

L_p -inequalities for the polar derivative of a lacunary-type polynomial

*N. A. Rather*¹, *L. Ali*¹, *S. Gulzar*²

¹ Department of Mathematics, University of Kashmir, Srinagar, 190006, India

² Department of Mathematics, Government College for Engineering & Technology, Safapora, Ganderbal, Kashmir, 193504, India

For citation: Rather N. A., Ali L., Gulzar S. L_p -inequalities for the polar derivative of a lacunary-type polynomial. *Vestnik of Saint Petersburg University. Mathematics. Mechanics. Astronomy*, 2021, vol. 8 (66), issue 3, pp. 502–510. <https://doi.org/10.21638/spbu01.2021.311>

In this paper, we extend an inequality concerning the polar derivative of a polynomial in L_p -norm to the class of lacunary polynomials and thereby obtain a bound that depends on some of the coefficients of the polynomial as well.

Keywords: L_p -inequalities, polar derivative, polynomials.

1. Zygmund type inequalities for polynomials. Let $f(x)$ be a real polynomial of degree at most n then according to a well-known classical result in approximation theory due to A. Markov [1],

$$\max_{-1 \leq x \leq 1} |f'(x)| \leq n^2 \max_{-1 \leq x \leq 1} |f(x)|.$$

The above inequality is best possible because for Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$, $\max_{-1 \leq x \leq 1} |T_n(x)| = 1$ and $|T'_n(\pm 1)| = n^2$. This inequality has been generalized in several ways, in particular, S. Bernstein (for details see [2] or [3]) obtained its extension to complex polynomials. Let \mathcal{P}_n denote the space of all complex polynomials $P(z) = \sum_{j=0}^n a_j z^j$ of degree at most n . According to Bernstein's inequality, if $P \in \mathcal{P}_n$ then

$$\|P'\|_\infty \leq n \|P\|_\infty, \quad \text{where} \quad \|P\|_\infty := \max_{|z|=1} |P(z)|.$$

Define the standard Hardy space H^p norm for $P \in \mathcal{P}_n$ by

$$\|P\|_p := \left(\frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right)^{1/p}, \quad 0 < p < \infty.$$

It is well known that the supremum norm of the space H^∞ satisfies

$$\|P\|_\infty = \lim_{p \rightarrow \infty} \|P\|_p.$$

The other limiting case, also known as *Mahler measure* of a polynomial $P(z)$, is

$$\|P\|_0 := \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{i\theta})| d\theta \right).$$

If $P \in \mathcal{P}_n$, then

$$\|P'\|_p \leq n\|P\|_p. \quad (1)$$

Inequality (1) is due to Zygmund [4]. Zygmund obtained this inequality as an analogue to Bernstein's inequality. Arestov [5] showed that the inequality (1) remains valid for $0 < p < 1$ as well. Equality in (1) holds for $P(z) = \alpha z^n, \alpha \neq 0$.

For the class of polynomials $P \in \mathcal{P}_n$ having no zero in $|z| < 1$, inequality (1) can be sharpened. In fact, if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ for $|z| < 1$, then

$$\|P'\|_p \leq \frac{n}{\|1+z\|_p} \|P\|_p, \quad p \geq 1. \quad (2)$$

Inequality (2) was found out by De Bruijn [6]. Rahman and Schmeisser [7] proved the inequality (2) remains true for $0 < p < 1$ as well.

The estimates is sharp and equality in (2) holds for $P(z) = az^n + b, |a| = |b| \neq 0$.

Govil and Rahman [8] generalized inequality (2) and proved that if $P \in \mathcal{P}_n$ does not vanish in $|z| < k$ where $k \geq 1$, then

$$\|P'\|_p \leq \frac{n}{\|k+z\|_p} \|P\|_p, \quad p \geq 1. \quad (3)$$

As a refinement of inequality (3), it was shown by Rather [9] that if $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^n a_j z^j \neq 0$ for $|z| < k, k \geq 1$, then

$$\|P'\|_p \leq \frac{n}{\|\delta_1 + z\|_p} \|P\|_p, \quad p > 0. \quad (4)$$

where δ_1 is defined by

$$\delta_1 = k^2 \frac{\frac{1}{n} \frac{|a_1|}{|a_0|} + 1}{\frac{1}{n} \frac{|a_1|}{|a_0|} k^2 + 1}. \quad (5)$$

