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Minimal ideal triangulations of hyperbolic 3-manifolds with geodesic boundary via $\mathbb{Z}_2$-homology

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Contents

1. Introduction

In this paper, we will restrict our attention to connected compact 3-manifolds M with non-empty boundary. An ideal triangulation of M is *minimal* if it uses the smallest number of tetrahedra in an ideal triangulation of M. The number of ideal tetrahedra in a minimal ideal triangulation of M is denoted $c_\Delta(M)$ and termed the *triangulation complexity* of M.

The *triangulation complexity* $c_\Delta(M)$ of M is the minimal number of tetrahedra in any ideal triangulation of M. There are remarkably few examples of exact computations of triangulation complexity of manifolds. The following lower bound

$$c_\Delta(M) \geq \beta_0(\partial M;\mathbb{Z}_2) - \chi(M) \tag{1}$$

on complexity is known through work of Frigerio, Martelli and Petronio [1], where it is shown that the bound is attained by infinite families of manifolds $M_{g,k}$ with a totally geodesic boundary component of genus $g \geq 2$ and k cusps. In particular, a minimal ideal triangulation of $M_{g,0}$ has only a single edge [2]. An equivalent approach to complexity is via Matveev's theory of special spines. From this point of view, it is proved [3] that any ideal triangulation $\mathcal{T}$ with exactly two edges such that no 3-2 Pachner moves can be applied to $\mathcal{T}$ is minimal. Infinite families of such manifolds were described in [4, 5, 6]. Further research in [7] shows that poor ideal three-edge triangulations are minimal. Moreover, a census of connected compact 3-manifolds with non-empty boundary decomposed into at most 4 ideal tetrahedra was given in [8, 9].

In this paper, we present a new lower bound for the triangulation complexity of connected compact 3-manifolds M with non-empty boundary via $\mathbb{Z}_2$-homology (Theorem 3.1). It is shown (Theorem 3.3) that this bound is stronger than those given by (1). We use $\mathbb{Z}_2$-homology to study the combinatorics of minimal ideal triangulations of M for which our lower bound is achieved (Theorem 4.1). The class of such manifolds is denoted by $\mathcal{M}_h$.

Our next task is to distinguish manifolds in $\mathcal{M}_h$. We characterise edges of minimal triangulations that yields a natural partition of the set of minimal triangulations into four classes. A natural question to ask does a manifold in $\mathcal{M}_h$ admits minimal triangulations of different classes. We prove (Theorem 6.1) that the answer is negative.

Finally we prove that each manifold in $\mathcal{M}_h$, with a few exceptions, is a hyperbolic manifold with totally geodesic boundary components and some cusps (Theorem 7.2). The hyperbolicity of manifolds in $\mathcal{M}_h$ provides several results concerning the Matveev's complexity and the hyperbolic volume of these manifolds (Theorem 7.8 and Theorem 7.9).

2. Triangulations and special spines

In this paper, we will translate freely back and forth between an ideal triangulation and its dual special spine. We begin by recalling some definitions.

2.1. **Ideal triangulations.** Let $\widehat{\mathcal{T}}$ be a cell complex made out of a pairwise disjoint collection of 3-simplices by gluing all of their 2-dimensional faces in pairs via simplicial maps. The simplices prior to identification, and their vertices, edges, and faces, are called *model cells*. Denote by $\widehat{\mathcal{T}}^{(i)}$ the i-skeleton of $\widehat{\mathcal{T}}$.

Note that $\widehat{\mathcal{T}}$ may actually not be a 3-manifold, because the link of a vertex could be any surface, and the link of the midpoint of an edge could be a projective plane. Throughout this paper we assume that $\widehat{\mathcal{T}}$ has no singularities at the midpoints of its edges. Under this assumption, $\widehat{\mathcal{T}}$ with small open stars of its vertices removed is a compact manifold with non-empty boundary, denoted M. Then $\widehat{\mathcal{T}} \backslash \widehat{\mathcal{T}}^{(0)}$ is denoted $\mathcal{T}$ and termed an *ideal triangulation* of M. We call $\widehat{\mathcal{T}}$ the *cell complex corresponding to* $\mathcal{T}$. This means in particular that the edges of $\mathcal{T}$ are in one-to-one correspondence with the edges of $\widehat{\mathcal{T}}$.

We fix the following notation. Let v be a vertex and e be an edge of $\widehat{\mathcal{T}}$. Denote by $\mathcal{V}(v)$ (resp., $\mathcal{E}(e)$) the union of model vertices (resp., edges) that are identified to form v (resp., e). The elements of $\mathcal{V}(v)$ (resp., $\mathcal{E}(e)$) are called *pre-images* of v (resp., e). Denote by $\mathtt{d}(\mathcal{T})$ the number of edges in $\mathcal{T}$. In the sequel we always refer to $\mathtt{d}$ itself as the number of edges tacitly implying that $\mathcal{T}$ is fixed.

2.2. **Partially truncated triangulation.** Let $\widetilde{\Delta}$ be a model tetrahedron and J be a subset of $\widetilde{\Delta}^{(0)}$, which is called the set of *ideal model vertices*. A *partially truncated model tetrahedron* $\widetilde{\Delta}^J$ corresponding to a pair $(\widetilde{\Delta}, J)$ is obtained from $\widetilde{\Delta}$ by removing the ideal model vertices and small open stars of non-ideal ones. We call *lateral model hexagon* and *truncation model triangle* the intersection of $\widetilde{\Delta}^J$ respectively with a face of $\widetilde{\Delta}$ and with the link in $\widetilde{\Delta}$ of a non-ideal vertex. The model edges of the triangulation model triangles, which also belongs to the lateral model hexagons, are called *boundary model edges*, and the other edges of $\widetilde{\Delta}^J$ are called *internal model edges*. Note that, if $J \neq \emptyset$, a lateral model hexagon of $\widetilde{\Delta}^J$ may not quite be a hexagon, because some of its (closed) boundary model edges may be missing.

Let $\mathcal{T}$ be an ideal triangulation of a compact 3-manifold M with non-empty boundary, and let $\widehat{\mathcal{T}}$ be the corresponding cell complex. Let I be a subset of $\widehat{\mathcal{T}}^{(0)}$, the elements of I are called *ideal vertices*. Define

$$\mathcal{V}(I) = \bigcup_{v \in I} \mathcal{V}(v).$$

For each model tetrahedron $\widetilde{\Delta}$ define a set of ideal model vertices $J = \widetilde{\Delta}^{(0)} \cap \mathcal{V}(I)$, and let $\widetilde{\Delta}^J$ be a partially truncated model tetrahedron corresponding to a pair $(\widetilde{\Delta}, J)$. Define a *partially truncated triangulation* $\widehat{\mathcal{T}}^I$ corresponding to a pair $(\widehat{\mathcal{T}}, I)$ as a gluing of some $\widetilde{\Delta}^J$'s along the lateral model hexagons induced by a simplicial pairing of the model faces of $\widetilde{\Delta}$'s.

Remark 2.1. We will use the following facts down in the sequel.

- The internal edges of $\widehat{\mathcal{T}}^I$ are in one-to-one correspondence with the with the edges of $\mathcal{T}$.
- The vertices of $\widehat{\mathcal{T}}$ are in one-to-one correspondence with the components of ∂M.
- $|\widehat{\mathcal{T}}^I|$ is homeomorphic to M with non-empty boundary components corresponding to ideal vertices of $\widehat{\mathcal{T}}$ removed.

Note, that if $I = \widehat{\mathcal{T}}^0$, then $\widehat{\mathcal{T}}^I$ coincides with ideal triangulation $\mathcal{T}$. If $I = \emptyset$, then $\widehat{\mathcal{T}}^I$ is actually a truncated triangulation of M, which we denote by $\mathcal{T}^\star$.

2.3. **Special spines.** A *spine* of a compact 3-manifold M with non-empty boundary is a compact polyhedron $P \subset M$ such that $M \setminus P$ is homeomorphic to $\partial M \times [0,1)$. A spine P carries much information about M. In particular, P is homotopy equivalent to M and hence determines the homotopy type of M.

We will restrict our class of spines to those called *special (or, standard) spines*. A compact two-dimensional polyhedron P is said to be *simple* if the link of every point x in P is homeomorphic either to a circle (such a point x is called *nonsingular*), to a graph consisting of two vertices and three edges joining them (such a point x is called a *triple point*), or to the complete graph K_4 with four vertices (such a point x is called a *true vertex*). Connected components of the set of all nonsingular points are called 2-*components* of P, while connected components of the set of all triple points are called *triple lines* of P. The set of singular points of P (that is, the union of all triple lines and all true vertices) is called a *singular graph* of P. A simple polyhedron is *special* if each of its triple line is an open 1-cell, and each of its 2-component is an open 2-cell. A singular graph of a special polyhedron has at least one true vertex and is a 4-regular graph. Therefore, it is natural to call the triple lines of a special polyhedron *edges*.

For each 2-component ξ of a special polyhedron P, there is a characteristic map $f : D^2 \to P$, which carries the interior of the disc D^2 onto ξ homeomorphically and which restricts to a local embedding on $S^1 = \partial D^2$. We will call the curve $f|_{\partial D^2} : \partial D^2 \to P$ (and its image $f|_{\partial D^2}(\partial D^2)$) the *boundary curve* $\partial\xi$ of ξ.

2.4. **Duality between ideal triangulations and special spines.** An ideal triangulation $\mathcal{T}$ of a compact 3-manifold M with non-empty boundary determines in a natural way a dual special polyhedron, which is in fact a spine of M. For each model tetrahedron Δ_i of $\mathcal{T}$, let R_i denote the union of the links of all four vertices of Δ_i in the first barycentric subdivision. Since the face-pairings are simplicial, gluing Δ_i's induces gluing the corresponding R_i's together. We get a special spine P of M. In fact, the assignment $\mathcal{T} \to P$ induces a bijection between ideal triangulations (considered up to equivalence) and special spines (considered up to homeomorphisms) [10].

3. Lower bounds for complexity of manifolds with boundary

3.1. **Lower bounds via $\mathbb{Z}_2$-homology.**

Theorem 3.1. *Let M be a connected compact 3-manifold with non-empty boundary. Then*

$$c_\Delta(M) \geq \beta_1(M, \mathbb{Z}_2).$$

By $\mathtt{d}(P)$ and $\mathtt{v}(P)$ denote the number of 2-components and the number of true vertices of a special polyhedron P, respectively.

Lemma 3.2. *Let P be a special spine of a connected compact 3-manifold M with non-empty boundary. Then we have:*

(i) $\mathtt{d}(P) - (\beta_2(M, \mathbb{Z}_2) + 1) = \mathtt{v}(P) - \beta_1(M, \mathbb{Z}_2)$;
(ii) $\mathtt{d}(P) \geq \beta_2(M, \mathbb{Z}_2) + 1$.

Proof. Since the singular graph of a special spine is 4-regular, $\chi(P) = \mathtt{d}(P) - \mathtt{v}(P)$. Since M is connected, we have $\beta_0(M, \mathbb{Z}_2) = 1$ and $\chi(M) = 1 - \beta_1(M, \mathbb{Z}_2) + \beta_2(M, \mathbb{Z}_2)$. The homotopy equivalence of P and M implies that $\chi(M) = \chi(P)$. Thus (i) holds.

In order to prove (ii) we consider a part of the chain complex of P with $\mathbb{Z}_2$-coefficients:

$$C_2 \xrightarrow{\partial} C_1.$$

Recall that $B_1 = \partial C_2$ is the group of 1-dimensional boundaries. Notice, that

$$\partial(\alpha_0 + \ldots + \alpha_{\mathtt{d}(P)-1}) = \gamma_0 + \ldots + \gamma_{2\mathtt{v}(P)-1}, \tag{2}$$

where $\alpha_0, \ldots, \alpha_{\mathtt{d}(P)-1}$ are the 2-components and $\gamma_0, \ldots, \gamma_{2\mathtt{v}(P)-1}$ are the edges of P. Hence $\dim B_1 \geqslant 1$. Since P is 2-dimensional polyhedron, we obtain $\mathtt{d}(P) = \dim C_2$ and $\dim(\operatorname{Ker}\partial) = \beta_2(P, \mathbb{Z}_2)$. Again, from homotopy equivalence of P and M we have $\beta_2(P, \mathbb{Z}_2) = \beta_2(M, \mathbb{Z}_2)$. The following computation provides (ii):

$$\mathtt{d}(P) = \dim C_2 = \dim(\operatorname{Ker}\partial) + \dim(\operatorname{Im}\partial) = \beta_2(P, \mathbb{Z}_2) + \dim B_1 \geq \beta_2(M, \mathbb{Z}_2) + 1. \tag{3}$$

□

Proof of Theorem 3.1. Let $\mathcal{T}$ be a minimal ideal triangulation of M. Consider the special polyhedron P that is dual to $\mathcal{T}$. It is clear that the number of tetrahedra in $\mathcal{T}$ is equal to $\mathtt{v}(P)$ due to the duality of P and $\mathcal{T}$. Hence $c_\Delta(M) = \mathtt{v}(P)$. Applying Lemma 3.2 to P we obtain $\mathtt{v}(P) \geqslant \beta_1(M, \mathbb{Z}_2)$. Hence $c_\Delta(M) \geqslant \beta_1(M, \mathbb{Z}_2)$. □

3.2. **Comparing two lower bounds.** Now we prove that the lower bound for $c_\Delta(M)$ in Theorem 3.1 is stronger than the Frigerio–Martelli–Petronio one (1).

