Saint-Petersburg State University

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Graduation qualification thesis

Toeplitz operators on the Paley-Wiener space

Bachelor's program
"Mathematics"
Specialization and code: 01.03.01 "Mathematics"
Cipher of the EP: SV.5000.2017

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Saint-Petersburg
2021

Abstract

The present thesis is devoted to operator theory. A classical result by R. Rochberg says that every bounded Toeplitz operator T on the Paley-Wiener space PW_a^2 has a bounded symbol φ. Moreover, one can choose φ so that $c \cdot \|\varphi\|_{L^\infty(\mathbb{R})} \leqslant \|T\| \leqslant \|\varphi\|_{L^\infty(\mathbb{R})}$. We prove this estimate for Toeplitz operators on Banach Paley-Wiener spaces PW_a^p, $1 < p < +\infty$.

Contents

1. Introduction

1.1. **Problem setting and the statement of main result.** Let $\mathcal{S}(\mathbb{R})$ denote the classical Schwartz class of smooth complex-valued functions $f \in C^\infty(\mathbb{R})$ such that for every pair of integers $n, m \geqslant 0$ we have

$$\sup_{x\in\mathbb{R}}(1+|x|)^n \cdot \Big|\frac{d^m f}{dx^m}(x)\Big| < +\infty.$$

Define the Fourier transform on $\mathcal{S}(\mathbb{R})$ by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i\xi x} f(x)\,dx, \qquad \xi \in \mathbb{R}.$$

Then the inverse Fourier transform is given by

$$\mathcal{F}^{-1}[f](x) = \check{f}(x) = \int_{\mathbb{R}} e^{2\pi i\xi x} f(\xi)\,d\xi, \qquad x \in \mathbb{R}.$$

Since $\mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$, the integrals above are well defined. It is well known that the Fourier transform is a homeomorphism of $\mathcal{S}(\mathbb{R})$ onto itself. It is also known that $\mathcal{F}$ extends from $\mathcal{S}(\mathbb{R})$ as a unitary operator on $L^2(\mathbb{R})$, see e.g., Section 2.2.2 in [8]. The support of $f \in \mathcal{S}(\mathbb{R})$ is defined by

$$\operatorname{supp} f = \operatorname{clos}\{x \in \mathbb{R} \mid f(x) \neq 0\}.$$

Take a positive real number a and denote

$$\mathcal{S}_a(\mathbb{R}) = \{f \in \mathcal{S}(\mathbb{R}) \mid \operatorname{supp} \hat{f} \subset [-a, a]\}.$$

For $1 \leqslant p < +\infty$, the Paley-Wiener space PW_a^p is a closed subspace of $L^p(\mathbb{R})$ defined by

$$\mathrm{PW}_a^p = \operatorname{clos}_{L^p(\mathbb{R})} \mathcal{S}_a(\mathbb{R}).$$

Observe that PW_a^2 is a Hilbert space. In fact, we have

$$\mathrm{PW}_a^2 = \{f \in L^2(\mathbb{R}) \mid \hat{f} = 0 \text{ a.e. on } \mathbb{R} \setminus [-a, a]\}.$$

Take a bounded measurable function $\mathfrak{m}$ on $\mathbb{R}$. The Fourier multiplier associated to symbol $\mathfrak{m}$ is the map defined by

$$f \longmapsto \mathcal{F}^{-1}\mathfrak{m}\mathcal{F}[f], \qquad f \in \mathcal{S}(\mathbb{R}).$$

As we will see in Proposition 2.1 below, the Fourier multiplier whose symbol is the indicator function $\chi_{[-a,a]}$ is a densely defined bounded operator on $L^p(\mathbb{R})$ for every $1 < p < +\infty$. Since $\chi_{[-a,a]}^2 = \chi_{[-a,a]}$, this operator is, in fact, a linear bounded projector to PW_a^p. It will be denoted by $\mathbb{P}_a$.

Let $\mathcal{P}(\mathbb{R})$ denote the set of all complex-valued functions defined on $\mathbb{R}$ that grow not faster than polynomials:

$$\mathcal{P}(\mathbb{R}) = \{f\colon \mathbb{R} \to \mathbb{C} \mid \exists n \in \mathbb{N} : \sup_{x\in\mathbb{R}} |f(x)| \cdot (1+|x|)^{-n} < +\infty\}.$$

Let $1 < p < +\infty$. Toeplitz operator $T_\varphi : \mathrm{PW}_a^p \to \mathrm{PW}_a^p$ with symbol $\varphi \in \mathcal{P}(\mathbb{R})$ is the mapping densely defined by

$$T_\varphi : \ f \mapsto \mathbb{P}_a[\varphi \cdot f], \qquad f \in \mathcal{S}_a(\mathbb{R}).$$

Since $\mathcal{P}(\mathbb{R}) \cdot \mathcal{S}_a(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$, we have $\varphi \cdot f \in \mathcal{S}(\mathbb{R})$ for every $f \in \mathcal{S}_a(\mathbb{R})$. Hence, T_φ is well defined. In case

$$\sup\{\|T_\varphi[f]\|_{L^p(\mathbb{R})} \mid f \in \mathcal{S}_a(\mathbb{R}), \|f\|_{L^p(\mathbb{R})} = 1\} < +\infty,$$

the operator T_φ admits a unique bounded extension to PW_a^p. This extension will be denoted by the same letter T_φ.

The symbol of a Toeplitz operator on PW_a^p is not unique. We say that a Toeplitz operator T_φ on PW_a^p has a bounded symbol ψ if $T_\varphi = T_\psi$ for a function $\psi \in L^\infty(\mathbb{R})$. Clearly, any bounded symbol $\varphi \in L^\infty(\mathbb{R})$ determines the bounded Toeplitz operator T_φ on PW_a^p, and

$$\|T_\varphi\|_{\mathrm{PW}_a^p \to \mathrm{PW}_a^p} \leqslant \|\varphi\|_{L^\infty(\mathbb{R})} .$$

The class of all bounded Toeplitz operators on PW_a^p will be denoted by $\mathcal{T}(a,p)$. It is easy to see that some unbounded symbols φ can produce bounded Toeplitz operators on PW_a^p. For instance, this is the case for the symbol

$$\varphi(z) = z \cdot e^{2\pi i z \cdot 2a}, \quad z \in \mathbb{C}.$$

Indeed, for every $f \in \mathcal{S}_a(\mathbb{R})$ we have $\operatorname{supp} \mathcal{F}[\varphi \cdot f] \subset [a, 3a]$. Thus, $\mathbb{P}_a[\varphi \cdot f] = 0$ and $T_\varphi = 0$ as an operator on PW_a^p. It makes interesting the question about existence of a bounded symbol for every bounded Toeplitz operator on PW_a^p. In case $p = 2$ the affirmative answer to this question was given by R. Rochberg [13] in 1987.

Our aim in the present thesis is to prove the following theorem.

Theorem 1.1. Let $1 < p < +\infty$. Every Toeplitz operator T_φ on PW_a^p with symbol $\varphi \in \mathcal{S}(\mathbb{R})$ has a bounded symbol ψ such that

$$\|\psi\|_{L^\infty(\mathbb{R})} \leqslant c\left(p + \frac{1}{p-1}\right) \cdot \|T_\varphi\|_{\mathrm{PW}_a^p \to \mathrm{PW}_a^p} ,$$

for a universal constant $c > 0$.

We expect that this theorem can be used to prove existence of a bounded symbol for every bounded Toeplitz operator on PW_a^p, $1 < p < +\infty$.

