be denoted by $V'$ and let $|V'| = n'$. Let the vertices $V'$ be ordered according to the embedding on the spine and $V' = \{v_1, \ldots, v_{n'}\}$. We add dummy points to set $S$ so that the new set of points $S'$ has size $n'$ and $S'$ is still one-sided. Let the points of the set $S' = \{p_1, p_2, \ldots, p_{n'}\}$ be ordered in ascending order of the $y$-coordinate. Then each vertex from $V'$ corresponds to a point from $S'$. When adding dummy vertices, we require that the point $p_i$ is a dummy if and only if the vertex $v_i$ is the dummy.

Now, let the left page, on which it is allowed to cover $S$ sub-edges, be nested inside the convex hull $CH(S')$, and the right page behind the convex hull of the point set $S'$. Inside $CH(S')$ all edges are drawn as straight-line segments. Since the edges did not intersect in TBE, the edges also do not intersect in the new embedding, since the set of points $S'$ is convex. All the edges of the other page behind the convex hull $CH(S')$ are embedded with one bend.

Potentially, it can turn out to be a 2-bend embedding. Suppose we removed all the dummy points. Then one bend will be outside the convex hull and one bend may occur in the place where the dummy point used to be. Let there be a $S$-edge $(u, v)$ in TBE. Let the bend points be the points $p_1$ (when the dummy top is removed) and $p_2$ (outside the convex hull). In order for there to be no bend at the point $p_1$, it is necessary to have the same slope for the segments $(u, p_1)$ and $(p_1, p_2)$.

Therefore, embedding consists of the following steps (Fig. 9):

1. Embedding the left page inside the convex hull $CH(S')$ (red).
2. Embedding all $S$-sub-edges of the right page (blue).
3. Embedding all edges that are located inside some $S$ sub-edge on the right page (green).
4. Embedding all edges that are not located inside a $S$ sub-edge on the right page (orange).

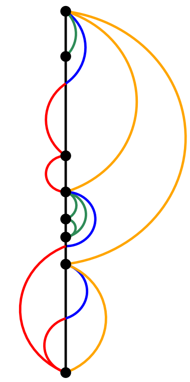

Figure 9: 4 steps for embedding of nice TBE.

**Step 1**. All edges and sub-edges of the left page embed inside the convex hull $CH(C)$ as straight segments. Since the vertices of the graph have the same order on the set $S'$ as on the spine, the drawn edges do not intersect.

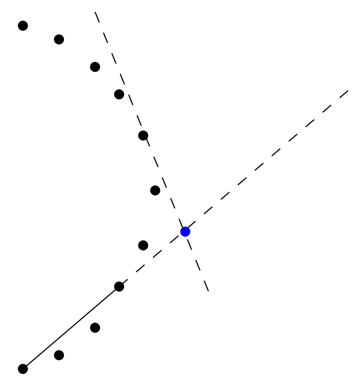

Figure 10: Step 2: building lines $l_1$ and $l_2$.

**Step 2**. Let there be some $S$-edge $(u, v)$, which is divided into two sub-edges $(u, c)$ and $(c, v)$. Let the points into which the vertices $u, c, v$ are embedded are $p_u, p_c, p_v$, respectively. W. l. o. g., let $(u, c)$ lie on the left page and have already been drawn in step 1. Let us draw a line $l_1$ containing the segment $(p_u, p_c)$. We also draw a line $l_2$ containing the point $p_v$ and not intersecting $CH(S')$. Then the lines intersect outside the convex hull at some point $p$. Hence, the sub-edge $(c, v)$ can be drawn as the union of the segments $(p_c, p)$ and $(p, p_v)$. The resulting $S$-edge has exactly one bend. Moreover, the bend is located at the point $p$, which, by construction, satisfies the condition of location in height: the $y$-coordinate of the point $p$ is between the $y$-coordinates of the points $p_c$ and $p_v$.

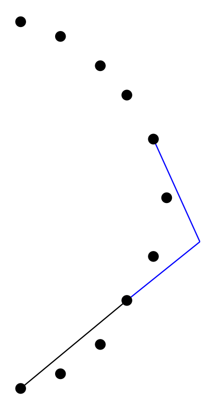

Figure 11: Step 2: building sub-edge $(c, v)$.

Since TBE is nice, all sub-edges on the right page do not overlap. This means that for any two sub-edges $(u_1, v_1)$ and $(u_2, v_2)$ constructed at this step, it is true that either $u_1 < v_1 < u_2 < v_2$ or $u_2 < v_2 < u_1 < v_1$ in order of $y$-coordinates. This means that no two sub-edges intersect.

**Step 3**. Consider some sub-edge $(u, v)$ of a $S$-edge. Let there be some edges inside it. Consider some edge $(a, b)$ among them. Let the sub-edge $(u, v)$ on the plane consist of two segments $(p_u, p)$ and $(p, p_v)$ with slopes $\sigma_1$ and $\sigma_2$ respectively. Then we construct the edge $(p_a, p_b)$ as follows. Let us draw a straight line with slope $\sigma_1$ from the point $p_a$, and draw a straight line with slope $\sigma_2$ from the point $p_b$. The lines intersect outside the convex hull at some point $z$. The edge $(p_a, p_b)$ is the union of the segments $(p_a, z)$ and $(z, p_b)$. Let do a similar construction with all edges inside the sub-edge $(u, v)$. All constructed edges are not intersect the edge $(u, v)$ and have exactly 1 bend. However, if among these edges there are edges with the same endpoint, then they intersect in the embedding. It is necessary to fix a sufficiently small $\epsilon > 0$ and correct the slope of the constructed edges. That is, slightly rotate the edges by $\epsilon$ in the right direction so that they do not intersect. See fig. 12 and fig. 13.