Theorem 3.3. *Let M be a connected compact 3-manifold with non-empty boundary. Then*

$$\beta_1(M, \mathbb{Z}_2) \geqslant \beta_0(\partial M; \mathbb{Z}_2) - \chi(M).$$

The desired inequality is derived from the following lemma.

Lemma 3.4. *Let M be a connected compact 3-manifold with non-empty boundary. Then*

$$\beta_2(M; \mathbb{Z}_2) + 1 \geq \beta_0(\partial M; \mathbb{Z}_2).$$

Proof. Consider a part of the long exact sequence for the pair $(M, \partial M)$ with $\mathbb{Z}_2$-coefficients:

$$H_1(M, \partial M; \mathbb{Z}_2) \xrightarrow{\varphi} H_0(\partial M; \mathbb{Z}_2) \xrightarrow{\psi} H_0(M; \mathbb{Z}_2).$$

Since $\operatorname{Ker}\psi = \operatorname{Im}\varphi$ and M is connected, we have

$$\begin{aligned}\beta_0(\partial M; \mathbb{Z}_2) = \dim(\operatorname{Im}\psi) + \dim(\operatorname{Im}\varphi) \leq \\ \leq \dim(H_0(M; \mathbb{Z}_2)) + \dim(H_1(M, \partial M; \mathbb{Z}_2)) = \\ = 1 + \beta_1(M, \partial M; \mathbb{Z}_2).\end{aligned}$$

Lefschetz duality gives a natural isomorphism $H_1(M, \partial M; \mathbb{Z}_2) \cong H^2(M; \mathbb{Z}_2)$. Since the homology group $H_2(M; \mathbb{Z}_2)$ is finitely generated, then the vector spaces $H_2(M; \mathbb{Z}_2)$ and $H^2(M; \mathbb{Z}_2)$ are finite-dimensional and mutually dual. In particular, they have the same dimension. Hence, $\beta_2(M; \mathbb{Z}_2) = \beta_1(M, \partial M; \mathbb{Z}_2)$. □

Proof of Theorem 3.3. Applying Lemma 3.4 we have:

$$\beta_1(M, \mathbb{Z}_2) = \beta_2(M; \mathbb{Z}_2) + 1 - \chi(M) \geqslant \beta_0(\partial M; \mathbb{Z}_2) - \chi(M)$$

□

4. 3-manifolds for which the lower bound in Theorem 3.1 is achieved and their minimal triangulations

Let $\mathcal{M}_h$ denote the set of connected compact 3-manifolds M with non-empty boundary, which have an ideal triangulation $\mathcal{T}$ with $\beta_1(M, \mathbb{Z}_2)$ ideal tetrahedra. By theorem 3.1, $\mathcal{T}$ is a minimal ideal triangulation of M.

We introduce two infinite sets $\mathscr{T}_o$ and $\mathscr{T}_e$ of ideal triangulations. Let $\mathcal{T}$ be an ideal triangulation, and let e be its edge. We say that e is *even* (resp., *odd*) if each model face contains even (resp., odd) number of pre-images of e. The set $\mathscr{T}_e$ consists of all the ideal triangulations with at least two edges, one of which is odd, and the others are even. The set $\mathscr{T}_o$ consists of all the ideal triangulations with odd edges only. By definition, $\mathscr{T}_o \cap \mathscr{T}_e = \emptyset$.

Theorem 4.1. *Let $\mathcal{T}$ be an ideal triangulation of a connected compact 3-manifold M with non-empty boundary. Then the following are equivalent:*

- *$\mathcal{T}$ is in the union $\mathscr{T}_o \cup \mathscr{T}_e$.*

- *M is in $\mathcal{M}_h$ and $\mathcal{T}$ is minimal.*

To prove Theorem 4.1 we will need the following lemma.

Lemma 4.2. *Let $\mathcal{T}$ be an ideal triangulation having only odd and even edges. Then $\mathcal{T}$ is connected and belongs to the union $\mathscr{T}_o \cup \mathscr{T}_e$. Moreover, if $\mathcal{T} \in \mathscr{T}_o$ then it has exactly one or exactly three edges.*

Proof. Since each model face contains exactly three model edges, $\mathcal{T}$ has at least one odd edge. By definition, an odd edge has pre-images in every ideal model tetrahedron of $\mathcal{T}$. Thus $\mathcal{T}$ is connected. Further arguments are obvious. □

Proof of Theorem 4.1. Let P be a special spine of M that is dual to $\mathcal{T}$. Recall that $\mathtt{d}(P)$ and $\mathtt{v}(P)$ denote the number of 2-components and the number of true vertices of P, respectively. Let $\alpha_0, \ldots, \alpha_{\mathtt{d}(P)-1}$ be the 2-components and $\gamma_0, \ldots, \gamma_{2\mathtt{v}(P)-1}$ be the edges of P. Consider a part of the chain complex of P with $\mathbb{Z}_2$-coefficients:

$$C_2 \xrightarrow{\partial} C_1.$$

Recall that $B_1 = \partial C_2$ is the group of 1-dimensional boundaries. We claim that the following are equivalent:

(a) $\mathcal{T}$ is in the union $\mathscr{T}_o \cup \mathscr{T}_e$.
(b) $\mathcal{T}$ has only odd and even edges.
(c) $\partial\alpha_i = 0$ or $\partial\alpha_i = \gamma_0 + \ldots + \gamma_{2\mathtt{v}(P)-1}$ for every $i \in \{0, \ldots, \mathtt{d}(P) - 1\}$.
(d) $\dim B_1 = 1$.
(e) $\mathtt{d}(P) = \beta_2(M, \mathbb{Z}_2) + 1$.
(f) $\mathtt{v}(P) = \beta_1(M, \mathbb{Z}_2)$.
(g) $\mathcal{T}$ consists of $\beta_1(M, \mathbb{Z}_2)$ tetrahedra.
(h) M is in $\mathcal{M}_h$ and $\mathcal{T}$ is minimal.

Indeed, the implication (b)⇒(a) is the statement of Lemma 4.2. The reverse implication (a)⇒(b) is clear by definition. The equivalences (b)⇔(c) and (f)⇔(g) come from the duality between P and $\mathcal{T}$. The implication (c)⇒(d) is evident, while the equality (2) implies the implication (d)⇒(c). The equivalence (d)⇔(e) is clear by (3), and (e)⇔(f) is a direct corollary of Lemma 3.2. And the final equivalence (g)⇔(h) is obtained by Theorem 3.1 and by the definition of $\mathcal{M}_h$. This completes the proof of the theorem. □

5. Boundary of manifolds in $\mathcal{M}_h$

Now we want to describe the boundary components of manifolds in $\mathcal{M}_h$ through the combinatorics of their minimal ideal triangulations. Let $\mathcal{M}_o$ (resp., $\mathcal{M}_e$) denote the set of all connected compact 3-manifolds with boundary, which admit an ideal triangulation in $\mathscr{T}_o$ (resp., $\mathscr{T}_e$). Theorem 4.1 provides that sets $\mathcal{M}_o$ and $\mathcal{M}_e$ cover $\mathcal{M}_h$.

Throughout this section, let $\mathcal{T}$ be an ideal triangulation of a connected compact 3-manifold M with non-empty boundary and let $\widehat{\mathcal{T}}$ be the corresponding cell complex. Additional hypotheses will be stated.

Note, that the vertices of $\widehat{\mathcal{T}}$ are in one-to-one correspondence with the components of ∂M. Let v be a vertex of $\widehat{\mathcal{T}}$. By $\partial_v M$ denote the component of ∂M corresponding to v, and by $\deg(v)$ denote the degree of v considered as a vertex of a graph $\widehat{\mathcal{T}}^{(1)}$.

Firstly, we give an abstract lemma (Lemma 5.1), that provide an explicit formula for an Euler characteristics of $\partial_v M$ corresponding to a vertex v of $\widehat{\mathcal{T}}$. After that in sections 5.1

and 5.2 we define an edge-labeling, types of model tetrahedra, and a combinatorial data for ideal triangulations in $\mathscr{T}_e$, that allows us to describe $\widehat{\mathcal{T}}^{(1)}$ in the case when $\mathcal{T} \in \mathscr{T}_e$. In particular, we obtain a full description of the vertices of $\widehat{\mathcal{T}}$. Then in section 5.3 we apply Lemma 5.1 to each vertex of $\mathcal{T}$ under an assumption that $\mathcal{T} \in \mathscr{T}_e$. Finally, in section 5.4 we apply Lemma 5.1 to each vertex of $\mathcal{T}$ under an assumption that $\mathcal{T} \in \mathscr{T}_o$.

Lemma 5.1. *Let v be a vertex of $\widehat{\mathcal{T}}$. Then*

$$\chi(\partial_v M) = \deg(v) - \frac{\#\mathcal{V}(v)}{2}. \tag{4}$$

Proof. Consider truncated triangulation $\mathcal{T}^\star$ of M, that obtained from $\widehat{\mathcal{T}}$ by removing the small open stars of its vertices. Note, that external model triangles of $\mathcal{T}^\star$ are glued to form the triangulation of ∂M. Denote by s_0, s_1, and s_2 the number of vertices, edges and triangles in the triangulation of $\partial_v M$. Then $\chi(\partial_v M) = s_0 - s_1 + s_2$. It is clear that $2s_1 = 3s_2$ and the external model triangles are in one-to-one correspondence to the model vertices. Thus $s_2 - s_1 = -\frac{\#\mathcal{V}(v)}{2}$.

It remains to check that $s_0 = \deg(v)$. To prove this we claim that each internal edge of $\mathcal{T}^\star$ has distinct endpoints. On the contrary, suppose that the endpoints of some internal edge of $\mathcal{T}^\star$ coincides. Then $\widehat{\mathcal{T}}$ has a singularity in the midpoint of the corresponding edge. This contradicts the fact that $\widehat{\mathcal{T}} \setminus \widehat{\mathcal{T}}^{(0)}$ is an ideal triangulation of M. Thus the vertices of $\mathcal{T}^\star$ are in one-to-one correspondence with the endpoints of internal edges of $\mathcal{T}^\star$ and $s_0 = \deg(v)$. $\square$

5.1. **Edge-labelling and types of model tetrahedra.** In this section we describe the combinatorics of $\mathcal{T} \in \mathscr{T}_e$. Define an edge-labelling of $\mathcal{T}$ to be the numeration of its edges from 0 to $\mathtt{d}-1$ such that the single odd edge of $\mathcal{T}$ is denoted by e_0 and the even edges of $\mathcal{T}$ are denoted by $e_1, \dots, e_{\mathtt{d}-1}$. We reserve a parameter i taking values in $\{1, \dots, \mathtt{d}-1\}$ to indicate the labels of edges of $\mathcal{T}$.

It follows that each model face $\widetilde{\sigma}$ falls into one of the following categories that are determined by the model edges contained in $\widetilde{\sigma}$.

- Type **A** The model edges of $\widetilde{\sigma}$ are the pre-images of e_0.
- Type $\mathbf{B_i}$ Two model edges of $\widetilde{\sigma}$ are the pre-images of e_i and the third model edge is the pre-image of e_0, where $i \in \{1, \dots, \mathtt{d}-1\}$.

A basic combinatorial argument provides each model tetrahedron $\widetilde{\Delta}$ to fall into one of the following categories determined by the types of model faces contained in it. We say that $\widetilde{\Delta}$ is of type $\mathbf{B_i}$ for some $i \in \{1, \dots, \mathtt{d}-1\}$ (resp., **A**) if it contains model faces of type $\mathbf{B_i}$ (resp., **A**) only. We also say that $\widetilde{\Delta}$ is of type $\mathbf{AB_i}$ for $i \in \{1, \dots, \mathtt{d}-1\}$ if it contains three model faces of type $\mathbf{B_i}$ and one model face of type **A** (see Figure 1).

5.2. **Combinatorial data.** The set-up and notation of the previous subsection are continued. Denote by a (resp., b_i or m_i for each $i \in \{1, \dots, \mathtt{d}-1\}$) the number of model tetrahedra of type **A** (resp., $\mathbf{B_i}$ or $\mathbf{AB_i}$). For an ideal triangulation $\mathcal{T}$, we define the *combinatorial data* $\mathcal{D}(\mathcal{T})$ to be a vector

$$(a, m_1, b_1, \dots, m_{\mathtt{d}-1}, b_{\mathtt{d}-1})$$

considered up to relabelling of the even edges of $\mathcal{T}$. Using $\mathcal{D}(\mathcal{T})$, define

$$\mathtt{w}(\mathcal{T}) := \#\{i \in \mathbb{N}, \text{ such that } i < d \text{ and } b_i > 0\}.$$

Clearly, $\mathtt{w}(\mathcal{T})$ is preserved under the relabelling of edges of $\mathcal{T}$.

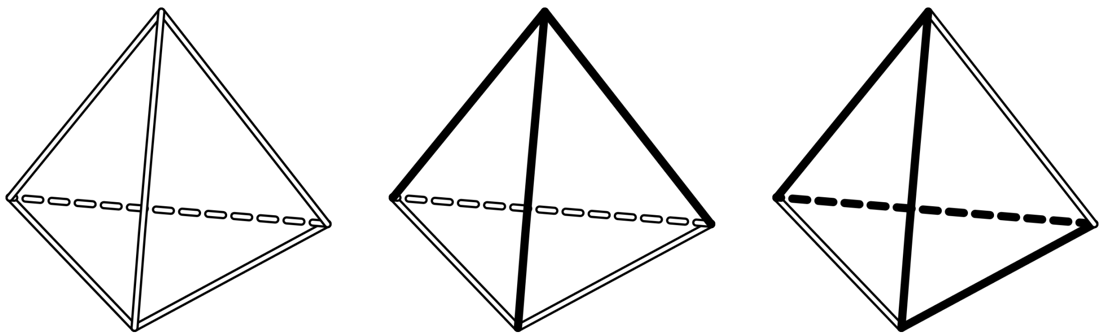


FIGURE 1. Model tetrahedra of types $\mathbf{A}$, $\mathbf{AB_i}$, and $\mathbf{B_i}$ respectively, where bold black edges are the pre-images of e_i (for some $i \in \{1, \dots, \mathtt{d}-1\}$) and the others are the preimages of e_0.