2. Extension of Zygmund type inequalities to polar derivatives. By Gauss–Lucas theorem (see [10]), if all the zeros of a polynomial $P \in \mathcal{P}_n$ of degree n lie in a half plane then its critical points are also contained therein. Since we may map a half plane onto a closed disk through a bilinear transformation $z = \phi(w) = \frac{aw+b}{aw+d}, a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Let $g(w) = (cw + d)^n P\left(\frac{aw+b}{cw+d}\right)$ be transformation of $P(z)$ under ϕ , then if ξ is a critical point of g , then $\phi(\xi)$ is either ∞ or a zero of the polynomial $nP(z) + \left(\frac{a}{c} - z\right)P'(z)$. This property guides us to the polynomial

$$D_\alpha[P](z) := nP(z) + (\alpha - z)P'(z),$$

called the polar derivative of P with respect to a complex number α (for details see [10, p. 44]). Note that $D_\alpha[P](z)$ is of degree at most $n - 1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha \rightarrow \infty} \frac{D_\alpha[P](z)}{\alpha} = P'(z)$$

uniformly with respect z for $|z| \leq R, R > 0$.

Aziz and Rather [11] extended inequality (2) to the polar derivative of a polynomial and proved that if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$, and $p \geq 1$,

$$\|D_\alpha[P]\|_p \leq n \left(\frac{|\alpha| + 1}{\|1 + z\|_p} \right) \|P\|_p. \quad (6)$$

Aziz et al. [12] also obtained an analogue of inequality (3) to the polar derivative and proved that if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ for $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $p \geq 1$,

$$\|D_\alpha[P]\|_p \leq n \left(\frac{|\alpha| + k}{\|k + z\|_p} \right) \|P\|_p. \quad (7)$$

Later N. A. Rather [13, 14] showed that inequalities (6) and (7) remain valid for $0 < p < 1$ as well.

Recently, Rather et al. [15] extended (4) to the polar derivative which among other things also include a refinement of (7) and proved if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $0 \leq p < \infty$,

$$\|D_\alpha P\|_p \leq n \left(\frac{|\alpha| + \delta_1}{\|\delta_1 + z\|_p} \right) \|P\|_p, \quad (8)$$

where δ_1 is given by (5).

Let $\mathcal{P}_{n,\mu} \subset \mathcal{P}_n$ be a class of lacunary type polynomials $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j$, where $1 \leq \mu \leq n$.

As a generalization of inequality (8), they [15] also proved that if $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $0 \leq p < \infty$,

$$\|D_\alpha[P]\|_p \leq n \left(\frac{|\alpha| + \delta_\mu}{\|\delta_\mu + z\|_p} \right) \|P\|_p, \quad (9)$$

where

$$\delta_\mu = k^{\mu+1} \left\{ \frac{\left(\frac{\mu}{n}\right) \frac{|a_\mu|}{|a_0|} k^{\mu-1} + 1}{\left(\frac{\mu}{n}\right) \frac{|a_\mu|}{|a_0|} k^{\mu+1} + 1} \right\} \quad (\geq k^\mu).$$

3. Main results. Our main result is a compact generalization of all the above results for the class of polynomials not vanishing in $|z| < k$, $k \geq 1$. Here, we present our main result:

Theorem. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$, $0 \leq p < \infty$ and $0 \leq t \leq 1$,*

$$\left\| |D_\alpha[P]| + nmt \left(\frac{|\alpha| - 1}{1 + \delta_{\mu,t}} \right) \right\|_p \leq n \left(\frac{|\alpha| + \delta_{\mu,t}}{\|z + \delta_{\mu,t}\|_p} \right) \|P\|_p \quad (10)$$

where $m = \min_{|z|=k} |P(z)|$ and

$$\delta_{\mu,t} = k^{\mu+1} \left\{ \frac{\left(\frac{\mu}{n}\right) \frac{|a_\mu|}{|a_0| - tm} k^{\mu-1} + 1}{\left(\frac{\mu}{n}\right) \frac{|a_\mu|}{|a_0| - tm} k^{\mu+1} + 1} \right\}. \quad (11)$$

For $t = 0$, (10) reduces to (9). If in above theorem, we let $p \rightarrow \infty$, we obtain the following Corollary.

