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Graduation qualification thesis

Equilibrium existence in the case of non-divisible objects

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1 Abstract

In this paper I studied the problem of equilibrium existence in the Fisher market model with indivisible goods and two buyers. A major breakthrough was paper [1] published in 2017. Nikita Kalinin showed that there are cases not covered by the main theorems in this paper. Here I prove existence of equilibrium in some open cases that are not proved in [1].

The structure of this paper is following. In the second section there is a description of the problem, in third section the main results of [1] are presented, in the forth section there are given the new key definitions and motivations for them, finally, in the last section the main results of this paper are proved.

2 Introduction

Let us assume that there are two buyers and n non-divisible objects. Buyers are characterized by their budgets b_1 and b_2. All objects are numbered with integer numbers from 1 to N, so the set $\overline{N} = \{1, ..., N\}$ represents the set of objects.

Definition 2.1. *Each buyer values every object with some positive number. Such values are represented by two additive functions $v_1, v_2 : 2^{\overline{N}} \to \mathbb{R}_+$.*

Value functions are additive in the following sense:

$$\text{If } A, B \subset \overline{N} \text{ and } A \cap B = \emptyset, \text{ then } v_i(A \cup B) = v_i(A) + v_i(B),\ i = 1, 2$$

Definition 2.2. *Price is an additive function $p : 2^{\overline{N}} \to \mathbb{R}$.*

Definition 2.3. *Given the price function, each buyer wants to maximise the value of objects that he buys, but he cannot spend more money than he has. The set of objects $A \subset \overline{N}$ is called i-lucky, $i = 1, 2$, if*

$$v_i(A) = \max_{p(B) \le b_i} v_i(B)$$

i-lucky set maximises the value for the player i on all subsets that the buyer i can afford to buy.

Definition 2.4. *The price function is called equilibrium if there exists set $A \subset \overline{N}$ such that A is 1-lucky and $\bar{A} = \overline{N} \backslash A$ is 2-lucky.*

Note that for equilibrium price $p(\overline{N}) = p(A) + p(\bar{A}) \le b_1 + b_2$

Geometric representation

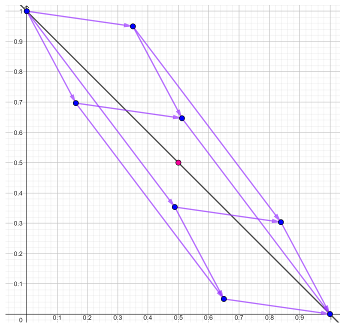
First of all, without loss of generality we can assume that

1. $v_1(\overline{N}) = v_2(\overline{N}) = 1$
2. $b_1 + b_2 = 1$

Otherwise, budgets, prices, and values can be normalized.

Then budgets and values can be represented on plane as follows.

Every point represents some set of objects $A \subset \overline{N}$, and it has coordinates $(v_1(A), v_2(\bar{A}))$. Also for all $A \subset \overline{N}$ we consider vectors from the point corresponding A to the point corresponding $A \cup \{i\}, i \in \bar{A}$. These points form the projection of an n-dimensional cube on the plane. Finally, there is one special point (b_1, b_2) that correspond to the budgets. Due to the normalization this point lies on the line $x + y = 1$.



Geometric representation of 3 objects

Open problem:
Are there generic budgets and values non-zero measure so that there is no equilibrium price?

3 Previously known results

The case with 4 players is well-known. If the number of objects is less than 4, then equilibrium always exists. Otherwise there is an example of values and budgets where there is no equilibrium as it is stated in Theorem 3.1. Cases with 3 and 2 buyers are open.

Theorem 3.1. *[2] If the number of buyers is at least 4, then there exist values and budgets such that there is no equilibrium.*

The example consists of 4 objects and 4 buyers.The budgets of buyers, a, b, c, d, satisfy the following inequalities:

$$2a > 2b > a + c > b + d > 2c > a > c + d > 2d > b > c > d$$

.

