Saint Petersburg State University

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Graduation qualification thesis

About conservativity of pure and weight-exact functors

Bachelor's program
"Mathematics"
Specialization and code: 01.03.01 "Mathematics"
Cipher of the EP: SV.5000.2016

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Saint Petersburg
2020

Abstract

In this paper we construct families of weight-conservative functors related to localizations of triangulated categories; in particular, our motivic functors are conservative and weight-exact with respect to the corresponding Chow weight structures. Families of this sort were shown to be useful in papers of T. Bachmann, M. Bondarko, and G. Tabuada, where they were applied to the study of Picard groups of (motivic) triangulated categories.

Key words and phrases Triangulated category, weight structure, pure functor, weight complex, weight-exact functor, conservativity, motives, pure functors, Voevodsky motives, Chow motives, Artin motives.

Contents

Introduction

In [BoT17] and [Bac16] certain functors from triangulated categories were shown to be useful for the study of their Picard groups. All the categories considered in these papers were endowed with so-called weight structures (as independently define by M. Bondarko and D. Pauksztello), and the ("pure") homological functors constructed there killed all objects of strictly negative and strictly positive weights. The advantages of functors of this type is that they have a simple description in terms of the heart $\underline{Hw}$ of the corresponding weight structure w (see Theorem 2.1.2 of [Bon18]); their properties imply that the corresponding group $\operatorname{Pic}(\underline{C})$ is isomorphic to $\operatorname{Pic}(\underline{Hw}) \bigoplus \mathbb{Z}$ if w is bounded, the tensor product of $\underline{C}$ restricts to $\underline{Hw}$, and $\underline{Hw}$ is semi-simple and *local* with respect to the tensor product (see Theorem 1.1 of [BoT17]).

The case where $\underline{Hw}$ is not semi-simple is more complicated (in general). To deal with it, in the current paper we construct certain pure functors corresponding to additive subcategories of $\underline{Hw}$; they are closely related to Verdier localizations. In some cases we also construct closely related exact functors F_i that are *weight-conservative*, i.e., their exists a weight structures v on the target such that the "w-range" of an object M of $\underline{C}$ equals the v-range of $(\prod F_i)(M)$ (cf. Definition 1.4.2 below; note that this assumption is stronger than conservativity if v is *non-degenerate*). It often happens that F_i respect tensor products; thus they can be used for the calculation of Picard groups.

These families of weight-conservative functors exist for certain subcategories of Voevodsky motives. As a particular case of this formalism, we recover the "fixed point functors" from Artin motives that were originally introduced in [Bac16].

Now we describe the contents of this text.

In §1 we recall some of the theory of weight structures, and use it for the construction of certain pure and weight-conservative functors corresponding to subcategories and localizations.

In §2 we demonstrate that our theory can be applied to (geometric) Voevodsky motives over a field; the corresponding adjoint functors come from base field extensions. In particular, we obtain certain "fixed point functors" from the subcategories of effective geometric Voevodsky motives whose dimension is bounded by some constant. We obtain an especially nice result in the dimension 0 case, i.e., for derived categories of Artin motives.

1 General weight structure theory

In this section we recall some of the theory of weight structures on triangulated categories. Moreover, we prove that certain (*pure*) homological functors can be used to calculate the weights of objects, and discuss the relation of this statement to *weight-conservativity*.

1.1 Some categorical notation and lemmas

- Given a category C and $X, Y \in \operatorname{Obj} C$ we will write $C(X,Y)$ for the set of morphisms from X to Y in C.
- For categories C', C we write $C' \subset C$ if C' is a full subcategory of C.
- Given a category C and $X, Y \in \operatorname{Obj} C$, we say that X is a *retract* of Y if id_X can be factored through Y.[1]
- A (not necessarily additive) subcategory $\underline{H}$ of an additive category C is said to be *retraction-closed* in C if it contains all retracts of its objects in C.
- For any $(C, \underline{H})$ as above the full subcategory $\operatorname{Kar}_C(\underline{H})$ of C whose objects are all retracts of (finite) direct sums of objects $\underline{H}$ in C will be called the *retraction-closure* of $\underline{H}$ in C; note that this subcategory is obviously additive and retraction-closed in C.
- The symbol $\underline{C}$ below will always denote some triangulated category; usually it will be endowed with a weight structure w. The symbols $\underline{C}'$ and $\underline{D}$ will also be used for triangulated categories only.
- We will write $K(\underline{B})$ for the homotopy category of (cohomological) complexes over $\underline{B}$. Its full subcategory of bounded complexes will be denoted by $K^b(\underline{B})$. We will write $M = (M^i)$ if M^i are the terms of the complex M.

