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1 Introduction

The main object of the thesis is the one-dimensional Dirac operator D_Q on the positive half-line $\mathbb{R}_+ = [0, \infty)$,

$$D_Q = J\frac{d}{dr} + Q,$$

where Q is a symmetric 2×2 zero-trace real potential of the form $\left(\begin{smallmatrix} -q & p \\ p & q \end{smallmatrix}\right)$ and $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is a square root of the minus identity matrix. The basics of the spectral theory of the Dirac operator can be found in the book [20] by B. Levitan and I. Sargsjan.

Spectral properties of the operator D_Q usually depend on the regularity of its potential. Assuming that both functions p, q belong to $L^2(\mathbb{R}_+)$, M. Krein proved [19] that the absolutely continuous spectrum of D_Q fills the whole real line $\mathbb{R}$. Similar results were proved later for Schrödinger operator $-\frac{d}{dx^2} + q$ with $q \in L^r$, $1 \leqslant r \leqslant 2$, see [9], [10], [6], [5], [7].

The potential Q is called exponentially decreasing if the relation

$$|p(r)| + |q(r)| = O\left(e^{-ar}\right), \quad r \to \infty,$$

holds for some $a > 0$. If the latter holds for every $a > 0$, then the potential is called super-exponentially decreasing. Let m_Q be the Weyl function of the operator D_Q. It is known that in general m_Q is an analytic function in the upper half-plane $\mathbb{C}_+ = \{z\colon \operatorname{Im} z > 0\}$. If Q has a compact support or Q is super-exponentially decreasing, then m_Q has a meromorphic continuation through the real line into the lower half-plane, see [24] and the references within. When Q is exponentially decreasing with some rate of decay $a > 0$, the Weyl function m_Q extends meromorphically only into the half-plane

$$\Omega_\delta = \{z\colon \operatorname{Im} z > -\delta\}$$

for some $\delta = \delta(a) > 0$, see also [17], [29], [24], [13]. The meromorphic continuation of the Weyl function is crucial for the theory of scattering resonances. In fact, resonances can be defined as the poles of m_Q, for details see [12], [15], [16], [18].

In the thesis we use methods of the theory of orthogonal polynomials on the unit circle and a recent approach developed by R. Bessonov and S. Denisov in [1], [2], [4] based on entropy estimates. We present a new class of potentials for which the Weyl function of the corresponding Dirac operator extends meromorphically through the real line $\mathbb{R}$. This class is described by the exponential decay of the entropy function from [1], [2], [4]. One can show that the exponentially decreasing potentials belong to this class. It also contains some nontrivial non-decreasing examples, see Section 1.2 below.

The case of decaying potentials in the setting of orthogonal polynomials on the unit circle was treated in the paper [21] by P. Nevai and V. Totik, see also [25], [28] and Chapter 7 in [26]. A similar problem for Jacobi matrices was solved [8] by D. Damanik and B. Simon, see Section 13.7 in [27].

1.1 The main result

It what follows we assume that the entries of the potential Q, i.e., functions p and q, are real-valued and $p, q \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. The latter means that $p, q \in L^1([0, r])$ for every $r > 0$. Consider the boundary value problem for the operator D_Q

$$JN'(t, \lambda) + Q(t)N(t, \lambda) = \lambda N(t, \lambda), \quad N(0, \lambda) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad t \geqslant 0. \tag{1}$$

Express its solution N in the form

$$N(t,\lambda) = \begin{pmatrix} \theta_+(t,\lambda) & \varphi_+(t,\lambda) \\ \theta_-(t,\lambda) & \varphi_-(t,\lambda) \end{pmatrix}.$$

For any potential Q there exist two Borel measures σ_Q, $\hat{\sigma}_Q$ on the real line such that

$$\int \frac{d\sigma_Q(x)}{1+x^2} < \infty, \qquad \int \frac{d\hat{\sigma}_Q(x)}{1+x^2} < \infty,$$

and the mappings defined by

$$\mathcal{F}(f_1, f_2)(\lambda) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f_1(r)\theta_+(x,\lambda) + f_2(x)\theta_-(x,\lambda)\, dx, \tag{2}$$

$$\mathcal{G}(f_1, f_2)(\lambda) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f_1(r)\varphi_+(x,\lambda) + f_2(x)\varphi_-(x,\lambda)\, dx, \tag{3}$$

are the isometric operators from $L^2(\mathbb{R}_+, \mathbb{C}^2)$ onto $L^2(\mathbb{R}, d\sigma_Q)$ and $L^2(\mathbb{R}, d\hat{\sigma}_Q)$ respectively, i.e,

$$\int_0^\infty |f_1(x)|^2 + |f_2(x)|^2\, dx = \int_{\mathbb{R}} |\mathcal{F}(f_1, f_2)(\lambda)|^2\, d\sigma_Q(\lambda) = \int_{\mathbb{R}} |\mathcal{G}(f_1, f_2)(\lambda)|^2\, d\hat{\sigma}_Q(\lambda).$$

Measures σ_Q and $\hat{\sigma}_Q$ are called the spectral measure and the dual spectral measure of D_Q. Simple calculations show that $f = (f_1, f_2)^T$ is a solution of $Jf + Qf = \lambda f$ if and only if $\hat{f} = (f_2, -f_1)^T$ is a solution of $J\hat{f} - Q\hat{f} = \lambda\hat{f}$. Hence $\hat{\sigma}_Q$ coincides with σ_{-Q}. The Weyl function of D_Q is an analytic function in the upper half-plane $\mathbb{C}_+ = \{z\colon\ \mathrm{Im}\, z > 0\}$ defined by the relation

$$m(z) = -\lim_{t\to\infty} \frac{\varphi_+(t,z)}{\theta_+(t,z)}, \quad z \in \mathbb{C}_+. \tag{4}$$

A Borel measure $\sigma = w\,dx + \sigma_s$ on the real line $\mathbb{R}$ belongs to the Szegő class if

$$\int_{\mathbb{R}} \frac{d\sigma(x)}{1+x^2} < \infty, \qquad \int_{\mathbb{R}} \frac{\log w(x)}{1+x^2} dx > -\infty. \tag{5}$$

Because of the inequality $\log w < w$, the second integral can't diverge to $+\infty$ and therefore for any measure in the Szegő class we have $\frac{\log w}{1+x^2} \in L^1(\mathbb{R})$. Furthermore, in this case there exists the outer function Π in $\mathbb{C}_+$ such that $\Pi(i) > 0$ and

$$|\Pi(x)|^2 = \frac{1}{\sigma'(x)} \tag{6}$$

for Lebesgue almost every point x on the real line (see Section 4 in [14]). The function Π can be explicitly defined by the Cauchy integral, see formula (14) below. Sometimes, by the analogy with the Szegő function in the theory of orthogonal polynomials (see Section 2.4 in the book [26]), the function $D = \Pi^{-1}$ is called the Szegő function of σ. We will call Π the inverse Szegő function of σ.

It is known that σ_Q belongs to the Szegő class on the real line if and only if $\hat{\sigma}_Q$ does (see Lemma 2.4 in [4] or Lemma 8.7 in [11]). Moreover, in this case there exists a constant $\gamma \in [0, 2\pi)$ such that

$$m(z) = e^{i\gamma} \frac{\hat{\Pi}(z)}{\Pi(z)}, \quad z \in \mathbb{C}_+, \tag{7}$$

where Π and $\hat{\Pi}$ are the inverse Szegő functions of σ_Q and $\hat{\sigma}_Q$ respectively. We recall the proof of this fact in Section 2.3.

Let $N_Q(t) = N_Q(t, 0)$ denote the solution of (1) for $\lambda = 0$. Let also

$$H_Q(t) = N_Q^*(t) N_Q(t), \quad E_Q(r) = \det \int_r^{r+2} H_Q(t)\, dt - 4. \tag{8}$$

As we will see in Section 3.2, the matrix H_Q is a Hamiltonian of the canonical system corresponding to D_Q and E_Q is a leading term of the entropy function $\mathbf{K}_H$ of this Hamiltonian (see (29) below). The main result of the thesis provides a sufficient condition in terms of E_Q for the situation when the Weyl function m admits a meromorphic continuation into the domain $\Omega_\delta = \{z\colon \operatorname{Im} z > -\delta\}$ for some $\delta > 0$.

Theorem 1. *Let $p, q \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be real-valued functions and $Q = \left(\begin{smallmatrix} -q & p \\ p & q \end{smallmatrix}\right)$. Assume that there exists $\delta > 0$ such that $E_Q(r) = O\left(e^{-\delta r}\right)$ as $r \to \infty$. Then*

(1) the spectral measure of D_Q is absolutely continuous and belongs to the Szegő class (5);

(2) the Weyl function of D_Q continues meromorphically into $\Omega_{\delta/8}$.

In particular, if $E_Q(r) = O\left(e^{-\delta r}\right)$ for every $\delta > 0$, then the Weyl function of D_Q is meromorphic in the whole complex plane $\mathbb{C}$.

In the case of orthogonal polynomials on the unit circle, the possibility of the meromorphic continuation of the Carthéodory function is equivalent to the exponential decay of the corresponding recurrence coefficients $\{a_n\}_{n\geqslant 0}$. This result is known as Nevai-Totik theorem [21]. The entropy function for orthogonal polynomials is defined on nonnegative integers and has the following form:

$$\mathcal{K}(n) = -\log \prod_{k \geqslant n} (1 - |a_k|^2), \quad n \geqslant 0,$$

see [3]. Note that the exponential decay of the recurrence coefficients a_n is equivalent to the exponential decay of the entropy function $\mathcal{K}(n)$. From that point of view Theorem 1 can be regarded as the half of the theorem by P. Nevai, V. Totik. The other half of this theorem remains an open problem.

