UDC 512.55 Вестник СПбГУ. Математика. Механика. Астрономия. 2021. Т. 8 (66). Вып. 3
MSC 16D10, 16D70

On a question concerning $D4$-modules

S. Das

Department of Mathematics, KPR Institute of Engineering and Technology, Coimbatore-641407, India

For citation: Das S. On a question concerning $D4$-modules. *Vestnik of Saint Petersburg University. Mathematics. Mechanics. Astronomy*, 2021, vol. 8 (66), issue 3, pp. 467–474. https://doi.org/10.21638/spbu01.2021.308

An R-module M is called a *D4-module* if 'whenever M_1 and M_2 are direct summands of M with $M_1 + M_2 = M$ and $M_1 \cong M_2$, then $M_1 \cap M_2$ is a direct summand of M'. Let $M = \oplus_{i \in I} M_i$ be a direct sum of submodules M_i with $Hom(M_i, M_j) = 0$ for distinct $i, j \in I$. We show that M is a $D4$-module if and only if for each $i \in I$ the module M_i is a $D4$-module. This settles an open question concerning direct sums of $D4$-modules. Our approach is independent of the solution obtained by D'Este, Keskin Tütüncü and Tribak recently.

Keywords: SIP-modules, $D4$-modules.

1. Introduction. By a ring we mean an associative ring with an identity element; modules are unitary.

A module M is said to be a *SIP-module* (*SSP-module*) if the intersection (respectively, the sum) of two direct summands of M is a direct summand of M. Kaplansky observed that over a commutative principal ideal domain every free module is a SIP-module (see [1, Exercise 51(a), p. 49].) SIP-modules and SSP-modules have been extensively studied (see, for example, [2–4] and [5]).

For $1 \leq i \leq 4$, a module M is called a *Di-module* if it satisfies the condition Di noted below.

$D1$. For every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2$ is small in M_2.

$D2$. If $A \leq M$ such that M/A is isomorphic to a direct summand of M, then A is a direct summand of M.

$D3$. If M_1 and M_2 are direct summands of M with $M_1 + M_2 = M$, then $M_1 \cap M_2$ is a direct summand of M.

$D4$. If M_1 and M_2 are direct summands of M with $M_1 + M_2 = M$ and $M_1 \cong M_2$, then $M_1 \cap M_2$ is a direct summand of M.

(For a detailed background of these notions, we refer to [6, Chapter 4] and to [7].)

A module M is also called a *lifting module* if it satisfies condition $D1$ (see [8] for detailed information regarding these modules). We recall the characterization "the ring R is semiperfect if and only if R is lifting as a right (or left) R-module" (see [9, Theorem 1.2.13]). Now let R be a commutative domain with zero Jacobson radical which is not a field, and hence is not semiperfect. Then, by the above results, ${}_RR$ is a projective module which is not a $D1$-module. We have, however, projective $\Longrightarrow$ quasi-projective $\Longrightarrow$

$D2$-module $\implies$ $D3$-module $\implies$ $D4$-module (see [6, Proposition 4.38 and Lemma 4.6]). Note that for all proper subgroups N of the (indecomposable) Prüfer p-group $M = Z_{p^\infty}$, the group M/N is isomorphic to M. Hence it is $D3$ (as a $\mathbb{Z}$-module) but not $D2$. In fact, there are rings over which every cyclic module is $D3$ but not all cyclic modules are $D2$ (see [10, Example 6.4]).

There is no known example of a module which is $D4$ but not $D3$ [11] (see also [12, p. 2]).

Let A and B be right R-modules. A homomorphism $f \in Hom_R(A, B)$ is said to be (*von Neumann*) *regular* (briefly, *regular*) if for some homomorphism $g \in Hom_R(B, A)$, we have the relation $f = fgf$. It is well-known that a homomorphism $f \in Hom_R(A, B)$ is regular if and only if $Ker(f)$ is a direct summand in A and $Im(f)$ is a direct summand in B.

Recall that a module M is called a *Rickart module* if the kernel of any endomorphism $f \in End_R(M)$ is a direct summand in M. It follows from [13, Proposition 2.16] that every Rickart module is a SIP-module. A module M is called a dual Rickart module if the image of any endomorphism $f \in End_R(M)$ is a direct summand in M. It follows from [14, Proposition 2.11] that every dual Rickart module is a SSP-module.

