Proofs for results presented in paper "Dynamically stable partitions in networks with the costs dependent on neighborhood composition" by Ping Sun and Elena Parilina

Appendix A: Proof of Theorem 1

Proof. 1. When cost function (1) is applied, from Lemma 2, when any player $i \neq 1$ is selected to make an action, she chooses to stay in or deviate to the group player 1 belongs to. Therefore, each player, except player 1, chooses not to deviate if she is in the same group with player 1.

(a) If from stage t, we formulate the action-making order of players such that the players $1,\dots,n$ choose their actions consequently. Then after player n makes the action at stage $n+t$, a partition Δ^{n+t}, in which all players belong to one group, is generated. Since $f(1)-f(0) \geq \frac{n-2}{n-1}$, all players prefer not to deviate indicating Δ^{n+t} is dynamically stable. It is also clear that any partition in which player 1 belongs to the group different from all other players is not dynamically stable;

(b) when all players belong to one certain group, player 1 chooses to deviate to another group when $f(1)-f(0) < \frac{n-2}{n-1}$. As a result, there is no dynamically stable partition under such condition.

2. When cost function (2) is applied, we prove the existence of a dynamically stable partition by formation of the order of players making actions based on any given initial partition such that a dynamically stable partition emerges. We assume four items in the proof without loss of generality (implied by the infinite and random dynamics):

- Let $f(1)-\frac{n-2}{n-1} < f(0) < f(1)-\frac{\eta^0_{k^*}}{n-1}$, where $k^* \in \arg\max\limits_{k\neq\pi^0(1)} \eta^0_k$ is met, and the case when $f(0)$ and $f(1)$ satisfy any other relation can be verified in the same way;
- Players $1,\dots,n$ are ordered to choose their actions;
- Condition
$$\min_{l\neq\pi^0(1)}\Big[(\eta^0_{\pi^0(1)}-\eta^0_l-1)\Big(f(1)-f(0)-\frac{\eta^0_{\pi^0(1)}+\eta^0_l-2}{n-1}\Big)\Big] \geq 0 \qquad \text{(A.1)}$$
is satisfied, indicating that player 1 decides to stay in the group $N^0_{\pi^0(1)}$ rather than to deviate to another group at stage 1 given partition Δ^0. If this condition is not satisfied, the partition, immediately generated after player 1 makes an action, can be regarded as the initial one. Let l^* be one of the solutions of the minimization problem in (A.1);
- When a player has a multiple choice, we assume a choice of a certain action.

We first prove that $\eta^0_{k^*} < n-2$ with condition that $(\eta^0_{\pi^0(1)}-\eta^0_{l^*}-1)\Big(f(1)-f(0)-\frac{\eta^0_{\pi^0(1)}+\eta^0_{l^*}-2}{n-1}\Big) \geq 0$. Since $m \geq 3$ and $\eta^0_k > 0$ for $k=1,\dots,m$, then $\eta^0_{k^*} \leq n-2$. If $\eta^0_{k^*} = n-2$, then $\eta^0_{\pi^0(1)} = 1$ and $\eta^0_l = 1$

for $l \neq k^*, l \neq \pi^0(1)$, thus we obtain

$$(\eta^0_{\pi^0(1)} - \eta^0_l - 1)\Big(f(1) - f(0) - \frac{\eta^0_{\pi^0(1)} + \eta^0_l - 2}{n-1}\Big) = f(0) - f(1) < 0$$

which contradicts the given condition. As a result, $\eta^0_{k^*} < n-2$.

Player 1 chooses to remain in the current group labeled $\pi^0(1)$ at stage 1 under stable equilibrium. Consider player $2 \in N^0_k$ who chooses an action at stage 2, and if $k = \pi^0(1)$, then

$$C_2(g^s, \Delta^1[2,k^*]) - C_2(g^s, \Delta^1) = f(1) - \frac{\eta^0_{k^*}}{n-1} - f(0) < 0,$$

and for any $k' \neq k^*, k$,

$$C_2(g^s, \Delta^1[2,k']) = f(1) - \frac{\eta^0_{k'}}{n-1} \geq C_2(g^s, \Delta^1[2,k^*]).$$

If $k \neq \pi^0(1), k^*$,

$$C_2(g^s, \Delta^1[2,k^*]) - C_2(g^s, \Delta^1) = \frac{\eta^0_k - 1 - \eta^0_{k^*}}{n-1} < 0,$$

and for any $k' \neq k^*, k$,

$$C_2(g^s, \Delta^1[2,k']) = f(1) - \frac{\eta^0_{k'}}{n-1} \geq C_2(g^s, \Delta^1[2,k^*]).$$

If $k = k^*$, then

$$C_2(g^s, \Delta^1) = f(1) - \frac{\eta^0_{k^*} - 1}{n-1} \leq f(0) = C_2(g^s, \Delta^1[2, \pi^0(1)]),$$

and if $\eta^0_{k^*} > \eta^0_{k'}$ for any $k' \neq k^*, \pi^0(1)$, then

$$C_2(g^s, \Delta^1) \leq f(1) - \frac{\eta^0_{k'}}{n-1} = C_2(g^s, \Delta^1[2,k']),$$

while if $\exists\, \bar{k} \neq k^*, \pi^0(1)$, $\eta^0_{k^*} = \eta^0_{\bar{k}}$, then

$$C_2(g^s, \Delta^1[2,\bar{k}]) = f(1) - \frac{\eta^0_{\bar{k}}}{n-1} < C_2(g^s, \Delta^1),$$
$$C_2(g^s, \Delta^1[2,\bar{k}]) \leq C_2(g^s, \Delta^1[2,k']).$$

Summarizing, player 2 chooses to stay or deviate to the group labeled k^* or $\bar{k}$. Without loss of generality, let k^* be the choice of player 2 at stage 2, then the group labeled k^* becomes the unique group containing the maximum number of players among all groups except the one labeled $\pi^0(1)$ in partition Δ^2.

We apply the same analysis for players $3, \ldots, n$, then we make a conclusion that player $i \geq 3$ chooses to stay or deviate to the group labeled k^* at stage i. Consequently, partition Δ^n satisfies $N^n_{k^*} = \{2, \ldots, n\}$, and $N^n_{\pi^0(1)} = \{1\}$. Now verify the stability of Δ^n:

- for player 1,

$$C_1(g^s,\Delta^n[1,k]) - C_1(g^s,\Delta^n) = \begin{cases} 0, & k \neq k^*, \\ f(0) - f(1) + \frac{n-2}{n-1} > 0, & k = k^*, \end{cases}$$

- for any player $i \neq 1$,

$$C_i(g^s,\Delta^n[i,k]) - C_i(g^s,\Delta^n) = \begin{cases} \frac{n-2}{n-1} > 0, & k \neq \pi^0(1), \\ f(0) - f(1) + \frac{n-2}{n-1} > 0, & k = \pi^0(1). \end{cases}$$

Therefore, partition Δ^n is dynamically stable.

We have proved that there exists a dynamically stable partition, and now we characterize $\bar{\Delta}$, i.e., it satisfies certain conditions in the following cases:

(a) Suppose $\exists\, i \neq 1$ such that $\bar{\pi}(i) \neq \bar{\pi}(1)$. Since

$$C_i(g^s,\bar{\Delta}[i,\bar{\pi}(1)]) - C_i(g^s,\bar{\Delta}) \leq f(0) - \left(f(1) - \frac{n-2}{n-1}\right) < 0,$$

player i chooses to deviate to the group labeled $\bar{\pi}(1)$ when she is selected to make an action based on structure $\bar{\Delta}$, contradicting that $\bar{\Delta}$ is dynamically stable. Thus, $\bar{\pi}(i) = \bar{\pi}(j)$ for any $i,j \in N$.

(b) We proceed the proof by showing that any of the following cases can never happen:

i. Let $\bar{\pi}(i) = \bar{\pi}(1)$ for any player $i \in N$. If it is the case, then for player 1,

$$C_1(g^s,\bar{\Delta}[1,k]) - C_1(g^s,\bar{\Delta}) = f(1) - \frac{n-2}{n-1} - f(0) < 0,\ k \neq \bar{\pi}(1),$$

which is a contradiction.

ii. $\exists\, i,j \in N \setminus \{1\}$ such that $\bar{\pi}(i) = \bar{\pi}(1)$, $\bar{\pi}(j) \neq \bar{\pi}(1)$. Then

$$C_i(g^s,\bar{\Delta}[i,\bar{\pi}(j)]) - C_i(g^s,\bar{\Delta}) = f(1) - \frac{\bar{\eta}_{\bar{\pi}(j)}}{n-1} - f(0) \geq 0,$$
$$C_j(g^s,\bar{\Delta}[j,\bar{\pi}(1)]) - C_j(g^s,\bar{\Delta}) = f(0) - f(1) + \frac{\bar{\eta}_{\bar{\pi}(j)} - 1}{n-1} \geq 0,$$

which can not be satisfied simultaneously.

iii. Let for any $i \neq 1$, $\bar{\pi}(i) \neq \bar{\pi}(1)$, and $\exists\, i,j \neq 1$, $\bar{\pi}(i) \neq \bar{\pi}(j)$, then

$$C_i(g^s,\bar{\Delta}[i,\bar{\pi}(j)]) - C_i(g^s,\bar{\Delta}) = \frac{\bar{\eta}_{\bar{\pi}(i)} - \bar{\eta}_{\bar{\pi}(j)} - 1}{n-1} \geq 0,$$
$$C_j(g^s,\bar{\Delta}[j,\bar{\pi}(i)]) - C_j(g^s,\bar{\Delta}) = \frac{\bar{\eta}_{\bar{\pi}(j)} - \bar{\eta}_{\bar{\pi}(i)} - 1}{n-1} \geq 0,$$

which can not be satisfied simultaneously.

(c) If the initial partition Δ^0 satisfies $\pi^0(i) = \pi^0(j)$ for any $i,j \in N$, or $\pi^0(i) = \pi^0(j) \neq \pi^0(1)$ for any $i,j \in N \setminus \{1\}$, then it can be easily verified that Δ^0 is dynamically stable when $f(1) - f(0) = \frac{n-2}{n-1}$ is satisfied. Then, it suffices to demonstrate that any other structure of partition is not dynamically stable:

i. $\exists\, i, j \in N \setminus \{1\}$ such that $\bar{\pi}(i) = \bar{\pi}(1)$, $\bar{\pi}(j) \neq \bar{\pi}(1)$, then

$$C_j(g^s, \bar{\Delta}[j, \bar{\pi}(1)]) - C_j(g^s, \bar{\Delta}) = f(0) - f(1) + \frac{\bar{\eta}_{\bar{\pi}(j)} - 1}{n-1} < 0,$$

which is a contradiction.

ii. $\exists\, i, j \in N \setminus \{1\}$ such that $\bar{\pi}(i) \neq \bar{\pi}(1)$, $\bar{\pi}(j) \neq \bar{\pi}(1)$ and $\bar{\pi}(i) \neq \bar{\pi}(j)$, then

$$C_i(g^s, \bar{\Delta}[i, \bar{\pi}(1)]) - C_i(g^s, \bar{\Delta}) = f(0) - f(1) + \frac{\bar{\eta}_{\bar{\pi}(i)} - 1}{n-1} < 0,$$

which is a contradiction.