1.2. **Summary of known results.** We use notation $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ for the unit circle. Let m denote the Lebesgue measure on $\mathbb{T}$ normalized by $m(\mathbb{T}) = 1$. Define the Fourier coefficients of $f \in L^1(\mathbb{T})$ by

$$\hat{f}(n) = \int_{\mathbb{T}} f(z) \bar{z}^n \, dm(z).$$

For $1 \leqslant p < +\infty$, a function f on $\mathbb{T}$ is said to belong to the Hardy space H^p in the unit disk if $f \in L^p(\mathbb{T})$ and $\hat{f}(n) = 0$ for all integer $n < 0$. The space H^p is a closed subspace of $L^p(\mathbb{T})$. Denote by P_+ the orthogonal projection in $L^2(\mathbb{T})$ to the subspace H^2. The classical Toeplitz operator $T_\varphi : \ H^2 \to H^2$ with symbol $\varphi \in L^\infty(\mathbb{T})$ is defined by

$$T_\varphi : \ f \mapsto \mathrm{P}_+[\varphi \cdot f], \qquad f \in H^2.$$

In 1964, A. Brown and P. Halmos [3] described basic algebraic properties of Toeplitz operators on H^2. In particular, they proved that the Toeplitz operator T_φ on H^2 with a bounded symbol φ satisfies

$$\|T_\varphi\|_{H^2 \to H^2} = \|\varphi\|_{L^\infty(\mathbb{T})} ,$$

see Corollary to Theorem 5 in [3]. This formula implies that the symbol of a Toeplitz operator on H^2 is unique.

For Toeplitz operators on the Paley-Wiener space PW_a^2, the classical treatment of their basic properties is due to R. Rochberg [13]. In 1987, he considered boundedness and compactness, as well as Schatten classes $\mathcal{S}^p$ membership. As we mentioned above, he proved that every bounded Toeplitz operator on PW_a^2 has a bounded symbol. In this thesis, we will apply his methods to prove a similar result for Toeplitz operators on PW_a^p.

Toeplitz operators on the Paley-Wiener space are in fact examples of general truncated Toeplitz operators that we defined below. A function $\theta \in H^2$ is called inner if $|\theta| = 1$ m-almost everywhere on the unit circle $\mathbb{T}$. With each non-constant inner function θ we associate the subspace $K_\theta^2 = H^2 \ominus \theta H^2$ of $L^2(\mathbb{T})$. Such subspaces are called model subspaces [11]. Denote by P_θ the orthogonal projector from $L^2(\mathbb{T})$ onto K_θ^2. Truncated Toeplitz operator $T_\varphi : K_\theta^2 \to K_\theta^2$ with symbol $\varphi \in L^2(\mathbb{T})$ is densely defined by the following expression

$$T_\varphi :\ f \mapsto \mathrm{P}_\theta[\varphi \cdot f], \qquad f \in K_\theta^2 \cap L^\infty(\mathbb{T}).$$

As an example, if $\theta = z^n$, then truncated Toeplitz operator on K_θ^2 are Toeplitz matrices of size $n \times n$:

$$T_\varphi = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{-2} & c_{-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{-n+1} & c_{-n+2} & c_{-n+3} & \cdots & c_0 \end{pmatrix}, \qquad c_k = \int\limits_{\mathbb{T}} \varphi \bar{z}^k \, dm,$$

where we identify the operator with its matrix in the orthonormal basis $\{z^k\}_{k=0}^{n-1}$ of K_θ^2. Similarly, Toeplitz operators on the Paley-Wiener space are closely related to truncated Toeplitz operators on the model subspace $K_{\theta_a}^2$ of the Hardy space H_+^2 in the upper-half plane $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ associated with the inner function $\theta_a = e^{2\pi i a z}, a > 0$. In fact, $\mathrm{PW}_a^2 = \bar{\theta}_a K_{\theta_a^2}^2$, see [11].

General theory of truncated Toeplitz operators has been started with D. Sarason's paper [14] appeared in 2007. It plays the same role for truncated Toeplitz operators as A. Brown and P. Halmos paper [3] plays for classical Toeplitz operators. D. Sarason posed a number of important questions on truncated Toeplitz operators including the problem of existence of a bounded symbol of a general bounded truncated Toeplitz operator.

In 2010, A. Baranov, I. Chalendar, E. Fricain, J. Mashreghi, and D. Timotin [2] constructed an inner function θ and a bounded truncated Toeplitz operator on K_θ^2 that has no bounded symbol. In 2011, A. Baranov, R. Bessonov, and V. Kapustin [1] characterized inner functions θ such that every bounded Toeplitz operator on K_θ^2 has a bounded symbol. In particular, this is the case for so-called one-component inner functions. An inner function θ is called one-component if the set $\{z \mid |\theta| < \varepsilon\}$ is a connected subset (of the unit disk or the upper half-plane of the complex plane) for some $0 < \varepsilon < 1$. Since the set $\{z \in \mathbb{C}_+ \mid |\theta_a(z)| < \varepsilon\}$ is connected for every $0 < \varepsilon < 1$, this result generalizes aforementioned theorem by R. Rochberg.

In 2011, M. Carlsson [4] proved an estimate similar to the one we want to prove in Theorem 1.1. Instead of Toeplitz operators on PW_a^2 he dealt with Wiener-Hopf operators on $L^2[0, 2a]$. Following [4], define truncated Wiener-Hopf operator W_φ on $L^2[0, 2a]$ with symbol $\varphi \in \mathcal{S}(\mathbb{R})$ by

$$W_\varphi[f](x) = \int_{\mathbb{R}} \hat{\varphi}(y) f(x+y)\, dy, \qquad x \in [0, 2a],$$

where f is extended by zero to $\mathbb{R} \setminus [0, 2a]$. One can consider more general symbols φ including tempered distributions, for simplicity of presentation we limit ourselves by the case $\varphi \in \mathcal{S}(\mathbb{R})$. M. Carlsson obtained the following estimate

$$\frac{1}{3} \cdot \|\varphi\|_{L^\infty(\mathbb{R})} \leqslant \|W_\varphi\|_{L^2[0,2a] \to L^2[0,2a]} \leqslant \|\varphi\|_{L^\infty(\mathbb{R})},$$

see Theorem 1.1 in [4]. We claim that

$$\frac{1}{3} \cdot \|\varphi\|_{L^\infty(\mathbb{R})} \leqslant \|T_\varphi\|_{\mathrm{PW}_a^2 \to \mathrm{PW}_a^2} \leqslant \|\varphi\|_{L^\infty(\mathbb{R})}.$$

Indeed, let $\mathcal{U}_t$ denote the translation operator $\mathcal{U}_t[f] = f(\cdot + t)$ on $L^2(\mathbb{R})$. For every $f \in L^2[0, 2a]$ we have

$$\begin{aligned}
\mathcal{F}\theta_a T_\varphi \bar{\theta}_a \mathcal{F}^{-1}[f] &= \mathcal{F}\theta_a T_\varphi \mathcal{F}^{-1}\mathcal{U}_a[f] \\
&= \mathcal{U}_{-a}\mathcal{F} T_\varphi \mathcal{F}^{-1}\mathcal{U}_a[f] \\
&= \mathcal{U}_{-a}\chi_{[-a,a]}\mathcal{F}[\varphi \cdot \mathcal{F}^{-1}\mathcal{U}_a[f]] \\
&= \chi_{[0,2a]}\mathcal{U}_{-a}[\hat{\varphi} * \mathcal{U}_a[f]] \\
&= \chi_{[0,2a]} \cdot (\hat{\varphi} * f) \\
&= W_{\tilde{\varphi}}[f],
\end{aligned}$$

where $\tilde{\varphi}(x) = \varphi(-x)$ for $x \in \mathbb{R}$ and $*$ denotes the convolution of functions in $L^1(\mathbb{R})$:

$$(f_1 * f_2)(x) = \int_{\mathbb{R}} f_1(x-y) f_2(y)\, dy, \quad x \in \mathbb{R}.$$

