Санкт-Петербургский государственный университет

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Выпускная квалификационная работа

Вещественные числа Гурвица

Уровень образования: бакалавриат
Направление 01.03.01 "Математика"
Основная образовательная программа СВ.5000.2018 "Математика"

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Санкт-Петербург
2022

Saint Petersburg State University

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Graduation qualification thesis
Real Hurwitz numbers

Bachelor's program
"Mathematics"
Specialization and code: 01.03.01 "Mathematics"
Cipher of the EP: SV.5000.2018

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Saint Petersburg
2022

1 Introduction

Hurwitz numbers were introduced by A. Hurwitz in 1891 ([**H91**]). In general, they enumerate branched covers of the Riemann sphere with prescribed ramification data. Equivalently, they count factorizations in symmetric group. In this paper we study real Hurwitz numbers, which enumerate branched covers preserving real structure of the surface. Some results about real Hurwitz numbers can be found in [**MR15**], [**GMR16**] and [**KLN18**].

1.1 Outline of the content

In section 2, we recall the notion of Zonal polynomials and study the properties of the Gelfand pairs formed by symmetric group and its hyperoctahedral subgroup.

In section 3, we recall some basic facts about complex Hurwitz numbers and define real Hurwitz numbers in terms of both branched covers and factorizations in symmetric group. One significant distinction between complex and real Hurwitz numbers is the dependence on the location of branch points. In Section 3.3, we present r-real Hurwitz numbers, which are invariant under change of branch points positions.

It is well known, that generating series for simple single Hurwitz numbers can be expressed in the Schur basis. In Section 4, we will prove analogous fact for real Hurwitz numbers, but using Zonal polynomials instead of Schur functions. In [**CD20**], [**BCD21**] and [**BF21**], generating series for different types of Hurwitz numbers were already expressed in Zonal polynomials, so we present one more type of Hurwitz numbers, which can be computed in these terms.

In section 5, we derive cut-and-join equation for special kind of real Hurwitz numbers. Using this equation, Hurwitz numbers can be computed recursively.

1.2 Notations

Let $\mathcal{S}_d$ be the symmetric group of order d and $\mathcal{P}_d$ be the set of partitions of d. We identify partitions with the corresponding Young diagrams, and write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ for $\lambda \in \mathcal{P}_d$ with parts $\lambda_1, \lambda_2, \dots, \lambda_k$. Sometimes we will also use another notation $\lambda = (1^{p_1} 2^{p_2} \cdots d^{p_d})$ meaning that partition λ has exactly p_i parts equal to i.

We call partition λ *even* if all its parts $\lambda_1, \dots, \lambda_k$ are even. We call partition *co-even* if all numbers $p_1, p_2, \dots, p_d$ are even, i.e. every part occurs an even number of times. It is clear that if λ is an even partition, then its conjugate partition λ' is co-even, and vice versa (conjugate means that the Young diagram of λ' is obtained by transposing the diagram of λ).

Denote by $\mathcal{C}(\sigma) \vdash d$ the cycle type of the element σ. Recall that for a box $\square = (i, j) \in \lambda$ in the j^{th} row and i^{th} column $(1 \le j \le \ell(\lambda), 1 \le i \le \lambda_j)$, its content is equal to $i - j$ and its 2-deformation is defined as follows:

$$c_2(\square) = 2(i-1) - (j-1).$$

In section 4, we use function

$$w(\lambda) = \sum_{\square \in \lambda} d + 2c_2(\square). \tag{1.1}$$

Observe that $c\left((2i-1, j)\right) + c\left((2i, j)\right) = 4i - 2j - 1 = 2c_2\left((i, j)\right) + 1$, so

$$w(\lambda) = \sum_{\square \in 2\lambda} c(\square).$$

We say that a skew Young diagram is a horizontal strip if every column contains at most one box. Denote by $\mathrm{HS}(d)$ the set of horizontal strips containing d boxes.

We write $\mathcal{I}_d$ for the set of all involutive permutations on d elements and $\mathcal{I}_d^r$ for involutions with r fixed points (obviously, $\mathcal{I}_d^r \neq \varnothing$ implies $d - r$ is even). Denote by $\Lambda = \bigoplus_{d \in \mathbb{N}_0} \Lambda^d$ the graded algebra of symmetric functions with scalar product $(\cdot,\cdot)$, which is defined for power-sums by

$$(p_\lambda, p_\mu) = \delta_{\lambda,\mu} z_\lambda,$$

where $z_\lambda = \prod i^{r_i} i!$ with $\lambda = 1^{r_1} 2^{r_2} \dots d^{r_d}$.

It is well known that the space of central functions $C(\mathcal{S}_d)$ is isomorphic to the space of degree d symmetric functions Λ^d. This isomorphism is called Frobenius characteristic map and denoted by $\mathrm{ch} : C(\mathcal{S}_d) \to \Lambda^d$. To each characteristic function ψ_λ of conjugacy class λ it assigns the power-sum $\mathrm{ch}(\psi_\lambda) = \frac{1}{z_\lambda} p_\lambda$, so for any central function f its image can be expressed as

$$\mathrm{ch}(f) = \frac{1}{d!} \sum_{\sigma \in \mathcal{S}_d} f(\sigma) p_{\mathcal{C}(\sigma)} = \sum_{\mu \vdash d} \frac{1}{z_\mu} f(\mu) p_\mu. \tag{1.2}$$

Under this isomorphism, the irreducible character χ^λ is mapped to the Schur function s_λ.

2 Hyperoctahedral subgroup and zonal polynomials

2.1 Gelfand pair $(\mathcal{S}_d, \mathcal{H}_d)$

In this subsection we assume that d is even number.

Definition 2.1. The pair (G, K) of group and its subgroup is called a *Gelfand pair* if for any irreducible representation V of G, the space V^K of K-invariant vectors in V is no-more-than-1-dimensional.

The definition of Gelfand pairs is closely related to the space of double K-invariant functions

Definition 2.2. A function $f : G \to \mathbb{Q}$ is said to be *double K-invariant* if

$$f(kxk') = f(x) \qquad \text{for all } x \in G, k, k' \in K.$$

We denote the space of all double K-invariant functions by $C(G; K)$

We equip the space $C(G; K)$ with the multiplication given by convolution.

Lemma 2.3 ([**Mac95**], VII, 1.1)**.** *Let G be a group and $K \leq G$. The pair (G, K) is a Gelfand pair if and only if the algebra $C(G; K)$ is commutative.*

Consider now the symmetric group $\mathcal{S}_d$ and the involution

$$\gamma_0 = (1\ 2)\,(3\ 4) \cdots (d-1\ d) \in \mathcal{S}_d.$$

Hyperoctahedral subgroup $\mathcal{H}_d$ is the centralizer $C_{\mathcal{S}_d}(\gamma_0)$. The cardinality of $\mathcal{H}_d$ is equal to $2^{d/2}(d/2)! = d!!$. Although the standard notation for the hyperoctahedral subgroup of $\mathcal{S}_d$ is $\mathcal{H}_{\frac{d}{2}}$, in this paper we will use $\mathcal{H}_d$ instead. Double $\mathcal{H}_d$-classes of $\mathcal{S}_d$ are indexed with partitions of $\frac{d}{2}$. To see this, observe that the permutation

$$[\sigma^{-1}, \gamma_0] = (\sigma(1)\ \sigma(2))\,(\sigma(3)\ \sigma(4)) \cdots (\sigma(d-1)\ \sigma(d)) \cdot (1\ 2)\,(3\ 4) \cdots (d-1\ d)$$

has the cycle type of the form $\nu \cup \nu = (\nu_1, \nu_1, \nu_2, \nu_2, \dots, \nu_k, \nu_k)$ (i.e. every part occurs an even number of times). So we assign to each double $\mathcal{H}_d$-coset the partition $\mathcal{H}(\sigma) = (\nu_1, \nu_2, \dots, \nu_k) \vdash d$, called the *coset-type* of σ. This correspondence between double $\mathcal{H}_d$-classes of $\mathcal{S}_d$ and partitions of $d/2$ turns out to be a bijection.

Also, the the hyperoctahedral group $\mathcal{H}_d$ is isomorphic to a wreath product $\mathcal{S}_2 \wr \mathcal{S}_{\frac{d}{2}}$. For more detailed information about Gelfand pairs and hyperoctahedral group $\mathcal{H}_d$ in symmetric functions theory, see e.g. [**Mac95**] (VII).

Theorem 2.4 ([**Mac95**], VII, 2.2)**.** *The pair $(\mathcal{S}_d, \mathcal{H}_d)$ is a Gelfand pair for all even d.*

Take any partition $\lambda \vdash d$ and consider the function from $\mathcal{S}_d$ to $\mathbb{Q}$:

$$\sigma \mapsto \frac{1}{|\mathcal{H}_d|} \sum_{h \in \mathcal{H}_d} \chi^\lambda(\sigma \cdot h) \tag{2.1}$$

The right $\mathcal{H}_d$-invariance of the above function is trivial, while the left $\mathcal{H}_d$-invariance holds due to the centrality of χ^λ, so this function belongs to $C(\mathcal{S}_d; \mathcal{H}_d)$.