Lemma 5.2. *Assume that $\mathcal{T}$ is in $\mathscr{T}_e$. Then for each $i < \mathtt{d}(\mathcal{T})$ the following hold:*

(i) $m_i + b_i > 0$*;*
(ii) m_i *is an even number;*
(iii) if $\mathtt{d}(\mathcal{T}) > 2$ *then* $m_i \geqslant 2$*;*

Proof. Note that for each $i \in \{1, \dots, \mathtt{d}-1\}$ $\mathcal{E}(e_i)$ is contained in the union of all model tetrahedra of types $\mathbf{B_i}$ and $\mathbf{AB_i}$, thus (i) holds.

It is clear, that the model tetrahedra could be glued only by the model faces of the same type. Hence for each $i \in \{1, \dots, \mathtt{d}-1\}$ a model tetrahedron of type $\mathbf{B_i}$ could be glued to the model tetrahedra of types $\mathbf{B_i}$ and $\mathbf{AB_i}$ only. Note that a model tetrahedron of type $\mathbf{B_i}$ (resp., $\mathbf{AB_i}$) has even (resp., odd) number of model faces of type $\mathbf{B_i}$ (resp., $\mathbf{AB_i}$), thus (ii) holds. Finally (iii) follows from the connectivity of $\mathcal{T}$ (see Lemma 4.2). $\square$

5.3. **Boundary of $M \in \mathcal{M}_e$.** Define $G_{x,y}$ to be a connected graph with $x + y$ edges and $y+1$ vertices such that one vertex of $G_{x,y}$ has multiple adjacenties, while the others are the degree-one vertices. It is clear that $G_{x,y}$ contains precisely x loops.

Assume that M is in $\mathcal{M}_e$. By definition, there exists a minimal ideal triangulation $\mathcal{T}$ of M in $\mathscr{T}_e$. Fix an edge-labelling of $\mathcal{T}$.

Lemma 5.3. *Assume that $\mathcal{T}$ is in $\mathscr{T}_e$. Then $\widehat{\mathcal{T}}^{(1)}$ is isomorphic to $G_{x,y}$ with $x = \mathtt{w}(\mathcal{T}) + 1$ and $y = \mathtt{d}(\mathcal{T}) - \mathtt{w}(\mathcal{T}) - 1$. Moreover, the loops of $\widehat{\mathcal{T}}^{(1)}$ are e_0 and the edges e_i for $i \in \{1, \dots, \mathtt{d}(\mathcal{T}) - 1\}$ such that $b_i > 0$.*

Proof. To establish the conclusion we first prove three claims, each showing that, under certain conditions, some model vertices of a model tetrahedron (or a model face) are identified to the same vertex of $\widehat{\mathcal{T}}$.

Claim 1 If the three model edges incident to a model face $\widetilde{\sigma}$ are identified to form the same edge of $\widehat{\mathcal{T}}$, then the three model vertices incident to $\widetilde{\sigma}$ are identified to form the same vertex of $\widehat{\mathcal{T}}$.

Claim 2 If each pair of opposite model edges incident to a model tetrahedron $\widetilde{\Delta}$ are identified to form the same edge of $\widehat{\mathcal{T}}$, then all the model vertices incident to $\widetilde{\Delta}$ are identified to form the same vertex of $\widehat{\mathcal{T}}$.

Claim 3 Each model tetrahedron of type $\mathbf{A}$ or $\mathbf{B_i}$ (for some $i \in \{1,\dots,\mathtt{d}-1\}$) in $\widehat{\mathcal{T}}$ has its model vertices identified to form the same vertex of $\widehat{\mathcal{T}}$.

Indeed, let the three model edges incident to a model face $\widetilde{\sigma}$ be identified to form an edge e of $\widehat{\mathcal{T}}$. Suppose that e has distinct endpoints that are denoted by u and v. It follows that each model edge incident to $\widetilde{\sigma}$ has endpoints on the pre-images of u and v, a contradiction. This proves Claim 1.

Let us prove Claim 2. On the contrary, suppose that there is an edge e of $\widehat{\mathcal{T}}$ with distinct endpoints, say u and v, that has pre-images in $\widetilde{\Delta}$. By hypothesis, there is a pair of opposite model edges in $\widetilde{\Delta}$ that are identified to form e. It follows that the model vertices of $\widetilde{\Delta}$ are pre-images of u and v. This contradicts our hypothesis and completes the proof of Claim 2.

Finally, Claim 3 follows directly from Claim 1 and Claim 2.

Now let $\mathcal{T}$ be in $\mathscr{T}_e$. Recall, that an edge-labelling of $\mathcal{T}$ is fixed. Moreover, each model tetrahedron is of type $\mathbf{A}$, $\mathbf{B_i}$, or $\mathbf{AB_i}$ for some $i \in \{1,\dots,\mathtt{d}-1\}$. Since all model tetrahedra of types $\mathbf{A}$, $\mathbf{B_i}$, and $\mathbf{AB_i}$ (for each $i \in \{1,\dots,\mathtt{d}-1\}$) has pre-images e_0, applying Claim 1 (to a model tetrahedron of type $\mathbf{AB_i}$) or Claim 3 (to a model tetrahedron of type $\mathbf{A}$ or $\mathbf{B_i}$) we obtain that the ends of e_0 coincide; denote this vertex by v_0. It will be a vertex of $\widehat{\mathcal{T}}^{(1)}$ with multiple adjacencies.

Fix $i \in \{1,\dots,\mathtt{d}-1\}$. If $b_i > 0$, then $\widehat{\mathcal{T}}$ contains at least one model tetrahedron of type $\mathbf{B_i}$, which has pre-images of e_i. Hence, by Claim 3, ends of e_i coincide with v_0. But if $b_i = 0$, then e_i has one endpoint in v_0 and the other is a degree-one vertex. This completes the proof. □

Corollary 5.4. *Assume that $\mathcal{T}$ is in $\mathscr{T}_e$. Then ∂M has exactly $\mathtt{d}(\mathcal{T}) - \mathtt{w}(\mathcal{T})$ connected components.*

Now we apply Lemma 5.1 to M.

Lemma 5.5. *Assume that $\mathcal{T}$ is in $\mathscr{T}_e$. Let v_0 be a vertex of $\widehat{\mathcal{T}}^{(1)}$ with multiple adjacenties, and let e_j (for some $j \in \{1,\dots,\mathtt{d}(\mathcal{T})-1\}$) be an even edge of $\widehat{\mathcal{T}}$. If $c_j = 0$ then e_j has one end in v_0 and the other in a degree-one vertex, say v. We have:*

$$\chi(\partial_v M) = 1 - \frac{m_j}{2}; \tag{5}$$

$$\chi(\partial_{v_0} M) = (\mathtt{d} + 1 + \mathtt{w}(\mathcal{T})) - \frac{1}{2}\left[4a + 4\sum_{i=1}^{\mathtt{d}-1}(m_i + b_i)\mathbb{1}_{b_i>0} + 3\sum_{i=1}^{\mathtt{d}-1} m_i(1 - \mathbb{1}_{b_i>0})\right], \tag{6}$$

where

$$\mathbb{1}_{b_i>0} = \begin{cases} 1 & \textit{if } b_i > 0 \\ 0 & \textit{otherwise} \end{cases}$$

stands for the characteristic function.

Proof. Due to Corollary 5.4, $\widehat{\mathcal{T}}^{(1)}$ has $\mathtt{w}(\mathcal{T})+1$ loops that are incident to v_0 and $\mathtt{d}-\mathtt{w}(\mathcal{T})-1$ edges with one end in v_0. Thus $\deg(v_0) = \mathtt{d} + 1 + \mathtt{w}(\mathcal{T})$.

To finish the proof we need to describe full pre-images of v and v_0. We refer to the proof of 5.3 with Claims 1,2, and 3 to obtain the following. A model vertex $\widetilde{v}$ belongs to $\mathcal{V}(v)$ if and only if $\widetilde{v}$ is a vertex of a model tetrahedron of type $\mathbf{AB_j}$ opposite to the model face of type $\mathbf{A}$, while $\mathcal{V}(v_0)$ consists of:

- all model vertices of model tetrahedra of type $\mathbf{A}$;

- all model vertices of model tetrahedra of types $\mathbf{B_i}$ and $\mathbf{AB_i}$ for each $i \in \{1, \ldots, \mathtt{d}-1\}$ such that $b_i > 0$;
- and model vertices of model tetrahedra of type $\mathbf{AB_i}$ that are contained in model face of type $\mathbf{A}$ for each $i \in \{1, \ldots, \mathtt{d}-1\}$ such that $b_i = 0$.

Using the combinatorial data $\mathcal{D}(\mathcal{T})$ and the description of $\mathcal{V}(v)$ and $\mathcal{V}(v_0)$ we obtain (5) and (6). □

Lemma 5.5 provides the criteria for a component of ∂M to be of zero Euler characteristic, which we will use later in section 7.

Corollary 5.6. *Assume that $\mathcal{T}$ is in $\mathscr{T}_e$. Let v_0 be a vertex of $\widehat{\mathcal{T}}^{(1)}$ with multiple adjacenties, and let e_j (for some $j \in \{1, \ldots, \mathtt{d}(\mathcal{T})-1\}$) be an even edge of $\widehat{\mathcal{T}}$. If $c_j = 0$ then e_j has one end in v_0 and the other in a degree-one vertex, say v. We have:*

(i) $\chi(\partial_v M) \leqslant 0$, *moreover* $\chi(\partial_v M) = 0$ *if and only if* $m_j = 2$ *;*
(ii) $\chi(\partial_{v_0} M) < 0$ *if* $c_\Delta(M) \geqslant 3$.

Proof. Item (i) follows directly from Lemma 5.5.

To prove (ii) we apply (6) from Lemma 5.5 that could be rewritten as follows:

$$\chi(\partial_{v_0} M) = (2-2a) + 2\sum_{i=1}^{\mathtt{d}-1}(1-m_i-b_i)\mathbb{1}_{b_i>0} + \sum_{i=1}^{\mathtt{d}-1}(1-\frac{3m_i}{2})(1-\mathbb{1}_{b_i>0}).$$

We will check all possible values of combinatorial data $\mathcal{D}(\mathcal{T})$ which satisfy lemma 5.2. Item (i) of lemma 5.2 provides that terms $1-m_i-b_i$ and $1-\frac{3m_i}{2}$ are negative or equals zero for each $i \in \{1, \ldots, \mathtt{d}-1\}$. The first term $2-2a$ is also negative or equals zero then $a \geqslant 1$. Otherwise $a = 0$ and the following situations are possible.

- $d > 2$ and $c_j > 0$ for some $j \in \{1, \ldots, \mathtt{d}-1\}$. Then by item (iii) in lemma 5.2 we have $\chi(\partial_{v_0} M) < 2 + 2(1-c_j-m_j) < 0$
- $d > 2$ and $b_i = 0$ for all $i \in \{1, \ldots, \mathtt{d}-1\}$. Then by item (iii) in lemma 5.2 we have $\chi(\partial_{v_0} M) \leqslant 2 + (1-\frac{3m_1}{2}) + (1-\frac{3m_2}{2}) < 0$
- $d = 2$, $m_1 = 0$ and $c_1 \geqslant 3$, then $\chi(\partial_{v_0} M) = 2 + 2(1-c_1) < 0$

Thus (ii) is proved. □

5.4. **Boundary of $M \in \mathcal{M}_o$.**

Lemma 5.7. *Assume that M is in $\mathcal{M}_o$ and $\mathcal{T}$ is in $\mathscr{T}_o$. Then ∂M is connected and*

$$\chi(\partial M) = 2\mathtt{d}(\mathcal{T}) - 2c_\Delta(M).$$

Proof. Let us show that ∂M is connected. Since the vertices of $\widehat{\mathcal{T}}$ are in one-to-one correspondence with the components of ∂M, it is sufficient to show that $\widehat{\mathcal{T}}$ has exactly one vertex. Since $\mathcal{T}$ belongs to $\mathscr{T}_o$ it has exactly one or exactly three edges, that are odd, thus the two situations possible.

Suppose $\mathtt{d}(\mathcal{T}) = 3$, then $\widehat{\mathcal{T}}$ has exactly three edge, say e_1, e_2, and e_3. Then each model tetrahedron has three pairs of its opposite model edges identified to form edges e_1, e_2, and e_3. Due to Claim 2 form the proof of Lemma 5.3 each model tetrahedron has its model vertices identified to form one vertex of $\widehat{\mathcal{T}}$. Thus $\widehat{\mathcal{T}}$ has exactly one vertex.

Now suppose $\mathtt{d}(\mathcal{T}) = 1$, then $\widehat{\mathcal{T}}$ has exactly one edge, say e_0. Then each model tetrahedron has its model edges identified to form e_0. Applying Claim 2 form the proof of Lemma 5.3 we obtain that $\widehat{\mathcal{T}}$ has exactly one vertex.