Corollary 3.1. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $0 \leq t \leq 1$,*

$$\|D_\alpha[P]\|_\infty \leq \frac{n}{1 + \delta_{\mu,t}} \{(|\alpha| + \delta_{\mu,t})\|P\|_\infty - (|\alpha| - 1)tm\}$$

where $\delta_{\mu,t}$ is given by (11).

If we divide both sides of inequality (10) by $|\alpha|$ and let $|\alpha| \rightarrow \infty$, we obtain the following refinement of inequality (4).

Corollary 3.2. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $0 \leq p < \infty$ and $0 \leq t \leq 1$,*

$$\left\| |P'| + \frac{mnt}{1 + \delta_{\mu,t}} \right\|_p \leq \frac{n}{\|z + \delta_{\mu,t}\|_p} \|P\|_p \quad (12)$$

where $m = \min_{|z|=k} |P(z)|$. The result is best possible as shown by the polynomial $P(z) = (z^\mu + k^\mu)^{n/\mu}$.

Inequality (12) also includes a refinement of (3). By taking $k = 1$ and $\mu = 1$ in (12), the following improvement of inequality (2), which holds uniformly for $0 \leq t \leq 1$, follows immediately.

Corollary 3.3. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$ then for $0 \leq p < \infty$ and $0 \leq t \leq 1$,*

$$\left\| |P'| + \frac{mnt}{2} \right\|_p \leq \frac{n}{\|1 + z\|_p} \|P\|_p \quad (13)$$

where $m = \min_{|z|=1} |P(z)|$. The result is sharp and equality in (13) holds for $P(z) = z^n + 1$.

4. Lemmas. For the proof of above theorem, we need the following lemmas. The first lemma is due to [16].

Lemma 4.1. *If $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zeros in $|z| < k$ where $k \geq 1$, then for $0 \leq t \leq 1$,*

$$\delta_{\mu,t} |P'(z)| \leq |Q'(z)| - tmn \quad \text{for } |z| = 1$$

and $\delta_{\mu,t} \geq k^\mu \geq 1$ where $\delta_{\mu,t}$ is given by (11), $Q(z) = z^n \overline{P(1/\bar{z})}$ and $m = \min_{|z|=k} |P(z)|$.

Lemma 4.2. *If A, B and C are non-negative real numbers such that $B + C \leq A$, then for every real number β ,*

$$|(A - C) + e^{i\beta}(B + C)| \leq |A + e^{i\beta}B|.$$

The above lemma is due to Aziz and Rather [17] and the next lemma is due to [15].

Lemma 4.3. *If a, b are any two positive real numbers such that $a \geq bc$ where $c \geq 1$, then for any $x \geq 1, p > 0$ and $0 \leq \beta < 2\pi$,*

$$(a + bx)^p \int_0^{2\pi} |c + e^{i\beta}|^p d\beta \leq (c + x)^p \int_0^{2\pi} |a + be^{i\beta}|^p d\beta.$$

We also need the following lemma due to Aziz and Rather [18].

Lemma 4.4. *If $P \in \mathcal{P}_n$ and $Q(z) = z^n \overline{P(1/\bar{z})}$, then for every $p > 0$ and β real, $0 \leq \beta < 2\pi$,*

$$\int_0^{2\pi} \int_0^{2\pi} |P'(e^{i\theta}) + e^{i\beta} Q'(e^{i\theta})|^p d\theta d\beta \leq 2\pi n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta.$$

5. Proof of Theorem. By hypothesis $P \in \mathcal{P}_n$ does not vanish in $|z| < k$ where $k \geq 1$ and $Q(z) = z^n \overline{P(1/\bar{z})}$, therefore, by Lemma 4.1, we have for $|z| = 1$,

$$\delta_{\mu,t} |P'(z)| \leq |Q'(z)| - tmn = |Q'(z)| - tmn \left(\frac{1 + \delta_{\mu,t}}{1 + \delta_{\mu,t}} \right).$$