Proposition 2.5 ([**Mac95**])**.** *The function defined in (2.1) is non-zero if and only if the partition λ is even. For $\lambda = 2\nu$, this function is called zonal spherical function and denoted by ω^ν*

$$\omega^\nu(\sigma) = \frac{1}{|\mathcal{H}_d|} \sum_{h \in \mathcal{H}_d} \chi^{2\nu}(\sigma \cdot h).$$

Zonal spherical functions $\{\omega^\nu\}_{\nu \vdash \frac{d}{2}}$ form a basis for the space $C(\mathcal{S}_d; \mathcal{H}_d)$.

The correspondence between double $\mathcal{H}_d$-invariant functions (see definition 2.2) and symmetric functions can be obtained analogously to (1.2). Namely, define a mapping

$$\begin{aligned} \mathrm{ch}' : C(\mathcal{S}_d; \mathcal{H}_d) &\to \Lambda^{d/2} \\ f &\mapsto \sum_{\sigma \in \mathcal{S}_d} f(\sigma)\, p_{\mathcal{H}(\sigma)}. \end{aligned}$$

Lemma 2.6 ([**Mac95**])**.** *The mapping $\mathrm{ch}' : C(\mathcal{S}_d; \mathcal{H}_d) \to \Lambda^{d/2}$ is an isomorphism of $\mathbb{Q}$-algebras.*

Finally, we can introduce an analogue of Schur functions.

Definition 2.7. For each partition ν of d, we define

$$Z_\nu = |\mathcal{H}_d|^{-1}\, \mathrm{ch}'(\omega^\nu),$$

i.e.

$$Z_\nu = |\mathcal{H}_d| \sum_{\mu \vdash \frac{d}{2}} \frac{1}{z_{2\mu}} \omega^\nu_\mu\, p_\mu,$$

where ω^ν_μ is the value of ω^ν at elements of the coset-type μ. Symmetric functions Z_ν are called zonal polynomials.

More generally, suppose $\gamma \in \mathcal{I}_d$ (now d is not necessarily even) is any involution. It can be written as the product of disjoint transpositions

$$\gamma = (a_1\ b_1)\,(a_2\ b_2)\ \cdots\ (a_k\ b_k).$$

To each permutation $\sigma \in \mathcal{S}_d$ we attach an undirected graph $\Gamma_\gamma(\sigma)$ with vertices $1, 2, \ldots, d$ and edges $\varepsilon_i, \varepsilon_i^\sigma$ $(1 \le i \le k)$, where ε_i joins vertices a_i, b_i and ε_i^σ joins $\sigma(a_i), \sigma(b_i)$. We also assume that edges ε_i are red, while ε_i^σ are blue. The graph $\Gamma_\gamma(\sigma)$ consists of even cycles (whose half-sizes we denote by $\nu_1, \ldots, \nu_k$) and disjoint paths. If the first and last edges of the path are of the different colour (in particular, such path can have only one vertex), then its length (i.e. the number of vertices) is odd. We denote the length of such paths by $\rho_1, \ldots, \rho_r$. Further, suppose that paths with both red ends have lengths $\hat\rho_1, \ldots, \hat\rho_{\hat r}$, and likewise lengths of paths with both blue ends are $\check\rho_1, \ldots, \check\rho_{\check r}$. The tuple of partitions $(\nu, \rho, \hat\rho, \check\rho) = \mathcal{H}_\gamma(\sigma)$ is called *γ-coset-type* of σ. Note that if γ is fixed-point-free, then $\rho = \hat\rho = \check\rho = \varnothing$.

In the next sections, we will be counting the number of permutations with given γ-coset-type. Let p_1, p_2, $\ldots$, q_1, $q_2, \ldots$, $\hat q_1$, $\hat q_2$, $\ldots$, $\check q_1$, $\check q_2$, $\ldots$ be independent variables. To each permutation $\sigma \in S_d$ we assign

$$p_{\mathcal{H}_\gamma(\sigma)} = p_\nu \cdot q_\rho \cdot \hat q_{\hat\rho} \cdot \check q_{\check\rho} = p_{\nu_1} \cdots p_{\nu_k} \cdot q_{\rho_1} \cdots q_{\rho_r} \cdot \hat q_{\hat\rho_1} \cdots \hat q_{\hat\rho_{\hat r}} \cdot \check q_{\check\rho_1} \cdots \check q_{\check\rho_{\check r}} \in \Lambda[q, \hat q, \check q]. \tag{2.2}$$

Here, we interpret variable p_ν as the corresponding power-sum in Λ.

2.2 Gelfand pair $(\mathcal{S}_d, \mathcal{H}_{d-1})$

Now, assume d is an odd number and the involution γ has exactly one fixed point. Then centralizer $C_{\mathcal{S}_d}(\gamma) \leq \mathcal{S}_d$ is isomorphic to $\mathcal{H}_{d-1}$ ($\mathcal{H}_{d-1}$ acts on non-fixed points of γ).

Theorem 2.8. *For an odd number d, the pair $(\mathcal{S}_d, \mathcal{H}_{d-1})$ is a Gelfand pair.*

Proof. We'll show that the restriction of any irreducible $\mathcal{S}_d$-representation to $\mathcal{H}_{d-1}$ contains the trivial representation $1_{\mathcal{H}_{d-1}}$ with multiplicity at most one. Let χ_λ be an irreducible character of $\mathcal{S}_d$ corresponding to the partition λ. Since $\mathcal{H}_{d-1} \leq \mathcal{S}_{d-1} \leq \mathcal{S}_d$, we may first restrict χ_λ to $\mathcal{S}_{d-1}$

$$\operatorname{Res}^{\mathcal{S}_d}_{\mathcal{S}_{d-1}} \chi_\lambda = \bigoplus_{\substack{\mu \vdash d-1 \\ \mu \subset \lambda}} \chi_\mu$$

and observe that $\langle \operatorname{Res}^{\mathcal{S}_{d-1}}_{\mathcal{H}_{d-1}} \chi_\mu, 1_{\mathcal{H}_{d-1}} \rangle = 1$ if μ is even, and 0 otherwise. Therefore, the multiplicity of the trivial character in $\operatorname{Res}^{\mathcal{S}_d}_{\mathcal{H}_{d-1}} \chi_\lambda$ is the number of ways to remove one box from λ and obtain an even partition, which is clearly at most one. ■

We established the following decomposition for induced character

$$1^{\mathcal{S}_d}_{\mathcal{H}_{d-1}} = \bigoplus_{\substack{\mu \vdash d \\ \text{at most 1 row of } \mu \text{ is even}}} \chi_\mu$$

Remark 2.9. We deduced that the centralizer of γ with at most one fixed point forms a Gelfand pair. Moreover, if γ has $k \geq 2$ fixed points, then $C_{\mathcal{S}_d}(\gamma) \cong \mathcal{H}_{d-k} \times \mathcal{S}_k$ and $(\mathcal{S}_d, \mathcal{H}_{d-k} \times \mathcal{S}_k)$ is not a Gelfand pair. Generally, (assuming $k - d$ is even) we have the decomposition

$$1^{\mathcal{S}_d}_{\mathcal{H}_{d-k} \times \mathcal{S}_k} = \bigoplus_{\mu \vdash d} \chi_\mu \cdot \#\{\nu \in \mathrm{HS}(k) \,|\, \mu/\nu \text{ is even}\}.$$

3 Hurwitz Numbers

3.1 Complex Hurwitz Numbers

We briefly recall the definition of complex Hurwitz numbers. Throughout this section we fix an integer d, which is the degree of coverings we will consider. Let C be a Riemann surface of genus g and fix a collection of points $\underline{x} = \{x_1, x_2, \ldots, x_k\} \subset \mathbb{CP}^1$.

Definition 3.1. Let $\lambda^1, \ldots, \lambda^k \in \mathcal{P}_d$ be partitions. *A complex Hurwitz covering* of type $(g; \lambda^1, \ldots, \lambda^k, \underline{x})$ is a degree d branched covering of $\mathbb{CP}^1$ by some genus g surface C such that the ramification profile over x_i is λ^i for $i = 1, 2, \ldots, k$.

We define the complex Hurwitz numbers $h^{\mathbb{C}}(g; \lambda^1, \ldots, \lambda^k)$ as the weighted number of complex Hurwitz coverings of the corresponding type:

$$h^{\mathbb{C}}(g; \lambda^1, \ldots, \lambda^k) = \sum_{[\pi]} \frac{1}{|\operatorname{Aut}(\pi)|},$$

where we sum over all equivalence classes of complex Hurwitz coverings of type $(g;\ \lambda^1, \ldots, \lambda^k)$, and $\operatorname{Aut}(\pi)$ is the automorphism group of π. It is a classical result that this number does not depend on the positions of points in $\underline{x}$. We also mention that the Riemann-Hurwitz formula implies

$$2g-2+\sum_{i=1}^{k}\ell(\lambda^i)=(k-2)d, \tag{3.1}$$

so g is uniquely determined by $\lambda^1,\dots,\lambda^k$.