□

Appendix B: Proof of Theorem 2

Proof. Given partition $\Delta = \{N_1, N_2\}$ where $\eta_1 - \eta_2 = k$, for any players $i \in N_1$ and $j \in N_2$, we have

$$C_i(g^c, \Delta[i,2]) - C_i(g^c, \Delta) = (k-1)\Big(f(1) - f(0) + \frac{2-n}{n-1}\Big), \tag{B.1}$$

$$C_j(g^c, \Delta[j,1]) - C_j(g^c, \Delta) = (k+1)\Big(f(0) - f(1) + \frac{n-2}{n-1}\Big). \tag{B.2}$$

Consider three different cases:

1. When $\eta_1^0 = \eta_2^0$, and
   (a) if $f(1) - f(0) \leq \frac{n-2}{n-1}$, then for any $i \in N_1^0$ and $j \in N_2^0$, both expressions (B.1) and (B.2) are nonnegative when $\Delta = \Delta^0$. Thus, we conclude that Δ^0 is dynamically stable, i.e., $\bar{\Delta} = \Delta^0$;
   (b) if $f(1) - f(0) > \frac{n-2}{n-1}$, without loss of generality, let $i \in N_1^0$ be the player who makes an action at stage 1, since expression (B.1) is negative when $\Delta = \Delta^0$, player i chooses to deviate to the group labeled 2. Then at any subsequent stage $l \geq 2$ starting with partition Δ^{l-1}, for any players $i \in N_1^{l-1}$ (if she exists) and $j \in N_2^{l-1}$, we directly obtain that expression (B.1) is negative and (B.2) is positive ($\Delta = \Delta^{l-1}$) since $k \leq -2$ for partition Δ^{l-1}, which indicates that player i chooses to deviate to the group labeled 2. Player j chooses to stay in the current group if they are selected to make an action at stage l. The dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_1 = 0, \bar{\eta}_2 = n$ may eventually emerge after the last player in the group labeled 1 is chosen to make an action at some stage.

2. When $\eta_1^0 > \eta_2^0$, and
   (a) if $f(1) - f(0) = \frac{n-2}{n-1}$, then for any players $i \in N_1^0$ and $j \in N_2^0$, expressions (B.1) and (B.2) are equal to zero when $\Delta = \Delta^0$. Therefore, Δ^0 is dynamically stable;
   (b) if $f(1) - f(0) > \frac{n-2}{n-1}$, then for any $i \in N_1^0$ and $j \in N_2^0$, expression (B.1) is nonnegative and (B.2) is negative ($\Delta = \Delta^0$) since $k \geq 1$ for partition Δ^0. As the dynamic process proceeds, each player in the group labeled 2 chooses to deviate to the group labeled 1, while each one in the group labeled 1 remains there since k, the cardinality difference between the two groups, is increasing. The dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_1 = n, \bar{\eta}_2 = 0$ eventually emerges after the last player in the group labeled 2 is chosen to make an action at a certain stage.

(c) if $f(1)-f(0)<\frac{n-2}{n-1}$, and

i. n is even, then for any $i\in N_1^0$ and $j\in N_2^0$, expression (B.1) is negative while expression (B.2) is positive ($\Delta=\Delta^0$) with $k\geq 2$ for partition Δ^0. Thus, as the process evaluates, each player in the group labeled 1 chooses to deviate to the group labeled 2, while each one in the group labeled 2 remains there when they are randomly selected until k, the cardinality difference between the two groups drops from 2 to 0. This happens after some player in the group labeled 1 deviates to the other group at some stage t. With respect to the emerged partition Δ^t such that $\eta_1^t=\eta_2^t=\frac{n}{2}$, expressions (B.1) and (B.2) >0 are positive ($\Delta=\Delta^t$) for $i\in N_1^t$ and $j\in N_2^t$, indicating Δ^t is dynamically stable;

ii. n is odd, if $k=1$ for Δ^0, then for any $i\in N_1^0$ and $j\in N_2^0$, expression (B.1) is zero and (B.2) is positive ($\Delta=\Delta^0$). Thus Δ^0 is dynamically stable satisfying $\eta_1^0=\frac{n+1}{2}$, $\eta_2^0=\frac{n-1}{2}$. If $k\geq 3$ for Δ^0, then the similar discussion can be performed as in Item i. A partition Δ^t such that $\eta_1^t=\frac{n+1}{2}$, $\eta_2^t=\frac{n-1}{2}$, is generated when k drops from 3 to 1 after some player in the group labeled 1 deviates to the other group at some stage t. We can easily verify that Δ^t is dynamically stable.

□

Appendix C: Proof of Theorem 3

Proof. Given partition $\Delta=\{N_1,N_2\}$ where $\eta_1-\eta_2=k\geq 0$ (without loss of generality), for any players $i\in N_1$ and $j\in N_2$, we have

$$C_i(g^c,\Delta[i,2])-C_i(g^c,\Delta)=\begin{cases} f(0)-f(1), & k=0,\\ 0, & k=1,\\ (k-1)(f(1)-f(0))+\frac{(n+k-2)(2-k)}{2(n-1)}, & k\geq 2,\end{cases} \tag{C.1}$$

$$C_j(g^c,\Delta[j,1])-C_j(g^c,\Delta)=(k+1)(f(0)-f(1))+\frac{k(n+k)}{2(n-1)}. \tag{C.2}$$

We examine all possible cases:

1. When $\eta_1^0=\eta_2^0$, assume player $i^{'}\in N_2^0$ is selected to make an action at stage 1,

(a) for player $i^{'}$, expression (C.2) is negative with $\Delta=\Delta^0$, $k=0$ and $j=i^{'}$. As a result, $i^{'}$ deviates to the group labeled 1 at stage 1, implying that the cardinality difference k increases from 0 to 2. At stage $t+1$ which begins with partition Δ^t, $t\geq 1$, if $2\leq k<x$ where k is the cardinality difference for Δ^t, and $x\leq n-2$ is an even such that $\frac{(x-2)(n+x-2)}{2(n-1)(x-1)}<f(1)-f(0)\leq\frac{x(n+x)}{2(n-1)(x+1)}$,

then for any players $i \in N_1^t$ and $j \in N_2^t$, we have

$$\begin{aligned}
C_i(g^c, \Delta^t[i,2]) - C_i(g^c, \Delta^t) &> \frac{(k-1)(x-2)(n+x-2)}{2(n-1)(x-1)} + \frac{(n+k-2)(2-k)}{2(n-1)} \\
&= \frac{kx^2 + xn + k^2 - kn - x^2 - k^2x}{2(n-1)(x-1)} \\
&= \frac{(x-k)(kx+n-x-k)}{2(n-1)(x-1)} > 0, \\
C_j(g^c, \Delta^t[j,1]) - C_j(g^c, \Delta^t) &< -\frac{(k+1)(x-2)(n+x-2)}{2(n-1)(x-1)} + \frac{k(n+k)}{2(n-1)} \\
&= \frac{(k-x+2)(kx+n+x-k-2)}{2(n-1)(x-1)} \leq 0.
\end{aligned}$$

While if $k = x$, then

$$C_i(g^c, \Delta^t[i,2]) - C_i(g^c, \Delta^t) > \frac{(x-k)(kx+n-x-k)}{2(n-1)(x-1)} = 0, \tag{C.3}$$

$$C_j(g^c, \Delta^t[j,1]) - C_j(g^c, \Delta^t) \geq -\frac{x(k+1)(n+x)}{2(n-1)(x+1)} + \frac{k(n+k)}{2(n-1)} = 0. \tag{C.4}$$

Therefore, we conclude that any player who belongs to the group labeled 2 chooses to deviate to the other group, and each player in the group labeled 1 chooses to keep his own group as long as the cardinality difference k between the two groups does not reach x. As players in the group labeled 2 deviate, the dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_2 = \frac{n-x}{2}$, $\bar{\eta}_1 = \frac{n+x}{2}$ appears immediately when k increases up to x since no player prefers to deviate which follows from (C.3) and (C.4).

(b) if $f(1) - f(0) > \frac{n-2}{n-1}$, from Item (a), it follows that player i' deviates to the group labeled 1 at stage 1, then k increases from 0 (for Δ^0) to 2 (for Δ^1). At stage $t+1$ starting with Δ^t, $t \geq 1$, if $2 \leq k < n$, then for any players $i \in N_1^t$ and $j \in N_2^t$, from (C.1) and (C.2), we get

$$\begin{aligned}
C_i(g^c, \Delta^t[i,2]) - C_i(g^c, \Delta^t) &> \frac{(k-1)(n-2)}{n-1} + \frac{(n+k-2)(2-k)}{2(n-1)} = \frac{k(n-k)}{2(n-1)} > 0, \\
C_j(g^c, \Delta^t[j,1]) - C_j(g^c, \Delta^t) &< -\frac{(n-2)(k+1)}{n-1} + \frac{k(n+k)}{2(n-1)} = \frac{(k+2)(k+2-n)}{2(n-1)} \leq 0.
\end{aligned}$$

Consequently, so long as the cardinality difference between two groups has not reached n (the group labeled 2 becomes empty), players in the group labeled 2 choose to deviate when they are selected to make actions. Finally, the dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_2 = 0$, $\bar{\eta}_1 = n$ appears immediately when the last player in the group labeled 2 finishes his deviation.

2. Without loss of generality, we assume x^0 is odd, but the proof, in which it is even, can be provided in a similar way.

(a) if $f(1) - f(0) < \frac{n+1}{4(n-1)}$, then for any $i \in N_1^0$ and $j \in N_2^0$, by (C.1) and (C.2), we get

$$\begin{aligned}
&C_i(g^c, \Delta^0[i,2]) - C_i(g^c, \Delta^0) \\
&= \begin{cases} 0, & x^0 = 1, \\ (x^0-1)(f(1)-f(0)) + \frac{(n+x^0-2)(2-x^0)}{2(n-1)} < \frac{(3-x^0)(n+2x^0-3)}{4(n-1)} \leq 0, & x^0 \geq 3, \end{cases}
\end{aligned}$$

$$C_j(g^c,\Delta^0[j,1]) - C_j(g^c,\Delta^0) > -\frac{(x^0+1)(n+1)}{4(n-1)} + \frac{x^0(n+x^0)}{2(n-1)} = \frac{(x^0-1)(2x^0+n+1)}{4(n-1)} > 0.$$

As a result, if $x^0 = 1$, the initial partition Δ^0 satisfying $\eta_1^0 = \frac{n+1}{2}$, $\eta_2^0 = \frac{n-1}{2}$ is dynamically stable. If $x^0 \geq 3$, then players from N_1^0 choose to deviate, players from N_2^0 choose to remain until their cardinality difference k reaches the value of 1, and the dynamically stable partition $\bar{\Delta}$ is such that $\bar{\eta}_1 = \frac{n+1}{2}$, $\bar{\eta}_2 = \frac{n-1}{2}$ is generated.

