UDC 517.977.56 Вестник СПбГУ. Прикладная математика. Информатика... 2021. Т. 17. Вып. 4
MSC 49N10

Optimal control of a differential-difference parabolic system with distributed parameters on the graph

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For citation: Zhabko A. P., Provotorov V. V., Shindyapin A. I. Optimal control of a differential-difference parabolic system with distributed parameters on the graph. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 2021, vol. 17, iss. 4, pp. 433–448. https://doi.org/10.21638/11701/spbu10.2021.411

In the paper be considered the problem of optimal control of the differential-difference equation with distributed parameters on the graph in the class of summable functions. Particular attention is given to the connection of the differential-differential system with the evolutionary differential system and the search conditions in which the properties of the differential system are preserved. This connection establishes a universal method of semi-digitization by temporal variable for differential system, providing an effective tool in finding conditions of uniqueness solvability and continuity on the initial data for the differential-differential system. A priori estimates of the norms of a weak solution of differential-differential system give an opportunity to establish not only the solvability of this system but also the existence of a weak solution of the evolutionary differential system. For the differential-difference system analysis of the optimal control problem is presented, containing natural in that cases a additional study of the problem with a time lag. This essentially uses the conjugate state of the system and the conjugate system for a differential-difference system — defining ratios that determine optimal control or the set optimal controls. The work shows courses to transfer the results in case of analysis of optimal control problems in the class of functions with bearer in network-like domains. The transition from an evolutionary differential system to a differential-difference system was a natural step in the study of applied problems of the theory of the transfer of solid mediums. The obtained results underlie the analysis of optimal control problems for differential systems with distributed parameters on a graph, which have interesting analogies with multiphase problems of multidimensional hydrodynamics.

Keywords: differential-difference system, conjugate system, oriented graph, optimal control, delay.

1. Introduction. The problems of optimal control of differential systems with distributed parameters on the graph were considered by the authors in the works [1–4]. In addition related problems were also studied: stability on Lyapunov and Neumann, stabilization of weak solutions, temporal delay [5–9]. The transition to differential-difference systems was the next natural step of the study, namely, an attempt to move closer to solving applied problems that have their own specifics. Particular attention is give to the relations of the differential-difference system with the differential system and the search for conditions for which the properties of the differential system are preserved. The semi-digitization method used is a universal method that provides an effective tool for finding conditions of uniqueness solvability and continuity on the initial data for a differential-differential system. The analysis of the problem of optimal control of the differential-difference system contain a additional study of the problem with temporary delay in such cases. The work also shows ways to transfer the results in case of analysis of optimal control problems with carrier in network-like domains.

2. Basic concepts, definitions and affirmations. Let Γ is a oriented bounded graph whose edges are parameterized by a segment $[0,1]$; Γ_0 is a set of all ribs that do not contain their endpoints: $\overline{\Gamma}_0 = \Gamma$, $\Gamma_T = \Gamma_0 \times (0,T)$.

We will use standard notations for the spaces of Lebesque and Sobolev:

- $L_p(\Gamma)$ $(p = 1, 2)$ is a Banach space of measurable functions on Γ_0, integrable with degree of order p (similarly defined the space $L_p(\Gamma_T)$);
- $W_2^1(\Gamma)$ is the space of functions from $L_2(\Gamma)$, with generalized derivative of order 1 also from $L_2(\Gamma)$;
- $L_{2,1}(\Gamma_T)$ is the space of functions from $L_1(\Gamma_T)$ with norm defined by ratio $\|u\|_{L_{2,1}(\Gamma_T)} = \int\limits_0^T (\int\limits_\Gamma u^2 dx)^{\frac{1}{2}} dt$;
- $W_2^{1,0}(\Gamma_T)$ is the space of functions from $L_2(\Gamma_T)$ with generalized derivative of order 1 for x belonging to space $L_2(\Gamma_T)$ (similarly defined the space $W_2^1(\Gamma_T)$).

In the domain Γ_T consider the parabolic equation

$$\frac{\partial y(x,t)}{\partial t} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial y(x,t)}{\partial x}\right) + b(x)y(x,t) = f(x,t), \quad x,t \in \Gamma_T, \tag{1}$$

with measurable and limited by Γ_0 coefficients $a(x)$, $b(x)$; $f(x,t) \in L_{2,1}(\Gamma_T)$.

Semi-digitization by temporal variable t (Rothe method [10]) applied to the equation (1) reduce to a differential-difference equation

$$\frac{1}{\tau}(y(k) - y(k-1)) - \frac{d}{dx}\left(a(x)\frac{dy(k)}{dx}\right) + b(x)y(k) = f_\tau(k), \quad k = 1, 2, ..., M, \tag{2}$$

where $y(k) := y(x;k)$ and $f_\tau(k) = \frac{1}{\tau}\int\limits_{(k-1)\tau}^{k\tau} f(x,t)dt \in L_2(\Gamma)$, $k = 1, 2, ..., M$.

Let's introduce the spaces of the states $y(x,t)$ of the equation (1) and $y(k) := y(x;k)$ $(k = 1, 2, ..., M)$ equation (2). Let's designate through $\Omega_a(\Gamma)$ a set of differentiable functions $y(x)$ that satisfy the relations

$$\sum_{\gamma\in R(\xi)} a(1)_\gamma \frac{dy(1)_\gamma}{dx} = \sum_{\gamma\in r(\xi)} a(0)_\gamma \frac{dy(0)_\gamma}{dx}$$

in all nodes $\xi \in J(\Gamma)$ (in here $R(\xi)$ and $r(\xi)$ as the sets of the edges γ respectively oriented "to node ξ" and "from node ξ", symbol $\theta(\cdot)_\gamma$ designated the narrowing of the function $\theta(\cdot)$ on the edge γ) and $u(x)|_{\partial\Gamma} = 0$. The closing of the set $\Omega_a(\Gamma)$ in norm $W_2^1(\Gamma)$ relabel $W_0^1(a;\Gamma)$.

Let the next $\Omega_a(\Gamma_T)$ is the set of functions $y(x,t) \in W_2^{1,0}(\Gamma_T)$, whose traces $u(x,t_0)$ are defined in sections of the domain Γ_T the plane $t = t_0$ $(t_0 \in (0,T))$ as a function of class $W_0^1(a;\Gamma)$. Closing the set $\Omega_a(\Gamma_T)$ by the norm $W_2^{1,0}(\Gamma_T)$ mark through $W_0^{1,0}(a;\Gamma_T)$: $W_0^{1,0}(a;\Gamma_T) \subset W_2^{1,0}(\Gamma_T)$. If closing the set $\Omega_a(\Gamma_T)$ realize by the norm $W_2^1(\Gamma_T)$, then we get space $W_0^1(a;\Gamma_T)$: $W_0^1(a;\Gamma_T) \subset W_2^1(\Gamma_T)$.