1.2 Examples

Even though the condition

$$E_Q(r) = O\left(e^{-\delta r}\right), \quad r \to \infty, \tag{9}$$

in Theorem 1 can be explicitly written out in terms of the potential, the question whether or not given Q satisfies (9) is still hard to answer. Here is a simple sufficient condition.

Theorem 2. *Let p be a real-valued function on $\mathbb{R}_+$ such that*

$$\sup_{t \geqslant r} \left| \int_r^t p(s)\, ds \right| = O\left(e^{-\delta r}\right), \quad r \to \infty \tag{10}$$

for some $\delta > 0$. Then the potential $Q = \left(\begin{smallmatrix} 0 & p \\ p & 0 \end{smallmatrix}\right)$ satisfies assertion (9). In particular, the conclusions of Theorem 1 hold for such Q.

Condition (10) shows that the possibility of meromorphic continuation of Weyl function depends on the size of antiderivative of the potential but not on the size of the potential itself. For more details see Section 5.

Finally, let us give an explicit example of a "large" potential with a meromorhic Weyl function. Consider function $p(x) = e^x \sin\left(e^{2x}\right)$. Clearly, we have

$$\sup_{t\geqslant r}\left|\int_r^t p(x)\,dx\right| = \sup_{t\geqslant r}\left|\int_r^t e^x \sin\left(e^{2x}\right)\,dx\right| = \sup_{t\geqslant r}\left|\int_{e^{2r}}^{e^{2t}} \frac{\sin(s)}{2\sqrt{s}}\,ds\right| = O\left(e^{-r}\right), \quad r \to \infty.$$

Therefore, by Theorems 1 and 2, the Weyl function of the operator D_Q with

$$Q = \begin{pmatrix} 0 & e^x \sin\left(e^{2x}\right) \\ e^x \sin\left(e^{2x}\right) & 0 \end{pmatrix}$$

is meromorphic in $\{z\colon \operatorname{Im} z > -1/8\}$. A similar argument for $p(x) = xe^{x^2}\sin\left(e^{2x^2}\right)$ gives an example of a nondecaying potential corresponding to a meromorphic Weyl function in the whole complex plane.

1.3 Structure of the thesis

In Sections 2 and 3 we introduce basic objects of the theory of Krein systems and the theory of canonical systems – the main tools of the thesis. Sections 3.3 and 3.4 are devoted to the regularized Krein system and its properties. In Section 4 we prove Theorem 1. In Section 5 we consider the case of the off-diagonal potential and prove Theorem 2.

2 Krein systems

2.1 General definitions

Let $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be a complex-valued function on the positive half-line $\mathbb{R}_+$. Krein system with coefficient a is the following system of differential equations:

$$\begin{cases} \frac{\partial}{\partial r} P(r,\lambda) = i\lambda P(r,\lambda) - \overline{a(r)} P_*(r,\lambda), & P(0,\lambda) = 1, \\ \frac{\partial}{\partial r} P_*(r,\lambda) = -a(r)P(r,\lambda), & P_*(0,\lambda) = 1. \end{cases} \tag{11}$$

After seminal work [19] of M. Krein, the solutions of Krein system (11) are called the continuous analogs of polynomials orthogonal on the unit circle (see books [30] and [26]). Using Krein systems, one can transfer methods from the theory of orthogonal polynomials on the unit circle to the spectral problems for self-adjoint differential operators. Detailed account of this approach can be found in the paper [11] by S. Denisov.

For any Krein system (11) there exists the unique Borel measure σ_a on the real line [11] such that $\displaystyle\int_{\mathbb{R}} \frac{d\sigma_a(x)}{1+x^2} < \infty$ and the mapping

$$\mathcal{U}_{\sigma_a} : f \mapsto \frac{1}{\sqrt{2\pi}} \int_0^\infty f(r)P(r,\lambda)\,dr, \tag{12}$$

initially defined on simple measurable functions with compact support, can be continuously extended to an isometry from $L^2(\mathbb{R}_+)$ to $L^2(\mathbb{R}, \sigma_a)$. This measure is called the spectral measure of Krein system (11).

Theorem A (Krein theorem, Section 8 in [10], [31]). *Let σ_a be the spectral measure of Krein system* (11) *and let P, P_* be its solutions. Then the following assertions are equivalent:*

(*a*) *σ_a belongs to Szegő class* (5) *on the real line,*

(*b*) *for some point λ_0 in $\mathbb{C}_+ = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$ we have*

$$\int_0^\infty |P(r,\lambda_0)|^2\, dr < \infty,$$

(*c*) *there exists a sequence $r_n \to \infty$ and a number $\gamma \in [0, 2\pi)$ such that for every $\lambda \in \mathbb{C}_+$ the limit*

$$\Pi(\lambda) = e^{-i\gamma} \lim_{n\to\infty} P_*(r_n, \lambda)$$

exists and defines an analytic in $\mathbb{C}_+$ function with $\Pi(i) > 0$.

If the equivalent assertions in Krein theorem hold then (see Lemma 8.6 in [11]) Π is the inverse Szegő function of σ_a defined by (6). In other words, Π is an outer function in $\mathbb{C}_+$ such that

$$|\Pi(x)|^2 = \frac{1}{\sigma_a'(x)} \tag{13}$$

almost everywhere on $\mathbb{R}$, and

$$\Pi(\lambda) = \exp\left[-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{1}{s-\lambda} - \frac{s}{s^2+1}\right)\log \sigma_a'(s)\, ds\right], \quad \lambda \in \mathbb{C}_+. \tag{14}$$

Let us notice that $[(\lambda + i)\Pi]^{-1}$ belongs to the Hardy space $H^2(\mathbb{C}_+)$, therefore Π has nontangentional boundary values at almost every point on $\mathbb{R}$. Condition (*c*) of Theorem A can be enhanced in the following way.

Lemma 1 (Section 8, [11]). *Assume that σ_a belongs to the Szegő class on the real line and the sequence $r_n \to \infty$ is such that $P(r_n, \lambda_0) \to 0$ as $n \to \infty$ for some point $\lambda_0 \in \mathbb{C}_+$. Then there exists constant $\gamma \in [0, 2\pi)$ and a subsequence n_k such that $P_*(r_{n_k}, \lambda) \to e^{i\gamma}\Pi(\lambda)$ as $k \to \infty$ for every $\lambda \in \mathbb{C}_+$.*

Proof. The statement immediately follows from Lemma 8.5 and the proof of Lemma 8.6 from [11]. □

2.2 Reproducing kernels and the minimization problem

Let $PW_{[0,r]}$ denote the Paley-Wiener space of entire functions f that can be represented in the form

$$f(z) = \int_0^r \varphi(s) e^{izs}\, ds, \quad z \in \mathbb{C}, \quad \varphi \in L^2[0,r].$$

The function

$$k_r(z', z) = \frac{1}{2\pi}\int_0^r P(s,z)\overline{P(s,z')}\, ds \tag{15}$$

is the reproducing kernel in $PW_{[0,r]}$ at the point z', see Lemma 8.1 in [11]. In other words, for every $f \in PW_{[0,r]}$ we have

$$f(z') = \langle f, k_r(z', \cdot)\rangle_{L^2(\sigma_a)} = \int_{\mathbb{R}} f(x)\overline{k_r(z', x)}\, d\sigma_a(x).$$

For $r > 0$, define

$$\mathbf{m}_r(z) = \mathbf{m}_r(\sigma_a, z) = \inf\left\{\frac{1}{2\pi|f(z)|^2}\int_{-\infty}^{\infty}|f(t)|^2\, d\sigma_a(t)\;\middle|\; f \in PW_{[0,r]}, f(z) \neq 0\right\}, \quad z \in \mathbb{C}.$$

The function $\mathbf{m}_r$ is the analog of the Christoffel function in the theory of orthogonal polynomials (see Section 1.8 in [26]). Lemma 8.2 in [11] says that

$$\mathbf{m}_r(z) = (2\pi k_r(z, z))^{-1} = \left(\int_0^r |P(s, z)|^2\, ds\right)^{-1}, \quad z \in \mathbb{C}. \tag{16}$$

The functions P, P_* satisfy the following Christoffel-Darboux formula:

$$P(r, \lambda)\overline{P(r, \mu)} - P_*(r, \lambda)\overline{P_*(r, \mu)} = i(\lambda - \overline{\mu})\int_0^r P(s, \lambda)\overline{P(s, \mu)}\, ds, \tag{17}$$

$$|P_*(r, \lambda)|^2 - |P(r, \lambda)|^2 = 2\operatorname{Im}\lambda\int_0^r |P(s, \lambda)|^2\, ds, \tag{18}$$

see Lemma 3.6 in [11]. Furthermore, a simple calculation shows that

$$P(r, \lambda) = e^{i\lambda r}\overline{P_*(r, \overline{\lambda})}, \qquad P_*(r, \lambda) = e^{i\lambda r}\overline{P(r, \overline{\lambda})}, \qquad r \geqslant 0, \quad \lambda \in \mathbb{C}. \tag{19}$$

Together with Theorem A, relation (18) gives

$$|\Pi(\lambda)|^2 = 2\operatorname{Im}\lambda\int_0^\infty |P(s, \lambda)|^2\, ds. \tag{20}$$

Define $\mathbf{m}_\infty(z) = \inf_r \mathbf{m}_r(z)$. It follows that m_∞ admits representation

$$\mathbf{m}_\infty(z) = \left(\int_0^\infty |P(r, z)|^2\, dr\right)^{-1} = \frac{2\operatorname{Im} z}{|\Pi(z)|^2}, \quad z \in \mathbb{C}_+. \tag{21}$$