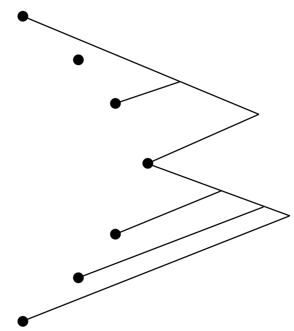

Figure 12: Step 3, 4: before rotation.

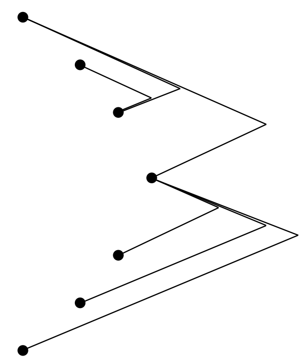

Figure 13: Step 3, 4: after rotation.

**Step 4**. All remaining edges do not lie under any sub-edge. Consider $\sigma$, the largest absolute value among all the slopes of the edges constructed at steps 2 and 3. Take the value $\sigma' > \sigma$. Consider some undrawn edge $(u, v)$. Let us construct a line from the point $p_u$ with slope $+\sigma'$ and a line from the point $p_v$ with slope $-\sigma'$. The lines intersect at some point $p$. Hence, the edge $(u, v)$ can be drawn as the union of $(p_u, p)$ and $(p, p_v)$. Let do this for all the remaining edges.

If some of the edges constructed at this step have the same endpoints, then these edges intersect. However, this can be corrected similarly to step 3. It is necessary to slightly rotate each edge in the right direction so that there are no more intersections.

## 5 *Outerplanar st-graph*

Let $P$ be a path in $G^*$ from face incident to $s$ to face incident to $t$. Denote $G^o = \cup_{f \in P} f$. Consider the $st$-fan decomposition $F_1, F_2, \ldots, F_k$ of the graph $G^o$. Recall that, by the remarks in section 3, we can choose a path $P$ such that $F_1$ and $F_k$ are one-sided. Denote by $G_i^o$ the union of the $F_1, \ldots, F_i$. If edge $(x, y)$ belongs to fan $F_i$, we say that corresponding appendage is **located at a height** $i$. By $G_j$ $(1 \leq j \leq k)$ denote the union of $G_j^o$ and all appendages at height $j$. It is true that

$$\cup_{i=1}^{k} G_j = G_k = G.$$

The proof is by induction on $G_i$ $(1 \leq i \leq k)$. Notice that we first embedded $G_i^o$ and then augment it to $G_i$. We build sequentially a TBE for $G_i$ while preserving the following invariants:

($\mathcal{I}_1$) For each $F_i$, the edge $e_i$ that separates $F_i$ and $F_{i+1}$ is free in topological book embedding (fig. 14).

($\mathcal{I}_2$) Let $e_i = (u_i, v_i)$ in topological book embedding of $G_i$. Then there are no other edges $(u, v)$ on the same page such that $u_i \leq u \leq v \leq v_i$.

($\mathcal{I}_3$) Embedding of $G_i$ is nice on both pages. Moreover, if $F_i$ has an $S$-edge that intersects the spine at the point $z_i$, and the edges $e_i = (u_i, v_i)$, then in the TBE the vertices $u_i$ and $v_i$ are above the point $z_i$.

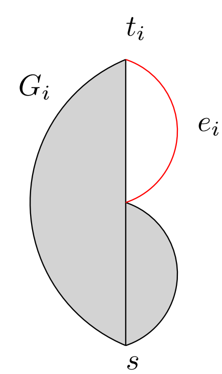


Figure 14: Invariant.

We also do the following for convenience. Let TBE come from the previous step of induction. The edge $e_{i-1}$ can be on the left or right page. For simplicity, we assume that the edge $e_{i-1}$ lies on the right page. If this is not the case, then TBE need to be mirrored. Since we require that the embedding be nice for both pages, mirroring the TBE is a legitimate action.

First, consider fans $F_i$ from $st$-fan decomposition. We have 2 cases depending on whether $e_i$ and $e_{i-1}$ lie on different sides of $(s_i, t_i)$ or not and 1 special case.

**Case 1: Special case**.

By Fact 1 $s_i$ is a head of $e_{i-1}$ and $t_i$ is a tail of $e_i$. Also by definition $e_i \neq (s_i, t_i)$, therefore we have 1 special case, where $e_{i-1} = (s_i, t_i)$. Then $F_i$ is a one-sided, since $e_{i-1}$ is an outer edge of fan. Since fan is one-sided, there is a path of the outer face of $F_i$ that does not contain the edge $(s_i, t_i)$. Let $s_i, u_1, \dots, u_m, t_i$ be such path. Since $t_i$ is a tail of $e_i$ and $e_i$ is an outer edge, $e_i = (u_m, t_i)$ (fig. 15). In this case, fan is a **triangle**. When the path $P$ passes through $e_{i-1}$, it arrives a face $\langle s_i, t_i, u_m \rangle$. Therefore, this face contains the edge $e_i$. This means that the path $P$ must immediately exit through the edge $e_i$, since it is impossible to leave face $\langle s_i, t_i, u_m \rangle$, walk along some another faces, and then return to the face $\langle s_i, t_i, u_m \rangle$.

