Санкт-Петербургский государственный университет

***МАМАЕВ Даниил Антонович***

**Выпускная квалификационная работа**

# ***«On Morse index retrieval»***

| | |
|---|---|
| Уровень образования: | магистратура |
| Направление | *01.04.01 «Математика»* |
| Основная образовательная программа | *ВМ.5832.2019 «Современная математика»* |

Научный руководитель:
**д.ф-м.н., профессор МКН СПбГУ**
**Панина Гаянэ Юрьевна**

Рецензент:
**Priv.-Doz., lect. in maths at U of Liverpool**
**Карпенков Олег Николаевич**

Санкт-Петербург
2021

# ON MORSE INDEX RETRIEVAL

DANIIL MAMAEV

*Chebyshev Laboratory, St. Petersburg State University, 14th Line V.O., 29, Saint Petersburg 199178 Russia*

Abstract. A smooth function $f$ in a neighbourhood of the unit sphere $S^{n-1}$ is said to be extendible with index $\lambda$ if it can be extended to a function $F$ in the unit ball $B^n$ such that $F$ has a unique critical point $p$ and the Morse index of $p$ is equal to $\lambda$. It is easy to see that a function $f$ cannot be extendible with both index $\lambda$ and $\mu$ if one of them is odd and the other is even. We prove that on the level of Morse-Barannikov complexes there are no obstructions to the existence of $f$ that is extendible with all possible indices of the same parity. We also prove that for any two indices that differ by two there indeed exists a function $f$ extendible with either of the indices.

Contents



*E-mail address*: dan.mamaev@gmail.com.

*Date*: May 30, 2021(Last Typeset).

## 1. Introduction

Consider a smooth function $f$ in a neighbourhood of the unit sphere $S^{n-1}$ inside the unit ball $B^n_{\mathbf{0}}(1)$. Its smooth extensions inside the unit ball generically have only non-degenerate critical points. In [1] S.Barannikov, following a question by V.I. Arnold, gave a lower bound for the number of critical points of such extensions making use of what later became known as Morse-Barannikov complexes. In the present paper we consider a related problem.

**Problem 1.1.** *Suppose we are given a smooth function $f$ in a neighbourhood $U$ of the unit sphere $S^{n-1}$ inside the unit ball $B^n_{\mathbf{0}}(1)$. Let $F\colon B^n_{\mathbf{0}}(1) \to \mathbb{R}$ be a smooth extension of $f$ such that the origin $\mathbf{0}$ is the unique critical point of $F$ and $\operatorname{Hess}_{\mathbf{0}} F$ is non-degenerate. What information about the Morse index $\mu_F(\mathbf{0})$ can be retrieved from $f$?*

We say that a function $f$ defined in a neighbourhood $U$ of the unit sphere $S^{n-1}$ inside the unit ball $B^n_{\mathbf{0}}(1)$ *admits* index $\lambda$ if there exists a smooth extension $F\colon B^n_{\mathbf{0}}(1) \to \mathbb{R}$ of $f$ such that the origin is the unique critical point of $F$ and its Morse index is equal to $\lambda$. There are cases when a function admits only one index. For instance, if $\operatorname{grad} f|_{S^{n-1}}$ always points inside the ball, then $\mathbf{0}$ must be the point of global maximum of $F$ hence $\mu_F(\mathbf{0}) = n$. In general, the parity of $\mu_F(\mathbf{0})$ can be retrieved from $f$ (see Proposition 4.1).

The main results of the paper are formulated in Proposition 4.6 and Theorem 6.1. In Proposition 4.6 we prove that *on the level of Morse-Barannikov complex there are no obstructions to the existence of a function $f$ that admits all the indices of the same parity.* In Theorem 6.1 we prove that *for any $0 \le \lambda \le n-2$ there exists a function $f$ define in a neighbourhood of $S^{n-1}$ that admits both index $\lambda$ and $\lambda+2$.*

We firmly believe that the following statement must hold.

**Conjecture 1.2.** *For any $n \ge 2$ there exist functions $f_0$ and $f_1$ in a neighbourhood of $S^{n-1}$ such that $f_0$ admits indices $0, 2, \dots, 2\cdot[n/2]$ and $f_1$ admits indices $1, 3, \dots, 2\cdot[(n+1)/2]-1$.*

Theorem 6.1 settles the conjecture in dimensions 2 and 3; the fact that the proof of the theorem is constructive indicates that the usual artefacts of higher dimensions such as exotic smooth structures should not come into our way.

The paper is organised as follows. In Section 2 we briefly review the basics of Morse and Cerf's theories, the goal of this section is mainly to fix the notation. In Section 3 we define the Morse-Barannikov complex (our definition differs slightly from the original one) and restate the results of Cerf's theory in the terms of that complex. In Section 4 we apply the Morse-Barannikov complex to Problem 1.1. Namely, in Proposition 4.1 we prove that the parity of $\mu_F(\mathbf{0})$ can be retrieved from $f$, in Proposition4.5 we construct Morse-Barannikov complexes admitting two indices that differ by two, and in Proposition 4.6 we construct Morse-Barannikov complexes that admit all possible indices of the same parity.

Sections 5 and 6 are devoted to the proof of Theorem 6.1. In Section 5 we develop a number of tools to perform the metamorphoses of functions in a controllable manner. In Section 6 we first explain the two-dimensional case and then prove the theorem.

In Sections 5 and 6 we do not use the Morse-Barannikov complexes directly, so the content of Sections 3 and 4 is not strictly necessary to understand the last two sections.

**Acknowledgement.** I am deeply indebted to Gaiane Panina for posing the problem and supervising my research. I am also grateful to Serguei Barannikov for useful comments. The first known to me example of a function $f$ in dimension 2 which is extendible with the index 0 as well as with the index 2 was constructed by Semën Podkorytov (private communication).

## 2. Preliminaries: Morse and Cerf's theories

In this section we mainly follow two books by Milnor [4], [5] and a more modern exposition by Nicolaescu [6].

Throughout the text smooth means $C^\infty$, a manifold means a manifold with boundary, and a closed manifold means a compact manifold without boundary. Let $M^n$ be a smooth $n$-dimensional manifold with the boundary $\partial M$. For a boundary point $p \in \partial M$ the tangent space $T_p\partial M$ divides the tangent space $T_pM$ into two semi-spaces that consist of the vectors pointing inside or outside the manifold; these (open) semi-spaces are denoted by $T_p^{in}M$ and $T_p^{out}M$ respectively.

For a smooth fibre bundle $E \to M$ and a subset $X \subseteq M$ we denote by $C^\infty(X, E)$ the space of smooth sections of $E$ over $X$. That is, the space of (global) smooth vector fields on $M$ is $C^\infty(M, TM)$, the space of smooth real-valued functions $C^\infty(M, M \times \mathbb{R})$ is abbreviated to $C^\infty(M)$.

Given a vector field $V \in C^\infty(M, TM)$ and a smooth function $f \in C^\infty(M)$ the *derivative of $f$ along $V$* is the function $Vf \in C^\infty(M)$ given by $Vf(p) = d_pf(V(p))$ for any $p \in M$.

2.1. **Morse functions.** Let $f\colon M \to \mathbb{R}$ be a smooth function. A point $p \in M$ is called a *critical point* of $f$ if the differential $d_pf\colon T_pM \to \mathbb{R}$ vanishes, otherwise $p$ is called a *regular point* of $f$. The point $p$ is critical if and only if in (any hence all) local coordinates $(x^1, \dots, x^n)$ around $p$ the partial derivatives $\frac{\partial f}{\partial x^i}$ vanish at $p$. $\operatorname{Crit} f$ denotes the set of all critical points of $f$.

Let $p$ be a critical point of the function $f$. The *Hessian form* of $f$ at $p$ is a symmetric bilinear form $\operatorname{Hess}_p f\colon T_pM \times T_pM \to \mathbb{R}$ defined on a pair of tangent vectors $u, v \in T_pM$ by

$$\operatorname{Hess}_p f(u, v) = \big(U(Vf)\big)(p),$$

where $U$ and $V$ are arbitrary extensions of $u$ and $v$ to local vector fields around $p$. In local coordinates $(x^1, \dots, x^n)$ around $p$ one has

$$\operatorname{Hess}_p f\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial^2 f}{\partial x^i \partial x^j}(p).$$

The $n \times n$ matrix $H_pf$ with entries $(H_pf)_{i,j} = \frac{\partial^2 f}{\partial x^i \partial x^j}(p)$ is called the *Hessian matrix* of $f$ at $p$ in the local coordinates $(x^1, \dots, x^n)$.

A point $p \in \operatorname{Crit} f$ is called a *degenerate* critical point of the function $f$ if the Hessian form $\operatorname{Hess}_p f$ is degenerate (that is, there exists a vector $v \in T_pM$ such that $\operatorname{Hess}_p f(v, \_)\colon T_pM \to \mathbb{R}$ vanishes). Otherwise $p$ is called a *non-degenerate* (or *Morse*) critical point of the function $f$. The point $p$ is degenerate if and only if in local coordinates

$(x^1, \dots, x^n)$ around $p$ the Hessian matrix $H_p f$ has a zero eigenvalue. $\mathrm{Morse}\, f$ denotes the set of all Morse critical points of $f$.

For a point $p \in \mathrm{Morse}\, f$ its *Morse index* $\mu_f(p)$ is the maximal dimension of a subspace of $T_pM$ on which the Hessian form $\mathrm{Hess}_p\, f$ is negative definite, that is,

$$\mu_f(p) = \max\left\{\dim V \colon V \leq T_pM \text{ and } \forall v \in V \setminus \{0\} \;\; \mathrm{Hess}_p\, f(v,v) < 0\right\}.$$

The Morse index $\mu_f(p)$ is equal to the number of negative eigenvalues of the Hessian matrix $H_p f$ in local coordinates $(x^1, \dots, x^n)$ around $p$.

**Definition 2.1.** A smooth function $f \colon M \to \mathbb{R}$ on the smooth manifold $M$ with the boundary $\partial M$ is called a *Morse function* if

(1) $\mathrm{Crit}\, f \subset M \setminus \partial M$, that is, there are no critical points of $f$ in the boundary $\partial M$,
(2) $\mathrm{Morse}\, f = \mathrm{Crit}\, f$, that is, the critical points of $f$ are non-degenerate, and
(3) $\mathrm{Morse}\, f|_{\partial M} = \mathrm{Crit}\, f|_{\partial M}$, that is, the critical points of the restriction of $f$ to the boundary $\partial M$ are non-degenerate.

2.2. **Flow lines.**

**Definition 2.2.** Let $f$ be a Morse function on a smooth manifold $M$.

(1) A local coordinate system $(x_1, \dots, x_n)$ in a neighbourhood $U_p$ of a point $p \in \mathrm{Crit}\, f$ is *adapted to* $f$ if
$$f = f(p) - \left(x^1\right)^2 - \ldots - \left(x^\lambda\right)^2 + \left(x^{\lambda+1}\right)^2 + \ldots + (x^n)^2 \text{ in } U_p$$
(2) A vector field $V \in C^\infty(M, TM)$ is called a *gradient-like vector field for* $f$ if $Vf(p) > 0$ for all non-critical $p \in M$, and for any critical point $p$ of $f$ there exists a neighbourhood $U_p$ of $p$ and a coordinate system $(x^1, \dots, x^n)$ in $U_p$ adapted to $f$ such that $V/2 = -x^1 \frac{\partial}{\partial x^1} - \ldots - x^\lambda \frac{\partial}{\partial x^\lambda} + x^{\lambda+1} \frac{\partial}{\partial x^{\lambda+1}} + \ldots + x^n \frac{\partial}{\partial x^n}$ on $U_p$.
(3) A Riemannian metric $g \in C^\infty(M, S^2T^*M)$ is *adapted to* $f$ if for any critical point $p$ of $f$ there exists a neighbourhood $U_p$ of $p$ and a coordinate system $(x^1, \dots, x^n)$ in $U_p$ adapted to $f$ such that $g = \left(dx^1\right)^2 + \ldots + (dx^n)^2$ on $U_p$.

