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About Construction of Realizability Arias of Salesman Strategies in Dynamic Salesmen Problem

Anastasiya V. Gavrilova[1] **and Yaroslavna B. Pankratova**[2]

[1] *Foresight, Beloostrovskaya St.,17- 2*
Saint-Petersburg 197342, Russia
E-mail: anastasia.gavrilova@fsight.ru
[2] *St. Petersburg State University,*
7/9 Universitetskaya nab., Saint Petersburg 199034, Russia
E-mail: y.pankratova@spbu.ru

Abstract The dynamic travelling salesman problem, where we assume that all objects can move with constant velocity, is considered. To solve this NP-hard problem we use a game-theoretic approach and well-known solution concepts of pursuit games. We identify the realizability areas of salesman strategies depending on the initial positions of customers and their velocities. We present different cases of realizability areas of salesman strategies constructing in Python program with several numbers of customers.

Keywords: dynamic travelling salesman problem, non-zero sum game, Nash equilibrium, realizability areas.

1. Introduction

There are many different modifications of traveling salesman problem (TSP). The classical traveling salesman problem is to find a route of a given number of cities, visiting each city exactly once and returning to the starting city where the length of this tour is minimized. The first example of the travelling salesman problem was from Euler in 1759 whose problem was to move a knight to every position on a chess board exactly once, then in a book written by German salesman B.F. Voigt in 1832 (Michalewicz, 1994). The origins of the TSP in mathematics are not really known - all we know for certain is that it happened around 1931 (Michalewicz, 1994).

The classical traveling salesman problem has always been attractive to the researcher despite the obvious difficulties of solving it (Lawler et al., 1986; Reinelt, 1994). This problem has had wide application in various fields. Some instances of the vehicle routing problem can be modeled as a travelling salesman problem. Here the problem is to find which customer should be served by which vehicle and the minimum number of vehicles needed to serve each customer. There are different variations of this problem including finding the minimum time to serve all customers. Now, in the modern world when the efficiency and dynamism of unmanned aerial vehicles or drones appeared and can be used, new areas of application of the traveling salesman problem are opening up. Various modifications of this task are applied to problems in which drones are used. For example, in the paper de Freitas, Penna, 2020 a variant of the traveling salesman problem (TSP), called the flying assistant traveling salesman problem, was presented, related to the delivery of parcels using drones.

We consider the dynamic travelling salesman problem (DTSP) (Sergeev, 2008; Tarashnina et al., 2017), allowing all considered objects (the salesman and m customers) to move on a plane with constant velocities. We apply a game-theoretical

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approach for solving the DTSP. In fact, we propose to use some methods of pursuit game theory for this purpose (Petrosjan and Shirjaev, 1981; Petrosjan, 1983; Kleimenov, 1993; Tarashnina, 1998; Pankratova, 2007). This means that each agent is considered as a player that has his own aim and his profit is described by a payoff function. The players may use admissible strategies and interact with each other. Here we find a solution of DTSP as a Nash equilibrium in a non-zero-sum game of pursuit. In other words, we define strategies of all players that provide the minimal length of the salesman route (Tarashnina et al., 2017) and construct the realizability arias for such strategies. We identify the realizability areas of salesman strategies depending on the initial locations of clients and their velocities. Several special cases of salesman behavior are investigated. A Python program is made to build realizability areas of salesman strategies. Such areas are obtained for the cases of one salesman and different numbers of clients.

2. The game

We have m customers $C_1, \ldots, C_m$ who are initially located in different cities and move on a plane with constant velocities, and a salesman S who wants to meet all of them. The players start their motion at the moment $t = 0$ at initial positions $z_1^0, \ldots, z_m^0, z^0$. At each instant t they may choose directions of their motion. Let α be the velocity of salesman S, β_j be the velocity of customer C_j, $j = 1, \ldots, m$, $\alpha < \beta_j$.

Suppose that the salesman never meets the same customer twice and does not return to the starting point (he she stays in the last meeting point). Thus, the salesman tries to find the shortest route that passes through the customers' current positions once and each customer also wants to meet the salesman as soon as possible.

In contrast to the classical problem, where customers are located at fixed points and may not move, here they move with constant velocities.

A strategy of salesman S is denoted by

$$u_S(t, z_1^t, \ldots, z_m^t, z^t) = u_S.$$

The salesman uses piecewise open-loop strategies.