(b) if $\frac{(x-2)(n+x-2)}{2(n-1)(x-1)} < f(1) - f(0) < \frac{x(n+x)}{2(n-1)(x+1)}$, where $3 \leq x \leq n-2$ is odd, then based on any partition $\Delta = \{N_1, N_2\}$ such that $k \geq 1$, for any $i \in N_1$ and $j \in N_2$,

$$C_i(g^c,\Delta[i,2]) - C_i(g^c,\Delta)$$
$$= \begin{cases} 0, & k=1, \\ (k-1)(f(1)-f(0)) + \frac{(n+k-2)(2-k)}{2(n-1)} > \frac{(k+n-2)(k-2)(k+x-2)}{2(n-1)(x-1)} > 0, & 3 \leq k \leq x, \\ (k-1)(f(1)-f(0)) + \frac{(n+k-2)(2-k)}{2(n-1)} < G(n,k,x)\big|_{k=x+2} = 0 & k > x, \end{cases}$$

where $G(n,k,x) = \frac{-(x+1)k^2 + \left[(x+2)^2 - n\right]k + (x+2)(n-x-2)}{2(n-1)(x+1)}$,

$$C_j(g^c,\Delta[j,1]) - C_j(g^c,\Delta)$$
$$= \begin{cases} (k+1)(f(0)-f(1)) + \frac{k(n+k)}{2(n-1)} < \frac{-(k+1)x^2 + \left[(k+2)^2 - n\right]x + (k+2)(n-k-2)}{2(n-1)(x-1)}\Big|_{x=k+2} = 0, & k < x, \\ (k+1)(f(0)-f(1)) + \frac{k(n+k)}{2(n-1)} > \frac{(k-x)(kx+n+k+x)}{2(n-1)(x+1)} \geq 0, & k \geq x. \end{cases}$$

Thus, when the cardinality difference between two groups is smaller than x, players in the group labeled 2 choose to deviate, and players in the group labeled 1 remain in the current one. When such a difference is larger than x, the reverse happens. And no player prefers to deviate when such a difference increases or decreases reaching x, indicating the dynamically stable partition $\bar{\Delta}$ is defined by $\bar{\eta}_1 = \frac{n+x}{2}$, $\bar{\eta}_2 = \frac{n-x}{2}$.

(c) if $f(1) - f(0) = \frac{x(n+x)}{2(n-1)(x+1)}$, where $1 \leq x \leq n-2$, and

i. $x^0 \leq x$, based on any partition $\Delta = \{N_1, N_2\}$ such that $k \geq 1$, for any $i \in N_1$ and $j \in N_2$, we have

$$C_i(g^c,\Delta[i,2]) - C_i(g^c,\Delta)$$
$$= \begin{cases} 0, & k=1, \\ \frac{-(x+1)k^2 + \left[(x+2)^2 - n\right]k + (x+2)(n-x-2)}{2(n-1)(x+1)} > \frac{4\left[(x-1)^2 - 5 + n\right]}{2(n-1)(x+1)} > 0, & 3 \leq k < x, \\ \frac{(n-x)^2 + 2(x^2-1)}{2(n-1)(x+1)} > 0, & 3 \leq k = x, \end{cases}$$

$$C_j(g^c,\Delta[j,1]) - C_j(g^c,\Delta) = \begin{cases} \frac{(k-x)(kx+n+k+x)}{2(n-1)(x+1)} < 0, & k < x, \\ \frac{(k-x)(kx+n+k+x)}{2(n-1)(x+1)} = 0, & k = x. \end{cases}$$

As a result, starting from partition Δ^0, in which the cardinality difference is not larger than x, players belonging to the group labeled 2 deviate to the other group ($x^0 < x$) or all players remain in their current groups ($x^0 = x$) until such a difference increasingly reaches x. Then the dynamically stable partition $\bar{\Delta}$ with $\bar{\eta}_1 = \frac{n+x}{2}$, $\bar{\eta}_2 = \frac{n-x}{2}$ emerges.

ii. $x^0 > x$, based on any partition $\Delta = \{N_1, N_2\}$ such that $k \geq x+2$, for any $i \in N_1$ and $j \in N_2$, we have

$$C_i(g^c, \Delta[i,2]) - C_i(g^c, \Delta) = \begin{cases} 0, & k = x+2, \\ \frac{-(x+1)k^2 + \left[(x+2)^2 - n\right]k + (x+2)(n-x-2)}{2(n-1)(x+1)} < 0, & k > x+2, \end{cases}$$

$$C_j(g^c, \Delta[j,1]) - C_j(g^c, \Delta) = \frac{(k-x)(kx+n+k+x)}{2(n-1)(x+1)} > 0.$$

As a result, starting from partition Δ^0, in which the cardinality difference is larger than x, players belonging to the group labeled 1 deviate to the other group ($x^0 > x+2$) or all players remain in their current groups ($x^0 = x+2$) until such a difference decreasingly reaches $x+2$. Then the dynamically stable partition $\bar{\Delta}$ with $\bar{\eta}_1 = \frac{n+x+2}{2}$, $\bar{\eta}_2 = \frac{n-x-2}{2}$ emerges.

(d) if $f(1) - f(0) > \frac{n-2}{n-1}$, then based on any partition $\Delta = \{N_1, N_2\}$ such that $k \geq 1$, for any $i \in N_1$, by (C.1) we get

$$C_i(g^c, \Delta[i,2]) - C_i(g^c, \Delta) = \begin{cases} 0, & k = 1, \\ (k-1)(f(1) - f(0)) + \frac{(n+k-2)(2-k)}{2(n-1)} > \frac{k(n-k)}{2(n-1)} \geq 0, & k \geq 2, \end{cases}$$

and for any $j \in N_2$, when $k < n$ by (C.2) we obtain

$$C_j(g^c, \Delta[j,1]) - C_j(g^c, \Delta) = (k+1)(f(0) - f(1)) + \frac{k(n+k)}{2(n-1)} < \frac{(k+2)(k+2-n)}{2(n-1)} \leq 0.$$

As a result, starting from partition Δ^0, players belonging to the group labeled 2 deviate to the group labeled 1 until such a group becomes empty, then the dynamically stable partition $\bar{\Delta}$ with $\bar{\eta}_1 = n$, $\bar{\eta}_2 = 0$ emerges.

□

Appendix D: Proof of Theorem 4

Proof. We prove the existence of a dynamically stable partition constructing for any initial partition Δ^0 an action-making order of players, after which the dynamically stable partition appears. Examinations of different structures of the initial partition are respectively conducted under various conditions for cost functions (1) and (2).

First, we consider the cost function (1). There are three possible cases:

- $f(1) - f(0) < 1 - \frac{2}{n}$, and if

  1. $\eta^0_{A,1} = \eta^0_{B,2} = \frac{n}{2}$, then for any players $i \in A$ and $j \in B$,

  $$C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0) = C_j(g^b, \Delta^0[j,1]) - C_j(g^b, \Delta^0) = \frac{n}{2}\left(f(0) - f(1) + 1 - \frac{2}{n}\right) > 0.$$

  Therefore, no player prefers to deviate to the other group given the initial partition Δ^0, i.e., Δ^0 is dynamically stable.

  It is obtained that given any initial partition, as long as a certain partition Δ, which satisfies $\eta_{A,1} = \eta_{B,2} = \frac{n}{2}$ or $\eta_{A,2} = \eta_{B,1} = \frac{n}{2}$, emerges after some players make actions in the sequence which we particularly design, it is dynamically stable.

2. $\eta^0_{A,1} = \eta^0_{B,1} = \frac{n}{2}$, then for any partition $\Delta = \{N_1, N_2\}$ such that $\eta_{A,1} = \frac{n}{2}$, it is easy to obtain that for any player $i \in B \cap N_1$,

$$C_i(g^b, \Delta[i,2]) - C_i(g^b, \Delta) = \frac{n}{2}\Big(f(1) - f(0) - 1 + \frac{2}{n}\Big) < 0.$$

We select players from set B in the sequence to make actions at the first $\frac{n}{2}$ stages, then they all choose to deviate. Therefore, $\Delta^{\frac{n}{2}}$, which satisfies $\eta^{\frac{n}{2}}_{A,1} = \eta^{\frac{n}{2}}_{B,2} = \frac{n}{2}$, is dynamically stable.

3. $\eta^0_{A,1} = \frac{n}{2}$, $\eta^0_{B,1} > 0$, $\eta^0_{B,2} > 0$, then we choose players from set $N^0_{B,1}$ in the sequence to make actions, then they all choose to deviate to the group labeled 2 from the proof above. Thus, $\Delta^{\eta^0_{B,1}}$ satisfying $\eta^{\eta^0_{B,1}}_{A,1} = \eta^{\eta^0_{B,1}}_{B,2} = \frac{n}{2}$ is dynamically stable.

4. $\eta^0_{A,1}, \eta^0_{A,2}, \eta^0_{B,1}, \eta^0_{B,2} > 0$, and four possible cases need to be separately discussed:

   (a) $\eta^0_{A,1} > \eta^0_{A,2}$ and $\eta^0_{B,1} > \eta^0_{B,2}$, then for any player $i \in N^0_{A,1}$,

   $$C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0) = \big(2\eta^0_{B,1} - \frac{n}{2}\big)(f(1) - f(0)) + 2\eta^0_{B,1}\big(\frac{2}{n} - 1\big) + \frac{n}{2} - 1,$$

   since $\eta^0_{B,1} > \frac{n}{4}$. Therefore, we have

   $$C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0) < \big(2\eta^0_{B,1} - \frac{n}{2}\big)(1 - \frac{2}{n}) + 2\eta^0_{B,1}\big(\frac{2}{n} - 1\big) + \frac{n}{2} - 1 = 0.$$

   Then, we choose players from set $N^0_{A,1}$ to make actions at the first $\eta^0_{A,1}$ stages, at which all players deviate to the group labeled 2. Then $\Delta^{\eta^0_{A,1}}$ has the structure feature, which is similar to the initial partition given in Item 3, i.e., all players from set A belong to the certain group and there are players from set B in both groups. Therefore, we make construction in the same way as in Item 3, obtaining the dynamically stable partition $\bar{\Delta}$ satisfying $\bar{\eta}_{A,2} = \bar{\eta}_{B,1} = \frac{n}{2}$.

   (b) $\eta^0_{A,1} > \eta^0_{A,2}$ and $\eta^0_{B,1} < \eta^0_{B,2}$, then for any player $i \in N^0_{A,2}$, with $\eta^0_{B,2} > \frac{n}{4}$, we obtain

   $$C_i(g^b, \Delta^0[i,1]) - C_i(g^b, \Delta^0) = \big(2\eta^0_{B,2} - \frac{n}{2}\big)(f(1) - f(0)) + 2\eta^0_{B,2}\big(\frac{2}{n} - 1\big) + \frac{n}{2} - 1 < 0.$$

   We choose the players from set $N^0_{A,2}$ to make actions at the first $\eta^0_{A,2}$ stages, at which all the players deviate to the other group. Then we make a construction in the same way as in Item 3, obtaining the dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_{A,1} = \bar{\eta}_{B,2} = \frac{n}{2}$.

   (c) $\eta^0_{A,1} = \eta^0_{A,2} = \frac{n}{4}$ and $\eta^0_{B,1} > \eta^0_{B,2}$, then for any player $i \in N^0_{A,1}$, we get

   $$C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0) = \big(2\eta^0_{B,1} - \frac{n}{2}\big)(f(1) - f(0)) + 2\eta^0_{B,1}\big(\frac{2}{n} - 1\big) + \frac{n}{2} - 1 < 0,$$

   since $\eta^0_{B,1} > \frac{n}{4}$. Choosing players from set $N^0_{A,1}$ to make their actions at the first $\eta^0_{A,1}$ stages, at which all the players deviate to the other group. Then we make a constriction in the same way as in Item 3, obtaining the dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_{A,2} = \bar{\eta}_{B,1} = \frac{n}{2}$.

