Supplementary material for the paper
"Can partial cooperation between developed and developing countries be stable?"
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Appendix A

First, the objective of player 1 does not depend on the stock variable and we can easily obtain that the maximal value of her objective is reached when $e_1 = \alpha_1$.

Second, players 2 and 3 are of type I, we take the second player to illustrate calculations. The player 2's optimization problem is

$$W_2^{\pi_1} = \int_0^\infty e^{-\rho t}(\alpha_2 e_2(t) - \frac{1}{2}e_2^2(t) - \frac{1}{2}\beta_2 S^2(t))dt \to \max_{e_2(t)\geq 0}. \tag{1}$$

Assuming the linear-quadratic form of the value functions $V_2(S) = \frac{1}{2}x_2S^2 + y_2S + z_2$ and $V_3(S) = \frac{1}{2}x_3S^2 + y_3S + z_3$, we write down the HJB equation for (1), which looks like

$$\rho V_2(S) = \max_{e_2}\Big\{\big(\alpha_2 e_2 - \frac{1}{2}e_2^2 - \frac{1}{2}\beta_2 S^2\big) + V_2'(S)[\mu(e_1 + e_2 + e_3) - \varepsilon S]\Big\}. \tag{2}$$

Maximizing the expression in the RHS of equation (2), we obtain $e_2 = \alpha_2 + \mu V_2'(S)$, and the corresponding strategy for player 3 is $e_3 = \alpha_3 + \mu V_3'(S)$. Taking into account the derivatives $V_j'(S) = x_jS + y_j, j = 2, 3$, and substituting these expressions into (2), we obtain an equation:

$$\begin{aligned}\rho\Big(\frac{1}{2}x_2S^2 + y_2S + z_2\Big) =& \frac{1}{2}[\alpha_2 + \mu(Sx_2 + y_2)]^2 + \mu\alpha_1(x_2S + y_2) + \mu(x_2S + y_2)[\alpha_3 + \mu(x_3S + y_3)] - \\ &- \frac{1}{2}\beta_2 S^2 - \varepsilon S(x_2S + y_2).\end{aligned}$$

By identification, two linear quadratic equations containing x_2, x_3 can be written as

$$\begin{aligned}&\mu^2x_2^2 - (2\varepsilon + \rho)x_2 + 2\mu^2x_2x_3 - \beta_2 = 0,\\ &\mu^2x_3^2 - (2\varepsilon + \rho)x_3 + 2\mu^2x_2x_3 - \beta_3 = 0.\end{aligned}$$

Correspondingly, the expressions for x_2 and x_3 are the solutions of the following equations:

$$3\mu^4x_2^4 - 4(2\varepsilon + \rho)\mu^2x_2^3 + [4\mu^2\beta_3 + (2\varepsilon + \rho)^2 - 2\mu^2\beta_2]x_2^2 - \beta_2^2 = 0, \tag{3}$$
$$3\mu^4x_3^4 - 4(2\varepsilon + \rho)\mu^2x_3^3 + [4\mu^2\beta_2 + (2\varepsilon + \rho)^2 - 2\mu^2\beta_3]x_3^2 - \beta_3^2 = 0. \tag{4}$$

We cannot write an explicit solution of the system of equations (3) and (4) with respect to x_2, x_3, therefore, we introduce this system with new variables values:

$$
\begin{aligned}
y_2 &= \frac{\mu\alpha_{123}x_2(\rho+\varepsilon-\mu^2x_{23}+\mu^2x_3)}{(\rho+\varepsilon-\mu^2x_{23})^2-\mu^4x_2x_3},\\
y_3 &= \frac{\mu\alpha_{123}x_3(\rho+\varepsilon-\mu^2x_{23}+\mu^2x_2)}{(\rho+\varepsilon-\mu^2x_{23})^2-\mu^4x_2x_3},\\
z_2 &= \frac{\alpha_2^2+2\mu y_2\alpha_{123}+2\mu^2y_2y_3}{2\rho},\\
z_3 &= \frac{\alpha_3^2+2\mu y_3\alpha_{123}+2\mu^2y_2y_3}{2\rho},
\end{aligned}
$$

where $x_{23} = x_2 + x_3$.