A strategy of customer C_j is a function of time, players' positions and a velocity-vector of the salesman at a current time instant, i.e.

$$u_{C_j}(t, z_1^t, \ldots, z_m^t, z^t, \mathbf{u}_S^t) = u_{C_j},$$

where $z_1^t, \ldots, z_m^t, z^t$ are current positions of players and $\mathbf{u}_S^t$ is a vector-velocity of S at time instant t. In this game we suppose that the customers use the parallel pursuit strategy (Π-strategy) (Petrosjan, 1965). Denote by $\mathcal{U}_S$ and $\mathcal{U}_{C_j}$ the sets of admissible strategies of the players, $j = 1, \ldots, m$.

The game is played as follows: at the initial moment of time the salesman informs customers $C_1, \ldots, C_m$ about a chosen direction of his motion. After that, S meets the customers on his route if they cross it. The game is finished when the salesman meets the last customer. S aspires to minimize the total meeting time, i.e. to meet all customers for the minimal time. At the same time each customer wants to minimize his own meeting time.

The payoff function of customer C_j is

$$K_{C_j}(z_1^0, \ldots, z_m^0, z^0, u_{C_1}, \ldots, u_{C_m}, u_S) = -T_j, \tag{1}$$

where T_j is a meeting time of S and customer C_j.

The payoff function of salesman S is

$$K_S(z_1^0, \ldots, z_m^0, z^0, u_{C_1}, \ldots, u_{C_m}, u_S) = -\max\{T_1, \ldots, T_m\}. \tag{2}$$

The objective of each player in the game is to maximize his own payoff function. So, we define this problem in a normal form

$$\Gamma(z_1^0, \ldots, z_m^0, z^0) = \langle N, \{\mathcal{U}_i\}_{i \in N}, \{K_i\}_{i \in N} \rangle, \tag{3}$$

where $N = \{C_1, \ldots, C_m, S\}$ is the set of players, $\mathcal{U}_i$ is the set of admissible strategies of player i, and K_i is a payoff function of player i defined by (1) and (2), $i \in N$. The constructed game depends on initial positions of the players. Let us fix players' initial positions and consider the game $\Gamma(z_1^0, \ldots, z_m^0, z^0)$.

3. Nash equilibria

Since we consider solution concepts from the pursuit game theory we need some notions from pursuit games that will help to find a solution of the DTSP.

Definition 1. (Petrosjan, 1965) The parallel pursuit strategy (Π-strategy) is a kind of motion of a customer C regard the motion of salesman S which provides a segment $C^t S^t$ connecting current players' positions C^t and S^t at each time instant $t > 0$ to be parallel to the initial segment $C^0 S^0$ and its length strictly decreases.

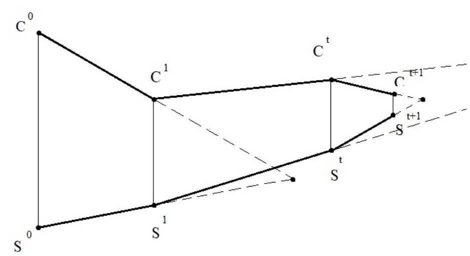

Fig. 1.

It is known that if customer C_j uses the parallel pursuit strategy and salesman S uses any admissible strategy from $\mathcal{U}_S$, then all possible meeting points of the salesman and customer C_j cover the Apollonius disk (Petrosjan, 1983). In particular, if the salesman moves along a straight line, then a meeting point of S and C_j lies on the Apollonius circle.

Definition 2. The Apollonius circle $A(z_j^0, z^0)$ for initial positions $C_j^0 = z_j^0$ and $S^0 = z^0$ of customer C_j and salesman S, respectively, is the set of points M such that

$$\frac{|S^0 M|}{\alpha} = \frac{|C_j^0 M|}{\beta_j},$$

where $\beta_j > \alpha > 0$ (see Fig. 2).