Equivalently,

$$\delta_{\mu,t} \left(|P'(z)| + \frac{mnt}{1 + \delta_{\mu,t}} \right) \leq |Q'(z)| - \frac{mnt}{1 + \delta_{\mu,t}} \quad \text{for } |z| = 1. \quad (14)$$

Setting $A = |Q'(e^{i\theta})|$, $B = |P'(e^{i\theta})|$ and $C = \frac{mnt}{1 + \delta_{\mu,t}}$ in Lemma 4.2 and noting by (14) that $B + C \leq \delta_{\mu,t}(B + C) \leq A - C \leq A$ since $\delta_{\mu,t} \geq 1$. Therefore, by Lemma 4.2 for each real β , we get

$$\left| \left(|Q'(e^{i\theta})| - \frac{mnt}{1 + \delta_{\mu,t}} \right) + e^{i\beta} \left(|P'(e^{i\theta})| + \frac{mnt}{1 + \delta_{\mu,t}} \right) \right| \leq ||Q'(e^{i\theta})| + e^{i\beta}|P'(e^{i\theta})||.$$

This implies for each $p > 0$,

$$\int_0^{2\pi} |F(\theta) + e^{i\beta} G(\theta)|^p d\theta \leq \int_0^{2\pi} ||Q'(e^{i\theta})| + e^{i\beta}|P'(e^{i\theta})||^p d\theta \quad (15)$$

where

$$F(\theta) = |Q'(e^{i\theta})| - \frac{mnt}{1 + \delta_{\mu,t}} \quad \text{and} \quad G(\theta) = |P'(e^{i\theta})| + \frac{mnt}{1 + \delta_{\mu,t}}. \quad (16)$$

Integrating (15) both sides with respect to β from 0 to 2π and using properties of definite integrals, we get

$$\begin{aligned} \int_0^{2\pi} \int_0^{2\pi} |F(\theta) + e^{i\beta} G(\theta)|^p d\theta d\beta &\leq \int_0^{2\pi} \int_0^{2\pi} ||Q'(e^{i\theta})| + e^{i\beta}|P'(e^{i\theta})||^p d\theta d\beta = \\ &= \int_0^{2\pi} \int_0^{2\pi} |P'(e^{i\theta}) + e^{i\beta} Q'(e^{i\theta})|^p d\theta d\beta. \end{aligned}$$

By using Lemma 4.4 this implies,

$$\int_0^{2\pi} \int_0^{2\pi} |F(\theta) + e^{i\beta} G(\theta)|^p d\theta d\beta \leq 2\pi n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \quad (17)$$

Since $\delta_{\mu,t} \geq 1$, we have

$$|Q'(e^{i\theta})| - \frac{mnt}{1 + \delta_{\mu,t}} \delta_{\mu,t} \leq |Q'(e^{i\theta})| - \frac{mnt}{1 + \delta_{\mu,t}}.$$

On adding $t \left(|P'(e^{i\theta})| + \frac{mnt}{1 + \delta_{\mu,t}} \right)$ on both sides, where $0 \leq t \leq 1$, we get

$$\begin{aligned} |Q'(e^{i\theta})| - \frac{mnt}{1 + \delta_{\mu,t}} \delta_{\mu,t} + t \left(|P'(e^{i\theta})| + \frac{mnt}{1 + \delta_{\mu,t}} \right) &\leq \\ &\leq |Q'(e^{i\theta})| - \frac{mnt}{1 + \delta_{\mu,t}} + t \left(|P'(e^{i\theta})| + \frac{mnt}{1 + \delta_{\mu,t}} \right). \end{aligned}$$

This further implies for each $p > 0$,

$$\begin{aligned} \int_0^{2\pi} \left\{ |Q'(e^{i\theta})| - \frac{mnt}{1 + \delta_{\mu,t}} \delta_{\mu,t} + t \left(|P'(e^{i\theta})| + \frac{mnt}{1 + \delta_{\mu,t}} \right) \right\}^p d\theta &\leq \\ &\leq \int_0^{2\pi} \left\{ |Q'(e^{i\theta})| - \frac{mnt}{1 + \delta_{\mu,t}} + t \left(|P'(e^{i\theta})| + \frac{mnt}{1 + \delta_{\mu,t}} \right) \right\}^p d\theta. \end{aligned}$$