There is an equivalent definition of complex Hurwitz numbers via symmetric groups.

Definition 3.2. Let $\lambda^1,\dots,\lambda^k\in\mathcal{P}_d$ be partitions. *A factorization* of type $(\lambda^1,\dots,\lambda^k)$ is a tuple $(\tau_1,\dots,\tau_k)$ of permutations in $\mathcal{S}_d$ such that

- $\mathcal{C}(\tau_i)=\lambda^i$ for $i=1,2,\dots,k$;
- $\tau_1\tau_2\cdots\tau_m=1$.

We denote by $\mathcal{F}^{\mathbb{C}}(\lambda^1,\dots,\lambda^k)$ the set of all factorizations of type $(\lambda^1,\dots,\lambda^k)$.

Theorem 3.3 (Hurwitz, [**H91**]). *For integer g and partitions $\lambda^1,\dots,\lambda^k$ satisfying (3.1),*

$$h^{\mathbb{C}}(g;\lambda^1,\dots,\lambda^k)=\frac{1}{d!}|\mathcal{F}^{\mathbb{C}}(\lambda^1,\dots,\lambda^k)|.$$

The important special case of the complex Hurwitz numbers is the Hurwitz numbers of the type $(\lambda,(1^{d-2}2^1),\dots,(1^{d-2}2^1))$, where the partition $(1^{d-2}2^1)$ repeats m times. These numbers are called *simple single Hurwitz numbers*. To light the notation, we denote by $h_m^{\mathbb{C}}(\lambda)$ the corresponding simple single Hurwitz number. We can collect all these numbers to a generating series

$$\mathbb{H}^{\mathbb{C}}=\sum_{m\in\mathbb{N}_0,\lambda\vdash d}h_m^{\mathbb{C}}(\lambda)p_{\lambda_1}\cdots p_{\lambda_k}\frac{u^m}{m!},$$

where u and $p_1,p_2,\dots$ are independent variables. We can think of p_i as the corresponding power-sums, then $\mathbb{H}^{\mathbb{C}}$ is an element of $\Lambda[[u]]$.

Proposition 3.4. *The series $\mathbb{H}^{\mathbb{C}}$ can be rewritten in the Schur basis*

$$\mathbb{H}^{\mathbb{C}}=\sum_{m\in\mathbb{N}_0,\lambda\vdash d}h_m^{\mathbb{C}}(\lambda)p_\lambda\frac{u^m}{m!}=\frac{1}{d!}\sum_{\lambda\vdash d}\dim(\lambda)s_\lambda e^{\sum c(\square)u},$$

where $\sum c(\square)$ is the sum of the contents of all boxes in λ.

3.2 Real Hurwitz Numbers

Now we define the real Hurwitz numbers and give equivalent definition in terms of factorizations in symmetric group.

Suppose that C is a Riemann surface of genus g and $\iota:C\to C$ is an orientation-reversing involution. We call a pair (C,ι) a *real surface*. Call a branched covering $\pi:C_1\to C_2$ of real surfaces $(C_1,\iota_1),(C_2,\iota_2)$ *real* if it respects the real structures, i.e. $\pi\circ\iota_1=\iota_2\circ\pi$. We say that two real branched coverings $\pi_1:(C_1,\iota_1)\to(C,\iota),\pi_2:(C_2,\iota_2)\to(C,\iota)$ are *equivalent* if there exists isomorphism $f:(C_1,\pi_1)\to(C_2,\pi_2)$ of complex covers such that $f\circ\iota_1=\iota_2\circ f$.

Real Hurwitz numbers count real branched coverings of $(\mathbb{CP}^1,\mathrm{conj})$, where $\mathrm{conj}:z\mapsto\overline{z}$ is complex conjugation. Suppose that $\pi:(C,\iota)\to(\mathbb{CP}^1,\mathrm{conj})$ is a real branched covering and let $y\in\mathbb{CP}^1\setminus\mathbb{RP}^1$ be a non-real branch point of π. It can be easily seen that $\mathrm{conj}(y)$ is also a branch point of π. Moreover, ramification profiles over y and $\mathrm{conj}(y)$ are the same.

Definition 3.5. Let $\lambda^1, \ldots, \lambda^k$ and $\mu^1, \ldots, \mu^m$ be partitions of integer d, $x_1 < x_2 < \cdots < x_k \in \mathbb{RP}^1 \subset \mathbb{CP}^1$ be an increasing sequence of points, and $y_1, y_2, \ldots, y_m \in \mathbb{CP}^1 \setminus \mathbb{RP}^1$ be a sequence of distinct non-conjugated points. A *real Hurwitz cover* of type $(g; \lambda^1, \ldots, \lambda^k, \underline{x}; \mu^1, \ldots, \mu^m, \underline{y})$ is a degree d real branched covering of $(\mathbb{CP}^1, \mathrm{conj})$ by some genus g real surface (C, ι) such that the ramification profiles over x_i and y_j are λ^i and μ^j respectively.

We define the real Hurwitz numbers $h^{\mathbb{R}}(g; \lambda^1, \ldots, \lambda^k; \mu^1, \ldots, \mu^m)$ as the weighted number of real Hurwitz coverings of the corresponding type:

$$h^{\mathbb{R}}(g; \lambda^1, \ldots, \lambda^k; \mu^1, \ldots, \mu^m) = \sum_{[\pi,\iota]} \frac{1}{|\operatorname{Aut}^{\mathbb{R}}(\pi, \iota)|}, \tag{3.2}$$

where we sum over all equivalence classes of real Hurwitz coverings of type $(g;\ \lambda^1, \ldots, \lambda^k, \underline{x};\ \mu^1, \ldots, \mu^m, \underline{y})$. Note that the number of such coverings doesn't depend on the positions of points in $\underline{x}$ and $\underline{y}$ themselves, but only on the order of points in $\underline{x}$. In this case Riemann-Hurwitz formula (3.1) reads

$$2g - 2 + \sum_{i=1}^{k} \ell(\lambda^i) + 2\sum_{j=1}^{m} \ell(\mu^j) = (k + 2m - 2)d. \tag{3.3}$$

It will be helpful for us to give an equivalent definition in terms of permutations of symmetric group.

Definition 3.6. Let $\lambda^1, \ldots, \lambda^k, \mu^1, \ldots, \mu^m$ be partitions of integer d and m be a natural number. *Real factorization* π of type $(\lambda^1, \ldots, \lambda^k; \mu^1, \ldots, \mu^m)$ is a tuple $(\gamma; \tau_1, \ldots, \tau_k; \sigma_1, \ldots, \sigma_m)$ of permutations in $\mathcal{S}_d$ such that

- $\mathcal{C}(\tau_i) = \lambda^i$ and $\mathcal{C}(\sigma_j) = \mu^j$ for $i = 1, \ldots, k$, $j = 1, \ldots, m$;
- $\gamma\tau_1\tau_2\cdots\tau_i$ is an involutive permutation for $i = 0, 1, \ldots, m$ (in particular $\gamma^2 = 1$);
- $\gamma\left(\sigma_1\sigma_2\cdots\sigma_m\right)^{-1}\gamma\tau_1\tau_2\cdots\tau_k\sigma_1\sigma_2\cdots\sigma_m = 1$.

We denote by $\mathcal{F}(\lambda^1, \ldots, \lambda^k; \mu^1, \ldots, \mu^m)$ the set of all real factorizations of the type $(\lambda^1, \ldots, \lambda^k; \mu^1, \ldots, \mu^m)$.

The next theorem establishes the correspondence between these two definitions.

Theorem 3.7. *For any partitions $\lambda^1, \ldots, \lambda^k, \mu^1, \ldots, \mu^m$ and integer g satisfying (3.3) the following equality holds*

$$h^{\mathbb{R}}(g; \lambda^1, \ldots, \lambda^k; \mu^1, \ldots, \mu^m) = \frac{1}{d!}|\mathcal{F}(\lambda^1, \ldots, \lambda^k; \mu^1, \ldots, \mu^m)|.$$

Proof. The proof is similar to the proof of the corresponding theorem for complex Hurwitz numbers. Let $\pi : (C, \iota) \to (\mathbb{CP}^1, \mathrm{conj})$ be the real Hurwitz covering of the above type. Pick $x \in (x_k, x_1)$ and consider the loops as in the picture 1.

The preimage $\pi^{-1}(x)$ of the point x consists of d distinct points $\{p_1, \ldots, p_d\}$. The monodromy actions of the loops $t_1, \ldots, t_k, s_1, \ldots, s_m, \overline{s}_1, \ldots, \overline{s}_m$ on this set is represented by permutations $\tau_1, \ldots, \tau_k, \sigma_1, \ldots, \sigma_m, \overline{\sigma}_1, \ldots, \overline{\sigma}_m$ respectively. Let γ be the involutive permutation corresponding to the action of ι on $\pi^{-1}(x)$. Next, since $\overline{s}_j = \mathrm{conj}(s_j^{-1})$, we have $\overline{\sigma}_j = \gamma\sigma_j^{-1}\gamma$ for $j = 1, \ldots, m$. Further, it's not hard to see that

$$\mathrm{conj}(t_i \circ t_{i-1} \circ \cdots \circ t_1) = (t_i \circ t_{i-1} \circ \cdots \circ t_1)^{-1},$$

which yields

$$\gamma\tau_1\tau_2\cdots\tau_i\gamma = (\tau_1\tau_2\cdots\tau_i)^{-1},$$

so the second condition of 3.6 is satisfied. The last condition is implied by the fact that the product of all loops depicted in 1 is trivial. By this construction we obtained the map from the set of isomorphism classes of real Hurwitz covers $\mathcal{R}$ to $\mathcal{F} := \mathcal{F}(\lambda^1, \ldots, \lambda^k, \mu^1, \ldots, \mu^m)$.

