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Computation Problems for Envy Stable Solutions of Allocation Problems with Public Resources

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Abstract We consider generalizations of TU games with restricted cooperation in partition function form and propose their interpretation as allocation problems with several public resources. Either all resources are goods or all resources are bads. Each resource is distributed between points of its set and permissible coalitions are subsets of the union of these sets. Each permissible coalition estimates each allocation of resources by its gain/loss function, that depends only on the restriction of the allocation on that coalition. A solution concept of "fair" allocation (envy stable solution) was proposed by the author in (Naumova, 2019). This solution is a simplification of the generalized kernel of cooperative games and it generalizes the equal sacrifice solution for claim problems. An allocation belongs to this solution if there do not exist special objections at this allocation between permissible coalitions. For several classes of such problems we describe methods for computation selectors of envy stable solutions.
Keywords: Wardrop equilibrium, envy stable solution, games with restricted cooperation, equal sacrifice solution.

1. Introduction

We consider a problem of "fair" distribution several public resources between elements of a finite set N. We suppose that either all distributing resources are public goods or all resources are public bads.

There is a partition τ of N such that each resource is distributed between elements of its $B \in \tau$. Similar allocations arise when different coalitions from τ correspond either to different financial sources or to different moments of distribution of one resource.

There is a family $\mathcal{A}$ of permissible coalitions which estimate allocations. Each coalition in $\mathcal{A}$ estimates an allocation by its gain/loss function G_S and the result of the estimation depends only on the restriction of the allocation on this coalition. Note that permissible coalitions may intersect (persons from the same region, persons of the same age, persons of the same sex can form permissible coalitions), but this does not influence to gains/losses of coalitions because we suppose that each resource is public one. A distributer needs to allocate resources as fair as possible.

One of the problems of a distributer is to find an allocation that seems fair for permissible coalitions. Our model includes an undirected graph Γ, where permissible coalitions are its nodes, and two permissible coalitions can compare their gains/losses at an allocation iff they are adjacent in Γ.

Envy stable allocations for such problems were introduced by the author in (Naumova, 2019).

These solutions are modifications of the kernel of cooperative games (see Sudholter and Peleg, 2007) and they generalize the equal sacrifice solution for claim

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problems (see Moulin, 2002 for example). We define objections at an allocation between coalitions that are adjacent in Γ, and an allocation is envy stable iff there are no objections at it.

Special cases of envy stable solutions w.r.t. (τ, Γ) were considered by the author (in Naumova, 2011, Naumova, 2012) for the cases, when $\tau = \{N\}$, the gain functions are either excesses or proportional excesses of cooperative games with restricted cooperation, and either all different coalitions in $\mathcal{A}$ are adjacent in Γ or all coalitions with empty intersections are adjacent in Γ.

Conditions on Γ and τ that ensure the existence of the envy stable solution for each collection of gain/loss functions of coalitions are described in (Naumova, 2019). The obtained conditions generalize the results (in Naumova, 2011, Naumova, 2012).

In this paper we describe methods for computation selectors of envy stable solutions for two classes of gain functions. For the case when the existence condition on Γ and τ is fulfilled, all $G_S(0) = 0$ and G_S are linear functions, methods of linear programming and Bregman's iterative method (Bregman, 1967) are suitable. The results of Bregman's method seem more fair because they seem fair not only for admissible coalitions but also for players.

The values of gain functions G_S of the second class depend on total amounts of resources. We propose computation methods that are correct under additional conditions on Γ, τ that are stronger than the condition which guarantees the existence result of envy stable solution.

The paper is organized as follows. In section 2 we give basic definitions and describe conditions on Γ, τ that guarantees existence result for envy stable solution. In section 3 we describe Bregman's iterative method for some linear programming problems. In section 4 for some linear gain functions, we formulate linear programming problem and compare its solutions by linear programming methods and by Bregman's iterative method. In section 5 we consider special cases of distribution of a unique resource. In section 6 we describe iterative methods for obtaining selectors of envy stable solutions under different restrictions on the functions G_S and the graph Γ. The obtained results are discussed in section 7.

