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Equilibrium Supply-Demand Allocation in a Single-Commodity Network*

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Abstract This paper is devoted to the recent findings in the analytical research of supply-demand allocation in a single-commodity network with distant (in space) suppliers and consumers. The allocation problem is formulated as an equilibrium flow assignment problem with affine functions of demand, supply, and logistic costs in a network represented by a digraph with suppliers and consumers located in nodes. We offer a brief overview of supply-demand relocation patterns obtained for elastic, shortage, and overproduction cases. Such kinds of results seem valuable since they allow one to develop different competitive distribution models to facilitate the decision-making of supply chain managers. In particular, supply chain managers can use available patterns to design decision-making strategies that mitigate risks concerning disruption or ripple effects.

Keywords: nonlinear optimization, distribution network, relocation, homogeneity.

1. Introduction

Samuelson (1952) constructed the net social pay-off function and offered the first mathematical formulation of the equilibrium flow assignment problem in a single-commodity network. The principle of equilibrium underlying such a model is the following: “the difference between the demand price of the consumer and the supply price of the supplier is equal to the cost of transporting a unit of product flow between supplier-consumer with a positive product flow and is less than the cost of transporting a unit of product between supplier-consumer with zero product flow". Takayama and Judge (1964) generalized this model for the multi-commodity network and, nowadays, this model is called the spatial equilibrium model. Worth mentioning this model considers relationships between supply, demand and logistic costs. Florian and Los (1982) studied general optimality conditions for this program.

Today, the problem of flow assignment in a single-commodity network does not lose its relevance. Vasin et al. (2020) and other researchers discuss the model implementation when investigating real logistic networks. Vasin and Daylova (2017) concentrate on models under imperfect competition, Bramoulle and Kranton (2002) and McNew (1996) study the integration of distribution networks under a perfect

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one. On the one hand, Barrett and Li (2002) and Novikov (2014) point out the non-identity of equilibrium models for distribution networks and integrative models with non-zero commodity flow. On the other hand, Kiselev and Yurchenko (2021) and Stephens et al. (2012) show that equilibrium spatial models demonstrate explainability and methodological potential for the commodity flow analysis and pricing in logistic networks.

This work is devoted to the recent findings in the analytical research of supply-demand allocation in a single-commodity network with distant (in space) suppliers and consumers. The allocation problem is formulated as an equilibrium flow assignment problem with affine functions of demand, supply, and logistic in a network represented by a digraph with suppliers and consumers located in nodes. We offer a brief overview of supply-demand relocation patterns obtained for elastic, shortage, and overproduction cases. In particular, Section 2 contains explicit allocation patterns for elastic demand and supply. Sections 3 and 4 deal with a shortage case and an overproduction one, respectively. Conclusions are given in the last section of the paper.

2. Network of Homogeneous Flows

Enke (1942) raised the issue that the then widespread principles of economics of Alfred Marshall (1920), do not pay enough attention to the problem of the spatial distance of market participants from each other. However, if logistic costs make a significant contribution to the cost of the product, then it is impossible to consider the same price of the product for the buyer and seller. In this case, one should talk separately about the supply price and the demand price of the product as well as the cost of transporting a unit of product from the supplier to the consumer, defined as the difference between the demand and supply prices. Based on such an understanding of pricing under conditions of spatial market equilibrium, Enke (1951) proposed the first model of the spatial equilibrium assignment of the product flow between a set of suppliers and a set of consumers.

Consider the set of suppliers M and the set of customers N, which are associated with commodity production, distribution, and consumption. We denote by s_i the supply of $i \in M$, and by λ_i – the price of a unit of the ith supply, $\lambda = (\lambda_1, \dots, \lambda_m)^{\mathrm{T}}$. By d_j we denote the demand of $j \in N$, and by μ_j – the price of a unit of the jth demand, $\mu = (\mu_1, \dots, \mu_n)^{\mathrm{T}}$. Finally, let $x_{ij} \geq 0$ be the commodity volume between a pair (i, j), while $c_{ij}(x_{ij})$ is the delivery cost of a unit of x_{ij}. Let us also introduce the indicator of delivery status:

$$\delta_{ij} = \begin{cases} 1 \text{ for } x_{ij} > 0, \\ 0 \text{ for } x_{ij} = 0, \end{cases} \quad \forall (i, j) \in M \times N.$$