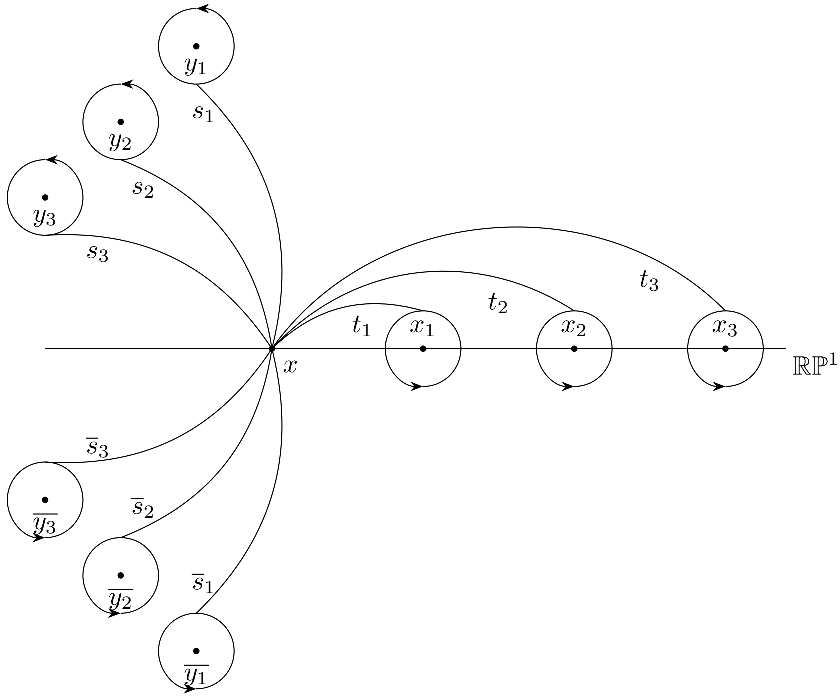

Figure 1: Generators of $\pi_1(\mathbb{CP}^1 \setminus \{\underline{x}, \underline{y}, \underline{\overline{y}}\}, x)$

Now we'll get the inverse map $\mathcal{F} \to \mathcal{R}$. Given permutations $\tau_1, \ldots, \tau_k$, $\sigma_1, \ldots, \sigma_m$, $\overline{\sigma}_1 = \gamma\sigma_1^{-1}\gamma, \ldots, \overline{\sigma}_m = \gamma\sigma_m^{-1}\gamma$, we construct complex branched covering $\pi : C \to \mathbb{CP}^1$ with branch points $x_1, \ldots, x_k \in \mathbb{RP}^1$, $y_1, \ldots, y_m$, $\overline{y_1}, \ldots, \overline{y_m} \in \mathbb{CP}^1 \setminus \mathbb{RP}^1$ and prescribed monodromy action on $\pi^{-1}(x) = \{p_1, \ldots, p_d\}$. It remains only to define the involution $\iota : C \to C$. Take any point $p \in C$ and choose a path α in $\mathbb{CP}^1 \setminus \{\underline{x}, \underline{y}, \underline{\overline{y}}\}$ from x to $\pi(p)$, put $\beta = \operatorname{conj}(\alpha)$. Lift α to a path $\tilde{\alpha}$ with endpoint p and let p_n be its starting point. Lift β to a path $\tilde{\beta}$ with starting point $p_{\gamma(n)}$ and set $\tau(p) = \tilde{\beta}(1)$. Now we'll show that this map is well-defined (i.e. does not depend on the choice of α). It is sufficient to verify this only for generators of $\pi_1(\mathbb{CP}^1 \setminus \{\underline{x}, \underline{y}, \underline{\overline{y}}\})$.

Indeed, if $p = p_n \in \pi^{-1}(x)$ and $\alpha = t_i \circ \cdots \circ t_1$, then the starting point of $\tilde{\alpha}$ is indexed with $(\tau_1 \cdots \tau_i)^{-1}(n)$ and the starting point of $\tilde{\beta}$ is thus labelled with $\gamma(\tau_1 \cdots \tau_i)^{-1}(n)$. Next, since $\operatorname{conj}(\alpha) = \alpha^{-1}$, the endpoint of $\tilde{\beta}$ has index $(\tau_1 \cdots \tau_i)^{-1}\gamma(\tau_1 \cdots \tau_i)^{-1}(n)$, which is by the third condition of 3.6 equal to $\gamma(n)$. Otherwise, if $\alpha = s_j$, then the starting point of $\tilde{\alpha}$ is indexed with $\sigma_j^{-1}(n)$ and the starting point of $\tilde{\beta}$ is thus labelled with $\gamma\sigma_j^{-1}(n)$. Next, since $\operatorname{conj}(s_j) = (\overline{s}_j)^{-1}$, the endpoint of $\tilde{\beta}$ has index $(\gamma\sigma_j\gamma)\gamma\sigma_j^{-1}(n) = \gamma(n)$.

It is a direct check that two constructed maps are mutually inverse and the second map $\mathcal{F} \to \mathcal{R}$ is clearly independent on the labelling of $\pi^{-1}(x)$, so we have a map $\mathcal{F}/\mathcal{S}_d \to \mathcal{R}$. Standard argument (e.g. see [**CM16**], 7.3.1 for complex case) shows that this map is bijection with $|\operatorname{Stab}_{\mathcal{S}_d}(T)| = |\operatorname{Aut}^{\mathbb{R}}(T)|$, and the theorem follows. ■

In this paper we're interested in so-called *conjugate-invariant simple single Hurwitz numbers* which restrict us to the case $k = 1$ and $\mu^1 = \cdots = \mu^m = 1^{d-2}2^1$ (this means that there is exactly one real branch point and all non-real branch points are simple).

Remark 3.8. In this case, the third condition of the real factorization is equivalent to the following:

$$\tau = [\gamma, \sigma_1 \cdots \sigma_m].$$

Moreover, it automatically implies $(\gamma\tau)^2 = 1$.

In order to make the notation lighter, we write $h_m^{\mathbb{R}}(\lambda)$ for real Hurwitz numbers of the type (λ; $1^{d-2}2^1$,

$\ldots, 1^{d-2}2^1)$, where partition $1^{d-2}2^1$ repeats m times. We also call (m,λ) the type of such factorization and denote the set of all real factorizations of type (m,λ) by $\mathcal{F}(m,\lambda)$.

We are studying the properties of the following generating series, which can be considered as the element of $\Lambda[u]$:

$$\mathbb{H}^{\mathbb{R}} = \sum_{\lambda \vdash d,\, m\in\mathbb{N}_0} h^{\mathbb{R}}_m(\lambda) p_\lambda \frac{u^m}{m!}.$$

To compute this series, we need a more subtle approach to combinatorics of real factorizations. Consider the series

$$\widetilde{\mathbb{H}}^{\mathbb{R}} = \frac{1}{d!} \sum_{\substack{\lambda\vdash d, m\in\mathbb{N}_0\\ \pi\in\mathcal{F}(m,\lambda)}} p_{\mathcal{H}_\gamma(\sigma_1\cdots\sigma_m)} \frac{u^m}{m!} \in \Lambda[q,\hat{q},\check{q},u]. \tag{3.4}$$

Note that $\mathbb{H}^{\mathbb{R}}$ can be derived by setting

$$\begin{aligned} p_\nu &\mapsto p_\nu^2;\\ q_\rho &\mapsto p_\rho;\\ \hat{q}_{\hat{\rho}} &\mapsto p_{\hat{\rho}};\\ \check{q}_{\check{\rho}} &\mapsto p_{\check{\rho}} \end{aligned}$$

in (3.4). This follows from the observation that assuming $\gamma = (a_1\ b_1)\,(a_2\ b_2)\cdots(a_k\ b_k)$ and denoting $\sigma = \sigma_1\cdots\sigma_m$, we can express the cycle type of commutator $\tau = [\gamma,\sigma]$ as

$$\mathcal{C}([\gamma,\sigma]) = \mathcal{C}\left((\sigma^{-1}\gamma\sigma)\cdot\gamma\right) = \mathcal{C}\left((\sigma(a_1)\ \sigma(b_1))\,(\sigma(a_2)\ \sigma(b_2))\cdots(\sigma(a_k)\ \sigma(b_k))\cdot(a_1\ b_1)\,(a_2\ b_2)\cdots(a_k\ b_k)\right).$$

Hence the cycle type λ of τ is uniquely determined by γ-coset-type $\mathcal{H}_\gamma(\sigma)$.