**Lemma 2.3** (Morse)**.** *Let $f$ be a smooth function on a smooth manifold $M$ and $p$ be a non-degenerate critical point of $f$. Then there exists a neighbourhood $U$ of $p$ and local coordinates $(x_1, \dots, x_n)$ on $U$ adapted to $f$.*

*Remark* 2.4. Let $f$ be a Morse function on a smooth manifold $M$ with boundary $\partial M$. Riemannian metrics on $M$ adapted to $f$ and gradient-like vector fields for $f$ are closely related. Namely,

(1) Given a Riemannian metric $g$ on $M$ adapted to $f$ (their existence is easily deduced by a partition of unity argument), the vector field $V = \mathrm{grad}_g\, f$ is called the *gradient-like vector field for $f$ associated to $g$*.
(2) Given a gradient-like vector field $V$ for $f$ one can define a Riemannian metric $g$ on $M$ adapted to $f$ such that $\mathrm{grad}_g\, f = V$ in the following fashion. Fix neighbourhoods $U_p$ from the definition of the gradient-like vector field. Define Riemannian metrics $g_p \in C^\infty(U_p, S^2T^*M)$ by $g_p = \left(dx^1\right)^2 + \ldots + (dx^n)^2$. Take a positive definite

bilinear form $\widetilde{g} \in C^\infty(M \setminus \operatorname{Crit} f, S^2(\ker df)^*)$ and define a Riemannian metric $g_{reg} \in C^\infty(M \setminus \operatorname{Crit} f, S^2T^*M)$ by

$$g_{reg}(u + aV(x), v + bV(x)) = \widetilde{g}(u, v) + ab \cdot (Vf)(x) \text{ for } u, v \in \ker d_x f \text{ and } a, b \in \mathbb{R}.$$

The metric $g$ is obtained from $g_p$ and $g_{reg}$ using a partition of unity subordinate to the open cover $\{U_p\}_{p\in\operatorname{Crit} f} \cup \{M \setminus \operatorname{Crit} f\}$ is the one we need.

Now let $M$ be a closed manifold, $f\colon M \to \mathbb{R}$ be a Morse function and $V \in C^\infty(M, TM)$ be a gradient-like vector field for $f$. Denote by $\Phi_t$ the flow on $M$ determined by $-V$, that is,

$$\left.\frac{d}{dt}\right|_{t=t_0} \Phi_t(x) = -V(\Phi_{t_0}(x)) \text{ and } \Phi_0(x) = x \text{ for any } x \in M \text{ and } t_0 \in \mathbb{R}.$$

For any point $x \in M$ the limits $\Phi_{\pm\infty}(x) = \lim_{t\to\pm\infty} \Phi_t(x)$ exist and are critical points of $f$. For a point $x \in M$ the curve $\gamma_V^x = \Phi_{\_}(x)\colon \mathbb{R} \to M$ is called the *parametrised flow line* through $x$ (with respect to $V$) and its image is called the (unparametrised) *flow line* through $x$ (with respect to $V$).

For a point $p \in \operatorname{Crit} f$ we set

$$W_p^\pm = W_p^\pm(V) := \Phi_{\pm\infty}^{-1}(p) = \left\{x \in M\colon \lim_{t\to\pm\infty} \Phi_t(x) = p\right\}.$$

The sets $W_p^+$ and $W_p^-$ are called the *stable* and *unstable manifolds* of $p$ with respect to $V$ respectively.

**Definition 2.5.** Let $f\colon M \to \mathbb{R}$ be a Morse function on a smooth closed manifold $M$ and $V \in C^\infty(M, TM)$ be a gradient-like vector field for $f$. $V$ is called a *Morse-Smale vector field* adapted to $f$ if for any $p, q \in \operatorname{Crit} f$ the unstable manifold $W_p^-(V)$ intersects the stable manifold $W_q^+(V)$ transversally.

**Theorem 2.6** (Smale, [8])**.** *For any Morse function $f$ on a smooth closed manifold $M$ there exists a Morse-Smale vector field on $M$ adapted to $f$.*

Let $f\colon M \to \mathbb{R}$ be a Morse function on the closed manifold $M^n$ and let $V \in C^\infty(M, TM)$ be a Morse-Smale vector field adapted to $f$. Consider two points $p, q \in \operatorname{Crit} f$. The intersection $W_q^p = W_p^- \cap W_q^+$ consists of the flow lines with source $q$ and target $p$ and its dimension is

$$\dim W_q^p = n - (\operatorname{codim}_M W_p^- + \operatorname{codim}_M W_q^+) = n - (\mu_f(p) + n - \mu_f(q)) = \mu_f(p) - \mu_f(q).$$

The space $W_q^p$ is endowed with a free action of $\mathbb{R}$ given by the flow $\Phi$. The quotient $\mathcal{M}_q^p = W_q^p/\mathbb{R}$ is obviously in bijection with the flow lines going from $p$ to $q$ and is thus called the *moduli space of flow lines* from $p$ to $q$.

**Proposition 2.7.**

(1) *Let $f(q) < r < f(p)$. Then $W_q^p \cap f^{-1}(r)$ is a smooth submanifold of $M$ of dimension $\mu_f(p) - \mu_f(q) - 1$.*

(2) *The moduli space* $\mathcal{M}_q^p$ *is a smooth manifold diffeomorphic to any of* $W_q^p \cap f^{-1}(r)$ *with* $f(q) < r < f(p)$.

Now let $p$ and $q$ be critical points of indices $\lambda$ and $\lambda - 1$ respectively. Then $\mathcal{M}_q^p$ is a 0-dimensional compact manifold, that is, a finite set with discrete topology. For each flow line $\bar\gamma \in \mathcal{M}_q^p$ we define its *sign* $\operatorname{sgn}\bar\gamma$ to be $\pm 1$ depending on the orientation of a frame at a point $x \in \bar\gamma \cap f^{-1}(r)$, consisting of positively oriented frames of $W_p^- \cap f^{-1}(r)$ and $W_q^+ \cap f^{-1}(r)$ at point $x$ together with $V(x)$.

2.3. **Cerf's theory.** For a more detailed exposition of Cerf's theory see [2] and [7].

Let $f, g\colon M \to \mathbb{R}$ be two smooth functions on a closed manifold $M$. They are called *equivalent* if there are diffeomorphisms $R\colon M \to M$ and $L\colon \mathbb{R} \to \mathbb{R}$ such that $f = L \circ g \circ R^{-1}$. A Morse function is called *non-resonant* if all of its critical values are distinct. A Morse function is called *simply resonant* if the number of its distinct critical values differs by one from the number of its critical points. A smooth function $f\colon M \to \mathbb{R}$ is called a *birth-death* function if all of its critical points but one are Morse, all the critical values are distinct, and there is a local coordinate system around the only non-Morse point $p$ in which the function is expressed as

$$f = f(p) - \left(x^1\right)^2 - \ldots - \left(x^\lambda\right)^2 + \left(x^{\lambda+1}\right)^2 + \ldots + \left(x^{n-1}\right)^2 + \left(x^n\right)^3.$$

**Proposition 2.8.**

(1) *Let* $f$ *be a non-resonant Morse function. Then there is a neighbourhood* $U$ *of* $f$ *such that each* $g \in U$ *is a non-resonant Morse function equivalent to* $f$.
(2) *Let* $f$ *be a simply resonant Morse function. Then there is a neighbourhood* $U = U_> \sqcup U_= \sqcup U_<$ *of* $f$ *such that* $U_=$ *is a codimension-one submanifold of* $U$ *consisting of simply resonant Morse functions equivalent to* $f$, *while* $U_>$ *and* $U_<$ *are open subsets consisting of equivalent non-resonant Morse functions.*
(3) *Let* $f$ *be a birth-death function. Then there is a neighbourhood* $U = U_0 \sqcup U_1 \sqcup U_2$ *of* $f$ *such that such that* $U_1$ *is a codimension-one submanifold of* $U$ *consisting of birth-death Morse functions equivalent to* $f$, *while* $U_0$ *and* $U_2$ *are open subsets consisting of equivalent non-resonant Morse functions.* $\#\operatorname{Crit} g = \#\operatorname{Crit} f - 1$ *for* $g \in U_0$ *and* $\#\operatorname{Crit} g = \#\operatorname{Crit} f + 1$ *for* $g \in U_2$.

Let us denote by $\mathcal{F}_0$ the set of non-resonant Morse functions, by $\mathcal{F}_1^\alpha$ the set of birth-death functions and by $\mathcal{F}_1^\beta$ the set of simply resonant functions. $\mathcal{F}_0$ is an open dense subset of $C^\infty(M, \mathbb{R})$ (in $C^2$-topology), $\mathcal{F}_1^\alpha$ and $\mathcal{F}_1^\beta$ are codimension-one (Frechet) submanifolds of $C^\infty(M, \mathbb{R})$.

**Proposition 2.9.** *Let* $f_0, f_1\colon M \to \mathbb{R}$ *be two non-resonant Morse functions. Then there exists a path* $\gamma\colon [0,1] \to \mathcal{C}^\infty(M, \mathbb{R})$ *such that*

(1) $\gamma(0) = f_0$ *and* $\gamma(1) = f_1$;
(2) $\gamma(t)$ *is a non-resonant Morse function, simply resonant Morse function or a birth-death function for all* $t \in [0,1]$;
(3) $\gamma(t)$ *intersects* $\mathcal{F}_1^\alpha$ *and* $\mathcal{F}_1^\beta$ *transversally (and thus in a finite number of points).*

*Moreover, a generic path satisfying (1) satisfies (2) and (3).*

**Definition 2.10.** A Morse function $F\colon M \to \mathbb{R}$ on the smooth manifold $M$ with the boundary $\partial M$ is called *non-resonant* if all the critical values of $F$ and of $F|_{\partial M}$ are distinct.

## 3. The Morse-Barannikov complex

3.1. **Definition and first examples.** Let $F \in C^\infty(U)$ be a Morse function in a neighbourhood $U$ of the boundary $\partial M$ of a smooth manifold $M^n$, $V \in C^\infty(U, TM)$ be a gradient-like vector field for $F$, and $W \in C^\infty(\partial M, T\partial M)$ be a Morse-Smale vector field adapted to $f = F|_{\partial M}$.

**Definition 3.1.** The *Morse-Barannikov complex* (*MB-complex* for short) associated to the triple $(F, V, W)$ is a decorated graph in the plane drawn as follows.

(1) For each $p \in \operatorname{Crit} f$ draw a vertex with coordinates $(\mu_f(p), f(p))$ marked by $p$ with an arrow pointing upwards if $V(p) \in T^{out}_p M$ and downwards if $V(p) \in T^{in}_p M$.
(2) For each pair of critical points $p, q \in \operatorname{Crit} f$ with $f(p) > f(q)$ and $\mu_f(q) + 1 = \mu_f(p) = \lambda$ count the numbers $+^p_q$ and $-^p_q$ of the points in $\mathcal{M}^p_q$ with sign $+1$ and $-1$ respectively.
(3) If $+^p_q > 0$, then draw a graph of a convex increasing function $a\colon [\lambda - 1, \lambda] \to \mathbb{R}$ with $a(\lambda - 1) = f(q)$ and $a(\lambda) = f(p)$ and write the number $+^p_q$ near the graph (we omit the number if $+^p_q = 1$).
(4) If $+^p_q > 0$, then draw a graph of a concave increasing function $b\colon [\lambda - 1, \lambda] \to \mathbb{R}$ with $b(\lambda - 1) = f(q)$ and $b(\lambda) = f(p)$ and write the number $-^p_q$ near the graph (we omit the number if $-^p_q = 1$).