2. Allocation model

For $N = \{1, \ldots, n\}$, let $\mathcal{A}$ be a collection of some subsets of N, $\mathcal{A}$ covers N, τ be a partition of N, $c_B > 0$ for each $B \in \tau$. Denote $C = \{c_B\}_{B\in\tau}$. Then

$$X = X(\tau, C) = \{x \in R^n : x_i \geq 0, \sum_{i\in B} x_i = c_B, B \in \tau\}$$

is the **set of allocations**. For $S \in \mathcal{A}$, $x \in X$, let $x_S = \{x_i\}_{i\in S}$, and G_S be a continuous strictly increasing in each variable function defined on

$$X_S = \{x_S : x \in X\}.$$

We suppose that either all resources are public goods or all resources are public bads and the function G_S is a gain/loss utility function of the coalition S. Thus, for each allocation x, each x_i can be used by each $S \in \mathcal{A}$ such that $i \in S$, and $G_S(x_S)$ is the gain/loss of S at the allocation x.

A distributer needs to take an allocation in X that seems "fair" for a collection of coalitions $\mathcal{A}$. Such problems arise either if different coalitions from τ have different resources or if different coalitions from τ correspond to different financial sources.

Cooperative transferable utility games generate special allocation problems as follows.

Denote $x(S) = \sum_{i \in S} x_i$.

Let (N, v) be a TU-cooperative game with $v(N) > 0$, $\tau = \{N\}$, $\mathcal{A}$ be a collection of permited coalitions of N. Then (N, v) generates $c = v(N)$, X is the set of imputations and $G^1_S(x_S) = x(S) - v(S)$.

If moreover $v(S) > 0$ for each $S \in \mathcal{A}$, then (N, v) also generates $G^2_S(x_S) = x(S)/v(S)$. These models were considered in (Naumova, 2011, Naumova, 2012).

We consider the following concept of fair distribution.

Let Γ be an undirected graph, where $\mathcal{A}$ is the set of nodes and the nodes of each link are different.

An allocation x belongs to $\{G_S\}_{S \in \mathcal{A}}$ **- envy stable solution w.r.t.** Γ if for each link $\{P, Q\}$ of Γ, $G_P(x_P) > G_Q(x_Q)$ implies $x(P) = 0$.

This notion has the following interpretation. Objections are possible only between the ends of links of Γ. For each link $\{P, Q\}$ of Γ, if we distribute goods, then Q envies to P at x, but it can't get anything from P, and if we distribute bads, then P envies to Q at x, but P has no bads.

This definition generalizes equal sacrifice solution for claim problems.

Existence condition on $\mathcal{A}$ for envy free solution with respect to undirected Γ was obtained in (Naumova, 2019).

Theorem 1. *Let $\mathcal{A} \subset 2^N$, τ, undirected Γ be given. For all continuous strictly increasing in each variable functions G_S ($S \in \mathcal{A}$), the $\{G_S\}_{S \in \mathcal{A}}$ - envy stable solution w.r.t. Γ is a nonempty set if and only if $\mathcal{A}$ satisfies the following condition*
C0(Γ, τ). If in each component of Γ its arbitrary node is taken out from $\mathcal{A}$, then each $B \in \tau$ is not covered by the remaining elements of $\mathcal{A}$.

The property C0(Γ, τ) is illustrated by the following examples.

Example 1. Let $\mathcal{A}$ cover N and all nodes of Γ be adjacent. Then C0(Γ, τ) is fulfulled iff $\mathcal{A}$ minimally covers any $B \in \tau$.

Example 2. $K, L \in \mathcal{A}$ are adjacent in Γ iff $K \cap L = \emptyset$. $|N| = 4$, $\tau = \{N\}$, $\mathcal{A}$ consists of no more than 5 two-person coalitions. Then C0(Γ, τ) is fulfilled.

Example 3. $K, L \in \mathcal{A}$ are adjacent in Γ iff $K \cap L = \emptyset$. $|N| = 4$, $\tau = \{N\}$, $\mathcal{A}$ consists of all two-person coalitions.Then $C0(\Gamma, \tau)$ is not fulfilled. Indeed, if we take off the coalitions $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, then the remaining elements of $\mathcal{A}$ cover N.