The global stability of the steady state requires $\mu^2x_{23} - \varepsilon < 0$, thus the negative roots x_2 and x_3 from (3) and (4) are chosen. Subsequently, the expression of the equilibrium stock $S^{nc}(t)$ is obtained as a solution of the dynamical system defined in Section 2 and it is given in Proposition 1. So as the steady state by setting the dynamic system equation equal to zero.

Appendix B

In the cooperative scenario, since all players jointly maximize the total profit defined in Section 3.2, the optimization problem is given by

$$
W^{\pi_2} = W_1^{\pi_2} + W_2^{\pi_2} + W_3^{\pi_2} = \sum_{i=1}^{3}\int_0^{\infty} e^{-\rho t}(\alpha_i e_i(t) - \frac{1}{2}e_i^2(t) - \frac{1}{2}\beta_i S^2(t))dt \to \max_{\substack{e_i\geq 0\\ i\in N}}. \tag{5}
$$

To solve an optimization problem (5), the HJB equation can be written as

$$
\rho V_c(S) = \max_{e_1,e_2,e_3}\Big\{\sum_{i=1}^{3}(\alpha_i e_i - \frac{1}{2}e_i^2 - \frac{1}{2}\beta_i S^2) + V_c'(S)[\mu(e_1+e_2+e_3) - \varepsilon S]\Big\}. \tag{6}
$$

Maximizing the expression in the RHS of equation (6), we write the first-order condition and find the optimal control $e_i = \alpha_i + \mu V_c'(S)$. Assuming the linear-quadratic form of $V_c(S)$, we set $V_c(S) = \frac{1}{2}x_cS^2 + y_cS + z_c$. Then substituting the corresponding variables in (6) brings

$$
\begin{aligned}
\rho\Big(\frac{1}{2}x_cS^2 + y_cS + z_c\Big) =& \frac{1}{2}[\alpha_1 + \mu(x_cS+y_c)]^2 + \frac{1}{2}[\alpha_2 + \mu(x_cS+y_c)]^2 + \frac{1}{2}[\alpha_3 + \mu(x_cS+y_c)]^2 - \\
&- \frac{1}{2}(\beta_1+\beta_2+\beta_3)S^2 - \varepsilon S(x_cS+y_c).
\end{aligned}
$$

By the procedure of identification, we obtain the system of equations:

$$
\begin{aligned}
&3\mu^2x_c^2 - (2\varepsilon+\rho)x_c - \beta_{123} = 0,\\
&y_c = \frac{\mu x_c\alpha_{123}}{\rho+\varepsilon-3\mu^2x_c},\\
&z_c = \frac{\sum_{i=1}^{3}(\alpha_i+\mu y_c)^2}{2\rho}.
\end{aligned} \tag{7}
$$

Since the global stability of the steady state is always satisfied under $3\mu^2 x_c - \varepsilon < 0$, then we take the negative root from (7), that is

$$x_c = \frac{2\varepsilon + \rho - \sqrt{(2\varepsilon+\rho)^2 + 12\mu^2\beta_{123}}}{6\mu^2}.$$

Substituting all necessary variables into the dynamical system, and solving this differential equation, we obtain the expression for $S^c(t)$ and S^c_∞ as presented in the proposition.

Appendix C

Since the invulnerable player 1 maximizes her own payoff, she behaves in the same way as in the noncooperative scenario as her objective does not depend on the stock variable. Next, we consider the payoff function of a coalition of the two other players, who are two vulnerable players, and this coalition solves the problem:

$$W^{\pi_3} = W_2^{\pi_3} + W_3^{\pi_3} = \sum_{i=2}^{3} \int_0^\infty e^{-\rho t}(\alpha_i e_i(t) - \frac{1}{2}e_i^2(t) - \frac{1}{2}\beta_i S^2(t))dt \to \max_{e_2,e_3}.$$