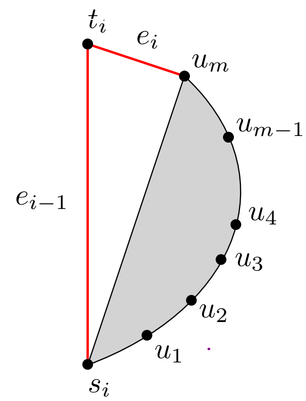

Figure 15: $F_i$ is a triangle.

**Case 2: Let $e_i$ and $e_{i-1}$ lie on the same side of $(s_i, t_i)$.**

In this case $F_i$ is also one-sided. Let $s_i, u_1, u_2, \dots, u_m, t_i$ are vertices on the outer face of $F_i$. Then $e_i = (u_m, t_i)$ and $e_{i-1} = (s_i, u_1)$. The path $P$ passes from the previous fan to the current fan along the edge $e_{i-1}$ to the face containing the vertices $s_i$ and $u_1$, that is, to the face $\langle s_i, u_1, u_2 \rangle$. Further, the path $P$ successively passes through all the faces of the current fan until it arrives the final face $\langle s_i, t_i, u_m \rangle$ containing the edge $e_i$. From the face $\langle s_i, t_i, u_m \rangle$ the path $P$ passes to the next fan along the edge $e_i$. We call this **right one-sided fan**.

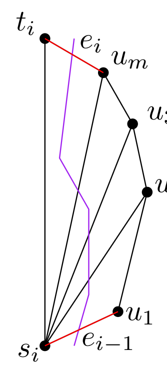

Figure 16: $F_i$ is a right one-sided fan.

**Case 3: Let $e_i$ and $e_{i-1}$ lie on the different side of $(s_i, t_i)$.**

Let $s_i, u_1, u_2, \dots, u_m, t_i$ and $s_i, v_1, v_2, \dots, v_l, t_i$ be two paths in $G$ from $s_i$ to $t_i$ of the outer face $F_i$. Consider a non-one-sided fan whose $e_i$ and $e_{i-1}$ lie on different sides with respect to the edge $(s_i, t_i)$. By the Fact 1 $e_{i-1} = (s_i, v_1)$ and $e_i = (u_m, t_i)$. Let the fan be divided by the edge $(s_i, t_i)$ into two halves. Denote by $f_{i_1}, \dots, f_{i_b}$ the faces of the fan of the half where the edge $e_{i-1}$ lies. The dual path $P$ starts from face $\langle s_i, v_1, v_2 \rangle$, passes along all faces $f_{i_1}, \dots, f_{i_b}$, and then continues in the second half of fan to the face $\langle s_i, t_i, u_m \rangle$. The face $\langle s_i, t_i, u_m \rangle$ contains the edge $e_i$ through which the path $P$ goes to the next fan, fig. 17. We call this **two-sided fan**.

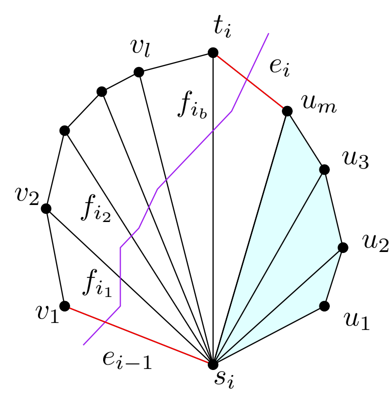

Figure 17: Two-sided fan.

Thus, to prove the theorem 4, it is necessary to consider 3 cases of fans: right one-sided, triangle and two-sided.

**Base case**.

As stated in section 3, we can assume that $F_1$ is an one-sided fan. Let on one side there are vertices $x_1, \ldots, x_m$. The edge $e_1$ has a sink $t_1$, then $e_1 = (x_m, t_1)$. By the definition of $st$-fan decomposition, the fan $F_1$ has no edge $e_0$. In addition, we define the edge $e_0 = (s_1, x_1)$.

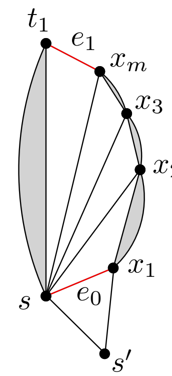


Figure 18: Base case.

Change our graph $G$: add a dummy vertex $s'$ and draw the edges $(s', s)$ and $(s', x_1)$. We call the new graph $G_{new}$. Note that since the vertex $s$ in the graph $G$ is source, then the new vertex $s'$ is a source in the graph $G_{new}$. Note that if a new face $\langle x_1, s, s' \rangle$ is added to the graph $G^o$, then the $s't$-primary outerpath of the graph $G_{new}$ is obtained. Hence, embedding of the graph $G_{new}$ is equivalent to embedding of the graph $G$ in which the vertex $s'$ has been removed.

Denote $G^o_{new} = G^o \cup \langle x_1, s, s' \rangle$. Then the $s't$-fan decomposition of the graph $G^o_{new}$ is the $st$-fan decomposition of the graph $G$ combined with $F_0 = \langle x_1, s, s' \rangle$. Thus, it suffices to prove the base case for $F_0$.

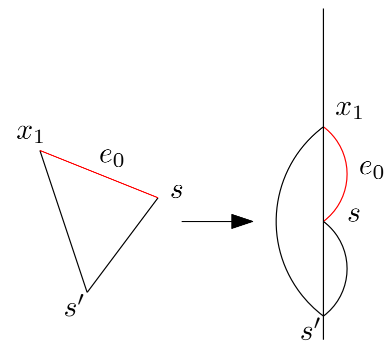


Figure 19: Base case in TBE.