Example 4. $K, L \in \mathcal{A}$ are adjacent in Γ iff $K \cap L = \emptyset$.
$N = \{1, \ldots 6\}$, $\tau = \{\{1, 2, 3, 4\}, \{5, 6\}\}$,
$\mathcal{A} = \{\{1, 2, 5\}, \{3, 4, 6\}, \{1, 3\}, \{2, 4\}\}$. Then C0($\Gamma, \tau$) is fulfilled.

3. Bregman's iterative method

Bregman (1967) proposed iterative methods for finding admissible points of systems of linear equations with nonnegative coordinates. In particular, for $x, y \in R^n$ with $x_i, y_i > 0$, he defined a nonsymmetric "distance function"

$$\rho(x, y) = \sum_{i=1}^{n} x_i (\ln(x_i/y_i) - 1).$$

Note that ρ is not a metric. If L is a hyperplane such that $L \cap R^n_+ \neq \emptyset$, then for each y with $y_i > 0$, there exists a unique solution of the problem

$$\rho(x, y) \to \min_{x \in L}.$$

This solution is called **the entropy projection of** y **on** L and can be found by Lagrange method.

Bregman, 1967 proved that if L^0 is an intersection of several hyperplanes and $L^0 \cap R^n_{++} \neq \emptyset$, then the solution of the problem

$$\rho(x, y) \to \min_{x \in L^0 \cap R^n_{++}}$$

is unique and it can be obtained as a limit of iterative entropy projections of y on these hyperplanes.

For some hyperplanes, the entropy projection on them is easily calculated because

$$\frac{d}{dx_i}\rho(x, y) = \ln x_i - \ln y_i.$$

Example 5. $L = \{x \in R^n : x(S) = a\}$, $a > 0$, and z is an entropy projection of y on L. Then $z_i = y_i$ for $i \notin S$, $z_i = \frac{a}{y(S)} y_i$ for $i \in S$.

Example 6. Let $L = \{x : x(P)/a_P = x(Q)/a_Q\}$, where $P, Q \subset N$, $P \cap Q = \emptyset$, $a_P > 0$, $a_Q > 0$, z be the entropy projection of y on L.

Denote $K(P, Q, y) = a_P y(Q)/(a_Q y(P))$. Then
$z_i = y_i$ for $i \in N \setminus (P \cup Q)$,
$z_i = K(P, Q, y)^{a_Q/(a_P + a_Q)}$ for $i \in P$,
$z_i = K(P, Q, y)^{-a_P/(a_P + a_Q)}$ for $i \in Q$.

4. Computation methods for linear functions

Let $G_S(x_S) = \sum_{i \in S} a_i^S x_i$, where $a_i^S > 0$. Let $\mathcal{A}$ satisfy Condition C0 then an allocation x of envy stable solution is an admissible point of the following system

$$\sum_{i \in S} a_i^S x_i - \sum_{i \in Q} a_i^Q x_i = 0 \quad \text{for each link } \{P, Q\} \text{ of } \Gamma,$$

$$\sum_{i \in B} x_i = c(B) \quad \text{for each } B \in \tau,$$

$$x_i \geq 0 \quad \text{for each } i \in N.$$

A solution of this system can be obtained by methods of linear programming.

Simplex method gives the results that do not look fair from the player's point of view even for trivial problems.

Example 7. $N = \{1, 2, 3, 4\}$, $\mathcal{A} = \{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}\}$, $\tau = \{N\}$, coalitions are adjacent in Γ iff they do not intersect, $G_S(x_S) = x(S)/v(S)$, where $v(1, 2) = 10$, $v(3, 4) = 8$, $v(1, 3) = 6$, $v(2, 4) = 12$, $c = v(N) = 18$.

Simplex method gives $x^0 = (0, 10, 6, 2)$, and $x_1^0 = 0$ seems unfair for players. The internal point method gives $x^1 = (3.5,\ 6.5,\ 2.5,\ 5.5)$ that seems fair both for coalitions and for players.