Definition. The allocation pattern x is called *equilibrium* if

$$\begin{array}{ll} \lambda_i + c_{ij}(x_{ij}) = \mu_j \text{ for } & x_{ij} > 0, \\ \lambda_i + c_{ij}(x_{ij}) \geq \mu_j \text{ for } & x_{ij} = 0, \end{array} \quad \forall (i, j) \in M \times N.$$

Thus, if the sum of the supplier's price and the delivery costs for a customer exceeds his/her demand price, then the supplier will face with the cancelled delivery.

In the case of fixed supply and demand, an equilibrium allocation pattern can be obtained as a solution to the following optimization problem (see, for instance, (Nagurney, 1993; Patriksson, 1994)):

$$\min_x \sum_{i\in M}\sum_{j\in N}\int_0^{x_{ij}} c_{ij}(u)du$$

subject to

$$\sum_{j\in N} x_{ij} = s_i \quad \forall i \in M,$$

$$\sum_{i\in M} x_{ij} = d_j \quad \forall j \in N,$$

$$x_{ij} \geq 0 \quad \forall i, j \in M \times N,$$

under

$$\sum_{i\in M} s_i = \sum_{j\in N} d_j.$$

From practical perspectives, fixed supply and demand seem to be not very valuable. In this paper, we offer a brief overview of supply-demand relocation patterns obtained for elastic, shortage, and overproduction cases. Such kinds of results are valuable since they allow one to develop different competitive distribution models to facilitate the decision-making of supply chain managers. In particular, supply chain managers can use available patterns to design decision-making strategies that mitigate risks concerning disruption or ripple effects.

3. Elastic Demand and Supply

Krylatov and Lonyagina (2022) obtained allocation patterns for elastic demand and supply. They assume that in a single-commodity network, there is a continuously differentiable dependence between the purchase price and demand

$$\mu_j = p_j(d_j), \quad \forall j \in N,$$

and a continuously differentiable dependence between the sale price and the supply

$$\lambda_i = r_i(s_i), \quad \forall i \in M,$$

while the overall demand is equal to the overall supply, i.e.,

$$\sum_{i\in M} s_i = \sum_{j\in N} d_j.$$

In such a case, an equilibrium allocation pattern can be obtained as a solution to the following optimization problem:

$$\min_{d,s,x}\left(\sum_{i\in M}\int_0^{s_i} r_i(v)dv + \sum_{i\in M}\sum_{j\in N}\int_0^{x_{ij}} c_{ij}(u)du - \sum_{j\in N}\int_0^{d_{ij}} p_j(z)dz\right) \tag{1}$$

subject to

$$\sum_{j\in N} x_{ij} = s_i, \quad \forall i \in M, \tag{2}$$

$$\sum_{i\in M} x_{ij} = d_j, \quad \forall j \in N, \tag{3}$$

$$x_{ij} \geq 0 \quad \forall\, (i,\ j) \in M \times N, \tag{4}$$

$$s_i \geq 0, \quad \forall i \in M, \tag{5}$$

$$d_j \geq 0, \quad \forall j \in N. \tag{6}$$

According to (Krylatov and Lonyagina, 2022), if functions of demand, supply, and logistic costs are affine functions:

$$\mu_j = p_j(d_j) = p_j^0 - k_j^p d_j, \quad p_j^0 > 0,\ k_j^p > 0, \quad \forall j \in N, \tag{7}$$

$$\lambda_i = r_i(s_i) = r_i^0 + k_i^r s_i, \quad r_i^0 \geq 0, \quad k_i^r > 0, \quad \forall i \in M, \tag{8}$$

$$c_{ij}(x_{ij}) = c_{ij}^0 + k_{ij}^t x_{ij}, \quad t_{ij}^0 \geq 0, \quad k_{ij}^t > 0, \quad \forall (i,\ j) \in M \times N, \tag{9}$$