Theorem 4.1 provides that $\mathcal{T}$ is a minimal ideal triangulation of M. Thus $\widehat{\mathcal{T}}$ has exactly $c_\Delta(M)$ model tetrahedra. Let v_0 be the single vertex of $\widehat{\mathcal{T}}$, then $\deg(v_0)$ equals twice the number of edges of $\widehat{\mathcal{T}}$ and $\#\mathcal{V}(v_0)$ equals to the number of model vertices, that equals to $4c_\Delta(M)$. This finishes the proof. $\square$

6. Exact cover of $\mathcal{M}_h$ with four subclasses

By Lemma 4.2, the set $\mathscr{T}_o$ can be divided into two pairwise disjoint subsets. One of them consists of one-edged ideal triangulations and is denoted by $\mathscr{T}_o^1$. The other consists of ideal triangulations with exactly three edges and is denoted by $\mathscr{T}_o^2$. The set $\mathscr{T}_e$ can also be divided into two pairwise disjoint subsets. We say that an ideal triangulation $\mathcal{T} \in \mathscr{T}_e$ is in $\mathscr{T}_e^1$ if $\mathtt{w}(\mathcal{T}) = 0$. Otherwise $\mathcal{T}$ is in $\mathscr{T}_e^2$.

The partition of $\mathscr{T}_o \cup \mathscr{T}_e$ induces a cover of $\mathcal{M}_h$ with four subsets $\mathcal{M}_o^1$, $\mathcal{M}_o^2$, $\mathcal{M}_e^1$ and $\mathcal{M}_e^2$, where $\mathcal{M}_o^1$ (resp., $\mathcal{M}_o^2$, $\mathcal{M}_e^1$, $\mathcal{M}_e^2$) denote the set of connected compact 3-manifolds with boundary admitting an ideal triangulation in $\mathscr{T}_o^1$ (resp., $\mathscr{T}_o^2$, $\mathscr{T}_e^1$, $\mathscr{T}_e^2$). Now we show that this cover is exact.

Theorem 6.1. *The sets $\mathcal{M}_o^1$, $\mathcal{M}_o^2$, $\mathcal{M}_e^1$, and $\mathcal{M}_e^2$ are pairwise disjoint and cover $\mathcal{M}_h$.*

Proof. Let $\mathcal{T}_o^1$, $\mathcal{T}_o^2$, $\mathcal{T}_e^1$, and $\mathcal{T}_e^2$ be ideal triangulations in $\mathscr{T}_o^1$, $\mathscr{T}_o^2$, $\mathscr{T}_e^1$, and $\mathscr{T}_e^2$, respectively. We need to prove that the corresponding manifolds M_o^1, M_o^2, M_e^1, and M_e^2 are pairwise non-homeomorphic. Since these ideal triangulations are minimal (Theorem 4.1), we may assume they consist of the same number, say n, of tetrahedra; otherwise the manifolds are non-homeomorphic.

Recall if we are given an ideal triangulation $\mathcal{T}$ of a connected compact 3-manifold M with non-empty boundary, we obtain the dual special spine P of M. As we noticed above in the proof of Lemma 3.2, $\chi(M) = \chi(P) = \mathtt{d}(P) - \mathtt{v}(P)$. Since the true vertices, edges and 2-components of P are in one-to-one correspondence with the tetrahedra, faces and edges of $\mathcal{T}$, respectively, we have $\chi(M) = \mathtt{d}(\mathcal{T}) - \mathtt{t}(\mathcal{T})$. Then we apply the last equality to show that M_o^1 is different from M_o^2, M_e^1, and M_e^2. In fact, $\chi(M_o^1) = 1 - n$, while $\chi(M_o^2) = 3 - n$, $\chi(M_e^1) = \mathtt{d}(\mathcal{T}_e^1) - n$, and $\chi(M_e^2) = \mathtt{d}(\mathcal{T}_e^2) - n$, where by definition $\mathtt{d}(\mathcal{T}_e^1) \geq 2$ and $\mathtt{d}(\mathcal{T}_e^2) \geq 2$. Hence, $M_o^1 \cap M_o^2 = \emptyset$, $M_o^1 \cap M_e^1 = \emptyset$, and $M_o^1 \cap M_2^2 = \emptyset$.

Now there are three cases to consider, depending on the equality between the Euler characteristic.

First, if $\chi(M_o^2) = \chi(M_e^1)$, then $\mathtt{d}(\mathcal{T}_e^1) = 3$. By Lemma 5.4, ∂M_e^1 has three boundary components, while ∂M_o^2 has only one. Hence, $M_o^2 \cap M_e^1 = \emptyset$.

Second, if $\chi(M_e^1) = \chi(M_e^2)$, then $\mathtt{d}(\mathcal{T}_e^1) = \mathtt{d}(\mathcal{T}_e^2)$. By Lemma 5.4, ∂M_e^1 has more boundary components than ∂M_e^2. Hence, $M_e^1 \cap M_e^2 = \emptyset$.

Consider the last case assuming $\chi(M_o^2) = \chi(M_e^2)$. It follows that $\mathtt{d}(\mathcal{T}_e^2) = 3$. Now we switch from the ideal triangulations $\mathcal{T}_o^2$ and $\mathcal{T}_e^2$ to its dual special spines, denoted P and Q, respectively.

To prove that the manifolds M_o^2, M_e^2 are non-homeomorphic, we use the ε-invariant of Matveev – Ovchinnikov – Sokolov (see [10, Chapter 8.1.3]), which is the homologically trivial part of the order 5 Turaev–Viro invariant. We give the definition of the ε-invariant following [10]. Let R be a special spine of a connected compact 3-manifold M with non-empty boundary. Denote by $\mathcal{F}(R)$ the set of all simple subpolyhedra of R including R and the empty set. Set $\varepsilon = (1 + \sqrt{5})/2$, a solution of the equation $\varepsilon^2 = \varepsilon + 1$. With each

$K \in \mathcal{F}(R)$ we associate its ε-*weight* by the formula

$$w_\varepsilon(K) = (-1)^{\mathtt{v}(K)}\varepsilon^{\chi(K)-\mathtt{v}(K)},$$

where $\mathtt{v}(K)$ is the number of true vertices of K and $\chi(K)$ is its Euler characteristic. Set

$$t(M) = \sum_{K\in\mathcal{F}(R)} w_\varepsilon(K).$$

As shown in [10], $t(M)$ is an invariant of M.

To complete the proof of the theorem we show that $t(M_o^2) \neq t(M_e^2)$. Let us calculate $t(M_o^2)$ by its special spine P. By the compactness of a simple subpolyhedron if it contains a point of a 2-component, then it contains the whole of it. Thus, to describe a simple subpolyhedron of P it is enough to indicate which 2-components of P it includes (its triple points and true vertices then will be determined uniquely). Since $\mathcal{T}_o^2$ has exactly three edges, and each of these edges is odd, P has exactly three 2-components, denoted ξ_1, ξ_2, ξ_3, and each boundary curve $\partial\xi_i$ passes exactly once along each edge of P. Hence, $\mathcal{F}(P) = \{\emptyset, P, P_1, P_2, P_3\}$, where $P_i = P\backslash\xi_i$. It is easy to see that $\mathtt{v}(P) = n$, $\chi(P) = 3-n$, $\mathtt{v}(P_i) = 0$, and $\chi(P_i) = 2-n$, $1 \leq i \leq 3$. Summing up the ε-weights $w_\varepsilon(\emptyset) = 1$, $w_\varepsilon(P) = (-1)^n\varepsilon^{3-2n}$, and $w_\varepsilon(P_i) = \varepsilon^{2-n}$, we get

$$t(M_o^2) = 1 + (-1)^n\varepsilon^{3-2n} + 3\varepsilon^{2-n}.$$

Now we calculate $t(M_e^2)$ by the special spine Q. Since $\mathcal{T}_e^2$ has exactly one odd edge and two even edges, Q has exactly three 2-components, denoted ξ_0, ξ_1, and ξ_2. Let ξ_0 corresponds to the odd edge, while ξ_1 and ξ_2 correspond to the even ones. We claim Q has exactly three proper simple subpolyhedra. Indeed, two of these polyhedra are connected closed surfaces, denoted Q_1, Q_2 such that each Q_i, $i = 1$ or 2, contains ξ_i and do not contain the other 2-components of Q. Therefore, $Q_1 \cap Q_2 = \emptyset$. The third polyhedron, denoted Q_3, is the union $Q_1 \cup Q_2$ (i.e. a closed surface too).

So we have $\mathcal{F}(Q) = \{\emptyset, Q, Q_1, Q_2, Q_3\}$, $\mathtt{v}(Q) = n$, $\chi(Q) = 3-n$, and $\mathtt{v}(Q_i) = 0$, $1 \leq i \leq 3$. Summing up the ε-weights we get

$$t(M_e^2) = 1 + (-1)^n\varepsilon^{3-2n} + \sum_{i=1}^{3}\varepsilon^{\chi(Q_i)}.$$

For each i, $1 \leq i \leq 3$, we claim $\chi(Q_i) > 2-n$. Indeed, let v_i^+, e_i^+, and d_i^+ denote the number of true vertices, edges, and 2-components of P, respectively, belonging to Q_i. By construction, $d_1^+ = d_2^+ = 1$ and $d_3^+ = 2$. We set $v_i^- = \mathtt{v}(Q) - v_i^+$ and $e_i^- = \mathtt{e}(Q) - e_i^+$, where $\mathtt{e}(Q)$ is the number of edges of Q. Since $\mathtt{e}(Q) = 2\mathtt{v}(Q)$, we have

$$\chi(Q_i) = v_i^+ - e_i^+ + d_i^+ = (2-v) + (e_i^- - v_i^-) + (d_i^+ - 2).$$

The inequality $e_i^- - v_i^- > 1$ can be easily proved by induction on v_i^- by using the fact that the surface Q_i does not contain all the edges of the special spine Q. It follows that $\chi(Q_i) > 2-n$. Hence, $3\varepsilon^{2-n} < \sum_{i=1}^3 \varepsilon^{\chi(Q_i)}$, and we have $t(M_o^2) \neq t(M_e^2)$. This then completes the proof. $\square$

Let $\mathcal{M}_b$ denote the set of connected compact 3-manifolds M with non-empty boundary, which have an ideal triangulation $\mathcal{T}$ with $\beta_0(\partial M; \mathbb{Z}_2) - \chi(M)$ ideal tetrahedra. Due to the lower bound (1), $\mathcal{T}$ is a minimal ideal triangulation of M.

Due to Theorem 3.3, $\mathcal{M}_b$ is contained in $\mathcal{M}_h$. The partition of $\mathcal{M}_h$ into four subsets $\mathcal{M}_o^1$, $\mathcal{M}_o^2$, $\mathcal{M}_e^1$ and $\mathcal{M}_e^2$ provides the following.

Theorem 6.2. *The set $\mathcal{M}_b$ is the disjoint union of $\mathcal{M}_o^1$ and $\mathcal{M}_e^1$.*

Proof. We already know, that $\mathcal{M}_b$ is contained in $\mathcal{M}_h$, thus it is covered with four sets $\mathcal{M}_o^1$, $\mathcal{M}_o^2$, $\mathcal{M}_e^1$, and $\mathcal{M}_e^2$ which are disjointed. Our goal is to show, that for manifolds in $\mathcal{M}_o^2$ and $\mathcal{M}_e^2$ the lower bound (1) is strict.

Let M be in $\mathcal{M}_o^2 \cup \mathcal{M}_e^2$. Let $\mathcal{T}$ be a minimal ideal triangulation of M and P be a spine of P dual to $\mathcal{T}$. Since M belongs to $\mathcal{M}_h$, the lower bound from Theorem 3.1 is achieved. Thus we are left to show that inequality in Theorem 3.3 is strict:

$$\beta_2(M;\mathbb{Z}_2) + 1 > \beta_0(\partial M;\mathbb{Z}_2). \tag{7}$$

Let d be the number of 2-components of P. From duality between P and $\mathcal{T}$ we have $\mathtt{d}(\mathcal{T}) = d$. Since P is a spine of M we have $\beta_2(M;\mathbb{Z}_2) = \beta_2(P;\mathbb{Z}_2)$. Now we refer to the proof of Theorem 4.1, where we stated and proved that items (e) and (g) are equivalent. Thus $\beta_2(M;\mathbb{Z}_2) = d - 1$

Suppose M belongs to $\mathcal{M}_o^2$. Hence $\mathcal{T}$ belongs to $\mathscr{T}_o^2$ and $d = \mathtt{d}(\mathcal{T}) = 3$. From Corollary 5.4 ∂M is connected. Thus $\beta_0(\partial M;\mathbb{Z}_2) = 1$ and inequality (7) holds.

Suppose M belongs to $\mathcal{M}_e^2$. Hence $\mathcal{T}$ belongs to $\mathscr{T}_e^2$ and $\mathtt{w}(\mathcal{T}) > 0$. From Corollary 5.4 ∂M has exactly $\mathtt{d}(\mathcal{T}) - \mathtt{w}(\mathcal{T})$ components. Thus $\beta_0(\partial M;\mathbb{Z}_2) = d - \mathtt{w}(\mathcal{T})$ and inequality (7) holds.