Now for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $0 \leq t \leq 1$, we have

$$\begin{aligned} |D_\alpha[P](e^{i\theta})| + nmt \left(\frac{|\alpha| - \delta_{\mu,t}}{1 + \delta_{\mu,t}} \right) &\leq |\alpha| |P'(e^{i\theta})| + |Q'(e^{i\theta})| + nmt \left(\frac{|\alpha| - \delta_{\mu,t}}{1 + \delta_{\mu,t}} \right) = \\ &= |\alpha| \left(|P'(e^{i\theta})| + \frac{mnt}{1 + \delta_{\mu,t}} \right) + \left(|Q'(e^{i\theta})| - \frac{mnt}{1 + \delta_{\mu,t}} \right). \end{aligned}$$

By integrating both sides with respect to θ from 0 to 2π , for each $p > 0$, we get

$$\begin{aligned} \int_0^{2\pi} \left\{ |D_\alpha[P](e^{i\theta})| + nmt \left(\frac{|\alpha| - \delta_{\mu,t}}{1 + \delta_{\mu,t}} \right) \right\}^p d\theta &\leq \\ &\leq \int_0^{2\pi} \left\{ |\alpha| \left(|P'(e^{i\theta})| + \frac{mnt}{1 + \delta_{\mu,t}} \right) + \left(|Q'(e^{i\theta})| - \frac{mnt}{1 + \delta_{\mu,t}} \right) \right\}^p d\theta. \end{aligned}$$

Multiply both sides by $\int_0^{2\pi} |\delta_{\mu,t} + e^{i\beta}|^p d\beta$, we obtain

$$\begin{aligned} \int_0^{2\pi} |\delta_{\mu,t} + e^{i\beta}|^p d\beta \int_0^{2\pi} \left\{ |D_\alpha[P](e^{i\theta})| + nmt \left(\frac{|\alpha| - \delta_{\mu,t}}{1 + \delta_{\mu,t}} \right) \right\}^p d\theta &\leq \\ \leq \int_0^{2\pi} \left\{ |\alpha| \left(|P'(e^{i\theta})| + \frac{mnt}{1 + \delta_{\mu,t}} \right) + \left(|Q'(e^{i\theta})| - \frac{mnt}{1 + \delta_{\mu,t}} \right) \right\}^p d\theta &\times \int_0^{2\pi} |\delta_{\mu,t} + e^{i\beta}|^p d\beta. \end{aligned} \quad (18)$$

Further, since $\delta_{\mu,t} \geq 1$, $1 \leq \mu \leq n$, by Lemma 4.3 with $a = |Q'(e^{i\theta})| - \frac{mnt}{1+\delta_{\mu,t}}$, $b = |P'(e^{i\theta})| + \frac{mnt}{1+\delta_{\mu,t}}$, $c = \delta_{\mu,t}$ and $x = |\alpha|$, we get for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$,

$$\begin{aligned} & \left\{ \left(|Q'(e^{i\theta})| - \frac{mnt}{1+\delta_{\mu,t}} \right) + |\alpha| \left(|P'(e^{i\theta})| + \frac{mnt}{1+\delta_{\mu,t}} \right) \right\}^p \int_0^{2\pi} |\delta_{\mu,t} + e^{i\beta}|^p d\beta \leq \\ & \leq (|\alpha| + \delta_{\mu,t})^p \int_0^{2\pi} \left| \left(|Q'(e^{i\theta})| - \frac{mnt}{1+\delta_{\mu,t}} \right) + e^{i\beta} \left(|P'(e^{i\theta})| + \frac{mnt}{1+\delta_{\mu,t}} \right) \right|^p d\beta. \end{aligned}$$