2.3 Connections with the Dirac operator

Consider a dual Krein system, i.e, Krein system (11) with the coefficient $-a$, and denote its solutions by $\hat{P}, \hat{P}_*$. It can be verified (see Section 4 in [11]) that $\hat{P}$ and $-\hat{P}_*$ solve the same differential system (11) as P and P_* but with the initial value $\binom{1}{-1}$. This can be rewritten in the form

$$X'(r, \lambda) = \begin{pmatrix} i\lambda & -\overline{a(r)} \\ -a(r) & 0\end{pmatrix} X(r, \lambda), \quad X(0, \lambda) = \begin{pmatrix} 1 & 1 \\ 1 & -1\end{pmatrix},$$

where

$$X(r, \lambda) = \begin{pmatrix} P(r, \lambda) & \hat{P}(r, \lambda) \\ P_*(r, \lambda) & -\hat{P}_*(r, \lambda)\end{pmatrix}.$$

Define

$$p_a(r) = -2\operatorname{Re} a(2r), \qquad q_a(r) = 2\operatorname{Im} a(2r), \qquad Q_a = \begin{pmatrix} -q_a(r) & p_a(r) \\ p_a(r) & q_a(r)\end{pmatrix}. \tag{22}$$

A calculation shows (see Chapter 13 in [11]) that

$$Y(r,\lambda) = \frac{e^{-i\lambda r}}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} X(2r,\lambda)$$

solves the differential system

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y'(r,\lambda) + Q_a Y(r,\lambda) = \lambda Y(t,\lambda), \quad Y(0,\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}.$$

This differential system differs from Dirac system (1) only in the definition of the square root of the minus identity matrix. It can be easily seen that $f = (f_1, f_2)^T$ solves $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) f + Q_a f = \lambda f$ if and ony if $f^\# = (f_1, -f_2)$ solves $Jf^\# - Q_a f^\# = \lambda f^\#$. Therefore, the fundamental solution of Dirac system (1) with the potential $-Q_a$ can be expressed in terms of $P, P_*, \hat{P}, \hat{P}_*$:

$$\begin{pmatrix} \theta_+(t,\lambda) & \varphi_+(t,\lambda) \\ \theta_-(t,\lambda) & \varphi_-(t,\lambda) \end{pmatrix} = \frac{e^{-i\lambda r}}{2}\begin{pmatrix} P(2r,\lambda) + P_*(2r,\lambda) & i\hat{P}(2r,\lambda) - i\hat{P}_*(2r,\lambda) \\ iP(2r,\lambda) - iP_*(2r,\lambda) & -\hat{P}(2r,\lambda) - \hat{P}_*(2r,\lambda) \end{pmatrix}. \tag{23}$$

Moreover, Theorem 13.1 in [11] says that in the notation of [11] the spectral measure of the Dirac operator with potential $-Q_a$ and of the Krein system with coefficient a differ by a multiplicative factor 2. Our normalization in definitions (2) and (12) of spectral measures differs from the one in [11]. Because of that Theorem 13.1 actually implies the coincidence of measures, i.e, we have

$$\sigma_{-Q_a} = \sigma_a.$$

On the other hand, if we start with a Dirac operator, we can easily construct a Krein system with the coefficient given by (22).

Let us prove relation (7) from the introduction. By (4) and (23), for every Dirac system we have

$$m(z) = -\lim_{r\to\infty}\frac{\varphi_+(r,z)}{\theta_+(r,z)} = -\lim_{r\to\infty}\frac{i\hat{P}(2r,z) - i\hat{P}_*(2r,z)}{P(2r,z) + P_*(2r,z)} = \lim_{r\to\infty}\frac{i\hat{P}_*(r,z) - i\hat{P}(r,z)}{P_*(r,z) + P(r,z)}, \quad z \in \mathbb{C}_+.$$

In the Szegő case, both $P(\cdot,z)$ and $\hat{P}(\cdot,z)$ are in $L^2(\mathbb{R}_+)$ and consequently there exists a sequence $r_n \to \infty$ such that $P(r_n,z) \to 0$ and $\hat{P}(r_n,z) \to 0$ as $n \to \infty$. Hence

$$m(z) = i\lim_{n\to\infty}\frac{\hat{P}_*(r_n,z)}{P_*(r_n,z)}.$$

Now (7) follows from the latter limit relation and Lemma 1.

3 Canonical systems

Approach that we will use is based on the reduction of Dirac system (1) and Krein system (11) to a canonical Hamiltonian system. Following [22], [23], let us introduce the key definitions of the theory of canonical systems.

Canonical Hamiltonian system is the differential equation

$$J\frac{\partial}{\partial t}M(t,z) = zH(t)M(t,z), \qquad M(0,z) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \tag{24}$$

where $t \geqslant 0$, $z \in \mathbb{C}$ and $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The Hamiltonian H is a matrix-valued mapping of the form

$$H = \begin{pmatrix} h_1 & h \\ h & h_2 \end{pmatrix} \tag{25}$$

with real entries from $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ satisfying

$$\operatorname{trace} H(t) > 0, \quad \det H(t) \geqslant 0, \qquad t \in \mathbb{R}_+.$$

Hamiltonian H is called singular if $\displaystyle\int_{\mathbb{R}_+} \operatorname{trace} H(s)\, ds = +\infty$. We will call Hamiltonian H trivial if it coincides with $\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0 & 1\\ 0 & 1\end{smallmatrix}\right)$ and nontrivial otherwise. The solution M of (24) is often represented as

$$M(t,z) = (\Theta(t,z), \Phi(t,z)) = \begin{pmatrix} \Theta_+(t,z) & \Phi_+(t,z)) \\ \Theta_-(t,z) & \Phi_-(t,z)) \end{pmatrix}. \tag{26}$$

The Weyl function of the canonical system is defined by

$$m(z) = \lim_{t\to\infty} \frac{w\Phi_+(t,z) + \Phi_-(t,z)}{w\Theta_+(t,z) + \Theta_-(t,z)}, \quad z \in \mathbb{C}_+, \quad w \in \mathbb{C} \cup \{\infty\}.$$

If the Hamiltonian is singular, this limit is correctly defined and does not depend on w. In particular, for $w = 0$ we have

$$m(z) = \lim_{t\to\infty} \frac{\Phi_-(t,z)}{\Theta_-(t,z)}, \quad z \in \mathbb{C}_+.$$

Furthermore, function m has a strictly positive imaginary part in $\mathbb{C}_+$ and therefore admits the representation

$$m(z) = \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{x - z} - \frac{x}{x^2 + 1} \right) d\sigma(x) + az + b, \tag{27}$$

where $a \in \mathbb{R}$ and $b \geqslant 0$ are constants and σ is a Borel measure satisfying $\int \frac{d\sigma}{x^2+1} < \infty$. The measure σ is called the spectral measure of the Hamiltonian H.

3.1 Reduction of the Dirac system to the canonical system

In this subsection we outline the formulas of the reduction omitting the calculations. For detailed explanation see [23] or Section 2.4 in [1]. Consider Dirac system (1) with the potential Q and define $N_Q(t) = N(t, 0)$. Then $M(t,z) = N_Q^{-1}(t) N(t,z)$ is the fundamental solution of canonical system (24) with the Hamiltonian

$$H_Q(t) = N_Q^*(t) N_Q(t).$$

Moreover, the spectral measure of the canonical system with Hamiltonian H_Q coincides with the spectral measure of the Dirac operator D_Q (see Section 2.4 in [1]).

3.2 Entropy of a canonical system

Consider canonical system (24). Let us define its entropy following the notation from [1], [2]. For $r \geqslant 0$, define Hamiltonian H_r as a shift of H, i.e., $H_r \colon x \mapsto H(r + x)$, and let $m_r, \sigma_r, w_r, a_r, b_r$ be Weyl function, spectral measure, density of the spectral measure and the coefficients in representation (27) of m_r. Next, define

$$\begin{aligned}
\mathcal{I}_H(r) &= \operatorname{Im} m_r(i) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\sigma_r(x)}{1 + x^2} + b_r, \\
\mathcal{R}_H(r) &= \operatorname{Re} m_r(i) = a_r, \\
\mathcal{J}_H(r) &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{\log(w_r(x))}{1 + x^2}\, dx.
\end{aligned}$$

The entropy of σ is defined by

$$\mathcal{K}_H(r) = \log \mathcal{I}_H(r) - \mathcal{J}_H(r), \quad r \geqslant 0.$$

Consider the Hamiltonian

$$\tilde{H}_r(t) = \begin{cases} H(t), & t \leqslant r, \\ M(r), & t > r, \end{cases} \qquad M(r) = \frac{1}{\mathcal{I}_H(r)} \begin{pmatrix} 1 & \mathcal{R}_H(r) \\ \mathcal{R}_H(r) & \mathcal{I}_H(r)^2 + \mathcal{R}_H(r)^2 \end{pmatrix}. \tag{28}$$

The Hamiltonian $\tilde{H}_r$ coincides with H on $[0, r]$ and is constant on (r, ∞). The matrix M is chosen so that (see Appendix in [1]) we have

$$\mathcal{K}_H(0) = \mathcal{K}_H(r) + \mathcal{K}_{\tilde{H}_r}(0),$$
$$\lim \mathcal{K}_H(r) = 0, \quad \lim \mathcal{K}_{\tilde{H}_r}(0) = \mathcal{K}_H(0), \quad r \to \infty.$$

In [2] R. Bessonov and S. Denisov described the class of Hamiltonians which spectral measures belong to the Szegő class (5). Define

$$\mathbf{K}_H = \sum_{n \geqslant 0} \left(\det \int_{\eta_n}^{\eta_{n+2}} H(t)\, dt - 4 \right), \quad \eta_n = \min \left\{ x \colon \int_0^x \sqrt{\det H(t)} dt = n \right\}. \tag{29}$$

This quantities are well-defined if $\sqrt{\det H} \notin L^1(\mathbb{R}_+)$. We call $\mathbf{K}_H$ the entropy of the Hamiltonian H. The main result of [2] is the following.