Finally it is easy to see, that for manifolds in $\mathcal{M}_o^1 \cup \mathcal{M}_e^1$ the lower bound (1) is achieved. □

7. Hyperbolicity of manifolds in $\mathcal{M}_h$

Let M be a compact 3-manifold with non-empty boundary. Define $\overline{M}$ to be M with boundary components of zero Euler characteristic removed. We say M is *hyperbolic* if $\overline{M}$ is a complete riemannian manifold of constant sectional curvature -1 with finite volume and totally geodesic boundary.

Remark 7.1. It was proven in [2, 14] that if $M \in \mathcal{M}_o^1$ and $c_\Delta(M) \geqslant 2$ or $M \in \mathcal{M}_o^2$ and $c_\Delta(M) \geqslant 4$, then M is a hyperbolic manifold with totally geodesic boundary.

Theorem 7.2. *If $M \in \mathcal{M}_e$ and $c_\Delta(M) \geqslant 3$, then M is a hyperbolic manifold with totally geodesic boundary components and some cusps.*

Throughout this section, let M be in $\mathcal{M}_e$ and $c_\Delta(M) \geqslant 3$, $\mathcal{T} \in \mathscr{T}_e$ be an ideal triangulation of M, and let $\widehat{\mathcal{T}}$ be the corresponding cell complex. We also fix an edge-labelling of $\mathcal{T}$ described in subsection 5.1 and define I to be a subset of $\widehat{\mathcal{T}}^{(0)}$ such that a vertex v is in I if and only if $\chi(\partial_v M) = 0$. Let $\widehat{\mathcal{T}}^I$ be a partially truncated triangulation corresponding to a pair $(\widehat{\mathcal{T}}, I)$ (see section 2.2 for exact definition). Additional hypotheses will be stated.

Now we introduce the strategy of the proof. Recall, that $\widehat{\mathcal{T}}^I$ is glued from partially truncated model tetrahedra $\widetilde{\Delta}^J$'s. Each $\widetilde{\Delta}^J$ corresponds to a pair $(\widetilde{\Delta}, J)$, where $\widetilde{\Delta}$ is a model tetrahedron and $J = \widetilde{\Delta}^{(0)} \cap \mathcal{V}(I)$. In section 7.1 we give several properties of $\widehat{\mathcal{T}}^I$ and $\widetilde{\Delta}^J$'s. In order to give $\overline{M}$ a hyperbolic structure with totally geodesic boundary we realize $\widetilde{\Delta}^J$'s as a special geometric blocks in $\mathbb{H}^3$ (section 7.2) and require that the structures match under the gluings (section 7.3). In section 7.4 we show the completeness of hyperbolic structure constructed above.

In addition, in section 7.5 we apply Theorem 7.2 to obtain the Matveev's complexity for manifolds in $\mathcal{M}_e$ and to show that two hyperbolic manifolds in $\mathcal{M}_e$ have equal volumes if they admit minimal ideal triangulation with the same combinatorial data.

7.1. **Properties of $\widehat{\mathcal{T}}^I$.**

Lemma 7.3. *Let $\widetilde{\Delta}$ be a model tetrahedron, and let $J = \widetilde{\Delta}^{(0)} \cap \mathcal{V}(I)$ be the set of ideal model vertices of $\widetilde{\Delta}$. Then the following holds.*

(i) If $\widetilde{\Delta}$ is of type $\mathbf{A}$ or $\mathbf{B_i}$ for some $i \in \{1, \dots, \mathtt{d}-1\}$, then $J = \emptyset$ (see Figures 2a and 2c respectively).

(ii) If $\widetilde{\Delta}$ is of type $\mathbf{AB_i}$ for some $i \in \{1, \dots, \mathtt{d}-1\}$ such that $m_i > 2$ or $b_i > 0$, then $J = \emptyset$ (see Figure 2d).

(iii) If $\widetilde{\Delta}$ is of type $\mathbf{AB_i}$ for some $i \in \{1, \dots, \mathtt{d}-1\}$ such that $m_i = 2$ and $b_i = 0$, then J consists of one model vertex of $\widetilde{\Delta}$ that is opposite to a model face of type $\mathbf{A}$ in $\widetilde{\Delta}$ (see Figure 2b).

Proof. By Lemma 5.3, $\widehat{\mathcal{T}}$ has one vertex with multiple adjacencies, say v_0, while the other vertices of $\widehat{\mathcal{T}}$ (if there are any) are of degree one. Under assumption of Theorem 7.2 we have $c_\Delta(M) \geqslant 3$, thus by Corollary 5.6 we have $\chi(\partial_{v_0} M) < 0$.

Suppose $\mathcal{T}$ has an odd edge e_j for some $j \in \{1, \dots, \mathtt{d}-1\}$ such that $c_j = 0$. Then e_j has one end in v_0 while the other in a degree-one vertex, say v. By Corollary 5.6 we have $\chi(\partial_v M) \leqslant 0$ and $\chi(\partial_v M) = 0$ if and only if $m_j = 2$. Suppose $m_j = 2$. Let us describe $\mathcal{V}(v)$.

Recall, that for each $i \in \{1, \dots, \mathtt{d}-1\}$ $\mathcal{E}(e_j)$ is contained in the union of all model tetrahedra of types $\mathbf{B_i}$ and $\mathbf{AB_i}$. Since we have $c_j = 0$ and $m_j = 2$ then $\mathcal{E}(e_j)$ is contained in the two model tetrahedra of type $\mathbf{AB_j}$. Since v is a degree-one vertex, then $\mathcal{V}(v)$ is contained in $\mathcal{E}(e_j)$. Further arguments are obvious. $\square$

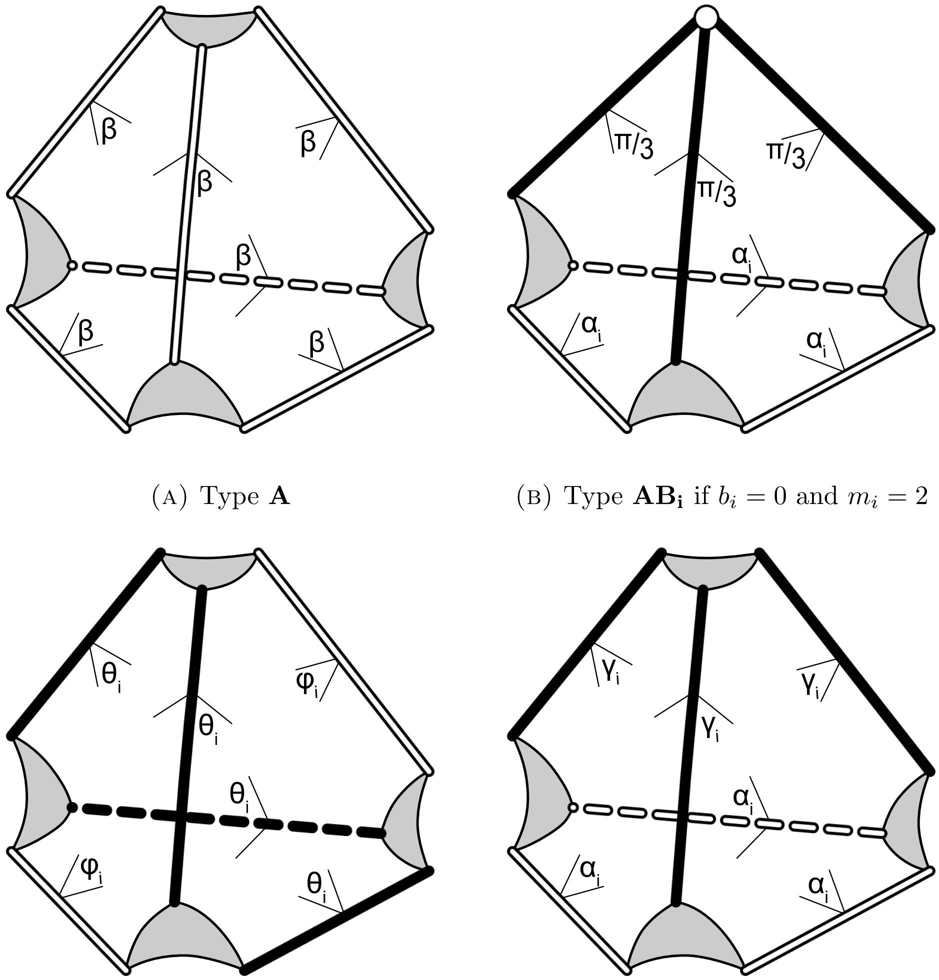
Remark 7.4. The partially truncated triangulation $\widehat{\mathcal{T}}^I$ satisfies the following.

- $\widehat{\mathcal{T}}^I$ is homeomorphic to $\overline{M}$.
- The truncation model triangles of the $\widetilde{\Delta}^J$'s give a triangulation of components of ∂M with negative Euler characteristic.
- The links of the ideal model vertices of $\widetilde{\Delta}^J$'s give a triangulation of components of ∂M with zero Euler characteristic.

Since the internal edges of $\widehat{\mathcal{T}}^I$ are in one-to-one correspondence with the edges of $\mathcal{T}$, the edge-labelling of $\mathcal{T}$ induces the labelling of internal edges of $\widehat{\mathcal{T}}^I$, which we denote by $e^I_0, \dots, e^I_{\mathtt{d}-1}$. Further, types of model faces, types of model tetrahedra, and the combinatorial data of $\widehat{\mathcal{T}}$ induce types of lateral model hexagons, types of partially truncated model tetrahedra, and the combinatorial data of $\widehat{\mathcal{T}}^I$ respectively.

Remark 7.5. The internal model edges of a given partially truncated model tetrahedron $\widetilde{\Delta}^J$ can be described as the pre-images of the internal edges of $\widehat{\mathcal{T}}^I$ depending on the type of $\widetilde{\Delta}^J$.

- If $\widetilde{\Delta}^J$ is of type $\mathbf{A}$ then all its internal model edges are the pre-images of e^I_0 (see Figure 2a.
- If $\widetilde{\Delta}^J$ is of type $\mathbf{B_i}$ (for some $i \in \{1, \dots, \mathtt{d}-1\}$) then two of its opposite internal model edges are the pre-images of e^I_0, and the others are the pre-images of e^I_i (see Figure 2c).
- If $\widetilde{\Delta}^J$ is of type $\mathbf{AB_i}$ (for some $i \in \{1, \dots, \mathtt{d}-1\}$) then the three internal model edges of $\widetilde{\Delta}^J$ that are incident to the lateral model hexagon of type $\mathbf{A}$ are the pre-images of e^I_0, and the others are the pre-images of e^I_i (see Figure 2d or Figure 2b if $m_i = 2$ and $b_i = 0$).



(A) Type **A** (B) Type $\mathbf{AB_i}$ if $b_i = 0$ and $m_i = 2$

(C) Type $\mathbf{B_i}$ (D) Type $\mathbf{AB_i}$

FIGURE 2. Model tetrahedra of types **A**, $\mathbf{B_i}$, and $\mathbf{AB_i}$ for some $i \in \{1, \ldots, \mathtt{d} - 1\}$, where the bold black internal edges are the pre-images of e_i^I and the others are the pre-images of e_0^I.

7.2. **Geometric model tetrahedra.** A *geometric realization* of a partially truncated model tetrahedron $\widetilde{\Delta}^J$ is an embedding of $\widetilde{\Delta}^J$ into $\mathbb{H}^3$ such that the truncation model triangles are geodesic triangles, the lateral model hexagons are geodesic polygons with ideal model vertices corresponding to missing edges, and the truncation model triangles and the lateral model hexagons lie at right angles to each other. A geometric realization of a partially truncated model tetrahedron is called a *geometric model tetrahedron*.

M. Fujii proved in [8] that a geometric realization of a truncated model tetrahedron (without ideal model vertices) is parameterized up to isometry by the set of dihedral angles along its internal model edges, which satisfy natural restrictions. R. Frigerio and C. Petronio generalize the result of Fujii to geometric realizations of partially truncated tetrahedra.

Now we give the geometric realization of partially truncated model tetrahedra $\widetilde{\Delta}^J$'s, that is determined by their types.

- If $\widetilde{\Delta}^J$ is of type $\mathbf{A}$ then we realize it in $\mathbb{H}^3$ with dihedral angle β along pre-images of e_0^I, where $\beta \in \mathbb{R}$ with $0 < \beta < \pi/3$ (see Figure 2a).
- If $\widetilde{\Delta}^J$ is of type $\mathbf{B_i}$ (for some $i \in \{1, \dots, \mathtt{d}-1\}$) then we realize it in $\mathbb{H}^3$ with dihedral angles ϕ_i and θ_i along pre-images of e_0^I and e_i^I respectively, where $\phi_i, \theta_i \in \mathbb{R}$ with $0 < \phi_i \leqslant \pi/3$ and $0 < \theta_i < \pi/3$ (see Figure 2c).
- If $\widetilde{\Delta}^J$ is of type $\mathbf{AB_i}$ (for some $i \in \{1, \dots, \mathtt{d}-1\}$) and either $m_i > 2$ or $b_i > 0$ then we realize $\widetilde{\Delta}^J$ in $\mathbb{H}^3$ with dihedral angles α_i and γ_i along pre-images of e_0^I and e_i^I respectively, where $\alpha_i, \gamma_i \in \mathbb{R}$ with $0 < \alpha_i, \gamma_i < \pi/3$ (see Figure 2d).
- If $\widetilde{\Delta}^J$ is of type $\mathbf{AB_i}$ (for some $i \in \{1, \dots, \mathtt{d}-1\}$), $m_i = 2$, and $a_i = 0$ then $\widetilde{\Delta}^J$ has an ideal vertex and we realize it in $\mathbb{H}^3$ with dihedral angles α_i and $\gamma_i = \pi/3$ along pre-images of e_0^I and e_i^I respectively, where $\alpha_i \in \mathbb{R}$ with $0 < \alpha_i < \pi/3$ (see Figure 2b).

