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A Categorical Characterization of a ①-Iteratively Defined State of Common Knowledge

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Abstract We present here a novel approach to the analysis of *common knowledge* based on category theory. In particular, we model the global epistemic state for a given set of agents through a hierarchy of beliefs represented by a presheaf construction. Then, by employing the properties of a categorical monad, we prove the existence of a state, obtained in an iterative fashion, in which all agents acquire common knowledge of some underlying statement. In order to guarantee the existence of a fixed point under certain suitable conditions, we make use of the properties entailed by Sergeyev's numeral system called grossone, which allows a finer control on the relevant structure of the infinitely nested epistemic states.

Keywords: common knowledge, category theory, grossone.

1. Introduction

Situations of *common knowledge* are those in which all agents in some context C know X, all agents in C know that all agents in C know X, all agents in C know that all agents in C know that all agents in C know X, and so on. Such situations are commonplace in ordinary experience, but due to their implicit infinite hierarchy of nested knowledge-states they pose certain challenges for formally rigorous epistemic modeling.

Common knowledge was first explicitly thematized as an auxiliary concept in a highly influential game-theoretical analysis of convention (Lewis, 1969). The theoretical approach introduced by Lewis was intended to capture the idea that an event constitutes common knowledge for a set of agents if each of the agents knows that all the other agents know that all the agents know that all the others know [...] that the event is the case. In subsequent work, Robert Aumann addressed this question in greater depth via a set-theoretical characterization of common knowledge such that common knowledge may be understood in terms of the meet of all the relevant individual information partitions (Aumann, 1976). An immediate extension of this characterization is the formalization of Lewis' original intuition as a hierarchical construction based on Aumann's set-theoretical framework (Geanakoplos, 1994). From then on, numerous works adopted Aumann's formalism, particularly in Game Theory (Osborne and Rubinstein, 1994).

Modal logics in which the modalities are intended to model *epistemic* operations and states constitute the core of a very active research program in Epistemic Logic (Hintikka, 1962). Starting in the 1990s, (Bicchieri, 1993), (Meyer and Hoek, 1995)

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and (Bonanno and Battigalli, 1999), among others, presented multi-modal systems in which common knowledge could be captured. (Fagin et al., 1995) presents a proof of the equivalence of this modal syntactic approach and the set-theoretical one.

An important contribution consisted in showing that the equivalence[1] between an event being common knowledge and being a *public event* could be used to define a fixed-point characterization of common knowledge (Barwise, 1988). In turn, (Heifetz, 1999) demonstrated the equivalence between this latter characterization and the hierarchical approach, for all finite levels in the iteration.

Category-theoretical formal approaches to common knowledge are lacking in the literature, despite the seeming naturalness of such a purely structural approach to the problem. This is unlike the closely related case of notions like that of *common belief*, where beliefs are represented by means of probability distributions (Brandenburger and Dekel, 1987). In the latter case, the notion of *coalgebra*) allows to ensure the existence of a fixed-point corresponding to an isomorphism between an object and its image under an endofunctor in a category of probability distributions (Moss and Viglizzo, 2004). One possible reason why this approach has not been translated to the common knowledge case is that the analogous endofunctor in Aumann's set-theoretical framework would be quite similar to a *powerset* functor, which does not have fixed-points.

In this paper we present an alternative categorical treatment in which a fixed-point yields states in which a ground proposition is commonly known. To ensure this result we need to be able to iterate an endofunctor such that it is, if not exactly idempotent, capable of being "smoothed out" or "unfolded" in a sufficiently idempotent-like way. This relatively vague idea may be made precise by way of the categorical construction called a *monad*. Making use of a monad construction will indeed help to resolve the problem indicated above such that analogues to the *powerset* functor typically lack fixed-points, but this mechanism alone will not yet be sufficient for modeling common knowledge categorically as a fixed-point in the intended manner. Any suitable structure, as already noticed in (Vassilakis, 1992), will require a more careful treatment that must involve infinite iterations, since if only finite iterations are allowed, there will not in general exist any fixed-points without further additions to the formalism.

A natural framework in which knowledge sequences can be studied without extra structure is given by the notion of *grossone*, first defined by Yaroslav Sergeyev (Sergeyev, 2017). In Sergeyev's approach the class of natural numbers is $\mathbb{N} = \{1, 2, 3, \ldots, ① - 2, ① - 1, ①\}$.

The basic idea is that *grossone* allows for a finer control on denumerable sequences than the classic Cantorian approach (for details, see Sergeyev's presentation (Sergeyev, 2013)). The *grossone* framework is based on the *Infinite Unit Axiom*, which postulates that:

1. *Infinity*: any finite number n is less than the grossone. That is $n < ①$ for every finite natural number n.
2. *Identity*: there are several relations linking ① to the identity elements 0 and 1, namely

$$0 \cdot ① = ① \cdot 0 = 0\ ;\ ① - ① = 0\ ;\ \frac{①}{①} = 1\ ;\ ①^0 = 1^{①} = 1\ ;\ 0^{①} = 0$$

[1] Already pointed out in (Lewis, 1969).