Thus an *n-dimensional MB-complex* consists of vertices that are characterised by a quadruple $(p, \mu, c, v)$, where $\mu = \mu(p) \in \{0, \dots, n-1\}$, $c = c(p) \in \mathbb{R}$, and $v = v(p) \in \{\uparrow, \downarrow\}$ and oriented edges that are represented by graphs of concave or convex increasing functions. We draw the lines $\{x = 0\}, \dots, \{x = n-1\}$ on which all the vertices sit for convenience, and we do not draw arrows for edges as every edge goes from left to right and from bottom to top.

The MB-complex defined above is a slight refinement of the framed Morse complex defined by Barannikov in [1]. In a sense, our definition is a step from the framed Morse complex to the category $\mathcal{C}_f$ from [3].

**Example 3.2.** Let $M = B^n_1(0)$ be the closed unit ball in $\mathbb{R}^n$, $F$ be the restriction of the height function in $\mathbb{R}^n$ to $M$ (i.e. $F(x^1, \dots, x^n) = x^n$), $V = \frac{\partial}{\partial x^n}$, $W(x)$ be the orthogonal projection of $V(x)$ to $T_x\partial M = T_x S^{n-1}$. Then $f = F|_{S^{n-1}}$ has only two critical points $t = (0, \dots, 0, 1)$ and $b = (0, \dots, 0, -1)$, the points of global maximum and minimum respectively. $\mu_f(t) = n - 1$, $\mu_f(b) = 0$. The MB-complex is simply $n$ lines $\{x = 0\}, \dots, \{x = n-1\}$ with two points: $(n-1, 1)$ and $(0, -1)$ with arrows pointing upwards and downwards respectively. If $n = 2$ then these two points are joint by two curves, a convex and a concave one, see Figure 1.

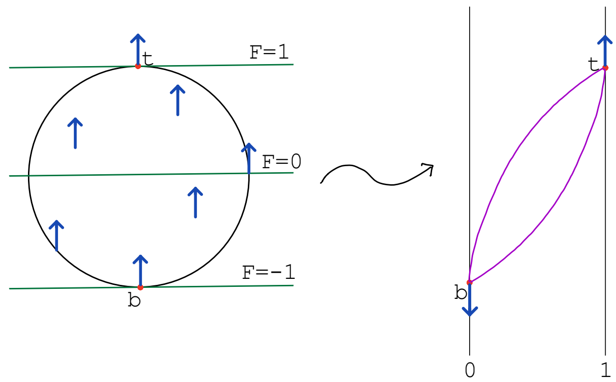



FIGURE 1. A trivial MB-complex in dimension 2

**Example 3.3.** Let $M = B_1^n(0)$ be the closed unit ball in $\mathbb{R}^n$ and let

$$F(x^1,\ldots,x^n) = a_1\left(x^1\right)^2 + \ldots a_n\left(x^n\right)^2 + \varepsilon\cdot\left(\left(x^1\right)^3 + \ldots + \left(x^n\right)^3\right)$$

with $a_1 < \ldots < a_k < 0 < a_{k+1} < \ldots < a_n$ and $100\varepsilon < |a_i - a_j|\ \forall i,j$. Then $\operatorname{Crit} f = \{\pm e_i\}$, around the critical point $\pm e_i$ the function $f$ looks like

$$f = a_i + \sum_{j\neq i}(a_j - a_i)\left(x^j\right)^2 \pm \varepsilon\cdot\left(1 - \sum_{j\neq i}\left(x^j\right)^2\right)^{3/2} \pm \varepsilon\cdot\sum_{j\neq i}\left(x^j\right)^3,$$

thus $\mu_f(\pm e_i) = i-1$. The MB-complex consists of $n$ lines $\{x=0\},\ldots,\{x=n-1\}$ with two points $(i-1, a_i\pm\varepsilon)$ on each. Arrow at a point $(i-1, a_i\pm\varepsilon)$ points upwards if $k+1\le i\le n$ and downwards if $1\le i\le k$. A point at the level $\lambda$ is connected to all the points at the levels $\lambda\pm 1$. See Figure 2.

3.2. **Metamorphoses of the MB-complex.** Here we will restate some results of Cerf's theory in the language of the MB-complex. The restatement is due to Barannikov (see [1]).

Let $M$ be a smooth manifold with the boundary $N_1 = \partial M$, $F\colon M\to\mathbb{R}$ be a Morse function on $M$, $N_0$ be a submanifold of $M$, $I\colon N_0\times[0,1]\to M$ be an isotopy joining $N_0$ and $N_1$. Take a Riemannian metric $g$ on $M$ such that $\operatorname{grad}_{g|_{N_t}} F|_{N_t}$ is a Morse-Smale vector field adapted to $f_t = F|_{N_t}$ for $t\in\{0,1\}$.

**Theorem 3.4** (Barannikov, [1])**.**

- *If there are no critical points of $F$ in $\operatorname{im} I$, then the MB-complex of $f_1$ can be obtained from that of $f_0$ by the following operations:*
  - (a) *An application of an orientation-preserving diffeomorphism $L\colon\mathbb{R}\to\mathbb{R}$ to each line $x=0,\ldots,x=n-1$. In this case, for each vertex $(p,\lambda,c,v)$ the value $L(c)-c$ is non-negative if $v=\uparrow$ and is non-positive if $v=\downarrow$.*
  - (b) *A vertical move of a vertex $(p,\lambda,c_0,v)$ along the direction of $v$. The new value $c$ should be such that if there is a path from $(p_1,\lambda_1,c_1,v_1)$ to $p_0$, then $c_1 < c$ and if there is a path from $p_0$ to $(p_2,\lambda_2,c_2,v_2)$, then $c < c_2$.*

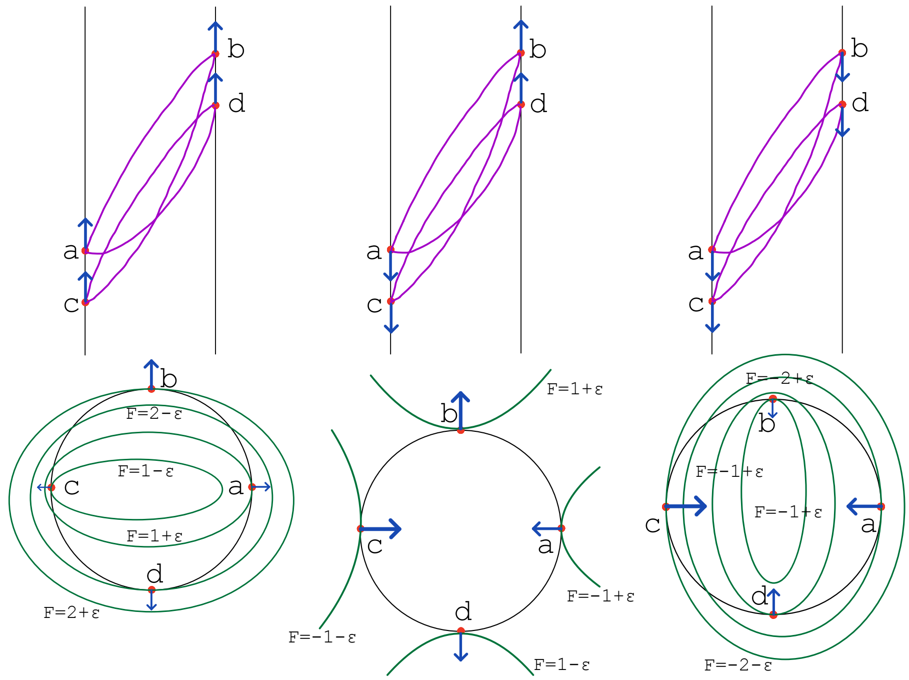



FIGURE 2. Standard two-dimensional MB-complexes of indices 0, 1, and 2.

(c) *An insertion of two critical points of neighbouring indices $\lambda$ and $\lambda - 1$ with arrows pointing in the same direction and one curve joining them, some other curves joining new points with the old ones can appear. This operation can only be performed if there are no old critical values of $f$ in between the two new critical values.*

(d) *A deletion of two critical points of neighbouring indices $\lambda$ and $\lambda - 1$ with arrows pointing in the same direction and one curve joining them. This operation can only be performed if there are no critical values of $f$ in between the critical values at the two deleted points.*

- *If there is one critical point of $F$ in $\operatorname{im} I$ and it has index $\lambda$, then, in addition to the operations (a)–(d), one of the following two operations should be performed once.*

($\lambda$.a) *A change of the direction of an arrow at a point at the level $\lambda$ from down to up.*

($\lambda$.b) *A change of the direction of an arrow at a point at the level $\lambda - 1$ from up to down.*

**Definition 3.5.** An MB-complex is called *trivial* if it can be obtained from the MB-complex in Example 3.2 by a metamorphose of type (1).

An MB-complex is called *standard of index $\mu$* if it can be obtained from the MB-complex in Example 3.3 with $k = \mu$ by a metamorphose of type (1).

## 4. Applications of MB-complexes to the problem

4.1. **Some information on Morse index from an MB-complex.** Let $f\colon U \to \mathbb{R}$ be a Morse function without critical points on the neighbourhood $U$ of $S^{n-1}$ inside $B^n_{\mathbf{0}}(1)$. For each $p \in S^{n-1}$ the vector $d_p f \neq 0$ and if $p$ is a critical point of $f|_{S^{n-1}}$, then $d_p f = T_p \partial S^{n-1}$ since otherwise $p$ would be a critical point of $f$. We introduce the following notation:

$$\operatorname{Crit}_\lambda(f) = \{p \in \operatorname{Crit} f\colon \mu_f(p) = \lambda\};$$
$$\operatorname{Crit}^{out}(f) = \{p \in \operatorname{Crit} f\colon\ d_p f|_{T_p^{out}M} > 0\},\ \ \operatorname{Crit}^{in}(f) = \{p \in \operatorname{Crit} f\colon\ d_p f|_{T_p^{in}M} > 0\};$$
$$\operatorname{Crit}^{out}_\lambda(f) = \operatorname{Crit}_\lambda(f) \cap \operatorname{Crit}^{out}(f),\ \ \operatorname{Crit}^{in}_\lambda(f) = \operatorname{Crit}_\lambda(f) \cap \operatorname{Crit}^{in}(f).$$

**Proposition 4.1.** *Let $F$ be a smooth extension of $f$ to $B^n_{\mathbf{0}}(1)$ such that $\operatorname{Crit} F = \operatorname{Morse} F = \mathbf{0}$. Then the parity of $\mu_F(\mathbf{0})$ can be retrieved from $f$. Namely,*

$$(-1)^{\mu_F(\mathbf{0})} = (-1)^n + \sum_{\lambda=0}^{n-1} (-1)^\lambda \cdot \#\operatorname{Crit}^{out}_\lambda(f) = 1 - \sum_{\lambda=0}^{n-1} (-1)^\lambda \cdot \#\operatorname{Crit}^{in}_\lambda(f)$$

*Proof.* Consider a Riemannian metric $g$ on $B^n_{\mathbf{0}}(1)$ that is adapted to $F$ and is standard in a neihbourhood of $S^{n-1}$. We can assume that $\operatorname{grad}_{g|_{S^{n-1}}} f$ is a Morse-Smale vector field adapted to $f$ (if this is not the case, then push slightly $S^{n-1}$ inside $B^n_{\mathbf{0}}(1)$). Take a small sphere $S_0$ inside $B^n_{\mathbf{0}}(1)$ such that $\mathbf{0}$ is not inside $S_0$ such that $\operatorname{grad}_{g|_{S_0}} F|_{S_0}$ is a Morse-Smale vector field adapted to $F|_{S_0}$, and the MB-complex associated to $(F|_{S_0}, W, \operatorname{grad}_{g|_{S_0}} F|_{S_0})$ is trivial.