Theorem B (Theorem 1.2, [2])**.** *The spectral measure of a singular Hamiltonian H belongs to the Szegő class* (5) *if and only if $\sqrt{\det H} \notin L^1(\mathbb{R}_+)$ and $\mathbf{K}_H < \infty$. Moreover, we have*

$$c_1 \mathcal{K}_H \leqslant \mathbf{K}_H \leqslant c_2 \mathcal{K}_H \cdot e^{c_2 \mathcal{K}_H}.$$

In the thesis we will be interested in the case when $\mathbf{K}_{H_r}$ (or, equivalently, $\mathcal{K}_H(r)$) decreases exponentially fast in r. Formally, this can be written as

$$\sum_{n \geqslant 0} \left(\det \int_{\eta_{r+n}}^{\eta_{r+n+2}} H(t)\, dt - 4 \right) = O\left(e^{-\delta r}\right), \quad r \to \infty \tag{30}$$

for some $\delta > 0$. Clearly, this is equivalent to the exponential decay of the first term of the latter sum, i.e.,

$$\det \int_{\eta_r}^{\eta_{r+2}} H(t)\, dt - 4 = O\left(e^{-\delta r}\right), \quad r \to \infty.$$

Notice that if $H = H_Q$ is a Hamiltonian constructed for the Dirac operator D_Q in Section 3.1, then $\det H(t) = 1$ for every $t \geqslant 0$, therefore $\eta_r = r$ for every $r \geqslant 0$. In this situation the latter assertion is equivalent to

$$\det \int_r^{r+2} H_Q(t)\, dt - 4 = O\left(e^{-\delta r}\right), \quad r \to \infty,$$

which is exactly the assertion $E_Q(r) = O\left(e^{-\delta r}\right)$ from Theorem 1.

3.3 Regularized Krein system

Fix a singular nontrivial Hamiltonian H. Define $\mathcal{K}_H, \mathcal{I}_H, \mathcal{J}_H$ and $\mathcal{R}_H$ as in the previous section. In order to simplify the exposition we will omit the index H later on. Following the notation from [1], let us define the regularized Krein system which will be our main tool for work with the entropy function. First of all, introduce the regularized solutions of the canonical system

$$\begin{pmatrix} \tilde{\Theta}_r^+ \\ \tilde{\Theta}_r^- \end{pmatrix} = \begin{pmatrix} 1/\sqrt{I(r)} & R(r)/\sqrt{I(r)} \\ 0 & \sqrt{I(r)} \end{pmatrix} \begin{pmatrix} \Theta_r^+ \\ \Theta_r^- \end{pmatrix},$$

where Θ_r^+ and Θ_r^- are as in (26). Then the function $\tilde{E}_r(z) = \tilde{\Theta}_r^+(z) + i\tilde{\Theta}_r^-(z)$ is Hermite-Biehler, i.e., it satisfies $|\tilde{E}_r(z)| > |\tilde{E}_r(\overline{z})|$ for all $z \in \mathbb{C}_+$. Furthermore, Lemma 4 in [1] states that the spectral measure of the Hamiltonian $\tilde{H}_r$ equals $|\tilde{E}_r(x)|^{-2}dx$. For $r \geqslant 0$, define

$$\tilde{P}_{2r}^* \colon z \mapsto e^{irz+iu(r)}\tilde{E}_r(z), \quad \tilde{P}_{2r} \colon z \mapsto e^{irz-iu(r)}\overline{\tilde{E}_r(\overline{z})}, \tag{31}$$

where $u(r)$ is given by $u(r) = \int_0^r \frac{R'(t)}{2I(t)}dt$. The solutions of Krein system (11) can be defined similarly if we start with the Hamiltonian associated with the Krein system and replace $M(r)$ with $H(r)$ in the definition of $\tilde{H}_r$ (28). Because of that $\tilde{P}_r$ and $\tilde{P}_r^*$ posses many important properties of P and P_* and therefore can be considered as the regularized Krein system. Description of the properties of these functions is presented in Section 4 in [1], we deal with some of them in the next subsection.

3.4 A closer look at the regularized Krein system

The argument in the proof of Theorem 1 is based on the regularized Krein system. In this subsection we discuss its properties in general situation and in the case of exponentially decaying entropy. Lemmas 2, 3 and 4 are respectively Lemmas 9, 8, 7 from [1], in Lemmas 3 and 4 we will need a slightly more accurate bounds than provided in [1] so we state them with proofs (which are almost identical to the ones in [1]).

Let H be a nontrivial singular Hamiltonian and let σ be its spectral measure. We assume that σ belongs to Szegő class (5) and therefore the inverse Szegő function Π is well defined by (14). Functions $\tilde{P}_r$ and $\tilde{P}_r^*$ are the regularized Krein system (31) defined in the previous subsection.

Lemma 2. *For $z \in \mathbb{C}_+$, we have*

$$\lim_{r\to\infty} \tilde{P}_r^*(z) = \Pi(z), \qquad \lim_{r\to\infty} \tilde{P}_r(z) = 0, \qquad \int_0^\infty |\tilde{P}_r(z)|^2\, dr < \infty.$$

Lemma 3. *There exists an absolute constant $C > 0$ such that*

$$\left|\frac{\mathcal{I}'(r)}{\mathcal{I}(r)}\right| + \left|\frac{\mathcal{R}'(r)}{\mathcal{I}(r)}\right| \leqslant C(\sqrt{|\mathcal{K}'(r)|} + |\mathcal{K}'(r)|).$$

Proof. Formulas (39) and (40) in [1] give

$$-\mathcal{K}' = \left(\mathcal{I}h_1 + \frac{1}{\mathcal{I}h_1} - 2\right) + \frac{1}{4}\left(\frac{\mathcal{R}'}{I}\right)^2 \frac{1}{\mathcal{I}h_1}, \tag{32}$$

$$\frac{\mathcal{I}'}{\mathcal{I}} = \mathcal{I}h_1 - \frac{1}{\mathcal{I}h_1} - \frac{1}{4}\left(\frac{\mathcal{R}'}{\mathcal{I}}\right)^2 \frac{1}{\mathcal{I}h_1}, \tag{33}$$

where h_1 is the upper-left entry of H, see (25). Function $\mathcal{K}$ is non-increasing hence $-\mathcal{K} = |\mathcal{K}'|$ for every $t \geqslant 0$. Two terms in the right hand side of the first equality are nonnegative and therefore we have

$$\mathcal{I}h_1 + \frac{1}{\mathcal{I}h_1} - 2 \leqslant |\mathcal{K}'|, \tag{34}$$

$$\frac{1}{4}\left(\frac{\mathcal{R}'}{\mathcal{I}}\right)^2 \frac{1}{\mathcal{I}h_1} \leqslant |\mathcal{K}'|. \tag{35}$$

Case 1. Assume r is such that $\mathcal{I}(r)h_1(r) \in \left[\frac{1}{2}, 2\right]$. Under this assumption, formula (35) gives

$$\left(\frac{\mathcal{R}'}{\mathcal{I}}\right)^2 \leqslant 4\mathcal{I}h_1|\mathcal{K}'| \leqslant 8|\mathcal{K}'|. \tag{36}$$

Moreover, the quantities

$$\left|1 - \frac{1}{\mathcal{I}h_1}\right|, \quad |1 - \mathcal{I}h_1|, \quad \text{and} \quad \sqrt{\mathcal{I}h_1 + \frac{1}{\mathcal{I}h_1} - 2}$$

are equivalent up to some absolute multiplicative constant. Then by (34) there exists an absolute constant $c > 0$ such that

$$\left|1 - \frac{1}{\mathcal{I}h_1}\right| + |1 - \mathcal{I}h_1| \leqslant c\sqrt{|\mathcal{K}'|},$$

$$\left|\frac{\mathcal{I}'}{\mathcal{I}}\right| \overset{(33)}{\leqslant} \left|1 - \frac{1}{\mathcal{I}h_1}\right| + |1 - \mathcal{I}h_1| + \left|\frac{1}{4}\left(\frac{\mathcal{R}'}{\mathcal{I}}\right)^2 \frac{1}{\mathcal{I}h_1}\right| \overset{(35)}{\leqslant} c\sqrt{|\mathcal{K}'|} + 8|\mathcal{K}'|. \tag{37}$$

Case 2. Assume r is such that $\mathcal{I}(r)h_1(r) \in \left(-\infty, \frac{1}{2}\right) \cup (2, +\infty)$. In this case we have the equivalence of the quantities

$$\mathcal{I}h_1 + \frac{1}{\mathcal{I}h_1} - 2 \quad \text{and} \quad \mathcal{I}h_1 + \frac{1}{\mathcal{I}h_1}$$

and therefore the inequalities

$$\mathcal{I}h_1 + \frac{1}{\mathcal{I}h_1} \leqslant c|\mathcal{K}'|,$$

$$\left(\frac{\mathcal{R}'}{\mathcal{I}}\right)^2 \overset{(35)}{\leqslant} 4\mathcal{I}h_1|\mathcal{K}'| \leqslant 4c|\mathcal{K}'|^2, \tag{38}$$

$$\left|\frac{\mathcal{I}'}{\mathcal{I}}\right| \overset{(33)}{\leqslant} \left|\frac{1}{\mathcal{I}h_1} + \mathcal{I}h_1\right| + \left|\frac{1}{4}\left(\frac{\mathcal{R}'}{\mathcal{I}}\right)^2 \frac{1}{\mathcal{I}h_1}\right| \overset{(35)}{\leqslant} c|\mathcal{K}'| + |\mathcal{K}'| \tag{39}$$

hold with some absolute constant c. Now lemma follows from (36), (37), (38), (39). $\square$

Lemma 4. *Fix $z \in \mathbb{C}$. Then the functions $r \mapsto \tilde{P}_r(z)$ and $r \mapsto \tilde{P}_r^*(z)$ are absolutely continuous and for almost all $r > 0$ we have*

$$\frac{\partial}{\partial r}\tilde{P}_r^*(z) = (z - i)f_1(r)\tilde{P}_r(z) + izf_2(r)\tilde{P}_r^*(z), \tag{40}$$

$$\frac{\partial}{\partial r}\tilde{P}_r(z) = iz\tilde{P}_r(z) + (z + i)\overline{f_1(r)}\tilde{P}_r^*(z) - iz\overline{f_2(r)}\tilde{P}_r(z), \tag{41}$$

where $f_2(r) = \frac{\mathcal{K}'(r/2)}{4}$ and $|f_1(r)| \leqslant c\left(\sqrt{|\mathcal{K}'(r/2)|} + |\mathcal{K}'(r/2)|\right)$ for some absolute constant $c > 0$.