Fan $F_0$ has no appendages and consists of only three edges: $(s', s)$, $(s', x_1)$, $(s, x_1)$. Let embed these three vertices in the only possible order: $s', s, x_1$. Let draw the edge $(s', x_1)$ on the left page, and the edges $(s', s), (s, x_1)$ on the right page (fig. 19). It is not difficult to verify that such an embedding satisfies all the invariants $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$.

**Induction step**.

**Fact 4.** *Consider some appendage $H$ with vertices $u_0, \ldots, u_m$. Let the TBE already contain vertices $u_0$ and $u_m$, and let the edge $(u_0, u_m)$ be drawn. Let there be no other vertices between the vertices $u_0$ and $u_m$. Then in the TBE we can place the $H$ appendage on one page.*

Fact 4 is a refinement of Lemma 3. The idea is that we first draw a primary outerpath from the graph $G$ containing all attachment edges. Sometimes there are no other vertices under these attachment edges. For such cases, we use Fact 4. However, there may be cases where an attachment edge is drawn, but there are vertices and edges underneath it. For such cases, we use the following lemma.

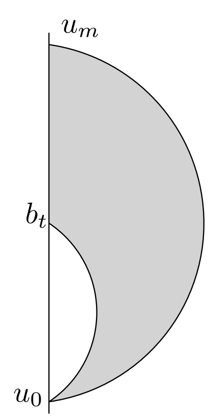


Figure 20: Lemma 4.

**Lemma 4.** *Consider some appendage $H$ with vertices $u_0, u_1, \ldots, u_m$. Let the TBE $\gamma$ already contain vertices $u_0$ and $u_m$ and edge $(u_0, u_m)$. Let there be other vertices in TBE between vertices $u_0$ and $u_m$, which we call $B$. Let the vertex $u_m$ not be connected to any vertex from $B$. Consider $b_t$, the topmost vertex of $B$. Then the appendage $H$ can be embedded so that $u_1, \ldots, u_{m-1}$ are above $b_t$. Similarly, if the vertex $u_0$ is not connected to any vertex from $B$. Consider $b_s$, the lowest vertex in $B$. Then the appendage can be embedded so that $u_1, \ldots, u_{m-1}$ are below $b_s$.*

Note that the topological book embedding potentially has $S$-edges. Since the proof is focused on only one of the two pages, for convenience we assume that the intersections of the $S$-edge and spine are also TBE vertices. We need this clarification in the further analysis of the case when $F_i$ two-sided fan.

*Proof.* We only prove the case when the vertex $u_m$ is not connected to any vertex from $B$. The case when the vertex $u_0$ is not connected to any vertex from $B$ is symmetric. We consider only the page on which the edge $(u_0, u_m)$ is drawn, the second page does not affect the embedding of the appendage. If there is no edge from vertex $u_0$ to vertex $b_t$, then we create a dummy edge $(u_0, b_t)$ on the same page. Then all the edges, which connect the set of vertices $B \cup \{u_0\}$, lie strictly inside the edge $(u_0, b_t)$, since the vertex $u_m$ is not connected to any vertex from $B$. It turns out a funnel in which we put the $H$ appendage.

We retract all vertices $B \cup \{u_0\}$ into one vertex $u_0$. We call the resulting TBE $\gamma_r$. Then the edge $(u_0, u_m)$ in $\gamma_r$ does not contain any vertices inside. Use the Fact 4 and put the appendage $H$ inside the edge $(u_0, u_m)$. Let $E'$ be all new edges and $E'_0$ be all new edges that incident to $u_0$. Now, in the original TBE $\gamma$, above the vertex $b_t$, we can put the vertices $u_1, \ldots, u_{m-1}$, as it is in $\gamma_r$. All edges $E' \setminus E'_0$ are transferred to the TBE $\gamma$. All edges of $E'_0$ are edges between $\{u_1, \ldots, u_{m-1}\}$ and $u_0$. All edges of $E'_0$ can be transferred to $\gamma$, since these edges do not intersect with any edge lying under the edge $(u_0, b_t)$. □

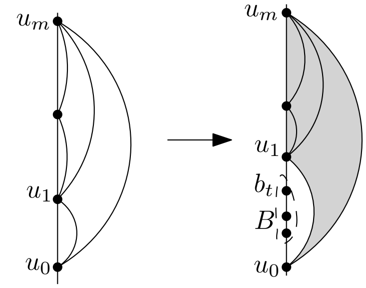


Figure 21: Proof of lemma 4.

*Embedding of a right one-sided fan.*

Let the vertices go in the order $s_i, v_1, \ldots, v_n, t_i$. The possible attachment edges are $(v_1, v_2), \ldots, (v_{n-1}, v_n)$ and $(s_i, t_i)$. The vertex order $s_i, v_1, \ldots, v_n, t_i$ is unique because $F_i$ is one-sided. All edges $E(F_i) \setminus e_i$ are drawn on one page, and $e_i$ on the other. It remains only to embed the appendages. The vertices $v_1, v_2, \ldots, v_n$ lie sequentially on the spine, so the corresponding appendages can be embed under attachment edges according to the Fact 4. Only the appendage attached to the edge $(s_i, t_i)$ remains. Let the appendage have vertices $s_i, u_1, u_2, \ldots, u_m, t_i$. Vertices $s_i$ and $t_i$ are already embedded. Consider a page on which $(s_i, t_i)$ is drawn. Vertex $t_i$ is not connected to any other vertex. Hence, this appendage satisfies the conditions of Lemma 4. We embed the appendage corresponding to the $(s_i, t_i)$ edge between $v_n$ and $t_i$ into this page. Then the vertices on the spine are going in the following order: $s_i, v_1, \ldots, v_p, u_1, \ldots, u_m, t_i$. See fig. 22.