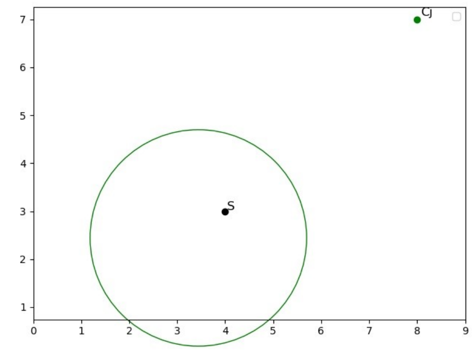

Fig. 2. The Apollonius circle for the game $\Gamma(z_j^0, z^0)$

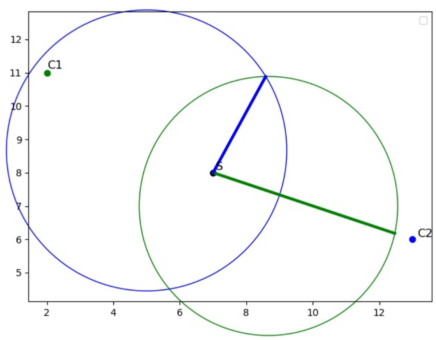

Fig. 3. The Apollonius circles for the game $\Gamma(z_1^0, z_2^0, z^0)$

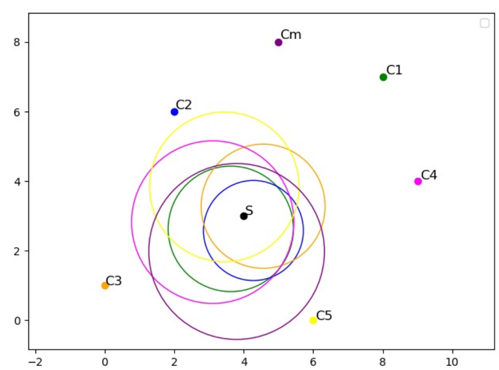

Fig. 4. The Apollonius circles for the game $\Gamma(z_1^0, \ldots, z_m^0, z^0)$

In Fig. 2 there are the Apollonius circles for all pairs of S and C_j, $j = 1, \ldots, m$. Denote by A_j the Apollonius disk corresponding to the Apollonius circle $A(z_j^0, z^0)$. The union of all Apollonius disks is denoted by A, i.e. $A = A_1 \cup \ldots \cup A_m$ and $\partial A = \partial(A_1 \cup \ldots \cup A_m)$ is a boundary of the set A. The boundary $\partial A = \partial^1 A$ is called the boundary of the first level. If we remove the boundary of the first level, then the remaining Apollonius disks form a new boundary, we call it the boundary of the second-level and denote by $\partial^2 A$, etc.

As we noted earlier, the meeting of Salesman and customer takes place on the Apollonius circle, then it is obvious that the meeting can occur at the intersection of two Apolonius circles (see Fig. 3). The goal of a traveling salesman is to minimize the total meeting time with all customers. Customers also seek to minimize their time of meeting with a traveling salesman. If we look at Fig. 3, it is clear that the total minimum meeting time of a traveling salesman with all clients is provided by the strategy of the traveling salesman, which prescribes him to move to the nearest intersection point of the Apollonius circles, provided that the clients use the Π-strategy. Analytical formulas for calculating the coordinates of the intersection of Apollonius circles are obtained in paper Pankratova et al., 2016. Next, it is interesting whether the idea of moving to the nearest point of intersection of Apollonius circles will be true for any number of customers. Unfortunately, this is not the case. As a result of the study of this issue, and using methods and ideas from Pankratova, 2007, Pankratova, 2010 the following theorem was formulated in the paper Tarashnina et al., 2017.

Theorem 1. *In the dynamic traveling salesman problem $\Gamma(z_1, z_2, z_3, z_4, z)$ there exists a Nash equilibrium. It is constructed as follows:*

- *The salesman chooses strategy u_S^* that prescribes to him one type of behavior u_S^1, u_S^2 or u_S^3 and gives the minimal meeting time.*
- *The customers use Π-strategy.*

Behavior u_S^1: Salesman S uses the type of behavior u_S^1, according to which he moves along a straight line towards customer C_j, that is, to the nearest point on the boundary of the union of all Apollonius disks A_j, $j = 1, \dots, m$ (Fig. 5 (green line)).

Behavior u_S^2: Salesman S uses the type of behavior u_S^2, according to which he moves along a straight line to the nearest intersection point of the Apollonius circles $A(z_j^0, z^0)$ and $A(z_k^0, z^0)$ $(j \neq k)$ that belongs to the boundary ∂A (Fig. 6 (blue line)).