   (d) $\eta^0_{A,1} = \eta^0_{A,2} = \frac{n}{4}$ and $\eta^0_{B,1} = \eta^0_{B,2} = \frac{n}{4}$, then for any players $i \in N^0_1$ and $j \in N^0_2$,

   $$C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0) = C_j(g^b, \Delta^0[j,1]) - C_j(g^b, \Delta^0) = 0,$$

   implying the initial partition Δ^0 is dynamically stable.

- $f(1)-f(0)=1-\frac{2}{n}$, then given any initial partition Δ^0, for any players $i\in N_1^0$ and $j\in N_2^0$,

$$C_i(g^b,\Delta^0[i,2])-C_i(g^b,\Delta^0)=C_j(g^b,\Delta^0[j,1])-C_j(g^b,\Delta^0)=0.$$

  Therefore, any initial partition is dynamically stable.

- $f(1)-f(0)>1-\frac{2}{n}$, then if

  1. $\eta_{A,1}^0=\eta_{B,2}^0=\frac{n}{2}$, and for any partition $\Delta=\{N_1,N_2\}$ such that $\eta_{A,1}=\frac{n}{2}$, it is easy to obtain that for any player $i\in B\cap N_2$,

  $$C_i(g^b,\Delta[i,1])-C_i(g^b,\Delta)=\frac{n}{2}\Big(f(0)-f(1)+1-\frac{2}{n}\Big)<0.$$

     We select players from set B in the sequence to make actions at the first $\frac{n}{2}$ stages, and all the players choose to deviate to the other group. Therefore, the realized partition $\Delta^{\frac{n}{2}}$ is such that $\eta_{A,1}^{\frac{n}{2}}=\eta_{B,1}^{\frac{n}{2}}=\frac{n}{2}$, thus, for any player $i\in N$,

  $$C_i(g^b,\Delta^{\frac{n}{2}}[i,2])-C_i(g^b,\Delta^{\frac{n}{2}})=\frac{n}{2}\Big(f(1)-f(0)-1+\frac{2}{n}\Big)>0,$$

     implying that $\Delta^{\frac{n}{2}}$ is dynamically stable.
     We obtain that given any initial partition, as long as a certain partition Δ such that $\eta_1=n$ or $\eta_2=n$ (i.e., all players belong to the same group) emerges after some players make actions in a designed sequence, it is dynamically stable.
  2. $\eta_{A,1}^0=\eta_{B,1}^0=\frac{n}{2}$, then from Item 1 above it follows that Δ^0 is dynamically stable.
  3. $\eta_{A,1}^0=\frac{n}{2}$, $\eta_{B,1}^0>0$, $\eta_{B,2}^0>0$. We choose players in set $N_{B,2}^0$ to make actions consequently, then they all choose to deviate to the group labeled 1, which follows from the statement in Item 1. Thus, $\Delta^{\eta_{B,2}^0}$ satisfying $\eta_1^{\eta_{B,2}^0}=n$ is dynamically stable.
  4. $\eta_{A,1}^0,\eta_{A,2}^0,\eta_{B,1}^0,\eta_{B,2}^0>0$, and four possible cases are to be discussed.
     (a) $\eta_{A,1}^0>\eta_{A,2}^0$ and $\eta_{B,1}^0>\eta_{B,2}^0$, then for any player $i\in N_{A,2}^0$,

  $$C_i(g^b,\Delta^0[i,1])-C_i(g^b,\Delta^0)=\big(2\eta_{B,2}^0-\frac{n}{2}\big)(f(1)-f(0))+2\eta_{B,2}^0\big(\frac{2}{n}-1\big)+\frac{n}{2}-1<0,$$

     since $\eta_{B,2}^0<\frac{n}{4}$.
     We choose players from set $N_{A,2}^0$ to make actions at the first $\eta_{A,2}^0$ stages, all these players deviate to the other group. Then $\Delta^{\eta_{A,2}^0}$ has the same structure as the initial partition in Item 3. Therefore, we make a construction in the same way as in Item 3, obtaining the dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_1=n$.
     (b) $\eta_{A,1}^0>\eta_{A,2}^0$ and $\eta_{B,1}^0<\eta_{B,2}^0$, then for any player $i\in N_{A,1}^0$,

  $$C_i(g^b,\Delta^0[i,2])-C_i(g^b,\Delta^0)=\big(2\eta_{B,1}^0-\frac{n}{2}\big)(f(1)-f(0))+2\eta_{B,1}^0\big(\frac{2}{n}-1\big)+\frac{n}{2}-1<0.$$

     We choose the players from set $N_{A,1}^0$ to make actions at the first $\eta_{A,1}^0$ stages, all these players deviate to the other group. Then we make construction in the same way as in Item 3, obtaining the dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_2=n$.

(c) $\eta^0_{A,1} = \eta^0_{A,2} = \frac{n}{4}$ and $\eta^0_{B,1} > \eta^0_{B,2}$, then for any player $i \in N^0_{A,2}$,

$$C_i(g^b, \Delta^0[i,1]) - C_i(g^b, \Delta^0) = \big(2\eta^0_{B,2} - \frac{n}{2}\big)(f(1) - f(0)) + 2\eta^0_{B,2}\big(\frac{2}{n} - 1\big) + \frac{n}{2} - 1 < 0.$$

We choose the players from set $N^0_{A,2}$ to make actions at the first $\eta^0_{A,2}$ stages, and they all choose an action to deviate to the other group. Then we make a construction in the same way as in Item 3 obtaining the dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_1 = n$.

(d) $\eta^0_{A,1} = \eta^0_{A,2} = \frac{n}{4}$ and $\eta^0_{B,1} = \eta^0_{B,2} = \frac{n}{4}$, then for any players $i \in N^0_1$ and $j \in N^0_2$,

$$C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0) = C_j(g^b, \Delta^0[j,1]) - C_j(g^b, \Delta^0) = 0,$$

implying that the initial partition Δ^0 is dynamically stable.

Before we start a discussion of the cases for cost function (2), we mention some statements required later on in the proof.

Given any initial partition Δ^0 such that $\eta^0_{A,1}, \eta^0_{A,2}, \eta^0_{B,1}, \eta^0_{B,2} > 0$, for any players $i \in N^0_{A,1}$ and $j \in N^0_{A,2}$, we have

$$\begin{aligned} &C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0) \\ &= (\eta^0_{B,1} - \eta^0_{B,2})(f(1) - f(0)) + \frac{2}{n}\Big(\eta^0_{B,2}\max\{\eta^0_{B,2} - 1, \eta^0_{A,1} - 1\} - \eta^0_{B,1}\max\{\eta^0_{B,1} - 1, \eta^0_{A,2}\}\Big), \end{aligned} \tag{D.1}$$

$$\begin{aligned} &C_j(g^b, \Delta^0[j,1]) - C_j(g^b, \Delta^0) \\ &= (\eta^0_{B,2} - \eta^0_{B,1})(f(1) - f(0)) + \frac{2}{n}\Big(\eta^0_{B,1}\max\{\eta^0_{B,1} - 1, \eta^0_{A,2} - 1\} - \eta^0_{B,2}\max\{\eta^0_{B,2} - 1, \eta^0_{A,1}\}\Big). \end{aligned} \tag{D.2}$$

If expression (D.1) is negative, then player i will deviate to the group labeled 2 when he is chosen to make an action at stage 1, implying $\eta^1_{A,1}$ and $\eta^1_{A,2}$ are respectively decreased and increased by unit 1 in comparison with $\eta^0_{A,1}$ and $\eta^0_{A,2}$. Observing equation (D.1), we may simply realize that for any player $i \in N^1_{A,1}$,

$$C_i(g^b, \Delta^1[i,2]) - C_i(g^b, \Delta^1) \leq C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0)$$

representing that when players from $N^0_{A,1}$ are selected in the sequence to make actions at the first $\eta^0_{A,1}$ stages, all these players choose to deviate if (D.1) < 0. The same conclusion can be also drawn for players from set $N^0_{A,2}$ if we observe equation (D.2).

We start examination of those cases when cost function (2) is defined for each player. Specifically, when

- $f(1) - f(0) < 1 - \frac{2}{n}$, and

  1. $\eta^0_{A,1} = \eta^0_{B,2} = \frac{n}{2}$, and for any players $i \in A$ and $j \in B$,

  $$C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0) = C_j(g^b, \Delta^0[j,1]) - C_j(g^b, \Delta^0) = \frac{n}{2}\Big(f(0) - f(1) + 1 - \frac{2}{n}\Big) > 0.$$

  Therefore, no player prefers to deviate to the other group given the initial partition Δ^0, i.e., Δ^0 is dynamically stable. And it is evident that given any initial partition, so long as a certain partition Δ which satisfies $\eta_{A,1} = \eta_{B,2} = \frac{n}{2}$ or $\eta_{A,2} = \eta_{B,1} = \frac{n}{2}$ emerges after some players make actions (in the sequence which we in particular formulate), it is dynamically stable.

2. $\eta^0_{A,1} = \eta^0_{B,1} = \frac{n}{2}$, then for any partition $\Delta = \{N_1, N_2\}$ such that $\eta_{A,1} = \frac{n}{2}$, it is easy to obtain that for any player $i \in N_{B,1}$,

$$C_i(g^b, \Delta[i,2]) - C_i(g^b, \Delta) = \frac{n}{2}\Big(f(1) - f(0) - 1 + \frac{2}{n}\Big) < 0.$$

Thus, if players from $N^0_{B,1}$ are selected consequently to make actions at the first $\frac{n}{2}$ stages, then they all choose to deviate to the other group. As a result, partition $\Delta^{\frac{n}{2}}$ which satisfies $\eta^{\frac{n}{2}}_{A,1} = \eta^{\frac{n}{2}}_{B,2} = \frac{n}{2}$ is dynamically stable.

3. $\eta^0_{A,1} = \frac{n}{2}$, $\eta^0_{B,1} > 0$, $\eta^0_{B,2} > 0$. We choose players from set $N^0_{B,1}$ consequently to make actions, then they all choose to deviate to the group labeled 2 from the proof in Item 2. Thus, $\Delta^{\eta^0_{B,1}}$, such that $\eta^{\eta^0_{B,1}}_{A,1} = \eta^{\eta^0_{B,1}}_{B,2} = \frac{n}{2}$, is dynamically stable.