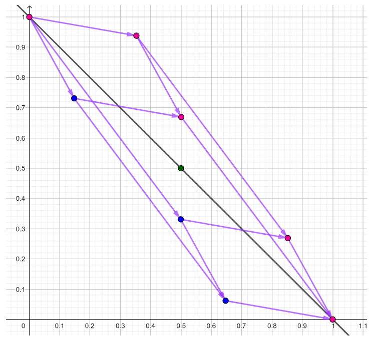
Values are following:

- Buyer with budget a values objects as 11, 7, 5, 3 respectively
- Buyer with budget b values objects as 9, 7, 6, 8 respectively
- Buyer with budget c values objects as 7, 9, 8, 6 respectively
- Buyer with budget d has any values.

Lemma 1. *[1]*
Suppose $A \subset \overline{N}$ is such that $v_1(A) = b_1, v_2(\bar{A}) = b_2$.
Then A is an equilibrium set if and only if for any subsets $X \subset A$ and $Y \subset \bar{A}$ or, conversely, $Y \subset A$ and $X \subset \bar{A}$, then

$$\textit{if } v_1(X) > v_1(Y) \textit{ and } v_2(X) > v_2(Y) \textit{ then } p(X) > p(Y)$$

Definition 3.2. *The set A is called Pareto optimal (PO) if there are no points in the top right quadrant of the point that represents A.*

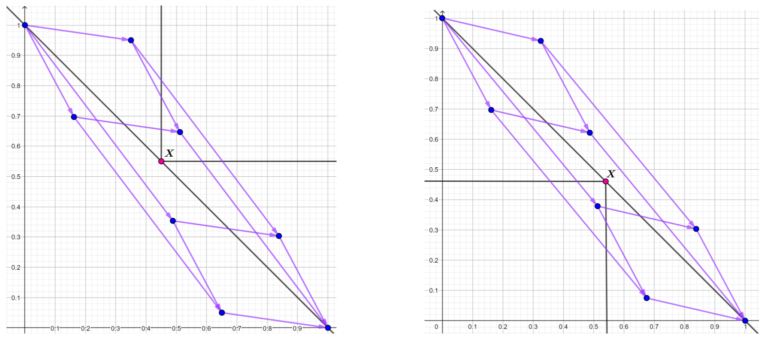


Example with 3 objects
Pink points are PO

If A is an equilibrium set, then A must be PO. Otherwise, suppose that B is to the top and right from an equilibrium set A. Then $v_1(B) > v_1(A)$ and $v_2(B) > v_2(A)$. Since B is not an equilibrium set, $p(B) > b_1, p(\bar{B}) > b_2$, then $p(\overline{N}) > b_1 + b_2$, that contradict existence of equilibrium set.

Many cases of values and budgets are covered by two following theorems.

Theorem 3.3. *[1] If there is a point in the top right quadrant from point X, then equilibrium exists. Moreover, equilibrium exists if there is a PO point in the left lower quadrant from the point (b_1, b_2).*



2 examples of values and budgets where Theorem 3.3 is applicable

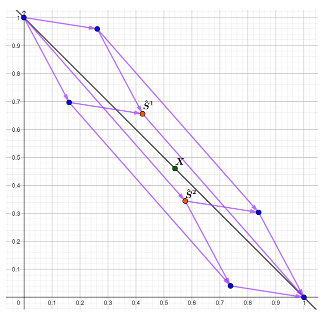
In this case price vector can be easily found by the slope from the point of budgets to the PO point From now on we consider only those values and budgets that are not covered by Theorem 1.