By Theorem 3.4, the MB-complex associated to $(f, V, \operatorname{grad}_{g|_{S^{n-1}}} f)$ can be obtained from the one associated to $(F|_{S_0}, W, \operatorname{grad}_{g|_{S_0}} F|_{S_0})$ by the metamorphoses (a)–(d) and a metamorphoses $(\mu_F(\mathbf{0}).a)$ or $(\mu_F(\mathbf{0}).b)$.

Note that the metamorphoses (a)–(d) do not change the alternating sums in the statement. A metamorphoses $(\mu_F(\mathbf{0}).a)$ increments $\#\operatorname{Crit}^{out}_{\mu_F(\mathbf{0})}$ while decrementing $\#\operatorname{Crit}^{in}_{\mu_F(\mathbf{0})}$. A metamorphoses $(\mu_F(\mathbf{0}).b)$ decrements $\#\operatorname{Crit}^{out}_{\mu_F(\mathbf{0})-1}$ while incrementing $\#\operatorname{Crit}^{in}_{\mu_F(\mathbf{0})-1}$. Thus

$$\sum_{\lambda=0}^{n-1} (-1)^\lambda \cdot \#\operatorname{Crit}^{out}_\lambda(f) = \sum_{\lambda=0}^{n-1} (-1)^\lambda \cdot \#\operatorname{Crit}^{out}_\lambda\left(F|_{S_0}\right) + (-1)^{\mu_F(\mathbf{0})} = (-1)^{n-1} + (-1)^{\mu_F(\mathbf{0})}$$

and

$$\sum_{\lambda=0}^{n-1} (-1)^\lambda \cdot \#\operatorname{Crit}^{in}_\lambda(f) = \sum_{\lambda=0}^{n-1} (-1)^\lambda \cdot \#\operatorname{Crit}^{in}_\lambda\left(F|_{S_0}\right) - (-1)^{\mu_F(\mathbf{0})} = (-1)^0 - (-1)^{\mu_F(\mathbf{0})}.$$

□

*Remark* 4.2. Given a gradient-like vector field $V \in C^\infty(S^{n-1}, TB^n_{\mathbf{0}}(1))$ for $f$ (along $S^{n-1}$) one can retrieve the parity of $\mu_F(\mathbf{0})$ easily, the way we did it has an advantage of looking only at a finite number of points. Let $(x^1, \dots, x^n)$ be a local coordinate system around $\mathbf{0}$ from Definition 2.2 and $S_0$ be a small sphere given by $(x^1)^2 + \ldots + (x^n)^2 = \varepsilon$. Connect

$S^{n-1}$ with $S_0$ by an isotopy $I\colon S^{n-1}\times[0,1]\to B^n_{\mathbf{0}}(1)$. Then $v_t(x)=V(I(x,t))/|V(I(x,t))|$ defines a homotopy between $v_1, v_0\colon S^{n-1}\to S^{n-1}$ hence $\deg v_1=\deg v_0$. It can be easily computed that $\deg v_0=(-1)^{\mu_F(\mathbf{0})+n}$, thus $V$ determines the parity of $\mu_F(\mathbf{0})$.

Sometimes one can retrieve the exact value of $\mu_F(\mathbf{0})$ from the MB-complex, one of the instances of that is provided below.

**Proposition 4.3.** *Let $F$ be a smooth extension of $f$ to $B^n_{\mathbf{0}}(1)$ such that $\operatorname{Crit} F=\operatorname{Morse} F=\mathbf{0}$. If a point $p\in\operatorname{Crit}_{n-1} f$ ($p\in\operatorname{Crit}_0 f$) with the largest (smallest) value of $f(p)$ lies in $\operatorname{Crit}^{in}_{n-1} f$ ($\operatorname{Crit}^{out}_0 f$), then $\mu_F(\mathbf{0})=n$ ($\mu_F(\mathbf{0})=0$).*

*Proof.* The version in brackets follows from the unbracketed one by taking $-F$ instead of $F$. We prove the version without brackets.

The global maximum $M$ of $F$ is attained somewhere. If it is attained at $S^{n-1}$, then $F(p)=M$, but $F$ grows on a curve in $B^n_{\mathbf{0}}(1)$ emanating from $p$ along $v$ for any $v\in T^{in}_p B^n_{\mathbf{0}}(1)$, a contradiction. Thus $M$ is attained at a critical point of $F$ in $B^n_{\mathbf{0}}(1)\setminus S^{n-1}$, but $\operatorname{Crit} F=\{\mathbf{0}\}$, so the origin is the point of global maximum of $F$ and since it is non-degenerate, we have $\mu_F(\mathbf{0})=n$. ☐

### 4.2. MB-complexes admitting different indices.

**Definition 4.4.** Let $C$ be an MB-complex. We say that $C$ *admits a critical point of index* $\lambda$ if $C$ can be obtained from a trivial MB-complex by metamorphoses (a)–(d) and one of the metamorphoses $(\lambda.a)$ or $(\lambda.b)$.

We will abbreviate the metamorphoses in the following way. A metamorphose is not completely determined by its abbreviation, yet it should be clear from the context what metamorphose we mean.

$p\uparrow q$ The vertex $p$ moves up and becomes higher than $q$.
$p\downarrow q$ The vertex $p$ moved down and becomes lower than $q$.
$+ab\uparrow$ Two vertices $a$ and $b$ with arrows up and a curve joining them are added.
$+ab\downarrow$ Two vertices $a$ and $b$ with arrows down and a curve joining them are added.
$-ab\uparrow$ Two vertices $a$ and $b$ with arrows up and one curve joining them are deleted.
$-ab\downarrow$ Two vertices $a$ and $b$ with arrows down and one curve joining them are deleted.
$\uparrow p\downarrow$ An arrow at point $p$ is changed from up to down.
$\downarrow p\uparrow$ An arrow at point $p$ is changed from down to up.

**Proposition 4.5.** *For any $n\geq 2$ and $0\leq\lambda\leq n-2$ there is an MB-complex $A^n_\lambda$ with four vertices admitting both indices $\lambda$ and $\lambda+2$.*

*Proof.* $A^n_\lambda$ is an MB-complex consisting of four vertices $m,d,u,M$ with $\mu(m)=0$, $\mu(d)=\lambda$, $\mu(u)=\lambda+1$, $\mu(M)=n-1$, $c(m)<c(d)<c(u)<c(M)$, and $v(m)=\downarrow$, $v(d)=\uparrow$, $v(u)=\downarrow$, $v(M)=\uparrow$.

$A^n_\lambda$ admits index $\lambda$ since it can be obtained from a trivial MB-complex by applying $+du\downarrow$ and then $\downarrow d\uparrow$.

$A^n_\lambda$ admits index $\lambda+2$ since it can be obtained from a trivial MB-complex by applying $+du\uparrow$ and then $\uparrow u\downarrow$.

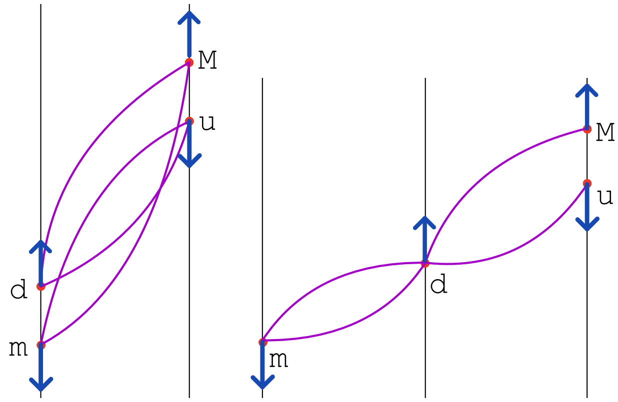



FIGURE 3. The MB-complexes $A_0^2$ and $A_1^3$

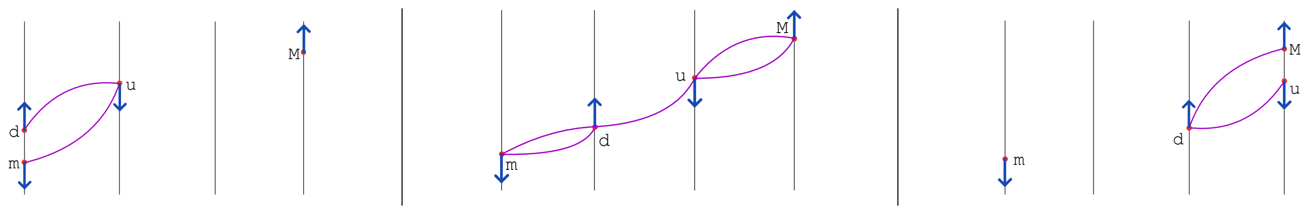



FIGURE 4. The MB-complexes $A_0^4$, $A_1^4$, and $A_2^4$

See Figures 3 and 4 for some examples. □

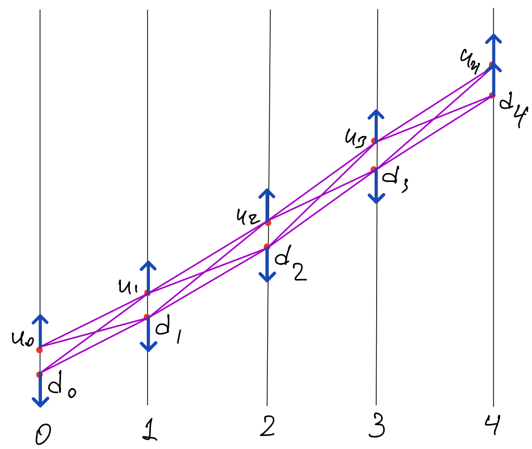
**Proposition 4.6.** *For any $n \geq 2$ there are two MB-complexes $C_0 = C_0^n$ and $C_1 = C_1^n$ such that $C_0$ admits critical points of indices $0, 2, \ldots, 2 \cdot [n/2]$ and $C_1$ admits critical points of indices $1, 3, \ldots, 2 \cdot [(n+1)/2] - 1$.*

*Proof.* Both $C_0$ and $C_1$ have two vertices at each level, and a vertex at level $\lambda$ is connected by a curve to each of the vertices at the neighbouring levels. The vertices at the level $\lambda$ are called $u_\lambda$ and $d_\lambda$; $u_\lambda$ is higher than $d_\lambda$) for all $\lambda$; the vertices at the level $\lambda$ are lower than the vertices at the level $\lambda + 1$. In $C_0$ all $u_\lambda$ have arrows up, and all $d_\lambda$ have arrows down except the case when $n$ is odd, then $d_{n-1}$ has arrow up. In $C_1$ the vertices $u_0$ and $d_0$ have arrows down and all other $u_\lambda$ have arrows up and all other $d_\lambda$ have arrows down except the case where $n$ is even, then $d_{n-1}$ has arrow up.

Let $0 \leq \lambda \leq n$ be an even number. We now describe the general algorithm for obtaining $C_0$ from a trivial MB-complex by metamorphoses (a)–(d) and one metamorphose of type ($\lambda$). See Figures 5 and 6 for examples.

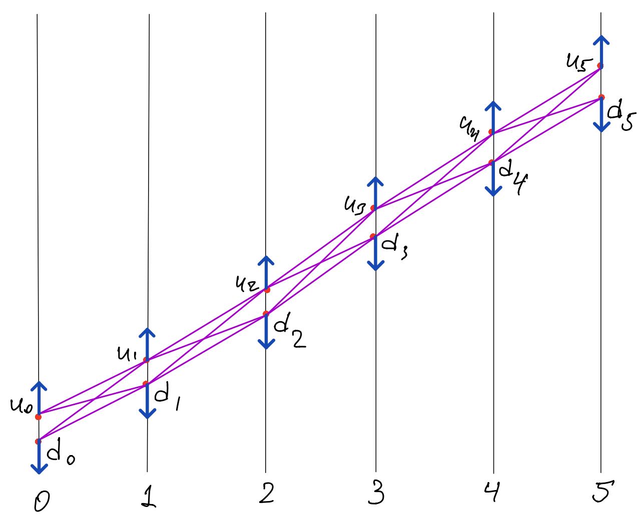
(1) $\lambda = 0$. First perform $+u_0 d_1 \downarrow$, then perform $u_{2k-1}u_{2k} \uparrow$ and $d_{2k}d_{2k+1} \downarrow$ until reaching $d_{n-1}$ or $u_{n-1}$ depending on the parity of $n$. Next, for each $k \geq 1$ perform $u_{2k} \uparrow d_{2k}$. Finally, perform $\downarrow u_0 \uparrow$.