3. *Divisibility*: for any finite number n, if $1 \leq k \leq n$, define $\mathbb{N}_{n,k} = \{k, k+n, k+2n, k+3n, \ldots\}$. The class $\{\mathbb{N}_{n,k}\}_{1 \leq k \leq n}$ satisfies that $\cup_{k=1}^{n} \mathbb{N}_{n,k} = \mathbb{N}$ and each $\mathbb{N}_{n,k}$ has a number of elements indicated by the numeral $\frac{①}{n}$.

Within this framework, Sergeyev proves (Sergeyev, 2008) the following result:

Theorem: *The number of elements in an infinite sequence is less or equal to ①.*

This result has an interesting consequence. Given a sequence $\{a_n\}$, it is not enough simply to provide a formula for each a_n. We must also determine the *first* and *last* elements in the sequence. Thus, given two sequences

$$\{a_n\} = \{5, 10, 15, \ldots, 5(① - 1), 5①\}$$
$$\{b_n\} = \{5, 10, 15, \ldots, 5(\frac{2①}{5} - 1), 5\frac{2①}{5}\}$$

even if $a_n = b_n = 5n$ they are different because they have a different number of elements.

The grossone approach has relevant implications in several applied fields (see for instance (Sergeyev, 2020) for some recent successful developments). Philosophical and logical aspects of the grossone have been extensively investigated (Lolli, 2015; Rizza, 2018; Rizza, 2019; Tohmé et al., 2020), and our work aims to extend the reach of applications in these fields of that idea.

Most of our analysis is independent of how we represent $\mathbb{N}$, either in the *grossone* or the Cantorian framework, except in the crucial step of defining common knowledge as "fixed point". For this, we apply the *grossone* formalism to ensure the existence of states in which a given proposition is common knowledge.

The advantages of resorting to the *grossone* formalism are rather straightforward. In our setting this means that the number of rounds of reasoning needed to reach common knowledge are undefined but finite. If we do not assume that our agents are capable of performing *supertasks*[2], to model this reasoning process in the Cantorian setting would make little sense, since reaching common knowledge is usually the *first* step in a decision-making setting. On the other hand, the usual proofs of existence of fixed points of monotonic operators on well-ordered sets in the Cantorian setting require transfinite induction (Echenique, 2005). Its application in the case of common knowledge would make sense only for agents able to carry out the corresponding reasoning processes in finite time. Thus, our *grossone* version drops the need of superhuman abilities in our agents and provides a simple and direct result, formalizing the intuition that at some point all the agents will reach a state in which some events are common knowledge.

The paper is organized as follows. In Section 2 we present the basic elements of category theory needed to make this paper self-contained. In Section 3 we define a category of knowledge hierarchies. Section 4 presents a monad **In** in this category. In Section 5 we show the existence of a fixed-point defined in terms of **In**, yielding a state in which a proposition is common knowledge.

[2]"A supertask may be defined as an infinite sequence of actions or operations carried out in a finite interval of time" (Manchak et al., 2016).

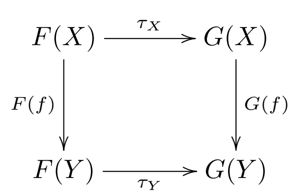
2. Our Categorical Toolbox

A *category* consists of a class of *objects* together with *morphisms* or arrows between objects. Given two objects a and b a morphism f between them will be denoted either $f : a \to b$ or $a \xrightarrow{f} b$. A category is subject to axioms of *identity* (every object a is equipped with an identity morphism, $a \xrightarrow{1_a} a$), *composition* (two morphisms, $a \xrightarrow{f} b$ and $b \xrightarrow{g} c$ compose to a unique morphism $a \xrightarrow{g \circ f} c$, where $\circ$ indicates the operation of composition) and *associativity* (paths of morphisms compose uniquely, i.e. given three arrows $f : a \to b, g : b \to c$ and $h : c \to d$, $h \circ (g \circ f) = (h \circ g) \circ f$, given the same morphism from a to d). For more details on the category-theoretical notions to be used in this paper see (Mac Lane, 2013), (Johnston, 2002) or (Spivak, 2014)).

A functor F is map between two categories (say $\mathbf{A}$ and $\mathbf{B}$), sending objects to objects and morphisms to morphisms. If for any morphism $a \to a'$ in $\mathbf{A}$, $F(a \to a')$ is mapped to a morphism $F(a) \to F(a')$ in $\mathbf{B}$, F is said *covariant*. If instead $F(a \to a')$ maps to $F(a) \leftarrow F(a')$, F is *contravariant*. A contravariant functor can be seen as a covariant functor $F : \mathbf{A}^{Op} \to \mathbf{B}$, where $\mathbf{A}^{Op}$ is obtained from $\mathbf{A}$ by reversing the direction of all its morphisms.