Remark 3.9. Consider the series $\widetilde{\mathbb{H}}^{\mathbb{R}}$ and put $q_i = \hat{q}_i = \check{q}_i = 0$ for all $i \in \mathbb{N}$. Observe that this specialization corresponds to real factorizations with fixed-point-free involution γ (that is, the graph $\Gamma_\gamma(\sigma)$ contains only even cycles). In further computations we will use the notation

$$\widetilde{\mathbb{H}}^{\mathbb{R}}_r = \frac{1}{d!} \sum_{\substack{\lambda\vdash d,\, m\in\mathbb{N}_0\\ \pi\in\mathcal{F}_r(m,\lambda)}} p_{\mathcal{H}_\gamma(\sigma_1\cdots\sigma_m)} \frac{u^m}{m!}.$$

Here, $\mathcal{F}_r(m,\lambda)$ is the set of real factorizations of type (m,λ) with involution γ having r fixed points. Clearly,

$$\begin{aligned} \widetilde{\mathbb{H}}^{\mathbb{R}} &= \widetilde{\mathbb{H}}^{\mathbb{R}}_0 + \widetilde{\mathbb{H}}^{\mathbb{R}}_1 + \ldots + \widetilde{\mathbb{H}}^{\mathbb{R}}_d\\ \widetilde{\mathbb{H}}^{\mathbb{R}}_0 &= \widetilde{\mathbb{H}}^{\mathbb{R}}|_{q_i=\hat{q}_j=\check{q}_l=0}\\ \widetilde{\mathbb{H}}^{\mathbb{R}}_d &= \frac{1}{d!} q_1^{d/2} \exp\left(\frac{d(d-1)}{2}u\right). \end{aligned}$$

Example 3.10. Put $r = 0$ and $m \in \{0,1\}$. For $m = 0$, all real factorizations have the type $\left(0,(1^d)\right)$, so $|\mathcal{F}_0\left(0,(1^d)\right)| = (d-1)!!$ and $|\mathcal{F}_0(0,\lambda)| = 0$ for any other λ.

Suppose now $m = 1$. Take any real factorization $(\gamma;\sigma_1)$, then the transposition σ_1 either commutes with γ (the number of such transpositions is $d/2$) or the commutator $[\gamma,\sigma_1]$ consists of two cycles of length 2 and $d-4$ fixed points. Therefore,

$$|\mathcal{F}_0\left(1,(1^d)\right)| = \frac{d}{2}\cdot(d-1)!!, \qquad |\mathcal{F}_0(1,\left(1^{d-4}\,2^2\right))| = \frac{d(d-2)}{2}\cdot(d-1)!!$$

and $|\mathcal{F}_1(0,\lambda)| = 0$ for any other λ.

Thus, we get the following expansion of $\widetilde{\mathbb{H}}_0^{\mathbb{R}}$

$$\widetilde{\mathbb{H}}_0^{\mathbb{R}} = \frac{p_1^{d/2}}{d!!} + \left(\frac{p_1^{d/2}}{2 \cdot (d-2)!!} + \frac{p_1^{d/2-2} p_2}{2 \cdot (d-4)!!} \right) u + O(u^2).$$

Lemma 3.11. *The series $\widetilde{\mathbb{H}}^{\mathbb{R}}$ is invariant under involution which swaps $\hat{q}_i$ and $\check{q}_i$ for all i.*

Proof. Indeed, let $(\gamma; \sigma_1, \sigma_2, \ldots, \sigma_m)$ be any factorization of type (m, λ), put $\sigma = \sigma_1 \cdots \sigma_m$. Then $(\sigma^{-1}\gamma\sigma;$ $\sigma_m, \sigma_{m-1}, \ldots, \sigma_1)$ is a real factorization of the same type, and graph $\Gamma_{\sigma^{-1}\gamma\sigma}(\sigma^{-1})$ is obtained from $\Gamma_\gamma(\sigma)$ by changing the colour of all edges. Thus, we see that coefficients of $p_\nu \cdot q_\rho \cdot \hat{q}_{\hat{\rho}} \cdot \check{q}_{\check{\rho}}$ and $p_\nu \cdot q_\rho \cdot \hat{q}_{\check{\rho}} \cdot \check{q}_{\hat{\rho}}$ in $\widetilde{\mathbb{H}}^{\mathbb{R}}$ are equal. ■

3.3 r-real Hurwitz Numbers

Now we consider another type of Hurwitz numbers. Real Hurwitz numbers defined in (3.7) depend on the order of points $x_1, x_2, \ldots, x_k$. For example, take $d = 3, k = 4, m = 0$, then

$$h^{\mathbb{R}}\left((1^1\,2^1), (3^1), (1^1\,2^1), (3^1)\right) = 0 \quad \text{and} \quad h^{\mathbb{R}}\left((1^1\,2^1), (1^1\,2^1), (3^1), (3^1)\right) \neq 0.$$

To avoid this dependence, one can add another restriction on Hurwitz covers. We say that real Hurwitz cover (π, ι) is *r-real Hurwitz cover*, if for any non-branched point $x \in \mathbb{RP}^1 \setminus \{x_1, \ldots, x_k\}$, the involution corresponding to the action of ι on $\pi^{-1}(x)$ has exactly r fixed points. Analogously we may define the *r-real Hurwitz numbers* as the weighted sum

$$h^{r,\mathbb{R}}(g; \lambda^1, \ldots, \lambda^k; \mu^1, \ldots, \mu^m) = \sum_{[\pi,\iota]} \frac{1}{\operatorname{Aut}^{\mathbb{R}}(\pi, \iota)},$$

where the sum is taken over all equivalence classes of r-real Hurwitz coverings of type $(g;\ \lambda^1, \ldots, \lambda^k, \underline{x};$ $\mu^1, \ldots, \mu^m, \underline{y})$.

In the same way the r-real Hurwitz numbers can be identified with the number of factorizations.

Definition 3.12. *r-real factorization* π is a real factorization $(\gamma, \tau_1, \ldots, \tau_k, \sigma_1, \ldots, \sigma_m)$ such that involution $\gamma\tau_1 \cdots \tau_i$ has exactly r fixed points for $i = 0, 1 \ldots, k$.

Denote the set of all r-real factorizations of type $(\lambda^1, \ldots, \lambda^k; \mu^1, \ldots, \mu^m)$ by $\mathcal{F}^r(\lambda^1, \ldots, \lambda^k; \mu^1, \ldots, \mu^m)$. Then

$$h^{r,\mathbb{R}}(g; \lambda^1, \ldots, \lambda^k; \mu^1, \ldots, \mu^m) = \frac{1}{d!} |\mathcal{F}^r(\lambda^1, \ldots, \lambda^k; \mu^1, \ldots, \mu^m)|.$$

Define the operator D_λ as the composition

$$D_\lambda : \mathbb{C}[\mathcal{I}_d^r] \xrightarrow{\cdot c_\lambda} \mathbb{C}[\mathcal{S}_d] \xrightarrow{\text{pr}} \mathbb{C}[\mathcal{I}_d^r],$$

where the first map is right multiplication by the sum of permutations of cycle type λ and the second map is projection to the subspace $\mathbb{C}[\mathcal{I}_d^r]$. Also define the operator $\widetilde{D}_\mu : \mathbb{C}[\mathcal{I}_d^r] \to \mathbb{C}[\mathcal{I}_d^r]$ as follows

$$\widetilde{D}_\mu(\gamma) = \sum_{\mathcal{C}(\sigma)=\mu} \sigma\gamma\sigma^{-1}.$$

It is not hard to see that

$$h^{r,\mathbb{R}}(\lambda^1, \ldots, \lambda^k; \mu^1, \ldots, \mu^m) = \frac{1}{d!} \operatorname{tr}\left(\widetilde{D}_{\mu^m} \circ \cdots \circ \widetilde{D}_{\mu^1} \circ D_{\lambda^k} \circ \cdots \circ D_{\lambda^1}\right). \tag{3.5}$$

Lemma 3.13. *For any partitions $\lambda^1, \lambda^2 \in \mathcal{P}_d$,*

$$[\widetilde{D}_{\lambda^1}, \widetilde{D}_{\lambda^2}] = [D_{\lambda^1}, \widetilde{D}_{\lambda^2}] = [D_{\lambda^1}, D_{\lambda^2}] = 0.$$

Proof. To see that first commutator is zero, write

$$\widetilde{D}_{\lambda_1}\widetilde{D}_{\lambda^2}(\gamma) = \sum_{\mathcal{C}(\sigma_2)=\lambda^2}\sum_{\mathcal{C}(\sigma_1)=\lambda^1} \sigma_1\sigma_2\gamma\sigma_2^{-1}\sigma_1^{-1} = \sum_{\mathcal{C}(\sigma_2)=\lambda^2}\sum_{\mathcal{C}(\sigma_1)=\lambda^1} (\sigma_1\sigma_2\sigma_1^{-1})\sigma_1\gamma\sigma_1^{-1}(\sigma_1\sigma_2\sigma_1^{-1}) =$$
$$= \sum_{\mathcal{C}(\sigma_1)=\lambda^1}\sum_{\mathcal{C}(\sigma_2)=\lambda^2} \sigma_2\sigma_1\gamma\sigma_1^{-1}\sigma_2^{-1} = \widetilde{D}_{\lambda^2}\widetilde{D}_{\lambda^1}(\gamma).$$