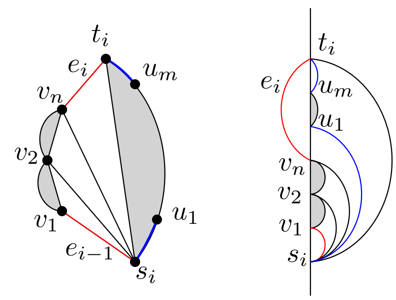


Figure 22: If $F_i$ is a right one-sided fan.

In this case, $e_i = (v_n, t_i)$ is a free edge ($\mathcal{I}_1$), since other edges on the same page can only be from $G^o_{i-1}$ and they are located only below the vertex $v_1$ ($v_1$ is the top in $G^o_{i-1}$), which is below the vertex $v_n$. It should be noted that all drawn edges of $F_i$ with appendages at a height $i$ (except $e_i$) are on the page where the edge $(s_i, t_i)$ is located. Then by construction, on the opposite page there is an edge $e_i = (v_n, t_i)$ satisfying the condition ($\mathcal{I}_2$). It is worth noting that $e_i$ and $e_{i-1}$ are on different pages and there are no $S$-edges. Hence, the niceness of embedding $G_{i-1}$ implies the niceness of embedding $G_i$ ($\mathcal{I}_3$).

*Embedding of a two-sided fan.*

Fan is divided into two halves of vertices with respect to the edge $(s_i, t_i)$. The half containing the edge $e_{i-1}$ can have multiple vertices. Let the vertices of this half be in the order $x_1, x_2, \ldots, x_p$. The second half containing the edge $e_i$ has a single vertex $u_m$. Then possible attachment edges are $(x_1, x_2), (x_2, x_3), \ldots, (x_{p-1}, x_p), (x_p, t_i)$ and $(s_i, u_m)$ (fig. 23).

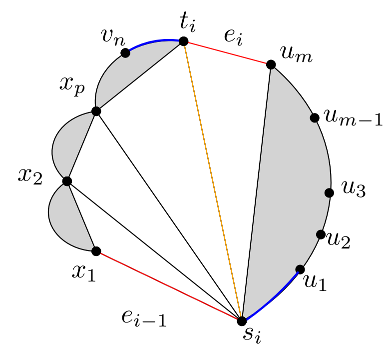


Figure 23: When $F_i$ is a two-sided fan in $G$.

We embed such a fan as follows (see fig. 24). Vertex order is $s_i, x_1, \ldots, x_p, u_m, t_i$. All edges from $s_i$ to other vertices of fan $x_1, x_2, \ldots, x_p$ (except $t_i$) are on the same page as the edge $e_{i-1}$. The edges of the outer face $(x_1, v_2), (x_2, x_3), \ldots, (x_{p-1}, x_p)$ and the edge $e_i = (u_m, t_i)$ are connected on the same page. The edge $(x_p, t_i)$ on the second page. The edge $(s_i, t_i)$ is drawn as an $S$-edge that intersects the spine between the vertices $x_p$ and $u_m$. Moreover, the bottom sub-edge of $(s_i, t_i)$ (which ends with $s_i$) is located on the same page as the edge $e_{i-1}$, and the upper sub-edge of $(s_i, t_i)$ is on a page other than the page of edge $e_i$.

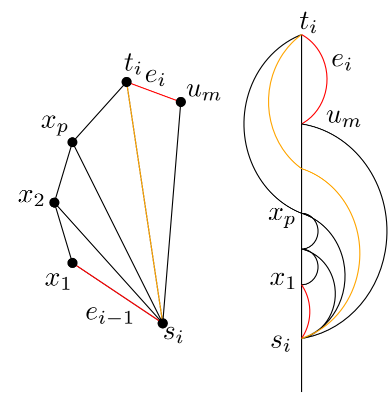


Figure 24: Embedding of a two-sided fan.

Edges $(t_i, x_1), (x_1, x_2), \ldots, (x_{p-1}, x_p)$ potentially attachment. However, the vertices $t_i, x_1, \ldots, x_p$ are consecutive on the spine. Hence, under the given attachment edges, one can embed the corresponding appendages according to Fact 4. Complexity arises only with attachment edges $(s_i, u)$ and $(x_p, t_i)$. Let these edges correspond to the appendages $H_1$ and $H_2$.

Let $S$-edge $(s_i, t_i)$ intersect the spine at some point $z$ in $F_i$ embedding. There is an appendage $H_1$ with vertices $s_i = u_0, u_1, \ldots, u_m$ such that $s_i$ and $u_m$ are already embedded in TBE, and only two edges are drawn from the vertex $u_m$: $(s_i, u_m)$ and $(u_m, t_i)$. Then this appendage satisfies the conditions of Lemma 4. Then $H_1$ can be embedded in TBE so that the vertices go in the following order: $z, u_1, \ldots, u_m$.