Given two functors $F, G : \mathbf{C} \to \mathbf{D}$, a *natural transformation* $\tau : F \to G$ is such that:

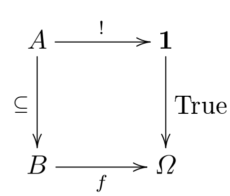
- For each object X in category $\mathbf{C}$, there exists a morphism in $\mathbf{D}$, $\tau_X : F(X) \to G(X)$.
- Given a morphism in $\mathbf{C}$, $f : X \to Y$ the following diagram commutes:



  meaning that $G(f) \circ \tau_X = \tau_Y \circ F(f)$.

If a contravariant functor F has as codomain the category of sets, i.e. $F : \mathbf{A}^{Op} \to \mathbf{Set}$, F is called a *presheaf.* Given a fixed category $\mathbf{A}$, the category in which the objects are all the preasheaves over $\mathbf{A}$ while the morphisms are the natural transformations between preasheaves is called a *topos*.

The paradigmatic example of a topos is $\mathbf{Set}$ itself, hinting at the fact that rich structures can be constructed inside any topos. For instance, each topos has an object Ω called *subobject classifier*, which in the case of $\mathbf{Set}$ is the set $\{0, 1\}$ which for every pair of sets A, B with $A \subseteq B$ makes the following diagram commutative:



where $!_A : A \to \mathbf{1}$ indicates the unique morphism from A to the *terminal* object $\mathbf{1}$,[3] True picks up the largest element in Ω, namely 1, and f is the function that assigns 1 to the fact that A is indeed a subset of B.

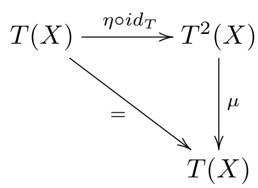
In a general topos, we replace $\subseteq$ with a *monomorphism*, which is a concept that abstracts away the notion of a *one-to-one* or *injective* function. In the case of a category of presheaves, we say that given $F, G : \mathbf{A}^{Op} \to \mathbf{Set}$, $F \to G$ is a monomorphism if $F(a) \subseteq G(a)$ for each object a in $\mathbf{A}$. In turn $\mathbf{1}$ is such that $\mathbf{1}(a)$ has a single element for each a in $\mathcal{A}$. The subobject classifier Ω consists of all the *sieves* on each object a in $\mathbf{A}$.[4]

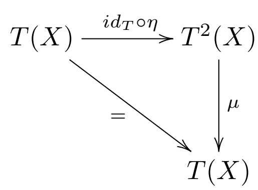
In this paper we will build a *complete partial order* where the elements are objects in a topos and the order obtain from morphisms among them.

A concept that will be relevant for our analysis is that of a *monad* which consists of:

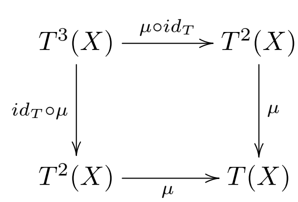
- An endofunctor (a functor from a category to itself) $T : \mathbf{C} \to \mathbf{C}$.
- Two natural transformations:
  - *unit map*: $\eta : id_{\mathbf{C}} \to T$ (where $id_{\mathbf{C}}$ is the identity in $\mathbf{C}$).
  - *multiplication map*: $\mu : T^2 \to T$ (where T^2 is the functor obtained composing T with itself.

such that the following diagrams commute (Spivak, 2014)), pp. 436):





and



A salient example of a monad is, in **Set** the *powerset* endofunctor, that yields for every set X the class of its subsets. For any given set X the multiplication map sends every subset X_2 of a subset X_1 of X to X_1. Another example is the *list*

[3] In **Set** the terminal object consists of a *singleton* $\{*\}$ since there exists a single function from any set to it.

[4] A *sieve* S on a in $\mathbf{A}$ is the class of all the morphisms $f : a' \to a$ in the category, such that, if there exists a morphism $g : a'' \to a'$ in $\mathbf{A}$, $f \circ g : a'' \to a$ belongs also to S.

endofunctor on **Set**, that given any X builds a list of its elements. The corresponding multiplication map for a set X collapses a list of lists of X's elements to a plain list of the elements of X.

3. A Category of Knowledge Hierarchies

Given a set $\mathcal{A}$ of agents, we define a category $\mathcal{A}_{seq}$ such that

- Given $\mathbb{N} = \{1, 2, 3, \ldots, ① - 2, ① - 1, ①\}$, we define $Obj(\mathcal{A}_{seq})$ as the class of sequences of elements of $\mathcal{A}$ of length less than ①. Each object $\bar{a}$ over $\mathcal{A}$, can be understood as a *word* in the alphabet on A.[5]
- Each morphism $\bar{a} \xrightarrow{f} (\bar{x}, \bar{a})$ between two objects $\bar{a}$ and $(\bar{x}, \bar{a})$ is the right inclusion of a sequence $\bar{a}$ into a sequence with prefix $\bar{x} \in Obj(\mathcal{A}_{seq})$ and suffix $\bar{a}$.

