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Generalized localizations of coefficient rings for (triangulated) categories

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1. Intro

T is a triangulated category closed under coproducts.

Exact functor $L: T \longrightarrow T$ is called a localisation functor [Krause] if there exists a natural transformation $\eta: Id \longrightarrow L$ such that $L\eta$ is invertible and $L\eta = \eta L$

An object A is said to be L-local if $LA \cong A$

Our goal is to describe the concrete construction of LA for $A \in Obj(T)$ for the following case: We consider a multiplicatively closed set S of natural transformation between Id and some functor F such that for every $\tau \in S$

- ⊝ $\tau(FA) = F(\tau(A))$ for any A from $Obj(T)$.
- ⊝ T is closed under countable coproducts.
- ⊝ F is an exact triangulated functor.
- ⊝ When τ is applied to any distinguished triangle, it can be completed to a 4×4 diagram where every row and column is distinguished. Example of such diagram is given in the proof of Theorem 1

For this S we consider localization functors L such that L-local objects correspond to $A \in Obj(T)$ with property: $\forall \tau \in S \;\; \tau(A)$ is invertible. This construction will attempt to generalise "inverting integers in triangulated categories"[1] and "localisation of coefficient rings"[2], where S corresponds to a subset of $\mathbb{Z}$ closed under multiplication.

Main results of this text are the following ones:

- In section 3 for some natural transformation τ and each A we construct LA as specific homotopy limit A', such that $\tau(A')$ is invertible.
- In section 4 we generalise it to the case of countable set of natural transformations.
- In section 5 we list some categorical properties of our construction.

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2. Preliminaries

Composition of morphisms is written in the diagrammatic order.

We use the standard definition of triangulated category and the following lemma cited from [Neeman] (Proposition 1.1.20):

Lemma 1. If in the morphism of distinguished triangles

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

both f and g are isomorphisms, then so is h.

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3. Calculation of LA. Powers of a natural transformation

3.1. **Context.** Let F be an endofunctor on a triangulated category T. Let $\tau : Id \longrightarrow F$ be a natural transformation. We define S as the set of powers of τ with the properties as written above in the previous section.

Let D be the full subcategory of objects X such that $\forall \tau \in S \;\; \tau(X)$ is invertible (our local objects).

For every object A we construct LA along with $\phi : A \longrightarrow LA$ such that:

⊝ $LA \in D$.

⊝ For every morphism $\alpha : A \longrightarrow B$ with B in D, there exists

$$\psi : LA \longrightarrow B$$

such that $\alpha = \phi\psi$

[1]see Appendix A.2 of [Kelly]

[2]see section 5.4 of [Bondarko] and section 5.6 of [Bondarko2]

3.2. **Construction.** We start with the following sequence:

$$0 \longrightarrow A \xrightarrow{\tau(A)} FA \xrightarrow{\tau(FA)} F^2A...$$

A map $FA \longrightarrow F^2A$ can be defined in two ways:

- $\tau(FA)$
- $F(\tau(A))$

It would be natural to require these possibilities to be the same. This is why we require $\tau(FA) = F(\tau(A))$

In the context of additive categories, cokernels and colimits which could be used for the construction [3] may not exist. Triangulated categories closed under countable coproducts happen to be a fitting extension, because one can use homotopy colimits instead of colimits. Homotopy colimits are constructed per [Neeman] in the following way:

Let

$$f_k : F^kA \longrightarrow F^kA \oplus F^{k+1}A$$

be defined as $(id, -\tau(F^kA))$.
We define f as the direct sum of f_k for all k. It's the same as $id - \tau(\coprod F^k)$. There exists C, which extends f to the distinguished triangle:

$$\coprod F^kA \xrightarrow{f} \coprod F^kA \longrightarrow C \xrightarrow{\phi} F^kA \xrightarrow{f} [1]$$

Then C is called a homotopy colimit of the sequence $(F^iA, \tau(F_iA))$ and written as $hocolimF^iA$.

3.3. **Localisation property.**
Theorem 1. $C \in D$
Proof of 1. It's enough to prove that $\tau(C) : C \longrightarrow FC$ is an isomorphism. Denote $\bigoplus F^kA$ as M. Then $\tau(M)$ is equal to the direct sum of $\tau(F^kA) : F^kA \longrightarrow F^{k+1}A$

$$\tau(M) : \coprod F^kA \longrightarrow F(\coprod F^kA)$$

We can apply F to

$$M \xrightarrow{f} M \xrightarrow{\pi} C \xrightarrow{\phi} M[1]$$

and connect two triangles with τ.