Let the function $y(x,t) \in W_0^{1,0}(a;\Gamma_T)$ satisfy the initial and boundary conditions

$$y\,|_{t=0} = \varphi(x), \;\; \varphi(x) \in L_2(\Gamma), \quad y\,|_{x\in\partial\Gamma_T} = 0, \tag{3}$$

and the functions $y(k)$ satisfy the conditions

$$y(0) = \varphi(x), \quad y(k) \mid_{x \in \partial \Gamma} = 0, \;\; k = 1, 2, ..., M. \tag{4}$$

Definition 1. *A weak solution to the initial boundary value problem* (1), (3) *of class* $W_2^{1,0}(\Gamma_T)$ *is called a function* $y(x,t) \in W_0^{1,0}(a; \Gamma_T)$ *that satisfies the integral identity*

$$-\int\limits_{\Gamma_T} y(x,t)\frac{\partial \eta(x,t)}{\partial t}dxdt + \ell_T(y,\eta) = \int\limits_{\Gamma} \varphi(x)\eta(x,0)dx + \int\limits_{\Gamma_T} f(x,t)\eta(x,t)dxdt$$

for any $\eta(x,t) \in W_0^1(a, \Gamma_T)$ *that is zero at* $t = T$. *Here* $\ell_T(y,\eta)$ *is bilinear form, defined by the ratio*

$$\ell_T(y,\eta) = \int\limits_{\Gamma_T} \left(a(x)\frac{\partial y(x,t)}{\partial x}\frac{\partial \eta(x,t)}{\partial x} + b(x)y(x,t)\eta(x,t)\right) dxdt.$$

Definition 2. *A weak solution to a boundary value problem* (2), (4) *is called functions* $u(k) = W_0^1(a,\Gamma)$ $(k = 0, 1, 2, ..., M)$, $u(0) = \varphi(x)$ $(\varphi(x) \in L_2(\Gamma))$, *satisfying an integral identity*

$$\int\limits_{\Gamma} y(k)_t \, \eta(x)dx + \ell(y(k),\eta) = \int\limits_{\Gamma} f_\tau(k)\, \eta(x)dx, \;\; k = 1, 2, ..., M,$$

for any $\eta(x) \in W_0^1(a,\Gamma)$, *equality* $y(0) = \varphi(x)$ *in* (4) *is understood almost everywhere,* $y(k)_t = \frac{1}{\tau}(y(k) - y(k-1))$; $\ell(y(k),\eta)$ *is bilinear form, defined by the ratio*

$$\ell(y(k),\eta) = \int\limits_{\Gamma} \left(a(x)\frac{dy(x;k)}{dx}\frac{d\eta(x)}{dx} + b(x)y(x;k)\eta(x)\right) dx.$$

Remark 1. Definition 2 shows that for each fixed $k = 1, 2, ..., M$ ratio (2), (4) is a boundary problem in space $W_0^1(a,\Gamma)$ for the elliptical equation (2) relatively $y(k)$.

Lemma 1. *Let* $\varphi(x) \in L_2(\Gamma)$ *and the conditions be fulfilled*

$$0 < a_* \leqslant a(x) \leqslant a^*, \;\; |b(x)| \leqslant \beta, \;\; x \in \Gamma_0. \tag{5}$$

Solution of system (2), (4), *i. e. functions* $y(k)$ $(k = 1, 2, ..., M)$, *when small enough* τ *are uniquely defined as elements of space* $W_0^1(a;\Gamma)$.

P r o o f. In the works [11, 12] establishes the basis property in the spaces $W_0^1(a;\Gamma)$ and $L_2(\Gamma)$ the system of generalized eigenfunctions of the one-dimensional elliptical operator Λ, generated by differential expression $\Lambda\phi = -\frac{d}{dx}\left(a(x)\frac{d\phi(x)}{dx}\right) + b(x)\phi(x)$. At the same time, if the conditions (5) by fulfilled, then eigenvalues of operator Λ are real, positive (except, maybe, the finitely number of the first) and have the finite-to-one. They can be numbered in the order of increasing modules, taking into account the multiplicity: $\{\lambda_i\}_{i \geqslant 1}$; respectively numbered and generalized eigenfunctions $\{\phi_i(x)\}_{i \geqslant 1}$. For the problem $\Lambda\phi = \lambda\phi + g$, $g \in L_2(\Gamma)$, there is an alternative to Fredholm. Based on this when $k = 1$ we get an uniqueness resolution relative to $y(1)$ the boundary problem

$$\Lambda y(1) = -\frac{1}{\tau}y(1) + f_\tau(1) + \frac{1}{\tau}y(0), \quad y(0) = \varphi(x),$$

for $\tau < \tau_0$ and a small enough positive τ_0. The same statement it remains true in any $k = 2, 3, ..., M$, granting the definition of functions $y(2)$, $y(3)$,..., $y(M)$ by the recurrent ratio

$$\Lambda y(k) = -\frac{1}{\tau}y(k) + f_\tau(k) + \frac{1}{\tau}y(k-1).$$

Below, at receiving a priori estimates norms of function $y(k)$, will indicate the boundary τ_0 of the change τ. Lemma is proven.

The method of proof of the existence of a weak solution of differential-difference system (2), (4) has a sequence of advantages. This method is based on finding a priori estimates for norm of function $y(k)$ does not dependent on τ. Specifically, it establishes the conditions of existence and uniqueness of the solution, the continuity of the solution on the initial data (the latter guarantees the stability of obtaining a solution to a different problem).

For determine a weak solution $y(k)$, $k = 1, 2, ..., M$, a differential-difference equation (2) will get an a priori estimate that does not depend on τ.

Theorem 1. *Let the conditions* (5) *be fulfilled and let them* $\varphi(x) \in L_2(\Gamma)$. *Under* $\tau \leqslant \tau_0 < \frac{1}{4\beta}$ *and any* $k = 1, 2, ..., M$ *for functions* $u(k)$ *correctly fair estimates*

$$\|y(k)\|_{2,\Gamma} \leqslant e^{4\beta T} \left(\|\varphi\|_{2,\Gamma} + 2\|f_\tau(k)\|_{2,1,\Gamma}\right) \tag{6}$$

and

$$\|y(m)\|^2_{2,\Gamma} + 2a_*\tau \sum_{k=1}^{m} \|\tfrac{dy(k)}{dx}\|^2 + \tau^2 \sum_{k=1}^{m} \|y(k)_t\|^2_{2,\Gamma} \leqslant C(\|\varphi\|^2_{2,\Gamma} + \|f_\tau(m)\|^2_{2,1,\Gamma}), \tag{7}$$

not dependent on the step τ*; constant* C *depends only on* a_*, β *and* T.

P r o o f. Here are the main arguments, the full proof is presented in the work [13].

From equality $y(k-1)^2 = (y(k) - \tau y(k)_t)^2 = y(k)^2 + \tau^2 y(k)_t^2 - 2\tau y(k) y(k)_t$ follows

$$2\tau y(k) y(k)_t = y(k)^2 + \tau^2 (y(k)_t)^2 - y(k-1)^2. \tag{8}$$

In the integral identity of the Definition 2 we will put $\eta(x) = 2\tau y(k)$ and, taking into account the ratios (5), (8), we get inequality

$$\begin{gathered}\int\limits_\Gamma y(k)^2 dx - \int\limits_\Gamma y(k-1)^2 dx + \tau^2 \int\limits_\Gamma (y(k)_t)^2 dx + 2a_*\tau \int\limits_\Gamma (\tfrac{dy(k)}{dx})^2 dx \leqslant \\ \leqslant -2\tau \int\limits_\Gamma b(x) y(k)^2 dx + 2\tau \int\limits_\Gamma f_\tau(k) y(k) dx\end{gathered}$$

and then, when $k = 1, 2, ..., M$,

$$\begin{gathered}\|y(k)\|^2_{2,\Gamma} - \|y(k-1)\|^2_{2,\Gamma} + \tau^2 \|y(k)_t\|^2_{2,\Gamma} + 2a_*\tau \|\tfrac{dy(k)}{dx}\|^2 \leqslant \\ \leqslant \varrho\tau \|y(k)\|^2_{2,\Gamma} + 2\tau \|f_\tau(k)\|_{2,\Gamma} \|y(k)\|_{2,\Gamma},\end{gathered} \tag{9}$$

where $\varrho = 2\beta$; here and below through $\|\cdot\|_{2,\Gamma}$ the marked norm in space $L_2(\Gamma)$. From inequality (9) follow

$$\|y(k)\|^2_{2,\Gamma} - \|y(k-1)\|^2_{2,\Gamma} \leqslant \varrho\tau \|y(k)\|^2_{2,\Gamma} + 2\tau \|f_\tau(k)\|_{2,\Gamma} \|y(k)\|_{2,\Gamma}. \tag{10}$$