The HJB equation in this case is as follows:

$$\rho V_{c_1}(S) = \max_{e_2,e_3}\Big\{\sum_{i=2}^{3}(\alpha_i e_i - \frac{1}{2}e_i^2 - \frac{1}{2}\beta_i S^2) + V'_{c_1}(S)[\mu(e_1+e_2+e_3) - \varepsilon S]\Big\}. \quad (8)$$

Maximizing the expression in the RHS of equation (8), we obtain strategies: $e_j = \alpha_j + \mu V'_{c_1}(S), j = 2,3$. Assuming a linear-quadratic form of V_{c_1}, i.e., $V_{c_1}(S) = \frac{1}{2}x_{c_1}S^2 + y_{c_1}S + z_{c_1}$, we get:

$$\begin{aligned}\rho\Big(\frac{1}{2}x_{c_1}S^2 + y_{c_1}S + z_{c_1}\Big) =& \frac{1}{2}[\alpha_2 + \mu(Sx_{c_1} + y_{c_1})^2 + \frac{1}{2}[\alpha_3 + \mu(Sx_{c_1} + y_{c_1}]^2 + \mu\alpha_1(x_{c_1}S + y_{c_1}) - \\ &- \frac{1}{2}\beta_{23}S^2 - \varepsilon S(x_{c_1}S + y_{c_1}).\end{aligned}$$

By identification, we obtain

$$\begin{aligned}&2\mu^2 x_{c_1}^2 - (2\varepsilon+\rho)x_{c_1} - \beta_{23} = 0, \\ &y_{c_1} = \frac{\mu x_{c_1}\alpha_{123}}{\rho + \varepsilon - 2\mu^2 x_{c_1}}, \\ &z_{c_1} = \frac{(\alpha_2 + \mu y_{c_1})^2 + (\alpha_3 + \mu y_{c_1})^2 + 2\mu y_{c_1}\alpha_1}{2\rho}.\end{aligned} \quad (9)$$

As in Appendix A, we also need to take negative root of x_{c_1} from (9) for satisfying the global stability of the solution, thus $x_{c_1} = \frac{2\varepsilon+\rho-\sqrt{(2\varepsilon+\rho)^2+8\mu^2\beta_{23}}}{4\mu^2}$, then $S^{pc_1}(t)$ is obtained as a solution of the dynamical system with initial condition $S(0) = S_0$.

Appendix D

We consider $\{\{I, II\}, \{I\}\}$, in which player 3 acts as a singleton. There are two optimization problems to solve. First, for the coalition of players 1 and 2, we formulate their joint optimization problem

$$W^{\pi_{4_1}} = W_1^{\pi_{4_1}} + W_2^{\pi_{4_1}} = \sum_{i=1}^{2} \int_0^\infty e^{-\rho t}(\alpha_i e_i(t) - \frac{1}{2}e_i^2(t) - \frac{1}{2}\beta_i S^2(t))dt \to \max_{e_1,e_2}.$$

Player 3 aims to maximize

$$W_3^{\pi_{4_1}} = \int_0^{\infty} e^{-\rho t}(\alpha_3 e_3(t) - \frac{1}{2}e_3^2(t) - \frac{1}{2}\beta_3 S^2(t))dt \to \max_{e_3}.$$

Following the method of previous cases, we write the HJB equation:

$$\rho V_{c_2}(S) = \max_{e_1,e_2}\Big\{\sum_{i=1}^{2}(\alpha_i e_i - \frac{1}{2}e_i^2 - \frac{1}{2}\beta_i S^2) + V'_{c_2}(S)[\mu(e_1+e_2+e_3) - \varepsilon S]\Big\}, \tag{10}$$

$$\rho V_{3_{c_2}}(S) = \max_{e_3}\Big\{(\alpha_3 e_3 - \frac{1}{2}e_3^2 - \frac{1}{2}\beta_3 S^2) + V'_{3_{c_2}}(S)[\mu(e_1+e_2+e_3) - \varepsilon S]\Big\}. \tag{11}$$