4. $\eta^0_{A,1}, \eta^0_{A,2}, \eta^0_{B,1}, \eta^0_{B,2} > 0$, and we consider the following cases:

   (a) $\eta^0_{A,1} > \eta^0_{A,2}$ and $\eta^0_{B,1} > \eta^0_{B,2}$, then for any players $i \in N^0_{A,1}$ and $j \in N^0_{A,2}$, we have

$$\begin{aligned} &C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0) \\ &\qquad = \big(2\eta^0_{B,1} - \frac{n}{2}\big)(f(1) - f(0)) + \frac{2\eta^0_{B,1}(2 - \eta^0_{B,1} - \eta^0_{A,1})}{n} - 1 + \eta^0_{A,1}, \end{aligned} \tag{D.3}$$

$$C_j(g^b, \Delta^0[j,1]) - C_j(g^b, \Delta^0) = \big(\frac{n}{2} - 2\eta^0_{B,1}\big)(f(1) - f(0)) + \frac{2\eta^0_{B,1}(\eta^0_{B,1} + \eta^0_{A,1} - 1)}{n} - \eta^0_{A,1}. \tag{D.4}$$

   Then the following expression can be easily obtained:

$$C_j(g^b, \Delta^0[j,1]) - C_j(g^b, \Delta^0) = -\Big(C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0)\Big) + \frac{2\eta^0_{B,1}}{n} - 1. \tag{D.5}$$

   Two cases are considered:

   i. if expression (D.3) is nonnegative, then from (D.5) we get $C_j(g^b, \Delta^0[j,1]) - C_j(g^b, \Delta^0) < 0$. In this case we choose players from $N^0_{A,2}$ to make actions at the first $\eta^0_{A,2}$ stages, and they all deviate to the group labeled 1. Then the structure of $\Delta^{\eta^0_{A,2}}$ is consistent with the initial partition in Item 3 above, thus, we make a construction in the same way and obtain that the dynamically stable partition is $\bar{\Delta}$ such that $\bar{\eta}_{A,1} = \bar{\eta}_{B,1} = \frac{n}{2}$.

   ii. if expression (D.3) is negative, let players from $N^0_{A,1}$ choose actions at the first $\eta^0_{A,1}$ stages, then the structure of $\Delta^{\eta^0_{A,1}}$ is similar with the initial one from Item 3, thus, we make a construction in the same way and obtain that the dynamically stable partition $\bar{\Delta}$ is such that $\bar{\eta}_{A,2} = \bar{\eta}_{B,1} = \frac{n}{2}$.

   (b) $\eta^0_{A,1} > \eta^0_{A,2}$ and $\eta^0_{B,1} < \eta^0_{B,2}$. Without loss of generality, we assume that $\eta^0_{B,2} \geq \eta^0_{A,1}$, but the case when $\eta^0_{B,2} < \eta^0_{A,1}$ can be similarly examined. For any player $i \in N^0_{A,1}$,

$$\begin{aligned} &C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0) \\ &\qquad = \big(2\eta^0_{B,1} - \frac{n}{2}\big)(f(1) - f(0)) + \frac{n}{2} - 2\eta^0_{B,1} - 1 + \frac{2\eta^0_{B,1}(\eta^0_{B,1} + 1 - \eta^0_{A,2})}{n}. \end{aligned} \tag{D.6}$$

For any player $j \in N^0_{A,2}$, if $\eta^0_{B,2} - 1 \geq \eta^0_{A,1}$,

$$C_j(g^b, \Delta^0[j,1]) - C_j(g^b, \Delta^0) = -\Big(C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0)\Big) - \frac{2\eta^0_{B,1}}{n}, \tag{D.7}$$

and if $\eta^0_{B,2} = \eta^0_{A,1}$,

$$C_j(g^b, \Delta^0[j,1]) - C_j(g^b, \Delta^0) = -\Big(C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0)\Big) - 1. \tag{D.8}$$

Two possible cases are considered:

i. if expression (D.6) is nonnegative, then $C_j(g^b, \Delta^0[j,1]) - C_j(g^b, \Delta^0) < 0$. Players $N^0_{A,2}$ are chosen to make actions consequently, then the structure of $\Delta^{\eta^0_{A,2}}$ is consistent with the initial partition from Item 3. As a result, a dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_{A,1} = \bar{\eta}_{B,2} = \frac{n}{2}$ emerges after we design the action-making order of players' in the same way.

ii. if expression (D.6) is nonnegative, let players from $N^0_{A,1}$ choose actions one by one, and we formulate the action-making order of players in the same way as in Item 3. Finally, a dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_{A,2} = \bar{\eta}_{B,1} = \frac{n}{2}$ is generated.

(c) $\eta^0_{A,1} = \eta^0_{A,2} = \frac{n}{4}$ and $\eta^0_{B,1} > \eta^0_{B,2}$, then for any players $i \in N^0_{A,1}$ and $j \in N^0_{A,2}$, we have

$$C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0) = \big(2\eta^0_{B,1} - \frac{n}{2}\big)(f(1) - f(0)) + \frac{n}{4} - 1 + \frac{\eta^0_{B,1}(8 - 4\eta^0_{B,1} - n)}{2n}, \tag{D.9}$$

$$C_j(g^b, \Delta^0[j,1]) - C_j(g^b, \Delta^0) = -\Big(C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0)\Big) + \frac{2\eta^0_{B,1}}{n} - 1. \tag{D.10}$$

Then

i. if expression (D.9) is nonnegative, then expression (D.10) is negative. Players from set $N^0_{A,2}$ are chosen to make actions consequently, then the structure of $\Delta^{\eta^0_{A,2}}$ is consistent with the initial partition from Item 3. As a result, a dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_{A,1} = \bar{\eta}_{B,2} = \frac{n}{2}$ emerges.

ii. if expression (D.9) is negative, let players from $N^0_{A,1}$ choose their actions consequently. Then we design the action-making order of players' in the same way as in Item 3. Finally, a dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_{A,2} = \bar{\eta}_{B,1} = \frac{n}{2}$ is generated.

(d) $\eta^0_{A,1} = \eta^0_{A,2} = \frac{n}{4}$ and $\eta^0_{B,1} = \eta^0_{B,2} = \frac{n}{4}$, then for any player $i \in N^0_1$,

$$C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0) = -\frac{1}{2}.$$

Let players from set $N^0_{A,1}$ choose their actions, and we design the sequence in the same way as in Item 3. Finally, a dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_{A,2} = \bar{\eta}_{B,1} = \frac{n}{2}$ is generated.

- $f(1) - f(0) = 1 - \frac{2}{n}$. We first prove that any initial partition Δ^0 such that $\eta^0_{X,1} = \frac{n}{2}$ or $\eta^0_{X,2} = \frac{n}{2}$ for $X = A$ or B is dynamically stable. Without loss of generality, let Δ^0 be such that $\eta^0_{A,1} = \frac{n}{2}$, i.e., all players from set A belong to the group labeled 1. For any players $i \in N^0_{A,1}$ and $j \in N^0_{B,1}$ (if she exists), $k \in N^0_{B,2}$ (if she exists),

$$C_i(g^b,\Delta^0[i,2]) - C_i(g^b,\Delta^0) = \begin{cases} -\frac{2(\eta^0_{B,1})^2}{n} + \eta^0_{B,1} > 0, & \eta^0_{B,1}, \eta^0_{B,2} > 0, \\ 0, & \eta^0_{B,1} = 0, \\ 0, & \eta^0_{B,2} = 0, \end{cases}$$

$$C_j(g^b,\Delta^0[j,2]) - C_j(g^b,\Delta^0) = 0,$$
$$C_k(g^b,\Delta^0[k,1]) - C_k(g^b,\Delta^0) = 0.$$

So, no player prefers to deviate implying that Δ^0 is dynamically stable.

1. Given any initial partition Δ^0 such that $\eta^0_{A,1} = \eta^0_{B,2} = \frac{n}{2}$, or $\eta^0_{A,1} = \eta^0_{B,1} = \frac{n}{2}$, or $\eta^0_{A,1} = \frac{n}{2}$, $\eta^0_{B,1} > 0$, $\eta^0_{B,2} > 0$, we may directly conclude that it is dynamically stable which follows from the statement obtained above.
2. $\eta^0_{A,1}, \eta^0_{A,2}, \eta^0_{B,1}, \eta^0_{B,2} > 0$, and four cases should be discussed:
   (a) $\eta^0_{A,1} > \eta^0_{A,2}$ and $\eta^0_{B,1} > \eta^0_{B,2}$. For any player $i \in N^0_{A,2}$,

   $$C_i(g^b,\Delta^0[i,1]) - C_i(g^b,\Delta^0) = \frac{2}{n}(\eta^0_{B,1})^2 + \left[\frac{2}{n}(\eta^0_{A,1}+1) - 2\right]\eta^0_{B,1} + \frac{n}{2} - \eta^0_{A,1} - 1, \quad \text{(D.11)}$$

   since $\frac{n}{4} < \eta^0_{B,1} < \frac{n}{2}$. Simple calculations show that expression (D.11) is negative. Let players from $N^0_{A,2}$ make actions at the first $\eta^0_{A,2}$ stages, all these players deviate to the group labeled 1. Then partition $\Delta^{\eta^0_{A,2}}$ satisfying $\eta^{\eta^0_{A,2}}_{A,1} = \frac{n}{2}$ is dynamically stable.

   (b) $\eta^0_{A,1} > \eta^0_{A,2}$ and $\eta^0_{B,1} < \eta^0_{B,2}$. Without loss of generality, we assume $\eta^0_{B,2} \geq \eta^0_{A,1}$, but the case when $\eta^0_{B,2} < \eta^0_{A,1}$ can be similarly examined. For any player $i \in N^0_{A,1}$, $C_i(g^b,\Delta^0[i,2]) - C_i(g^b,\Delta^0)$ can be obtained by substituting $f(1) - f(0)$ with $1 - \frac{2}{n}$ in (D.6). Such a difference for any player $j \in N^0_{A,2}$ may also be obtained by the same substitution in (D.7) if $\eta^0_{B,2} - 1 \geq \eta^0_{A,1}$, and in (D.8) if $\eta^0_{B,2} = \eta^0_{A,1}$. Two cases should be studied:
      i. if expression (D.6) is nonnegative, then $C_j(g^b,\Delta^0[j,1]) - C_j(g^b,\Delta^0) < 0$. Players from $N^0_{A,2}$ are chosen to make actions, and we obtain that partition $\Delta^{\eta^0_{A,2}}$ satisfying $\eta^{\eta^0_{A,2}}_{A,1} = \frac{n}{2}$ is dynamically stable.
      ii. if expression (D.6) is negative, let players from $N^0_{A,1}$ choose actions one by one, then partition $\Delta^{\eta^0_{A,1}}$ satisfying $\eta^{\eta^0_{A,1}}_{A,2} = \frac{n}{2}$ is dynamically stable.

   (c) $\eta^0_{A,1} = \eta^0_{A,2} = \frac{n}{4}$ and $\eta^0_{B,1} > \eta^0_{B,2}$, then for any player $i \in N^0_{A,2}$,

   $$C_i(g^b,\Delta^0[i,1]) - C_i(g^b,\Delta^0) = \frac{2}{n}(\eta^0_{B,1})^2 + \left(\frac{2}{n} - \frac{3}{2}\right)\eta^0_{B,1} + \frac{n}{4} - 1,$$

   which is negative under the condition that $\frac{b}{4} < \eta^0_{B,1} < \frac{n}{2}$. Therefore, choosing players from $N^0_{A,2}$ to make actions consequently, all these players deviate to the group labeled 1. Then partition $\Delta^{\eta^0_{A,2}}$ satisfying $\eta^{\eta^0_{A,2}}_{A,1} = \frac{n}{2}$ is dynamically stable.