(2) $2 \leq \lambda \leq n-1$. Perform $u_{2k}u_{2k+1} \uparrow$ and $d_{2k+1}d_{2k+2} \downarrow$ until reaching $d_\lambda$. Then perform $u_\lambda d_{\lambda+1}$, next perform $u_{2k-1}u_{2k} \uparrow$ and $d_{2k}d_{2k+1} \downarrow$ until reaching $d_{n-1}$ or $u_{n-1}$ depending on the parity of $n$. After that perform $u_i \uparrow d_i$ for each $i$ with $c(u_i) < c(d_i)$



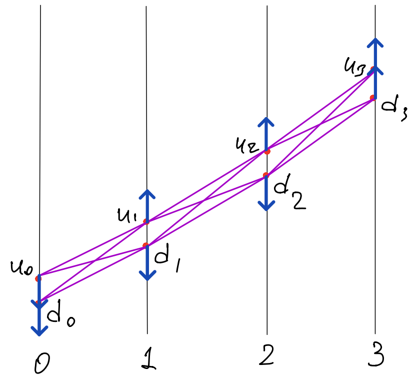

| ind 0 | ind 2 | ind 4 |
|---|---|---|
| $+u_0d_1\downarrow$ | $+u_0u_1\uparrow$ | $+u_0u_1\uparrow$ |
| $+u_1u_2\uparrow$ | $+d_1d_2\downarrow$ | $+d_1d_2\downarrow$ |
| $+d_2d_3\downarrow$ | $+u_2d_3\downarrow$ | $+u_2u_3\uparrow$ |
| $+u_3u_4\uparrow$ | $+u_3u_4\uparrow$ | $+d_3d_4\downarrow$ |
| $u_2\uparrow d_2$ | $u_1\uparrow d_1$ | $u_1\uparrow d_1$ |
| $u_4\uparrow d_4$ | $u_4\uparrow d_4$ | $u_3\uparrow d_3$ |
| $\downarrow u_0\uparrow$ | $\downarrow u_2\uparrow$ | $\downarrow d_4\uparrow$ |

FIGURE 5. $C_0$ in dimension 5 and metamorphoses from a trivial complex to $C_0$



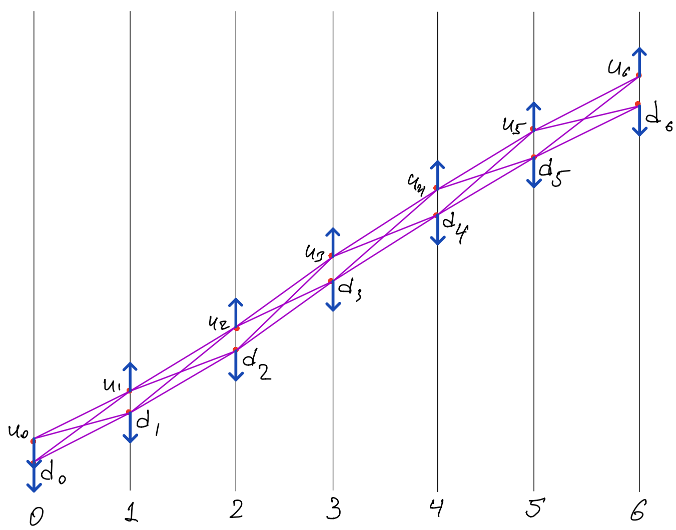

| ind 0 | ind 2 | ind 4 | ind 6 |
|---|---|---|---|
| $+u_0d_1\downarrow$ | $+u_0u_1\uparrow$ | $+u_0u_1\uparrow$ | $+u_0u_1\uparrow$ |
| $+u_1u_2\uparrow$ | $+d_1d_2\downarrow$ | $+d_1d_2\downarrow$ | $+d_1d_2\downarrow$ |
| $+d_2d_3\downarrow$ | $+u_2d_3\downarrow$ | $+u_2u_3\uparrow$ | $+u_2u_3\uparrow$ |
| $+u_3u_4\uparrow$ | $+u_3u_4\uparrow$ | $+d_3d_4\downarrow$ | $+d_3d_4\downarrow$ |
| $+d_4d_5\downarrow$ | $+d_4d_5\downarrow$ | $+u_4d_5\uparrow$ | $+u_4d_5\uparrow$ |
| $u_2\uparrow d_2$ | $u_1\uparrow d_1$ | $u_1\uparrow d_1$ | $u_1\uparrow d_1$ |
| $u_4\uparrow d_4$ | $u_4\uparrow d_4$ | $u_3\uparrow d_3$ | $u_3\uparrow d_3$ |
| $\downarrow u_0\uparrow$ | $\downarrow u_2\uparrow$ | $\downarrow u_4\uparrow$ | $\uparrow d_5\downarrow$ |

FIGURE 6. $C_0$ in dimension 6 and metamorphoses from a trivial complex to $C_0$




| ind 1 | ind 3 |
|---|---|
| $+u_0u_1 \uparrow$ | $+u_0d_1 \downarrow$ |
| $+d_1d_2 \downarrow$ | $+u_1u_2 \uparrow$ |
| $+u_2u_3 \uparrow$ | $+d_2d_3 \downarrow$ |
| $u_1 \uparrow d_1$ | |
| $u_3 \uparrow d_3$ | $u_2 \uparrow d_2$ |
| $\uparrow u_0 \downarrow$ | $\downarrow d_3 \uparrow$ |

FIGURE 7. $C_1$ in dimension 4 and metamorphoses from a trivial complex to $C_1$




| ind 1 | ind 3 | ind 5 | ind 7 |
|---|---|---|---|
| $+u_0u_1 \uparrow$ | $+u_0d_1 \downarrow$ | $+u_0d_1 \downarrow$ | $+u_0d_1 \downarrow$ |
| $+d_1d_2 \downarrow$ | $+u_1u_2 \uparrow$ | $+u_1u_2 \uparrow$ | $+u_1u_2 \uparrow$ |
| $+u_2u_3 \uparrow$ | $+d_2u_3 \uparrow$ | $+d_2d_3 \downarrow$ | $+d_2d_3 \downarrow$ |
| $+d_3d_4 \downarrow$ | $+d_3d_4 \downarrow$ | $+u_3u_4 \uparrow$ | $+u_3u_4 \uparrow$ |
| $+u_4u_5 \uparrow$ | $+u_4u_5 \uparrow$ | $+d_4u_5 \uparrow$ | $+d_4d_5 \downarrow$ |
| $+d_5d_6 \downarrow$ | $+d_5d_6 \downarrow$ | $+d_5d_6 \downarrow$ | $+u_5u_6 \uparrow$ |
| $u_1 \uparrow d_1$ | $u_2 \uparrow d_2$ | $u_2 \uparrow d_2$ | $u_2 \uparrow d_2$ |
| $u_3 \uparrow d_3$ | $u_3 \uparrow d_3$ | $u_4 \uparrow d_4$ | $u_4 \uparrow d_4$ |
| $u_5 \uparrow d_5$ | $u_5 \uparrow d_5$ | $u_5 \uparrow d_5$ | $u_6 \uparrow d_6$ |
| $\uparrow u_0 \downarrow$ | $\uparrow d_2 \downarrow$ | $\uparrow d_4 \downarrow$ | $\uparrow d_6 \downarrow$ |

FIGURE 8. $C_1$ in dimension 7 and metamorphoses from a trivial complex to $C_1$

(these are odd $i$ before $\lambda$ and even $i$ after $\lambda$). Finally, perform $\downarrow u_\lambda \uparrow$ if $\lambda < n-1$ or $\downarrow d_{n-1} \uparrow$ otherwise.

(3) $\lambda = n$. Perform $u_{2k}u_{2k+1} \uparrow$ and $d_{2k+1}d_{2k+2} \downarrow$ until reaching $d_{n-2}$. Then perform $u_{n-2}d_{n-1} \downarrow$. Next for each odd $1 \le i \le n-3$ perform $u_i \uparrow d_i$. Finally, perform $\uparrow d_{n-1} \downarrow$.

Let $1 \le \lambda \le n$ be an odd number. We now describe the general algorithm for obtaining $C_1$ from a trivial Morse-Barannikov by metamorphoses (a)–(d) and one metamorphose of type ($\lambda$). See Figures 7 and 8 for examples.

(1) $\lambda = 1$. Perform $u_{2k}u_{2k+1} \uparrow$ and $d_{2k+1}d_{2k+2} \downarrow$ starting from $k=0$ until reaching $d_{n-1}$ or $u_{n-1}$ depending on the parity of $n$. Then perform $u_i \uparrow d_i$ for each odd $i$. Finally, perform $\uparrow u_0 \downarrow$.

(2) $3\lambda \le n-1$. First perform $u_0 d_1 \downarrow$. Then perform $u_{2k-1}u_{2k} \uparrow$ and $d_{2k}d_{2k+1} \downarrow$ until reaching $u_{\lambda-1}$. Next perform $d_{\lambda-1}u_\lambda \uparrow$. After that perform $d_{2k-1}d_{2k} \downarrow$ and $u_{2k}u_{2k+1} \uparrow$ until reaching $d_{n-1}$ or $u_{n-1}$ depending on the parity of $n$. Then perform $u_i \uparrow d_i$ for each $i$ with $c(u_i) < c(d_i)$ (these are even $i$ before $\lambda$ and odd $i$ after $\lambda$). Finally, perform $\uparrow d_{\lambda-1} \downarrow$.

(3) $\lambda = n$. First perform $u_0 d_1 \downarrow$. Then perform $u_{2k-1}u_{2k} \uparrow$ and $d_{2k}d_{2k+1} \downarrow$ until reaching $d_{n-1}$. Next perform $u_i \uparrow d_i$ for all even $i \ge 2$. Finally, perform $\downarrow d_{n-1} \uparrow$.

□

## 5. Toolbox: metamorphoses of Morse functions

**Definition 5.1.** Let $U \subset B^n_{\mathbf{0}}(1)$ be a neighbourhood of $S^{n-1}$ and $f\colon U \to \mathbb{R}$ be a Morse function without critical points. We say that $f$ *admits index* $\lambda$ if there exists a Morse function $F\colon B^n_{\mathbf{0}}(1) \to \mathbb{R}$ such that

(1) $F|_U = f$,
(2) $\operatorname{Crit} F = \{\mathbf{0}\}$, and
(3) $\mu_F(\mathbf{0}) = \lambda$.

*Remark* 5.2. It is clear from Theorem 3.4, that if $f$ admits index $\lambda$, then the MB-complex associated to $(f, \operatorname{grad} f, W)$ admits index $\lambda$ ($W \in C^\infty(\partial M, T\partial M)$ is a Morse–Smale vector field adapted to $f$).

Note that if $f$ is extendable with index $\lambda$, then so is $L \circ f$ for any orientation-preserving diffeomorphism $L$ of $\mathbb{R}$ and $f \circ R^{-1}\colon R(U) \to \mathbb{R}$ for any diffeomorphism $R$ of $B^n_{\mathbf{0}}(1)$.