then the equilibrium allocation in problem (1)–(6) is obtained by the following pattern:

$$d_j = \begin{cases} \frac{p_j^0 - \mu_j}{k_j^p}, \text{ if } p_j^0 > \mu_j, \\ 0, \qquad\ \text{if } p_j^0 \leq \mu_j, \end{cases} \quad \forall j \in N,$$

$$s_i = \begin{cases} \frac{\lambda_i - r_i^0}{k_i^r}, \text{ if } \lambda_i > r_i^0, \\ 0, \qquad\ \text{if } \lambda_i \leq r_i^0, \end{cases} \quad \forall i \in M,$$

$$x_{ij} = \begin{cases} \frac{\mu_j - \lambda_i - c_{ij}^0}{k_{ij}^t}, \ \text{if } \mu_j - \lambda_i > t_{ij}^0, \\ 0, \qquad\qquad \text{if } \mu_j - \lambda_i \leq t_{ij}^0, \end{cases} \quad \forall (i,\ j) \in M \times N,$$

where λ and μ satisfies the following matrix equation:

$$\begin{pmatrix} -B_r & B \\ -B^T & B_p \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} q_r \\ q_p \end{pmatrix}, \tag{10}$$

where B_r, B and B_p are such that

$$B_r = \operatorname{diag}\left\{ \sum_{j\in N} \frac{\delta_{1j}^t}{k_{1j}^t} + \frac{\delta_1^r}{k_1^r}, \ldots, \sum_{j\in N} \frac{\delta_{mj}^t}{k_{mj}^t} + \frac{\delta_m^r}{k_m^r} \right\},$$

$$B = \left\{ \delta_{ij}^t / k_{ij}^t \right\}_{i\in M, j\in N},$$

$$B_p = \operatorname{diag}\left\{ \sum_{i\in M} \frac{\delta_{i1}^t}{k_{i1}^t} + \frac{\delta_1^p}{k_1^p}, \ldots, \sum_{i\in M} \frac{\delta_{in}^t}{k_{in}^t} + \frac{\delta_n^p}{k_n^p} \right\},$$

and vectors q_r and q_p:

$$q_r = \left(\sum_{j\in N} \frac{c_{1j}^0 \delta_{1j}^t}{k_{1j}^t} - \frac{r_1^0 \delta_1^r}{k_1^r}, \ldots, \sum_{j\in N} \frac{c_{mj}^0 \delta_{mj}^t}{k_{mj}^t} - \frac{r_m^0 \delta_m^r}{k_m^r} \right)^T,$$

$$q_p = \left(\sum_{i\in M} \frac{c_{i1}^0 \delta_{i1}^t}{k_{i1}^t} + \frac{p_1^0 \delta_1^p}{k_1^p}, \ldots, \sum_{i\in M} \frac{c_{in}^0 \delta_{in}^t}{k_{in}^t} + \frac{p_n^0 \delta_n^p}{k_n^p} \right)^T.$$

Moreover, if $(d,\ s,\ x)$ is a solution to the problem (1)–(6), then there are $\hat{M} \subseteq M$ and $\hat{N} \subseteq N$ such that

$$d_j \begin{cases} > 0, \text{ for } j \in \hat{N}, \\ = 0, \text{ for } j \in N \backslash \hat{N}, \end{cases} \qquad s_i \begin{cases} > 0, \text{ for } i \in \hat{M}, \\ = 0, \text{ for } i \in M \backslash \hat{M}. \end{cases}$$

In such a case, the matrix equation (10) can be transformed as follows:

$$\begin{pmatrix} -\hat{B}_r & \hat{B} \\ -\hat{B}^T & \hat{B}_p \end{pmatrix} \begin{pmatrix} \hat{\lambda} \\ \hat{\mu} \end{pmatrix} = \begin{pmatrix} \hat{q}_r \\ \hat{q}_p \end{pmatrix}, \tag{11}$$

where $\hat{\mu}$ and $\hat{\lambda}$ contain only those components of the original vectors that correspond to the active (with non–zero optimal values) supply and demand variables, respectively. Fortunately, Krylatov and Lonyagina (2022) proved that in the case of m suppliers and 1 consumer (1 supplier and n consumers), the block matrix from the matrix equation (11) is invertible.