Definition 3.4. *Allocation is a pair $S = (A, \bar{A})$ $A = S_1,\ \bar{A} = S_2$*

Definition 3.5. *Truncated share is $b_i^- \in [0, 1]$ such that*

$$b_i^- = \max_{S-PO,\ v_i(S_i) \leq b_i} v_i(S_i)$$

$$\hat{S}^i = \hat{S}^i(b_i) \text{ such that } b_i^- = v_i(\hat{S}^i)$$

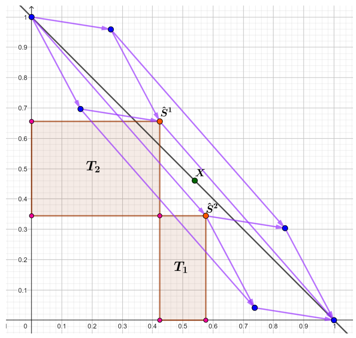


Truncated shares

This way $\hat{S}^1$ is the first PO point to the left from the point of budgets. And $\hat{S}^2$ is the closest PO point for the point of budgets below it.

Definition 3.6. *$T_1 = T_1(b_1, v_1, v_2)$ is the set of allocations S such that the first coordinate is between $\hat{S}^1$ and $\hat{S}^2$ and the second is not more than $\hat{S}^2$. In other words,*
$v_1(\hat{S}^1_1) < v_1(\hat{S}_1) < v_1(\hat{S}^2_1)$
$0 < v_2(S_2) < v_2(\hat{S}^2_2)$

Similarly, we can define T_2 as allocations S that satisfy
$v_2(\hat{S}^2_2) < v_1(\hat{S}_2) < v_1(\hat{S}^1_2)$
$0 < v_1(S_1) < v_1(\hat{S}^1_1)$



Representation of T_1, T_2

Definition 3.7. *$R_i(v_1, v_2)$ - set of pairs of budgets (b_1, b_2) such that there exists index $1 \leq r \leq n$*

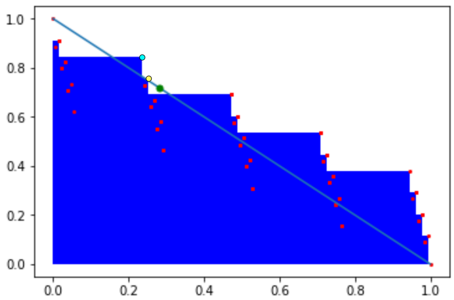
$$\frac{b_i}{v_i(S(r+1)_i)} = \frac{1 - b_i}{1 - v_i(S(r))_i}$$

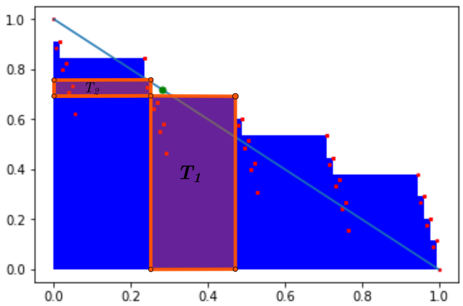
where $S(r)$ is defined as follows: each $S(r)$ is PO allocation and $S(r+1)_i > S(r)_i$

For each PO point there is at most one pair (b_1, b_2) that lies in $R_i(v_1, v_2)$. Thus, every set $R_i(v_1, v_2)$ has zero measure.

Theorem 3.8. *[1] Assume there is no such point as in Theorem 3.3.*
If for some buyer i $(b_1, b_2) \notin R_i(v_1, v_2)$ and the set $T_i = T_i(b_i, v_1, v_2)$ is empty, then equilibrium exists.

Example 3.9. *Though these theorems cover many cases of values and budgets, still there are cases not covered by these two theorems. One of the cases shown by Nikita Kalinin consists of three types of objects and their copies.*





Red points represent values and the green point represents budgets. Points inside the blue area are not PO, PO allocations only at the edge of it.

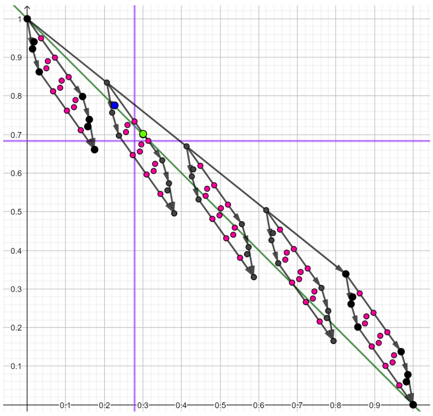
There are 3 types of objects. There are four objects of the first type, three of the second, and one object of the third type.