Let, as before, $K^2_{\theta_a^2} = H^2_+ \ominus \theta_a^2 H^2_+$. Above argument says that the following diagram is commutative:

$$\begin{tikzcd}
L^2[0,2a] \arrow[r, "W_{\tilde{\varphi}}"] \arrow[d, "\mathcal{F}^{-1}"'] & L^2[0,2a] \\
K^2_{\theta_a^2} \arrow[d, "\bar{\theta}_a"'] & K^2_{\theta_a^2} \arrow[u, "\mathcal{F}"'] \\
\mathrm{PW}_a^2 \arrow[r, "T_\varphi"] & \mathrm{PW}_a^2 \arrow[u, "\theta_a"']
\end{tikzcd}$$

Since the operator $h \mapsto \mathcal{F}\theta_a[h]$ is unitary, we have $\|W_{\tilde{\varphi}}\| = \|T_\varphi\|$. Since also $\|\varphi\|_{L^\infty(\mathbb{R})} = \|\tilde{\varphi}\|_{L^\infty(\mathbb{R})}$, the claim follows. Thus, in case $p = 2$, one can take $c = 1$ in Theorem 1.1 of the present thesis.

More information about truncated Toeplitz operators can be found in survey [5] by I. Chalendar, E. Fricain and D. Timotin.

1.3. **Plan of the proof.** In this section, we describe the structure of the present thesis. Let $1 < p < +\infty$. In Section 2 we show that projector $\mathbb{P}_a$ on $L^p(\mathbb{R})$ is bounded and admits an integral representation with the following kernel

$$\operatorname{sinc}_a(z) = \frac{\sin(2\pi a z)}{\pi z}, \quad z \in \mathbb{C},\ a > 0.$$

In Section 2.2 we show that every Toeplitz operator on PW_a^p also admits an integral representation with the $\operatorname{sinc}_a(\cdot)$ kernel.

Further, we fix some $\varphi \in \mathcal{S}(\mathbb{R})$. In Section 2.3 we define three smooth and compactly supported functions with the help of which we construct left, central, and right parts of the symbol φ. Next, given a Toeplitz operator T_φ on PW_a^p we construct Toeplitz operators $T_{\mathfrak{L}}$, $T_{\mathfrak{C}}$, and $T_{\mathfrak{R}}$. In Proposition 2.4, we prove the existence of a universal constant $c > 0$ such that

$$\|T_{\mathfrak{L}}\| + \|T_{\mathfrak{C}}\| + \|T_{\mathfrak{R}}\| \leqslant c \cdot \|T_\varphi\|.$$

In the beginning of the Section 3, we start with some preliminaries and prove auxiliary statements. Then we prove the upper bound for the norm of the central part of the symbol. In addition, in Section 4 we define Hankel operators with bounded symbols on the Hardy space in the upper half-plane H_+^p and sketch a proof of the Nehari theorem. Furthermore, we show that any Hankel operator with symbol $\bar{\theta}_a^2\varphi_*$ such that $\varphi_* \in \mathcal{S}(\mathbb{R})$ and $\operatorname{supp}\hat{\varphi}_* \subset \mathbb{R}_+$ corresponds to a Toeplitz operator on PW_a^p. Finally, in Section 5 we prove the main result of the present thesis.

2. Riesz projector and related operators. Splitting the symbol

2.1. **Riesz projector.** Let $1 \leqslant p < +\infty$. The Hardy space H_+^p in the upper half-plane $\mathbb{C}_+$ can be defined by

$$H_+^p = \operatorname{clos}_{L^p(\mathbb{R})}\{f \in \mathcal{S}(\mathbb{R}) \mid \operatorname{supp}\hat{f} \subset \mathbb{R}_+\}.$$

Let also

$$H_-^p = \operatorname{clos}_{L^p(\mathbb{R})}\{f \in \mathcal{S}(\mathbb{R}) \mid \operatorname{supp}\hat{f} \subset \mathbb{R}_-\}.$$

Basic theory of Hardy spaces can be found in [6], [7], [9], and [10].

Define the Riesz projector $\mathbb{P}_+$ to be the Fourier multiplier associated to symbol $\chi_{\mathbb{R}_+}$,

$$\chi_{\mathbb{R}_+}(x) = \begin{cases} 1, & x \geqslant 0, \\ 0, & x < 0. \end{cases}$$

For $1 < p < +\infty$, $\mathbb{P}_+$ extends from $\mathcal{S}(\mathbb{R})$ to a linear bounded operator on $L^p(\mathbb{R})$, see e.g., Lecture 19.2 and 19.3 in [10]. Since $\chi_{\mathbb{R}_+}^2 = \chi_{\mathbb{R}_+}$, we have $\mathbb{P}_+^2 = \mathbb{P}_+$, that is, $\mathbb{P}_+$ operator is a linear bounded projector to H_+^p in $L^p(\mathbb{R})$. Set

$$A_p = \|\mathbb{P}_+\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})}.$$

It is known that

$$A_p \leqslant \tfrac{A}{p-1}, \qquad p \to 1,$$

$$A_p \leqslant Ap, \quad p \to +\infty,$$

for a universal constant $A > 0$, see [7]. Consider an inner function $\theta_a = e^{2\pi i a z}$, $a > 0$. Recall that $\mathcal{U}_t$ is the translation operator $f \mapsto f(\cdot + t)$ and $\mathbb{P}_a$ is the projector to PW_a^p.

Proposition 2.1. We have $\|\mathbb{P}_a\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} \leqslant 2A_p$.

Proof. We have

$$\mathcal{U}_{2a}[\chi_{\mathbb{R}_+}] - \mathcal{U}_{-a}[\chi_{\mathbb{R}_+}] = \chi_{[-2a,+\infty]} - \chi_{[a,+\infty]} = \chi_{[-a,a]}.$$

By the definition of Fourier transform, $\mathcal{F}^{-1}\mathcal{U}_a = \bar{\theta}_a\mathcal{F}^{-1}$ for every $a > 0$. Hence,

$$\begin{aligned}\mathbb{P}_a = \mathcal{F}^{-1}\chi_{[-a,a]}\mathcal{F} &= \mathcal{F}^{-1}\chi_{[-2a,+\infty]}\mathcal{F} - \mathcal{F}^{-1}\chi_{[a,+\infty]}\mathcal{F}\\ &= \mathcal{F}^{-1}\mathcal{U}_{2a}\chi_{\mathbb{R}_+}\mathcal{U}_{-2a}\mathcal{F} - \mathcal{F}^{-1}\mathcal{U}_{-a}\chi_{\mathbb{R}_+}\mathcal{U}_a\mathcal{F}\\ &= \bar{\theta}_a^2\mathbb{P}_+\theta_a^2 - \theta_a\mathbb{P}_+\bar{\theta}_a.\end{aligned}$$

The result follows. □

Let $C_0^\infty(\mathbb{R})$ be the space of all complex-valued smooth functions on $\mathbb{R}$ with compact support. Note that $\mathbb{P}_a(L^p(\mathbb{R})) = \mathrm{PW}_a^p$. Indeed, since PW_a^p is a closed subspace of $L^p(\mathbb{R})$, it is enough to prove that $\mathbb{P}_a(E) \subset \mathrm{PW}_a^p$ for some subset $E \subset L^p(\mathbb{R})$ such that $\mathrm{clos}_{L^p(\mathbb{R})} E = L^p(\mathbb{R})$ and $\mathcal{S}_a(\mathbb{R}) \subset E$. This holds for

$$E = \{f \mid \exists g \in C_0^\infty(\mathbb{R}) :\ f = \check{g},\ \pm a \notin \operatorname{supp} g\}.$$

Whence, operator $\mathbb{P}_a$ is a bounded projector onto Paley-Wiener space PW_a^p.