Finally, Krylatov and Lonyagina (2022) proved that in the case of 1 consumer and m suppliers (with affine transport functions (9), demand functions (7) and supply functions (8)) renumbered according to

$$r_1^0 + c_1^0 \leq \ldots \leq r_m^0 + c_m^0,$$

if there exists $\hat{m}$, $1 \leq \hat{m} \leq m$ such that

$$\begin{cases} p^0 - r_\tau^0 - c_\tau^0 > \sum\limits_{i=1}^{\hat{m}} \frac{k^p}{k_i^r + k_i^t} \left(r_\tau^0 + c_\tau^0 - r_i^0 - c_i^0 \right), \quad \forall \tau \leq \hat{m}, \\ \\ p^0 - r_\tau^0 - c_\tau^0 \leq \sum\limits_{i=1}^{\hat{m}} \frac{k^p}{k_i^r + k_i^t} \left(r_\tau^0 + t_\tau^0 - r_i^0 - t_i^0 \right), \quad \forall \tau > \hat{m}, \end{cases}$$

then the equilibrium flows distribution in the problem (1)–(6) is achieved by implementing the following pattern:

$$x_\tau \begin{cases} > 0, \text{ for } \tau \leq \hat{m}, \\ = 0, \text{ for } \tau > \hat{m}, \end{cases}$$

while the vector $\hat{x} = (x_1, \ldots, x_{\hat{m}})^T$ is:

$$\hat{x} = \hat{D}_r^{-1} \left(\left(e_{\hat{m}} - \hat{B}_r^{-1} \hat{B} \right) \frac{\hat{q}_p - \hat{B}^T \hat{B}_r^{-1} \hat{q}_r}{\hat{B}_p - \hat{B}^T \hat{B}_r^{-1} \hat{B}} + \hat{B}_r^{-1} \hat{q}_r - \hat{c}_r \right),$$

where $e_{\hat{m}} = (1, \ldots, 1)^T$ over $\dim e_{\hat{m}} = \hat{m} \times 1$,

$$\hat{D}_r = \operatorname{diag} \left\{ \frac{1}{k_1^t}, \ldots, \frac{1}{k_{\hat{m}}^t} \right\} \quad \text{and} \quad \hat{c}_r = \left(c_1^0, \ldots, c_{\hat{m}}^0 \right)^T.$$

On the contrary, in the case of 1 supplier and n consumers (with linear logistic costs (9), demand functions (7) and supply functions (8)) renumbered according to

$$p_1^0 - c_1^0 \geq \ldots \geq p_n^0 - c_n^0,$$

if there exists $\hat{n}$, $1 \leq \hat{n} \leq n$ such that

$$\begin{cases} p^0_\tau - r^0 - c^0_\tau > \sum\limits_{j=1}^{\hat{n}} \frac{k^r}{k^p_j + k^t_j} \left(p^0_j - t^0_j - p^0_\tau + t^0_\tau\right), & \forall \tau \leq \hat{n}, \\ p^0_\tau - r^0 - c^0_\tau \leq \sum\limits_{j=1}^{\hat{n}} \frac{k^r}{k^p_j + k^t_j} \left(p^0_j - t^0_j - p^0_\tau + t^0_\tau\right), & \forall \tau > \hat{n}, \end{cases}$$

then the equilibrium flows distribution in the problem (1)–(6) is achieved by implementing the following pattern:

$$x_\tau \begin{cases} > 0, \text{ for } \tau \leq \hat{n}, \\ = 0, \text{ for } \tau > \hat{n}, \end{cases}$$

where the vector $\hat{x} = (x_{11}, \ldots, x_{1\hat{n}})^T$ is:

$$\hat{x} = \hat{D}_p^{-1} \left(\left(\hat{B}_p^{-1} \hat{B}^T - e_{\hat{n}} \right) \frac{\hat{q}_r - \hat{B} \hat{B}_p^{-1} \hat{q}_p}{\hat{B}_r - \hat{B} \hat{B}_p^{-1} \hat{B}^T} + \hat{B}_p^{-1} \hat{q}_p - \hat{c}_p \right),$$

and $e_{\hat{n}} = (1, \ldots, 1)^T$ over $\dim e_{\hat{n}} = \hat{n} \times 1$,

$$\hat{D}_p = \operatorname{diag} \left\{ \frac{1}{k^t_1}, \ldots, \frac{1}{k^t_{\hat{n}}} \right\} \quad \text{and} \quad \hat{c}_p = \left(c^0_1, \ldots, c^0_{\hat{n}}\right)^T.$$