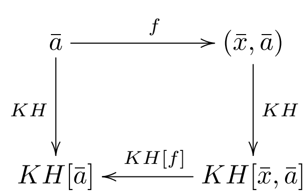
A contravariant functor $KH : \mathcal{A}_{seq}^{op} \longrightarrow \mathbf{Set}$ assigns a sequence $\bar{a}$ in $\mathcal{A}_{seq}$ to a *knowledge hierarchy.*

Given $\bar{a} = (a_1, a_2, \ldots a_n)$ we have that:

$$KH[\bar{a}] = \{\gamma : \gamma \text{ is a state where } a_1 \text{ knows that } a_2 \ldots \text{knows that } a_n \text{ knows } P \in \Lambda\}$$

where Λ is a class of *objective facts of the world.* Each $KH[\bar{a}]$ is a *section* of KH at $\bar{a}$.

Given a morphism $\bar{a} \xrightarrow{f} (\bar{x}, \bar{a})$ where $\bar{x} = (x_1, x_2, \ldots, x_m)$ and $\bar{a} = (a_1, a_2, \ldots, a_n)$, the following diagram commutes:



A morphism like $KH[f]$, a *restriction* along f, has a clear interpretation: for each $\gamma \in KH[\bar{x}, \bar{a}]$ it assigns a $\gamma' \in KH[\bar{a}]$ such that γ' corresponds to the "a_1 knows that $a_2 \ldots$ knows that a_n knows γ" fragment of the hierarchy "x_1 knows that x_2 knows that x_m knows a_1 knows that $a_2 \ldots$ knows that a_n knows γ".

Figure 1 depicts a contravariant functor on 2, 1 and empty sequences on $\mathcal{A} = \{a, b\}$.

Consider then the category of contravariant functors from $\mathcal{A}_{seq}$ to **Set**, $\mathcal{KH}$:

- Each $KH \in Obj(\mathcal{KH})$ is a contravariant functor $KH : \mathcal{A}_{seq}^{op} \longrightarrow \mathbf{Set}$ with the properties described above.
- Given two objects KH, KH' of $\mathcal{KH}$, a morphism $KH \xrightarrow{\tau} KH'$ is a *natural transformation.* That is, given a morphism $\bar{a} \xrightarrow{f} (\bar{x}, \bar{a})$ in $\mathcal{A}_{seq}$, the following

[5]This recasts in the grossone framework the Computer Science concept of *streams.* See (Manes, and Arbib, 1986,pp. 276), (Barwise and Moss, 1996, pp. 34) or (Milewski, 2017, pp. 234).

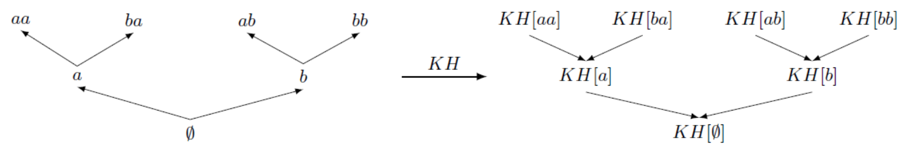


Fig. 1. Example of a contravariant functor KH

diagram commutes:

$$\begin{array}{ccc} KH'[\bar{a}] & \xleftarrow{KH'[f]} & KH'[\bar{x},\bar{a}] \\ \downarrow{\scriptstyle \tau_{\bar{a}}} & & \downarrow{\scriptstyle \tau_{\bar{x},\bar{a}}} \\ KH[\bar{a}] & \xleftarrow{KH[f]} & KH[\bar{x},\bar{a}] \end{array}$$

With this specification, $\mathcal{KH}$ is a category of presheaves on **Set** and thus a *topos*.

In this topos, as briefly discussed in Section 2, $KH \xrightarrow{\tau} KH'$ is a *monomorphism* if $KH[\bar{a}] \subseteq KH'[\bar{a}]$ for each $\bar{a}$ in $\mathcal{A}_{seq}$. In turn, the terminal object $\mathbf{1}$ is such that $\mathbf{1}[\bar{a}]$ includes a single state of knowledge for each $\bar{a}$.

The *subobject classifier* Ω is defined as follows. For each $\bar{a} = (a_1, \dots, a_n)$, $\Omega[\bar{a}]$ includes all the *sieves* on $\bar{a}$. Any such a sieve S includes a class of morphisms $\bar{a}' \to \bar{a}$, where $\bar{a}'$ is a subsequence of $(a_1, \dots, a_n)$ as well as all the morphisms that can be composed with morphisms already in S in such a way that the composition has codomain a. A trivial result is the following:

Proposition 1 *If given a monomorphism $KH \xrightarrow{\tau} KH'$ the following diagram commutes*

$$\begin{array}{ccc} KH & \xrightarrow{!} & \mathbf{1} \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle True} \\ KH' & \xrightarrow[f]{} & \Omega \end{array}$$

then, for every $\hat{a}$ in $\mathcal{A}_{seq}$ of length ①, True selects in $\Omega[\hat{a}]$ the sieve of all morphisms $\bar{a} \xrightarrow{f} (\bar{x}, \bar{a})$ for every subsequences $\bar{a}$ and $(\bar{x}, \bar{a})$ in the sequence $\hat{a}$. This sieve corresponds to a single state $\gamma_{①}$.