Our next aim is to derive an integral formula for $\mathbb{P}_a$.

Corollary 2.2. For $1 < p < +\infty$, the projector $\mathbb{P}_a$ admits the following integral representation:

$$\mathbb{P}_a[f](x) = \int_{\mathbb{R}} \mathrm{sinc}_a(x-y)f(y)\,dy, \qquad f \in L^p(\mathbb{R}). \tag{2.1}$$

Proof. Let us first show that function $\mathrm{sinc}_a \in L^p(\mathbb{R})$ for every $1 < p \leqslant +\infty$. Indeed, this follows from the estimate

$$|\,\mathrm{sinc}_a(x)| \leqslant \frac{1}{\pi\cdot|x|}, \qquad x \in \mathbb{R},$$

and boundedness of sinc_a near the origin. Therefore, the integral in (2.1) converges and defines the function on $\mathbb{R}$. Since

$$\check{\chi}_{[-a,a]}(x) = \int_{-a}^{a} e^{2\pi i\xi x}\,d\xi = \frac{e^{2\pi ixa} - e^{-2\pi ixa}}{2\pi ix} = \frac{\sin(2\pi xa)}{\pi x} = \mathrm{sinc}_a(x), \tag{2.2}$$

formula (2.1) holds for every $f \in \mathcal{S}(\mathbb{R})$ by the definition of $\mathbb{P}_a$. Take an arbitrary function $f \in L^p(\mathbb{R})$ and consider a sequence $\{f_n\} \subset \mathcal{S}(\mathbb{R})$ such that $f_n \to f$ in $L^p(\mathbb{R})$. Then $\mathbb{P}_a[f_n] \to \mathbb{P}_a[f]$ in $L^p(\mathbb{R})$ and one can choose a subsequence $\{f_{n_k}\}$ such that $\mathbb{P}_a[f_{n_k}](x) \to \mathbb{P}_a[f](x)$ for almost every $x \in \mathbb{R}$. On the other hand,

$$\mathbb{P}_a[f_{n_k}](x) = \int_{\mathbb{R}} \mathrm{sinc}_a(x-y)f_{n_k}(y)\,dy$$

converges to $\int_{\mathbb{R}} \mathrm{sinc}_a(x-y)f(y)\,dy$ for every $x \in \mathbb{R}$, by Holder's inequality. Hence, (2.1) holds for every $f \in L^p(\mathbb{R})$. □

2.2. **Toeplitz operators on PW_a^p as integral operators.** It is easy to see from Corollary 2.2 that every Toeplitz operator on PW_a^p with symbol $\varphi \in \mathcal{P}(\mathbb{R})$ admits the following representation

$$T_\varphi[f](x) = \int\limits_{\mathbb{R}} \operatorname{sinc}_a(x-y) f(y)\varphi(y)\, dy, \qquad f \in \mathrm{PW}_a^p. \tag{2.3}$$

Given a function h on $\mathbb{R}$, let $h|_A$ denote the restriction of h to a subset $A \subset \mathbb{R}$.

Proposition 2.3. If $\varphi \in \mathcal{S}(\mathbb{R})$ is such that $\hat{\varphi}|_{[-2a,2a]} = 0$, then $T_\varphi = 0$.

Proof. Take $f \in \mathcal{S}_a(\mathbb{R})$. By definition,

$$\mathcal{F}T_\varphi[f](x) = \chi_{[-a,a]}(x) \int\limits_{\mathbb{R}\setminus[-2a,2a]} \hat{f}(x-y)\hat{\varphi}(y)\, dy.$$

For $x \in [-a,a]$ and y such that $|y| > 2a$, we have $|x-y| > a$. Hence, for such x and y we have $\hat{f}(x-y) = 0$ because $\operatorname{supp} \hat{f} \subset [-a,a]$. $\square$

2.3. **Splitting procedure. Norm estimates.** Consider a function $\psi_{\mathfrak{L}} \in C_0^\infty(\mathbb{R})$ such that $\psi_{\mathfrak{L}} \geqslant 0$, $\operatorname{supp} \psi_{\mathfrak{L}} = [-4, -\frac{1}{4}]$ and $\psi_{\mathfrak{L}}|_{[-2,-\frac{1}{2}]} = 1$. Set $\psi_{\mathfrak{R}}(x) = \psi_{\mathfrak{L}}(-x)$ and define $\psi_{\mathfrak{C}} = \chi_{[-\frac{1}{2},\frac{1}{2}]}(1 - \psi_{\mathfrak{L}} - \psi_{\mathfrak{R}})$. Then $\psi_{\mathfrak{L}}$, $\psi_{\mathfrak{C}}$, $\psi_{\mathfrak{R}}$ are smooth compactly supported functions such that $\psi_{\mathfrak{L}} + \psi_{\mathfrak{C}} + \psi_{\mathfrak{R}} = 1$ on $[-2, 2]$, see Figure 1 below.

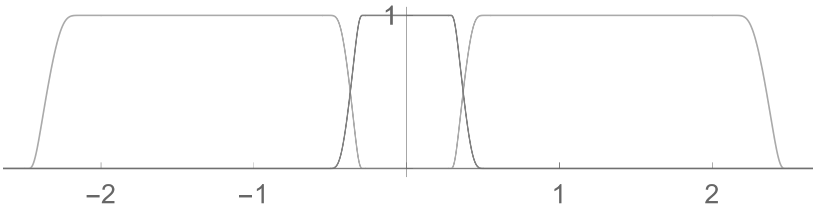

FIGURE 1. Graphs of functions $\psi_{\mathfrak{L}}, \psi_{\mathfrak{C}}, \psi_{\mathfrak{R}}$.

For $a > 0$ define $\psi_{\mathfrak{C},a} \colon x \mapsto \psi_{\mathfrak{C}}(xa)$ and $\psi_{\mathfrak{L},a}$, $\psi_{\mathfrak{R},a}$ similarly. Consider a Toeplitz operator $T_\varphi :\ \mathrm{PW}_a^p \to \mathrm{PW}_a^p$ with symbol $\varphi \in \mathcal{S}(\mathbb{R})$. Define $\varphi_{\mathfrak{C}} = \mathcal{F}^{-1}\psi_{\mathfrak{C},a}\mathcal{F}[\varphi]$ and let $T_{\mathfrak{C}} = T_{\varphi_{\mathfrak{C}}}$. Analogously, define $\varphi_{\mathfrak{L}}$, $\varphi_{\mathfrak{R}}$, $T_{\mathfrak{L}}$, $T_{\mathfrak{R}}$ using functions $\psi_{\mathfrak{L}}$, $\psi_{\mathfrak{R}}$. We call $T_{\mathfrak{L}}$, $T_{\mathfrak{C}}$, $T_{\mathfrak{R}}$ the left, central, and right parts of T_φ, respectively.