Note also that if $F\colon M \to \mathbb{R}$ is a Morse function and $A$ is a closed subset of $M$ not meeting $\operatorname{Crit} F$, then a vector field $V \in C^\infty(A, TM)$ satisfying $VF > 0$ can be extended to a gradient-like vector field $\widetilde{V} \in C^\infty(M, TM)$ for $F$ and a Riemannian metric $g \in C^\infty(A, S^2T^*M)$ along $A$ can be extended to a Riemannian metric $g \in C^\infty(M, S^2T^M)$ adapted to $F$. In particular, if $F\colon B^n_{\mathbf{0}}(1) \to \mathbb{R}$ is a Morse function, then we can assume that the Riemannian metric adapted to $F$ is the standard metric coming from $\mathbb{R}^n$ outside an arbitrary neighbourhood $U$ of $\operatorname{Crit} F$.

5.1. **Flips.**

**Definition 5.3.** Let $F\colon B^n_{\mathbf{0}}(1) \to \mathbb{R}$ be a Morse function without critical points and $p \in S^{n-1}$ be a critical point of $F|_{S^{n-1}}$. A Morse function $\widetilde{F}\colon B^n_{\mathbf{0}}(1) \to \mathbb{R}$ is a *flip of $F$ at $p$ of index $\lambda$* if

(1) $\widetilde{F} = F$ outside some neighbourhood $U_p$ of $p$ in $B^n_{\mathbf{0}}(1)$;
(2) $\operatorname{grad} F(p)$ and $\operatorname{grad} \widetilde{F}(p)$ point in opposite directions;
(3) $\widetilde{F}\Big|_{S^{n-1}} = F|_{S^{n-1}}$;
(4) $\operatorname{Crit} \widetilde{F} = \{\mathbf{0}\}$ and $\mu_{\widetilde{F}}(\mathbf{0}) = \lambda$.

The following statement is a refinement of [1, Lemma 1] by Barannikov.

**Lemma 5.4.** *Let $F\colon B^n_{\mathbf{0}}(1) \to \mathbb{R}$ be a Morse function without critical points and $p \in S^{n-1}$ be a critical point of $f = F|_{S^{n-1}}$ with $\mu_f(p) = \lambda$ and $\operatorname{grad} F(p) \in T^{in}_p B^n_{\mathbf{0}}(1)$. Then there exists a flip $\widetilde{F}$ of $F$ at $p$ of index $\lambda$.*

*Proof.* Without loss of generality we can assume that $F(p) = 0$. Choose a local coordinate system $(x^1, \dots, x^{n-1}, y)$ in a neighbourhood $U_1$ of $p$ in such a way that

(1) $U_1$ is given by $y \le 0$,
(2) $x = (x^1, \dots, x^{n-1})$ is a coordinate system in $U_1 \cap S^{n-1}$ adapted to $p$ (with respect to $f$), and
(3) $F(x, y) = f(x) - 2y$ (to satisfy that first choose $\widetilde{y}$ such that (1) and (2) hold, and then define $2y(x, \widetilde{y}) = f(x) - F(x, \widetilde{y})$).

Let $a > 0$ be such that $U_2 = B^{n-1}_{\mathbf{0}}(a) \times [-a, +\infty)$ satisfies $U_2 \cap \{y \le 0\} \subset U_1$. Note that for $(x, y) \in U_2$ with $y \le 0$ we have

$$F(x, y) = -\sum_{i=1}^{\lambda} \left(x^i\right)^2 + \sum_{i=\lambda+1}^{n-1} \left(x^i\right)^2 - 2y.$$

By modifying $F$ on $\{(x, y) \in B^{n-1}_{\mathbf{0}}(a) \times (-a/5, 0)\colon a/5 < |x| < 4a/5\}$ we can obtain a smooth function $F_1$ without critical points such that $F_1(x, y) = F(x, 0)$ for $2a/5 \le |x| \le 3a/5$ and $-a/10 < y \le 0$. Now we extend this function to the smooth function

$$F_2\colon B^{n-1}_{\mathbf{0}}(a) \times (-a, 0] \cup \{(x, y) \in U_2\colon 2a/5 \le |x| \le 3a/5\} \to \mathbb{R}$$

by $F_2(x, y) = F_2(x, 0)$ for $2a/5 \le |x| \le 3a/5$ and $y > 0$.

Define a function $G\colon B^{n-1}_{\mathbf{0}}(a) \times [0, +\infty)$ by

$$G(x, y) = 1 - \sum_{i=1}^{\lambda} \left(x^i\right)^2 + \sum_{i=\lambda+1}^{n-1} \left(x^i\right)^2 + (y - 1)^2.$$

Note that $F_2$ and $G$ satisfy $F_2(x, 0) = G(x, 0)$ for $x \in B^{n-1}_{\mathbf{0}}(a)$ and $\operatorname{grad} F_2(x, 0) = \operatorname{grad} G(x, 0)$ for $|x| \le a/5$. So there is a smooth extension $F_3\colon U_2 \to \mathbb{R}$ of $F_2$ such that

(1) $F_3(x, y) = G(x, y)$ for $|x| \le a/5$ and $y \ge 1/2$ and

(2) $\operatorname{Crit} F_3 = \operatorname{Crit} G = \{(0,\dots,0,1)\}$.

Now let $\widetilde{y}\colon B^{n-1}_{\mathbf{0}}(a) \to \mathbb{R}$ be a smooth function satisfying

(1) $\widetilde{y}(x) = 2$ for $|x| \leq a/5$,
(2) $\widetilde{y}(x) = 0$ for $|x| \geq a/2$, and
(3) $F_3(x, \widetilde{y}(x)) = F(x,0)$ for all $x \in B^{n-1}_{\mathbf{0}}(a)$.

Take a diffepmorphism $\Phi$ of $U_2$ such that

(1) $\Phi(x,y) = (x,y)$ for $|x| \geq 2a/5$ and
(2) $\Phi(x,\widetilde{y}) = (x,0)$ for all $x \in B^{n-1}_{\mathbf{0}}(a)$.

The function $F_4(x,y) = F_3(x, \Phi^{-}1(x,y))$ is almost the one we need. The only difference is that the critical point $q$ of $F_4$ is not at the origin, so $\widetilde{F} = F_4 \circ \Psi$ where $\Psi$ is a diffeomorphism of $B^n_{\mathbf{0}}$ that maps $\mathbf{0}$ to $q$ and is the identity near $S^{n-1}$. □

**Corollary 5.5.** Let $F\colon B^n_{\mathbf{0}}(1) \to \mathbb{R}$ be a Morse function without critical points and $p \in S^{n-1}$ be a critical point of $f = F|_{S^{n-1}}$ with $\mu_f(p) = \lambda - 1$ and $\operatorname{grad} F(p) \in T^{out}_p B^n_{\mathbf{0}}(1)$. Then there exists a flip $\widetilde{F}$ of $F$ at $p$ of index $\lambda$.

*Proof.* It follows from the proposition that there exists a flip $\widetilde{-F}$ of $-F$ at $p$ of index $n-\lambda$. Then $-\left(\widetilde{-F}\right)$ is a flip for $F$ at $p$ of index $\lambda$. □

5.2. **Standard births.**

**Definition 5.6.** Let $F\colon B^n_{\mathbf{0}}(1) \to \mathbb{R}$ be a Morse function, $V \in C^\infty(S^{n-1}, TS^{n-1})$ be a Morse–Smale vector field adapted to $f = F|_{S^{n-1}}$, $p \in S^{n-1}$ be a regular point of $f$, $U_p$ be a neighbourhood of $p$ in $B^n_{\mathbf{0}}(1)$, $\gamma = \gamma^p_V$ be the flow line through $p$, and $\varepsilon > 0$ be such that $\gamma(t) \in U_p$ for $|t| < 3\varepsilon$. We say that a Morse function $\widetilde{F}\colon B^n_{\mathbf{0}}(1) \to \mathbb{R}$ is *obtained from $F$ by a standard birth in $U_p$ of index $\lambda$* if there exists a Morse-Smale vector field $\widetilde{V} \in C^\infty(S^{n-1}, TS^{n-1})$ for $\widetilde{f} = \widetilde{F}\Big|_{S^{n-1}}$ such that

(1) $\widetilde{F} = F$ and $\widetilde{V} = V$ outside $U_p$;
(2) $\operatorname{Crit} \widetilde{f} = \operatorname{Crit} f \cup \{p_-, p_+\}$ where $p_\pm = \gamma(\pm\varepsilon) \in U_p$;
(3) $\mu_{\widetilde{f}}(p_+) = \lambda$, $\mu_{\widetilde{f}}(p_-) = \lambda + 1$, $\gamma^p_{\widetilde{V}}(-\varepsilon, \varepsilon)$ is the unique flow line between $p_+$ and $p_-$, and $\operatorname{im} \gamma^{\gamma(-2\varepsilon)}_{\widetilde{V}} \cup \operatorname{im} \gamma^p_{\widetilde{V}} \cup \operatorname{im} \gamma^{\gamma(2\varepsilon)}_{\widetilde{V}} = \operatorname{im} \gamma^p_V$.
(4) $\operatorname{grad} F(x)$ and $\operatorname{grad} \widetilde{F}(x)$ both lie in either $T^{in}_x B^n_{\mathbf{0}}(1)$ or $T^{out}_x B^n_{\mathbf{0}}(1)$ for any $x \in S^{n-1}$.

The construction we present here is essentially the one known classically and explained by Cerf in [2, III.1]. The only difference is that we have an additional dimension, that is, we need to extend the modification of a function on $S^{n-1}$ to its tubular neighbourhood. This is done straightforwardly, yet we write the construction in some detail as we later need its additional property, namely, that one can relate standard births at two points on the same flow line.

**Lemma 5.7.** *Let $F\colon B^n_{\mathbf{0}}(1) \to \mathbb{R}$ be a Morse function, $V \in C^\infty(S^{n-1}, TS^{n-1})$ be a Morse–Smale vector field adapted to $f = F|_{S^{n-1}}$, $p \in S^{n-1}$ be a regular point of $f$ with* $\operatorname{grad} F(p) \notin$

*$T_pS^{n-1}$, and $\lambda \in \{0,\dots,n-2\}$ be a number. Then for any sufficiently small neighbourhood $U_p$ of $p$ and sufficiently small $\varepsilon > 0$ there exists a Morse function $\widetilde{F}\colon B^n_{\mathbf{0}}(1) \to \mathbb{R}$ obtained from $F$ by a standard birth in $U_p$ of index $\lambda$.*

*Proof.* Without loss of generality we can assume that $F(p) = 0$. Choose a local coordinate system $(x^1,\dots,x^{n-1},y)$ in a neighbourhood $U_1$ of $p$ such that

(1) $U_1$ is given by $y \leq 0$;
(2) $F(x^1,\dots,x^{n-1},y) = x^{n-1} + \sigma y$ with $\sigma = 1$ if $\operatorname{grad} F(p) \in T^{out}_p B^n_{\mathbf{0}}(1)$ and $\sigma = -1$ if $\operatorname{grad} F(p) \in T^{in}_p B^n_{\mathbf{0}}(1)$;
(3) the flow line through $p$ is given by $\gamma(t) = (0,\dots,0,t,0)$.

Let $a > 0$ be such that $U_2 = B^{n-1}_{\mathbf{0}}(a) \times (-a,a)$ satisfies $U_2 \cap \{y \leq 0\} \subset U_1$. Let *omega*: $\mathbb{R} \to \mathbb{R}$ be a smooth function that is equal to 1 for $|x| < a/4$, equal to 0 for $|x| > a/2$, symmetric with respect to 0 and monotone on $[0,+\infty)$. Define a function

$$G(x,y) = -\sum_{i=1}^{\lambda}\left(x^i\right)^2 + \sum_{i=\lambda+1}^{n-2}\left(x^i\right)^2 + \left(x^{n-1}\right)^3 + (1-2\omega(|(x,y)|))\varepsilon_1 x^{n-1} + \sigma y$$

and a diffeomorphism

$$\Phi(x,y) = (x^1,\dots,x^{n-2}, -\sum_{i=1}^{\lambda}\left(x^i\right)^2 + \sum_{i=\lambda+1}^{n-2}\left(x^i\right)^2 + \left(x^{n-1}\right)^3 + \varepsilon_1 x^{n-1}, y),$$

where $\varepsilon_1 = 3\left(\frac{3\varepsilon}{4}\right)^{2/3}$.