2. Results. We begin with the recall of some results from [15].

Lemma 1 [15, Lemma 2.1]. *Let M be a right R-module, $f, g \in End_R(M)$ be regular homomorphisms, and let*

$$M = Ker(f) \oplus A = Im(f) \oplus B, M = Ker(g) \oplus A' = Im(g) \oplus B'.$$

Then the following assertions hold:

(a) $Im(fg) = f(A \cap (Im(g) + Ker(f)))$;

(b) $Ker(fg) = (g|_{A'})^{-1}(Im(g) \cap Ker(f)) + Ker(g)$.

Lemma 2 [15, Lemma 2.2]. *Let M be a right R-module, π be the projection onto the first direct summand with respect to the decomposition $M = A_1 \oplus A_2$, and let π' be the projection onto the first direct summand with respect to the decomposition $M = B_1 \oplus B_2$. Then the following assertions hold:*

(a) $Im(\pi'\pi) = (A_1 + B_2) \cap B_1$;

(b) $Ker(\pi'\pi) = (A_1 \cap B_2) + A_2$.

Proposition 1 [15, Theorem 2.3]. *For a right R-module M, the following conditions are equivalent.*

1. *M is a SSP-module.*

2. *For any two regular homomorphisms $f, g \in End_R(M)$, the module $Im(fg)$ is a direct summand of the module M.*

Proposition 2 [15, Theorem 2.4]. *For a right R-module M, the following conditions are equivalent.*

1. *M is a SIP-module.*

2. *For any two regular homomorphisms* $f, g \in End_R(M)$*, the module* $Ker(fg)$ *is a direct summand of the module* M*.*

Next we note examples of finite abelian groups which are not $D4$.

Example. Consider $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ as a $\mathbb{Z}$-module. Then $A = (\bar{1}, \bar{3})\mathbb{Z}$ and $B = (\bar{0}, \bar{3})\mathbb{Z}$ are isomorphic direct summands of M. However, $A \cap B$ is not a direct summand of M. In fact, for any prime p, consider $M = \mathbb{Z}/p^m\mathbb{Z} \oplus \mathbb{Z}/p^n\mathbb{Z}$ with $n > m$ as a $\mathbb{Z}$-module, then M is not a $D4$-module, since there is an epimorphism $\mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathbb{Z}/p^m\mathbb{Z}$ whose kernel is not a direct summand of $\mathbb{Z}/p^n\mathbb{Z}$.

The following theorem is an analogue of [15, Theorem 3.3].

Theorem 1. *For a right* R*-module* M*, consider the following statements.*

1. M *is a D3-module.*

2. *For any two regular endomorphisms* $f, g \in End_R(M)$*, if* $Im(fg)$ *is a direct summand of the module* M*, then the module* $Ker(fg)$ *is a direct summand of the module* M*.*

3. *For any two regular endomorphisms* $f, g \in End_R(M)$ *satisfying the following:*

(i) $Im(fg)$ *is a direct summand of the module* M*,*

(ii) $Ker(f) \cong Im(g)$*,*

then the module $Ker(fg)$ *is a direct summand of the module* M*.*

4. M *is a D4-module.*

5. *For any two regular endomorphisms* $f, g \in End_R(M)$ *satisfying the following:*

(i) $Im(fg)$ *is a direct summand of the module* M*,*

(ii) $N + Ker(f) \cong Im(g)$ *for any direct summand* N *of* M *such that* $N \cap Ker(f) = 0$*,*

then the module $Ker(fg)$ *is a direct summand of the module* M*.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