Proposition 2.4. Let $1 < p < +\infty$. Consider a Toeplitz operator T_φ on PW_a^p with symbol $\varphi \in \mathcal{S}(\mathbb{R})$. We have $T_\varphi = T_{\mathfrak{L}} + T_{\mathfrak{C}} + T_{\mathfrak{R}}$ and

$$c\Big(\|T_{\mathfrak{L}}\| + \|T_{\mathfrak{C}}\| + \|T_{\mathfrak{R}}\|\Big) \leqslant \|T_\varphi\| \leqslant \|T_{\mathfrak{L}}\| + \|T_{\mathfrak{C}}\| + \|T_{\mathfrak{R}}\|,$$

for a universal constant $c > 0$.

Proof. Since $\psi_{\mathfrak{L},a} + \psi_{\mathfrak{C},a} + \psi_{\mathfrak{R},a} = 1$ on $[-2a, 2a]$, we have $\hat{\varphi} = \hat{\varphi}_{\mathfrak{L}} + \hat{\varphi}_{\mathfrak{C}} + \hat{\varphi}_{\mathfrak{R}}$ on $[-2a, 2a]$. Hence, we have $T_\varphi = T_{\mathfrak{L}} + T_{\mathfrak{C}} + T_{\mathfrak{R}}$ by Lemma 2.3, and thus

$$\|T_\varphi\| \leqslant \|T_{\mathfrak{L}}\| + \|T_{\mathfrak{C}}\| + \|T_{\mathfrak{R}}\|.$$

Let us check the opposite inequality. Take $f \in \mathcal{S}_a(\mathbb{R})$. By Fubini–Tonelli Theorem and (2.3), we have

$$\begin{aligned}
T_{\mathfrak{C}}[f](x) &= \int \operatorname{sinc}_a(x-y)f(y)\cdot\left[\int \varphi(y-t)\check{\psi}_{\mathfrak{C},a}(t)\,dt\right]dy \\
&= \int \check{\psi}_{\mathfrak{C},a}(t)\int \operatorname{sinc}_a(x-y)f(y)\varphi(y-t)\,dy\,dt \\
&= \int \check{\psi}_{\mathfrak{C},a}(t)\int \operatorname{sinc}_a(x-t-\xi)f(\xi+t)\varphi(\xi)\,d\xi\,dt \\
&= \int \check{\psi}_{\mathfrak{C},a}(t)\cdot \mathcal{U}_{-t}T_\varphi\mathcal{U}_t[f](x)\,dt.
\end{aligned}$$

Set $p = |\check{\psi}_{\mathfrak{C},a}|/\left\|\check{\psi}_{\mathfrak{C},a}\right\|_{L^1(\mathbb{R})}$, then

$$\int\limits_{\mathbb{R}} p(x)\,dx = 1.$$

Jensen's inequality gives

$$\Phi\left(\int\limits_{\mathbb{R}} h(x)p(x)\,dx\right) \leqslant \int\limits_{\mathbb{R}} \Phi(h(x))p(x)\,dx,$$

for every convex function $\Phi:\ \mathbb{R}\to\mathbb{R}_+$ and every h such that $hp \in L^1(\mathbb{R})$. Choosing $\Phi = |x|^p$, we obtain

$$\begin{aligned}
\|T_{\mathfrak{C}}[f]\|^p_{L^p(\mathbb{R})} &= \int\left|\int \check{\psi}_{\mathfrak{C},a}(t)\cdot\mathcal{U}_{-t}T_\varphi\mathcal{U}_t[f](x)\,dt\right|^p dx \\
&\leqslant \left\|\check{\psi}_{\mathfrak{C},a}\right\|^p_{L^1(\mathbb{R})}\int\left|\int |\mathcal{U}_{-t}T_\varphi\mathcal{U}_t[f](x)|\cdot p(t)\,dt\right|^p dx \\
&\leqslant \left\|\check{\psi}_{\mathfrak{C},a}\right\|^{p-1}_{L^1(\mathbb{R})}\int\int |\mathcal{U}_{-t}T_\varphi\mathcal{U}_t[f](x)|^p\cdot p(t)\,dt\,dx \\
&\leqslant \left\|\check{\psi}_{\mathfrak{C},a}\right\|^{p-1}_{L^1(\mathbb{R})}\int |\check{\psi}_{\mathfrak{C},a}(t)|\int |\mathcal{U}_{-t}T_\varphi\mathcal{U}_t[f](x)|^p\,dx\,dt \\
&= \left\|\check{\psi}_{\mathfrak{C},a}\right\|^{p-1}_{L^1(\mathbb{R})}\int |\check{\psi}_{\mathfrak{C},a}(t)|\cdot\|\mathcal{U}_{-t}T_\varphi\mathcal{U}_t[f]\|^p_{L^p(\mathbb{R})}\,dt \\
&\leqslant \|T_\varphi\|^p\cdot\|f\|^p_{L^p(\mathbb{R})}\cdot\left\|\check{\psi}_{\mathfrak{C},a}\right\|^p_{L^1(\mathbb{R})}.
\end{aligned}$$

Hence, $\|T_{\mathfrak{C}}\| \leqslant \|T_\varphi\|\cdot\left\|\check{\psi}_{\mathfrak{C},a}\right\|_{L^1(\mathbb{R})}$. Similar arguments apply to $T_{\mathfrak{L}}$, $T_{\mathfrak{R}}$ and give us the estimate

$$\|T_{\mathfrak{L}}\| + \|T_{\mathfrak{C}}\| + \|T_{\mathfrak{R}}\| \leqslant \left(\left\|\check{\psi}_{\mathfrak{L},a}\right\|_{L^1(\mathbb{R})} + \left\|\check{\psi}_{\mathfrak{C},a}\right\|_{L^1(\mathbb{R})} + \left\|\check{\psi}_{\mathfrak{R},a}\right\|_{L^1(\mathbb{R})}\right)\cdot\|T_\varphi\|.$$

It remains to note that the constant in the right hand side does not depend on a because

$$\left\|\check{\psi}_{\mathfrak{C},a}\right\|_{L^1(\mathbb{R})} = \left\|\check{\psi}_{\mathfrak{C}}\right\|_{L^1(\mathbb{R})}$$

and similar identities hold for $\check{\psi}_{\mathfrak{L},a}$, $\check{\psi}_{\mathfrak{R},a}$. $\square$

3. Reproducing kernels. Central part of the symbol

3.1. **Paley-Wiener space as a Banach space of entire functions.** In Section 2.1 we prove that $\mathbb{P}_a(L^p(\mathbb{R})) = \mathrm{PW}_a^p$ for $1 < p < +\infty$. In addition, Corollary 2.2 says that for every function $f \in \mathrm{PW}_a^p$ we have

$$f(x) = \mathbb{P}_a[f](x) = \int\limits_{\mathbb{R}} \mathrm{sinc}_a(x-y) f(y)\, dy, \tag{3.1}$$

almost everywhere on $\mathbb{R}$. Note that the right hand side is an entire function with respect to x. This follows from the fact that the integral

$$\int\limits_{\mathbb{R}} \frac{\partial}{\partial z} \mathrm{sinc}_a(z-y) f(y)\, dy = \int\limits_{\mathbb{R}} \frac{2a\cos(2\pi a(z-y)) - \mathrm{sinc}_a(z-y)}{z-y} f(y)\, dy$$

converges uniformly in a neighborhood of any point $z \in \mathbb{C}$. This shows that any function $f \in \mathrm{PW}_a^p$ can be naturally identified with an entire function using (3.1). In other words, for every $f \in \mathrm{PW}_a^p$ one can find an entire function $g : \mathbb{C} \to \mathbb{C}$ such that $g \in L^p(\mathbb{R})$ and $f = g$ almost everywhere on $\mathbb{R}$. In particular, for every $z \in \mathbb{C}$ and $f \in \mathrm{PW}_a^p$ the value $f(z)$ is well defined.