Proof. Lemma 8 in [1] provides a differential equation for $\tilde{P}^*$:

$$\frac{\partial}{\partial r}\tilde{P}^*_{2r}(z) = -\frac{1}{2}(z-i)e^{2iu(r)}\left(\frac{R'(r)}{I(r)} + i\frac{I'(r)}{I(r)}\right)\tilde{P}_{2r}(z) + \frac{i}{2}z\mathcal{K}'(r)\tilde{P}^*_{2r}(z).$$

Formula (40) easily follows from this relation and Lemma 3 by the change of variables. To establish (41), notice that

$$\tilde{P}^*_r(z) = e^{irz}\overline{\tilde{P}_r(\overline{z})}, \quad \tilde{P}_r(z) = e^{irz}\overline{\tilde{P}^*_r(\overline{z})}. \tag{42}$$

because the function u in definition (31) of the regularized Krein system is real-valued. Therefore

$$\begin{aligned}\frac{\partial}{\partial r}\tilde{P}_r(z) &= \frac{\partial}{\partial r}\left(e^{irz}\overline{\tilde{P}^*_r(\overline{z})}\right) = iz\cdot e^{irz}\overline{\tilde{P}^*_r(\overline{z})} + e^{irz}\cdot\frac{\partial}{\partial r}\overline{\tilde{P}^*_r(\overline{z})}\\ &= iz\tilde{P}_r(z) + e^{irz}\overline{\left((\overline{z}-i)f_1(r)\tilde{P}_r(\overline{z}) + i\overline{z}f_2(r)\tilde{P}^*_r(\overline{z})\right)}\\ &= iz\tilde{P}_r(z) + (z+i)\overline{f_1(r)}\tilde{P}^*_r(z) - iz\overline{f_2(r)}\tilde{P}_r(z)\end{aligned}$$

holds and the proof is complete. □

Corollary 1. The following differential equations hold for the absolute values of $\tilde{P}_r$ and $\tilde{P}^*_r$:

$$\frac{\partial}{\partial r}|\tilde{P}^*_r(z)|^2 = 2\operatorname{Re}\left((z-i)f_1(r)\tilde{P}_r(z)\overline{\tilde{P}^*_r(z)}\right) - 2\operatorname{Im} z f_2(r)|\tilde{P}^*_r(z)|^2, \tag{43}$$

$$\frac{\partial}{\partial r}|\tilde{P}_r(z)|^2 = -2\operatorname{Im} z|\tilde{P}_r(z)|^2 + 2\operatorname{Re}\left((z+i)\overline{f_1(r)}\tilde{P}^*_r(z)\overline{\tilde{P}_r(z)}\right) + 2\operatorname{Im} z f_2(r)|\tilde{P}_r(z)|^2. \tag{44}$$

Proof. The proof is a straightforward calculation. We have

$$\begin{aligned}\frac{\partial}{\partial r}|\tilde{P}^*_r(z)|^2 &= 2\operatorname{Re}\left(\overline{\tilde{P}^*_r(z)}\frac{\partial}{\partial r}\tilde{P}^*_r(z)\right) = 2\operatorname{Re}\left(\overline{\tilde{P}^*_r(z)}\left((z-i)f_1(r)\tilde{P}_r(z) + izf_2(r)\tilde{P}^*_r(z)\right)\right)\\ &= 2\operatorname{Re}\left((z-i)f_1(r)\tilde{P}_r(z)\overline{\tilde{P}^*_r(z)}\right) - 2\operatorname{Im} z\operatorname{Re} f_2(r)|\tilde{P}^*_r(z)|^2;\\ \frac{\partial}{\partial r}|\tilde{P}_r(z)|^2 &= 2\operatorname{Re}\left(\overline{\tilde{P}_r(z)}\frac{\partial}{\partial r}\tilde{P}_r(z)\right)\\ &= 2\operatorname{Re}\left(\overline{\tilde{P}_r(z)}\left(iz\tilde{P}_r(z) + (z+i)\overline{f_1(r)}\tilde{P}^*_r(z) - iz\overline{f_2(r)}\tilde{P}_r(z)\right)\right)\\ &= -2\operatorname{Im} z|\tilde{P}_r(z)|^2 + 2\operatorname{Re}\left((z+i)\overline{f_1(r)}\tilde{P}^*_r(z)\overline{\tilde{P}_r(z)}\right) + 2\operatorname{Im} z\operatorname{Re} f_2(r)|\tilde{P}_r(z)|^2.\end{aligned}$$

Function f_2 is real-valued hence the claim follows. □

Consider Krein system (11). Let H be a Hamiltonian constructed from this Krein system via the reductions in Sections 2.3 and 3.1 so that the spectral measure of the Krein system coincides with the spectral measure of the canonical system with the Hamiltonian H. Denote this measure by σ. The next lemma connects the Krein system and its regularized version.

Lemma 5. *For $z_0 \in \mathbb{C}_+$ and $r \geqslant 0$ we have*

$$2\operatorname{Im} z_0\int_r^\infty |P(x,z_0)|^2\,dx = |\Pi(z_0)|^2 - \left(|\tilde{P}^*_r(z_0)|^2 - |\tilde{P}_r(z_0)|^2\right). \tag{45}$$

Proof. Let σ be a spectral measure of the Krein system. Recall that the reproducing kernel in the space $PW_{[0,r]}$ with norm inherited from $L^2(d\sigma)$ at the point $z_0 \in \mathbb{C}_+$ is given by (15),

$$k_r(z_0,z) = \frac{1}{2\pi}\int_0^r P(x,z)\overline{P(x,z_0)}\,dx.$$

On the other hand, the reproducing kernel admits the following representation in terms of the regularized Krein system (see formula (48) in [1]):

$$k_r(z_0, z) = -\frac{1}{2\pi i}\frac{\tilde{P}_r^*(z)\overline{\tilde{P}_r^*(z_0)} - \tilde{P}_r(z)\overline{\tilde{P}_r(z_0)}}{z - \overline{z_0}}.$$

The two latter equalities give

$$-\frac{1}{2\pi i}\frac{\tilde{P}_r^*(z)\overline{\tilde{P}_r^*(z_0)} - \tilde{P}_r(z)\overline{\tilde{P}_r(z_0)}}{z - \overline{z_0}} = \frac{1}{2\pi}\int_0^r P(x,z)\overline{P(x,z_0)}\,dx, \quad z \in \mathbb{C}.$$

If we replace z by z_0 there, we obtain

$$|\tilde{P}_r^*(z_0)|^2 - |\tilde{P}_r(z_0)|^2 = 2\,\mathrm{Im}\, z_0 \int_0^r |P(x,z_0)|^2\,dx.$$

The statement of the lemma now follows from (20). □

4 Proof of Theorem 1

4.1 An auxiliary lemma

In the proof we will need the following lemma.

Lemma 6. *Let b, c, d be positive numbers and let f be an absolutely continuous function on $\mathbb{R}_+$ such that for every $r \geqslant 0$ we have*

$$\left\|f' + bf\right\|_{L^1[r,\infty)} \leqslant ce^{-dr}. \tag{46}$$

Take an arbitrary $\Delta < \min(b,d)$. Assume that

$$f(r) \geqslant e^{-\Delta r} \tag{47}$$

on some interval $I = [A, B]$. Then $|I| \leqslant C$ for a positive number C depending only on b, c, d, Δ and $f(0)$.

Remark. Notice that the bound for Δ is sharp for the function $f(r) = e^{\min(b,d)r}$.