Proof: It follows from the definition of the subobject classifier and the True natural transformation. □

4. The Unfolding Monad

We define an *endofunctor* $\mathbf{In} : \mathcal{KH} \to \mathcal{KH}$ such that $\mathbf{In}(KH) = KH'$ specified as follows. Given $\bar{a}$ in $\mathcal{A}_{seq}$ and an element $\gamma \in KH[\bar{a}]$, $\mathbf{In}$ assigns $\gamma' \in KH'[\bar{a}]$, with

$$\gamma' = \langle \gamma_{-1}, \gamma_{-2}, \dots, \gamma_{-n} \rangle$$

where:

– $\bar{a} = (a_1, \ldots, a_n)$, $\bar{a}_{-1} = (a_2, \ldots, a_n)$, $\bar{a}_{-2} = (a_3, \ldots, a_n)$, …, $\bar{a}_{-n} = ()$, such that

$$\bar{a}_{-n} \xrightarrow{f_n} \bar{a}_{-(n-1)} \xrightarrow{f_{n-1}} \ldots \bar{a}_{-1} \xrightarrow{f_1} \bar{a}$$

– Then:

$$KH[\bar{a}_{-n}] \xleftarrow{KH[f_n]} KH[\bar{a}_{-(n-1)}] \xleftarrow{KH[f_{n-1}]} \ldots KH[\bar{a}_{-1}] \xleftarrow{KH[f_1]} KH[\bar{a}]$$

– For each $i = 1, \ldots, n-1$, each $\gamma_{-(i+1)} \in KH[f_{i+1}](KH(\bar{a}_{-i}))$.

Now consider the following natural transformations:

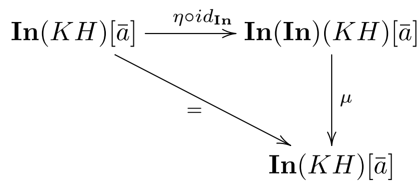
– The *unit map* $\eta : id_{\mathcal{KH}} \to \mathbf{In}$, such that for given $\bar{a}$, $\eta_{\bar{a}}[\gamma] = \gamma'$ for $\gamma \in KH[\bar{a}]$ and $\gamma' \in \mathbf{In}(KH)[\bar{a}]$.
– The *multiplication map* $\mu : \mathbf{In} \circ \mathbf{In} \to \mathbf{In}$, such that given $\bar{a}$, if we have:
  - $\gamma \in KH[\bar{a}]$,
  - $\langle \gamma_{-1}, \gamma_{-2}, \ldots, \gamma_{-n} \rangle \in \mathbf{In}(KH)[\bar{a}]$ and
  - $\langle \langle \gamma_{-2}, \gamma_{-3}, \ldots, \gamma_{-n} \rangle, \langle \gamma_{-3}, \ldots, \gamma_{-n} \rangle, \ldots, \langle \gamma_{-n} \rangle \rangle \in \mathbf{In}(\mathbf{In})(KH)[\bar{a}]$.

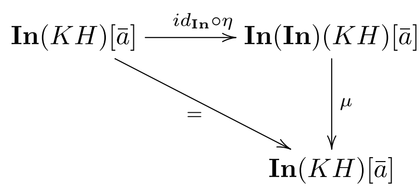
  Then $\mu_{\bar{a}}[\langle \langle \gamma_{-2}, \gamma_{-3}, \ldots, \gamma_{-n} \rangle, \langle \gamma_{-3}, \ldots, \gamma_{-n} \rangle, \ldots, \langle \gamma_{-n} \rangle \rangle] = \langle \gamma_{-1}, \gamma_{-2}, \ldots, \gamma_{-n} \rangle$

We have that:

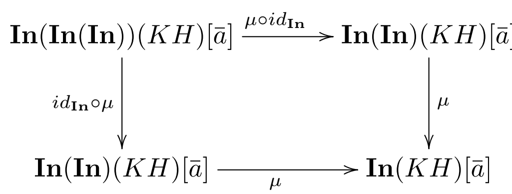
Proposition 2 **In**, *with η and μ constitutes a monad.*

Proof: as indicated in Section 2, we have to show that the following diagrams commute:





and



Consider the first diagram above (the second one is analogous). Take $\langle \gamma_{-1}, \gamma_{-2}, \ldots, \gamma_{-n} \rangle \in \mathbf{In}(KH)[\bar{a}]$ and apply first the identity of **In**, which yields exactly the same element and then $\eta_{[\bar{a}]}$, which gives $\langle \langle \gamma_{-2}, \gamma_{-3}, \ldots, \gamma_{-n} \rangle, \langle \gamma_{-3}, \ldots, \gamma_{-n} \rangle, \ldots, \langle \gamma_{-n} \rangle \rangle \in \mathbf{In}(\mathbf{In})(KH)[\bar{a}]$. If we apply now $\mu_{[\bar{a}]}$ on this element we obtain $\langle \gamma_{-1}, \gamma_{-2}, \ldots, \gamma_{-n} \rangle \in \mathbf{In}(KH)[\bar{a}]$, indicating that the diagram commutes.