Therefore, the conditions on active commodity flows in a single-commodity network were obtained explicitly under affine mappings of elastic demand and supply. However, when supply manager faces such uncertainties as shortage or overproduction, supply and demand can no longer consider elastic.

4. Shortage Case

Krylatov et al. (2022) studied the problem of reallocating supply that results from the order promising process under shortage. They assume that the available supply is less than overall demand:

$$s < \sum_{i \in N} d_i.$$

In other words, a supply manager faces competitive supply relocation in a distribution network under a shortage. Thus, they introduced $\Delta > 0$ as a shortage supply value, i.e.,

$$\sum_{i \in N} d_i - s = \Delta,$$

while $\epsilon_i \geq 0$ as the difference between i-th demand and its actual delivery volume, i, $i \in N$, $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$:

$$\sum_{i \in N} d_i - x_i = \sum_{i \in N} \epsilon_i = \Delta.$$

Therefore, the allocation pattern, which satisfies the following optimization problem:

$$\min_x \sum_{i \in N} \int\limits_0^{x_i} c_i(u) du \tag{12}$$

subject to

$$\begin{array}{l} \sum\limits_{i\in N} x_i = s, \\ x_i \geq 0, \quad \forall\, i \in N \\ \epsilon_i \geq 0, \quad \forall i \in N \\ \sum\limits_{i\in N} \epsilon_i = \Delta \end{array}. \tag{13}$$

is the equilibrium allocation pattern under shortage supply. According to (Krylatov et al., 2022), if storage costs increase when the volume of the order increases:

$$c_i(z) = c_i^0 + k_i z, \quad c_i^0 \geq 0, \;\; k_i > 0, \;\; \forall i \in N,$$

then the equilibrium allocation of shortage supply in problem (12)–(13) is obtained by the following pattern:

$$x_i = \begin{cases} \frac{\mu_i - \lambda - c_i^0}{k_i}, & \text{if } \mu_i - \lambda > c_i^0, \\ 0, & \text{if } \mu_i - \lambda \leq c_i^0, \end{cases} \quad \forall i \in N,$$

where λ and μ satisfy

$$\begin{array}{l} \sum_{i\in N} \frac{\mu_i - \lambda - c_i^0}{k_i}\delta_i = s, \\ \frac{\mu_i - \lambda - c_i^0}{k_i}\delta_i = d_i - \epsilon_i, \quad \forall i \in N \\ \sum\limits_{i\in N} \epsilon_i = \Delta \\ \mu_i = \eta, \quad \text{if } \epsilon_i > 0 \\ \mu_i = \eta - \beta_i, \quad \text{if } \epsilon_i = 0 \end{array}$$

Moreover, let us order customers as follows:

$$c_1^0 + k_1 d_1 \geq c_2^0 + k_2 d_2 \geq \cdots \geq c_n^0 + k_n d_n.$$

If there is $\bar{n}$ such that

$$\begin{array}{l} \sum_{i=1}^{\bar{n}} \frac{\left(c_i^0 + k_i d_i\right) - \left(c_\tau^0 + k_\tau d_\tau\right)}{k_i} < \Delta, \quad \forall \tau = 1, \ldots, \bar{n}, \\ \sum\limits_{i=1}^{\bar{n}} \frac{\left(c_i^0 + k_i d_i\right) - \left(c_\tau^0 + k_\tau d_\tau\right)}{k_i} \geq \Delta, \quad \forall \tau = \bar{n}+1, \ldots, n, \end{array}$$

then for $i = \overline{1, \bar{n}}$:

$$x_i = \begin{cases} 0, & \text{if } c_i^0 \geq \frac{\sum\limits_{l=1}^{\bar{n}} \frac{c_l^0 + d_l k_l}{k_l} - \Delta}{\sum_{l=1}^{\bar{n}} \frac{1}{k_l}}, \\ \frac{\sum_{l=1}^{\bar{n}} d_l - \Delta + \sum_{l=1}^{\bar{n}} \frac{c_l^0 - c_i^0}{k_l}}{k_i \sum_{l=1}^{\bar{n}} \frac{1}{k_l}}, & \text{if } c_i^0 < \frac{\sum_{l=1}^{\bar{n}} \frac{c_l^0 + d_l k_l}{k_l} - \Delta}{\sum_{l=1}^{\bar{n}} \frac{1}{k_l}}, \end{cases}$$

and $x_i = d_i$ for all $i = \overline{\bar{n}+1, n}$.

5. Overproduction Case

Krylatov et al. (2022) studied the reallocating supply problem that result from the order promising process under overproduction. They assumed that available supply is more than the overall demand:

$$d < \sum_{i\in M} s_i.$$

In other words, a supply manager faces competitive supply relocation in a distribution network under overproduction. Thus, they introduced $\Delta > 0$ as an overproduction value:

$$\sum_{i \in M} s_i - d = \Delta,$$

while $\epsilon_i \geq 0$ as the difference between i-th demand and its actual delivery volume, i, $i \in M$, $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_m)$:

$$\sum_{i \in M} (s_i - x_i) = \sum_{i \in M} \epsilon_i = \Delta.$$

In terms of a single-commodity network, the allocation pattern, which satisfies the following optimization problem:

$$\min_x \sum_{i \in M} \int\limits_0^{x_i} c_i(u) du, \tag{14}$$

subject to

$$\begin{gathered} \sum_{i \in M} x_i = d, \\ x_i = s_i - \epsilon_i, \quad \forall i \in M, \\ x_i \geq 0, \quad \forall\, i \in M, \\ \epsilon_i \geq 0, \quad \forall i \in M, \\ \sum_{i \in M} \epsilon_i = \Delta. \end{gathered} \tag{15}$$

is the equilibrium allocation pattern under overproduction. According to (Krylatov et al., 2022), if delivery costs increase when the volume of the order increases:

$$c_i(z) = c_i^0 + k_i z, \quad c_i^0 \geq 0, \;\; k_i > 0, \;\; \forall i \in M,$$

then the equilibrium deliveries allocation of overproduction in problem (14)–(15) is obtained be the following pattern:

$$x_i = \begin{cases} \frac{\mu - \lambda_i - c_i^0}{k_i}, & \text{if } \mu - \lambda_i > c_i^0, \\ 0, & \text{if } \mu - \lambda_i \leq c_i^0, \end{cases} \quad \forall i \in M,$$

where λ and μ satisfy

$$\begin{array}{l} \sum_{i \in M} \frac{\mu - \lambda_i - c_i^0}{k_i} \delta_i = d, \\ \frac{\mu - \lambda_i - c_i^0}{k_i} \delta_i = s_i - \epsilon_i, \quad \forall i \in M \\ \sum\limits_{i \in M} \epsilon_i = \Delta \\ \lambda_i = \eta, \quad \text{if } \epsilon_i > 0 \\ \lambda_i = \eta + \beta_i, \quad \text{if } \epsilon_i = 0 \end{array}$$

Moreover, let us order suppliers as follows:

$$c_1^0 + k_1 s_1 \geq c_2^0 + k_2 s_2 \geq \cdots \geq c_m^0 + k_m s_m.$$

If there is $\bar{m}$ such that

$$\begin{aligned} &\textstyle\sum_{i=1}^{\bar{m}} \frac{\left(c_i^0+k_i s_i\right)-\left(c_\tau^0+k_\tau s_\tau\right)}{k_i} < \Delta, \quad \forall \tau = 1,\ldots,\bar{m}, \\ &\sum_{i=1}^{\bar{m}} \frac{\left(c_i^0+k_i s_i\right)-\left(c_\tau^0+k_\tau s_\tau\right)}{k_i} \geq \Delta, \quad \forall \tau = \bar{m}+1,\ldots,m, \end{aligned}$$