Let's say that $\|y(k)\|_{2,\Gamma} + \|y(k-1)\|_{2,\Gamma} > 0$. Dividing inequality (10) by expression $\|y(k)\|_{2,\Gamma} + \|y(k-1)\|_{2,\Gamma}$, granting $\|y(k)\|_{2,\Gamma} / \left(\|y(k)\|_{2,\Gamma} + \|y(k-1)\|_{2,\Gamma}\right) \leqslant 1$, reduce to an estimate

$$\|y(k)\|_{2,\Gamma} \leqslant \tfrac{1}{1-\varrho\tau} \|y(k-1)\|_{2,\Gamma} + \tfrac{2\tau}{1-\varrho\tau} \|f_\tau(k)\|_{2,\Gamma}, \tag{11}$$

under $\tau \leqslant \tau_0 < \frac{1}{2\varrho}$. If $\|y(k)\|_{2,\Gamma} + \|y(k-1)\|_{2,\Gamma} = 0$, then out of (10) it should by $0 \leqslant \varrho\tau \|y(k)\|_{2,\Gamma} + 2\tau \|f_\tau(k)\|_{2,\Gamma}$. The obtained inequality also leads to an estimate (11) on which we receive

$$\|y(k)\|_{2,\Gamma} \leqslant \frac{1}{1-\varrho\tau}\|y(k-1)\|_{2,\Gamma} + \frac{2\tau}{1-\varrho\tau}\|f_\tau(k)\|_{2,\Gamma} \leqslant$$
$$\leqslant \frac{1}{(1-\varrho\tau)^k}\|y(0)\|_{2,\Gamma} + 2\tau\sum_{s=1}^{k}\frac{1}{(1-\varrho\tau)^{k-s+1}}\|f_\tau(s)\|_{2,\Gamma} \leqslant$$
$$\leqslant \frac{1}{(1-\varrho\tau)^k}\left(\|y(0)\|_{2,\Gamma} + 2\tau\sum_{s=1}^{k}\|f_\tau(s)\|_{2,\Gamma}\right) \leqslant$$
$$\leqslant e^{2\varrho T}\left(\|y(0)\|_{2,\Gamma} + 2\|f_\tau(k)\|_{2,1,\Gamma}\right), \quad \|f_\tau(k)\|_{2,1,\Gamma} = \tau\sum_{s=1}^{k}\|f_\tau(s)\|_{2,\Gamma},$$

the latter inequality is a consequence of the ratios $\frac{\varrho\tau}{1-\varrho\tau}k \leqslant \frac{\varrho T}{1-\varrho\tau} \leqslant 2\varrho T$ at $\tau < \frac{1}{2\varrho}$ and $\frac{1}{(1-\varrho\tau)^k} \leqslant e^{2\varrho T}$. Thus, the estimate (6). By summing up k the inequality (10) by 1 to $m \leqslant M$ and using estimates (9), we come to inequality

$$\|y(m)\|^2_{2,\Gamma} + 2a_*\tau\sum_{k=1}^{m}\|\frac{dy(k)}{dx}\|^2 + \tau^2\sum_{k=1}^{m}\|y(k)_t\|^2_{2,\Gamma} \leqslant C(\|\varphi\|^2_{2,\Gamma} + \|f_\tau(m)\|^2_{2,1,\Gamma}),$$
$$k = 1, 2, ..., M,$$

here the constant C depends only on a_*, β and T. The latter proves the correctness of the estimate (7).

Corollary 1. From the inequalities (6) and (7) follows continuity in the spaces $L_2(\Gamma)$ and $W^1_0(a, \Gamma)$ solutions of the differential-difference system (2), (4) according to the source data $\varphi(x)$, $f_\tau(k)$.

Corollary 2. Inequality (7) makes it possible to establish the convergence of the Rothe method for the initial boundary value problem (1), (3) in space $W^{1,0}_0(a; \Gamma_T)$. Let's designate it through $u_M(x, t)$ a function equal $y(k)$ at $t \in ((k-1)\tau, k\tau]$, $k = 1, 2, ..., M$. It is clear that $u_M(x, t)$ it belongs to the space $W^{1,0}_0(a; \Gamma_T)$ and satisfy inequality (7). Occur estimate

$$\|y_M\|_{2,\Gamma_T} + \|\frac{\partial y_M}{\partial x}\|_{2,\Gamma_T} \leqslant C^*, \tag{12}$$

where C^* is constant, independent of τ; here and below through $\|\cdot\|_{2,\Gamma_T}$ the marked norm in space $L_2(\Gamma_T)$. By analogy, we'll introduce a function $f_M(x, t)$ equal to $f_\tau(k)$ under $t \in ((k-1)\tau, k\tau]$, $k = 1, 2, ..., M$. Let it $M \to \infty$. Because of the estimate (12), from the sequence $y_M(x, t)$ can distinguish a sub-sequence that is weakly convergent in norm $W^{1,0}_2(\Gamma_T)$ to function $y(x, t) \in W^{1,0}_0(a; \Gamma_T)$. It is not difficult what $y(x, t)$ is a weak solution to the initial boundary value problem (1), (3), i. e. satisfies the identity of Definition 1. To do this, it is enough to establish this identity for a function $\eta(x, t) \in C^1(\Gamma_{T+\tau})$ that satisfy the conditions of agreement in all internal nodes of the graph Γ at any $t \in (0, T)$ and conditions $\eta|_{\partial\Gamma_T} = 0$, $\eta|_{t\in[T,T+\tau]} = 0$ (see above the definition of space $W^1_0(a; \Gamma_T)$). Functions $\eta(k)$ are defined by $\eta(x, t)$ equality $\eta(k) = \eta(x, k\tau)$, $k = 1, 2, ..., M$, in addition $\eta(k)_{t'} = \frac{1}{\tau}[\eta(k+1) - \eta(k)]$ (note that the difference quotient $\eta(k)_{t'}$ and $\eta(k)_t = \frac{1}{\tau}[\eta(k) - \eta(k-1)]$ are right and left approximations derivative $\frac{\partial\eta}{\partial t}$ at the point $t = k\tau$, respectively). As done above for $y(k)$, by $\eta(k)$ defined sectionally continuous by t function $\eta_M(x, t)$, $\frac{\partial\eta_M(x,t)}{\partial x}$, $\frac{\partial\eta_M(x,t)}{\partial t}$. It's easy to verify that $\eta_M(x, t)$, $\frac{\partial\eta_M(x,t)}{\partial x}$, $\frac{\partial\eta_M(x,t)}{\partial t}$ uniformly converge at $M \to \infty$ on $\overline{\Gamma}_T$ to the functions $\eta(x, t)$, $\frac{\partial\eta(x,t)}{\partial x}$, $\frac{\partial\eta(x,t)}{\partial t}$, respectively, where $\eta_M(x, t) = 0$, $t \in [T, T + \tau]$.

3. The problem of optimal control. Turn to the problem of optimal control of differential-difference system (2), (4). Let's designate through U the control space (set depending on the nature of the applied tasks, everywhere below $U = L_2(\Gamma)$) and let the linear operator $B : U \to L_2(\Gamma)$ be set.