But for another example the internal point method also gives a vector that seems unfair for players.

Example 8. $N = \{1,2,3,4,5,6\}$, $\mathcal{A} = \{\{1,2\},\ \{3,4,5,6\},\ \{1,3\},\ \{2,4,5,6\}\}$, $\tau = \{N\}$, coalitions are adjacent in Γ iff they do not intersect, $G_S(x_S) = x(S)/v(S)$, where $v(1,2) = 10$, $v(3,4,5,6) = 8$, $v(1,3) = 6$, $v(2,4,5,6) = 12$, $c = v(N) = 18$.

Simplex method gives $x^2 = (0,10,6,0,0,2)$. The internal point method gives $x^3 = (3.5,\ 6.5,\ 2.5,\ 0,\ 0,\ 5.5)$. The results of these methods seem unfair for players because the players 4,5,6 are symmetric.

The advantage of Bregman's iterative method over simplex method and internal point method is that it can ensure equal results for "symmetric" players of N if their coordinates in y coincide.

Example 9. Consider the allocation problem from Example 8, where "symmetric" players 4, 5, 6 get nonequal distriburtion results by simplex method and internal point method. If $y = (1,\ 1,\ 1,\ 1/3,\ 1/3,\ 1/3)$ then Bregman's metod gives $x^4 = (3.5,\ 6.5,\ 2.5,\ 1.83,\ 1.83,\ 1.83)$.

Moreover, Bregman's method permits to take into account significances of points in N (if y_i describes the significance of $i \in N$).

However, Bregman's iterative method works if the solution of the problem exists. Theorem 1 gives only sufficient condition for existence of admissible points of the linear system. We can attempt to find an admissible point of the linear system by linear programming methods and use Bregman's method if such point exists.

5. Gain functions of total amounts of one resource

In this section we consider two cases, where the values of gain functions G_S depend on total amounts of resources. If $|\tau| > 1$, this is possible when one resource is distributed from different sources. We propose computation methods that are correct under additional conditions on Γ, τ. These conditions are stronger than the condition which guarantees the existence result of envy stable solution.

5.1. $\mathcal{A}$ is a union of several partitions

Let $\tau = \{N\}$, coalitions in $\mathcal{A}$ be adjacent in Γ iff they do not intersect, and each component of Γ be a partition of N. Let the condition $\mathrm{C0}(\Gamma,\tau)$ be fulfilled. (Such collections of coalitions $\mathcal{A}$ are called totally mixed colections of coalitions in (Naumova, 2011).) Let, moreover, $G_S(x_S) = g_S(x(S))$, where g_S are strictly increasing continuous functions. Then points of envy stable solution can be found in two steps.

Step 1. For each component $\mathcal{B}_k$ of Γ, determine envy stable solution for the problem, where $\mathcal{B}_k$ is the collection of admissible coalitions. The solutions of all such problems assign a unique collection of numbers $\{a_S\}_{S\in\mathcal{A}}$ such that $\sum_{S\in\mathcal{B}_k} a_S = c(N)$ for each component $\mathcal{B}_k$ of Γ and if $P, Q \in \mathcal{B}_k$, then $g_P(a_p) > g_Q(a_Q)$ implies $a_P = 0$.

Step 2. By Bregman's method, find a point x that satisfies the following conditions.
$\sum_{i\in S} x_i = a_S$ for each $S \in \mathcal{A}$, $x_i \geq 0$ for all $i \in N$.
Such point x exists in view of Theorem 1.

5.2. Several sources of the same resource

Now consider the case when $|\tau| > 1$, but for each permissible coalition, the value of its gain/loss function depends on the total amount of resources obtained by this coalition. Such problems arise when the same resource is distributed from several sources.

The following result was obtained in (Naumova, 2019, Proposition 1 and Theorem 4). We take an optimization problem, which solution is Wardrop equilibria defined in (Wardrop, 1952) for road traffic problems (see Krylatov and Zakharov, 2017 or Mazalov, 2010 for example), and give conditions on Γ, τ that ensure inclusion of Wardrop equilibria in envy stable solution. Thus for (Γ, τ) described in the following theorem, we reduce the problem of finding a selector of envy stable solution to an optimization problem.