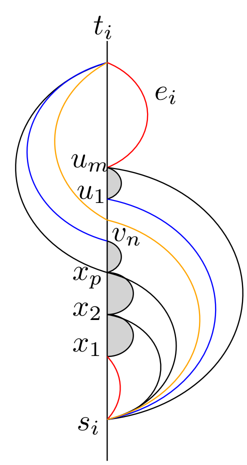


Figure 25: Embedding of a two-sided fan $F_i$ in TBE.

There is an appendage $H_2$ attached to the edge $(x_p, t_i)$ with vertices $x_p = v_0, v_1, v_2, ..., v_n, t_i$. Vertices $x_p$ and $t_i$ are already embedded in TBE. There are only one edge that have head $x_p$. Then appendage $H_2$ satisfies the condition of Lemma 4, then $H_2$ can be embedded in TBE so that the vertices go in the following order: $x_p, v_1, \ldots, v_n, z$.

The edge $e_i = (u_m, t_i)$ is free because the edge $(x_p, t_i)$, corresponding to this edge $H_2$ and $S$-sub-edge $(z, t_i)$ are on the opposite page by Lemma 4. All other vertices of this fan are located strictly lower than the vertex $u_m$, therefore, the edge between these vertices also does not cover the edge $e_i$ ($\mathcal{I}_1$). Also, the vertices $u_m$ and $t_i$ are consecutive, so this case also satisfies the second invariant ($\mathcal{I}_2$).

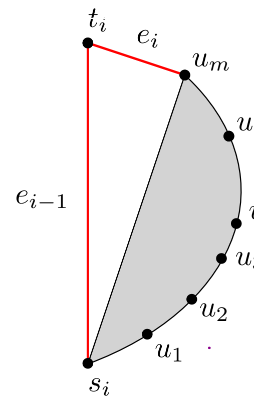

Figure 26: Appendage of a triangle.

Let us prove that niceness of the $G_i$. Since $G^o_{i-1}$ satisfies the invariant ($\mathcal{I}_3$), it suffices to check that the $S$-edge $(s_i, t_i)$ does not cover a possible $S$-edge below. Let the $S$-edge of the fan $F_i$ intersect the spine at the point $z_i$. Let $F_{i-1}$ have an $S$-edge $(u_{i-1}, v_{i-1})$. Then, since $e_{i-1} = (s_i, x_1)$, the $S$-edge intersects the spine at the point $z_{i-1}$ lying below the vertex $s_i$. It is obvious that $S$-sub-edge $(u_{i-1}, z_{i-1})$ lies strictly below the $S$-edge of $F_i$, and hence does not cover any $S$-sub-edge of the $F_{i-1}$ fan and vice versa, no $S$-sub-edge of the $F_{i-1}$ fan covers the $S$-sub-edge $(u_{i-1}, z_{i-1})$. Since $F_i$ satisfies the ($\mathcal{I}_1$) invariant, no edge covers $e_{i-1}$, hence the $S$-sub-edge is located on the opposite side of $e_{i-1}$ page. Hence $S$-sub-edges $(u_{i-1}, z_{i-1})$ and $(s_i, z_i)$ are on opposite pages (and cannot cover), and $S$-sub-edges $(u_{i-1}, z_{i-1})$ and $(z_i, t_i)$ on the same page. But by the construction of $z_{i-1} \le s_i \le z_i$, these $S$-sub-edges do not cover.

*Embedding of a triangle.*

Let $s_i, t_i$ and $u$ are vertices of $F_i$. On the spine, these vertices must be embedded in the order $s_i, u, t_i$. Edges $e_{i-1} = (s_i, t_i)$ and $e_i = (u, t_i)$ are not attachment. The third edge $(u, t_i)$ can potentially be attachment (fig. 26). We embedd the edge $e_i = (u, t_i)$ so that the vertices $u$ and $t_i$ are consecutive.

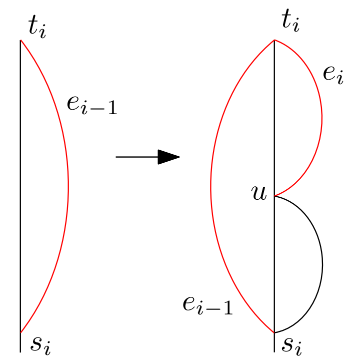

Figure 27: Embedding of $F_i$ in TBE of $G$.

The idea of the triangle case is to be able to flip the edge $e_{i-1}$ to the opposite page. We consider two cases: when the vertices of the $e_{i-1}$ edge are consecutive in the embedding $G_{i-1}$ (*simple case*) and when the vertices are not consecutive (*hard case*). Note that the simple case includes options if $F_{i-1}$ is a triangle or a one-sided fan. The hard case is if $F_{i-1}$ is a two-sided fan.

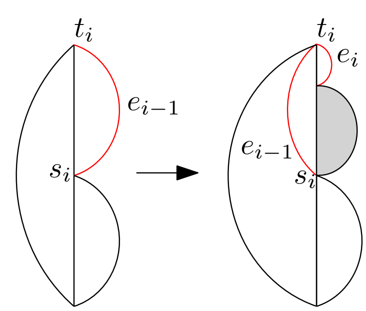

Figure 28: $F_i$ is a triangle: simple case.