Let's designate through $y(k;v(k)) := y(x,k;v(k))$, $v(k) := v(x;k) \in U$ $(k = 0,\ 1, ...,$ $M)$, the solution of the system

$$\frac{1}{\tau}[y(k;v(k)) - y(k-1;v(k-1))] - \frac{d}{dx}\left(a(x)\frac{dy(k;v(k))}{dx}\right) + b(x)y(k;v(k)) =$$
$$= f_\tau(k) + Bv(k),\ \ k = 1,2,...,M, \tag{13}$$

$$y(0;v(0)) = \varphi(x),\ \ y(k;v(k)) \mid_{x\in\partial\Gamma} = 0,\ \ k = 1,2,...,M. \tag{14}$$

Functions $y(k;v(k))$ describe the state of the system (13), (14), the observation is set by a line operator $C : W_0^1(a;\Gamma) \to L_2(\Gamma)$, i. e. $w(k;v(k)) := w(x,k;v(k)) = Cy(k;v(k))$. As ensues from consequence 1 of Theorem 1, linear display $v(k) \to y(k;v(k))$ of space U into space $W_0^1(a;\Gamma)$ continuously for any $k = 1,2,...,M$.

Definition 3. *A weak solution of the differential-difference system* (13), (14) *is called functions* $y(k;v(k)) = W_0^1(a,\Gamma)$ $(k = 0,1,2,...,M)$, $y(0;v(0)) = \varphi(x)$ $(\varphi(x) \in L_2(\Gamma))$, *satisfying an integral identity*

$$\int_\Gamma y(k;v(k))_t\, \eta(x)dx + \ell(y(k;v(k)),\eta) = \int_\Gamma f_\tau(k)\,\eta(x)dx + (Bv(k),\eta)$$
$$(k = 1,2,...,M)$$

for any $\eta(x) \in W_0^1(a,\Gamma)$*; equality* $y(0;v(0)) = \varphi(x)$ *is understood almost everywhere.*

Let's define the minimizing functional by ratio

$$\Psi(v) := \Psi(v(1),v(2),...,v(M)) = \tau\sum_{n=1}^{M}\Psi_k(v(k)),$$
$$\Psi_k(v(k)) = \|Cy(k;v(k)) - w_0(k)\|^2_{L_2(\Gamma)} + (Nv(k),v(k))_U, \tag{15}$$

where $w_0(k)$ $(k = 1,2,...,M)$ are given elements of space $L_2(\Gamma)$ and $N : U \to U$ is linear positively defined Hermite operator for which the conditions are met

$$(N(v(k),(v(k))_U \geqslant \varsigma\|v(k)\|^2_U,\ \ \varsigma > 0\ \ \forall\, v(k) \in U,\ \ k = 1,2,...,M; \tag{16}$$

here and everywhere below the symbol $(\cdot,\cdot)$ is denoted a scalar work in space $L_2(\Gamma)$, unless it is specified specifically. The functional $\Psi(v)$ is determined by an operator $v \to y(v)$ that establishes for all $k = 1,2,...,M$ the connection of the control of the effect $v(k)$ with the state $y(k;v(k))$ of the system (13), (14) and the operator $y(k;v(k)) \to Cy(k;v(k))$ of the transition from state $y(k;v(k))$ to observation $Cy(k;v(k))$.

Let's mark through U_∂ a convex closed subset of set U.

The problem of optimal control system (13), (14) is to determine

$$\inf_{v\in U_\partial}\Psi(v),\quad v = \{v(k),k = 1,2,...,M\}.$$

Theorem 2. *Let the conditions of the Theorem* 1 *be fulfilled. The task of optimal system control* (13), (14) *has the only solution* $v^* \in U_\partial$*, i. e.* $\Psi(v^*) = \min\limits_{v\in U_\partial}\Psi(v)$*, here* $v^* = \{v^*(k),k = 1,2,...,M\} \in U_\partial$ *is the optimal control of the system* (13), (14).

P r o o f. By virtue of the approval of the statement 1 of Theorem 1 linear mapping $v \to y(v)$ of the space of admissible control $\mathbb{U}$ in the space of the states $W_0^1(a,\Gamma)$ of the system (13), (14) continuously. The functional $\Psi(v)$ is determined by the transition operator $v \to y(v)$ from control effect v to state $y(v)$ of system (13), (14) and the transition operator $y(v) \to Cy(v)$ from state $y(v)$ to observation $Cy(v)$. Further proof uses the property of the coercive of the quadratic component of functional $\Psi(v)$ on the convex closed set U_∂. Namely, based on the notation (15) for any $k = 1, 2, ..., M$, we have

$$\Psi_k(v(k)) = \|Cy(k; v(k)) - w_0(k)\|^2_{L_2(\Gamma)} + (Nv(k), v(k))_U =$$

$$= \|C(y(k; v(k)) - y(0; v(0))) + Cy(0; v(0)) - w_0(k)\|^2_{L_2(\Gamma)} + (Nv(k), v(k))_U =$$

$$= \mathfrak{F}_k(v(k), v(k)) - 2\mathfrak{L}_k(v(k)) + \|Cy(0; v(0)) - w_0(k)\|^2_{L_2(\Gamma)},$$

where

$$\mathfrak{F}_k(v(k), (v(k)) = (C(y(k; v(k)) - y(0; v(0))),\ C(y(k; v(k)) - y(0; v(0)))) + (Nv(k), v(k))_U$$

is a square form on U,

$$\mathfrak{L}_k(v(k)) = (w_0(k) - Cy(0; v(0)), C(y(k; v(k)) - y(0; v(0))))$$

determines the linear form on U. Hence and (16) follow the view

$$\Psi(v) = \mathfrak{F}(v, v) + \mathfrak{L}(v),\ \ \mathfrak{F}(v, v) = \tau \sum_{n=1}^{M} \mathfrak{F}_k(v(k), v(k)),\ \ \mathfrak{L}(v) = \tau \sum_{n=1}^{M} \mathfrak{L}_k(v(k)).$$

Conditions (16) guarantee the coercive of a square form $\mathfrak{F}(v, v)$. Further reasoning almost literally repeats the given in the work [14, p. 13].

Remark 2. In the case $N = 0$, it can be shown that when the conditions of the Theorem 1 are met, there is a nonempty closed and convex subset $U^0_\partial \subset U_\partial$ such that

$$\Psi(v^*) = \inf_{v \in U_\partial} \Psi(v) \ \ \forall v \in U^0_\partial.$$

The proof of this fact is similar to the presented in the work [14, Theorem 5.2, p. 47].

Next, let's dwell on a detailed study of the conditions of optimal control and get the ratios that determine optimal control. To simplify the representations of distinct transformations, further operations is taken simultaneously for all states $y(k; u(k))$ and control $u(k)$, $k = 1, 2, ..., M$; notations $y(k; u(k))$, $y(k; u(k))_t$ and $u(k)$ are replace by $y(u)$, $y(u)_t$ and u, respectively.

Pre-proving the following auxiliary statements (see also [14, p. 16, 56]).

Lemma 2. *Let the conditions of the Theorem* 1 *be fulfilled and* $u^* = \{u^*(k), k = 1, 2, ..., M\} \in U_\partial$ *is the minimizing element of functional* $\Psi(v)$*, then inequality*

$$\Psi'(u^*)(v - u^*) \geqslant 0 \tag{17}$$

is fulfilled for any $v \in U_\partial$*; derivative* $\Psi'(u^*)$ *is understood in the sense of Frechet.*

P r o o f. Since u^* is a minimizing element of functional $\Psi(v)$, for any $v \in U_\partial$ and any number $\theta \in (0, 1)$ is true inequality $\Psi(u^*) \leqslant \Psi((1 - \theta)u^* + \theta v)$. This means that

$$\frac{1}{\theta}[\Psi((1 - \theta)u^* + \theta v) - \Psi(u^*)] = \frac{1}{\theta}[\Psi(u^* + \theta(v - u^*)) - \Psi(u^*)] \geqslant 0$$

and $\Psi'(u^*)(v - u^*) \geqslant 0$ at $\theta \to 0$ whence it should be (17). The inverse statement is also true. Indeed, let for certain fixed $u \in U_\partial$ fairly inequality $\Psi'(u)(v - u) \geqslant 0$ for any $v \in U_\partial$.