Proof. Denote $\varepsilon(r) = f'(r) + bf(r)$. Then

$$f'(r) = \varepsilon(r) - bf(r),$$
$$\frac{f'(r)}{f(r)} = -b + \frac{\varepsilon(r)}{f(r)}.$$

Integrating over some finite segment $[r_1, r_2] \subset I$, we get

$$\log f(r_2) - \log f(r_1) = -b(r_2 - r_1) + \int_{r_1}^{r_2} \frac{\varepsilon(r)}{f(r)}\,dr,$$
$$f(r_2) = f(r_1)e^{-b(r_2-r_1)}\exp\left[\int_{r_1}^{r_2}\frac{\varepsilon(r)}{f(r)}\,dr\right]. \tag{48}$$

Without loss of generality, we can assume that I is a maximal interval satisfying (47). Let us show that I is not a half-line, i.e., that $B < \infty$. Assume the converse. Denote $I_n = [A+n, A+n+1]$. We have

$$\begin{aligned}\int_A^\infty \left|\frac{\varepsilon(r)}{f(r)}\right| dr &= \sum_{n=0}^\infty \int_{I_n} \left|\frac{\varepsilon(r)}{f(r)}\right| dr \leqslant \sum_{n=0}^\infty \sup_{r\in I_n} |f(r)^{-1}| \, \|\varepsilon\|_{L^1(I_n)} \\ &\leqslant \sum_{n=0}^\infty \left(e^{\Delta(A+n+1)} \cdot ce^{-d(A+n)}\right) = e^{-(d-\Delta)A} \sum_{n=0}^\infty \left(e^{\Delta(n+1)} \cdot ce^{-dn}\right). \end{aligned} \tag{49}$$

We assumed that $\Delta < d$ hence the latter series converges and there exists $C_1 > 0$ depending on Δ, d and c such that

$$\int_A^\infty \left|\frac{\varepsilon(r)}{f(r)}\right| dr \leqslant C_1(d, \Delta, c) e^{-(d-\Delta)A}.$$

Then by (48) for any $N > A$ we have

$$f(N) \leqslant f(A) e^{-b(N-A)} M,$$

where $M = \exp\left[\int_A^\infty \left|\frac{\varepsilon(r)}{f(r)}\right| dr\right] < \infty$. From this we see that $f(N) \geqslant e^{-\Delta r}$ does not hold for a large N which gives us a contradiction. Therefore B is finite and we can substitute A and B into (48). We have

$$f(B) = f(A) e^{-b(B-A)} \exp\left[\int_A^B \frac{\varepsilon(r)}{f(r)} dr\right].$$

By the same argument as in (49) for the interval $[A, B]$ instead of $[A, +\infty)$, we obtain

$$f(B) \leqslant f(A) e^{-b(B-A)} \exp\left[C_1(d, \Delta, c)\right]. \tag{50}$$

Interval I is a maximal interval satisfying (47) hence the continuity of f gives

$$f(B) = e^{-\Delta B}, \qquad \begin{cases} f(A) = e^{-\Delta A}, & A \neq 0, \\ f(A) = f(0), & A = 0. \end{cases}$$

In both cases we have $f(A) \leqslant (1 + |f(0)|) e^{-\Delta A}$ and (50) implies

$$\begin{aligned} e^{-\Delta B} &\leqslant (1 + |f(0)|) e^{-\Delta A} e^{-b(B-A)} \exp\left[C_1(d, \Delta, c)\right], \\ e^{(b-\Delta)(B-A)} &\leqslant (1 + |f(0)|) \exp\left[C_1(d, \Delta, c)\right]. \end{aligned}$$

The difference $b - \Delta$ is positive, therefore, is follows that

$$|I| = B - A \leqslant \frac{\log(1 + |f(0)|) + C_1(d, \Delta, c)}{b - \Delta}.$$

This inequality completes the proof. □

4.2 Proof of Theorem 1

Proof. The Hamiltonian H_Q defined by (8) coincides with the Hamiltonian constructed in Section 3.1. Let σ be the spectral measure of D_Q. In the proof we estimate the rate of exponential decay of many quantities. Let us denote $\Delta_E = \delta$ to emphasize that this number corresponds to the the rate of decay of E_Q. At the end of Section 3.2 we have proved that the assertion of the theorem

$$E_Q(r) = \det \int_r^{r+2} H_Q(t)\,dt - 4 = O\left(e^{-\Delta_E r}\right), \quad r \to \infty,$$

is equivalent to

$$K_{H_r} = \sum_{n \geqslant 0} \left(\det \int_{r+n}^{r+n+2} H_Q(t)\,dt - 4 \right) = O\left(e^{-\Delta_E r}\right), \quad r \to \infty.$$

By Theorem B, it follows that σ is in the Szegő class and, moreover, we have

$$\mathcal{K}_H(r) = O\left(e^{-\Delta_E r}\right), \quad r \to \infty.$$

The rest of the proof is divided into 5 steps.

i. Establish the exponentially fast convergence of $\tilde{P}_r$ and $\tilde{P}_r^*$. More precisely, show that for every fixed point $z_0 \in \mathbb{C}_+$ we have

$$|\tilde{P}_r^*(z_0) - \Pi(z_0)| = O\left(e^{-r\Delta_E/4}\right), \quad r \to \infty, \tag{51}$$

and that for every $\Delta_{\tilde{P}} < \min(\operatorname{Im} z_0, \Delta_E/8)$ there exists an increasing unbounded sequence r_n of positive numbers with $\sup |r_{n+1} - r_n| < \infty$ such that

$$|\tilde{P}_{r_n}(z_0)| = O\left(e^{-\Delta_{\tilde{P}} r_n}\right), \quad n \to \infty. \tag{52}$$

ii. Show that for every $z_0 \in \mathbb{C}_+$ and $\Delta_P < \min(\operatorname{Im} z_0, \Delta_E/8)$ we have

$$\int_r^{\infty} |P(x, z_0)|^2\,dx = O\left(e^{-2\Delta_P r}\right), \quad r \to \infty.$$

iii. Use Christoffel-Darboux formula (17) to extend Π into the domain $\Omega_{\Delta_E/8}$.

iv. Use the Bernstein-Szegő approximation to show that σ has no singular part.

v. Extend m meromorphically into $\Omega_{\Delta_E/8}$ using relation (7).

Step i. Lemma 2 states that the functions $\tilde{P}_r^*(z_0)$ and $\tilde{P}_r(z_0)$ both converge as $r \to \infty$ and therefore they are uniformly bounded for $r \in \mathbb{R}_+$. From this observation we see that formulas (40) and (44) can be rewritten in the form

$$\frac{\partial}{\partial r} \tilde{P}_r^*(z_0) = g_1(r), \tag{53}$$

$$\frac{\partial}{\partial r} |\tilde{P}_r(z_0)|^2 = -2 \operatorname{Im} z_0 |\tilde{P}_r(z_0)|^2 + g_2(r), \tag{54}$$

where g_1 and g_2 are some functions satisfying

$$|g_1(r)| + |g_2(r)| \leqslant C_1(z_0) \left(\sqrt{|\mathcal{K}'(r/2)|} + |\mathcal{K}'(r/2)| \right).$$

Notice that

$$\int_r^\infty \sqrt{|\mathcal{K}'(s/2)|}\,ds = 2\int_{r/2}^\infty \sqrt{|\mathcal{K}'(s)|}\,ds = 2\sum_{n=0}^\infty \int_{r/2+n}^{r/2+n+1} \sqrt{|\mathcal{K}'(s)|}\,ds$$
$$\leqslant 2\sum_{n=0}^\infty \sqrt{\int_{r/2+n}^{r/2+n+1} |\mathcal{K}'(s)|\,ds} = 2\sum_{n=0}^\infty \sqrt{\mathcal{K}_H\left(r/2+n\right)} = O\left(e^{-r\Delta_E/4}\right),$$
$$\int_r^\infty |\mathcal{K}'(s/2)|\,ds = 2\int_{r/2}^\infty |\mathcal{K}'(s)|\,ds = 2\mathcal{K}\left(r/2\right) = O\left(e^{-r\Delta_E/2}\right).$$

Therefore we have

$$\int_r^\infty |g_1(s)|\,ds = O\left(e^{-r\Delta_E/4}\right), \quad \int_r^\infty |g_2(s)|\,ds = O\left(e^{-r\Delta_E/4}\right), \quad r\to\infty.$$

Now (51) follows from the latter equality, (53) and Lemma 2:

$$\left|\tilde{P}_r^*(z_0) - \Pi(z_0)\right| = \left|\int_r^\infty g_1(s)\,ds\right| = O\left(e^{-r\Delta_E/4}\right), \quad r\to\infty.$$

Formula (54) is equivalent to

$$\frac{\partial}{\partial r}|\tilde{P}_r(z_0)|^2 + 2\operatorname{Im} z_0 |\tilde{P}_r(z_0)|^2 = g_2(r).$$

We see that Lemma 6 can be applied to

$$f = |\tilde{P}_r(z_0)|^2, \quad b = 2\operatorname{Im} z_0, \quad d = \Delta_E/4,$$

and the sequence r_n required in (52) can be chosen such that $\sup|r_{n+1}-r_n| \leqslant C$, where C is the constant from Lemma 6.

Step ii. Formula (45) in Lemma 5 is equivalent to

$$2\operatorname{Im} z_0 \int_r^\infty |P(x,z_0)|^2\,dx = \left(|\Pi(z_0)|^2 - |\tilde{P}_r^*(z_0)|^2\right) + |\tilde{P}_r(z_0)|^2. \tag{55}$$

From the results of Step i we know that for every $\Delta_P < \min(\operatorname{Im} z, \Delta_E/8)$ there exists an increasing sequence $r_n \to \infty$ such that

$$\sup_n |r_{n+1}-r_n| < \infty \tag{56}$$

and

$$\left(|\Pi(z_0)|^2 - |\tilde{P}_{r_n}^*(z_0)|^2\right) + |\tilde{P}_{r_n}(z_0)|^2 = O\left(e^{-2r_n\Delta_P}\right), \quad n\to\infty.$$

Consequently, we have

$$2\operatorname{Im} z_0 \int_{r_n}^\infty |P(x,z_0)|^2\,dx = O\left(e^{-2r_n\Delta_P}\right), \quad n\to\infty.$$

Next, take arbitrary $r\in[r_n, r_{n+1}]$ then

$$\int_r^\infty |P(x,z_0)|^2\,dx \leqslant \int_{r_n}^\infty |P(x,z_0)|^2\,dx = O(e^{-2\Delta_P r_n}) = O(e^{-2\Delta_P r}), \quad r\to\infty.$$

The last equality holds because of the inequality $|r - r_n| < |r_{n+1} - r_n|$ and assumption (56).