We infer that $V_{c_2}(S) = \frac{1}{2}x_{c_2}S^2 + y_{c_2}S + z_{c_2}$, $V_{3_{c_2}}(S) = \frac{1}{2}x_{3_{c_2}}S^2 + y_{3_{c_2}}S + z_{3_{c_2}}$, consequently, the optimal strategies can be constructed as

$$\begin{aligned} e_j(t) &= \alpha_j + \mu(x_{c_2}S(t) + y_{c_2}), \quad j \in 1,2, \\ e_3(t) &= \alpha_3 + \mu(x_{3_{c_2}}S(t) + y_{3_{c_2}}). \end{aligned}$$

Then by substituting $V_{c_2}(S), V'_{c_2}(S), V_{3_{c_2}}(S), V'_{3_{c_2}}(S)$ into (10) and (11), we obtain

$$\begin{aligned} \rho\Big(\frac{1}{2}x_{c_2}S^2 + y_{c_2}S + z_{c_2}\Big) =& \frac{1}{2}[\alpha_1 + \mu(Sx_{c_2} + y_{c_2}]^2 + \mu(x_{c_2}S + y_{c_2})[\alpha_3 + \mu(x_{3_{c_2}}S + y_{3_{c_2}})] - \\ & - \frac{1}{2}\beta_{12}S^2 - \varepsilon S(x_{c_2}S + y_{c_2}) + \frac{1}{2}[\alpha_2 + \mu(Sx_{c_2} + y_{c_2}]^2, \\ \rho\Big(\frac{1}{2}x_{3_{c_2}}S^2 + y_{3_{c_2}}S + z_{3_{c_2}}\Big) =& \frac{1}{2}[\alpha_3 + \mu(Sx_{3_{c_2}} + y_{3_{c_2}})]^2 + \mu(x_{3_{c_2}}S + y_{3_{c_2}})[\alpha_{12} + 2\mu(x_{c_2}S + y_{c_2})] - \\ & - \frac{1}{2}\beta_3 S^2 - \varepsilon S(x_{3_{c_2}}S + y_{3_{c_2}}). \end{aligned}$$

The following system is obtained by identification method:

$$\begin{aligned} & 12\mu^4 x_{c_2}^4 - 8(2\varepsilon+\rho)\mu^2 x_{c_2}^3 + \Big((2\varepsilon+\rho)^2 + 4\mu^2\beta_3 - 4\mu^2\beta_2\Big)x_{c_2}^2 - \beta_2^2 = 0, \\ & 3\mu^4 x_{3_{c_2}}^4 - 4(2\varepsilon+\rho)\mu^2 x_{3_{c_2}}^3 + \Big((2\varepsilon+\rho)^2 + 8\mu^2\beta_2 - 2\mu^2\beta_3\Big)x_{3_{c_2}}^2 - \beta_3^2 = 0, \\ & y_{c_2} = \frac{\mu\alpha_{123}x_{c_2}(\rho+\varepsilon-2\mu^2 x_{c_2})}{(\rho+\varepsilon-2\mu^2 x_{c_2} - \mu^2 x_{3_{c_2}})^2 - 2\mu^4 x_{c_2}x_{3_{c_2}}}, \\ & y_{3_{c_2}} = \frac{\mu\alpha_{123}x_{c_2}(\rho+\varepsilon-\mu^2 x_{3_{c_2}})}{(\rho+\varepsilon-2\mu^2 x_{c_2} - \mu^2 x_{3_{c_2}})^2 - 2\mu^4 x_{c_2}x_{3_{c_2}}}, \\ & z_{c_2} = \frac{\alpha_1^2 + \alpha_2^2 + 2\mu y_{c_2}(\alpha_{123} + \mu y_{c_2} + \mu y_{3_{c_2}})}{2\rho}, \\ & z_{3_{c_2}} = \frac{\alpha_3^2 + \mu y_{3_{c_2}}(2\alpha_{123} + 4\mu y_{c_2} + \mu y_{3_{c_2}})}{2\rho}. \end{aligned}$$

We get the negative roots of x_{c_2} and $x_{3_{c_2}}$ as usual, and obtain $S^{pc_2}(t)$ and $S_{\infty}^{pc_2}$ afterwards.