*Simple case*. When $F_i$ is a triangle and $F_{i-1}$ is a two-sided fan or a triangle: the idea of turning the edge $e_{i-1}$ to the second page work without problems. Let $e_{i-1} = (s_i, t_i)$, $s_i$ and $t_i$ are consecutive. We rotate the edge $e_{i-1}$ to the second page, and draw the other two edges $e_i$ and $(s_i, u)$ in sequence on the page where the edge $e_{i-1}$ used to be (fig. 27). Now we need to look carefully on embedding of appendage. If $F_i$ is a triangle the only appendage is attached to the edge $(s_i, u)$. After the embedding $G^o_i$, the vertices $s_i$ and $u$ are consecutive. Hence, by Fact 4, the appendage with vertices $u_1, u_2, u_m = u$ can be embedded under the edge $(s_i, u)$ (fig. 28). The $\mathcal{I}_2$ invariant is obviously preserved, since in this case the vertices of $G^o_i$ are $V(G^o_{i_1}) \cup \{u\}$, therefore, there are no other vertices and edges between the vertices $u$ and $t_i$. Also, the edge $e_i$ is not covered by any other edge, because the edge $e_{i-1}$ is not covered by any other edge in $G^o_{i-1}$ ($\mathcal{I}_1$). There are no $S$-edges, then ($\mathcal{I}_3$)

invariant is also preserved.

*Hard case*. Case where $F_i$ is a triangle and $F_{i-1}$ is a right one-sided fan is difficult in that it is impossible to rotate the edge of $e_{i-1}$: from right one-sided fan $F_{i-1}$ there could be an appendage $H'$. Let $H'$ has vertices $v_1, \ldots, v_n$.

*Step 1*. Consider an appendage $H$ with vertices $s_i, u_1, \ldots, u_m$, which is attached to the attachment edge $(s_i, u_m)$. Vertex $u_m$ is not yet embedded in TBE. Place it anywhere above $v_n$. Since there is an invariant $(\mathcal{I}_2)$, there are no edges under the edge $e_{i-1}$ on the same page. Therefore, the appendage $H$ satisfies the condition of Lemma 4. Now we can embedd the $H$ appendage. Since $v_n < u_1$, the $H'$ appendage is located strictly below the $H$ appendage, so there is a gap between them. Strictly speaking, each appendage is located in its own funnel, and there is a distance between the funnels.

*Step 2*. Take a point $q$ in this gap. Delete the old edge $e_{i-1}$ and redraw it as an $S$-edge that intersects the spine at the point $q$. It remains only to embedd the edges $(s_i, u_m)$ and $(u_m, t_i)$. Vertices $s_i, u_m, t_i$ are already in TBE. Also, the $S$-edge has a point of intersection below the vertex $u_m$. Therefore, these edges can be drawn on the same page where the $e_{i-1}$ edge used to be (fig. 30). Then the edge $e_i$ is also free, similar to the case where $F_{i-1}$ is a triangle or a two-sided fan $(\mathcal{I}_1)$. And also the vertices $u_m$ and $t_i$ are consecutive. Then this case satisfies the condition $(\mathcal{I}_2)$.

In this case, an $S$-edge $(s_i, t_{i-1})$ is formed. Use the fact that $F_{i-1}$ is a one-sided fan. When constructing $F_{i-1}$, no $S$-edge was formed. But it could have been formed during the construction of $F_{i-2}$. Then the $S$-edge of $F_{i-2}$ can have tail at $s_i = t_{i-2}$ or lower. This means that the old $S$-edges do not intersect with the new $S$-edge.

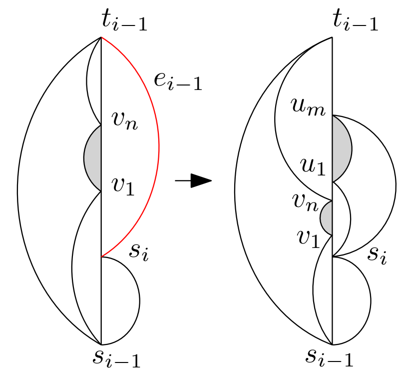


Figure 29: $F_i$ is a triangle, $F_{i-1}$ is a one-sided fan: Step 1.

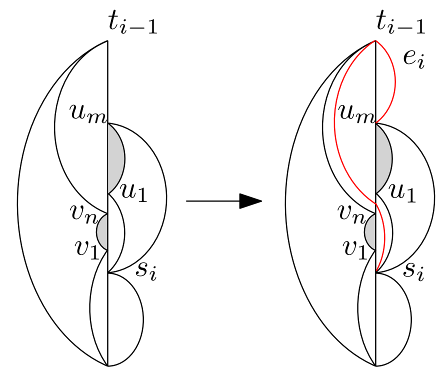


Figure 30: $F_i$ is a triangle, $F_{i-1}$ is a one-sided fan: Step 2.

## References

[1] M. Alzohairi, I. Rival. Series-Parallel Planar Ordered Sets Have Pagenumber Two. In Stephen C. North, editor, Graph Drawing, GD '96, volume 1190 of LNCS, pages 11–24. Springer, 1996.

[2] P. Angelini, F. Frati, M. Geyer, M. Kaufmann, T. Mchedlidze, A. Symvonis. Upward geometric graph embeddings into point sets. 18th International Symposium on Graph Drawing (GD 2010), Lect. Notes Comput. Sci., vol. 6502, Springer (2010), pp. 25-37.

[3] E. Arseneva et al. Upward Point Set Embeddings of Paths and Trees. Workshop on Algorithms and Computation (2020).