For the second commutator, write

$$D_{\lambda^1}\widetilde{D}_{\lambda^2}(\gamma) = \sum_{\mathcal{C}(\sigma_2)=\lambda^2}\sum_{\substack{\mathcal{C}(\sigma_1)=\lambda^1\\ \sigma_1\cdot\sigma_2\gamma\sigma_2^{-1}\in\mathcal{I}_d^r}} \sigma_1\cdot\sigma_2\gamma\sigma_2^{-1} = \sum_{\mathcal{C}(\sigma_2)=\lambda^2}\sum_{\substack{\mathcal{C}(\sigma_1)=\lambda^1\\ \sigma_1\cdot\sigma_2\gamma\sigma_2^{-1}\in\mathcal{I}_d^r}} \sigma_2(\sigma_2^{-1}\sigma_1\sigma_2\cdot\gamma)\sigma_2^{-1} =$$
$$= \sum_{\substack{\mathcal{C}(\sigma_1)=\lambda^1\\ \sigma_1\cdot\gamma\in\mathcal{I}_d^r}}\sum_{\mathcal{C}(\sigma_2)=\lambda^2} \sigma_2(\sigma_1\cdot\gamma)\sigma_2^{-1} = \widetilde{D}_{\lambda^2}D_{\lambda^1}(\gamma).$$

For the last identity, we use that D_λ preserves the cycle type. For any $\eta \in \mathcal{S}_d$, we can write

$$[\eta\gamma\eta^{-1}]D_{\lambda^1}D_{\lambda^2}(\gamma) = \#\{\gamma' \in \mathcal{I}_d^r \,|\, \mathcal{C}(\gamma\gamma') = \lambda^1,\, \mathcal{C}(\gamma'\eta\gamma\eta^{-1}) = \lambda^2\},$$

where D_{λ_1} mutiplies γ by $\gamma\gamma'$, D_{λ_2} multiplies by $\gamma'\eta\gamma\eta^{-1}$ ($[\eta\gamma\eta^{-1}]$ means the coefficient of $\eta\gamma\eta^{-1}$). Similarly,

$$[\eta\gamma\eta^{-1}]D_{\lambda^2}D_{\lambda^1}(\gamma) = \#\{\gamma' \in \mathcal{I}_d^r \,|\, \mathcal{C}(\gamma\gamma') = \lambda^2,\, \mathcal{C}(\gamma'\eta\gamma\eta^{-1}) = \lambda^1\}.$$

We claim that the map $\gamma \mapsto \eta\gamma\eta^{-1}$ establish bijection between these sets. Indeed,

$$\mathcal{C}(\gamma\gamma') = \lambda^1 \Leftrightarrow \mathcal{C}(\gamma'\gamma) = \lambda^1 \Leftrightarrow \mathcal{C}((\eta\gamma'\eta^{-1})\eta\gamma\eta^{-1}) = \lambda^1,$$
$$\mathcal{C}(\gamma'\eta\gamma\eta^{-1}) = \lambda^2 \Leftrightarrow \mathcal{C}((\eta\gamma\eta^{-1})\gamma') = \lambda^2.$$

Therefore, $D_{\lambda^1}D_{\lambda^2}(\gamma) = D_{\lambda^2}D_{\lambda^1}(\gamma)$, and we're done. ∎

This lemma together with (3.5) immediately implies the following result.

Proposition 3.14. *For any partitions $\lambda^1, \dots, \lambda^k, \mu^1, \dots, \mu^m$, the r-real Hurwitz number $h^{r,\mathbb{R}}(\lambda^1, \dots, \lambda^k; \mu^1, \dots, \mu^m)$ does not depend on the order of real branch points. In other words,*

$$h^{r,\mathbb{R}}(\lambda^1, \dots, \lambda^k; \mu^1, \dots, \mu^m) = h^{r,\mathbb{R}}(\lambda^{\sigma(1)}, \dots, \lambda^{\sigma(k)}; \mu^1, \dots, \mu^m)$$

for any permutation σ acting on $\{1, 2, \dots, k\}$. ∎

This result is natural in view of geometric and combinatorial correspondence because it claims that r-real Hurwitz numbers are independent on the positions of branch points.

Now we connect the notion of 0-real Hurwitz numbers and the algebra of double $\mathcal{H}_d$ invariant functions $C(\mathcal{S}_d; \mathcal{H}_d)$. First, observe that all partitions λ^i must be co-even, otherwise the set of 0-real factorizations is empty. So, suppose $\lambda^i = \tilde{\lambda}^i \cup \tilde{\lambda}^i$, $1 \le i \le k$ and define the characteristic function $\psi_{\tilde{\lambda}} \in C(\mathcal{S}_d; \mathcal{H}_d)$

$$\psi_{\tilde{\lambda}}(\eta) = \begin{cases} 1/|\mathcal{H}_d| & \text{if } \mathcal{H}(\eta) = \tilde{\lambda}; \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.15. *For co-even partitions $\lambda^1 = \tilde{\lambda}^1 \cup \tilde{\lambda}^1, \ldots, \lambda^k = \tilde{\lambda}^k \cup \tilde{\lambda}^k$, holds the equation*

$$h^{0,\mathbb{R}}(\lambda^1, \ldots, \lambda^k) = \frac{1}{d!!} \left(\psi_{\tilde{\lambda}^1} \psi_{\tilde{\lambda}^2} \cdots \psi_{\tilde{\lambda}^k} \right) (\mathrm{id}).$$

Proof. Take any two involutions $\gamma, \gamma' \in \mathcal{I}_d^r$, then γ' can be written as $\eta^{-1}\gamma\eta$ (the choice of η is unique up to left multiplication by the element of $C_{\mathcal{S}_d}(\gamma)$). Therefore, we can express the action of D_λ in the following way

$$D_\lambda(\gamma) = \frac{1}{|C_{\mathcal{S}_d}(\gamma)|} \sum_{\substack{\eta \in \mathcal{S}_d \\ \mathcal{C}([\gamma,\eta]) = \lambda}} \eta^{-1}\gamma\eta.$$

Thus, we may write

$$D_{\lambda^k} \circ \cdots \circ D_{\lambda^1}(\gamma) = \frac{1}{|C_{\mathcal{S}_d}(\gamma)|^k} \sum_{\substack{\eta_1 \in \mathcal{S}_d \\ \mathcal{C}([\gamma,\eta_1]) = \lambda^1}} \cdots \sum_{\substack{\eta_k \in \mathcal{S}_d \\ \mathcal{C}([\gamma,\eta_k]) = \lambda^k}} (\eta_k \cdots \eta_1)^{-1} \gamma \eta_k \cdots \eta_1. \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$h^{0,\mathbb{R}}(\lambda^1, \ldots, \lambda^k) = \frac{1}{d!} \cdot \frac{(d-1)!!}{|\mathcal{H}_d|^k} \cdot \#\{\eta_1, \ldots, \eta_k \in \mathcal{S}_d \,|\, \eta_1 \cdots \eta_k \in \mathcal{H}_d, \mathcal{C}\left([\gamma_0, \eta_i]\right) = \lambda^i \text{ for } i = 1, \ldots, k\} \tag{3.7}$$

for some fixed involution $\gamma_0 \in \mathcal{I}_d^0$ (we used that $|\mathcal{I}_d^0| = (d-1)!!$ and $C_{\mathcal{S}_d}(\gamma_0) = \mathcal{H}_d$). Now clearly,

$$\frac{1}{|\mathcal{H}_d|^k} \#\{\eta_1, \ldots, \eta_k \in \mathcal{S}_d \,|\, \eta_1 \cdots \eta_k \in \mathcal{H}_d, \mathcal{C}\left([\gamma_0, \eta_i]\right) = \lambda^i \text{ for } i = 1, \ldots, k\} = (\psi_{\tilde{\lambda}^1} \psi_{\tilde{\lambda}^2} \cdots \psi_{\tilde{\lambda}^k})(\mathrm{id}).$$

■

4 Real Hurwitz numbers in terms of zonal polynomials

4.1 $\widetilde{\mathbb{H}}_0^{\mathbb{R}}$ expansion in zonal polynomials

Theorem 4.1. *The function $\widetilde{\mathbb{H}}_0^{\mathbb{R}}$ has the expansion in the basis of zonal polynomials*

$$\widetilde{\mathbb{H}}_0^{\mathbb{R}} = \frac{1}{d!} \sum_{\nu \vdash \frac{d}{2}} \dim(2\nu) Z_\nu e^{w(\nu)u},$$

where $w(\nu) = d + 2 \sum_{\square \in \nu} c_2(\square)$.