With respect to the third diagram consider the following element in $\mathbf{In}(\mathbf{In}(\mathbf{In}))(KH)[\bar{a}]$:

$$\langle\langle\langle\gamma_{-3},\gamma_{-4},\ldots,\gamma_{-n}\rangle,$$
$$\langle\gamma_{-4},\ldots,\gamma_{-n}\rangle,\ldots,\langle\gamma_{-n}\rangle\rangle,\langle\langle\gamma_{-4},\ldots,\gamma_{-n}\rangle,\ldots,\langle\gamma_{-n}\rangle\rangle,\ldots,\langle\langle\gamma_{-n}\rangle\rangle\rangle$$

if we apply $\mu_{\bar{a}} \circ id_{\mathbf{In}(KH)[\mathbf{a}]}$ (or $id_{\mathbf{In}(KH)[\mathbf{a}]} \circ \mu_{\bar{a}}$) we obtain

$$\langle\langle\gamma_{-2},\gamma_{-3},\ldots,\gamma_{-n}\rangle,\langle\gamma_{-3},\ldots,\gamma_{-n}\rangle,\ldots,\langle\gamma_{-n}\rangle\rangle \in \mathbf{In}(\mathbf{In})(KH)[\bar{a}]$$

Then, the application of μ yields $\langle\gamma_{-1},\gamma_{-2},\ldots,\gamma_{-n}\rangle \in \mathbf{In}(KH)[\bar{a}]$, showing that the diagram commutes. □

The monad $(\mathbf{In},\eta,\mu)$ yields, for every presheaf KH in $\mathcal{KH}$ the class of *unfoldings* of its elements. In other words, for any $\gamma \in KH[\bar{a}]$ it gives the *fiber* over γ corresponding to the knowledge hierarchy below it. Notice that this hierarchy is unique, i.e. $\mathbf{In}^{-1}(\gamma)$ is a singleton. By the same token, μ yields also a single element in $\mathbf{In}(\mathbf{In})(KH)[\bar{a}]$, since it is completely defined by the knowledge (and the order) of the agents in $\bar{a}$.

Figure 2 illustrates how the monad acts on a presheaf KH at 3, 2, 1 and ground-sequences on $\mathcal{A} = \{a,b\}$:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\uparrow & & \downarrow & & & & \downarrow & & \\
aba & & KH[aba] & = & \{\gamma_{aba}\} & & \mathbf{In}(KH)[aba] & = & <\{\gamma_{ba},\gamma_a,p\}> \\
\uparrow & & \downarrow & & & & \downarrow & & \\
ba & & KH[ba] & = & \{\gamma_{ba}\} & & \mathbf{In}(KH)[ba] & = & <\{\gamma_{ba},p\}> \\
\uparrow & & \downarrow & & & & \downarrow & & \\
a & & KH[a] & = & \{\gamma_a\} & & \mathbf{In}(KH)[a] & = & <p> \\
\uparrow & & \downarrow & & & & \downarrow & & \\
\emptyset & & KH[\emptyset] & = & p & & \mathbf{In}(KH)[\emptyset] & = & p
\end{array}$$

Fig. 2. Representation of **In** on $\mathcal{A} = \{a,b\}$

5. Common Knowledge

Given the monad $(\mathbf{In},\eta,\mu)$, any KH in $\mathcal{KH}$ is such that $\eta_{\bar{a}} : KH[\bar{a}] \to \mathbf{In}(KH)[\bar{a}]$ yields $\eta_{\bar{a}}[\gamma] = \langle\gamma_{-1},\ldots,\gamma_{-n}\rangle$.

Any $\hat{\gamma}$ such that $\eta_{\bar{a}}[\hat{\gamma}] = \hat{\gamma}$ is a *fixed point*. This would mean that $KH[\bar{a}] \subseteq \mathbf{In}(KH)[\bar{a}]$. But this is not possible in particular for any $\bar{a} \in A^n$ with $n <$ ① since the sequence $\bar{a}_{-1},\ldots,\bar{a}_{-n}$ has length $n-1$ (if we discard the final empty sequence).

This means that for $\hat{\gamma} \in KH[\hat{a}]$ to be a fixed point it is necessary to ask for the satisfaction of certain conditions. Consider the following two:

- *Uniformity*: $\hat{a} = (\ldots,\mathbf{a},\mathbf{a},\ldots,\mathbf{a})$ where $\mathbf{a} \in \mathcal{A}^{|\mathcal{A}|}$ such that it does not include repeated agents.