Due to the convexity of the mapping $v \to \Psi(v)$ (see proof of the Theorem 2) for any $v \in U_\partial$ has a place

$$\frac{1}{\theta}[\Psi((1-\theta)u+\theta v)-\Psi(u)] = \frac{1}{\theta}[\Psi(u^*+\theta(v-u^*))-\Psi(u^*)] \leqslant \Psi(v)-\Psi(u),$$

which means $0 \leqslant \Psi'(u)(v-u) \leqslant \Psi(v)-\Psi(u)$ at $\theta \to 0$. It follows $\Psi(v) \geqslant \Psi(u)$ for any $v \in U_\partial$, i. e. u is a minimizing element of functional $\Psi(v)$.

Lemma 3. *For all $v, u \in U_\partial$ take place a ratio*

$$y'(u)(v-u) = y(v)-y(u), \tag{18}$$

here $y'(u)$ is derivative in the sense of Frechet mapping $u \to y(u)$.

P r o o f. Based on Definition 3, for control $u(k), v(k) \in U_\partial$ $(k = 0, 1, ..., M)$ is a ratio

$$\begin{gathered}\tfrac{1}{\tau}\int\limits_\Gamma [(y(k;v(k))-y(k;u(k)))-(y(k-1;v(k-1))-y(k-1;u(k-1)))]\,\eta(x)dx+\\ +\ell(y(k;v(k))-y(k;u(k)),\eta) = (B(v(k)-u(k),\eta)_U\end{gathered} \tag{19}$$

for any function $\nu(x) \in W_0^1(a,\Gamma)$. On the other hand, we have

$$\begin{gathered}\tfrac{1}{\tau}\int\limits_\Gamma [(y(k;u(k)+\vartheta(v(k)-u(k)))-y(k;u(k)))-\\ -(y(k-1;u(k-1)+\vartheta(v(k-1)-u(k-1)))-y(k-1;u(k-1)))]\,\eta(x)dx+\\ +\ell(y(k;u(k)+\vartheta(v(k)-u(k)))-y(k;u(k)),\eta) = \vartheta(B(v(k)-u(k),\eta)_U\end{gathered}$$

for any $\vartheta \in (0,1)$ and any function $\eta(x) \in W_0^1(a,\Gamma)$. By dividing both parts of the received ratio by ϑ and calculating the limit at $\vartheta \to 0$, come to the ratio

$$\begin{gathered}\tfrac{1}{\tau}\int\limits_\Gamma [y'(k;u(k))(v(k)-u(k))-y'(k-1;u(k-1))(v(k-1)-u(k-1))]\,\eta(x)dx+\\ +\ell(y'(k;u(k))(v(k)-u(k)),\eta) = (B(v(k)-u(k),\eta)_U\end{gathered} \tag{20}$$

for any function $\eta(x) \in W_0^1(a,\Gamma)$. Comparing the left parts of the ratios (19) and (20), come to the equality

$$y'(k;u(k))(v(k)-u(k)) = y(k;v(k))-y(k;u(k)),\ \ k = 0,1,...,M,$$

that complete the proof.

Let $u(k)$ is the optimal control for each fixed $k = 1, 2, ..., M$, then by virtue of (17) and (18) have

$$\begin{gathered}\tfrac{1}{2}\Psi_k'(u(k))(v(k)-u(k)) =\\ = (Cy(k;u(k))-w_0(k),\,Cy'(k;u(k))(v(k)-u(k)))+(Nu(k),\,v(k)-u(k))_U =\\ = (Cy(k;u(k))-w_0(k),\,C(y(k;v(k))-y(k;u(k))))+(Nu(k),\,v(k)-u(k))_U \geqslant 0\end{gathered} \tag{21}$$

for any $v(k) \in U_\partial$.

Denote through C^* the operator, conjugate to C, then the ratio (21) takes the form of

$$(C^*(Cy(k;u(k))-w_0(k)),\,y(k;v(k))-y(k;u(k)))+(Nu(k),\,v(k)-u(k))_U \geqslant 0, \tag{22}$$

so, based on the notation (15) of functional $\Psi(v)$ and ratio (17), we come to inequality

$$\tau\sum_{k=1}^{M}[(C^*(Cy(k;u(k))-w_0(k)),\,y(k;v(k))-y(k;u(k)))+(Nu(k),\,v(k)-u(k))_U] \geqslant 0 \tag{23}$$

for any $v(k) \in U_\partial$. Thus, inequality (23) is a necessary condition for optimal control of the system (13), (14).

A more detailed description of the conditions of optimal control can be obtained using the conjugate state for the system (13), (14). In space $W^1_0(a;\Gamma)$, we introduce the notation of a conjugate state $p(k;v(k))$ $(k = 1, 2, ..., M)$ and a conjugate system to a system (13), (14), for which we will use the obvious equality

$$\tau \sum_{k=1}^{M} \theta(k)_t \vartheta(k) = -\tau \sum_{k=0}^{M-1} \theta(k)\vartheta(k)_{t'} - \theta(0)\vartheta(0) + \theta(M)\vartheta(M)$$

for arbitrary functions $\theta(k)$ and $\vartheta(k)$ (similar to the formula of integration by parts by variable t for functions $\theta(t)$ and $\vartheta(t)$), based on which we define the conjugate state $p(k;v(k))$ $(k = 1, 2, ..., M)$ to control $v(k)$ $(k = 1, 2, ..., M)$ as a solution to a conjugate problem

$$\begin{gathered} -\tfrac{1}{\tau}[p(k+1;v(k+1)) - p(k;v(k))] - \tfrac{d}{dx}\left(a(x)\tfrac{dp(k;\,v(k))}{dx}\right) + b(x)p(k;v(k)) = \\ = C^*(Cy(k;v(k)) - w_0(k)), \;\; k = 0, 1, ..., M-1, \end{gathered} \tag{24}$$

$$p(M;v(M)) = 0, \;\; p(k;v(k))\,|_{x\in\partial\Gamma} = \; 0, \;\; k = 0, 1, ..., M-1. \tag{25}$$

Lemma 4. *The solution of the system* (24), (25) *at small enough τ is uniquely defined as elements of space $W^1_0(a;\Gamma)$.*

P r o o f. To be sure of this, it is enough to renumber the ratios of the system (24), (25) and apply the statement of Lemma 1. Indeed, by changing the numbering by law $l = M - k$, $k = M, M-1, ..., 1, 0$, we get that l change from 0 until M and we come to the system

$$\begin{gathered} -\tfrac{1}{\tau}[\widetilde{p}(l-1;v(l-1)) - \widetilde{p}(l;v(l))] - \tfrac{d}{dx}\left(a(x)\tfrac{d\widetilde{p}(l;\,v(l))}{dx}\right) + b(x)\widetilde{p}(l;v(l)) = \\ = C^*(Cy(l;v(l)) - w_0(l)), \;\; l = 1, 2, ..., M, \end{gathered}$$

$$\widetilde{p}(0;v(0)) = 0, \;\; \widetilde{p}(l;v(l))\,|_{x\in\partial\Gamma} = 0, \;\; l = 1, 2, ..., M,$$

for which correctly of the Lemma 1.