[3] In the case of categories closed under countable colimits, localisation can be constructed as a colimit of the sequence:

$$0 \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} ...$$

In abelian categories it can be represented as a cokernel of

$$\coprod A_i \xrightarrow{f} \coprod A_i$$

$$f(a_i) = (a_i, -xa_i)$$

$$
\begin{array}{ccccccc}
M & \longrightarrow & M & \longrightarrow & C & \longrightarrow & M[1] \\
\downarrow{\scriptstyle \tau(M)} & & \downarrow{\scriptstyle \tau(M)} & & \downarrow{\scriptstyle \tau(C)} & & \downarrow{\scriptstyle \tau(M)[1]} \\
FM & \longrightarrow & FM & \longrightarrow & FC & \longrightarrow & FM[1]
\end{array}
$$

The 4th property from the box in Section 1 of τ allows extending this diagram to the 4×4 one.

$$
\begin{array}{ccccccc}
M & \xrightarrow{f} & M & \xrightarrow{\pi} & C & \xrightarrow{\phi} & M[1] \\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau[1]} \\
FM & \xrightarrow{Ff} & FM & \xrightarrow{F\pi} & FC & \xrightarrow{F\phi} & FM[1] \\
\downarrow{\scriptstyle h_M} & & \downarrow{\scriptstyle h_M} & & \downarrow{\scriptstyle h_C} & & \downarrow{\scriptstyle h_M[1]} \\
K & \xrightarrow{\alpha} & K & \xrightarrow{\beta} & D & \xrightarrow{\gamma} & K[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle [1]} \\
? & \longrightarrow & ? & \longrightarrow & ? & \longrightarrow & ?[1]
\end{array}
$$

Let's denote the left 4×2 rectangle in the diagram above (which we consider as a morphism of triangles) by $(*)$:

$$
\begin{array}{ccccccc}
M & \xrightarrow{\tau} & FM & \xrightarrow{h_M} & K & \longrightarrow & M[1] \\
\downarrow{\scriptstyle (id-\tau)} & & \downarrow{\scriptstyle (id-\tau)} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle (id-\tau)[1]} \\
M & \longrightarrow & FM & \longrightarrow & FK & \longrightarrow & M[1]
\end{array}
$$

Denote the following diagram as $(**)$:

$$
\begin{array}{ccccccc}
M & \xrightarrow{\tau(M)} & FM & \xrightarrow{h_M} & K & \xrightarrow{p_M} & M[1] \\
\downarrow{\scriptstyle \tau(M)} & & \downarrow{\scriptstyle \tau(FM)} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \tau(M)[1]} \\
M & \xrightarrow{\tau(M)} & FM & \xrightarrow{h_M} & K & \xrightarrow{p_M} & M[1]
\end{array}
$$

The left square of (**) commutes by the definition of a natural transformation. Let's show that the remaining squares commute on the level of summands:

$$\begin{array}{ccccccc}
F^iA & \xrightarrow{\tau(F^iA)} & F^{i+1}A & \xrightarrow{h_i} & K_i & \xrightarrow{p_i} & F^iA[1] \\
\downarrow{\scriptstyle \tau(F^iA)} & & \downarrow{\scriptstyle \tau(F^{i+1}A)} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \tau(F^iA)[1]} \\
F^{i+1}A & \xrightarrow{\tau(F^{i+1}A)} & F^{i+2}A & \xrightarrow{h_{i+1}} & K_{i+1} & \xrightarrow{p_{i+1}} & F^{i+1}A[1]
\end{array}$$

The bottom triangle is distinguished. It follows that $\tau(F^{i+1}A)h_{i+1} = 0$ as the composition of consequent morphisms in a distinguished triangle. This equality corresponds to the commutativity of the central square. The rotation of the diagram gives the equality $p_i\tau(F^iA)[1] = 0$ which implies the commutativity of the right square and finishes the proof of the commutativity of $(**)$. It means that (**) is a morphism of triangles. The sum $(*) + (**)$ corresponds to the following diagram:

$$\begin{array}{ccccccc}
M & \longrightarrow & FM & \longrightarrow & K & \longrightarrow & M[1] \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle id[1]} \\
M & \longrightarrow & FM & \longrightarrow & K & \longrightarrow & M[1]
\end{array}$$