– *Exchangeability*: $\mathbf{a} \equiv \mathbf{a}'$ where $\mathbf{a} = (a_1, \ldots, a_{|\mathcal{A}|})$ and $\mathbf{a}' = (a_{p(1)}, a_{p(2)}, \ldots, a_{p(|\mathcal{A}|)})$, where $p : \{1, \ldots, |\mathcal{A}|\} \to \{1, \ldots, |\mathcal{A}|\}$ is a bijection. This means that two sequences of all the agents are the same, modulo permutations of their names.

Notice that, according to the definition of $\mathcal{A}_{seq}$, a uniform and exchangeable sequence $\hat{a}$ will have length less than ①. We can denote it $\hat{a}_{①}$.

Proposition 3 *If $\hat{a}_{①}$ satisfies uniformity and exchangeability, any $\hat{\gamma}$ such that $\eta_{\hat{a}_{①}}[\hat{\gamma}] = \hat{\gamma}$ represents a state in which a given $P \in \Lambda$ is* common knowledge

Proof: Since $\hat{\gamma}$ is a fixed point of $\eta_{\hat{a}_{①}}$, it means that there exists a $P \in \Lambda$ such that the $\hat{\gamma}$ is identical to the fiber over $\hat{\gamma}$, which includes the following states of knowledge:

– a_1 knows P,
– ...,
(*) **a** know P,
– ...,
(**) **a** know that **a** know P,
– ...,
(***) **a** know that **a** know that **a** know P,
– ...

where the starred statements can be translated as $(*)$ "everybody knows P", $(**)$ "everybody knows that everybody knows P", $(***)$ "everybody knows that everybody knows that everybody knows P", etc. That is, P is common knowledge. □

The existence of such $\hat{\gamma}$ is predicated on two properties:

Proposition 4 *There exists an object $\hat{a}$ of $\mathcal{A}_{seq}$ satisfying uniformity and exchangeability.*

Proof: Trivial. By definition, $Obj(\mathcal{A}_{seq})$ includes any possible sequence of agents of length $<$ ①, in particular $\hat{a}_{①}$. □

The remaining property is the existence of a fixed point of $\eta_{\hat{a}_{①}}$. We will use here the fact that $\mathcal{KH}$ is a *topos*. As shown in (Johnston, 2002), internal categories can be defined inside a topos.[6] We will define such internal category $\mathcal{K}_P$, which can be trivially described as a partially ordered set. The first step in this construction requires considering a sequence in $\mathcal{A}_{seq}$, denoted $\hat{a}^{\downarrow}_{①}$:

$$a_n \to (a_{n-1}\ a_n) \to (a_{n-2}\ a_{n-1}\ a_n) \to \mathbf{a} \to \ldots \to (\mathbf{a}\ \mathbf{a}) \to \ldots \to \hat{a}_{①}$$

such that $\hat{a}_{①}$ satisfies uniformity and exchangeability. Given a $P \in \Lambda$ we consider the class of presheaves $\overline{\mathcal{KH}} = \{KH \in \mathcal{KH} : KH[\bar{a}] \neq \emptyset,$ for every $\bar{a} \in \hat{a}^{\downarrow}_{①}$ with $KH[\emptyset] = \{P\}\}$. This class is non-empty in $\mathcal{KH}$, since always exists a KH

[6] An important proviso is that there does not exist a one-to-one function between $\mathcal{KH}$ and the internal category $\mathcal{K}_P$. The cardinality of the objects in the latter is strictly less than that of $\mathcal{KH}$, since we have assumed that the length of $\hat{a}_{①}$ is strictly less than ①.

such that each section is non-empty.

We can now define a category $\mathcal{K}_P$, in which its set of objects is $\mathcal{K}_P^0 = \{KH[\emptyset], KH[a_n], KH[a_{n-1}a_n], \quad \ldots, KH[\mathbf{a}], KH[a_n\mathbf{a}], \ldots, KH[\mathbf{aa}], \ldots KH[\hat{a}_{\textcircled{1}}], \text{ for each } KH \in \overline{\mathcal{KH}}\}$. The data of $\mathcal{K}_P$ includes a set $\mathcal{K}_P^1$ of morphisms among the objects of $\mathcal{K}_P^0$, inherited from the category $\mathcal{KH}$.

For each morphism $f : \mathbf{a} \rightarrow \bar{x}\mathbf{a}$, we define $KH[\mathbf{a}] \preceq KH[\bar{x}\mathbf{a}]$ for every $KH \in \overline{\mathcal{KH}}$. This means that any element in $KH[\mathbf{a}]$ is the tail in the unfolding of an element in $KH[\bar{x}\mathbf{a}]$.