Proof. First of all, denoting $\mu = \mathcal{C}(\sigma_1 \cdots \sigma_m)$, we obtain

$$\widetilde{\mathbb{H}}^{\mathbb{R}} = \frac{1}{d!} \sum_{\substack{\lambda \vdash d,\, m \in \mathbb{N}_0 \\ \pi \in \mathcal{F}(m,\lambda)}} p_{\mathcal{H}_\gamma(\sigma_1 \cdots \sigma_m)} \frac{u^m}{m!} = \sum_{\mu \vdash d,\, m \in \mathbb{N}_0} h_m(\mu) \frac{u^m}{m!} \sum_{\gamma \in \mathcal{I}_d} p_{\mathcal{H}_\gamma(\sigma_\mu)},$$

where σ_μ is an arbitrary permutation of cycle type μ. Therefore, the power series $\widetilde{\mathbb{H}}^{\mathbb{R}}$ can be derived by applying the map $\Phi : \Lambda \to \Lambda[q, \hat{q}, \check{q}]$

$$\Phi : p_\mu \mapsto \sum_{\gamma \in \mathcal{I}_d} p_{\mathcal{H}_\gamma(\sigma_\mu)} \tag{4.1}$$

to the power series $\mathbb{H}^{\mathbb{C}}$. The key idea of the proof is to rewrite $\mathrm{pr}_\Lambda \circ \Phi$ (pr_Λ is the specialization homomorphism $q = \hat{q} = \check{q} = 0$) in the Schur basis

$$\Phi : s_\lambda = \sum_\mu \frac{1}{z_\mu} \chi^\lambda(\mu)\, p_\mu = \frac{1}{d!} \sum_{\sigma \in \mathcal{S}_d} \chi^\lambda(\sigma) p_{\mathcal{C}(\sigma)} \mapsto \frac{1}{d!} \sum_{\substack{\sigma \in \mathcal{S}_d \\ \gamma \in \mathcal{I}_d}} \chi^\lambda(\sigma) p_{\mathcal{H}_\gamma(\sigma)}. \tag{4.2}$$

Since any two fixed-point-free involutions are conjugate to each other, it is enough to compute the sum (4.2) only with

$$\gamma = \gamma_0 = (1\ 2)\,(3\ 4)\ \cdots\ (d-1\ d).$$

The number of fixed-point-free involutions in $\mathcal{S}_d$ is $(d-1)!!$, so we get

$$\frac{1}{d!} \sum_{\substack{\sigma \in \mathcal{S}_d \\ \gamma \in \mathcal{I}_d^0}} \chi^\lambda(\sigma)\, p_{\mathcal{H}_\gamma(\sigma)} = \frac{1}{|\mathcal{H}_d|} \sum_{\sigma \in \mathcal{S}_d} \chi^\lambda(\sigma)\, p_{\mathcal{H}(\sigma)}.$$

The multiplication by element of $\mathcal{H}_d$ preserves the coset-type of permutation, so we see that

$$\frac{1}{|\mathcal{H}_d|} \sum_{\sigma \in \mathcal{S}_d} \chi^\lambda(\sigma)\, p_{\mathcal{H}(\sigma)} = \frac{1}{|\mathcal{H}_d|^2} \sum_{\substack{\sigma \in \mathcal{S}_d \\ h \in \mathcal{H}_d}} \chi^\lambda(\sigma \cdot h)\, p_{\mathcal{H}(\sigma)}. \tag{4.3}$$

By 2.5, it follows that sum in (4.3) can be non-zero only if λ is even, so we may assume $\lambda = 2\nu$. Now, if we recall the definitions of zonal spherical functions and zonal polynomials, we get

$$\frac{1}{|\mathcal{H}_d|^2} \sum_{\substack{\sigma \in \mathcal{S}_d \\ h \in \mathcal{H}_d}} \chi^{2\nu}(\sigma \cdot h)\, p_{\mathcal{H}(\sigma)} = \frac{1}{|\mathcal{H}_d|} \sum_{\sigma \in \mathcal{S}_d} \omega^\rho(\sigma)\, p_{\mathcal{H}(\sigma)} = \frac{1}{|\mathcal{H}_d|} \operatorname{ch}'(\omega^\rho) = Z_\rho.$$

Finally, we're ready to rewrite Φ in the Schur basis:

$$s_\lambda \mapsto \begin{cases} Z_\nu & \text{if } \lambda = 2\nu \text{ for some } \nu; \\ 0 & \text{otherwise.} \end{cases}$$

Since $\sum\limits_{\square \in 2\nu} c(\square) = 2 \sum\limits_{\square \in \nu} c_2(\square) + d = w(\nu)$, applying this map to the series for Hurwitz numbers

$$\Phi(\mathbb{H}^{\mathbb{C}}) = \Phi\left(\frac{1}{d!} \sum_\lambda \dim(\lambda)\, s_\lambda\, e^{\sum c(\square) u} \right) = \frac{1}{d!} \sum_{\nu \vdash \frac{d}{2}} \dim(2\nu) Z_\nu e^{w(\nu) u}$$

concludes the proof. ■

Remark 4.2. Very similar object was studied in [**BF21**]. *Twisted Hurwitz numbers* count the number of fixed-point-free real factorizations $\mathcal{F}_0(m, \lambda)$ with additional restriction that none of the transpositions $\sigma_1, \ldots, \sigma_m$ coincide with transpositions of γ. Up to constant, generating series for twisted Hurwitz numbers equals $\widetilde{\mathbb{H}}_0^{\mathbb{R}} \cdot e^{-ud}$.
The same generating series is obtained in [**CD20**] for *non-orientable Hurwitz numbers*. However, the bijection between twisted Hurwitz numbers and non-orientable Hurwitz numbers is not yet known.

4.2 Explicit formula for $h_m^{\mathbb{R}}\left((1^d)\right)$

Proposition 4.3. *Let m be a natural number. Then the following equation for the number of real Hurwitz coverings with $2m$ simple non-real branch points holds*

$$h_m^{\mathbb{R}}\left((1^d)\right) = \frac{1}{d!}\sum_{\lambda\vdash d}\dim\lambda\left(\left\lfloor\frac{\lambda_1}{2}\right\rfloor - \left\lfloor\frac{\lambda_2-1}{2}\right\rfloor\right)\cdots\left(\left\lfloor\frac{\lambda_{k-1}}{2}\right\rfloor - \left\lfloor\frac{\lambda_k-1}{2}\right\rfloor\right)\cdot\left(\sum_{\square\in\lambda}c(\square)\right)^m.$$

Remark 4.4. Since $\lfloor\frac{x}{2}\rfloor = \lfloor\frac{x-1}{2}\rfloor$ for odd x, we can restrict the above sum to partitions that are strict on odd parts.

Proof. We have seen that the map Φ defined in (4.1) establishes the connection between $\mathbb{H}^{\mathbb{C}}$ and $\widetilde{\mathbb{H}}^{\mathbb{R}}$. Further, the condition $\lambda = 1^d$ means that the permutations σ and γ commute. Hence we have the following expression for $h_m^{\mathbb{R}}\left((1^d)\right)$

$$\sum_{m\in\mathbb{N}_0} h_m^{\mathbb{R}}\left((1^d)\right)\frac{u^m}{m!} = \Phi'(\mathbb{H}^{\mathbb{C}}),$$

where

$$\Phi'(p_\mu) = |C_{\mathcal{S}_d}(\sigma_\mu)\cap\mathcal{I}_d|,$$

i.e. the number of involutive permutations commuting with σ_μ. Again we write this map in the Schur basis

$$\Phi' : s_\lambda = \frac{1}{d!}\sum_{\sigma\in\mathcal{S}_d}\chi^\lambda(\sigma)p_{\mathcal{C}(\sigma)} \mapsto \frac{1}{d!}\sum_{\sigma\in\mathcal{S}_d}\chi^\lambda(\sigma)\cdot|C_{\mathcal{S}_d}(\sigma_\mu)\cap\mathcal{I}_d| = \frac{1}{d!}\sum_{\gamma\in\mathcal{I}_d}\sum_{\sigma\in C_{\mathcal{S}_d}(\gamma)}\chi^\lambda(\sigma).$$

The centralizer of involution γ is a subgroup conjugate to $\mathcal{H}_{2k}\times\mathcal{S}_{d-2k}$, where $d-2k$ is the number of fixed points of γ, $\mathcal{H}_{2k}$ acts on the first $2k$ elements of $\{1,2,\dots,d\}$ and $\mathcal{S}_{d-2k}$ acts on the remaining elements. Therefore we can write

$$\sum_{k\le\frac{d}{2}}\sum_{\sigma\in\mathcal{H}_{2k}\times\mathcal{S}_{d-2k}}\chi^\lambda(\sigma) = \sum_{k\le\frac{d}{2}}\sum_{\substack{\nu\subset\lambda\\ \sigma_1\in\mathcal{H}_{2k}\\ \sigma_2\in\mathcal{S}_{d-2k}}}\chi^\nu(\sigma_1)\cdot\chi^{\lambda/\nu}(\sigma_2) = \sum_{k\le\frac{d}{2}}\sum_{\substack{\nu\subset\lambda\\ \sigma_1\in\mathcal{H}_{2k}}}\chi^\nu(\sigma_1)\cdot\langle\chi^{\lambda/\nu},\mathrm{id}_{\mathcal{S}_{d-2k}}\rangle.$$