Remark 3. The resulting differential-difference system (24), (25) correspond to a differential problem conjugated with (1), (3) (see also [1, 4]).

For each fixed $k = 1, 2, ..., M$ transform inequality (22). Considering the ratios

$$\begin{gathered} -\tfrac{1}{\tau}\sum_{k=0}^{M-1}[p(k+1;u(k+1)) - p(k;u(k))][y(k;v(k)) - y(k;u(k))] = \\ = \tfrac{1}{\tau}\sum_{k=1}^{M}\{[y(k;v(k)) - y(k;u(k))] - [y(k-1;v(k-1)) - y(k-1;u(k-1))]\}\,p(k;u(k)), \end{gathered}$$

$$\sum_{k=0}^{M-1} \ell(p(k;u(k)),\, y(k;v(k)) - y(k;u(k))) = \sum_{k=1}^{M} \ell(y(k;v(k)) - y(k;u(k)),\, p(k;u(k))),$$

come to equality

$$\begin{gathered} \sum_{k=0}^{M-1}(C^*(Cy(k;v(k)) - w_0(k)),\, y(k;v(k)) - y(k;u(k))) = \\ = \sum_{k=1}^{M}(Bv(k) - Bu(k),\, p(k;u(k))) = \sum_{k=0}^{M-1}(Bv(k) - Bu(k),\, p(k;u(k))) = \\ = \sum_{k=0}^{M-1}(B^*p(k;u(k)),\, v(k) - u(k))_U \end{gathered}$$

(there are zero equality $y(0;v(0)) - y(0;u(0))$ and $p(M;u(M))$). Therefore, from the obtained equality flows

$$(C^*(Cy(k;v(k)) - w_0(k)),\, y(k;v(k)) - y(k;u(k))) = (B^*p(k;u(k)),\, v(k) - u(k))_U$$

for each fixed $k = 0, 1, ..., M-1$, then inequality (22) can be rewritten in the form

$$(B^*p(k;u(k)) + Nu(k),\, v(k) - u(k))_U \geqslant 0 \quad \forall\, v(k) \in U_\partial,\;\; k = 0, 1, ..., M, \tag{26}$$

and inequality (23) is transformed to form

$$\tau \sum_{k=0}^{M} (B^*p(k;u(k)) + Nu(k),\, v(k) - u(k))_U \geqslant 0 \quad \forall\, v(k) \in U_\partial,\;\; k = 0, 1, ..., M, \tag{27}$$

as above is taken into account $y(0;v(0)) - y(0;u(0)) = 0$ and $p(M;u(M)) = 0$.

Thus, the totality of ratios (13), (14), (24), (25) and (27) determines the optimal control $u(k)$ and corresponding states $p(k;u(k))$, $p(k;u(k))$, $k = 1, 2, ..., M$.

Private case. Let $U_\partial = U$, i. e. hence are no restrictions on control — a fairly often case in practice. Inequality (26), (27) take the form of equality

$$(B^*p(k;u(k)) + Nu(k),\, v(k) - u(k))_U = 0 \quad \forall\, v(k) \in U_\partial,\;\; k = 0, 1, ..., M,$$

$$\tau \sum_{k=0}^{M} (B^*p(k;u(k)) + Nu(k),\, v(k) - u(k))_U = 0 \quad \forall\, v(k) \in U_\partial,\;\; k = 0, 1, ..., M,$$

respectively. The latter provide an opportunity to determine the optimal control from the ratios $B^*p(k;u(k)) + Nu(k) = 0$, $k = 0, 1, ..., M$:

$$u(k) = -N^{-1}B^*p(k;u(k)),\;\; k = 0, 1, ..., M.$$

In this case, the state $y(k)$ of the system (13), (14) and the conjugate state $p(k)$ of the system (24), (25) for each fixed $k = 0, 1, ..., M$ are defined as weak solutions of problems

$$\begin{cases} \frac{1}{\tau}[y(k) - y(k-1)] - \frac{d}{dx}\left(a(x)\frac{dy(k)}{dx}\right) + b(x)y(k) + BN^{-1}B^*p(k) = f_\tau(k), \\ k = 1, 2, ..., M,\;\; y(0) = \varphi(x), \\ -\frac{1}{\tau}[p(k+1) - p(k)] - \frac{d}{dx}\left(a(x)\frac{dp(k)}{dx}\right) + b(x)p(k) - C^*Cy(k) = -C^*w_0(k), \\ k = 0, 1, ..., M-1,\;\; p(M) = 0, \end{cases}$$

in space $W_0^1(a;\Gamma)$, and optimal control — by formulas

$$u(k) = -N^{-1}B^*p(k),\;\; k = 0, 1, ..., M.$$

In the case $N = 0$ it is possible show that when the conditions of the Theorem 1 are fulfill, there is a nonempty closed and convex subset U_∂^0 of the set U_∂ that $\Psi(u) = \inf\limits_{v \in U_\partial} \Psi(v)$ for any $u \in U_\partial^0$ (see Remark 2).

Finally we get the following statements.

Theorem 3. *Let the conditions* (5) *be met.*

1\. *If the set* $\mathbb{U}_\partial$ *is bounded, then the optimal control* $u = \{u(k) \in U_\partial, k = 0, 1, ..., M\}$ *and it of the corresponding states* $y(k;u(k)), p(k;u(k)) \in W_0^1(a;\Gamma)$, $k = 0, 1, ..., M$, *are determined by the solution of the system*

$$\begin{cases} \frac{1}{\tau}[y(k;u(k)) - y(k-1;u(k-1))] - \frac{d}{dx}\left(a(x)\frac{dy(k;\,u(k))}{dx}\right) + b(x)y(k;u(k)) = \\ = f_\tau(k) + Bu(k), \ \ k = 1,2,...,M, \ \ y(0;u(0)) = \varphi(x), \\ -\frac{1}{\tau}[p(k+1;u(k+1)) - p(k;u(k))] - \frac{d}{dx}\left(a(x)\frac{dp(k;\,u(k))}{dx}\right) + b(x)p(k;u(k)) = \\ = C^*(Cy(k;u(k)) - w_0(k)), \ \ k = 0,1,...,M-1, \ \ p(M;u(M)) = 0, \\ (B^*p(k;u(k)) + Nu(k), v(k) - u(k))_U \geqslant 0 \ \ \forall v(k) \in U_\partial, \ \ k = 0,1,...,M. \end{cases}$$

2. *If $U_\partial = U$, then optimal control u is determined by formulas*

$$u(k) = -N^{-1}B^*p(k), \ \ k = 0,1,...,M,$$

and it's the corresponding states $y(k), p(k) \in W_0^1(a;\Gamma)$, $k = 0,1,...,M$, determined by the solution of the system

$$\begin{cases} \frac{1}{\tau}[y(k) - y(k-1)] - \frac{d}{dx}\left(a(x)\frac{dy(k)}{dx}\right) + b(x)y(k) + BN^{-1}B^*p(k) = f_\tau(k), \\ k = 1,2,...,M, \ \ y(0) = \varphi(x), \\ -\frac{1}{\tau}[p(k+1) - p(k)] - \frac{d}{dx}\left(a(x)\frac{dp(k)}{dx}\right) + b(x)p(k) - C^*Cy(k) = -C^*w_0(k), \\ k = 0,1,...,M-1, \ \ p(M) = 0. \end{cases}$$

At the same time: a) *if the operator $N \neq 0$, the optimal control $u \in \mathbb{U}_\partial$ is the uniquely;* b) *if $N = 0$, the optimal controls form a convex set $\mathbb{U}_\partial^0 \subset \mathbb{U}_\partial$.*

4. Optimal control of the differential-difference equation with delay. At first, in space $W_0^1(a;\Gamma)$ consider a differential-difference system with a constant delay without control:

$$\begin{gathered} \frac{1}{\tau}[y(k) - y(k-1)] - \frac{d}{dx}\left(a(x)\frac{dy(k)}{dx}\right) + b(x)y(k) + c(x)y(k-m) = f_\tau(k), \\ k = m+1, m+2, ..., M, \end{gathered} \tag{28}$$

$$y(k) = \varphi(k), \ \ k = 0,1,...,m, \ \ 1 \leqslant m < M, \tag{29}$$

the coefficient $c(x)$ is boundary measurable on Γ function $\varphi(0) \in L_2(\Gamma)$, $\varphi(k) \in W_0^1(a;\Gamma)$, $k = 1,2,...,m$.