then

$$x_i = \begin{cases} 0, & \text{if } c_i^0 \geq \dfrac{\sum\limits_{l=1}^{\bar{m}} \frac{c_l^0+s_l k_l}{k_l} - \Delta}{\sum\limits_{l=1}^{\bar{m}} \frac{1}{k_l}}, \\ \dfrac{\sum\limits_{l=1}^{\bar{m}} s_l - \Delta + \sum\limits_{l=1}^{\bar{m}} \frac{c_l^0 - c_i^0}{k_l}}{k_i \sum\limits_{l=1}^{\bar{m}} \frac{1}{k_l}}, & \text{if } c_i^0 < \dfrac{\sum\limits_{l=1}^{\bar{m}} \frac{c_l^0+s_l k_l}{k_l} - \Delta}{\sum\limits_{l=1}^{\bar{m}} \frac{1}{k_l}}, \end{cases} \quad \forall i = \overline{1,\bar{m}},$$

and $x_i = s_i$ for all $i = \overline{\bar{m}+1, m}$.

6. Conclusion

This paper is devoted to the recent findings in the analytical research of supply-demand allocation in a single-commodity network with distant (in space) suppliers and consumers. The allocation problem was formulated as an equilibrium flow assignment problem with affine functions of demand, supply, and logistic costs in a network represented by a digraph with suppliers and consumers located in nodes. We offered a brief overview of supply-demand relocation patterns obtained for elastic, shortage, and overproduction cases. Such kinds of results seem valuable since they allow one to develop different competitive distribution models to facilitate the decision-making of supply chain managers. In particular, supply chain managers can use available patterns to design decision-making strategies that mitigate risks concerning disruption or ripple effects.

The closed-form solution under elastic supply and demand contributes to the field of long-term management since elasticity means smooth market reaction to changes in prices and commodity flows. In other words, equality

$$\sum_{i\in M} s_i = \sum_{j\in N} d_j$$

is assumed to be held in the market at all times. On the contrary, the closed-form solutions under shortage or overproduction contribute to the field of short-term management since such effects appear at a certain moment and should be mitigated on time. For example, at moment t there can be $|M|$ suppliers who satisfy the demand of $|N|$ consumers, i.e., overall supply is equal to overall demand. However, if at moment $t+1$ new supplier q comes to the market with non-zero supply, then, at that moment, the market faces overproduction:

$$\sum_{i\in M\cup q} s_i > \sum_{j\in N} d_j.$$

No doubt that at moment $t+1$ suppliers from M have to cope with the effects of flow relocation and price changing.

To conclude, let us pay attention to the positioning of the considered results in supply chain management. The shortage, overproduction, and reliable supply chain planning are of concern to many researchers. Samuel and Mahanty (2003) studied shortage gaming as a leading contributor to the bullwhip phenomenon. Barron and Hermel (2017) considered different decision policies under shortage, while Najid et al. (2011) investigated an integrated production and maintenance planning model with time windows and shortage costs. Ji et al. (2021) developed machine learning techniques for reducing underproduction costs and overproduction costs. Grillo et al. (2016) supports the importance of shortage planning. Grillo et al. (2018) pointed out that the frequency of unexpected events increases when companies are characterized by a lack of homogeneity in the product, which renders having to execute the shortage planning process more frequently. In this paper, we have collected results which, in our opinion, are able to give a fresh look at the methodological approaches in the field.

References

Barrett, C. B. and Li, J. R. (2002). *Distinguishing between equilibrium and integration in spatial price analysis.* Am. J. Agric. Econ., **84(2)**, 292–307.

Barron, Y. and Hermel, D. (2017). *Shortage decision policies for a fluid production model with map arrivals.* International Journal of Production Research, **55(14)**, 3946–3969.

Bramoulle, Y. and Kranton, R. (2007). *Public goods in networks.* J. Econ. Theory, **135**, 478–494.