With this order relation $\mathcal{K}_P$ becomes a *complete* partially ordered set (*poset*), i.e. every subset of $\mathcal{K}_P$ as a *least upper bound*. To see this, consider any set $\{KH_1[\bar{a}_1], \ldots, KH_\alpha[\bar{a}_\alpha]\}$ in $\mathcal{K}_P^0$, for $\alpha < \textcircled{1}$. Take the least upper bound of $\{\bar{a}_1, \ldots, \bar{a}_\alpha\}$, an element in the sequence $\hat{a}^{\downarrow}_{\textcircled{1}}$, denoted $\bar{a}$.[7] Then take $\cup_{j=1}^{\alpha} KH_j[\bar{a}] \neq \emptyset$. There always exist a presheaf $\mathbf{KH} \in \overline{\mathcal{KH}}$ that satisfies the condition of having non-empty sections at all the elements in the sequence $\hat{a}^{\downarrow}$ and such that $\mathbf{KH}[\bar{a}] = \cup_{j=1}^{\alpha} KH_j[\bar{a}] \in \mathcal{K}_P$. It follows that $KH_j[\bar{a}_j] \preceq \mathbf{KH}[\bar{a}]$, for each $j = 1, \ldots, \alpha < \textcircled{1}$.

Notice that $\mathbf{KH}$ is such that $KH \xrightarrow{\tau} \mathbf{KH}$ is a monomorphism for every KH in $\overline{\mathcal{KH}}$. And thus, according to Proposition 1 at $\hat{a}_{\textcircled{1}}$ the identity $\mathbf{KH} \xrightarrow{1_{\mathbf{KH}}} \mathbf{KH}$ (a trivial monomorphism) corresponds to a single state $\hat{\gamma}_{\textcircled{1}}$.

We also have that $KH[a_\alpha] \preceq \mathbf{KH}[\hat{a}_{\textcircled{1}}]$ for every $KH[a_\alpha] \in \mathcal{K}_P^0$. Then, the restriction of the monad endofunctor on $\mathcal{K}_P$, $\mathbf{In} : \mathcal{K}_P \rightarrow \mathcal{K}_P$ must satisfy that $\mathbf{In}(\mathbf{KH})[\hat{a}_{\textcircled{1}}] \preceq \mathbf{KH}[\hat{a}_{\textcircled{1}}]$. But this means that for each $\hat{\gamma} \in \mathbf{KH}[\hat{a}_{\textcircled{1}}]$ its unfolding must be its own tail. But this is only possible if $\eta_{\hat{a}_{\textcircled{1}}}[\hat{\gamma}_{\textcircled{1}}] = \hat{\gamma}_{\textcircled{1}}$.

Since for each $P \in \Lambda$ a corresponding complete internal poset $\mathcal{K}_P$ can be defined, we have proven the following claim:

Proposition 5 *There exists a presheaf* $\mathbf{KH}$ *in* $\mathcal{KH}$ *and a state* $\hat{\gamma}_{\textcircled{1}} \in \mathbf{In}(\mathbf{KH})[\hat{a}_{\textcircled{1}}] = \mathbf{KH}[\hat{a}_{\textcircled{1}}]$ *such that in* $\hat{\gamma}_{\textcircled{1}}$, $P \in \Lambda \cap \mathbf{KH}[\emptyset]$ *is common knowledge.*

Figure 3 represents how a fixed-point arises on $\mathcal{A} = \{a, b\}$.

$$
\begin{array}{ccccc}
\hat{a}_{\textcircled{1}} & \hat{\gamma} \in KH(\hat{a}_{\textcircled{1}}) & \xrightarrow{\eta_{\hat{a}_{\textcircled{1}}}} & \mathbf{In}(KH)[\hat{a}_{\textcircled{1}}] & = \{\ldots, \hat{\gamma}, \ldots\} \\
\vdots & \vdots & & \vdots & \\
aba & KH[aba] & \xrightarrow{\eta_{aba}} & \mathbf{In}(KH)[aba] & \\
\uparrow & \downarrow & & \downarrow & \\
ba & KH[ba] & \xrightarrow{\eta_{ba}} & \mathbf{In}(KH)[ba] & \\
\uparrow & \downarrow & & \downarrow & \\
a & KH[a] & \xrightarrow{\eta_{a}} & \mathbf{In}(KH)[a] & \\
\uparrow & \downarrow & & \downarrow & \\
\emptyset & KH[\emptyset] & \xrightarrow{\eta_{\emptyset}} & \mathbf{In}(KH)[\emptyset] &
\end{array}
$$

Fig. 3. Fixed point of $\eta_{\hat{a}_{\textcircled{1}}}$

[7] $\hat{a}^{\downarrow}_{\textcircled{1}}$ is, by definition, a linear order, and thus trivially a complete poset.

6. Conclusions and Further Work

The category $\mathcal{KH}$ equipped with the monad $(\mathbf{In}, \eta, \mu)$ provides a categorical framework for treating the problem of common knowledge from the perspective of fixed-points of an endofunctor defined on presheaves over the category of hierarchies of a fixed set of agents. Our main result is that under the conditions of *uniformity* and *exchangeability*, a fixed point may be determined such that the associated proposition necessarily takes the form of common knowledge among the agents in the model. Crucial to the construction has been the use of the *grossone* notation in order to regulate the infinite hierarchical sequences of agents at issue.

In further research, we hope to extend the categorical framework for common knowledge established here in the direction of a formal theory of collective agency in games.

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