(d) $\eta^0_{A,1} = \eta^0_{A,2} = \frac{n}{4}$ and $\eta^0_{B,1} = \eta^0_{B,2} = \frac{n}{4}$, then for any player $i \in N^0_{A,1}$,

$$C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0) = -\frac{1}{2} < 0.$$

Let players from $N^0_{A,1}$ be chosen to make actions consequently, then partition $\Delta^{\eta^0_{A,1}}$ satisfying $\eta^{\eta^0_{A,1}}_{A,2} = \frac{n}{2}$ emerges and it is dynamically stable.

- $f(1) - f(0) > 1 - \frac{2}{n}$, and

1. $\eta^0_{A,1} = \eta^0_{B,2} = \frac{n}{2}$, then for any player $i \in A$,

$$C_i(g^b, \Delta[i,2]) - C_i(g^b, \Delta) = \frac{n}{2}\Big(f(0) - f(1) + 1 - \frac{2}{n}\Big) < 0.$$

Then partition $\Delta^{\frac{n}{2}}$, such that $\eta^{\frac{n}{2}}_{A,2} = \eta^{\frac{n}{2}}_{B,2} = \frac{n}{2}$, emerges after players from set A are selected consequently to make their actions. Thus, for any player $i \in N$,

$$C_i(g^b, \Delta^{\frac{n}{2}}[i,1]) - C_i(g^b, \Delta^{\frac{n}{2}}) = \frac{n}{2}\Big(f(1) - f(0) - 1 + \frac{2}{n}\Big) > 0,$$

which indicates $\Delta^{\frac{n}{2}}$ is dynamically stable.

It is evident that given any initial partition, as long as a certain partition Δ, such that $\eta_1 = n$ or $\eta_2 = n$ (i.e., all players belong to the same group), emerges after some players make actions consequently and it is dynamically stable.

2. $\eta^0_{A,1} = \eta^0_{B,1} = \frac{n}{2}$, then from the statement given in Item 1 it follows that Δ^0 is dynamically stable.

3. $\eta^0_{A,1} = \frac{n}{2}$, $\eta^0_{B,1} > 0$, $\eta^0_{B,2} > 0$, then for any player $i \in N^0_{B,2}$,

$$C_i(g^b, \Delta^0[i,1]) - C_i(g^b, \Delta^0) = \frac{n}{2}\Big(f(0) - f(1) + 1 - \frac{2}{n}\Big) < 0,$$

then choosing players from $N^0_{B,2}$ to make actions, partition $\Delta^{\eta^0_{B,2}}$ such that $\eta^{\eta^0_{B,2}}_1 = n$ is dynamically stable.

4. $\eta^0_{A,1}, \eta^0_{A,2}, \eta^0_{B,1}, \eta^0_{B,2} > 0$, and

(a) $\eta^0_{A,1} > \eta^0_{A,2}$ and $\eta^0_{B,1} > \eta^0_{B,2}$, then as above, the relation between player $i \in N^0_{A,1}$ and $j \in N^0_{A,2}$ is represented by equation (D.5). Since $f(1) - f(0) > 1 - \frac{2}{n}$ as well as $\frac{n}{4} < \eta^0_{B,1} < \frac{n}{2}$, we get

$$C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0) > -\frac{2(\eta^0_{B,1})^2}{n} + \frac{3\eta^0_{B,1}}{2} - \frac{n}{4} > 0.$$

As a result, $C_j(g^b, \Delta^0[j,1]) - C_j(g^b, \Delta^0) < 0$, and the players from $N^0_{A,2}$ are chosen to make actions, and they all deviate to the other group. We continue a construction given in Item 3, and obtain a dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_{A,1} = \bar{\eta}_{B,1} = \frac{n}{2}$.

(b) $\eta^0_{A,1} > \eta^0_{A,2}$ and $\eta^0_{B,1} < \eta^0_{B,2}$. Without loss of generality, we assume $\eta^0_{B,2} \geq \eta^0_{A,1}$, and the case when $\eta^0_{B,2} < \eta^0_{A,1}$ can be similarly examined. For any players $i \in N^0_{A,1}$ and $j \in N^0_{A,2}$, their relation is presented by (D.7) if $\eta^0_{B,2} - 1 \geq \eta^0_{A,1}$, and by (D.8) if $\eta^0_{B,2} = \eta^0_{A,1}$. If $C_j(g^b, \Delta^0[j,1]) - C_j(g^b, \Delta^0) < 0$, we choose players from $N^0_{A,2}$ to make their actions, then we make a construction in the same way as in Item 3. Finally, a dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_{A,1} = \bar{\eta}_{B,1} = \frac{n}{2}$ emerges. While if (D.6) < 0, let players from $N^0_{A,1}$ choose actions, then ordering them in the action-making sequence as in Item 3, we obtain that a dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_{A,2} = \bar{\eta}_{B,2} = \frac{n}{2}$ is generated.

(c) $\eta^0_{A,1} = \eta^0_{A,2} = \frac{n}{4}$ and $\eta^0_{B,1} > \eta^0_{B,2}$, then for any player $j \in N^0_{A,2}$, the expression (D.10) is negative. We choose players from $N^0_{A,2}$ to make actions, then continue with the construction like in Item 3. As a result, a dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_{A,1} = \bar{\eta}_{B,1} = \frac{n}{2}$ appears.

(d) $\eta^0_{A,1} = \eta^0_{A,2} = \frac{n}{4}$ and $\eta^0_{B,1} = \eta^0_{B,2} = \frac{n}{4}$, then for any player $i \in N^0_{A,1}$,

$$C_i(g^b, \Delta^0[i,2]) - C_i(g^b, \Delta^0) = -\frac{1}{2}.$$

Let players from set $N^0_{A,1}$ choose their actions, then taking into account Item 3, a dynamically stable partition $\bar{\Delta}$ such that $\bar{\eta}_{A,2} = \bar{\eta}_{B,2} = \frac{n}{2}$ is finally generated.

Thus, we obtain that in any case examined above, there exists a dynamically stable partition. Moreover, we have formulated the order of action-making for the players in such a way that a particular partition structure which meets the corresponding condition above appears and we have checked that it is dynamically stable. To complete the proof, it suffices to verify that any partition which does not satisfy the specified condition is not dynamically stable. Indeed, in the proof above, given any initial partition Δ^0 such that the condition for the dynamically stable partition is not met, it has been shown that there exists at least one player choosing to deviate when he is selected to make an action. This implies that Δ^0 is not dynamically stable. □

Appendix E: Proof of Corollary 1

Proof. With partition $\Delta = \{N_1, \ldots, N_m\}$, let $M_1 = \big\{k \in \{1, \ldots, m\} \mid N_{A,k} \neq \varnothing\big\}$, $M_2 = \big\{k \in \{1, \ldots, m\} \mid N_{B,k} \neq \varnothing\big\}$, and $m_1 = |M_1|$, $m_2 = |M_2|$. It is obvious that $m_1 \leq m$ and $m_2 \leq m$.

Consider initial partition Δ^0. If any player $i \in N$ chooses not to deviate to any other group when she is selected to make an action, then Δ^0 is exactly the dynamically stable partition. Otherwise, we particularly formulate the process (both the players' order of action-making and the specified choices of actions) in such a way that a dynamically stable partition appears after several stages when the process starts from the unstable partition. We should mention that both the players' order of action-making and specified action choices may materialize with a positive probability since the dynamic process is random and infinite.

Assume Δ^0 is not dynamically stable. Without loss of generality, let k be such that for any $i \in N^0_{A,k}$,

$$\min_{k'} C_i(g^b, \Delta^0[i,k']) - C_i(g^b, \Delta^0) < 0,$$

i.e., each player $i \in N^0_{A,k}$ chooses to deviate from the current group N^0_k to another group labeled $\bar{k}$ where $\bar{k} \in \arg\min_{k'} C_i(g^b, \Delta^0[i,k'])$ when he is selected to make an action (if such k can not be found with respect to set A, we consider set B instead). Let players from set $N^0_{A,k}$ be selected consequently. Moreover, if $\arg\min_{k'} C_i(g^b,\Delta^0[i,k']) \cap M^0_1 \neq \varnothing$, those players are assumed to deviate to one group labeled $\bar{k} \in \arg\min_{k'} C_i(g^b,\Delta^0[i,k']) \cap M^0_1$ unanimously. Otherwise, they uniformly deviate to any group $\bar{k}$ such that $\bar{k} \in \arg\min_{k'} C_i(g^b,\Delta^0[i,k'])$. Let $\hat{\Delta}$ be the partition generated after the procedures designed above are realized, i.e., after the last player from $N^0_{A,k}$ makes an action. We can easily obtain that $\hat{m}_1 \leq m^0_1$ and $\hat{m}_2 \leq m^0_2$. If $\hat{\Delta}$ is dynamically stable, then the whole process is completely designed, otherwise, $\hat{\Delta}$ is regarded as the initial partition and we repeat the above procedures.

We repeat the described procedures and stop the process when one of the two cases materialized: (i) a dynamically stable partition appears; (ii) a partition $\hat{\Delta}$ satisfying $\max\{\hat{m}_1,\hat{m}_2\} \leq 2$ emerges. We can easily realize that at least one of the two cases may terminate the process. If the process ends with a partition $\hat{\Delta}$ satisfying the specified condition, and we take it as an initial one, then by Theorem 4, a dynamically stable partition certainly appears. □

Appendix F: Proof of Theorem 5

Proof. We first show that all players from set A or B belong to the same group in a dynamically stable partition when condition $f(1)-f(0) < 1-\frac{2}{n}$ is met. The proof is carried out in such a way that we come to a contradiction for the partition which is supposed to be dynamically stable but does not meet the condition. Significantly, it is worthwhile remarked that the same conclusion can be drawn for $f(1)-f(0) \geq 1-\frac{2}{n}$ since one can find certain contradictions when he goes through the proof process assuming $f(1)-f(0) \geq 1-\frac{2}{n}$.

Suppose partition $\Delta = \{N_1,\ldots,N_m\}$ is dynamically stable, and let $P^{\Delta}_A = \{k \mid \exists\ i \in A, i \in N_{A,k}\}$ denote the set of labels for groups to which at least one player from set A belongs, $P^{\Delta}_B = \{k \mid \exists\ i \in B, i \in N_{B,k}\}$, $p^{\Delta}_A = |P^{\Delta}_A|$, $p^{\Delta}_B = |P^{\Delta}_B|$.

Several possible structures (except the given one) for partition Δ are consequently examined:

1. $P^{\Delta}_A \cap P^{\Delta}_B = \varnothing$. Let k_1, k_2 be such that $\eta_{A,k_1} = \max_{k\in P^{\Delta}_A} \eta_{A,k}$, $\eta_{B,k_2} = \max_{k\in P^{\Delta}_B} \eta_{B,k}$, and without loss of generality, $\eta_{A,k_1} \geq \eta_{B,k_2}$. Then for any player from set $N_{A,k'}$, $k' \neq k_1$, we have

$$C_i(g^b,\Delta[i,k_1]) - C_i(g^b,\Delta) = \sum_{k\in P^{\Delta}_B} 2\eta_{B,k}\Big[\frac{\max\{\eta_{B,k},\eta_{A,k'}\}-1-\eta_{A,k_1}}{n}\Big] < 0,$$

   a contradiction with the stability of partition Δ.