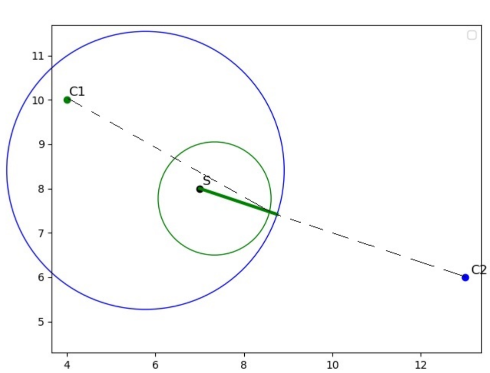

Fig. 5. Behavior u_S^1

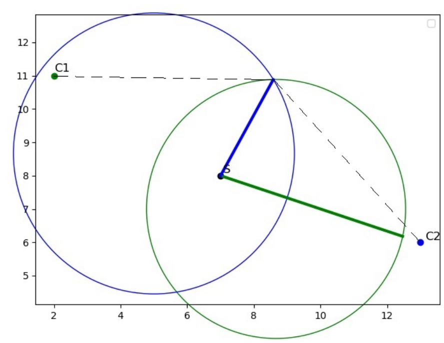

Fig. 6. Behavior u_S^2

Behavior u_S^3: Salesman S uses the type of behavior u_S^3, according to which he moves along a straight line to the nearest intersection point of the Apollonius circles $A(z_j^0, z^0)$ and $A(z_k^0, z^0)$ $(j \neq k)$ that belongs to the boundary $\partial^2 A$ (Fig. 7 (yellow line)) and then changes his direction and moves along a straight line towards the last customer C_l, $l \neq j \neq k$.

The question arises how to determine by the current positions and velocities the point on the Apollonius circle to which salesman should move to minimize the total meeting time with customers. In other word, we should define which type of behaviour the salesman should use to minimise the total meeting time. To answer this question in the next section we will define the realizability areas of the traveling salesman strategy.

4. Realizability arias

Denote by R the set of all possible initial positions of players $C_1 \dots, C_m$.

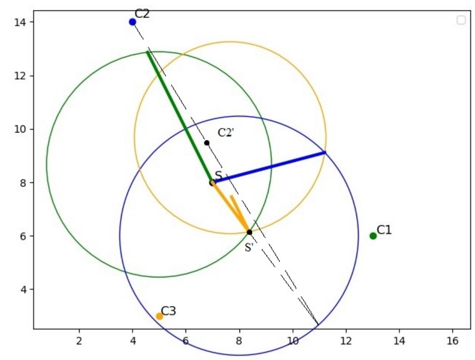

Fig. 7. Behavior u_S^1, u_S^2, u_S^3 for game $\Gamma(z_1^0, z_2^0, z_3^0, z^0)$

Definition 3. The realizability arias $R(u_S^j)$ $(j = 1, 2, 3)$ of salesman strategy u_S^j is a set of initial position of player S for which the type of behaviour u_S^j, $j = 1, 2, 3$ guaranties to salesman S the minimal meeting time with all customers, i.e. the minimal total time.

Having constructed such areas, it is possible to determine which strategy (behavior) a traveling salesman should adhere to minimize the total meeting time

4.1. Exmalple 1. Salesman and 2 clients

Consider example with one salesman S and two customers C_1 and C_2, and define all possible types of behaviour of S depending on initial positions and velocities.

To build realizability area of salesman, we fix the initial positions and velocities of the salesman S and the customer C_1, and the velocity of C_2. The initial location of the customer C_2 changes. Green colour corresponds to the first type of behavior, red to the second.

Fix the traveling salesman S at the point $[0; 0]$ with the velocity $\alpha = 6$, the customer C_1 at the point $[1; 0]$ with the velocity $\beta_1 = 8$ and fix the velocity of the client C_2, $\beta_2 = 7$ and construct the realizability areas of salesman using Python program.

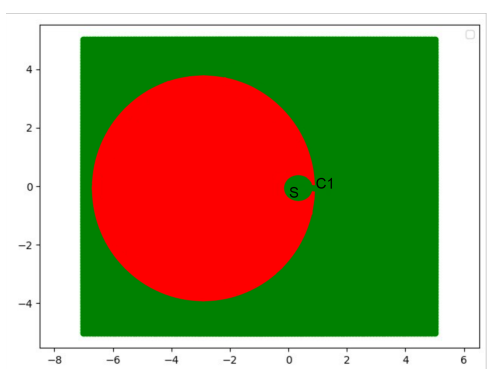

Fig. 8. Reailzability arias of Salesman according two types of behaviour

That is, given the position of the traveling salesman S and the customer C_1 and fixed velocities, if the client C_2 is in the green area, then the traveling salesman must use the first type of behavior (u_S^1), to achieve the minimum meeting time with all customers. If the customer C_2 is in the green area, then the minimum meeting time will be achieved when the traveling salesman uses the second type of behavior (u_S^2) (Fig. 8).