3.2. **Reproducing kernels in PW_a^p.**

Lemma 3.1. Let $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For each $z \in \mathbb{C}$ the linear functional $\phi_z :\ f \mapsto f(z)$ on PW_a^p is bounded and

$$\phi_z(f) = \int\limits_{\mathbb{R}} \mathrm{sinc}_a(z-y) f(y)\, dy, \qquad f \in \mathrm{PW}_a^p.$$

Moreover, for $x \in \mathbb{R}$ we have $\|\phi_x\| \leqslant \|\mathrm{sinc}_a\|_{L^q(\mathbb{R})}$.

Proof. By definition (see also the discussion in Section 3.1), we have

$$\phi_z(f) = f(z) = \int\limits_{\mathbb{R}} \mathrm{sinc}_a(z-y) f(y)\, dy, \qquad f \in \mathrm{PW}_a^p.$$

Then, by Holder's inequality for every $f \in \mathrm{PW}_a^p$ we have

$$|\phi_x(f)| \leqslant \|\mathcal{U}_{-z}[\mathrm{sinc}_a] \cdot f\|_{L^1(\mathbb{R})} \leqslant \|\mathcal{U}_{-z}[\mathrm{sinc}_a]\|_{L^q(\mathbb{R})} \cdot \|f\|_{L^p(\mathbb{R})}\,.$$

It follows that ϕ_z is bounded and $\|\phi_z\| \leqslant \|\mathcal{U}_{-z}[\mathrm{sinc}_a]\|_{L^q(\mathbb{R})}$. In particular, if $x \in \mathbb{R}$, then $\|\phi_x\| \leqslant \|\mathrm{sinc}_a\|_{L^q(\mathbb{R})}$. ☐

3.3. **Upper bound for the norm of the central part of the symbol.**

Proposition 3.2. Let $1 < p < +\infty$. Consider a Toeplitz operator T_φ on PW_a^p with symbol $\varphi \in \mathcal{S}(\mathbb{R})$. Let $T_{\mathfrak{C}}$ be its central part constructed in Section 2.3. Then we have

$$\|\varphi_{\mathfrak{C}}\|_{L^\infty(\mathbb{R})} \leqslant c_p \cdot \|T_{\mathfrak{C}}\|_{\mathrm{PW}_a^p \to \mathrm{PW}_a^p}\,,$$

for some constant $c_p > 0$ depending only on p.

Proof. Take $\varepsilon = \frac{a}{8}$ and fix some $x \in \mathbb{R}$. From formula (2.2) we see that $\operatorname{supp} \mathcal{F}[\mathrm{sinc}_\varepsilon(\cdot)] \subset [-\varepsilon, \varepsilon]$, therefore, $\mathrm{sinc}_\varepsilon \in \mathrm{PW}_a^p$. Recall that $\operatorname{supp} \hat{\varphi}_{\mathfrak{C}} = [-\frac{a}{2}, \frac{a}{2}]$, hence the support of

$$\mathcal{F}[\varphi_{\mathfrak{C}} \cdot \mathcal{U}_{-x}[\mathrm{sinc}_\varepsilon]] = (\psi_{\mathfrak{C},a}\hat{\varphi}) * (\chi_{[-\varepsilon,\varepsilon]} e^{-2\pi i x\xi})$$

is in $[-a,a]$ by properties of convolution ($\operatorname{supp} f * g \subset \operatorname{supp} f + \operatorname{supp} g$). We have

$$\begin{aligned}\phi_x(T_{\mathfrak{C}}\mathcal{U}_{-x}[\operatorname{sinc}_\varepsilon]) &= T_{\mathfrak{C}}\mathcal{U}_{-x}[\operatorname{sinc}_\varepsilon](x)\\ &= \mathbb{P}_a[\varphi_{\mathfrak{C}}\cdot\mathcal{U}_{-x}[\operatorname{sinc}_\varepsilon]](x)\\ &= \varphi_{\mathfrak{C}}(x)\cdot\mathcal{U}_{-x}[\operatorname{sinc}_\varepsilon](x)\\ &= \varphi_{\mathfrak{C}}(x)\cdot\operatorname{sinc}_\varepsilon(0)\\ &= 2\varepsilon\cdot\varphi_{\mathfrak{C}}(x).\end{aligned}$$

By Lemma 3.1, we have $\phi_x \in (\mathrm{PW}_a^p)^*$, therefore

$$\begin{aligned}|\varphi_{\mathfrak{C}}(x)| &\leqslant \frac{1}{2\varepsilon}\|\phi_x\|\cdot\|T_{\mathfrak{C}}\mathcal{U}_{-x}[\operatorname{sinc}_\varepsilon]\|_{L^p(\mathbb{R})}\\ &\leqslant \frac{1}{2\varepsilon}\|\operatorname{sinc}_a\|_{L^q(\mathbb{R})}\cdot\|T_{\mathfrak{C}}\|\cdot\|\operatorname{sinc}_\varepsilon\|_{L^p(\mathbb{R})}.\end{aligned}$$

It remains to note that

$$\frac{1}{2\varepsilon}\|\operatorname{sinc}_a\|_{L^q(\mathbb{R})}\cdot\|\operatorname{sinc}_\varepsilon\|_{L^p(\mathbb{R})} = 4\|\operatorname{sinc}_1\|_{L^q(\mathbb{R})}\cdot\left\|\operatorname{sinc}_{1/8}\right\|_{L^p(\mathbb{R})}$$

does not depend on a. □

Lemma 3.3. Let $1 < p < +\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. We have

$$\|\operatorname{sinc}_1\|_{L^q(\mathbb{R})}\cdot\left\|\operatorname{sinc}_{1/8}\right\|_{L^p(\mathbb{R})} \leqslant c\cdot\left(p+\frac{1}{p-1}\right),$$

for a universal constant $c > 0$.