Difference between morphisms of triangles is a morphism of triangles. Both of the triangles are distinguished. According to Lemma 1 α is an isomorphism. As

$$K \xrightarrow{\alpha} K \longrightarrow D \longrightarrow K[1]$$

is distinguished, it implies that $D = 0$.

$$C \xrightarrow{\tau} FC \longrightarrow D \longrightarrow C[1]$$

is distinguished. It follows that τ_C is an isomorphism as well. □

3.4. **Universality.**
Theorem 2. The "universal" property of C.
Let A, M and C be defined as above and let B be an object such that $\tau(B)$ is invertible. Then any morphism $g : A \longrightarrow B$ can be factored through C.
Proof of 2. We have two distinguished triangles:

$$M \xrightarrow{f} M \longrightarrow C \longrightarrow M[1]$$

$$0 \longrightarrow B \xrightarrow{id} B \longrightarrow 0$$

The morphism g produces $F^ig : F^i(A) \longrightarrow F^i(B)$ for every $i \geqslant 0$. As $\tau^i(B)$ is an isomorphism $B \longrightarrow F^i(B)$, we can define

$$g_i = (F^ig)(\tau^{-i}(F^iB))$$

from $F^i(A)$ to B where $\tau^{-i}(F^iB) = (\tau^i(B))^{-1}$. Define $g' : M \longrightarrow B$ to be a sum of g_i for every $i \geqslant 0$.

Now we will prove that $fg' = 0$. In other words, the left square of

$$\begin{array}{ccccccc} M & \longrightarrow & M & \xrightarrow{\pi} & C & \longrightarrow & M[1] \\ \downarrow & & \downarrow{\scriptstyle g'} & & & & \downarrow \\ 0 & \longrightarrow & B & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

commutes.

For every object X we required the equality

$$F(\tau(X)) = \tau(F(X))$$

If $\tau(X)^{-1}$ exists, then

$$1_{F(X)} = F(1_X) = F(\tau(X)^{-1}\tau(X)) = F(\tau(X)^{-1})F(\tau(X)) = F(\tau(X)^{-1})\tau(FX)$$

It means that for such X

$$F(\tau(X)^{-1}) = \tau(F(X))^{-1}$$

is also true. It implies that in the case of $X \in D$ similar equality holds for all integer powers of τ:

$$\forall i \in \mathbb{Z}: \ F(\tau(X)^i) = \tau(F(X))^i$$

Let's prove the following equality by induction:

$$\tau(F^{i-1}A)g_i = g_{i-1} \forall i > 0$$

Base: $i = 1$

$$\tau(A)g_1 = \tau(FA)F(g)\tau(B)^{-1}$$

Naturality of τ implies the commutativity of the following square

$$\begin{array}{ccc} A & \xrightarrow{\tau(A)} & F(A) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle Fg} \\ B & \xrightarrow{\tau(B)} & F(B) \end{array}$$

$$(g)(\tau(B)) = (\tau(A))(Fg)$$

Then, for every $i \geqslant 0$

$$F^i(g\tau(B)) = F^i(\tau(A)Fg)$$
$$F^i g\tau(F^i B) = \tau(F^i A)F^{i+1}g$$

Thus

$$\tau(A)F(g)\tau(B)^{-1} = g\tau(B)\tau(B)^{-1} = g$$

Inductive step for $k \geqslant 2$. Suppose that $\tau(F^{k-2}A)g_{k-1} = g_{k-2}$: Let's notice that for any $n \geqslant 1$

$$g_n = F^n g\tau^{-n}(F^n B) = (FF^{n-1}g)(\tau^{-n+1}(F^n B))(\tau^{-1}(FB)) =$$
$$= F(F^{n-1}g)(\tau^{-n+1}(F(F^{n-1}B)))(\tau^{-1}(FB)) =$$
$$= F(F^{n-1}g)F(\tau^{-n+1}(F^{n-1}B))(\tau^{-1}(FB)) =$$
$$= F(F^{n-1}g\tau^{-n+1}(F^{n-1}B))(\tau^{-1}(FB)) =$$

$$= F(g_{n-1})\tau^{-1}(FB)$$

Then, by the equality which we just got:

$$\tau(F^{k-1}A)g_k = \tau(F^{k-1}A)F(g_{k-1})\tau^{-1}(FB) =$$
$$= F(\tau(F^{k-2}A))F(g_{k-1})\tau^{-1}(FB) =$$
$$= F(\tau(F^{k-2}A)g_{k-1})\tau^{-1}(FB) =$$
$$= F(g_{k-2})\tau^{-1}(FB) = g_{k-1}$$