4.2. Exmalple 2. Salesman and 3 customers

To build the areas, we will fix the initial positions and velocities of the traveling salesman S and customers C_1, C_2, and the velocity of C_3. The initial position of the customer C_3 is changing. Green color corresponds to the first type of behavior, red — to the second, blue — to the third. In this case we have two possible situation depending on where we fix the initial position of the second customer C_2

In the first case, we fix the position of the player C_2 such that in the game $\Gamma(z_1, z_2, z)$ the strategy u_S^1 will guarantee the minimal total meeting time. We fix the traveling salesman S at the point $[0; 0]$ with velocity $\alpha = 6$, the customer C_1 at the point $[1; 0]$ with velocity $\beta_1 = 8$, the customer C_2 at the point $[3; 0]$ with velocity $\beta_2 = 7$ and fix the customer's velocity C_3 $\beta_3 = 9$ (Fig. 9).

If we move the client C_2, while keeping it in the same area, we will see that the area will also move, but its elements will not change: we fix the traveling salesman S at the point $[0; 0]$ with velocity $\alpha = 6$, the client C_1 at the point $[1; 0]$ with velocity $\beta_1 = 8$, the client C_2 at the point $[2; 1]$ with velocity $\beta_2 = 7$ and fix the velocity of the client C_3 $\beta_3 = 9$ (Fig. 10).

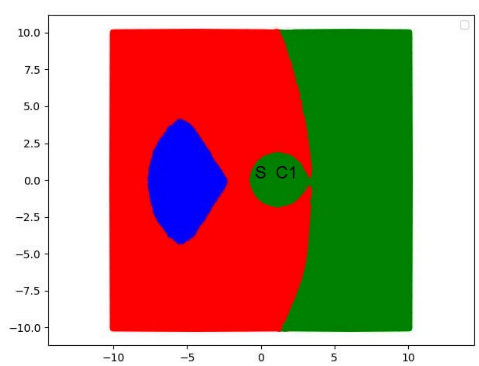

Fig. 9. Case 1.1

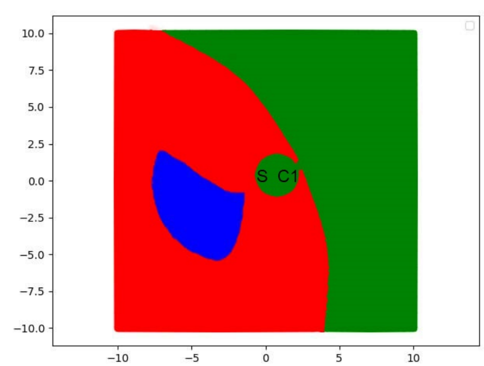

Fig. 10. Case 1.2

In the second case, consider the position of the player C_2 such that u_s^2 is strategy which gives the salesman the minimal total meeting time in the game $\Gamma(z_1, z_2, z)$. We fix the traveling salesman S at the point $[0; 0]$ with the velocity $\alpha = 6$, the customer C_1 is at the point $[1; 0]$with the velocity $\beta_1 = 8$, the customer C_2 at the point $[-2; 0]$ with velocity $\beta_2 = 7$. We also set the velocity of the client C_3 $\beta_3 = 9$ (Fig. 11).

Thus, at a given position of the traveling salesman and the customers C_1, C_2 and fixed velocities, if C_3 will be in the green area, then the traveling salesman must use the first type of behaviour (u_S^1), to achieve the minimum meeting time with all customers. If the initial position of C_3 belongs to the red area, then the minimum meeting time will be achieved when the traveling salesman uses the second type of

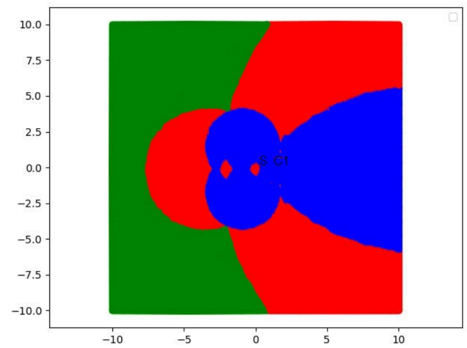

Fig. 11. Case 2

behavior (u_S^2), and if the initial position of customer C_3 will be in the blue region, the traveling salesman should use the third type of behavior (u_S^3).

5. Conclusion

In the considered dynamic traveling salesman problem we propose a new approach to finding a solution of this task. we introduce realizability areas of salesman strategies applying approach from Pankratova and Tarashnina, 2004. Such areas help us to make a desicion which type of behaviour we have to use to get minimal total meeting time with all customers. We consider several example of constructing the realizability areas of salesman strategies depending on initial positions and velocities of the players. In a future, it is interesting to get relations between initial positions of the players and velocities or classification table, which allow us to calculate "switch" points between type of behaviours.

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