Proof. (1) $\Leftrightarrow$ (2) follows from [15, Theorem 3.3].
(2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (4). Let $M = A \oplus A' = B \oplus B'$, where $A + B = M$ and $A \cong B$. Consider the natural projections $\pi_1 : A \oplus A' \longrightarrow A$ and $\pi_2 : B \oplus B' \longrightarrow B'$. Then by Lemma 2(a), $Im(\pi_2\pi_1) = B'$ is a direct summand of M. Therefore by assumption and Lemma 2(b), $Ker(\pi_2\pi_1) = (A \cap B) \oplus A'$ is a direct summand of M. This shows that $A \cap B$ is a direct summand of M, as required.
(4) $\Rightarrow$ (5). Let

$$M = Ker(f) \oplus A = Im(f) \oplus B = Ker(g) \oplus A' = Im(g) \oplus B'.$$

By Lemma 1(a), since $f|_A$ is an isomorphism $(Im(g) + Ker(f)) \cap A$ is a direct summand of M. Therefore, $A = N \oplus (Im(g) + Ker(f)) \cap A$, for some $N \leq A$. Since $(N + Ker(f)) + Im(g) = M$, $N + Ker(f) \cong Im(g)$ and M is a $D4$-module, we have $(N + Ker(f)) \cap$

$Im(g) = (Ker(f) \cap Im(g))$ is a direct summand of M. Since $g|_{A'} : A' \longrightarrow Im(g)$ is an isomorphism, we have $(g|_{A'})^{-1}(Im(g) \cap Ker(f))$ is a direct summand of M. Hence by Lemma 1(b), $Ker(fg)$ is a direct summand of M. □

Recall that a module M is called a *C3-module* if A and B are direct summands in M with $A \cap B = 0$, then $A \oplus B$ is a direct summand in M.

Following Ding et al. [16, Theorem 2.2(5)], a module M is called a *C4-module* if A and B are isomorphic direct summands in M with $A \cap B = 0$, then $A \oplus B$ is a direct summand in M. Clearly $C3$-modules are $C4$-modules. However, there are examples of $C4$-modules which are not $C3$.

The following theorem is an analogue of [15, Theorem 3.1].

Theorem 2. *For a right R-module M, consider the following statements.*

1. *M is C3-module.*

2. *For any two regular endomorphisms $f, g \in End_R(M)$, if $Ker(fg)$ is a direct summand of the module M, then the module $Im(fg)$ is a direct summand of the module M.*

3. *For any two regular endomorphisms $f, g \in End_R(M)$ satisfying the following:*

(i) *$Ker(fg)$ is a direct summand of the module M,*

(ii) *$Ker(f) \cong Im(g)$,*

then the module $Im(fg)$ is a direct summand of the module M.

4. *M is a C4-module.*

5. *For any two regular endomorphisms $f, g \in End_R(M)$ satisfying the following:*

(i) *$Ker(fg)$ is a direct summand of the module M,*

(ii) *$N \cong Im(g)$ for any direct summand N of $Ker(f)$,*

then the module $Im(fg)$ is a direct summand of the module M.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

Proof. (1) $\Leftrightarrow$ (2) follows from [15, Theorem 3.1].

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4). Let $M = A \oplus A' = B \oplus B'$, where $A \cap B = 0$ and $A \cong B$. Consider the natural projections $\pi_1 : A \oplus A' \longrightarrow A$ and $\pi_2 : B \oplus B' \longrightarrow B'$. Then by Lemma 2(b), $Ker(\pi_2\pi_1) = A'$ is a direct summand of M. Therefore by assumption and Lemma 2(a), $Im(\pi_2\pi_1) = (A + B) \cap B'$ is a direct summand of M. Since $A + B = B \oplus (A + B) \cap B'$, $A + B$ is a direct summand of M, as required.

(4) $\Rightarrow$ (5). Let

$$M = Ker(f) \oplus A = Im(f) \oplus B = Ker(g) \oplus A' = Im(g) \oplus B'.$$

By Lemma 1(b), $(g|_{A'})^{-1}(Im(g) \cap Kerf)$ is a direct summand of A'. Since $g|_{A'} : A' \longrightarrow Im(g)$ is an isomorphism and $Im(g)$ is a direct summand of the module M, we have that $Im(g) \cap Ker(f)$ is a direct summand of the module M. Therefore, $Ker(f) = N \oplus (Im(g) \cap Ker(f))$, for some $N \leq M$. Since $N \cap Im(g) = 0$, $N \cong Im(g)$ and M is a

$C4$-module, we have $N \oplus Im(g)$ is a direct summand of M. Since $Ker(f) \leq Im(g) \oplus N$, we have that

$$Im(g) \oplus N = Ker(f) \oplus (Im(g) + N) \cap A = Ker(f) \oplus (Im(g) + Ker(f)) \cap A.$$

Therefore, $(Im(g) + Ker(f)) \cap A$ is a direct summand of M. Hence by Lemma 1(a), $Im(fg)$ is a direct summand of M. □

We can now prove the following result which has already appeared in [17, Proposition 5.7 and Corollary 2.9]. The proof has been outlined by us for the sake of completeness.