Appendix E

$\bar{t} = 1$	Player 1 Invul. player	Player 2 Vul. player	Player 3 Vul. player
$\pi_1 = \{\{1\}, \{2\}, \{3\}\}$	4.167	-6.681	-6.382
$\pi_2 = \{\{1, 2, 3\}\}$	0.144	-0.307	-0.255
$\pi_3 = \{\{1\}, \{2, 3\}\}$	4.167	-0.495	-0.499
$\pi_{4_1} = \{\{1, 2\}, \{3\}\}$	0.190	-0.569	-4.736
$\pi_{4_2} = \{\{1, 3\}, \{2\}\}$	0.177	-5.082	-0.786
$\bar{t} = 5$	Player 1 Invul. player	Player 2 Vul. player	Player 3 Vul. player
$\pi_1 = \{\{1\}, \{2\}, \{3\}\}$	4.167	-20.911	-25.467
$\pi_2 = \{\{1, 2, 3\}\}$	0	0	0
$\pi_3 = \{\{1\}, \{2, 3\}\}$	4.167	0	0
$\pi_{4_1} = \{\{1, 2\}, \{3\}\}$	0	0	-19.807
$\pi_{4_2} = \{\{1, 3\}, \{2\}\}$	0	-15.464	0
$\bar{t} = 10$	Player 1 Invul. player	Player 2 Vul. player	Player 3 Vul. player
$\pi_1 = \{\{1\}, \{2\}, \{3\}\}$	4.167	-21.817	-26.682
$\pi_2 = \{\{1, 2, 3\}\}$	0	0	0
$\pi_3 = \{\{1\}, \{2, 3\}\}$	4.167	0	0
$\pi_{4_1} = \{\{1, 2\}, \{3\}\}$	0	0	-20.553
$\pi_{4_2} = \{\{1, 3\}, \{2\}\}$	0	-15.936	0

Table 1: Players' payoffs in subgames starting at time $\bar{t} = 1, 5, 10$ (first run)

$\bar{t} = 1$	Player 1 Invul. player	Player 2 Vul. player	Player 3 Vul. player
$\pi_1 = \{\{1\}, \{2\}, \{3\}\}$	4.167	-10.364	-16.938
$\pi_2 = \{\{1, 2, 3\}\}$	0.094	-0.277	-0.401
$\pi_3 = \{\{1\}, \{2, 3\}\}$	4.167	-0.578	-0.820
$\pi_{4_1} = \{\{1, 2\}, \{3\}\}$	0.171	-0.710	-13.159
$\pi_{4_2} = \{\{1, 3\}, \{2\}\}$	0.132	-6.830	-1.051
$\bar{t} = 5$	Player 1 Invul. player	Player 2 Vul. player	Player 3 Vul. player
$\pi_1 = \{\{1\}, \{2\}, \{3\}\}$	4.167	-25.443	-39.911
$\pi_2 = \{\{1, 2, 3\}\}$	0	0	0
$\pi_3 = \{\{1\}, \{2, 3\}\}$	4.167	0	0
$\pi_{4_1} = \{\{1, 2\}, \{3\}\}$	0	0	-29.138
$\pi_{4_2} = \{\{1, 3\}, \{2\}\}$	0	-15.550	0
$\bar{t} = 10$	Player 1 Invul. player	Player 2 Vul. player	Player 3 Vul. player
$\pi_1 = \{\{1\}, \{2\}, \{3\}\}$	4.167	-25.998	-40.755
$\pi_2 = \{\{1, 2, 3\}\}$	0	0	0
$\pi_3 = \{\{1\}, \{2, 3\}\}$	4.167	0	0
$\pi_{4_1} = \{\{1, 2\}, \{3\}\}$	0	0	-29.526
$\pi_{4_2} = \{\{1, 3\}, \{2\}\}$	0	-15.722	0

Table 2: Players' payoffs in subgames starting at time $\bar{t} = 1, 5, 10$ (second run)