The values are following:
v_1(*first object*) $= 0.22$, v_2(*first object*) $= 0.15$,
v_1(*second object*) $= 0.031$, v_2(*second object*) $= 0.09$
v_1(*third object*) $= 0.027$, v_2(*third object*) $= 0.13$

- If $b_1 < 0.25$ then the equilibrium allocation is the blue point and we set the price for the first type object as b_1 and for other objects we set any positive prices such that their union costs $b_2 - 3b_1$.
- If $\frac{1}{3} > b_1 > 0.25$, then the equilibrium allocation is the yellow point and we set the price for the first type object as $b_1 - \varepsilon$ with $b_1 > \varepsilon > 4b_1 - 1$, the second type object costs ε and the third type object costs $1 + \varepsilon - 4b_1$.

4 Repetition of several objects with a car

Example 4.1. *There are examples of budgets and values not covered by theorems above with the similar structure: several types of objects and copies of these objects.*

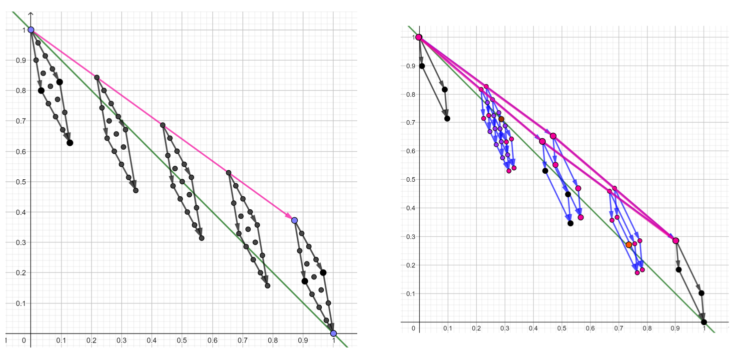


Budgets are represented by green point, red and black points represent values example with 4 types of objects

Order all objects by the angle with ordinate. Then values are following:
v_1(*first object*) $= 0.205$, v_2(*first object*) $= 0.162$,
v_1(*second object*) $= 0.035$, v_2(*second object*) $= 0.05$
v_1(*third object*) $= 0.02$, v_2(*third object*) $= 0.082$
v_1(*forth object*) $= 0.01$, v_2(*forth object*) $= 0.035$

To see that blue point is equilibrium set the price for the object appreciated the most by the first buyer as $b_1 - \varepsilon$ *with* $\frac{1}{3}(4b_1 - 1) < \varepsilon < b_1$, *price* ε *for the other object that the first buyer receives and* $b_2 - 3(b_1 - \varepsilon)$ *for the union of all other objects.*

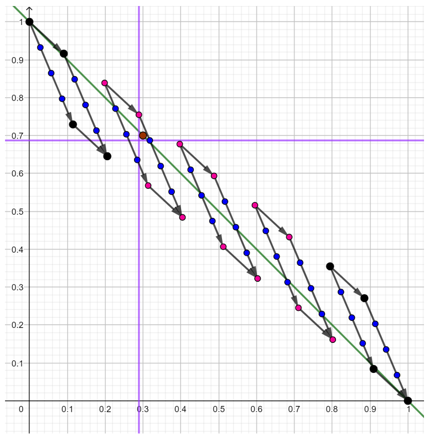
Definition 4.2. *Suppose that all objects can be divided into two groups the following way: any object from the first group the first buyer appreciates more than the union of all objects from the second group. In other words, let all objects be ordered such that* $v_1(i) > v_1(j)$ *if* $i > j$. *Suppose that exists* n *such that for any* $j \leq n$ $v_1(j) > v_1(\{n+1, ..., N\})$. *In this case we call any of the object from the first group* ***a car***. *Then* n *is the number of cars in total.*



Cars and their copies lie on the pink line

Conjecture 1
For any budgets and values with n cars not covered by theorems 3.3 and 3.8 there exists equilibrium.