[4] M. A. Bekos, M. Kaufmann, F. Klute, S. Pupyrev, C. Raftopoulou,

T. Ueckerdt. Four Pages Are Indeed Necessary for Planar Graphs, (2020). https://doi.org/10.48550/arXiv.2004.07630

[5] F. Bernhart, P. C. Kainen. The book thickness of a graph. Journal of Combinatorial Theory, Series B, 27(3):320–331, 1979. https://doi.org/10.1016/0095-8956(79)90021-2

[6] Di Battista, G., Tamassia, R.: Algorithms for plane representations of acyclic di- graphs. Theor. Comput. Sci. 61, 175–198 (1988). https://doi.org/10.1016/0304-3975(88)90123-5

[7] S. Bhore, G. Da Lozzo, F. Montecchiani, M. Nöllenburg (2021). On the Upward Book Thickness Problem: Combinatorial and Complexity Results. In: Purchase, H.C., Rutter, I. (eds) Graph Drawing and Network Visualization. GD 2021. Lecture Notes in Computer Science(), vol 12868. Springer, Cham. https://doi.org/10.1007/978-3-030-92931-2_18

[8] C. Binucci, E. Di Giacomo, W. Didimo, A. Estrella-Balderrama, F. Frati, S. Kobourov, G. Liotta. Upward straight-line embeddings of directed graphs into point sets. Comput. Geom., 43 (2) (2010), pp. 219-232.

[9] C. Binucci, G. Da Lozzo, E. Di Giacomo, W. Didimo, T. Mchedlidze, M. Patrignani. Upward Book Embeddings of st-Graphs. In Gill Barequet and Yusu Wang, editors, 35th International Symposium on Computational Geometry (SoCG 2019), volume 129 of Leibniz International Proceedings in Informatics (LIPIcs), pages 13:1–13:22, Dagstuhl, Germany, 2019. Schloss Dagstuhl–Leibniz-Zentrum für Informatik. http://drops.dagstuhl.de/opus/volltexte/2019/10417

[10] Bose, P. On embedding an outer-planar graph in a point set. In: DiBattista, G. (eds) Graph Drawing. GD 1997. Lecture Notes in Computer Science, vol 1353. Springer, Berlin, Heidelberg. https://doi.org/10.1007/3-540-63938-1_47

[11] S. Cabello. Planar embeddability of the vertices of a graph using a fixed point set is NP-hard. Journal of Graph Algorithms and Applications 10.2 (2006): 353-363. http://eudml.org/doc/55397

[12] O. Çagirici et al. On upward straight-line embeddings of oriented paths, (2017).

[13] S. Chaplick, H. Förster, M. Hoffmann, M. Kaufmann. Monotone Arc Diagrams with few Biarcs, (2020). https://arxiv.org/abs/2003.05332

[14] E. D. Giacomo, W. Didimo, G. Liotta, S. K. Wismath. Book Embeddability of Series-Parallel Digraphs. Algorithmica, 45(4):531–547, 2006. https://doi.org/10.1007/s00453-005-1185-7

[15] F. Giordano, G. Liotta, T. Mchedlidze, A. Symvonis, S. H. Whitesides. Computing Upward Topological Book Embeddings of Upward Planar Digraphs. Journal of Discrete Algorithms, 30:45–69, January 2015. https://doi.org/10.1016/j.jda.2014.11.006

[16] L. S. Heath, S. V. Pemmaraju, A. N. Trenk. Stack and Queue Layouts of Directed Acyclic Graphs: Part I. SIAM Journal on Computing, 28(4):1510–1539, 1999.

[17] P. Jungeblut, L. Merker, T. Ueckerdt. A Sublinear Bound on the Page Number of Upward Planar Graphs. Society for Industrial and Applied Mathematics, (2022). https://doi.org/10.48550/arXiv.2107.05227

[18] M. Kaufmann, R. Wiese. Embedding Vertices at Points: Few Bends Suffice for Planar Graphs, (1999). In: Kratochvíyl, J. (eds) Graph Drawing. GD 1999. Lecture Notes in Computer Science, vol 1731. Springer, Berlin, Heidelberg. https://doi.org/10.1007/3-540-46648-7_17

[19] T. Mchedlidze, A. Symvonis. Crossing-Free Acyclic Hamiltonian Path Completion for Planar st-Digraphs. In Yingfei Dong, Ding-Zhu Du, and Oscar H. Ibarra, editors, Algorithms and Computation, ISAAC 2009, volume 5878 of LNCS, pages 882–891. Springer, 2009.

[20] T. Mchedlidze, A. Symvonis. Crossing-Optimal Acyclic HP-Completion for Outerplanar st-Digraphs. Journal of Graph Algorithms and Applications, 15(3):373–415, 2011.

[21] M. Nöllenburg, S. Pupyrev. On Families of Planar DAGs with Constant Stack Number, (2021). https://arxiv.org/abs/2107.13658

[22] R. Nowakowski, A. Parker. Ordered sets, pagenumbers and planarity. Order 6, 209–218 (1989). https://doi.org/10.1007/BF00563521

[23] S. Pupyrev. Mixed linear layouts of planar graphs. In: Frati, F., Ma, K. (eds.) GraphDrawing and Network Visualization. LNCS, vol. 10692, pp. 197–209. Springer (2017). https://doi.org/10.1007/978-3-319-73915-1_17

[24] M. Yannakakis. Embedding Planar Graphs in Four Pages. Journal of Computer and System Sciences, 38(1):36–67, 1989. https://doi.org/10.1016/0022-0000(89)90032-9