For the evolutionary differential equation (1) the constant control $h = m\tau < T$ define two domains $\Gamma_{0,h} = \Gamma_0 \times (0,h)$ and $\Gamma_{h,T} = \Gamma_0 \times (h,T)$: $\Gamma_T = \Gamma_{0,h} \cup \Gamma_{h,T}$. Differential-difference system (28), (29) correspond to a evolutionary differential system

$$\frac{\partial y(x,t)}{\partial t} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial y(x,t)}{\partial x}\right) + b(x)y(x,t) + c(x)y(x,t-h) = f(x,t), \ \ x,t \in \Gamma_{h,T},$$

$$y(x,t) = \varphi(x,t), \ \ x,t \in \Gamma_{0,h}, \ \ y\,|_{x \in \partial\Gamma_T} = 0$$

(the system was considered in the work [1]).

Let's introduce a delay operator Z to represent the system (28), (29) in a more suitable form. Let $Z : W_0^1(a;\Gamma) \to W_0^1(a;\Gamma)$ is a linear continuous operator, defined by the ratio

$$Zy(k) = \begin{cases} y(k), \ \ k = m+1, m+2, ..., M, \\ 0, \ \ k = 1,2,...,m. \end{cases}$$

Let's set the function $F(k) \in W_0^1(a;\Gamma)$, $k = 1, 2, ..., M$, the ratio

$$F(k) = \begin{cases} f_\tau(k), & k = m+1, m+2, ..., M, \\ \frac{1}{\tau}[\varphi(k) - \varphi(k-1)] - \frac{d}{dx}\left(a(x)\frac{d\varphi(k)}{dx}\right) + b(x)\varphi(k), & k = 1, ..., m. \end{cases}$$

Then the differential-difference system (28), (29) will take the form

$$\begin{gathered} \frac{1}{\tau}[y(k) - y(k-1)] - \frac{d}{dx}\left(a(x)\frac{dy(k)}{dx}\right) + b(x)y(k) + c(x)Zy(k) = F(k), \\ k = 1, 2, ..., M, \end{gathered} \tag{30}$$

$$y(0) = \varphi(0) \in L_2(\Gamma). \tag{31}$$

It is not difficult to show that all the statements of the previous section are true for a differential-difference system (30), (31).

Next, let's look at the optimal control problem, in addition keep all the notations and concepts of Section 3. Consider a differential-different system with control $v(k) \in U$ ($k = 0, 1, ..., M$) the state of which $y(k; v(k)) \in W_0^1(a;\Gamma)$ ($k = 0, 1, ..., M$) is defined as the solution to the problem

$$\begin{gathered} \frac{1}{\tau}[y(k; v(k)) - y(k-1; v(k-1))] - \frac{d}{dx}\left(a(x)\frac{dy(k;v(k))}{dx}\right) + b(x)y(k; v(k)) + \\ + c(x)Zy(k; v(k)) = F(k) + Bv(k), \;\; k = 1, 2, ..., M, \end{gathered}$$

$$y(0; v(0)) = \varphi(0) \in L_2(\Gamma).$$

Optimizing functional $\Psi(v)$ is determined by ratio (15), the problem of optimal control system (30), (31) has a uniquely solution (see statement of the Theorem 2 for the system (30), (31)). The conjugate state $p(k; v(k))$ ($k = 1, 2, ..., M$) is defined by a system similar (24), (25) with the only difference that the conjugate system will contain an operator Z^* conjugate at Z:

$$Z^* p(k; v(k-m)) = \begin{cases} y(k), & k = m+1, m+2, ..., M, \\ 0, & k = 1, 2, ..., m. \end{cases}$$

The pairing system takes the form

$$\begin{gathered} -\frac{1}{\tau}[p(k+1; v(k+1)) - p(k; v(k))] - \frac{d}{dx}\left(a(x)\frac{dp(k;\, v(k))}{dx}\right) + b(x)p(k; v(k)) + \\ + c(x)Z^* p(k; v(k)) = C^*(Cy(k; v(k)) - w_0(k)), \;\; k = 0, 1, ..., M-1, \end{gathered}$$

$$p(M; v(M)) = 0, \;\; p(k; v(k)) \,|_{x \in \partial\Gamma} = 0, \;\; k = 0, 1, ..., M.$$

As it is easy to verify, the statements of the Theorem 3 remain correct.

Remark 4. Taken differential-difference system (2), (4) as an approximation of differential system (1), (3) is not the only (the two-layer scheme used has an approximation error $O(\tau)$). You can use the system as a more precise approximation

$$\begin{gathered} \frac{1}{\tau}[\tfrac{3}{2}(y(k+1) - y(k)) - \tfrac{1}{2}(y(k) - y(k-1))] - \frac{d}{dx}\left(a(x)\frac{dy(k+1)}{dx}\right) + b(x)y(k+1) = f_\tau(k), \\ k = 1, 2, ..., M, \end{gathered}$$

$$y(0) = \varphi(x), \;\; y(1) = \varphi_1(x), \;\; \varphi(x), \varphi_1(x) \in W_0^1(a;\Gamma),$$

$$y(k)\,|_{x\in\partial\Gamma}=0,\ k=0,1,\dots,M,$$

with an approximation error $O(\tau^2)$ (see also work [15]). The study of such a system is similar to the one presented above in Sections 2, 3.

5. Generalization for a many-dimensional case. The results (statements by Theorems 1, 2 and 3) can be extended to a many-dimensional case. In the Euclidean space $\mathbb{R}^n$, $n\geqslant 2$, let's look at a network-like bounded domain $\Im$, comprised of N domains $\Im_k$ ($k=\overline{1,N}$), pairwise united by means of M nodal place ω_j ($j=\overline{1,M}$, $M<N$): $\Im=\hat{\Im}\bigcup\hat{\omega}$, where $\hat{\Im}=\bigcup\limits_{k=1}^{N}\Im_k$, $\hat{\omega}=\bigcup\limits_{j=1}^{M}\omega_j$, moreover $\Im_k\bigcap\Im_l=\emptyset$ ($k\neq l$), $\omega_j\bigcap\omega_i=\emptyset$ ($j\neq i$), $\Im_k\bigcap\omega_j=\emptyset$ [16, 17]. Domains $\Im_k$ in nodal place ω_j share common boundaries in the form of adjoining surfaces S_j (meas $S_j>0$). At each nodal place ω_j the adjoining surface S_j consisting of m_j parts S_{ji} ($1\leqslant i\leqslant m_j\leqslant N-1$) has a representation $S_j=\bigcup\limits_{i=1}^{m_j}S_{ji}$ (meas $S_{ji}>0$). In addition S_j and S_{ji} are parts of boundary $\partial\Im_{k_0}$ and $\partial\Im_{k_i}$ of domains $\Im_{k_0}$ and $\Im_{k_i}$, respectively; S_{ji} is two-sided surface for each j,i: S_{ji}^- is interior surface, S_{ji}^+ is exterior surface. Thus, the nodal place ω_j is determined by the adjoining surface S_j, for which S_{ji} are also the adjoining surface $\Im_{k_0}$ to $\Im_{k_i}$, $i=\overline{1,m_j}$. The boundary $\partial\Im$ of the domain $\Im$ is called the union of the boundary $\partial\Im_k$ of domain $\Im_k$ ($k=\overline{1,N}$), which does not include the adjoining surface of all node places: $\partial\Im=\bigcup\limits_{k=1}^{N}\partial\Im_k\backslash\bigcup\limits_{j=1}^{M}S_j$. The domain $\Im$ has a network-like structure similar to that of the geometric graph (see also works [2, 16, 17]), each domain $\Im_k$ adjoins to one or two node places and has one or more of the surface adjoining other domains (to compare with the structure of the graph: each edge of the graph has two endpoints, of which one or both are conjugation nodes with the other edges).