Proposition 3. *For a right R-module M, the following conditions are equivalent.*

1. *M is a D4-module and a SSP-module.*
2. *M is a C3-module and a SIP-module.*
3. *M is a C4-module and a SIP-module.*
4. *M is a D3-module and a SSP-module.*
5. *M is an SSP-module and a SIP-module.*

Proof. (1) $\Longrightarrow$ (2). Let M be a SSP-module. It is clear that M is a $C3$-module. To see that M is a SIP-module, we shall use Proposition 2. Let $f, g \in End_R(M)$ be two regular endomorphisms such that

$$M = Ker(f) \oplus A = Im(f) \oplus B = Ker(g) \oplus A' = Im(g) \oplus B'.$$

We need to show that $Ker(fg)$ is a direct summand of M. By Lemma 1(b), enough to show that $Im(g) \cap Ker(f)$ is a direct sumand of M. To this end we shall follow the proof of [3, Proposition 1.4]. Let $\pi_1 : Im(g) \oplus B \longrightarrow Im(g)$ and $\pi_2 : Ker(f) \oplus A \longrightarrow Ker(f)$ be the natural projections. Define $\theta = ((\pi_1 - 1) \circ \pi_2)|_{Im(g)} : Im(g) \longrightarrow B'$. Then by [2, Proposition 1.4], $Im(\theta)$ is a direct summand of B'. Hence M being a D4-module (use [7, Theorem 2.2]), we have $Ker(\theta) = (Im(g) \cap Ker(f)) \oplus (Im(g) \cap A)$ is a direct summand of $Im(g)$. Thus $Im(g) \cap Ker(f)$ is a direct sumand of M, as desired.

(2) $\Longrightarrow$ (3) is clear.

(3) $\Longrightarrow$ (4). Let M be a SIP-module. It is clear that M is a $D3$-module. To see that M is a SSP-module, we shall use Proposition 1. Let $f, g \in End_R(M)$ be two regular endomorphisms such that

$$M = Ker(f) \oplus A = Im(f) \oplus B = Ker(g) \oplus A' = Im(g) \oplus B'.$$

We need to show that $Im(fg)$ is a direct summand of M. By Lemma 1(a), enough to show that $Im(g) + Ker(f)$ is a direct sumand of M. To this end we shall follow the proof of [5, Theorem 8]. Let $\pi_1 : Ker(f) \oplus A \longrightarrow Ker(f)$ and $\pi_2 : Im(g) \oplus B' \longrightarrow B'$ be the natural projections. Define $\phi = (\pi_2 \circ \pi_1)|_{Im(g)} : Im(g) \longrightarrow B'$. Then by [3, Proposition 1.4], $Ker(\phi)$ is a direct summand of B'. Hence M being a $C4$-module (use [16, Theorem 2.2]), we have $Im(\phi) = [Im(g) + Ker(f)] \cap [Im(g) + A] \cap B'$ is a direct summand of $Im(g)$. So we can write $M = Im(\phi) \oplus X$ for some $X \leq M$. Hence $B' = Im(\phi) \oplus (B' \cap X)$. Then we have $M = [Im(g) + Ker(f)] \oplus [(Im(g) + A) \cap (B' \cap X)]$, as required.

(4) $\Longrightarrow$ (5) follows from Proposition 2 and Theorem 1.

(5) $\implies$ (1) is clear. ☐

The following result extends [15, Lemma 4.2(2)].

Proposition 4. *Let M be a dual Rickart module. If M is a D4-module, then the product of any two regular elements in the ring $End_R(M)$ is a regular element.*

Proof. It follows from the hypothesis and Proposition 3 that M is a SSP-module and a SIP-module. Hence the result follows from [15, Theorem 2.7]. ☐

The following theorem was proved in [17].