Denote $\mathcal{A}_i = \{T \in \mathcal{A} : i \in T\}$.

A collection of coalitions $\mathcal{A}$ is a (Γ, τ)–**mixed collection of coalitions** if for each $B \in \tau$, $i \in B$, $Q \in \mathcal{A}_i$, and a link $\{Q, S\}$ of Γ, there exists $j \in B$ such that $\mathcal{A}_j = \mathcal{A}_i \cup \{S\} \setminus \{Q\}$.

Theorem 2. *Let Γ be an undirected graph, τ be a partition of N, $G_S(x_S) = g_S(x(S))$, $(S \in \mathcal{A})$. For all strictly increasing continuous functions g_S, the solutions of the problem*

$$\sum_{S\in\mathcal{A}} \int\limits_0^{z(S)} g_S(t)dt \to min_{\{z:\ z\in X\}}$$

are contained in the $\{G_S\}_{S\in\mathcal{A}}$ - envy stable solution w.r.t. Γ if and only if $\mathcal{A}$ is a (Γ, τ)–mixed collection of coalitions.

Example 10. $|N| = 4$, $\tau = \{N\}$, P and Q are adjacent in Γ iff $P \cap Q = \emptyset$, $\mathcal{A} = \{\{i, j\},\ \{k, l\};\ \{i, k\},\ \{j, l\}\}$, then $\mathcal{A}$ is a (Γ, τ)–mixed collection.

Other examples concerning mixed collections of coalitions are in the next section.

6. Dynamic systems for gain functions of total amounts for each resources

Here we restrict the class of gain/loss functions of coalitions as follows. Let τ be a partition of N, $\mathcal{A}$ be a collection of subsets of N. Denote by $\mathcal{G}^{\tau}_{\mathcal{A}}$ the set of G_S such that $S \in \mathcal{A}$ and $G_S(x_S) = g_S(\{x(S \cap B)\}_{B\in\tau})$ for some continuous strictly increasing in each of $|\tau|$ variables function g_S. In this case resources of different types are possible.

6.1. Positive and negative mixed collections of coalitions

Computation methods of this section permit two classes of collections of coalitions which are larger than the class of mixed collections of coalitions, introduced in the previous section.

A collection of coalitions $\mathcal{A}$ is a (Γ, τ)–**positive mixed collection of coalitions** if for each $B \in \tau$, $i \in B$, $P \in \mathcal{A}_i$, and a link $\{S, P\}$ of Γ, there exists $j \in B$ such that $\mathcal{A}_j \supset \mathcal{A}_i \cup \{S\} \setminus \{P\}$.

A collection of coalitions $\mathcal{A}$ is a (Γ, τ) – **negative mixed collection of coalitions** if for each $B \in \tau$, $i \in B$, $P \in \mathcal{A}_i$, and a link $\{P, S\}$ of Γ, there exists $j \in B$ such that $\mathcal{A}_j \subset \mathcal{A}_i \cup \{S\} \setminus \{P\}$.

Each (Γ,τ)–mixed collection of coalitions is a (Γ,τ)–positive mixed collection and a (Γ,τ)–negative mixed collection.

Example 11. $|N| = 5$, $K, L \in \mathcal{A}$ are adjacent in Γ iff $K \cap L = \emptyset$, $\tau = \{N\}$, $\mathcal{A} = \{\{i,j\},\ \{k,l,m\},\ \{i,k\},\ \{j,l\}\}$,
$\mathcal{A}$ is a (Γ,τ)–positive mixed collection of coalitions.

Example 12. $|N| = 4$, $K, L \in \mathcal{A}$ are adjacent in Γ iff $K \cap L = \emptyset$, $\tau = \{N\}$, $\mathcal{A} = \{\{i,j\},\ \{k\},\ \{i,m\}\}$,
$\mathcal{A}$ is **not** a (Γ,τ)–positive mixed collection of coalitions. Indeed, take $P = \{i,j\}$, $S = \{k\}$.