We use customary Lebesque spaces $L_q(\Omega)$, $q=1,2$, and the Sobolev space $W_2^1(\Omega)$, where U is a bounded domain in $\mathbb{R}^n$. For each fixed k ($1\leqslant k\leqslant N$) denote through $W_{2,0}^1(\Im_k)$ the closure in $W_2^1(\Im_k)$ a set infinitely differentiable on $\overline{\Im}_k$ functions equal to zero on $\partial\Im_k\subset\partial\Im$. Let $\Omega_a(\Im)$ is a set of functions $z:\Im\to\mathbb{R}^1, z|_{\Im_k}\in W_{2,0}^1(\Im_k)$ for each $k=1,2,...,N$, u satisfies the condition of agreement

$$z|_{S_{ji}^+}=z|_{S_{ji}^-},\ i=\overline{1,m_j};\quad \int\limits_{S_j\subset\partial\Im_{k_0}}a(x)\frac{\partial z(x)}{\partial\mathbf{n}_j}dx+\sum_{i=1}^{m_j}\int\limits_{S_{ji}\subset\partial\Im_{k_i}}a(x)\frac{\partial z(x)}{\partial\mathbf{n}_{ji}}dx=0,$$

for each node place ω_j on surfaces $S_j=\bigcup\limits_{i=1}^{m_j}S_{ji}$, $j=\overline{1,M}$; here vectors $\mathbf{n}_j$ and $\mathbf{n}_{ji}$ are outer normals to S_j and S_{ji}, respectively, $a(x)$ is measurable bounded function from $L_2(\Im)$.

Closing the set $\Omega_a(\Im)$ in norm $\|z\|_{\Im}^1=((z,z)_{\Im}^1)^{1/2}$, where

$$(z,\omega)_{\Im}^1=\sum_{k=1}^{N}\int\limits_{\Im_k}\left(z(x)\omega(x)+\sum_{\kappa=1}^{n}\frac{\partial z(x)}{\partial x_\kappa}\frac{\partial\omega(x)}{\partial x_\kappa}\right)dx,$$

let's call space $\widetilde{W}_0^1(a,\Im)$.

The space $\widetilde{W}_0^1(a,\Im)$ considers a differential-difference system, similar (2), (4):

$$\frac{1}{\tau}(z(k)-z(k-1))-\frac{\partial}{\partial x_\iota}\left(a(x)\frac{\partial z(k)}{\partial x_\kappa}\right)+b(x)z(k)=f(k),\quad k=1,2,...,M,$$

$$z(0)=\varphi(x),\quad y(k)\,|_{x\in\partial\Im}=0.$$

Here, through $\frac{\partial}{\partial x_\iota}\left(a(x)\frac{\partial z(k)}{\partial x_\kappa}\right)$ denoted the sum $\sum\limits_{\kappa,\iota=1}^{n}\frac{\partial}{\partial x_\iota}\left(a(x)\frac{\partial z(k)}{\partial x_\kappa}\right)$, measurable bounded functions $a(x)$, $b(x)$ meet the conditions (5) (Γ replaced by $\Im$); $f(k)\in L_2(\Im)$ $(k=1,2,...,M)$.

The main complexity in analysis such a differential-difference system and proving statements similar to presented in Sections 3 and 4 is to establish conditions that guarantee the spectral completeness and basis property of set of the generalized eigenfunctions of operator $z(k)=-\frac{\partial}{\partial x_\iota}\left(a(x)\frac{\partial z(k)}{\partial x_\kappa}\right)+b(x)z(k)$ in space $\widetilde{W}_0^1(a,\Im)$. The works [2, 16] shows ways to obtain such conditions.

6. Conclusion. The work presents the approach of approximation of the evolutionary differential system (1), (3) with distributed parameters on the graph using the method of semi-digitization by temporal variable. A priori estimates of norms of weak solution (6), (7) of differential-difference system (statement of the Theorem 1) make possibility to establish not only the solvability of this system but also the evolutionary system (corollary 2 of the theorem 1). For differential-difference system (1), (3) is presented analysis of the optimal control problem without lag (13), (14) and with lag (Section 4). This essentially uses the conjugate state of the system and the conjugate system for a differential-difference system. It should be noted that the results presented in the work can be used in the analysis of control problems [18, 19], stabilization [20–23] of differential systems, as well as in the study of network-like processes of applied character [24–26].

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Received: February 9, 2021.
Accepted: October 13, 2021.

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Оптимальное управление дифференциально-разностной параболической системой с распределенными параметрами на графе

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Для цитирования: *Zhabko A. P., Provotorov V. V., Shindyapin A. I.* Optimal control of a differential-difference parabolic system with distributed parameters on the graph // Вестник Санкт-Петербургского университета. Прикладная математика. Информатика. Процессы управления. 2021. Т. 17. Вып. 4. С. 433–448. https://doi.org/10.21638/11701/spbu10.2021.411

Рассматривается задача оптимального управления дифференциально-разностным уравнением параболического типа с распределенными параметрами на графе в классе суммируемых функций. При этом особое внимание уделяется связи дифференциально-разностной системы с эволюционной дифференциальной системой и поиску условий, при выполнении которых сохраняются свойства дифференциальной системы. Такую связь устанавливает используемый для дифференциальной системы универсальный метод полудискретизации по временной переменной, дающий эффективный инструмент при отыскании условий однозначной разрешимости и непрерывности по исходным данным для дифференциально-разностной системы. Априорные оценки норм слабого решения дифференциально-разностной системы позволяют установить не только разрешимость данной системы, но и существование слабого решения эволюционной дифференциальной системы. Для дифференциально-разностной системы представлен анализ задачи оптимального управления, содержащий естественное в таких случаях дополнительное исследование задачи с временным запаздыванием. При этом существенно используются сопряженное состояние системы и сопряженная система для дифференциально-разностной системы — получены соотношения, определяющие оптимальное управление или множество оптимальных управлений. Указаны пути переноса полученных результатов на случай анализа задач оптимального управления в классе функций с носителями на сетеподобных областях. Переход от эволюционной дифференциальной системы к дифференциально-разностной явился естественным шагом изучения прикладных задач теории переноса сплошных сред. Приведенные результаты лежат в основе анализа задач оптимального управления дифференциальными системами с распределенными параметрами на графе, выявлены интересные аналогии с многофазовыми задачами многомерной гидродинамики.

Ключевые слова: дифференциально-разностная система, сопряженная система, ориентированный граф, оптимальное точечное управление.

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