It finishes the proof of the equality. [4]

We wanted to prove that $fg' = 0$: For every $i \geqslant 0$ define the ith summand of fg' as $\eta_i : F^iA \longrightarrow B$. It can be written as the composition:

$$F^iA \overset{(id,-\tau)}{\longrightarrow} F^iA \oplus F^{i+1}(A) \overset{(g_i,g_{i+1})}{\longrightarrow} B$$

$$\eta_i = g_i - \tau(F^iA)g_{i+1}$$

But we had proved already that $\tau(F^iA)g_{i+1} = g_i$ and thus

$$\forall i \geqslant 0 : \ \eta_i = 0$$

It follows that $fg' = 0$ and the diagram

$$\begin{array}{ccccccc} M & \longrightarrow & M & \xrightarrow{\pi} & C & \longrightarrow & M[1] \\ \downarrow & & \downarrow{\scriptstyle g'} & & & & \downarrow \\ 0 & \longrightarrow & B & \xrightarrow{id} & B & \longrightarrow & 0 \end{array}$$

commutes. Then we can complete it to a full morphism of the triangles with an arrow $C \overset{\mu}{\longrightarrow} B$ such that

$$g' = \pi\mu$$

First summand of this equality is

$$g = \phi\mu$$

□

Note: in general, there is no such property as uniqueness of μ.

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4. Localisation by a family of natural transformations closed under multiplication

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4.1. **Generalization.** We had constructed the localisation for the case when a multiplicative family was just a set of powers of some transformation. In this section we construct the localisation by arbitrary countable multiplicatively closed set S. It means that for any $\tau, \nu \in S$ the composition $\tau\nu \in S$ as well.

---

[4] All equalities from the proof become much simpler if you don't write an argument of a natural transformation and assume that morphisms chain correctly.

4.2. **Construction.** For finitely generated multiplicative sets we can construct the localisation by the whole set as the localisation by all generators (one by one). To extend this process to infinitely generated sets, we will use a homotopy limit.

Theorem 3. Let S be a countable multiplicative set of natural transformations Id to F indexed by the natural numbers. Let S_i be the set of the first i elements from S. Let A_k be the localisation of A by S_i for every $i < k$. Construct the homotopy limit

$$\coprod_i A_i \xrightarrow{(id,-i)} \coprod_i A_i \longrightarrow H \longrightarrow ...$$

Where i is the "inclusion" $A_i \longrightarrow A_{i+1}$.
Then H is the localisation by S
Proof of 3. As $S = \cup S_i$, for any $\tau \in S$ exists κ such that $\tau \in S_\kappa$. If we choose a subsequence starting at A_κ, then, according to Lemma 1.7.1 from [Neeman] it will have a homotopy limit H isomorphic to the initial one. Then consider the diagram for τ applied to the shortened distinguished triangle:

$$\begin{array}{ccccccc} \coprod_{\kappa<\gamma} A_\gamma & \longrightarrow & \coprod_{\kappa<\gamma} A_\gamma & \longrightarrow & H & \longrightarrow & (\coprod_{\kappa<\gamma} A_\gamma)[1] \\ \downarrow \tau & & \downarrow \tau & & \downarrow \tau & & \downarrow \tau[1] \\ \coprod_{\kappa<\gamma} F(A_\gamma) & \longrightarrow & \coprod_{\kappa<\gamma} F(A_\gamma) & \longrightarrow & F(H) & \longrightarrow & (\coprod_{\kappa<\gamma} F(A_\gamma))[1] \end{array}$$

At all entries except H τ is an isomorphism. It means that at H it will be an isomorphism by Lemma 1. □

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5. Examples

In previous chapters we assumed some properties on a set of natural transformations. Now we will present a few examples which satisfy them.

5.1. **Example 1.** If S is a subset of $\mathbb{Z}$ closed under multiplication then for each element $s \in S$ we can define the natural transformation $Id \xrightarrow{\tau_s} Id$ as sid: sum of s identity maps. All properties are satisfied as Id functors preserves and commutes with everything.