Remark 7.6. Since a geometric realization of each partially truncated model tetrahedron is determined by its type we will identify partially truncated model tetrahedra $\widetilde{\Delta}^J$'s with their geometric realizations.

7.3. **Consistency.** In order to construct a hyperbolic structure on $\overline{M}$ we give geometric realization of the partially truncated model tetrahedra $\widetilde{\Delta}^J$'s, described in section 7.2, and require the structures to match under the gluings. Firstly, we should glue partially truncated model tetrahedra by the isometries of their lateral model hexagons such that the internal (resp., external) model edges match to the internal (resp., external) ones. An isometry between two lateral model hexagons exists if and only if the corresponding external model edges have equal lengths; we will call it by *length condition*. The gluing of the partially truncated model tetrahedra by the isometries of their lateral model hexagons allows us to extend the hyperbolic structure to $\overline{M}$ with the internal edges of $\widehat{\mathcal{T}}^I$ removed. To be able to extend the hyperbolic structure to the whole of $\overline{M}$, we should require the total dihedral angle around each internal edge of $\widehat{\mathcal{T}}^I$ to equal 2π; we will call it by *cone angle condition*. It was shown in [12] that the hyperbolic structure given on the partially truncated model tetrahedra of $\widehat{\mathcal{T}}^I$ extends to the whole of $\overline{M}$ if and only if both length condition and cone angle condition hold.

7.3.1. *Length condition.* Let us remind, that partially truncated model tetrahedra can be glued only by lateral model hexagons of the same type. Thus all *possible* gluings are given in table 1.

Type of a lateral model hexagon	Types of two partially truncated model tetrahedra
$\mathbf{A}$	$\mathbf{A}\&\mathbf{A}$, $\mathbf{A}\&\mathbf{AB_i}$, $\mathbf{AB_i}\&\mathbf{AB_i}$, $\mathbf{AB_i}\&\mathbf{AB_i}$
$\mathbf{B_i}$	$\mathbf{B_i}\&\mathbf{B_i}$, $\mathbf{AB_i}\&\mathbf{AB_i}$, $\mathbf{AB_i}\&\mathbf{B_i}$

TABLE 1. Possible gluings of partially truncated model tetrahedra of $\widehat{\mathcal{T}}^I$.

In this section we describe the length condition for each possible gluing, and in section 7.3.3 we show which gluings appear in $\widehat{\mathcal{T}}^I$. Since geometric realization of the partially truncated model tetrahedra of $\widehat{\mathcal{T}}^I$ depends only on their types, gluings of two partially truncated model tetrahedra with the same type produce trivial length conditions. It is worth noting, that the gluing of two partially truncated model tetrahedra along lateral model hexagons with ideal model vertices produces trivial length conditions due to the argument above. Thus it remains to consider gluings along lateral model hexagons without ideal model vertices. To compute the lengths of external model edges through the dihedral angles of a partially truncated model tetrahedron we use explicit formulas from [8].

Firstly we consider a gluing of two partially truncated model tetrahedra of types $\mathbf{AB_i}$ and $\mathbf{B_i}$ along a pair of their lateral model hexagons of type $\mathbf{B_i}$, where $i \in \{1, \ldots, \mathtt{d}-1\}$. Note, that each lateral model hexagon of type $\mathbf{B_i}$ has two external model edges with equal lengths, thus the length condition translates into equations:

$$\frac{\cos^2\gamma_i + \cos\gamma_i}{\sin^2\gamma_i} = \frac{\cos^2\theta_i + \cos\phi_i}{\sin^2\theta_i} \tag{8}$$

$$\frac{\cos\alpha_i\cos\gamma_i + \cos\alpha_i}{\sin\alpha_i\sin\gamma_i} = \frac{\cos\theta_i\cos\phi_i + \cos\theta_i}{\sin\theta_i\sin\phi_i} \tag{9}$$

Now consider gluings of partially truncated model tetrahedra along lateral model hexagons of type $\mathbf{A}$. Note, that each lateral model hexagons of type $\mathbf{A}$ has external model edges with equal lengths, thus each gluing will produce only one equation. More precisely, the length condition for a gluing of two partially truncated model tetrahedra of types $\mathbf{A}$ and $\mathbf{AB_i}$ (resp., $\mathbf{AB_i}$ and $\mathbf{AB_j}$) translates into an equation (10) (resp., (11)), where $i, j \in \{1, \ldots, \mathtt{d}-1\}$ with $i \neq j$.

$$\frac{\cos^2\beta + \cos\beta}{\sin^2\beta} = \frac{\cos^2\alpha_i + \cos\gamma_i}{\sin^2\alpha_i} \tag{10}$$

$$\frac{\cos^2\alpha_i + \cos\gamma_i}{\sin^2\alpha_i} = \frac{\cos^2\alpha_j + \cos\gamma_j}{\sin^2\alpha_j} \tag{11}$$

7.3.2. *Cone angle condition.* Let us describe the cone angle condition for the internal edges of $\widehat{\mathcal{T}}^I$, which are denoted by $e_0^I, e_1^I, \ldots, e_{\mathtt{d}-1}^I$.

Firstly we consider an internal edge e_i^I for some $i \in \{1, \ldots, \mathtt{d}-1\}$. If $m_i > 0$ and $b_i > 0$ then $\mathcal{E}(e_i^I)$ is contained in partially truncated model tetrahedra of types $\mathbf{AB_i}$ and $\mathbf{B_i}$ (see remark 7.5). More precisely, each partially truncated model tetrahedron of type $\mathbf{AB_i}$ (resp., $\mathbf{B_i}$) has a dihedral angle γ_i (resp., θ_i) along all three (resp., four) pre-images of e_i^I. Hence the cone angle condition for e_i^I provides an equation:

$$3m_i\gamma_i + 4b_i\theta_i = 2\pi. \tag{12}$$

By Lemma 5.2 we have $m_i + b_i > 0$. If $m_i = 0$ or $b_i = 0$ then the corresponding term of equation (12) vanishes. Note, that if $m_i = 2$ and $b_i = 0$ for some $i \in \{1, \ldots, \mathtt{d}-1\}$ then $\mathcal{E}(e_i^I)$ is contained in exactly two partially truncated model tetrahedra of type $\mathbf{AB_i}$ which have ideal model vertices. By definition, the dihedral angles along the pre-images of e_i^I equal $\gamma_i = \pi/3$. Thus the cone angle condition for e_i^I holds: $3m_i\gamma_i = 6(\pi/3) = 2\pi$.

Now, we consider the cone angle condition for e_0^I. This time each partially truncated model tetrahedron of $\widehat{\mathcal{T}}^I$ contains pre-images of e_0^I (see remark 7.5). If all components of

$\mathcal{D}(\mathcal{T})$ are greater than 0, then the cone angle condition for e_0^I provides an equation:

$$\sum_{i=1}^{\mathtt{d}-1}(3m_i\alpha_i + 2b_i\phi_i) + 6a\beta = 2\pi. \tag{13}$$

If some components of $\mathcal{D}(\mathcal{T})$ equal to 0, then the corresponding terms in equation (13) vanish.

7.3.3. *Algebraic criteria for hyperbolicity.* It is time to combine all equations together. Note that all consistency conditions depends only on the combinatorics of $\widehat{\mathcal{T}}^I$, so it is convenient to consider six cases (see table 2). Define $\mathtt{m}(\mathcal{T}) := m_1 + \ldots + m_{\mathtt{d}-1}$, where $\{m_i\}_{i=1}^{\mathtt{d}-1}$ are the components of $\mathcal{D}(\mathcal{T})$. Recall, that a is a component of $\mathcal{D}(\mathcal{T})$ that equals to the number of model tetrahedra of type **A**.

Cases	a	$\mathtt{m}(\mathcal{T})$	$\mathtt{w}(\mathcal{T})$	$\mathtt{d}(\mathcal{T})$
1	0	0	>0	2
2	0	>0	0	2
3	0	>0	0	>2
4	>0	>0	0	$\geqslant 2$
5	0	>0	>0	$\geqslant 2$
6	>0	>0	>0	$\geqslant 2$

TABLE 2. Possible anatomies of $\mathcal{T}$.

In cases 1 and 2 we have $d = 2$ and all partially truncated model tetrahedra are of the same type ($\mathbf{B_1}$ or $\mathbf{AB_1}$). Thus the length condition is trivial.

In case 1 cone angle condition translates into system of equations:

$$\begin{cases} 4c_1\theta_1 = 2\pi \\ 2c_1\phi_1 = 2\pi \end{cases}$$

Clearly $(\theta_1;\phi_1) = (\frac{\pi}{2c_1};\frac{\pi}{c_1})$ is the unique solution of this system. Moreover, $\theta_1 < \pi/3$ and $\phi_1 \leqslant \pi/3$ since $c_\Delta(M) \geqslant 3$, thus M is actually hyperbolic without singularities.

In case 2 cone angle condition translates into system of equations:

$$\begin{cases} 3m_1\gamma_1 = 2\pi \\ 3m_1\alpha_1 = 2\pi \end{cases}$$

Clearly $(\gamma_1;\alpha_1) = (\frac{2\pi}{3m_1};\frac{2\pi}{3m_1})$ is the unique solution of this system. Moreover, $\gamma_1 = \alpha_1 < \pi/3$ since $c_\Delta(M) \geqslant 3$, thus M is actually hyperbolic without singularities.

For the rest of this section we will consider case 6 while cases 3-5 could be considered in the same way. Note that in case 6 we have $a > 0$ and $m_i > 0$ for all $i \in \{1,\ldots,\mathtt{d}-1\}$ due to Lemma 5.2. For the aim of simplicity we denote $\mathtt{w} = \mathtt{w}(\mathcal{T})$ tacitly implying that $\mathcal{T}$ is fixed. Using an edge relabelling if needed we suppose that $b_i > 0$ if and only if $i \in \{1,\ldots,\mathtt{w}\}$. Hence equations (8) and (9) occur only for $i \in \{1,\ldots,\mathtt{w}\}$.

The connectivity of $\widehat{\mathcal{T}}^I$ provides that the system of equations corresponding to the gluings of partially truncated model tetrahedra along lateral model hexagons of type **A** is equivalent to:

$$\frac{\cos^2\beta + \cos\beta}{\sin^2\beta} = \frac{\cos^2\alpha_1 + \cos\gamma_1}{\sin^2\alpha_1} = \ldots = \frac{\cos^2\alpha_{\mathtt{d}-1} + \cos\gamma_{\mathtt{d}-1}}{\sin^2\alpha_{\mathtt{d}-1}} \tag{14}$$

Geometrically this means that all external model edges of lateral model hexagons of type **A** has equal lengths and this hexagons are, actually, isometric to each other. We combine all equations together and obtain the following system:

$$\begin{cases}
\dfrac{\cos^2\gamma_i+\cos\gamma_i}{\sin^2\gamma_i}=\dfrac{\cos^2\theta_i+\cos\phi_i}{\sin^2\theta_i} \quad for\ i\leqslant \mathtt{w} & (15a)\\
\dfrac{\cos\alpha_i\cos\gamma_i+\cos\alpha_i}{\sin\alpha_i\sin\gamma_i}=\dfrac{\cos\theta_i\cos\phi_i+\cos\theta_i}{\sin\theta_i\sin\phi_i} \quad for\ i\leqslant \mathtt{w} & (15b)\\
\dfrac{\cos^2\beta+\cos\beta}{\sin^2\beta}=\dfrac{\cos^2\alpha_1+\cos\gamma_1}{\sin^2\alpha_1}=\ldots=\dfrac{\cos^2\alpha_{\mathtt{d}-1}+\cos\gamma_{\mathtt{d}-1}}{\sin^2\alpha_{\mathtt{d}-1}} & (15c)\\
3m_i\gamma_i+4b_i\theta_i=2\pi, \quad for\ i\leqslant \mathtt{w} & (15d)\\
3m_i\gamma_i=2\pi, \quad for\ \mathtt{w}<i<\mathtt{d} & (15e)\\
\sum\limits_{i<\mathtt{d}}(3m_i\alpha_i)+\sum\limits_{i\leqslant\mathtt{w}}(2b_i\phi_i)+6a\beta=2\pi & (15f)
\end{cases}$$

Recall, that i is a natural parameter. We call the solution of system (15) *admissible* if it satisfies: $\{\alpha_i\}_{i<\mathtt{d}}$, $\{\theta_i\}_{i\leqslant\mathtt{w}}$, $\beta\in(0,\pi/3)$ and $\{\gamma_i\}_{i<\mathtt{d}}$, $\{\phi_i\}_{i\leqslant\mathtt{w}}\in(0,\pi/3]$

Theorem 7.7. *System* (15) *has an admissible solution.*

Note, that property of the solution to be admissible guarantees the existence of geometric model tetrahedra we use. Hence this solution determines a hyperbolic structure on M.

7.4. **Completeness.** To check completeness of the hyperbolic structure just described we have to determine the similarity structure it induces on the boundary tori and Klein bottles. By construction, each torus (Klein bottle) in ∂M is tiled by two equilateral Euclidean triangles. This shows us that the structure on the boundary tori (Klein bottles) are indeed Euclidean, so the hyperbolic structure constructed in the previous paragraph is complete, and corresponds by Mostow‘s rigidity theorem to the unique complete finite-volume hyperbolic structure with geodesic boundary on the topological manifold M with the boundary tori and Klein bottles removed.