Step iii. Fix any large number $h > \Delta_E/8$. It is clear that in this situation we have

$$\min\left(\operatorname{Im}(ih), \Delta_E/8\right) = \Delta_E/8$$

and therefore for any $\Delta_h < \Delta_E/8$ we have

$$\int_r^\infty |P(x, ih)|^2\, dx = O(e^{-2\Delta_h r}), \quad r \to \infty. \tag{57}$$

Substitute ih into Christoffel-Darboux formula (17). We have

$$i\frac{P_*(r,z)\overline{P_*(r,ih)} - P(r,z)\overline{P(r,ih)}}{z + ih} = \int_0^r P(x,z)\overline{P(x,ih)}\, dx. \tag{58}$$

Take an arbitrary increasing sequence $\rho_n \to \infty$ such that $P(\rho_n, ih) \to 0$ as $n \to \infty$. Then from Lemma 1 we know that there exist a subsequence n_k and $\gamma \in [0, 2\pi)$ such that

$$P_*(\rho_{n_k}, z) \to e^{i\gamma}\Pi(z), \qquad P(\rho_{n_k}, z) \to 0$$

uniformly on compact subsets in $\mathbb{C}_+$. Substituting ρ_{n_k} for r in (58) and taking the limit as $k \to \infty$, we get

$$i\frac{\Pi(z)\overline{\Pi(ih)}}{z + ih} = \lim_{k\to\infty}\int_0^{\rho_{n_k}} P(x,z)\overline{P(x,ih)}\, dx, \quad z \in \mathbb{C}_+,$$

or, equivalently,

$$\Pi(z) = \frac{z + ih}{i\overline{\Pi(ih)}}\lim_{k\to\infty}\int_0^{\rho_{n_k}} P(x,z)\overline{P(x,ih)}\, dx, \quad z \in \mathbb{C}_+. \tag{59}$$

Now we must only prove that the integral

$$\int_0^\infty P(x,z)\overline{P(x,ih)}\, dx \tag{60}$$

converges uniformly on compact subsets not only in $\mathbb{C}_+$ but in $\Omega_{\Delta_h} = \{z\colon \operatorname{Im} z > -\Delta_h\}$ because then the function defined by (59) in Ω_{Δ_h} will be the required extension of Π.

For two positive real numbers $A \leqslant B$, define

$$F_{A,B}(z) = \int_A^B P(r,z)\overline{P(r,ih)}\, dr.$$

For $z \in \mathbb{C}_+$, Cauchy-Schwartz inequality gives

$$\begin{aligned}|F_{A,B}(z)| &\leqslant \sqrt{\int_A^B |P(r,z)|^2\, dr}\sqrt{\int_A^B |P(r,ih)|^2\, dr} \\ &\overset{(57)}{\leqslant} O\left(e^{-A\Delta_h}\right)\cdot\sqrt{\int_0^\infty |P(r,z)|^2\, dr} \overset{(20)}{\leqslant} O\left(e^{-A\Delta_h}\right)\frac{|\Pi(z)|}{\sqrt{\operatorname{Im} z}}.\end{aligned}$$

Therefore, we have

$$|F_{A,B}(z)| \leqslant O(e^{-A\Delta_h})\frac{|\Pi(z)|}{\sqrt{\operatorname{Im} z}}, \qquad z \in \mathbb{C}_+, \tag{61}$$

where the constant in O does not depend on the point $z \in \mathbb{C}_+$. Next, consider $z \in \mathbb{C}_- \cap \Omega_{\Delta_h}$, where $\mathbb{C}_- = \{z\colon \operatorname{Im} z < 0\}$. From the reflection formula (19), $F_{A,B}$ admits the representaion

$$F_{A,B}(z) = \int_A^B e^{izr}\overline{P_*(r,\overline{z})P(r,ih)}\,dr.$$

Using the same Cauchy-Schwartz argument as for $z \in \mathbb{C}_+$, we get

$$\begin{aligned}|F_{A,B}(z)| &\leqslant \sqrt{\int_A^B e^{2|\operatorname{Im} z|r}|P_*(r,\overline{z})|^2\,dr}\sqrt{\int_A^B |P(r,ih)|^2\,dr}\\ &\leqslant O\left(e^{-A(\Delta_h-|\operatorname{Im} z|)}\right)\cdot\sqrt{\int_A^B |P_*(r,\overline{z})|^2\,dr}.\end{aligned} \tag{62}$$

The function $|P_*(r,\overline{z})|^2$ is not summable on $\mathbb{R}_+$, however, the integral on the finite segment can be estimated by (18) and (20). In other words, we have

$$\begin{aligned}\int_A^B |P_*(r,\overline{z})|^2\,dr &= \int_A^B \left(|P(r,\overline{z})|^2 + 2\operatorname{Im}(\overline{z})\int_0^r |P(s,\overline{z})|^2\,ds\right)dr\\ &= \int_A^B |P(r,\overline{z})|^2\,dr + 2\operatorname{Im}(\overline{z})\int_A^B\int_0^r |P(s,\overline{z})|^2\,ds\,dr\\ &\leqslant \int_0^\infty |P(r,\overline{z})|^2\,dr + 2\operatorname{Im}(\overline{z})\int_A^B\int_0^\infty |P(s,\overline{z})|^2\,ds\,dr\\ &= \left(\frac{1}{2\operatorname{Im}(\overline{z})} + B - A\right)2\operatorname{Im}(\overline{z})\int_0^\infty |P(s,\overline{z})|^2\,ds\\ &\leqslant \left(\frac{1}{2\operatorname{Im}(\overline{z})} + B - A\right)|\Pi(\overline{z})|^2.\end{aligned}$$

In addition, suppose that $|B-A| \leqslant 1$. Then, for $z \in \mathbb{C}_- \cap \Omega_{\Delta_h}$, we have

$$\begin{gathered}\frac{1}{2\operatorname{Im}(\overline{z})} + B - A \leqslant \frac{1+2\operatorname{Im}(\overline{z})}{2\operatorname{Im}(\overline{z})} \leqslant \frac{1+2\Delta_h}{2\operatorname{Im}(\overline{z})},\\ \int_A^B |P_*(r,\overline{z})|^2\,dr \leqslant \frac{1+2\Delta_h}{2\operatorname{Im}(\overline{z})}|\Pi(\overline{z})|^2.\end{gathered}$$

Substituting the latter bound into (62), we get

$$|F_{A,B}(z)| \leqslant O\left(e^{-A(\Delta_h-|\operatorname{Im} z|)}\right)\frac{|\Pi(\overline{z})|}{\sqrt{|\operatorname{Im} z|}},\quad z \in \mathbb{C}_- \cap \Omega_{\Delta_h}. \tag{63}$$

Take a connected compact set $K \subset \Omega_{\Delta_h}$. Let us show that there exist positive constants $C(K)$ and $\alpha(K)$ such that

$$|F_{A,B}(z)| \leqslant C(K)e^{-A\alpha(K)}, \tag{64}$$

uniformly for A, B and $z \in K$. Three different situations are possible:

$$K \subset \mathbb{C}_+,\quad K \subset \Omega_{\Delta_h} \cap \mathbb{C}_-,\quad K \cap \mathbb{R} \neq \emptyset.$$

In the first and in the second situations bound (64) easily follows from (61) and (63) respectively. If K intersects real line then take rectangle R with sides parallel to the real and imaginary axis of the complex plane such that

$$K \subset R \subset \Omega_{\Delta_h},\quad \operatorname{dist}(K,\partial R) > 0,$$

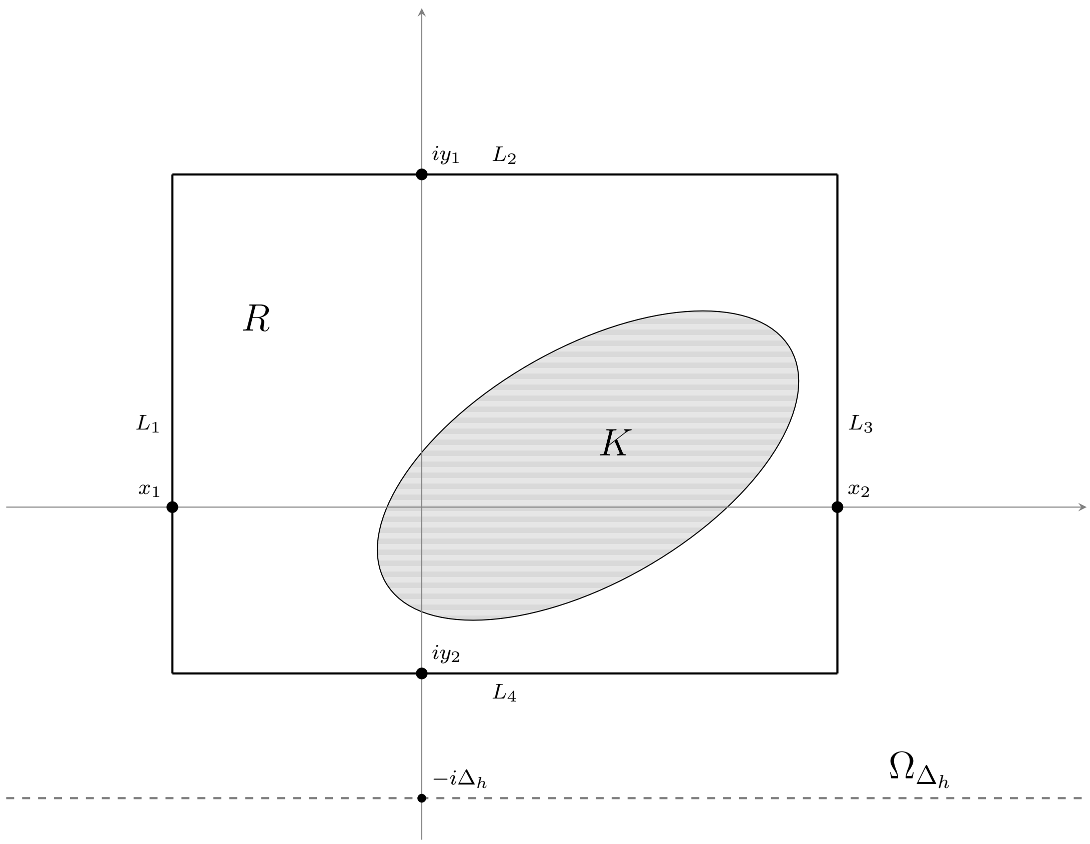

Figure 1: a compact K and a rectangle R in the proof of (64)

see Figure 1. Let L_1, L_2, L_3 and L_4 be the left, top, right and bottom sides of R respectively and let x_1, x_2, y_1, y_2 be such that

$$L_1 \subset \{z\colon \operatorname{Re} z = x_1\}, \qquad L_3 \subset \{z\colon \operatorname{Re} z = x_2\},$$
$$L_2 \subset \{z\colon \operatorname{Im} z = y_1\}, \qquad L_4 \subset \{z\colon \operatorname{Im} z = y_2\}.$$