Example 13. $\tau = \{N\}$. If $\mathcal{A}$ is a minimal covering of N, then $\mathcal{A}$ is a (Γ,τ) – negative mixed collection of coalitions for each Γ. Indeed, for a link $\{Q,S\}$, there exists $j \in N$ such that $\mathcal{A}_j = \{S\}$.

Example 14. $|N| = 5$, $K, L \in \mathcal{A}$ are adjacent in Γ iff $K \cap L = \emptyset$, $\tau = \{N\}$, $\mathcal{A} = \{\{1,2\},\ \{3,4,5\},\ \{1,3\},\ \{2,4\}\}$,
$\mathcal{A}$ is not a (Γ,τ)–negative mixed collection of coalitions. Indeed, take $P = \{3,4,5\}$, $i = 5$, $S = \{1,2\}$.

These collections were introduced in (Naumova, 2019) and it was proved that (Γ,τ)–negative mixed collections satisfy the condition $C0(\Gamma,\tau)$, and (Γ,τ)-positive mixed collections satisfy the condition $C0(\Gamma,\tau)$ under the additional condition (the nodes of links of Γ do not intersect).

We shall describe an iterative procedure that converge to points in envy stable solution for (Γ,τ)–negative mixed collections and (Γ,τ)-positive mixed collections of coalitions. But there exist collections of coalitions such that Condition C0 is fulfilled but such collections are neither weakly positive mixed nor weakly negative mixed.

Example 15. $\mathcal{A} = \{\{i,j\},\ \{k,l\},\ \{i,k,n\},\ \{j,l,m\},\ \{i,k,m\},\ \{j,l,n\}\}$. Then $\mathcal{A}$ satisfies Condition C0, but it is neither positive mixed nor negative mixed. Indeed, if we take $P = \{i,k,m\}$, $S = \{j,l,n\}$, then $\mathcal{A}_i \cup S \setminus P \not\subset \mathcal{A}_p$ for each p, and if we take $P = \{i,k,n\}$, $S = \{j,l,m\}$ then $\mathcal{A}_n \cup S \setminus P \not\supset \mathcal{A}_p$ for each p.

6.2. Dynamic systems

The following definitions and notations are due to Sudholter and Peleg, 2007.

Let X be a metric space and $d : X \times X \to R$ be a metric for X. A *dynamic system on* X is a set-valued function $\varphi : X \to 2^X$.

A *φ-sequence from* $x_0 \in X$ is a sequence $\{x_t\}_{t=0}^{\infty}$ such that $x_{t+1} \in \varphi(x_t)$ for all $t = 0, 1, \ldots$. A point $x \in X$ is called an *endpoint of* φ if $\varphi(x) = \{x\}$.

A set-valued function $\varphi : X \to X$ is *lower hemicontinuous at* $x \in X$ if for every open set $U \subset X$ such that $\varphi(x) \cap U \neq \emptyset$, there exists an open set $V \subset X$ such that $x \in V$ and $\varphi(z) \cap U \neq \emptyset$ for every $z \in V$.

φ is *lower hemicontinuous*, if it is lower hemicontinuous at each $x \in X$.

A *valuation for* φ is a continuous function $\Psi : X \to R$ such that

$$y \in \varphi(x) \Longrightarrow \Psi(x) - \Psi(y) \geq d(x,y) \text{ for all } x, y \in X.$$

Define the function $\rho_\varphi : X \to R \cup \{\infty\}$ by

$$\rho_\varphi(x) = \sup\{d(x,y) : y \in \varphi(x)\}.$$

A φ–sequence $\{x_t\}_{t=0}^{\infty}$ is *maximal* if there exists $\alpha > 0$ and a subsequence $\{x_{t_j}\}_{j=0}^{\infty}$ such that

$$d(x_{t_j}, x_{t_{j+1}}) \geq \alpha \rho_\varphi(x_{t_j}) \text{ for all } j.$$

The proof of convergence of φ–sequence that will be defined in this paper is based on the following results.