7.5. **Applications of hyperbolicity.**

7.5.1. *Matveev‘s complexity.* For any compact 3-manifold M, an $\mathbb{N}$-valued invariant $c(M)$ was defined by Matveev in [10] and called the *complexity* of M. Matveev also proved that, when M is hyperbolic, $c(M)$ equals to the $c_\Delta(M)$. Thus the following theorem holds.

Theorem 7.8. *Let M be a hyperbolic 3-manifold in $\mathcal{M}_h$. Then $c(M)=\beta_1(M,\mathbb{Z}_2)$.*

7.5.2. *Manifolds with same combinatorial data.* Note, that different 3-manifolds in $\mathcal{M}_e$ might have minimal ideal triangulation with the same combinatorial data. Due to Theorem 7.2 all 3-manifolds in $\mathcal{M}_e$, except finite number of 3-manifolds of small complexity, are hyperbolic with finite volume. Then the following Proposition holds.

Theorem 7.9. *Let M_1 and M_2 be two hyperbolic 3-manifolds in $\mathcal{M}_e$, which have minimal ideal triangulations with the same combinatorial data. Then M_1 and M_2 have equal volumes.*

Proof. Recall that hyperbolic structure on M_1 and M_2 are constructed using the partially truncated triangulations. Clearly, this partially truncated triangulations are glued from equal number of partially truncated model tetrahedra of each type. Note, that the geometric realisations of partially truncated model tetrahedra and the consistency conditions depend on the combinatorial data only. Due to the Mostow rigidity theorem consistency conditions has a unique solutions. Thus the hyperbolic manifolds M_1 and M_2 are glued from equal sets of geometric tetrahedra. This finishes the proof. $\square$

8. Proof of Theorem 7.7

In this section we work with system (15), so we recall that $\{\alpha_i,\ \gamma_i\}_{i<d}$, $\{\theta_i,\ \phi_i\}_{i\leqslant w}$ and β are the variables of the system (15) and a, $\{m_i, b_i\}_{i<d}$ are positive integers, which depends on the 3-manifold we are working with. Clearly $\gamma_i = \frac{2\pi}{3m_i}$ is a unique solution of equation (15f) for each $w < i < d$. We consider a one-parameter system with parameter $K \geqslant 0$:

$$
\begin{cases}
\dfrac{\sin^2\gamma_i}{1+\cos\gamma_i} = \dfrac{\sin^2\theta_i}{1+\cos\phi_i} \quad \mathit{for}\ i \leqslant w & \text{(16a)}\\
\dfrac{\cos\alpha_i\cos\gamma_i + \cos\alpha_i}{\sin\alpha_i\sin\gamma_i} = \dfrac{\cos\theta_i\cos\phi_i + \cos\theta_i}{\sin\theta_i\sin\phi_i} \quad \mathit{for}\ i \leqslant w & \text{(16b)}\\
\dfrac{\sin^2\alpha_i}{1+\cos\gamma_i} = K \quad \mathit{for}\ i \leqslant w & \text{(16c)}\\
3m_i\gamma_i + 4b_i\theta_i = 2\pi, \quad \mathit{for}\ i \leqslant w & \text{(16d)}\\
\dfrac{\sin^2\alpha_i}{1+\cos\frac{2\pi}{3m_i}} = K \quad \mathit{for}\ w < i < d & \text{(16e)}\\
\dfrac{\sin^2\beta}{1+\cos\beta} = K & \text{(16f)}
\end{cases}
$$

We study how solution of system (16) depends on the parameter K.

Definition 8.1. We call the solution of system (16) *almost admissible* if it satisfies: $\{\alpha_i\}_{i<d}$, $\{\theta_i,\ \phi_i,\ \gamma_i\}_{i\leqslant w}$ and $\beta \in (0, \pi/3]$.

Proposition 8.2. *There exists a unique $\widetilde{K} > 0$ (which we will call a* critical value *of parameter K) such that:*

(1) $\forall\ K \in [0, \widetilde{K}]$ system (16) has a unique almost admissible solution, that depends on K continuously. Denote these functions as $\{\alpha_i(K)\}_{i<d}$, $\{\theta_i(K),\ \phi_i(K),\ \gamma_i(K)\}_{i\leqslant w}$ and $\beta(K)$.
(2) $\{\theta_i(K),\ \gamma_i(K)\}_{i\leqslant w}$ take values in $(0, \pi/4]$ for all $K \in [0, \widetilde{K}]$.
(3) $\{\alpha_i(K)\}_{i<d}$, $\{\phi_i(K)\}_{i\leqslant w}$ and $\beta(K)$ are strictly increasing functions with respect to K which equal zero at $K = 0$.
(4) Functions $\{\alpha_i(K)\}_{i<d}$, $\{\phi_i(K)\}_{i\leqslant w}$ and $\beta(K)$ take values in $(0, \pi/3)$ for all $K < \widetilde{K}$. Moreover at least one of these functions reaches the value of $\pi/3$ at $K = \widetilde{K}$.

Let $\widetilde{K}$ be as in Proposition 8.2. Fix $\bar{K} < \widetilde{K}$. Then $\{\alpha_i(\bar{K})\}_{i<d}$, $\{\theta_i(\bar{K}),\ \phi_i(\bar{K}),\ \gamma_i(\bar{K})\}_{i\leqslant w}$ and $\{\gamma_i = \frac{2\pi}{3m_i}\}_{w<i<d}$ appear to be an admissible solution of system (15) if and only if the equation (15f) holds. Define a function of K with domain $[0, \widetilde{K}]$:

$$F(K) = \sum_{i<d} 3m_i\alpha_i(K) + \sum_{i\leqslant w} 2b_i\phi_i(K) + 6a\beta(K).$$

Clearly $F(K)$ is continuous and equation (15f) translates into equality $F(\bar{K}) = 2\pi$.

Proposition 8.3. *Let $\widetilde{K}$ be as in Proposition 8.2. Then there exists a unique $\bar{K} < \widetilde{K}$ such that $F(\bar{K}) = 2\pi$.*

Hence an admissible solution of system (15) exists. Summing up, Propositions 8.2 and 8.3 together give a proof of Theorem 7.7.

8.1. **Proof of Proposition 8.2.** Our goal is to evaluate each variable of system 16 as a function of parameter K and find the value of $\widetilde{K}$ explicitly. We will use the following variation of the inverse function theorem several times.

Lemma 8.4 (Theorem 3.10 in [13])**.** *Assume G is strictly increasing and continuous on an interval $[p, q]$. Let $r = G(p)$ and $s = G(q)$ and let H be the inverse of G. That is, for each y in $[r, s]$, let $H(y)$ be that x in $[p, q]$ such that $y = G(x)$. Then*

- *H is strictly increasing on $[r, s]$;*
- *H is continuous on $[r, s]$.*

Note, that system (16) breaks into independent sub-systems consisting of:

- a single equation (16f);
- a single equation (16e) for some $i \in \mathbb{N}$ such that $w < i < d$;
- equations (16a), (16b), (16c), and (16d) for some $i \in \mathbb{N}$ such that $i \leqslant w$.

To each sub-system we match it‘s own critical value of parameter K such that all the statements of Proposition 8.2 corresponding to the solution of a given sub-system hold. Then we define $\widetilde{K}$ as the minimal critical value among critical values of sub-systems of system (16). Clearly, Proposition 8.2 holds for such $\widetilde{K}$.

Firstly we consider equation (16f). Applying Lemma 8.4 to function $G_0(\beta) := \frac{\sin^2 \beta}{1+\cos \beta}$ defined on an interval $[0, \pi/3]$ we obtain that there exists a continuous and strictly increasing function $\beta(K)$ defined on an interval $[0, \widetilde{K}_0]$, where $\widetilde{K}_0 := G_0(\pi/3)$ is a critical value for given sub-system.

Now we fix $i \in \mathbb{N}$ such that $w < i < d$ and consider equation (16e) for given i. Applying Lemma 8.4 to function $G_i(\alpha_i) := \frac{\sin^2 \alpha_i}{1+\cos((2\pi)/(3m_i))}$ defined on an interval $[0, \pi/3]$ we obtain that there exists a continuous and strictly increasing function $\alpha_i(K)$ defined on an interval $[0, \widetilde{K}_i]$, there $\widetilde{K}_i := G_i(\pi/3)$ is a critical value for given sub-system.

Now we fix $i \in \mathbb{N}$ such that $i \leqslant w$ and consider equations (16a), (16b), (16c) and (16d) for given i. We try to evaluate α_i, γ_i, θ_i, and ϕ_i as continuous functions of K as follows.

1. From equations (16a) and (16d) we evaluate θ_i and γ_i as continuous functions of an argument ϕ_i. Denote this functions $\widehat{\theta}_i(\phi_i)$ and $\widehat{\gamma}_i(\phi_i)$.
2. From equation (16b) with θ_i and γ_i considered as a continuous functions on ϕ_i we evaluate α_i as a continuous function of an argument ϕ_i. We denote this function $\widehat{\alpha}_i(K)$ and show that it is strictly increasing with respect to ϕ_i.
3. Consider equation (16c) with γ_i and α_i considered as a continuous functions of an argument ϕ_i, and use Lemma 8.4 to obtain function $\phi_i(K)$.
4. Finally, define functions $\alpha_i(K) := \widehat{\alpha}_i(\phi_i(K))$, $\theta_i(K) := \widehat{\theta}_i(\phi_i(K))$, $\gamma_i(K) := \widehat{\gamma}_i(\phi_i(K))$ and the critical value $\widetilde{K}_i$.

We begin by considering equations (16a) and (16d) with ϕ_i considered as a parameter which takes values in $[0, \pi/3]$.

Lemma 8.5. *For each $\phi_i \in [0, \pi/3]$ there exist unique θ_i and γ_i in $(0, \pi/4]$ such that equations* (16a) *and* (16d) *hold.*

Proof. Fix $\phi_i \in [0, \pi/3]$. All functions appearing in the proof depend on ϕ_i but we will omit this dependence since ϕ_i supposed to be fixed. Define:

$$N_i := \frac{\sin^2 \gamma_i}{1 + \cos \gamma_i} = \frac{\sin^2 \theta_i}{1 + \cos \phi_i} \tag{17}$$

We apply Lemma 8.4 to functions $f_i(\theta_i) = \frac{\sin^2 \theta_i}{1+\cos \phi_i}$ and $g_i(\gamma_i) = \frac{\sin^2 \gamma_i}{1+\cos \gamma_i}$ defined on an interval $[0, \pi/4]$ and obtain continuous strictly increasing functions $\theta_i^\star(N_i)$ and $\gamma_i^\star(N_i)$. Note that domain of these functions depends on ϕ_i, but they always contain an interval $[0, 0.25]$.

Define:

$$S_i(N_i) := 3m_i\theta_i^\star(N_i) + 4b_i\gamma_i^\star(N_i) \tag{18}$$

Clearly $S_i(N_i)$ is continuous and strictly increasing function with respect to N_i as a linear combination of continuous and strictly increasing functions $\theta_i^\star(N_i)$ and $\gamma_i^\star(N_i)$ with positive coefficients. Moreover $S_i(0) = 0$. We claim, that there exists $\bar{N}_i \in [0, 0.25]$ such that $S_i(\bar{N}_i) = 2\pi$. Thus equations (16a) and (16d) hold for $\theta_i = \theta_i^\star(\bar{N}_i)$ and $\gamma_i = \gamma_i^\star(\bar{N}_i)$.

To prove the claim we apply the Bolzano‘s theorem of an intermediate value to the function $S_i(N_i)$ defined on an interval $[0, 0.25]$. It remains to check that $S_i(0.25) > 2\pi$. Direct computations provides that:

- $\gamma_i^\star(0.25) = \arccos 0.75 \in (0.23\pi, 0.25\pi)$,
- $\theta_i^\star(0.25) = \arcsin \sqrt{0.25(1 + \cos \phi_i)} \in (0.209\pi, 0.25\pi)$ for any $\phi_i \in [0, \pi/3]$.

From lemma 5.2 we have $m_i \geqslant 2$ and $b_i \geqslant 1$ since $i \leqslant w$. Thus $S_i(0.25) > 2\pi$. Finally θ_i and γ_i are unique for given $\phi_i \in [0, \pi/3]$ since function $S_i(N_i)$ is monotone. □

Lemma 8.5 allows us to define functions $\widehat{\theta}_i(\phi_i)$ and $\widehat{\gamma}_i(\phi_i)$ with domain $[0, \pi/3]$ that turn equations (16a) and (16d) into tautology.

Lemma 8.6. *Function $\widehat{\gamma}_i(\phi_i)$ (resp., $\widehat{\theta}_i(\phi_i)$) is is continuous and strictly increasing (resp., decreasing) with respect to ϕ_i.*

Proof. Function $\widehat{\gamma}_i(\phi_i)$ and $\widehat{\theta}_i(\phi_i)$ are analytic on the interval $(0, \pi/3)$ by the analytic implicit function theorem, thus they are continuous.