Example 4.3. *There exists an example of values and budgets not covered by theorems 3.3 and 3.8 with no car.*



Let object types be ordered as in definition of the car so that v_1(first object) $>$ v_1(second object) $>$ v_1(third object) $= 0.02$. Then values are following:
v_1(*first object*) $= 0.2$, v_2(*first object*) $= 0.16$,
v_1(*second object*) $= 0.08$, v_2(*second object*) $= 0.08$
v_1(*third object*) $= 0.03$, v_2(*third object*) $= 0.07$
Still in this example the equilibrium exists. The first buyer takes one of each object type. A car costs $b_1 - \varepsilon - \delta$, where ε is the price of a non-car object that the second buyer appreciates less than the other non-car object. $\varepsilon = \frac{1}{3}(4b_1 - 1)$, $b_1 - \varepsilon > \delta > \varepsilon$ and δ is the price of the third object.

Conjecture 2
For budgets and values are not covered by theorems 3.3 and 3.8 the following holds

$$|\hat{S}_1 \triangle \hat{S}_2| = 1$$

So there is one object difference between the two closest PO points to the point of budget.

5 Main results

Theorem 5.1. *Suppose that all cars are valued the same by both buyers so that $v_1(i) = v_1(1), v_2(i) = v_2(1)$ for $i \leq n$. Consider k such that $b_1 n < k < b_1(n+1)$. The distribution where the first buyer receives k cars and the second buyer receives everything else is an equilibrium.*

Proof. Set the price for one car $\dfrac{b_1}{k}$. Since the first buyer appreciates one car more than the union of all other objects, he would take k cars and he cannot take anything else. This way, the set of k cars is 1-lucky.

Since $b_1 n < k < b_1(n+1)$ and $b_2 = 1 - b_1$, it is easy to obtain

$$(n-k)\frac{b_1}{k} < b_2 < (n-k+1)\frac{b_1}{k}$$

That means the second buyer can acquire $(n-k)$ cars and after that he still would have some money left for non-car objects. However, from the right part of inequality, he cannot afford to buy $(n-k+1)$ cars.

For all non-car objects we chose prices such that all copies of one object cost the same and altogether all non-car objects cost $b_2 - (n-k)\frac{b_1}{k}$ - the money that the second buyer has after buying his share of cars.

Lemma 5.2. $(n-k)$ *cars together with all non-car objects is 2-lucky set*

Indeed, suppose that 2-lucky set consists of t cars and some subset A of the set of non-car objects. Since $b_2 < (n-k+1)\frac{b_1}{k}$, $t \leq n-k$, and $v_2(t \text{ cars}) < v_2(n-k \text{ cars})$. Moreover, $v_2(A) \leq v_2(\text{all non-car objects})$. This way, it is easy to see that $(n-k)$ cars together with all non-car objects is a 2-lucky set.

This concludes the proof of the Theorem. □

If $v_2(car) < v_2(non - car\ objects)$ then the interval $(b_1 n, b_1(n+1))$ that must contain integer k can be broadened.

Theorem 5.3. *Suppose that all cars are valued the same by both buyers so that* $v_1(i) = v_1(1), v_2(i) = v_2(1)$ *for* $i \leq n$. *Consider* k *such that* $b_1 n < k < \frac{b_1}{v_2(car)}$. *The distribution where the first buyer receives k cars and the second buyer receives everything else is an equilibrium.*

Proof. We will take prices such that the first and second buyer spend all their budgets for their sets of objects.

Again, set the price for one car $\dfrac{b_1}{k}$. Then the same argument as in the previous theorem works. Since the first buyer appreciates one car more than the union of all other objects, he would take k cars and he cannot take anything else. This way, the set of k cars is 1-lucky.

The second buyer spends $(n-k)\frac{b_1}{k}$ money for cars, and he is left with $b_2 - (n-k)\frac{b_1}{k}$ to spend for all non-car objects.