5.2. **Tensor product.** For this example we need to assume that category T is a tensor triangulated category with the distributive property for coproducts.

Category T is called a tensor triangulated category [5] if

- T has the structure of triangulated category.
- T has the structure of tensor category.
- Tensor product is additive and exact.

Given a countable set of objects $\{X_i\}$ and morphisms $1 \xrightarrow{t_i} X_i$ with "balancing" property $(*)$ $id \otimes t_i = t_i \otimes id$, we can define:

[5] For more precise definition, refer to `https://ncatlab.org/nlab/show/tensor+triangulated+category`

— Functors F_i which map $A \xrightarrow{f} B$ to $A \otimes X_i \xrightarrow{f \otimes id} B \otimes X_i$
— Natural transformations $\tau_i(X) : X \xrightarrow{id \otimes t_i} X \otimes X_i$

Let's check the properties:
Commutation. Let A be an object from category. Then, we want

$$\tau_i(F_i(A)) = F(\tau_i(A))$$

By definition:

$$\tau_i(F_i(A)) = A \otimes X_i \otimes 1 \xrightarrow{id \otimes id \otimes t_i} A \otimes X_i \otimes X_i$$

$$F_(\tau_i(A)) = A \otimes 1 \otimes X_i \xrightarrow{id \otimes t_i \otimes id} A \otimes X_i \otimes X_i$$

The property (*) required from the set $\{X_i\}$ assures that these morphisms are equal. The second and the third properties are yet again required by the definition. For the 4th property consider two triangles

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

$$1 \xrightarrow{t_i} X_i \longrightarrow H_i \longrightarrow 1[1]$$

Then, according to the definition of tensor triangulated category, all rows and columns of 4×4 square built as the tensor product of the triangles will be distinguished as needed. The commutativity of the smaller squares is expressed analogously to

$$f \otimes id_A \ \ id_B \otimes t_i = f \otimes t_i = id_A \otimes t_i \ \ f \otimes id_B$$

5.3. **Example of Not example.** The properties we require from natural transformations are quite strong. For example, property $\tau F = F\tau$ fails in the case of functor which sends an object A to $A \oplus A$ and a morphism f to a $f \oplus f$ and τ is set to the diagonal map.

$$F(A) = A \oplus A$$

$$\tau(A) = \Delta(A)$$

$$\tau(F(A)) = \tau(A \oplus A) = \Delta(A \oplus A)$$

$$F\tau(A) = F(\Delta(A)) = \Delta(A) \oplus \Delta(A)$$

The problem is that $\Delta(A) \oplus \Delta(A)$ and $\Delta(A \oplus A)$ are quite different morphisms: Let's fix in $Hom(A^{\oplus 2}, A^{\oplus 4})$ matrix units $e_{i,j}$ which are just the identity maps from the ith component to the jth one.

$$\Delta(A) \oplus \Delta(A), \Delta(A \oplus A) \in Hom(A^{\oplus 2}, A^{\oplus 4})$$

Then, $\Delta(A) \oplus \Delta(A) = e_{1,1} + e_{1,2} + e_{2,3} + e_{2,4}$ while $\Delta(A \oplus A) = e_{1,1} + e_{1,3} + e_{2,2} + e_{2,4}$.

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6. Categorical properties of the constructed localisation.

In this section we study how the construction of the localisation described above is related or similar to localisation functors. While homotopy limit is not a functor (it's not even unique), it's possible to notice the properties similar to the properties of localisation functors [6] if we look at all possible limits at the same time.

In the context of this section, T is a triangulated category with all countable colimits. As the localisation by any S is built as the homotopy colimit of the localisations by sets of powers, we limit ourselves to the case when S is the set of powers of some natural transformation τ. For an object A we denote the localisation by S via homotopy colimit as LA. D is the collection of objects A such that $\tau(A)$ is an isomorphism. The elements of D will be called as local objects.

6.1. **Definitions.**

Definition: For every object A from T we define $L'(A)$ as the family of all possible localizing homotopy limits.