2. $P^{\Delta}_A \cap P^{\Delta}_B = \{k\}$, $P^{\Delta}_A = \{k_1,k\}$, and $P^{\Delta}_B = \{k_2,k\}$. Without loss of generality, $\eta_{B,k} \geq \eta_{A,k}$, then $\eta_{B,k_2} \leq \eta_{A,k_1}$. Two cases are discussed:

   (a) $\eta_{B,k} \geq \eta_{A,k_1}$. For any $i \in N_{B,k_2}$, we have

$$\begin{aligned}&C_i(g^b,\Delta[i,k]) - C_i(g^b,\Delta)\\ &= \eta_{A,k_1}\Big(f(1)-\frac{2\eta_{B,k}}{n}\Big) + \eta_{A,k}f(0) - \eta_{A,k_1}\Big(f(1)-\frac{2\eta_{A,k_1}-2}{n}\Big) - \eta_{A,k}\Big(f(1)-\frac{2\eta_{A,k}-2}{n}\Big) \geq 0.\end{aligned}$$

   Since $f(1)-\frac{2\eta_{B,k}}{n} < f(1)-\frac{2\eta_{A,k_1}-2}{n}$, we obtain $f(0) > f(1)-\frac{2\eta_{A,k}-2}{n} \geq f(1)-\frac{2\eta_{B,k}-2}{n}$. Then,

i. if $\eta_{B,k} > \eta_{B,k_2}$, for any $j \in N_{A,k}$,

$$
\begin{aligned}
& C_j(g^b,\Delta[j,k_2]) - C_j(g^b,\Delta) \\
&= \eta_{B,k}\Big(f(1) - \frac{2\eta_{B,k}-2}{n}\Big) + \eta_{B,k_2} f(0) - \eta_{B,k_2}\Big(f(1) - \frac{2\eta_{A,k}-2}{n}\Big) - \eta_{B,k} f(0) \\
&= (\eta_{B,k} - \eta_{B,k_2})(f(1) - f(0)) + \frac{2(\eta_{B,k} - \eta_{B,k}\eta_{B,k} + \eta_{B,k_2}\eta_{A,k} - \eta_{B,k_2})}{n} \\
&< \frac{(\eta_{B,k} - \eta_{B,k_2})(2\eta_{B,k}-2)}{n} + \frac{2(\eta_{B,k} - \eta_{B,k}\eta_{B,k} + \eta_{B,k_2}\eta_{A,k} - \eta_{B,k_2})}{n} \\
&= \frac{2\eta_{B,k_2}(\eta_{A,k} - \eta_{B,k})}{n} \le 0,
\end{aligned}
$$

which contradicts the stability of partition Δ.

ii. if $\eta_{B,k} = \eta_{B,k_2} (= \eta_{A,k} = \eta_{A,k_1})$, for any $j \in N_{A,k}$,

$$
\begin{aligned}
& C_j(g^b,\Delta[j,k_1]) - C_j(g^b,\Delta) \\
&= \eta_{B,k_2}\Big(f(1) - \frac{2\eta_{A,k_1}}{n}\Big) + \eta_{B,k}\Big(f(1) - \frac{2\eta_{A,k_1}}{n}\Big) - \eta_{B,k_2}\Big(f(1) - \frac{2\eta_{A,k_1}-2}{n}\Big) - \eta_{B,k} f(0).
\end{aligned}
$$

Since $f(1) - \frac{2\eta_{A,k_1}}{n} < f(0)$, we easily get $C_j(g^b,\Delta[j,k_1]) - C_j(g^b,\Delta) < 0$, which contradicts the stability of partition Δ.

(b) $\eta_{B,k} < \eta_{A,k_1}$. For any $i \in N_{A,k}$,

$$
\begin{aligned}
& C_i(g^b,\Delta[i,k_1]) - C_i(g^b,\Delta) \\
&= \eta_{B,k_2}\Big(f(1) - \frac{2\eta_{A,k_1}}{n}\Big) + \eta_{B,k}\Big(f(1) - \frac{2\eta_{A,k_1}}{n}\Big) - \eta_{B,k_2}\Big(f(1) - \frac{2\eta_{B,k_2}-2}{n}\Big) - \eta_{B,k} f(0) \ge 0,
\end{aligned}
$$

from which, we easily obtain that $f(1) - \frac{2\eta_{B,k_2}}{n} \ge f(1) - \frac{2\eta_{A,k_1}}{n} > f(0)$. Then, for any player $j \in N_{B,k_2}$,

$$
C_j(g^b,\Delta[j,k]) - C_j(g^b,\Delta) = \eta_{A,k} f(0) - \eta_{A,k}\Big(f(1) - \frac{2\eta_{B,k_2}-2}{n}\Big) < 0,
$$

which contradicts the stability of partition Δ.

3. $P_A^\Delta \cap P_B^\Delta = \{k\}$, and $\max\{p_A^\Delta, p_B^\Delta\} \ge 3$. Without loss of generality, let $\{k,k_1,k_2\} \subseteq P_A^\Delta$ and $\{k,k_3\} \subseteq P_B^\Delta$. All possible items in this case are then considered:

(a) $\eta_{A,k_2} = \max\limits_{k' \in P_A^\Delta} \eta_{A,k'}$. Then for any $i \in N_{A,k_1}$, from $C_i(g^b,\Delta[i,k_2]) - C_i(g^b,\Delta) \ge 0$, we obtain that for any $k' \in P_B^\Delta$: $\eta_{B,k'} \ge \eta_{A,k_2} + 1 \ge \eta_{A,k} + 1$. Consider any player $i \in N_{A,k}$,

$$
C_i(g^b,\Delta[i,k_2]) - C_i(g^b,\Delta) = \eta_{B,k}\Big(f(1) - \frac{2\eta_{B,k}-2}{n}\Big) - \eta_{B,k} f(0) \ge 0,
$$

Therefore, $f(0) \le f(1) - \frac{2\eta_{B,k}-2}{n}$. Then we examine the following subcases:

i. $f(0) < f(1) - \frac{2\eta_{B,k}-2}{n}$, then for any $i \in N_{A,k_2}$,

$$C_i(g^b,\Delta[i,k]) - C_i(g^b,\Delta) = \eta_{B,k} f(0) - \eta_{B,k}\Big(f(1) - \frac{2\eta_{B,k}-2}{n}\Big) < 0,$$

which contradicts the stability of partition Δ.

ii. $f(0) = f(1) - \frac{2\eta_{B,k}-2}{n}$, and

A. $\exists\, \hat{k} \in P_B^{\Delta}, \hat{k} \neq k$ such that $\eta_{B,\hat{k}} \geq \eta_{B,k}$, then for any $i \in N_{B,k}$,

$$\begin{aligned} & C_i(g^b,\Delta[i,\hat{k}]) - C_i(g^b,\Delta) \\ &= \sum_{\substack{k' \in P_A^{\Delta} \\ k' \neq k}} \eta_{A,k'}\Big(f(1) - \frac{2\eta_{B,\hat{k}}}{n}\Big) + \eta_{A,k}\Big(f(1) - \frac{2\eta_{B,\hat{k}}}{n}\Big) - \sum_{\substack{k' \in P_A^{\Delta} \\ k' \neq k}} \eta_{A,k'}\Big(f(1) - \frac{2\eta_{B,k}-2}{n}\Big) \\ &- \eta_{A,k} f(0). \end{aligned}$$

Since $\frac{2\eta_{B,\hat{k}}}{n} > \frac{2\eta_{B,k}-2}{n}$ together with $f(0) = f(1) - \frac{2\eta_{B,k}-2}{n} > f(1) - \frac{2\eta_{B,\hat{k}}}{n}$, we get

$$C_i(g^b,\Delta[i,\hat{k}]) - C_i(g^b,\Delta) < 0,$$

which contradicts the stability of partition Δ.

B. for any $k' \in P_B^{\Delta}, k' \neq k$, $\eta_{B,k} > \eta_{B,k'}$, then for any $i \in N_{B,k_3}$,

$$\begin{aligned} & C_i(g^b,\Delta[i,k]) - C_i(g^b,\Delta) \\ &= \sum_{\substack{k' \in P_A^{\Delta} \\ k' \neq k}} \eta_{A,k'}\Big(f(1) - \frac{2\eta_{B,k}}{n}\Big) + \eta_{A,k} f(0) - \sum_{\substack{k' \in P_A^{\Delta} \\ k' \neq k}} \eta_{A,k'}\Big(f(1) - \frac{2\eta_{B,k_3}-2}{n}\Big) \\ &- \eta_{A,k}\Big(f(1) - \frac{2\eta_{B,k_3}-2}{n}\Big). \end{aligned}$$

Since $\frac{2\eta_{B,k}}{n} > \frac{2\eta_{B,k_3}-2}{n}$ as well as $f(0) = f(1) - \frac{2\eta_{B,k}-2}{n} < f(1) - \frac{2\eta_{B,k_3}}{n}$, we get

$$C_i(g^b,\Delta[i,k]) - C_i(g^b,\Delta) < 0,$$

which contradicts the stability of partition Δ.

(b) $\eta_{A,k} > \eta_{A,k'}$ for any $k' \in P_A^{\Delta}$, $k' \neq k$. Then for any $i \in N_{A,k_1}$, from $C_i(g^b,\Delta[i,k]) - C_i(g^b,\Delta) \geq 0$, we directly obtain that $f(0) \geq f(1) - \frac{2\eta_{B,k}-2}{n-2}$. Consider the following subcases:

i. $f(0) = f(1) - \frac{2\eta_{B,k}-2}{n}$. Then for any $i \in N_{A,k_2}$, from $C_i(g^b,\Delta[i,k]) - C_i(g^b,\Delta) \geq 0$, we get that for any $k' \in P_B^{\Delta}$, $k' \neq k$, $\eta_{B,k'} \geq \eta_{A,k} + 1$. If

A. $\exists\, \hat{k} \in P_B^{\Delta}, \hat{k} \neq k$ such that $\eta_{B,\hat{k}} \geq \eta_{B,k}$, then for any $i \in N_{B,k}$, we may directly get that $C_i(g^b,\Delta[i,\hat{k}]) - C_i(g^b,\Delta) < 0$, which contradicts the stability of partition Δ.