Proof. We have

$$\left\|\operatorname{sinc}_{1/8}\right\|_{L^p(\mathbb{R})} = 8^{-\frac{1}{q}}\|\operatorname{sinc}_1\|_{L^p(\mathbb{R})} \leqslant \|\operatorname{sinc}_1\|_{L^p(\mathbb{R})}.$$

Clearly, $|\operatorname{sinc}_1(x)| \leqslant 2$ for $|x| \leqslant \frac{1}{2\pi}$ and $|\operatorname{sinc}_1(x)| \leqslant \frac{1}{\pi|x|}$ for $|x| > \frac{1}{2\pi}$. Then, we obtain

$$\|\operatorname{sinc}_1\|^q_{L^q(\mathbb{R})} \leqslant \frac{2^q}{\pi} + \frac{2}{\pi}\int_{1/2}^{+\infty}\frac{dx}{x^q} = \frac{2^q}{\pi}\left(1+\frac{1}{q-1}\right).$$

Then,

$$\left(\frac{2^q}{\pi}\left(1+\frac{1}{q-1}\right)\right)^{\frac{1}{q}}\cdot\left(\frac{2^p}{\pi}\left(1+\frac{1}{p-1}\right)\right)^{\frac{1}{p}} = \frac{4}{\pi}\left(1+\frac{1}{q-1}\right)^{\frac{1}{q}}\cdot\left(1+\frac{1}{p-1}\right)^{\frac{1}{p}},$$

and, by Bernoulli's inequality, we have

$$\begin{aligned}\left(1+\frac{1}{q-1}\right)^{\frac{1}{q}}\cdot\left(1+\frac{1}{p-1}\right)^{\frac{1}{p}} &\leqslant \left(1+\frac{1}{(q-1)q}\right)\cdot\left(1+\frac{1}{(p-1)p}\right)\\ &\leqslant \left(1+\frac{1}{q-1}\right)\cdot\left(1+\frac{1}{(p-1)p}\right)\\ &= p\cdot\left(1+\frac{1}{(p-1)p}\right)\\ &= p+\frac{1}{p-1}.\end{aligned}$$

Summarizing, one can take $c = \frac{4}{\pi}$. □

4. Nehari Theorem. Right and left parts of the symbol

4.1. **Hankel operators on the Hardy space. Nehari Theorem.** Hankel operator $H_\varphi : H^2 \to \overline{zH^2}$ with symbol $\varphi \in L^2(\mathbb{T})$ can be densely defined by

$$H_\varphi : f \mapsto \mathrm{P}_-[\varphi \cdot f], \qquad f \in H^2 \cap L^\infty(\mathbb{T}),$$

where $\mathrm{P}_- = I - \mathrm{P}_+$. Consider p such that $1 < p < +\infty$. Similarly, one can define Hankel operator $H_\varphi : H^p_+ \to H^p_-$ with symbol $\varphi \in L^\infty(\mathbb{R})$ by

$$H_\varphi : f \mapsto \mathbb{P}_-[\varphi \cdot f], \qquad f \in H^p_+,$$

where $\mathbb{P}_- = I - \mathbb{P}_+$, I being the identity operator on $L^p(\mathbb{R})$. For an introduction to the theory of Hankel operators, see the book [12] by V. Peller. The following theorem, which characterizes bounded Hankel operators on H^2, is due to Z. Nehari.

Theorem 4.1 ([12], Theorem 1.3)**.** Let $\varphi \in L^2(\mathbb{T})$. The following statements are equivalent:

(1) H_φ is bounded on H^2;
(2) there exists $\psi \in L^\infty(\mathbb{T})$ such that $H_\psi = H_\varphi$ and $\|\psi\|_{L^\infty(\mathbb{T})} = \|H_\varphi\|_{H^2 \to \overline{zH^2}}$.

The following theorem can be proved in the same way as Nehari's theorem.

Theorem 4.2. Let $1 < p < +\infty$ and let $\varphi \in L^\infty(\mathbb{R})$. Then there exists a function $\psi \in L^\infty(\mathbb{R})$ such that $H_\psi = H_\varphi$ and, moreover, $\|\psi\|_{L^\infty(\mathbb{R})} \leqslant \|H_\varphi\|_{H^p_+ \to H^p_-}$.

Let us give a sketch of the proof of this result.

Proof. Consider a function $\varphi \in L^\infty(\mathbb{R})$. We have

$$\|H_\varphi\| = \sup\{\langle \varphi f, \mathbb{P}_-[g]\rangle \mid f \in H^p_+, g \in L^q(\mathbb{R}),\ \|f\|_{L^p(\mathbb{R})} \leqslant 1, \|g\|_{L^q(\mathbb{R})} \leqslant 1\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\langle f_1, f_2\rangle = \int_{\mathbb{R}} f_1 \bar{f}_2 \, dx.$$

Choosing $g \in H^q_-$ we see that

$$\|H_\varphi\| \geqslant \sup\{\langle \varphi, \overline{fh}\rangle \mid f \in H^p_+, h \in H^q_+,\ \|f\|_{L^p(\mathbb{R})} \leqslant 1, \|h\|_{L^q(\mathbb{R})} \leqslant 1\}.$$

Since every function F in the unit ball of H^1_+ can be represented in the form $F = fh$ for some $f \in H^p_+, h \in H^q_+$, we have

$$\|H_\varphi\| \geqslant \sup\{\langle \varphi, \overline{F}\rangle \mid F \in H^1_+,\ \|F\|_{L^1(\mathbb{R})} \leqslant 1\}.$$

Extending the linear functional $\Phi_\varphi : F \to \langle \varphi, \overline{F}\rangle$ from H^1_+ to $L^1(\mathbb{R})$ by Hahn-Banach theorem, we see that there exists a function $\psi \in L^\infty(\mathbb{R})$ such that $\|\psi\|_{L^\infty(\mathbb{R})} \leqslant \|H_\varphi\|$ and $\langle \varphi, \overline{F}\rangle = \langle \psi, \overline{F}\rangle$ for every $F \in H^1_+$. In particular, we have $\langle \varphi f, g\rangle = \langle \psi f, g\rangle$ for all $f \in H^p_+$, $g \in H^q_-$. In other words $H_\varphi = H_\psi$. $\square$

4.2. **Analytic Toeplitz operators on PW_a^p as Hankel operators.** We say that Toeplitz operator T_φ with symbol $\varphi \in \mathcal{S}(\mathbb{R})$ is an analytic if $\operatorname{supp}\hat{\varphi} \subset \mathbb{R}_+$. One can easily check that that for every $1 < p < +\infty$ and for every $a > 0$ we have

$$\mathbb{P}_a = \theta_a \mathbb{P}_- \bar{\theta}_a^2 \mathbb{P}_+ \theta_a.$$

This formula will be used in the proof of Lemma 4.3 below.

Lemma 4.3. Let $1 < p < +\infty$ and let $\varphi \in \mathcal{S}(\mathbb{R})$ be such that $\operatorname{supp}\hat{\varphi} \subset \mathbb{R}_+$. Then

$$H_{\bar{\theta}_a^2 \varphi} = \bar{\theta}_a T_\varphi \theta_a \mathbb{P}_- \bar{\theta}_a^2. \tag{4.1}$$

Proof. First note that for any function $g \in H_+^p$, there are functions $g_1 \in \mathrm{PW}_a^p$, $g_2 \in H_+^p$ such that $g = \theta_a g_1 + \theta_a^2 g_2$. We have

$$H_{\bar{\theta}_a^2 \varphi}[g] = \mathbb{P}_-[\bar{\theta}_a \varphi g_1 + \varphi g_2] = H_{\bar{\theta}_a^2 \varphi}[\theta_a g_1],$$

because $\varphi g_2 \in H_+^p$. We also have

$$\bar{\theta}_a T_\varphi \theta_a \mathbb{P}_- \bar{\theta}_a^2[g] = \bar{\theta}_a T_\varphi \theta_a \mathbb{P}_-[\bar{\theta}_a g_1 + g_2] = \bar{\theta}_a T_\varphi[g_1].$$

On the other hand, taking into account (4.1), we obtain

$$\bar{\theta}_a T_\varphi[g_1] = \bar{\theta}_a \mathbb{P}_a[\varphi g_1] = \mathbb{P}_- \bar{\theta}_a^2 \mathbb{P}_+[\theta_a \varphi g_1] = \mathbb{P}_-[\bar{\theta}_a \varphi g_1] = H_{\bar{\theta}_a^2 \varphi}[\theta_a g_1].$$

This proves the lemma. $\square$

5. Proof of the main result. Concluding remarks

5.1. **Proof of the main result.** Recall that we want to prove that every Toeplitz operator T_φ on PW_a^p, $1 < p < +\infty$, with symbol $\varphi \in \mathcal{S}(\mathbb{R})$ has a bounded symbol ψ such that

$$\|\psi\|_{L^\infty(\mathbb{R})} \leqslant c\left(p + \frac{1}{p-1}\right) \cdot \|T_\varphi\|_{\mathrm{PW}_a^p \to \mathrm{PW}_a^p},$$

for a universal constant $c > 0$.