Then $\widetilde{F} = G \circ \Phi^{-1}$ and $\widetilde{f} = \widetilde{F}\Big|_{\{y=0\}}$ have the following properties

(1) $\widetilde{F}(x,y) = x^{n-1} + \sigma y = F(x,y)$ if $|x| > a/2$ or $|y| > a/2$;
(2) $\operatorname{Crit} \widetilde{F} = \varnothing$;
(3) $\operatorname{Crit} \widetilde{f} = \{p_+, p_-\} = \{(0,\dots,0,\varepsilon,0),(0,\dots,0,-\varepsilon,0)\}$; $\mu_{\widetilde{f}}(p_+) = \lambda$, and $\mu_{\widetilde{f}}(p_-) = \lambda + 1$;
(4) $\frac{\partial \widetilde{F}}{\partial y}(x,y) = \sigma = \frac{\partial F}{\partial y}(x,y)$ for all $(x,y) \in U_2$.

Thus $\widetilde{F}$ and $\widetilde{V} = \operatorname{grad} \widetilde{f}$ are the desired function and vector field. □

*Remark* 5.8. From the construction it follows that $\widetilde{f}$ and $\widetilde{V}$ depend only on $f$ and $V$ and not on the extension of $f$ to $F$.

5.3. **Connecting two Morse functions.** Since two Morse functions with the same Morse-Barannikov complex are not necessarily equivalent, we need to find a way to somehow connect them. First we prove a technical lemma that allows us to obtain a Morse function without critical points in a neighbourhood of $S^{n-1}$ inside $\mathbb{R}^n$ from two Morse functions without critical points in neighbourhoods of $S^{n-1}$ inside $\{x \in \mathbb{R}^n \colon |x| \leq 1\}$ and $\{x \in \mathbb{R}^n \colon |x| \geq 1\}$.

**Lemma 5.9.** *Let $F$ be a continuous function on $S_\varepsilon = \{x \in \mathbb{R}^n \colon 1-\varepsilon < |x| < 1+\varepsilon\}$ which is a Morse function without critical points on $S^+_\varepsilon = \{x \in \mathbb{R}^n \colon 1 \leq |x| < 1+\varepsilon\}$ and*

$S_\varepsilon^- = \{x \in \mathbb{R}^n \colon 1-\varepsilon < |x| \leq 1\}$. *Suppose that* $\operatorname{grad}\left(F|_{S_\varepsilon^+}\right)(p) = \operatorname{grad}\left(F|_{S_\varepsilon^-}\right)(p)$ *for all* $p \in \operatorname{Crit} F|_{S^{n-1}}$. *Then there exists a Morse function without critical points* $\widetilde{F}$ *on* $S_\varepsilon$ *such that* $\widetilde{F}$ *is equal to* $F$ *in a neighbourhood of* $\partial S_\varepsilon$.

*Proof.* Let $F^\pm$ be smooth extensions of $F$ from $S_\varepsilon^\pm$ to $S_\varepsilon$ such that $\operatorname{grad}\left(F|_{S_\varepsilon^\pm}\right)(x) = \operatorname{grad}\left(F^\pm\right)(x)$ for all $x \in S^{n-1}$. Taking a smaller $\varepsilon$ we can assume that $F^\pm$ are Morse functions without critical points. Let $c = \min_{y \in S^{n-1}} \min\left\{\left|\operatorname{grad} F|_{S_\varepsilon^+}(y)\right|, \left|\operatorname{grad} F|_{S_\varepsilon^-}(y)\right|\right\}$, for $p \in \operatorname{Crit} F|_{S^{n-1}}$ let $U_p$ be a small cap on $S^{n-1}$ around $p$ such that $|\operatorname{grad} F^+(x) - \operatorname{grad} F^-(x)| < c/100$ and $|\operatorname{grad} F^\pm(x) - \operatorname{grad} F^\pm(p)| < c/100$ for any $x \in U_p$. Pick $\delta > 0$ such that the following conditions are satisfied

(1) $F^t_{\alpha^+,\alpha^-} = \alpha^+ \, F^+|_{tS^{n-1}} + \alpha^- \, F^-|_{tS^{n-1}}$ is a Morse function equivalent to $F|_{S^{n-1}}$ for any $t \in [1-\delta, 1+\delta]$ and $\alpha^+, \alpha^- > 0$ with $\alpha^+ + \alpha^- = 1$ and for any $q \in \operatorname{Crit} F^t_{\alpha^+,\alpha^-}$ the point $q/|q|$ lies in $U_p$ for the corresponding $p \in \operatorname{Crit} F|_{S^{n-1}}$;
(2) $|\operatorname{grad} F^\pm(x) - \operatorname{grad} F^\pm(x/|x|)| < c/100$ for any any $x \in S_\delta$.

Take a partition of unity $\{\varphi^+, \varphi^-\}$ on $S_\varepsilon$ subordinate to the open cover $\big\{\{x \in \mathbb{R}^n \colon 1-\delta \leq |x| \leq 1+\varepsilon\}, \{x \in \mathbb{R}^n \colon 1-\varepsilon \leq |x| \leq 1+\delta\}\big\}$ such that $\varphi^+$ constant on each $|x|S^{n-1}$, decreases in $|x|$ and $|\operatorname{grad}\varphi^+| < 2/\delta$. We define $\widetilde{F} = \varphi^+ F^+ + \varphi^- F^-$. We need to prove that $\widetilde{F}$ has no critical points.

Consider $x \in S_\varepsilon$. If $x \notin S_\delta$, then $\widetilde{F}$ coincides with $F^+$ or $F^-$ in a neighbourhood of $x$, therefore, $x$ is not a critical point of $\widetilde{F}$. Now let $x \in S_\delta$. If $x$ is not a critical point of $\widetilde{F}\Big|_{|x|S^{n-1}}$, then $x$ is obviously not a critical point of $\widetilde{F}$. If $x \in \operatorname{Crit} \widetilde{F}\Big|_{|x|S^{n-1}}$, then let $p$ be the corresponding critical point of $F|_{S^{n-1}}$. We have

$$\begin{aligned}\left|\operatorname{grad}\widetilde{F}(x)\right| &= \left|\operatorname{grad} F^-(x) + (F^+(x) - F^-(x)) \cdot \operatorname{grad}\varphi^+(x) + \varphi^+(x) \cdot (\operatorname{grad} F^+(x) - \operatorname{grad} F^-(x))\right| \\ &\geq \left|\operatorname{grad} F^-(p)\right| - \left|\operatorname{grad} F^-(p) - \operatorname{grad} F^-(x/|x|)\right| - \left|\operatorname{grad} F^-(x) - \operatorname{grad} F^-(x/|x|)\right| \\ &- \left|\operatorname{grad}\varphi^+(x)\right| \cdot |x - x/|x|| \cdot \max_{y \in [x, x/|x|]} \left|\operatorname{grad} F^+(y) - \operatorname{grad} F^-(y)\right| \\ &- \left|\operatorname{grad} F^+(x) - \operatorname{grad} F^-(x)\right|\end{aligned}$$

For $y \in [x, x/|x|]$ we have

$$\begin{aligned}\left|\operatorname{grad} F^+(y) - \operatorname{grad} F^-(y)\right| &\leq \left|\operatorname{grad} F^+(y) - \operatorname{grad} F^+(y/|y|)\right| \\ &+ \left|\operatorname{grad} F^+(x/|x|) - \operatorname{grad} F^-(x/|x|)\right| + \left|\operatorname{grad} F^-(y/|y|) - \operatorname{grad} F^-(y)\right| \\ &\leq 3c/100\end{aligned}$$

so

$$\left|\operatorname{grad}\widetilde{F}(x)\right| \geq c - c/100 - c/100 - \frac{2}{\delta} \cdot \delta \cdot \frac{3c}{100} - 3c/100 > 0$$

thus $x$ is not a critical point of $\widetilde{F}$. □

Now we connect two Morse functions in such a way that the result will satisfy the assumptions of the previous lemma.

**Lemma 5.10.** *Let $f_1, f_2\colon S^-_\varepsilon \to \mathbb{R}$ be two Morse functions without critical points. Suppose that* $\mathrm{Crit}\ f_1|_{S^{n-1}} = \mathrm{Crit}\ f_2|_{S^{n-1}} = C$*, for each $p \in C$ $f_1(p) \neq f_2(p)$ and the vectors* $\mathrm{grad}\, f_i(p)$ *both point out if $f_2(p) > f_1(p)$ and point in if $f_2(p) < f_1(p)$. If there exists a smooth vector field $V \in C^\infty(S^{n-1}, TS^{n-1})$ such that $Vf_i(x) > 0$ for any $x \in S^{n-1} \setminus C$, then there exists a Morse function without critical points $F\colon \{x \in \mathbb{R}^n \colon 1 \leq |x| \leq 2\} = A \to \mathbb{R}$ such that*

(1) *$F(i \cdot x) = f_i(x)$ for any $x \in S^{n-1}$;*
(2) *$i \cdot \mathrm{grad}\, F(i \cdot p) = \mathrm{grad}\, f_i(p)$ for any $p \in C$;*

*Proof.* For each $x \in S^{n-1}$ denote by $n_x$ the unit outer normal to $S^{n-1}$ at $x$. For $p \in C$ let $U_p$ be a small cap in $S^{n-1}$ around $p$ such that

$$\mathrm{sgn}(f_2(x) - f_1(x)) = \mathrm{sgn}\,\langle n_x, \mathrm{grad}\, f_1(x)\rangle = \mathrm{sgn}\,\langle n_x, \mathrm{grad}\, f_2(x)\rangle \text{ for all } x \in U_p.$$

Let $\varphi_p\colon \{x \in A\colon x/|x| \in U_p\} \to \mathbb{R}$ be a smooth function such that for any $x \in S^{n-1}$

(1) $\varphi_p(x) = f_1(x)$ and $\mathrm{grad}\,\varphi_p(x) = \mathrm{grad}\, f_1(x)$;
(2) $\varphi_p(2x) = f_2(x)$ and $2\,\mathrm{grad}\,\varphi_p(2x) = \mathrm{grad}\, f_2(x)$;
(3) $\varphi_p|_{[x,2x]}$ is strictly monotone.

Let $F_{reg}\colon A \to \mathbb{R}$ be a convex combination of $f_1$ and $f_2$:

$$F_{reg}(x) = (2 - |x|)f_1(x/|x|) + (|x| - 1)f_2(x/|x|).$$

Take a partition of unity $\{h_p\}_{p\in C} \cup \{h\}$ subordinate to an open cover $\{U_p\}_{p\in C} \cup \{U\}$ where $U = S^{n-1} \setminus C$ and define

$$F = hF_{reg} + \sum_{p\in C} h_p\varphi_p.$$

We need to check that $F$ has no critical points. Indeed, $\langle \mathrm{grad}\, F(x), x/|x|\rangle = 0$ only if $f_1(x) = f_2(x)$ and that can happen only outside each of $U_p$. But then for $y \in S^{n-1}$ we have

$$F(|x| \cdot y) = (2 - |x|)f_1(y) + (|x| - 1)f_2(y),$$

so

$$VF(|x| \cdot \_)(y) = ((2 - |x|))Vf_1(y) + (|x| - 1)Vf_2(y) > 0$$

thus $\mathrm{grad}\, F(x) \neq 0$. ☐

## 6. Functions in a neighbourhood of $S^{n-1}$ admitting different indices

With all the tools developed in the previous section we are ready to construct a function that admits two different indices. We first illustrate the idea with the two-dimensional case.