Since $\langle\chi^{\lambda/\nu},\mathrm{id}_{\mathcal{S}_{d-2k}}\rangle$ equals 1 if λ/ν is a horizontal strip and 0 otherwise, we have

$$\Phi'(s_\lambda) = \sum_{k\le\frac{d}{2}}\frac{1}{|\mathcal{H}_{2k}|}\sum_{\substack{\lambda/\nu\in\mathrm{HS}(2k)\\ \sigma_1\in\mathcal{H}_{2k}}}\chi^\nu(\sigma_1).$$

Using 2.5, we get

$$\Phi'(s_\lambda) = \sum_{\substack{\lambda/\nu\in\mathrm{HS}\\ \nu\text{ is even}}}\omega_{(1^n)}^{\nu/2} = \#\{\nu\subset\lambda\,:\,\nu\text{ is even and }\lambda/\nu\in\mathrm{HS}\},$$

since $\omega_{(1^n)}^{\rho} = 1$ for any ρ. The possible half-number of boxes in the i^{th} row of ν belongs to the set

$$\frac{\mu_i}{2}\in\{\lfloor\frac{\lambda_i}{2}\rfloor,\lfloor\frac{\lambda_i}{2}\rfloor-1,\dots,\lfloor\frac{\lambda_{i+1}-1}{2}\rfloor+1\}.$$

Thus, we conclude

$$\Phi'(s_\lambda) = \left(\left\lfloor\frac{\lambda_1}{2}\right\rfloor - \left\lfloor\frac{\lambda_2-1}{2}\right\rfloor\right)\cdots\left(\left\lfloor\frac{\lambda_{k-1}}{2}\right\rfloor - \left\lfloor\frac{\lambda_k-1}{2}\right\rfloor\right).$$

Combining this with $\mathbb{H}^{\mathbb{C}}$ expression in Schur basis, we get the desired equality. ■

5 Cut-and-join equation for $\widetilde{\mathbb{H}}_0^{\mathbb{R}}$ and $\widetilde{\mathbb{H}}_1^{\mathbb{R}}$

In this section we present the differential equation which is usually called 'cut-and-join'. It demonstrates how the γ-coset-type of permutation can change when multiplied by a transposition.

Definition 5.1. Using the notation $\partial_x = \frac{\partial}{\partial x}$ define the deformed Laplace-Beltrami operator $\mathcal{A}_0$ as

$$\mathcal{A}_0 = \sum_{i\geq 1} i^2 p_i \partial_{p_i} + \sum_{i,j\geq 1} \left[(i+j)p_i p_j \partial_{p_{i+j}} + 2ij p_{i+j} \partial_{p_i} \partial_{p_j}\right].$$

Remark 5.2. It is well known that Zonal polynomials Z_ν are eigenfunctions of Laplace-Beltrami operator with eigenvalues $w(\nu) = d + 2 \sum\limits_{\square\in\nu} c_2(\square)$. Thus, it can be easily checked that

$$\mathcal{A}_0 \, \widetilde{\mathbb{H}}_0^{\mathbb{R}} = \partial_u \widetilde{\mathbb{H}}_0^{\mathbb{R}}.$$

However, this equation also has combinatorial meaning, we will demonstrate it for real Hurwitz number with one fixed point.

Remark 5.3. The cut-and-join operator for twisted Hurwitz numbers studied in [**BF21**] coincide with

$$\sum_{i\geq 1} i(i-1) p_i \partial_{p_i} + \sum_{i,j\geq 1} \left[(i+j)p_i p_j \partial_{p_{i+j}} + 2ij p_{i+j} \partial_{p_i} \partial_{p_j}\right] = \mathcal{A}_0 - \sum_{i\leq 1} i p_i \partial_{p_i}.$$

This means that the eigenfunctions of this operator are the same (i.e. Zonal polynomials Z_ν), and the eigenvalues differ by $|\nu| = d/2$ from the eigenvalues of $\mathcal{A}_0$.

Theorem 5.4 (**Cut-and-join equation**)**.** *The power series* $\widetilde{\mathbb{H}}_1^{\mathbb{R}}$ *satisfy the equation*

$$\partial_u \widetilde{\mathbb{H}}_1^{\mathbb{R}} = \mathcal{A}_1 \, \widetilde{\mathbb{H}}_1^{\mathbb{R}}, \tag{5.1}$$

where the operator $\mathcal{A}_1$ *is given by*

$$\mathcal{A}_1 = \mathcal{A}_0 + \sum_{i\geq 1} \frac{i^2-1}{4} q_i \partial_{q_i} + \sum_{i,j\geq 1} \left[2ij q_{i+2j} \partial_{q_i} \partial_{p_j} + i q_i p_j \partial_{q_{i+2j}}\right]. \tag{5.2}$$

Proof. We start with considering the series

$$\partial_u \widetilde{\mathbb{H}}_1^{\mathbb{R}} = \sum_{\substack{\lambda\vdash d,\, m\in\mathbb{N} \\ \pi\in\mathcal{F}_k(m,\lambda)}} p_{\mathcal{H}_\gamma(\sigma_1\cdots\sigma_m)} \frac{u^{m-1}}{(m-1)!}$$

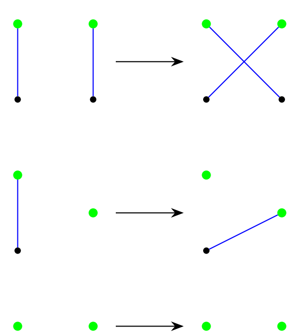

Figure 2: Mutating vertices $\sigma(a)$ and $\sigma(b)$ are coloured green

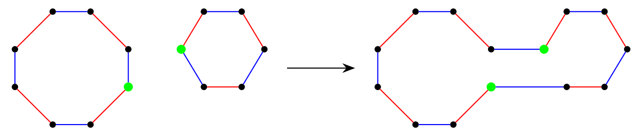

Figure 3: Mutating vertices belong to different even cycles. The produced term is $2ijp_{i+j}\partial_{p_i}\partial_{p_j}$.

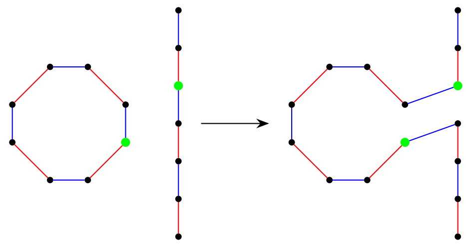

Figure 4: Mutating vertices belong to an even cycle and path. The produced term is $ijq_{i+2j}\partial_{q_i}\partial_{p_j}$.

and observing that this series can be derived by applying the operator which maps $p_{\mathcal{H}_\gamma(\sigma)}$ to $\sum\limits_{\sigma_m} p_{\mathcal{H}_\gamma(\sigma\sigma_m)}$, where the sum is taken over all transpositions $\sigma_m \in \mathcal{S}_d$. Now our purpose is to find out how the γ-coset-types of σ and $\sigma\sigma_m$ are related. Let

$$\gamma = (a_1\ b_1)\,(a_2\ b_2)\cdots(a_k\ b_k) \quad \text{and} \quad \sigma_m = (a\ b).$$

We call vertices $\sigma(a)$ and $\sigma(b)$ *mutating vertices*. The graph $\Gamma_\gamma(\sigma\sigma_m)$ can be obtained from $\Gamma_\gamma(\sigma)$ by switching blue edges incident to mutating vertices as shown in 2 (depending on whether a and b are fixed points of γ or not).

Recall that the graph $\Gamma_\gamma(\sigma)$ consists of some number of even cycles and one path of even length. Suppose, $\sigma(a)$ and $\sigma(b)$ are both vertices of the same cycle of length $2l$. If the distance between $\sigma(a)$ and $\sigma(b)$ in this cycle is even, then multiplying σ by σ_m split this cycle into two even cycles of total length $2l$ (this case produces the term $(i+j)p_ip_j\partial_{p_{i+j}}$ in (5.2)). If the distance is odd, then nothing happens (this case produces the term $i^2p_i\partial_{p_i}$). Four other possible cases are shown in the pictures 3, 4, 5 and 6. ■

Remark 5.5. Using a similar analysis of cases, it is not hard to write cut-and-join equations for $\widetilde{\mathbb{H}}^{\mathbb{R}}$. However, we do not provide it here due to very large number of its summands.

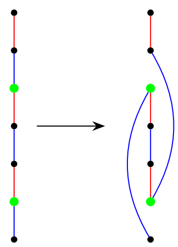

Figure 5: Both mutating vertices belong to one path, and the distance is odd. The produced term is $\frac{i^2-1}{4}q_i\partial_{q_i}$.

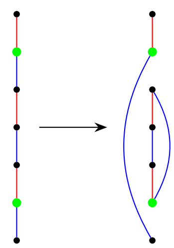

Figure 6: Both mutating vertices belong to one path, and the distance is even. The produced term is $iq_ip_j\partial_{q_{i+2j}}$.

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