Enke, S. (1951). *Equilibrium among spatially separated markets: solution by electric analogue.* Econometrica, **19(1)**, 40–47.

Enke, S. (1942). *Space and value.* Q. J. Econ., **56(4)**, 627–637.

Florian, M. and Los, M. (1982). *A new look at static spatial price equilibrium models.* Reg. Sci. Urban Econ., **12(4)**, 579–597.

Grillo, H., Alemany, M. and Ortiz, A. (2016). *A review of mathematical models for supporting the order promising process under lack of homogeneity in product and other sources of uncertainty.* Computers and Industrial Engineering, **91**, 239–261.

Grillo, H., Alemany, M., Ortiz, A. and Mula, J. (2018). *A fuzzy order promising model with non-uniform finished goods.* International Journal of Fuzzy Systems, **20**, 187–208.

Ji, B., Ameri, F. and Cho, H. (2021). *A non-conformance rate prediction method supported by machine learning and ontology in reducing underproduction cost and overproduction cost.* International Journal of Production Research, **59(16)**, 5011–5031.

Kiselev, A. O. and Yurchenko, N. I. (2021). *Game equilibria and transition dynamics in a dyad with heterogeneous agents.* Autom. Remote Control, **82(3)**, 549–564.

Kovalenko, A. G. (2018). *In search of equilibrium state at the spatial dispersed markets with imperfect competition of a uniform product.* Ekon. Mat. Metody, **54(1)**, 52–68 (in Russian).

Kovalenko, A. G., Khachaturov, V. R. and Kalimoldaev, M. N. (2012). *Models of a dispersed market of imperfect competition: problems of their development, application in the management of the regional economy.* Probl. Inf., **4(16)**, 18–23 (in Russian).

Krylatov, A. Y. and Lonyagina, Y. E. (2022). *Equilibrium flow assignment in a network of homogeneous goods.* Autom. Remote Control, **83**, 805–827.

Krylatov, A., Lonyagina, Y. and Raevskaya, A. (2022). *Competitive supply allocation in a distribution network under overproduction.* Lecture Notes in Computer Science (to appear).

Krylatov, A., Lonyagina, Y. and Raevskaya, A. (2022). *Competitive relocation of shortage supply in a distribution network.* IFAC-PapersOnLine (to appear).

Marshall, A. (1920). *Principles of Economics.* Macmillan: London.

McNew, K. (1996). *Spatial market integration: definition, theory, and evidence.* Agric. Resour. Econ. Rev., **25(1)**, 1–11.

Nagurney, A. (1993) *Network Economics: a Variational Inequality Approach* Kluwer Acad.: Amsterdam.

Najid, N., Alaoui-Selsouli, M. and Mohafid, A. (2011). *An integrated production and maintenance planning model with time windows and shortage cost.* International Journal of Production Research, **49(8)**, 2265–2283.

Novikov, D. A. (2014). *Games and networks.* Autom. Remote Control, **75(6)**, 1145–1154.

Patriksson, M. (1994). *The Traffic Assignment Problem: Models and Methods* Dover: New York.

Samuel, C. and Mahanty, B. (2003). *Shortage gaming and supply chain performance.* International Journal of Manufacturing Technology and Management, **5(5/6)**, 536–548.

Samuelson, P. A. (1952). *Spatial price equilibrium and linear programming.* Am. Econ. Rev., **42(3)**, 283–303.

Stephens, E. C., Mabaya, E., von Cramon-Taubadel, S. and Barrett, C. B. (2012). *Spatial price adjustment with and without trade.* Oxford Bull. Econ. Stat., **74(3)**, 453–469.

Takayama, T. and Judge, G. G. (1964). *Equilibrium among spatially separated markets: a reformulation.* Econometrica, **32(4)**, 510–524.

Vasin, A. A. and Daylova, E. A. (2017). *Two-node market under imperfect competition.* Autom. Remote Control, **78(9)**, 1709–1729.

Vasin, A., Grigoryeva, O. and Tsyganov, N. (2020). *A model for optimization of transport infrastructure for some homogeneous goods markets*, J. Global Optim., **76(3)**, 499–518.