Definition: Define C as the union of all $L'(A)$

Now, for every morphism $f : A \longrightarrow B$ consider the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{l_a} & L(A) \\ \downarrow{\scriptstyle f} & & \\ B & \xrightarrow{l_b} & L(B) \end{array}$$

The universal property of the localisation applied to fl_b gives us the non-unique morphism $L(A) \longrightarrow L(B)$, which completes the diagram to the commuting one.

Definition: Define $Mor(C)$ as the family of all such morphisms.

Theorem 4. C together with $Mor(C)$ forms a category ImL. □

Definition: Define $KerL$ as the full subcategory of objects A such that $L(A) \cong 0$

Now we can ask the following questions: If for A all transformations are already invertible, will it lie in ImL? Can the constructed localisation be realised as the abstract localisation by the subcategory $KerL$? Is there an equivalent localisation functor?

6.2. **Results.** Let's prove the properties which will help to answer these questions.

Theorem 5. $\forall c \in D; L(c) \cong c$. (It follows that $D = ImL$)

Proof of 5. Let ϕ_i be the sequence of isomorphisms $c_i \xrightarrow{phi_i} c_{i+1}$, where $c = c_0$ and $i \geqslant 0$. Let ψ_k be the composition $\phi_0\phi_1 \ldots \phi_k$. Then $\Psi = \coprod_{i \geqslant 0} \psi_i$ is an isomorphism $\coprod c \longrightarrow \coprod c_i$. Construct the commutative square where the horizontal morphisms are as in the construction of the homotopy limit.

---

[6]They are described extensively in [Krause]

$$\begin{array}{ccc} \coprod c & \longrightarrow & \coprod c \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi} \\ \coprod c_i & \longrightarrow & \coprod c_i \end{array}$$

Complete this diagram to the morphism of the distinguished triangles. *hocolim* of identity maps is isomorphic to the first object of the sequence[7]. So the top row will be completed with c. Lemma 1 implies that $c \cong hocolim\ c_i$. Substitution of c_i with $F^i c$ and ϕ_i with $\tau(F^i(c))$ implies that $c \cong hocolim F^i c \cong Lc$.

□

Important consequence of this theorem is

Theorem 6. ImL allows completions of a morphism to a triangle.
Proof of 6. For any morphism from $ImL : L(A) \longrightarrow L(B)$ we can complete it to a triangle in T

$$L(A) \longrightarrow L(B) \longrightarrow c \longrightarrow L(A)[1]$$

$\tau(L(A))$ and $\tau(L(B))$ are isomorphisms. Lemma 1 guarantees that $\tau(c)$ is an isomorphism. $L(c) \cong c$, so we got the triangle consisting from elements of ImL. □

Theorem 7. $KerL \perp ImL$
Proof of 7. $A \in Obj(KerL)$. $B \in Obj(ImL) = D$. $\phi \in Hom(A, B)$. As $B \in D$, ϕ can be decomposed into $A \longrightarrow L(A) \longrightarrow B$. But $L(A) \cong 0$, so $\phi = 0$. □

Theorem 8. Take $A \in Obj(T)$ and $\phi \in Hom(A, L(A))$. There is the distinguished triangle

$$A \xrightarrow{\phi} L(A) \xrightarrow{\psi} H(A) \longrightarrow A[1]$$

Then, $H(A) \in {}^{\perp}Obj(ImL)$ if and only if L(A) satisfies the universal property from Section 3.1 with uniqueness.
Proof of 8. For any C from ImL functor $Hom(-, C)$ is a homological functor which will turn the distinguished triangle from the statement of the theorem into the following long exact sequence:

$$\longleftarrow Hom(H(A)[-1], C) \longleftarrow Hom(A, C) \longleftarrow Hom(L(A), C)$$
$$\longleftarrow Hom(H(A), C) \longleftarrow Hom(A[1], C) \longleftarrow Hom(L(A)[1], C)) \longleftarrow$$

The universal property states that for any $f \in Hom(A, C)$ exists $g \in Hom(L(A), C)$. In other words $Hom(A[i], C) \longleftarrow Hom(L(A)[i], C)$ is surjective for every i. Then the long exact sequence splits into the short exact sequences of type:

$$0 \longleftarrow Hom(A, C) \longleftarrow Hom(L(A), C) \longleftarrow Hom(H(A), C) \longleftarrow 0$$

But then $Hom(H(A), C) = 0$ if and only if the left arrow is injective. The left arrow is injective if and only if choice of g above is unique. □

[7][Neehman] Lemma 1.6.6

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