Corollary 10.1.9.in Sudholter and Peleg, 2007 *Let X be a compact metric space and φ be a lower hemicontinuous set-valued function. If φ has a valuation, then every maximal φ–sequence converges to an endpoint of φ.*

In order to prove the existence of valuations for our models, we shall exploit the results in (Justman, 1977). The next definition use notations of that paper.

Let $U = \{u_i\}_{i\in M}$ be a finite set of real valued functions defined on X, φ be a dynamic system on X. A pair $\{U, \varphi\}$ is a **nucleolar framework w.r.t.** U**,** φ if for some $\epsilon > 0$ and all $x \in X$ if $y \in \varphi(x)$ then for some $i \in M$,
$u_i(x) - u_i(y) \geq \epsilon d(x, y)$,
for all $j \in M$, $u_j(y) > u_i(y)$ implies $u_j(y) \leq u_j(x)$.

Justman, 1977 proved in Theorem 2.2 (pp. 193-194) that if there exists $K > 0$ such that $u_i(x)-u_i(y) \geq -Kd(x, y)$ for all $x, y \in X$, $i \in M$, then φ has a valuation.

Theorem 3. *Let for $\{G_S\}_{S\in\mathcal{A}} \in \mathcal{G}_{\mathcal{A}}^{\tau}$ there exist $K > 0$, $\epsilon > 0$ such that*

$$\epsilon d(x, y) \leq |G_S(x) - G_S(y)| \leq Kd(x, y)$$

for all $x, y \in X(\tau, C)$, where $d(x, y) = \max_{k\in N} |x_k - y_k|$,
(Γ, τ) satisfy one of the following conditions:
$\mathcal{A}$ is a (Γ, τ) – negative mixed collection of coalitions;
$\mathcal{A}$ is a (Γ, τ) – positive mixed collection of coalitions and $S \cap Q = \emptyset$ for each link $\{S, Q\}$ of Γ.

Then there exists a dynamic system φ such that each maximal φ-sequence converges to an allocation in $\{G_S\}_{S\in\mathcal{A}}$ - envy stable solution w.r.t. Γ.

Proof. Let us construct dynamic systems that satisfy the conditions of Justman's theorem. (Here $M \leftrightarrow \mathcal{A}$, $U \leftrightarrow \{G_S\}_{S\in\mathcal{A}}$, $u_k(x) \leftrightarrow G_S(x)$.)

For $i, j \in B \in \tau$, $\beta \geq 0$, let $x(i, j, \beta)_i = x_i-\beta$, $x(i, j, \beta)_j = x_j+\beta$, $x(i, j, \beta)_k = x_k$ for $k \neq i, j$. Then $x(i, j, \beta)$ is an allocation if $\beta \leq x_i$.

For a link $\{P, Q\}$ of Γ and an allocation x with $G_P(x_P) > G_Q(x_Q)$, $B \in \tau$, $i \in P \cap B$, $j \in B$, let

$$\delta(i, j, x) = \min\{x_i, \sup\{\beta : G_P(x(i, j, \beta)_P) > G_Q(x(i, j, \beta)_Q)\}\}.$$

We define dynamic systems φ for the collection of $\{G_S\}_{S\in\mathcal{A}} \in \mathcal{G}_{\mathcal{A}}^{\tau}$ in 2 cases as follows.

Case 1. $\mathcal{A}$ is a (Γ, τ)–negative mixed collection of coalitions.

Let x be an allocation. If x belongs to the $\{G_S\}_{S\in\mathcal{A}}$ - envy stable solution w.r.t. Γ then take $\varphi(x) = \{x\}$. Else there exists a link $\{P^0, Q^0\}$ of Γ such that $G_{P^0}(x_{P^0}) > G_{Q^0}(x_{Q^0})$ and $x(P^0) > 0$.

Then $y \in \varphi(x)$ iff there exists a link $\{P, Q\}$ of Γ such that $G_P(x_P) > G_Q(x_Q)$ and $x(P) > 0$, for this link there exists $i_0 \in P\cap B$, where $B \in \tau$ with $x_{i_0} > 0$, $j_0 \in B$ with $\mathcal{A}_{j_0} \subset \mathcal{A}_{i_0} \cup \{Q\} \setminus \{P\}$ and $\beta > 0$ such that $y = x(i_0, j_0, \beta)$ and $\beta \leq \delta(i_0, j_0, x)$.