Theorem 3 [17, Theorem 5.6]. *Let $M = \oplus_{i \in I} M_i$ be a direct sum of submodules M_i. If $N = \oplus_{i \in I}(N \cap M_i)$ for every submodule N of M, then M is a D4-module if and only if for each $i \in I$, M_i is a D4-module.*

In [17], immediately after Theorem 3 the following question was asked.

Question (see [17, Question, p. 4494]). It is known that if $N = \oplus_{i \in I}(N \cap M_i)$ for every submodule N of M, then $Hom(M_i, M_j) = 0$ for every $i \neq j$ in I, so it is natural to ask if [17, Theorem 5.6] (that is the theorem above) remains true if one assumes that $Hom(M_i, M_j) = 0$ for every $i \neq j$ in I.

In the next proposition we show that Question above has a positive answer.

Proposition 5. *Let $M = \oplus_{i \in \mathbb{N}} M_i$ be a direct sum of submodules M_i in which $Hom(M_i, M_j) = 0$ for every $i \neq j$. Then the following assertions hold:*

(i) *if M is a D4-module, then for each $i \in I$, M_i is a D4-module,*

(ii) *if each M_i is a D4-module, then M is a D4-module.*

Proof. (i). Since a direct summand of a $D4$-module is a $D4$-module (see [7, Proposition 2.11]), for every $i \in \mathbb{N}$, M_i is a $D4$-module if M is a $D4$-module.

(ii). By hypothesis and [18, the paragraph before Corollary 16.5], we have

$$End_R(M) \cong \begin{pmatrix} End_R(M_1) & 0 & 0 & \cdot\cdot & \cdot\cdot \\ 0 & End_R(M_2) & 0 & \cdot\cdot & \cdot\cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdot\cdot & End_R(M_n) & \cdot\cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{\mathbb{N} \times \mathbb{N}}.$$

Take two regular elements f, g in $End_R(M)$ such that $Im(fg)$ is a direct summand of M and $Ker(f) \cong Im(g)$. Then $f = (f_i)_{i \in \mathbb{N}}$ and $g = (g_i)_{i \in \mathbb{N}}$ for some regular elements f_i and g_i in $End_R(M_i)$ such that $Im(f_i g_i)$ is a direct summand of M_i and $[X_i + Ker(f_i)] \cong Im(g_i)$ for any direct summand X_i of M_i such that $X_i \cap Ker(f_i) = 0$ for all $i \in \mathbb{N}$. But then each M_i is a $D4$-module. Therefore by Theorem 1, $Ker(f_i g_i)$ is a direct summand of M_i for all $i \in \mathbb{N}$. Hence $Ker(fg)$ is a direct summand of M, as required. ☐

Remark. Let $\{p_i\}_{i \in \mathbb{N}}$ be an infinite set of prime numbers and let p be a prime different from any of them. Then we have the following examples of $D4$-modules:

(i) $M = \mathbb{Z}_{p^\infty} \oplus (\oplus_{i \in \mathbb{N}} \mathbb{Z}/p_i\mathbb{Z})$ as a $\mathbb{Z}$-module, where $\mathbb{Z}_{p^\infty}$ is the Prüfer p-group;

(ii) $M = \mathbb{Q} \oplus (\oplus_{i \in \mathbb{N}} \mathbb{Z}/p_i\mathbb{Z})$ as a $\mathbb{Z}$-module.

The author is grateful to the referee for a detailed list of suggestions and comments that helped improve the article significantly. Also, the author would like to thank Professor Yasser Ibrahim for some fruitful conversations and Professor M. B. Rege for his encouragement and help in presentation.