First of all, we check that this value is positive.

$$b_2 - (n-k)\frac{b_1}{k} = 1 - b_1 - \frac{n-k}{k}b_1 = 1 - \frac{n}{k}b_1 > 0$$

That is equivalent to

$$nb_1 < k$$

Now we set prices for the set of all other objects such that $(n-k)$ cars together with all other objects form a 2-lucky set.

Note that all copies of one object must cost the same. Then if price for each non-car object is defined, then it's also possible to extend linearly prices for linear combinations of non-car objects the following way:
if X_i are non-car objects and $\alpha_i \in \mathbb{R}$ are corresponding coefficients, then

$$p\left(\sum_i \alpha_i X_i\right) = \alpha_i \sum_i p(X_i)$$

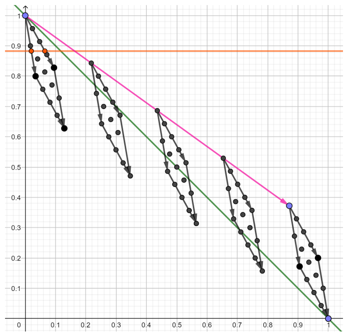
This is also true for values:

$$v_1\left(\sum_i \alpha_i X_i\right) = \alpha_i \sum_i v_1(X_i)$$

$$v_2\left(\sum_i \alpha_i X_i\right) = \alpha_i \sum_i v_2(X_i)$$

Two linear combinations of non-car objects $Z = \sum \alpha_i X_i$ and $T = \sum \beta_j Y_j$, where X_i, Y_j are non-car objects, are **on the same level** if

$$v_2(Z) = v_2(T)$$



Orange points lie on the same level

We define prices of non-car objects such that they are the same for any linear combinations of non-car objects on the same level. Since the price of all non-car objects is defined uniquely as $1 - \frac{n}{k} b_1$, then prices of all linear combinations of non-car objects are defined uniquely, hence prices of all subsets of the set of non-car objects are defined uniquely.

To check that $(n - k)$ cars together with all other objects is a 2-lucky set by lemma 1 it suffices to check that for any linear combination of non-car objects $Z = \sum \alpha_i X_i$ and any integer r:

$$\text{if } v_2(Z) < v_2(r \text{ cars}), \text{ then } p(v_2(Z)) < p(r \text{ cars}) = r\frac{b_1}{k} \tag{1}$$

By linearity it suffices to check (1) only for $r = 1$.

Now we set price $\frac{b_1}{k}$ for linear combinations on the level of one car (on linear combinations Z such that $v_2(Z) = v_2(Car)$) and check that in this case price that we obtain for all non-car object is not less than the money that second buyer has for non-car objects $1 - \frac{n}{k} b_1$.

Since the price is linear and it has the same value on all linear combinations on the same level, it is directly proportional to $1 - v_2(\cdot)$, then for any A, subset of the set of non-car objects,

$$p(A) = \frac{v_2(A)}{v_2(\text{car})} \frac{b_1}{k}$$

the union of all non-car objects has the price

$$\frac{v_2(\text{all non car objects})}{v_2(\text{car})} \frac{b_1}{k} = \frac{1 - nv_2(\text{car})}{v_2(\text{car})} \frac{b_1}{k}$$

We have to check that this value is greater than $1 - \frac{n}{k} b_1$

$$\frac{1 - nv_2(\text{car})}{v_2(\text{car})} \frac{b_1}{k} > 1 - \frac{n}{k} b_1$$

That is equivalent

$$k < \frac{b_1}{v_2(\text{car})}$$

This holds by the choice of k. □

References

[1] Moshe Babaio, Noam Nisan, Inbal Talgam-Cohen, Competitive Equilibria with Indivisible Goods and Generic Budgets, 10-20, 23 Mar 2017, arXiv:1703.08150v1 [cs.GT].

[2] Erel Segal-Halevi, Competitive Equilibrium For Almost All Incomes: Existence and Fairness, 26-27, 24 Jan 2020, arXiv:1705.04212v6 [cs.GT].