Proof. Define operators $T_{\mathfrak{L}}$, $T_{\mathfrak{C}}$, $T_{\mathfrak{R}}$ as in Section 2.3. By Proposition 2.4 we have

$$\|T_{\mathfrak{L}}\| + \|T_{\mathfrak{C}}\| + \|T_{\mathfrak{R}}\| \leqslant c \cdot \|T_\varphi\|,$$

for a universal constant $c > 0$. By Proposition 3.2 we have

$$\|\varphi_{\mathfrak{C}}\|_{L^\infty(\mathbb{R})} \leqslant c_p \cdot \|T_{\mathfrak{C}}\|,$$

for some constant $c_p > 0$ depending only on p. Let us prove an upper bound for the left and right parts of Toeplitz operators. By Nehari Theorem (see Theorem 4.2), there exists $\psi_r \in L^\infty(\mathbb{R})$ such that $H_{\psi_r} = H_{\bar{\theta}_a^2 \varphi_{\mathfrak{R}}}$, and, moreover,

$$\|\psi_r\|_{L^\infty(\mathbb{R})} \leqslant \|H_{\psi_r}\| = \left\|\bar{\theta}_a T_{\mathfrak{R}} \theta_a \mathbb{P}_- \bar{\theta}_a^2\right\| \leqslant A_p \|T_{\mathfrak{R}}\|,$$

where we used the fact that $\|\mathbb{P}_-\| = \|\mathbb{P}_+\| = A_p$. We claim that $T_{\mathfrak{R}} = T_{\theta_a^2 \psi_r}$. Since $H_{\psi_r} = H_{\bar{\theta}_a^2 \varphi_{\mathfrak{R}}}$, we have $H_{\psi_r}[\theta_a^2 f] = H_{\bar{\theta}_a^2 \varphi_{\mathfrak{R}}}[\theta_a^2 f] = 0$ for every $f \in H_+^p$. Therefore,

$$\mathbb{P}_+[\psi_r \theta_a^2 f] = \psi_r \theta_a^2 f - H_{\psi_r}[\theta_a^2 f] = \psi_r \theta_a^2 f, \quad f \in H_+^p.$$

Let $h \in \mathrm{PW}_a^p$ and let $f = \theta_a h$. Then $f \in H_+^p$ and we have

$$\begin{aligned} T_{\theta_a^2 \psi_r}[h] &= \mathbb{P}_a[\theta_a^2 \psi_r h] = \theta_a \mathbb{P}_- \bar{\theta}_a^2 \mathbb{P}_+[\theta_a^2 \psi_r f] = \\ &= \theta_a \mathbb{P}_-[\psi_r f] = \theta_a H_{\psi_r}[f] = \theta_a H_{\bar{\theta}_a^2 \varphi_{\mathfrak{R}}}[f]. \end{aligned}$$

By Lemma 4.3, we have $\theta_a H_{\bar{\theta}_a^2 \varphi_{\mathfrak{R}}}[g_2] = T_{\mathfrak{R}} \theta_a \mathbb{P}_-[\bar{\theta}_a^2 g_2] = T_{\mathfrak{R}} \theta_a \mathbb{P}_-[\bar{\theta}_a h] = T_{\mathfrak{R}}[h]$, and the claim follows.

Similarly, there exists $\psi_l \in L^\infty(\mathbb{R})$ such that

$$\|\psi_l\|_{L^\infty(\mathbb{R})} \leqslant A_p \|T_{\mathfrak{L}}\| \quad \text{and} \quad T_{\mathfrak{L}} = T_{\bar{\theta}_a^2 \psi_l}.$$

Setting $\psi = \bar{\theta}_a^2 \psi_l + \varphi_{\mathfrak{C}} + \theta_a^2 \psi_r$ we obtain

$$T_\varphi = T_{\mathfrak{L}} + T_{\mathfrak{C}} + T_{\mathfrak{R}} = T_{\bar{\theta}_a^2 \psi_l} + T_{\mathfrak{C}} + T_{\theta_a^2 \psi_r} = T_\psi,$$

by Proposition 2.4. Since

$$\begin{aligned}
\|\psi\|_{L^\infty(\mathbb{R})} &\leqslant \|\psi_l\|_{L^\infty(\mathbb{R})} + \|\varphi_{\mathfrak{C}}\|_{L^\infty(\mathbb{R})} + \|\psi_r\|_{L^\infty(\mathbb{R})} \\
&\leqslant A_p \|T_{\mathfrak{L}}\| + c_p \|T_{\mathfrak{C}}\| + A_p \|T_{\mathfrak{R}}\| \\
&\leqslant \tilde{c} \cdot (2A_p + c_p) \|T_\varphi\|,
\end{aligned}$$

we have

$$\|\psi\|_{L^\infty(\mathbb{R})} \leqslant c \cdot \left(p + \frac{1}{p-1}\right) \|T_\varphi\|,$$

by Lemma 3.3 and the estimate for the Riesz projector norm from Section 2.1. The theorem is proved. $\square$

5.2. **Concluding remarks.** In this section, we describe a possible application of our result to function theory.

In 2011, A. Baranov, R. Bessonov, and V. Kapustin [1] proved that the existence of a bounded symbol for every truncated Toeplitz operator on K_θ^2 is equivalent to the fact that every function $f \in H^1 \cap \theta^2 \overline{zH^1}$ admits a weak factorization.

Theorem 5.1 ([1], Theorem 2.4)**.** Let θ be an inner function on $\mathbb{T}$. The following assertions are equivalent:

(1) any bounded truncated Toeplitz operator on K_θ^2 has a bounded symbol;

(2) for any function $f \in H^1 \cap \theta^2 \overline{zH^1}$ there exist $x_k, y_k \in K_\theta^2$ with

$$\sum_k \|x_k\|_{L^2(\mathbb{T})} \cdot \|y_k\|_{L^2(\mathbb{T})} < +\infty \quad \text{such that} \quad f = \sum_k x_k y_k.$$

As we mentioned in Section 1.1, we expect that the main result of present thesis can be used to prove existence of a bounded symbol for every bounded Toeplitz operator on PW_a^p, $1 < p < +\infty$. Our work and Theorem 5.1 motivate the following conjecture.

Conjecture 5.2. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For any function $f \in \mathrm{PW}_{2a}^1$ there exist $x_k \in \mathrm{PW}_a^p$, $y_k \in \mathrm{PW}_a^q$ with

$$\sum_k \|x_k\|_{L^p(\mathbb{R})} \cdot \|y_k\|_{L^q(\mathbb{R})} < +\infty \quad \text{such that} \quad f = \sum_k x_k y_k.$$

Acknowledgement

I am grateful to R. Bessonov and S. Rukshin for constant attention to this work, various helpful suggestions and discussions, and for my mathematical education.

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