Then $\varphi(x) \neq \emptyset$ and x belongs to the $\{G_S\}_{S\in\mathcal{A}}$ - envy stable solution w.r.t. Γ iff x is an endpoint of φ.

Note that $j_0 \notin P$, hence $G_P(x_P) - G_P(y_P) \geq \epsilon\beta = \epsilon d(x,y)$. If for some $S \in \mathcal{A}$, $G_S(x_S) < G_S(y_S)$, then $S = Q$, and $G_Q(y_Q) \leq G_P(y_P)$ by the definition of $\delta(i_0, j_0, x)$. Thus, the conditions of Justman's theorem are verified in this case (the coalition P corresponfs to Justman's coordinate i).

Case 2. $\mathcal{A}$ is a (Γ, τ)–positive mixed collection of coalitions and $S \cap T = \emptyset$ for each link $\{S,T\}$ of Γ. We construct $\varphi(x)$ as follows. If x belongs to the $\{G_S\}_{S\in\mathcal{A}}$ - envy stable solution w.r.t. Γ then take $\varphi(x) = \{x\}$. Else $y \in \varphi(x)$ iff there exists a link $\{P,Q\}$ of Γ such that $G_P(x_P) > G_Q(x_Q)$ and $x(P) > 0$, for this link there exists $i_0 \in P \cap B$, where $B \in \tau$ with $x_{i_0} > 0$, $j_0 \in B$ with $\mathcal{A}_{j_0} \supset \mathcal{A}_{i_0} \cup \{Q\} \setminus \{P\}$ and $\beta > 0$ such that $y = x(i_0, j_0, \beta)$ and $\beta \leq \delta(i_0, j_0, x)$. Then $\varphi(x) \neq \emptyset$ and x belongs to the $\{G_S\}_{S\in\mathcal{A}}$ - envy stable solution w.r.t. Γ iff x is an endpoint of φ.

In this case $G_Q(x_Q) < G_Q(y_Q)$ because $i_0 \notin Q$ as if $P \cap Q = \emptyset$. If $G_S(x_S) > G_S(y_S)$ then $S = P$ and $G_P(y_P) \geq G_Q(y_Q)$. The conditions of Justman's theorem are realized for $u_S = -G_S$ and the coalition Q corresponds to Justman's coordinate i.

By Justman's theorem, the constructed dynamic systems φ have valuations and are lower hemicontinuous set-valued functions as if G_S are continuous functions. Then by Corollary 10.1.9.in Sudholter and Peleg, 2007, maximal φ–sequences converge to endpoints of φ. By constructions of φ, these endpoints belong to $\{G_S\}_{S\in\mathcal{A}}$ - envy stable solution w.r.t. Γ.

7. Conclusion

In this paper for generalized cooperative games with restricted cooperation and several public resources, we consider envy stable solutions that were introduced by the author in Naumova, 2019. The allocations of this solution seem "fair" for admissible coalitions. We propose computation methods for the following classes of the problem.

1. Each admissible coalition S has a linear gain function G_S that is strictly increasing in each variable and $G_S(0) = 0$. If envy stable solution exists, its selector can be found by linear programming methods or. by Bregman's iterative method. The results of linear programming methods may seem unfair for players, and the results of Bregman's method seem fair both for coalitions and for players.

2. We distribute one resource, a collection of admissible coalitions is a union of several partitions of the set of players and the existence result for envy stable solution is guaranteed. If we solve the problem for each partition separately, then the solution of the whole problem can be obtained by Bregman's method.

3. There are several sources of the same resource, the value of each gain function of a coalition depends on the total amount of resources obtained by this coalition, and the collection of admissible coalitions is mixed. Then a selector of envy stable solution is the solution of a special optimization problem.

4. There are several resources, the gain functions of coalitions depend on total amounts of each resource, and the collection of admissible coalitions is either positive mixed of negative mixed. Then we describe an iterative procedure (under some restrictions on gain functions) that converges to an envy stable allocation.

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