References

1. Kaplansky I. *Infinite Abelian Groups*. Ann Arbor, Univ. of Michigan Press (1969).
2. Garcia J. L. Properties of direct summands of modules. *Comm. Algebra* **17**, 73–92 (1989). https://doi.org/10.1080/00927878908823714
3. Hausen J. Modules with the summand intersection property. *Comm. Algebra* **17**, 135–148 (1989). https://doi.org/10.1080/00927878908823718
4. Wilson G. V. Modules with the direct summand intersection property. *Comm. Algebra* **14**, 21–38 (1986).
5. Alkan M., Harmanci A. On summand sum and summand intersection property of modules. *Turkish J. Math.* **26**, 131–147 (2002).
6. Mohamed S., Müller B. J. *Continuous and Discrete Modules*. In: London Math. Soc. Lecture Note Ser., vol. 147. Cambridge, Cambridge Univ. Press (1990).
7. Ding N., Ibrahim Y., Yousif M., Zhou Y. $D4$-modules. *J. Algebra Appl.* **16** (5), 1750166 (2017). https://doi.org/10.1142/S0219498817501663
8. Clark J., Lomp C., Vanaja N., Wisbauer R. *Lifting modules: supplements and projectivity in module theory*. Basel, Boston, Berlin, Birkhäuser, Verlag (2006).
9. Baba Y., Oshiro K. *Classical Artinian Rings and Related Topics*. World Scientific Publishing (2009).
10. Nguyen X. H., Yousif M., Zhou Y. Rings whose cyclics are $D3$-modules. *J. Algebra Appl.* **16** (8), 1750184 (2017). https://doi.org/10.1142/S0219498817501845
11. Ibrahim Y. *Email communication* (2020).
12. D'Este G., Keskin Tütüncü D., Tribak R. $D3$-modules versus $D4$-modules — Applications to quivers. *Glasgow Math. Journal*, 1–27 (2020). https://doi.org/10.1017/S0017089520000452
13. Lee G., Rizvi S. T., Roman C. S. Rickart modules. *Comm. Algebra* **38** (11) 4005–4027 (2010). https://doi.org/10.1080/00927872.2010.507232
14. Lee G., Rizvi S. T., Roman C. S. Dual Rickart modules. *Comm. Algebra* **39** (11), 4036–4058 (2011). https://doi.org/10.1080/00927872.2010.515639
15. Abyzov A. N., Tuganbaev A. A. Modules in which sums or intersections of two direct summands are direct summands. *J. Math. Sciences* **211** (3), 297–303 (2015). https://doi.org/10.1007/s10958-015-2605-0
16. Ding N., Ibrahim Y., Yousif M., Zhou Y. $C4$-modules. *Comm. Algebra* **45** (4), 1727–1740 (2017). https://doi.org/10.1080/00927872.2016.1222412
17. Altun-Ozarslan M., Ibrahim Y., Özcan A. Ç., Yousif M. $C4$- and $D4$-Modules via perspective direct summands. *Comm. Algebra* **46** (10), 4480–4497 (2018). https://doi.org/10.1080/00927872.2018.1448838
18. Anderson F. W., Fuller K. R. *Rings and Categories of Modules*. In: Graduate Texts in Math., vol. 13. New York, Springer-Verlag (1974).

Received: September 13, 2020
Revised: March 14, 2021
Accepted: March 19, 2021

Author's information:

Soumitra Das — Assistant Professor; soumitrad330@gmail.com

К вопросу о $D4$-модулях

С. Дас

Инженерно-технологический институт КПР, Коимбатур, 641407, Индия

Для цитирования: *Das S.* On a question concerning $D4$-modules // Вестник Санкт-Петербургского университета. Математика. Механика. Астрономия. 2021. Т. 8 (66). Вып. 3. С. 467–474. https://doi.org/10.21638/spbu01.2021.308

R-модуль M называется $D4$-*модулем*, если всякий раз, когда M_1 и M_2 являются прямыми слагаемыми M с $M_1 + M_2 = M$ и $M_1 \cong M_2$, то $M_1 \backslash M_2$ является прямым слагаемым M. Пусть $M = \bigoplus_{i \in I} M_i$ — прямая сумма подмодулей M_i с $Hom(M_i; M_j) = 0$ для различных $i, j \in I$. Показано, что M является $D4$-модулем тогда и только тогда, когда для каждого $i \in I$ модуль M_i является $D4$-модулем. Это решает открытый вопрос о прямых суммах $D4$-модулей. Наш подход не зависит от решения, полученного недавно Д'Эсте, Кескином Тютюнджу и Трибаком.

Ключевые слова: SIP-модули, $D4$-модули.

Статья поступила в редакцию 13 сентября 2020 г.;
после доработки 14 марта 2021 г.;
рекомендована в печать 19 марта 2021 г.

Контактная информация:

Сумитра Дас — soumitrad330@gmail.com