B. $\eta_{B,k} > \eta_{B,k'}$ for any $k' \in P_B^{\Delta}, k' \neq k$, then we may easily obtain that for any $i \in N_{B,k'}$, $C_i(g^b,\Delta[i,k]) - C_i(g^b,\Delta) < 0$, which contradicts the stability of partition Δ.

ii. $f(0) > f(1) - \frac{2\eta_{B,k}-2}{n}$, and if

A. $P_B^\Delta = \{k_3, k\}$, then $\eta_{B,k_3} < \eta_{A,k}$. Since if not, for any $i \in N_{A,k}$,

$$C_i(g^b, \Delta[i,k_2]) - C_i(g^b, \Delta) = \eta_{B,k}\Big(f(1) - \frac{\max\{2\eta_{B,k_3} - 2, 2\eta_{A,k_2}\}}{n}\Big) - \eta_{B,k} f(0) < 0.$$

Then consider two cases, $\eta_{B,k} \geq \eta_{A,k}$ or $\eta_{B,k} < \eta_{A,k}$. In the first one, for any $i \in N_{A,k}$,

$$\begin{aligned}
&C_i(g^b, \Delta[i,k_3]) - C_i(g^b, \Delta) \\
&= \eta_{B,k_3} f(0) + \eta_{B,k}\Big(f(1) - \frac{2\eta_{B,k}-2}{n}\Big) - \eta_{B,k_3}\Big(f(1) - \frac{2\eta_{A,k}-2}{n}\Big) - \eta_{B,k} f(0) \\
&\leq \eta_{B,k_3} f(0) + \eta_{B,k}\Big(f(1) - \frac{2\eta_{B,k}-2}{n}\Big) - \eta_{B,k_3}\Big(f(1) - \frac{2\eta_{B,k}-2}{n}\Big) - \eta_{B,k} f(0) \\
&= (\eta_{B,k} - \eta_{B,k_3})\Big(f(1) - \frac{2\eta_{B,k}-2}{n} - f(0)\Big) < 0,
\end{aligned}$$

which contradicts the stability of partition Δ. In the second one, for any $i \in N_{B,k_3}$ and $j \in N_{B,k}$,

$$\begin{aligned}
C_i(g^b, \Delta[i,k]) - C_i(g^b, \Delta) &= \sum_{\substack{k' \in P_A^\Delta \\ k' \neq k}} \eta_{A,k'}\Big(f(1) - \frac{2\eta_{B,k}}{n}\Big) + \eta_{A,k} f(0) \\
&\quad - \sum_{\substack{k' \in P_A^\Delta \\ k' \neq k}} \eta_{A,k'}\Big(f(1) - \frac{2\eta_{B,k_3}-2}{n}\Big) - \eta_{A,k}\Big(f(1) - \frac{2\eta_{A,k}-2}{n}\Big), \\
C_j(g^b, \Delta[j,k_3]) - C_j(g^b, \Delta) &= \sum_{\substack{k' \in P_A^\Delta \\ k' \neq k}} \eta_{A,k'}\Big(f(1) - \frac{2\eta_{B,k_3}}{n}\Big) + \eta_{A,k}\Big(f(1) - \frac{2\eta_{A,k}-2}{n}\Big) \\
&\quad - \sum_{\substack{k' \in P_A^\Delta \\ k' \neq k}} \eta_{A,k'}\Big(f(1) - \frac{2\eta_{B,k}-2}{n}\Big) - \eta_{A,k} f(0).
\end{aligned}$$

Therefore, the following relation can be obtained:

$$C_i(g^b, \Delta[i,k]) - C_i(g^b, \Delta) = -\Big[C_j(g^b, \Delta[j,k_3]) - C_j(g^b, \Delta)\Big] - \sum_{\substack{k' \in P_A^\Delta \\ k' \neq k}} \frac{4\eta_{A,k'}}{n}.$$

Then from $C_j(g^b, \Delta[j,k_3]) - C_j(g^b, \Delta) \geq 0$, we get $C_i(g^b, \Delta[i,k]) - C_i(g^b, \Delta) < 0$, which contradicts the stability of partition Δ.

B. $\{k_3, k_4, k\} \subseteq P_B^\Delta$. Without loss of generality, $\eta_{B,k_3} \leq \eta_{B,k_4}$, then we immediately obtain that for any player $i \in N_{B,k_3}$, $C_i(g^b, \Delta[i,k_4]) - C_i(g^b, \Delta) < 0$.

4. $|P_A^\Delta \cap P_B^\Delta| \geq 2$. Without loss of generality, let $\{k_1,k_2,k_3\} \subseteq P_A^\Delta$ and $\{k_1,k_2\} \subseteq P_B^\Delta$. For any $i \in N_{A,k_1}$, $j \in N_{A,k_2}$,

$$C_i(g^b,\Delta[i,k_2]) - C_i(g^b,\Delta) = \sum_{\substack{k' \in P_B^\Delta \\ k' \neq k_1,k_2}} \eta_{B,k'} \Big(f(1) - \frac{\max\{2\eta_{B,k'}-2, 2\eta_{A,k_2}\}}{n} \Big) + \eta_{B,k_2} f(0)$$
$$+ \eta_{B,k_1} \Big(f(1) - \frac{\max\{2\eta_{B,k_1}-2, 2\eta_{A,k_2}\}}{n} \Big) - \sum_{\substack{k' \in P_B^\Delta \\ k' \neq k_1,k_2}} \eta_{B,k'} \Big(f(1) - \frac{\max\{2\eta_{B,k'}-2, 2\eta_{A,k_1}-2\}}{n} \Big)$$
$$- \eta_{B,k_2} \Big(f(1) - \frac{\max\{2\eta_{A,k_1}-2, 2\eta_{B,k_2}-2\}}{n} \Big) - \eta_{B,k_1} f(0),$$

$$C_j(g^b,\Delta[j,k_1]) - C_j(g^b,\Delta) = \sum_{\substack{k' \in P_B^\Delta \\ k' \neq k_1,k_2}} \eta_{B,k'} \Big(f(1) - \frac{\max\{2\eta_{B,k'}-2, 2\eta_{A,k_1}\}}{n} \Big) + \eta_{B,k_1} f(0)$$
$$+ \eta_{B,k_2} \Big(f(1) - \frac{\max\{2\eta_{B,k_2}-2, 2\eta_{A,k_1}\}}{n} \Big) - \sum_{\substack{k' \in P_B^\Delta \\ k' \neq k_1,k_2}} \eta_{B,k'} \Big(f(1) - \frac{\max\{2\eta_{B,k'}-2, 2\eta_{A,k_2}-2\}}{n} \Big)$$
$$- \eta_{B,k_1} \Big(f(1) - \frac{\max\{2\eta_{B,k_1}-2, 2\eta_{A,k_2}-2\}}{n} \Big) - \eta_{B,k_2} f(0)$$

Respectively let

$$\bar{C}_i = \sum_{\substack{k' \in P_B^\Delta \\ k' \neq k_1,k_2}} \frac{\eta_{B,k'} \Big[\max\{2\eta_{B,k'}-2, 2\eta_{A,k_1}-2\} - \max\{2\eta_{B,k'}-2, 2\eta_{A,k_2}\} \Big]}{n},$$
$$\bar{C}_j = \sum_{\substack{k' \in P_B^\Delta \\ k' \neq k_1,k_2}} \frac{\eta_{B,k'} \Big[\max\{2\eta_{B,k'}-2, 2\eta_{A,k_2}-2\} - \max\{2\eta_{B,k'}-2, 2\eta_{A,k_1}\} \Big]}{n},$$
$$\bar{\bar{C}}_i = \eta_{B,k_1} \Big(f(1) - \frac{\max\{2\eta_{B,k_1}-2, 2\eta_{A,k_2}\}}{n} \Big) - \eta_{B,k_2} \Big(f(1) - \frac{\max\{2\eta_{A,k_1}-2, 2\eta_{B,k_2}-2\}}{n} \Big),$$
$$\bar{\bar{C}}_j = \eta_{B,k_2} \Big(f(1) - \frac{\max\{2\eta_{B,k_2}-2, 2\eta_{A,k_1}\}}{n} \Big) - \eta_{B,k_1} \Big(f(1) - \frac{\max\{2\eta_{B,k_1}-2, 2\eta_{A,k_2}-2\}}{n} \Big).$$

Thus

$$C_i(g^b,\Delta[i,k_2]) - C_i(g^b,\Delta) = \bar{C}_i + \bar{\bar{C}}_i + \eta_{B,k_2} f(0) - \eta_{B,k_1} f(0),$$
$$C_j(g^b,\Delta[j,k_1]) - C_j(g^b,\Delta) = \bar{C}_j + \bar{\bar{C}}_j + \eta_{B,k_1} f(0) - \eta_{B,k_2} f(0).$$

First, observe $\bar{C}_i$ and $\bar{C}_j$. It is easily obtained that $\bar{C}_i \leq -\bar{C}_j$. Then observe $\bar{\bar{C}}_i$ and $\bar{\bar{C}}_j$. Since

$$\begin{cases} \max\{2\eta_{B,k_1}-2, 2\eta_{A,k_2}\} \geq \max\{2\eta_{B,k_1}-2, 2\eta_{A,k_2}-2\}, \\ \max\{2\eta_{A,k_1}-2, 2\eta_{B,k_2}-2\} \leq \max\{2\eta_{B,k_2}-2, 2\eta_{A,k_1}\}, \end{cases}$$

then $\bar{\bar{C}}_i \leq -\bar{\bar{C}}_j$. If

$$\left\{\begin{array}{l} \max\{2\eta_{B,k_1}-2, 2\eta_{A,k_2}\} = \max\{2\eta_{B,k_1}-2, 2\eta_{A,k_2}-2\}, \\ \max\{2\eta_{A,k_1}-2, 2\eta_{B,k_2}-2\} = \max\{2\eta_{B,k_2}-2, 2\eta_{A,k_1}\}, \end{array}\right. \tag{F.1}$$

then

$$\left\{\begin{array}{l} \eta_{B,k_1} \geq \eta_{A,k_2}+1, \\ \eta_{B,k_2} \geq \eta_{A,k_1}+1. \end{array}\right. \tag{F.2}$$

When the same examination is conducted for $i^{'} \in N_{B,k_1}$ and $j^{'} \in N_{B,k_2}$, we similarly obtain

$$\left\{\begin{array}{l} \eta_{A,k_1} \geq \eta_{B,k_2}+1, \\ \eta_{A,k_2} \geq \eta_{B,k_1}+1. \end{array}\right. \tag{F.3}$$

Nevertheless, (F.2) and (F.3) can not hold at the same time.

While if at least one equality in the system (F.1) does not hold, then $\bar{\bar{C}}_i < -\bar{\bar{C}}_j$, from which it follows that

$$C_i(g^b, \Delta[i,k_2]) - C_i(g^b, \Delta) < -\Big[C_j(g^b, \Delta[j,k_1]) - C_j(g^b, \Delta)\Big].$$

With $C_j(g^b, \Delta[j,k_1]) - C_j(g^b, \Delta) \geq 0$, contradiction $C_i(g^b, \Delta[i,k_2]) - C_i(g^b, \Delta) < 0$ is obtained.

Based on the conclusion above, without loss of generality, we assume that all players from set A belong to the group labeled k. Below, we first demonstrate that all players from set B also belong to the group labeled k when $f(1) - f(0) > 1 - \frac{2}{n}$ is satisfied, and no player from B belongs to such a group when $f(1) - f(0) < 1 - \frac{2}{n}$.

When $f(1) - f(0) > 1 - \frac{2}{n}$, we suppose there exists a player $i \in N_{B,k_1}$, $k_1 \neq k$, then

$$C_i(g^b, \Delta[i,k]) - C_i(g^b, \Delta) = \frac{n}{4}\Big(f(1) + \frac{2}{n} - 1 - f(0)\Big) < 0,$$

which is a contradiction. Therefore, all players from both sets A and B belong to the group labeled k.

When $f(1) - f(0) < 1 - \frac{2}{n}$, suppose there exists player $i \in N_{B,k}$, then for any $k_1 \neq k$,

$$C_i(g^b, \Delta[i,k_1]) - C_i(g^b, \Delta) = \frac{n}{4}\Big(f(1) + \frac{2}{n} - 1 - f(0)\Big) < 0,$$

which is a contradiction. Therefore, no player from set B belongs to the group labeled k. □