Let us show that for arbitrary $0 \leqslant a < b \leqslant \pi/3$ we have $\widehat{\gamma}_i(a) < \widehat{\gamma}_i(b)$. On the contrary, suppose $\widehat{\gamma}_i(a) \geqslant \widehat{\gamma}_i(b)$. Recall, that functions $\widehat{\theta}_i(\phi_i)$ and $\widehat{\gamma}_i(\phi_i)$ turn equations (16a) and (16d) into tautology. Thus $\widehat{\theta}_i(a) \leqslant \widehat{\theta}_i(b)$ and we have:

$$\frac{\sin^2 \widehat{\gamma}_i(a)}{1 + \cos \widehat{\gamma}_i(a)} = \frac{\sin^2 \widehat{\theta}_i(a)}{1 + \cos a} < \frac{\sin^2 \widehat{\theta}_i(b)}{1 + \cos b} = \frac{\sin^2 \widehat{\gamma}_i(b)}{1 + \cos \widehat{\gamma}_i(b)}$$

But from the assumption we have $\frac{\sin^2 \widehat{\gamma}_i(a)}{1+\cos \widehat{\gamma}_i(a)} \geqslant \frac{\sin^2 \widehat{\gamma}_i(b)}{1+\cos \widehat{\gamma}_i(b)}$. Contradiction. Thus $\widehat{\gamma}_i(\phi_i)$ is strictly increasing with respect to ϕ_i. And equation (16d) provides that $\widehat{\theta}_i(\phi_i)$ is strictly decreasing with respect to ϕ_i. □

We rewrite equation (16b) as follows:

$$\cot(\alpha_i)\frac{1 + \cos \gamma_i}{\sin \gamma_i} = \frac{\cos \theta_i}{\sin \theta_i}\frac{1 + \cos \phi_i}{\sin \phi_i}$$

Using equation (16a) and substituting functions $\widehat{\gamma}_i(\phi_i)$ $\widehat{\theta}_i(\phi_i)$ we evaluate α_i as a continuous function of ϕ_i defined on an interval $[0, \pi/3]$:

$$\widehat{\alpha}_i(\phi_i) = \cot^{-1}\left(\frac{\sin 2\widehat{\theta}_i(\phi_i)}{2\sin\phi_i \sin\widehat{\gamma}_i(\phi_i)}\right) \tag{19}$$

Lemma 8.7. *$\widehat{\alpha}_i(\phi_i)$ is a strictly increasing function with respect to ϕ_i, and $\widehat{\alpha}_i(0) = 0$.*

Proof. By Lemma 8.6, $\widehat{\gamma}_i(\phi_i)$ is strictly increasing and $\widehat{\theta}_i(\phi_i)$ is strictly decreasing with respect to ϕ_i. Moreover, they take values in $(0, \pi/4]$. Thus $\sin\widehat{\gamma}_i(\phi_i)$ (resp., $\sin 2\widehat{\theta}_i(\phi_i)$) is a strictly increasing (resp., decreasing) with respect to ϕ_i. This finishes the proof. □

Consider equation (16c) with α_i and γ_i considered as functions of ϕ_i:

$$G_i(\phi_i) := \frac{\sin^2\widehat{\alpha}_i(\phi_i)}{1+\cos\widehat{\gamma}_i(\phi_i)} = K$$

Due to Lemmas 8.6 and 8.7 function $G_i(\phi_i)$ is continuous and strictly increasing with respect to ϕ_i. Again applying Lemma 8.4 to function $G_i(\phi_i)$ defined on an interval $[0, \pi/3]$ we obtain continuous and strictly increasing function $\phi_i(K)$ defined on an interval $[0, \widetilde{K}_i^{(1)}]$, where $\widetilde{K}_i^{(1)} = G_i(\pi/3)$. Define:

$$\alpha_i(K) := \widehat{\alpha}_i(\phi_i(K)),\ \theta_i(K) := \widehat{\theta}_i(\phi_i(K)),\ \text{ and } \gamma_i(K) := \widehat{\gamma}_i(\phi_i(K)).$$

Clearly these are continuous functions of K defined on an interval $[0, \widetilde{K}_i^{(1)}]$. Moreover, $\alpha_i(K)$ is a strictly increasing with respect to K and $\alpha_i(0) = 0$ due to Lemma 8.7. Note, that $\widetilde{K}_i^{(1)}$ may not satisfy all the properties of the critical value from Proposition 8.2 for sub-system of equations (16a), (16b), (16c), and (16d) for given i. Lemma 8.5 provides that $\theta_i(K)$ and $\gamma_i(K)$ take values in $(0, \pi/4]$, while $\alpha_i(K)$ may take values greater than $\pi/3$. If $\alpha_i(\widetilde{K}_i^{(1)}) > \pi/3$ then there exists $\widetilde{K}_i$ such that $\alpha_i(K) \leqslant \pi/3$ for all $K \in [0, \widetilde{K}_i]$ and $\alpha_i(\widetilde{K}_i) = \pi/3$. Otherwise define the critical value $\widetilde{K}_i := \widetilde{K}_i^{(1)}$.

Finally we define critical value $\widetilde{K}$ as follows:

$$\widetilde{K} := \min_{0\leqslant i\leqslant \mathtt{d}-1} \widetilde{K}_i$$

It is not hard to check, that Proposition 8.2 holds for such $\widetilde{K}$.

8.2. **Proof of Proposition 8.3.** Note, that $F(K)$ is a strictly increasing function with respect to K as a linear combination of strictly increasing functions with positive coefficients. Moreover $F(0) = 0$.

To prove the proposition we apply the Bolzano‘s theorem of an intermediate value to the function $F(K)$ defined on an interval $[0, \widetilde{K}]$. It remains to check that $F(\widetilde{K}) > 2\pi$. We finish the proof with two following lemmas.

Lemma 8.8. *If $\phi_i(\widetilde{K}) = \pi/3$ for some $i \in \{1, \ldots, w\}$, then $\alpha_i(\widetilde{K}) > \pi/4$.*

Proof. From equation (16a) with $\phi_i = \pi/3$ we obtain:

$$\sin(\theta_i) = \sqrt{\frac{3}{2}(1-\cos\gamma_i)} \quad \text{and} \quad \cos(\theta_i) = \sqrt{\frac{3}{2}\cos\gamma_i - \frac{1}{2}}$$

Hence the equation (19) with $\phi_i = \pi/3$ could be replaced with:

$$\widehat{\alpha}_i(\phi_i) = \cot^{-1}\Big(\frac{\sqrt{(1-\cos\gamma_i)(3\cos\gamma_i - 1)}}{\sin\gamma_i}\Big)$$

By direct computations we have: $\alpha_i > \pi/4$ for all $\gamma_i \in [0, \pi/3]$. □

Lemma 8.9. $F(\widetilde{K}) > 2\pi$.

Proof. We would strongly rely on the fourth item of Proposition 8.2 in the proof. We know, that at least one of functions $\{\alpha_i(K)\}_{i<d}$, $\{\phi_i(K)\}_{i\leqslant w}$ or $\beta(K)$ reaches the value of $\pi/3$ at $K = \widetilde{K}$. So we consider all the possible cases. Note that by assumption we have $a > 0$, $m_i \geqslant 2$ for each $i \in \{1, \ldots, \mathtt{d}-1\}$ and $b_i > 0$ if and only if $1 \leqslant i \leqslant w$, where $w, d \in \mathbb{N}$ with $1 \leqslant w \leqslant d$.

- If $\beta(\widetilde{K}) = \pi/3$, then:
$$F(\widetilde{K}) \geqslant 3m_1\alpha_1(\widetilde{K}) + 6a\beta(\widetilde{K}) \geqslant 6\alpha_1(\widetilde{K}) + 6\pi/3 > 2\pi$$
- If $\alpha_i(\widetilde{K}) = \pi/3$ for some $i \in \{1, \ldots, \mathtt{d}-1\}$, then:
$$F(\widetilde{K}) \geqslant 3m_i\alpha_i(\widetilde{K}) + 6a\beta(\widetilde{K}) \geqslant 6\pi/3 + 6\beta(\widetilde{K}) > 2\pi$$
- Otherwise $\phi_i(\widetilde{K}) = \pi/3$ for some $i \in \{1, \ldots, w\}$. Then by Lemma 8.8 we have:
$$F(\widetilde{K}) \geqslant 3m_i\alpha_i(\widetilde{K}) + 2b_i\phi_i(\widetilde{K}) \geqslant 6\pi/4 + 2\pi/3 > 2\pi$$

So we have $F(\widetilde{K}) > 2\pi$ and Lemma 8.9 is proved. □

Remark 8.10. In the proof of Lemma 8.9 we use some restrictions on the combinatorial data: $a > 0$, $m_i \geqslant 2$ for each $i \in \{1, \ldots, \mathtt{d}-1\}$ and $b_i > 0$ if and only if $1 \leqslant i \leqslant w$. This restrictions do not hold in general case. However one can prove that $F(\widetilde{K}) > 2\pi$ without this restrictions only using Lemma 5.2.

9. Examples

Due to Theorem 7.2 and Remark 7.1, almost all manifolds in $\mathcal{M}_h$, except finite number of manifolds of small complexity, are hyperbolic. In this section we give a full list of manifolds in $\mathcal{M}_h$, which do not satisfy the conditions of Theorem 7.2 and Remark 7.1 (Table 3). Some manifolds in the list are actually cusped hyperbolic manifolds, which were tabulated in [15], the others are included in computer programs "SnapPy" [16] and "3-Manifold Recognizer" [17].

M	$c_\Delta(M)$	Class	$\chi(M)$	$H_1(M,\mathbb{Z})$	Spine
m000	1	$\mathcal{M}_o^1$	0	$\mathbb{Z}$	Figure 3a
$L(4,1) \setminus B^3$	1	$\mathcal{M}_e^2$	1	$\mathbb{Z}_4$	Figure 3b
Seifert (S^2, (2,1), (2,1), (2,1), (1,-1))	2	$\mathcal{M}_o^2$	1	$\mathbb{Z}_2 + \mathbb{Z}_2$	Figure 3c
m002	2	$\mathcal{M}_e^1$	0	$\mathbb{Z}_2 + \mathbb{Z}$	Figure 3d
m001	2	$\mathcal{M}_e^2$	0	$\mathbb{Z}_2 + \mathbb{Z}$	Figure 3e
m025	3	$\mathcal{M}_o^2$	0	$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}$	Figure 3f

Table 3. List of exceptions.

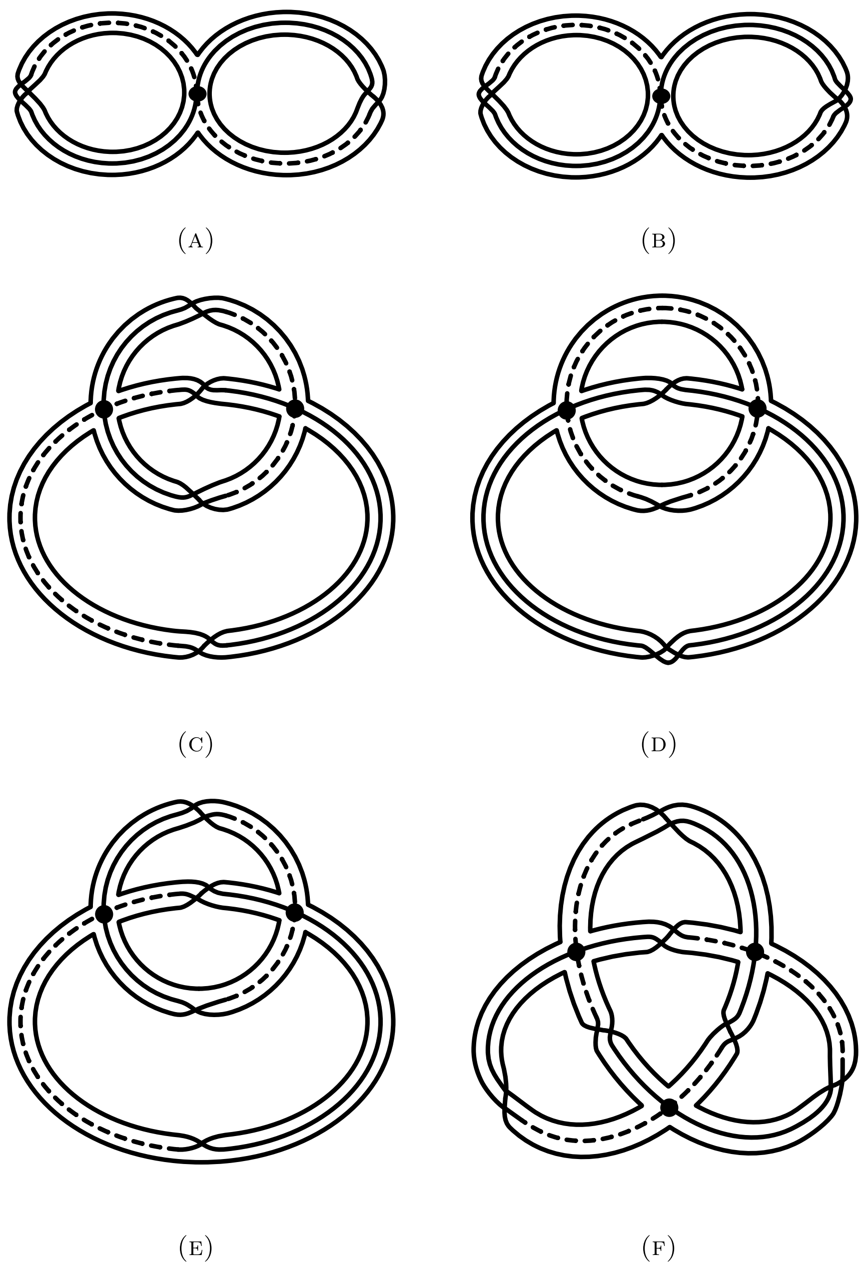


Figure 3. Special spines of non-hyperbolic manifolds in $\mathcal{M}_h$.

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