Again, integrating both sides with respect to θ from 0 to 2π , we obtain

$$\begin{aligned} & \int_0^{2\pi} \left\{ \left(|Q'(e^{i\theta})| - \frac{mnt}{1+\delta_{\mu,t}} \right) + |\alpha| \left(|P'(e^{i\theta})| + \frac{mnt}{1+\delta_{\mu,t}} \right) \right\}^p d\theta \int_0^{2\pi} |\delta_{\mu,t} + e^{i\beta}|^p d\beta \leq \\ & \leq (|\alpha| + \delta_{\mu,t})^p \int_0^{2\pi} \int_0^{2\pi} |F(\theta) + e^{i\beta}G(\theta)|^p d\beta d\theta \end{aligned}$$

where $F(\theta)$ and $G(\theta)$ are given by (16). Using this in inequality (18), we get

$$\begin{aligned} & \int_0^{2\pi} |\delta_{\mu,t} + e^{i\beta}| d\beta \int_0^{2\pi} \left\{ |D_\alpha[P](e^{i\theta})| + nmt \left(\frac{|\alpha| - \delta_{\mu,t}}{1 + \delta_{\mu,t}} \right) \right\}^p d\theta \leq \\ & \leq (|\alpha| + \delta_{\mu,t})^p \int_0^{2\pi} \int_0^{2\pi} |F(\theta) + e^{i\beta}G(\theta)|^p d\beta d\theta. \quad (19) \end{aligned}$$

By using (17) in (19), we obtain for each $p > 0$ and $|\alpha| \geq 1$

$$\begin{aligned} & \int_0^{2\pi} |\delta_{\mu,t} + e^{i\beta}| d\beta \int_0^{2\pi} \left\{ |D_\alpha[P](e^{i\theta})| + nmt \left(\frac{|\alpha| - \delta_{\mu,t}}{1 + \delta_{\mu,t}} \right) \right\}^p d\theta \leq \\ & \leq (|\alpha| + \delta_{\mu,t})^p 2\pi n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \end{aligned}$$

Equivalently,

$$\begin{aligned} & \left(\frac{1}{2\pi} \int_0^{2\pi} \left\{ |D_\alpha[P](e^{i\theta})| + nmt \left(\frac{|\alpha| - \delta_{\mu,t}}{1 + \delta_{\mu,t}} \right) \right\}^p d\theta \right)^{1/p} \leq \\ & \leq \frac{n(|\alpha| + \delta_{\mu,t})}{\left(\frac{1}{2\pi} \int_0^{2\pi} |\delta_{\mu,t} + e^{i\beta}| d\beta \right)^{1/p}} \left(\frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right)^{1/p} \end{aligned}$$

which immediately leads to (15) and this completes the proof of Theorem for $p > 0$. To obtain this result for $p = 0$, we simply make $p \rightarrow 0+$. \square

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Received: August 30, 2020
Revised: November 24, 2020
Accepted: March 19, 2021

Authors' information:

Nisar Ahmad Rather — Dr. Sci., Professor; dr.narather@gmail.com
Liyaqat Ali — PhD; liyaqat7897@gmail.com
Suhail Gulzar — PhD; sgmattoo@gmail.com

L_p -неравенства для полярной производной полинома лакунарного типа

Н. А. Ратхер¹, Л. Али¹, С. Гульзар²

¹ Кашмирский университет, Индия, 190006, Сринагар

² Государственный инженерный и технологический колледж,
Индия, 193504, Кашмир, Гандербал, Сафапора

Для цитирования: Rather N. A., Ali L., Gulzar S. L_p -inequalities for the polar derivative of a lacunary-type polynomial // Вестник Санкт-Петербургского университета. Математика. Механика. Астрономия. 2021. Т. 8 (66). Вып. 3. С. 502–510. <https://doi.org/10.21638/spbu01.2021.311>

В настоящей работе мы распространяем неравенство относительно полярной производной многочлена в L_p -норме на класс лакунарных многочленов и тем самым получаем оценку, которая также зависит от некоторых коэффициентов многочлена.

Ключевые слова: L_p -неравенства, полярная производная, многочлены.

Статья поступила в редакцию 30 августа 2020 г.;
после доработки 24 ноября 2020 г.;
рекомендована в печать 19 марта 2021 г.

Контактная информация:

Ратхер Нисар Ахмад — dr.narather@gmail.com

Али Лийакат — liyaqat7897@gmail.com

Гульзар Сухайл — sgmattoo@gmail.com