Let $F^0_1 \in C^\infty(B^2_{\mathbf{0}}(1))$ be a Morse function without critical points given by $F^0_1(x, y) = 2y(\Phi^0(x, y))$, where $\Phi^0$ is a diffeomorphism of the plane that maps $B^2_{\mathbf{0}}$ diffeomorphically onto the figure bounded by the red curve on Figure 9. Let $F^2_1 \in C^\infty(B^2_{\mathbf{0}}(1))$ be a Morse function without critical points given by $F^0_1(x, y) = y(\Phi^2(x, y))$, where $\Phi^2$ is a diffeomorphism of the plane that maps $B^2_{\mathbf{0}}$ diffeomorphically onto the figure bounded by

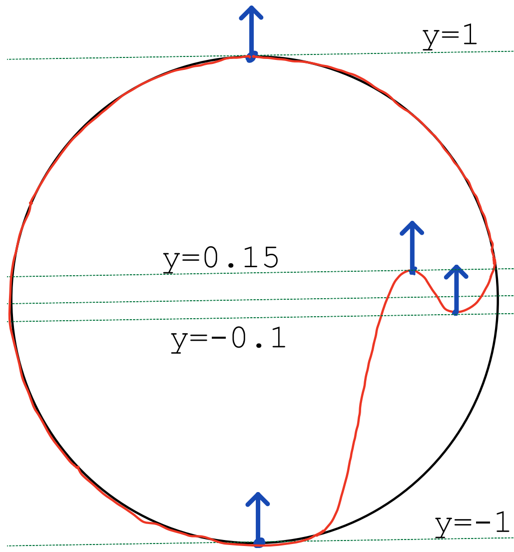



FIGURE 9. Insertion of a pair of critical points with arrows down in dimension 2

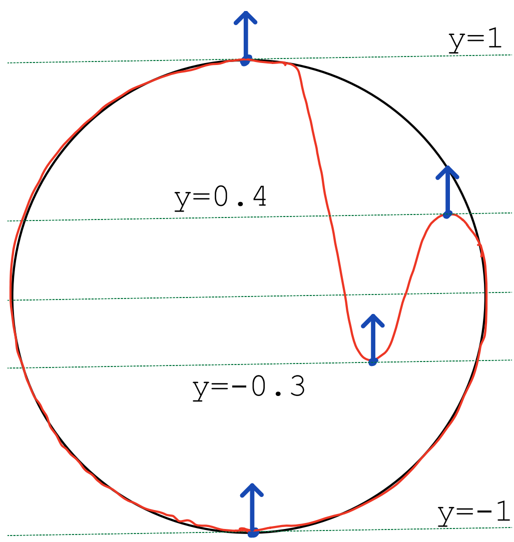



FIGURE 10. Insertion of a pair of critical points with arrows down in dimension 2

the red curve on Figure 10. Which choose the diffeomorphisms $\Phi^0$ and $\Phi^2$ in such a way that Crit $F_1^0|_{S^1} =$ Crit $F_1^2|_{S^1}$.

$F_2^0$ is obtained from $F_1^0$ by a flip of index 0 at the critical point $p^0$ with $F_1^0(p^0) = -0.2$. $F_2^2$ is obtained from $F_1^2$ by a flip of index 2 at the critical point $p^2$ with $F_1^2(p^2) = 0.4$. Now, using Lemma 5.10, we connect $F_2^0$ and $F_2^2$, let $G$ be the resulting function in $\{p \in \mathbb{R}^2 \colon 1 \leq |p| \leq 2\}$ (see Figure 11).

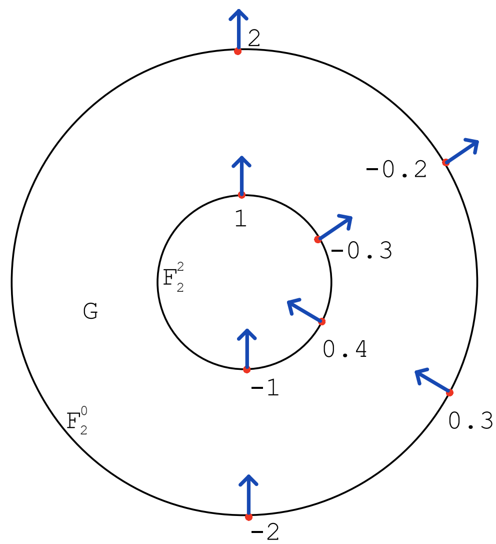



FIGURE 11. Connecting two functions in dimension 2

$G$ and $F_2^2$ satisfy the assumptions of Lemma 5.9, so there exists a Morse function function $F_3^2 \in C^\infty(B_{\mathbf{0}}^2(2))$ such that $\operatorname{Crit} F_3^2 = \{\mathbf{0}\}$, $F_3^2$ is equal to $F_2^2$ in a neighbourhood of the origin and is equal to $G$ in a neighbourhood of $2S^1$.

Now extend $F_2^0$ from $B_{\mathbf{0}}^2(2)$ to a function $F_3^0$ on $B_{\mathbf{0}}^2(2+\varepsilon)$ such that no new critical points occur and the Morse-Barannikov complexes for $F_2^0$ and $F_3^0$ near the boundary are the same. $F_3^2$ and $F_3^0$ satisfy the assumptions of Lemma 5.9 near $2S^1$, so there exists a Morse function $F$ on $B_{\mathbf{0}}^2(2+\varepsilon)$ such that $\operatorname{Crit} F\{\mathbf{0}\}$, $F$ is equal to $F_3^2$ in a neighbourhood of the origin and to $F_3^0$ in a neighbourhood $U$ of $(2+\varepsilon)S^1$.

We obtained a function $F|_U$ that can be extended to $B_{\mathbf{0}}^2(2+\varepsilon)$ in two different ways: by $F$ and by $F_3^0$. In the former case we have $\operatorname{Crit} F = \operatorname{Morse} F = \{\mathbf{0}\}$ and $\mu_F(\mathbf{0}) = 2$, and in the latter one we have $\operatorname{Crit} F_3^0 = \operatorname{Morse} F_3^0 = \{\mathbf{0}\}$ and $\mu_F(\mathbf{0}) = 0$. Now let us proceed with the general case.

**Theorem 6.1.** *Let $n \geq 2$ and $0 \leq \lambda \leq n-2$. Then there exists a Morse function $f\colon U \to \mathbb{R}$ in the neighbourhood $U$ of $S^{n-1}$ inside $B_{\mathbf{0}}^n(1)$ that admits both indices $\lambda$ and $\lambda+2$.*

*Proof.* First we construct functions $F^\lambda$ and $F^{\lambda+2}$ guided by the combinatorial procedure described in Proposition 4.5.

(1) $F_0^\lambda = F_0^{\lambda+2} = x^n$. The gradient-like vector fields $V = \operatorname{grad} x^n|_{S^{n-1}}$ for the standard round metric on the sphere, $\Phi_t$ is the flow on $S^{n-1}$ generated by $-V$. The critical points of $f_0^\lambda$ and $f_0^{\lambda+2}$ are $n = (0, \dots, 0, 1)$ and $s = (0, \dots, 0, -1)$.
(2) Let $\gamma$ be the flow line on $S^{n-1}$ through the point $p = (1, 0, \dots, 0)$. Denote $p_+ = \gamma(\varepsilon)$, $p_- = \gamma(-\varepsilon)$.
(3) Set $F_1^\lambda = F_0^\lambda \circ \Psi^\lambda$ and $F_1^{\lambda+2} = F_0^{\lambda+2} \circ \Psi^{\lambda+2}$, where $\Psi^\lambda$ and $\Psi^{\lambda+2}$ are diffeomorphisms of $B_{\mathbf{0}}^n(1)$ such that $\Psi^\lambda\big|_{S^{n-1}} = \Phi_{2\varepsilon}$ and $\Psi^{\lambda+1}\big|_{S^{n-1}} = \Phi_{-2\varepsilon}$.

(4) $F_2^\lambda$ and $F_2^{\lambda+2}$ are obtained by a standard birth at point $p$ from $F_1^\lambda$ and $F_1^{\lambda+2}$ respectively.
(5) $F_3^\lambda$ is a flip of $F_2^\lambda$ at $p_-$ of index $\lambda$, $F_3^\lambda$ is a flip of $F_3^{\lambda+2}$ at $p_+$ of index $\lambda+2$.
(6) Let $L\colon \mathbb{R} \to \mathbb{R}$ be an orientation preserving deffeomorphism such that

$$L(f_3^\lambda(s)) < f_3^{\lambda+2}(s) < f_3^{\lambda+2}(p_-) < L(f_3^\lambda(p_-)) < L(f_3^\lambda(p_+)) < f_3^{\lambda+2}(p_+) < f_3^{\lambda+2}(n) < L(f_3^\lambda(n)).$$

Set $F_4^\lambda = L \circ F_3^\lambda$ and $F_4^{\lambda+2} = F_3^{\lambda+2}$.
(7) Use Lemma 5.10 to connect $F_4^\lambda\big|_{S_\varepsilon^-}$ and $F_4^{\lambda+2}\big|_{S_\varepsilon^-}$ by a function $G\colon \{1 \le |x| \le 2\} \to \mathbb{R}$.
(8) Let $F_5$ be an extension of $F_4^\lambda$ to $B_{\mathbf{0}}^n(1+\varepsilon)$ such that no new critical points occur and the MB-complex of $f_4^\lambda$ does not change.
(9) The functions $F_4^{\lambda+2}\big|_{S_\varepsilon^-}$ and $G|_{S_\varepsilon^+}$ satisfy the assumptions of Lemma 5.9, let $F_5^{\lambda+2}$ be the resulting Morse function in $B_{\mathbf{0}}^n(2)$.
(10) The function $F_5^{\lambda+2}\big|_{2S_\varepsilon^-}$ and $F_5^\lambda\big|_{2S_\varepsilon^+}$ satisfy the assumptions of Lemma 5.9, let $F_6^{\lambda+2}$ be the resulting Morse function in $B_{\mathbf{0}}^n(2+2\varepsilon)$.

The functions $F_5^\lambda$ and $F_6^{\lambda+2}$ are equal in a neighbourhood of $\partial B_{\mathbf{0}}^n(2+2\varepsilon)$, $\operatorname{Crit} F_5^\lambda = \operatorname{Crit} F_6^{\lambda+2} = \{\mathbf{0}\}$, $\mu_{F_5^\lambda}(\mathbf{0}) = \lambda$, and $\mu_{F_6^{\lambda+2}}(\mathbf{0}) = \lambda+2$. Thus we constructed the desired functions. □

## 7. Conclusion

Now that we have constructed functions in a neighbourhood of spheres that admit two indices differing by two, we can outline a path to Conjecture 1.2. The combinatorial part is done in Proposition 4.6, and we believe one should follow the algorithm there. The obvious difficulty is that we do not yet developed a way to change the value of $f$ at a critical point in a controllable fashion. Some other technical difficulties will undoubtedly arise, yet we consider following the combinatorial procedure from the proposition promising and hope it will lead us to the proof of the conjecture.

## References

[1] Barannikov, S. The framed morse complex and its invariants. *American Mathematical Society Translations 2* (1994).
[2] Cerf, J. La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie. *Publications Mathématiques de l'IHÉS 39* (1970), 5–173.
[3] Cohen, R., Jones, J., and Segal, G. Morse theory and classifying spaces.
[4] Milnor, J., Siebenmann, L., and Sondow, J. *Lectures on the H-Cobordism Theorem*. Princeton University Press, 1965.
[5] Milnor, J., SPIVAK, M., and WELLS, R. *Morse Theory*, vol. 51 of *Annals of Mathematics Studies*. Princeton University Press, 1969.
[6] Nicolaescu, L. *An Invitation to Morse Theory*. Springer New York, 2011.
[7] Sharko, V. *Functions on Manifolds: Algebraic and Topological Aspects: Algebraic and Topological Aspects*. Translations of mathematical monographs. American Mathematical Society, 1993.
[8] Smale, S. On gradient dynamical systems. *Annals of Mathematics 74*, 1 (1961), 199–206.