Without loss of generality, it can be assumed that Π has nontangential boundary values at the points x_1 and x_2 on the real line and therefore

$$\sup_{z\in\partial R\cap\mathbb{C}_+} |\Pi(z)| < \infty, \qquad \sup_{z\in\partial R\cap\mathbb{C}_-} |\Pi(\overline{z})| < \infty.$$

Denote $\alpha = \Delta_h - |y_2| > 0$. Then, combining (61) and (63), we obtain

$$|F_{A,B}(z)| \leqslant O\left(e^{-A\alpha}\right)\frac{1}{\sqrt{|\operatorname{Im} z|}}, \quad z \in \partial R, \quad |B-A| \leqslant 1. \tag{65}$$

For an arbitrary point $z_0 \in K$ we have

$$F_{A,B}(z_0) = \frac{1}{2\pi i}\int_{\partial R}\frac{F_{A,B}(z)}{z-z_0}dz = \frac{1}{2\pi i}\sum_{n=1}^{4}\int_{L_n}\frac{F_{A,B}(z)}{z-z_0}dz,$$

Clearly, $|z-z_0| \geqslant \operatorname{dist}(K,\partial R) > 0$ holds and therefore

$$|F_{A,B}(z_0)| \leqslant \frac{1}{2\pi\operatorname{dist}(K,\partial R)}\int_{\partial R}|F_{A,B}(z)||dz| = \frac{1}{2\pi\operatorname{dist}(K,\partial R)}\sum_{n=1}^{4}\int_{L_n}|F_{A,B}(z)||dz|. \tag{66}$$

It remains to bound the integrals over the sides of R. We have

$$\int_{L_2} |F_{A,B}(z)||dz| = \int_{x_1}^{x_2} |F_{A,B}(x+iy_1)|dx \overset{(65)}{=} O\left(e^{-A\alpha}\right),$$
$$\int_{L_1} |F_{A,B}(z)||dz| = \int_{y_2}^{y_1} |F_{A,B}(x_1+iy)|dy \overset{(65)}{=} O\left(e^{-A\alpha}\right) \int_{y_2}^{y_1} \frac{1}{\sqrt{|\operatorname{Im} y|}}dy = O\left(e^{-A\alpha}\right).$$

Integrals over the segments L_3 and L_4 can be bounded similarly. Now (64) under the assumption $|B-A| \leqslant 1$ follows from (66). Finally, take two arbitrary numbers $A < B$ and let n be an integer number such that $n \leqslant B - A < n+1$. Then we can write

$$\begin{aligned} |F_{A,B}(z)| &= \left|\int_A^B P(r,z)\overline{P(r,ih)}\,dr\right| \\ &= \sum_{k=0}^{n-1} \left|\int_{A+k}^{A+k+1} P(r,z)\overline{P(r,ih)}\,dr\right| + \left|\int_{A+n}^{B} P(r,z)\overline{P(r,ih)}\,dr\right| \\ &= \sum_{k=0}^{n-1} O\left(e^{-(A+k)\alpha}\right) + O\left(e^{-(A+n)\alpha}\right) = O\left(e^{-A\alpha}\right). \end{aligned}$$

Convergence of integral (60) follows and Step iii is finished.

Step iv. Take a sequence ρ_n from the previous step. Recall that

$$P_*(\rho_n, ih) \to e^{i\gamma}\Pi(ih), \quad P(\rho_n, ih) \to 0. \tag{67}$$

On Step iii we have proved that

$$\Pi(z) = \frac{z+ih}{i\overline{\Pi(ih)}} \lim_{n\to\infty} \int_0^{\rho_n} P(x,z)\overline{P(x,ih)}\,dx, \quad z \in \Omega_{\Delta_h}. \tag{68}$$

Substituting ih in Christoffel-Darboux formula (17), we get

$$F_{0,\rho_n}(z) = \int_0^{\rho_n} P(x,z)\overline{P(x,ih)}\,dx = \frac{P_*(\rho_n,z)\overline{P_*(\rho_n,ih)} - P(\rho_n,z)\overline{P(\rho_n,ih)}}{z+ih}.$$

For $z \in \mathbb{R}$ we have $|P_*(\rho_n,z)| = |P(\rho_n,z)|$, therefore,

$$\frac{|P_*(\rho_n,z)|(|P_*(\rho_n,ih)| - |P(\rho_n,ih)|)}{|z+ih|} \leqslant |F_{0,\rho_n}(z)| \leqslant \frac{|P_*(\rho_n,z)|(|P_*(\rho_n,ih)| + |P(\rho_n,ih)|)}{|z+ih|}.$$

Reordering the multipliers, we obtain

$$\frac{\left|(z+ih)\int_0^{\rho_n} P(x,z)\overline{P(x,ih)}\,dx\right|}{|P_*(\rho_n,ih)| + |P(\rho_n,ih)|} \leqslant |P_*(\rho_n,z)| \leqslant \frac{\left|(z+ih)\int_0^{\rho_n} P(x,z)\overline{P(x,ih)}\,dx\right|}{|P_*(\rho_n,ih)| - |P(\rho_n,ih)|}.$$

Because of (67) and (68), the left hand side and the right hand side of this formula both converge to $|\Pi(z)|$ as $n \to \infty$. Thus we have

$$\lim_{n\to\infty} |P_*(\rho_n,z)| = |\Pi(z)|$$

and the convergence is uniform on compact subsets in $\mathbb{R}$. Note that Π does not have zeroes on $\mathbb{R}$ because otherwise Szegő condition (5) for measure σ would fail. Hence we have

$$\frac{1}{2\pi|P_*(\rho_n,x)|} \to \frac{1}{2\pi|\Pi(x)|^2}, \quad n \to \infty,$$

uniformly on compact subsets in $\mathbb{R}$. On the other hand, Theorem 6.2 in [11] states that

$$\frac{dx}{2\pi|P_*(r,x)|} \xrightarrow{w^*} d\sigma(x), \quad r\to\infty.$$

Therefore, σ is absolutely continuous with respect to the Lebesgue measure on the real line and

$$d\sigma(x) = \frac{dx}{2\pi|\Pi(x)|^2}.$$

Step v. As we have seen in the Introduction, the dual spectral measure $\hat{\sigma}_Q$ of D_Q and the spectral measure σ_{-Q} of D_{-Q} coincide. Moreover, for every $t \geqslant 0$ we have (see Lemma 3, [1])

$$H_{-Q}(t) = J^* H_Q(t) J, \qquad \mathbf{K}_{H_{-Q}}(t) = \mathbf{K}_{H_Q}(t), \qquad E_{-Q}(t) = E_Q(t).$$

Hence the result of Step iii can be applied for D_{-Q} as well as for D_Q. It follows that both Π and $\hat{\Pi}$ admit the analytical continuation into $\Omega_{\Delta_E/8}$ and therefore the Weyl function of D_Q can be meromophicaly extended into the same domain via relation (7). This concludes the proof of the whole theorem. □

5 Sufficient condition in the off-diagonal case

In this section we prove Theorem 2. Consider an off-diagonal potential $Q = Q_p = \left(\begin{smallmatrix} 0 & p \\ p & 0 \end{smallmatrix}\right)$. Then both matrix N_{Q_p} and the Hamiltonian H_{Q_p} defined by (8) are diagonal and can be calculated explicitly. We have

$$N_{Q_p}(t) = \begin{pmatrix} \exp(-g_0(t)) & 0 \\ 0 & \exp(g_0(t)) \end{pmatrix}, \qquad H_{Q_p}(t) = \begin{pmatrix} \exp(-2g_0(t)) & 0 \\ 0 & \exp(2g_0(t)) \end{pmatrix},$$

where g_r is defined by $g_r(t) = \int_r^t p(s)\,ds$. Moreover,

$$\begin{aligned} E_{Q_p}(r) = \det \int_r^{r+2} H_Q(t)\,dt - 4 &= \int_r^{r+2} e^{-2g_0(t)}\,dt \cdot \int_r^{r+2} e^{2g_0(t)}\,dt - 4 \\ &= \int_r^{r+2} e^{-2g_r(t)}\,dt \cdot \int_r^{r+2} e^{2g_r(t)}\,dt - 4. \end{aligned} \tag{69}$$

Assertion (10) of Theorem 2 can be rewritten in the form

$$\sup_{t\geqslant r} |g_r(t)| = O\left(e^{-\delta r}\right), \quad r\to\infty.$$

Therefore, in this situation we have

$$\sup_{t\geqslant r} \left|e^{2g_r(t)} - 1\right| = O\left(e^{-\delta r}\right), \qquad \sup_{t\geqslant r} \left|e^{-2g_r(t)} - 1\right| = O\left(e^{-\delta r}\right), \quad r\to\infty.$$

Now (69) gives

$$E_{Q_p}(r) = \int_r^{r+2} e^{-2g_r(t)}\,dt \cdot \int_r^{r+2} e^{2g_r(t)}\,dt - 4 = O\left(e^{-\delta r}\right), \quad r\to\infty.$$

This is exactly the claim of Theorem 2.

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