[1] Clearly, if C is triangulated or abelian, then X is a retract of Y if and only if X is its direct summand.

- We will say that an additive covariant functor from $\underline{C}$ into $\underline{A}$ is *homological* if it converts distinguished triangles into long exact sequences.

  For a homological functor H and $i \in \mathbb{Z}$ we will write H_i for the composition $H \circ [-i]$.

- We call a category $\frac{A}{B}$ the *factor* of an additive category A by its full additive subcategory B if $\operatorname{Obj}\left(\frac{A}{B}\right) = \operatorname{Obj} A$ and

  $(\frac{A}{B})(X,Y) = A(X,Y)/(\sum_{Z \in \operatorname{Obj} B} A(Z,Y) \circ A(X,Z))$.

  It is easily seen that $\frac{A}{B}$ isomorphic to $\frac{A}{\operatorname{Kar}_A B}$

- For a small additive category $\underline{H}$ we will write $Pshv(\underline{B})$ for the category of additive contravariant functors from $\underline{H}$ into abelian groups.

- For an arbitrary small additive category $\underline{A}$ denote by $\mathcal{Y}_{\underline{A}} : \underline{A} \to Pshv(\underline{A})$ the Yoneda embedding. Certainly, $\mathcal{Y}_{\underline{A}}(X)(Y)$ is the functor $\underline{A}(Y, X)$.

We have an obvious

Proposition 1.1.1. (The universal property of factor category) For an arbitrary additive category $\underline{A}$ and full subcategory $\underline{N} \subset \underline{A}$ take the obvious quotient functor $F : \underline{A} \to \underline{A}/\underline{N}$. Assume that $G : \underline{A} \to \underline{B}$ is an additive functor into another additive category $\underline{B}$ that maps every object of $\underline{N}$ to 0.

Then there exists a unique additive functor $H : \underline{A}/\underline{N} \to \underline{B}$ such that $G = H \circ F$.

The following statement is crucial for the weight-conservativity statements below.

Lemma 1.1.2. Let $\underline{H}$ be a small additive category and $\underline{N} \subset \underline{H}$ be a full additive subcategory of $\underline{H}$. Consider the quotient functor $F : \underline{H} \to \underline{H}/\underline{N} = \underline{H'}$ and the restriction functor $\mathcal{R} : Pshv(\underline{H}) \to Pshv(\underline{N})$ that comes from the embedding $\underline{N} \to \underline{H}$, and take the following composite functors:

$$\mathcal{A} = \mathcal{Y}_{\underline{H'}} \circ F : \underline{H} \to Pshv(\underline{H'})$$

and

$$\mathcal{B} = \mathcal{R} \circ \mathcal{Y}_{\underline{H}} : \underline{H} \to Pshv(\underline{N}).$$

Then for any morphism $\phi : X \to Y$ in $\underline{H}$ the following conditions are equivalent:

(i) there exists $\xi : Y \to X$ such that $\phi \circ \xi = \mathrm{id}_Y$

(ii) the morphisms $\mathcal{A}(\phi)$ and $\mathcal{B}(\phi)$ are epimorphic.

Proof. The implication $(i) \Rightarrow (ii)$ is obvious. Let us prove that (ii) implies (i).

Let $F(\phi) = \phi' : X' \to Y'$ be the image of ϕ under the quotient functor. By condition (ii), $\mathcal{A}(\phi) = \mathcal{Y}_{\underline{H'}}(\phi')$ is a surjection. Since a morphism of presheaves is epimorphic if and only if it surjective on the level of objects, $\mathcal{Y}_{\underline{H'}}(\phi')(Y')$ is surjective as well. The morphism $\mathcal{Y}_{\underline{H'}}(\phi')(Y') : \underline{H}'(Y', X') \to \underline{H}'(Y', Y')$ is just the composition map $\omega \to \phi' \circ \omega$. Hence there exists $\theta \in \underline{H}'(Y', X')$ such that

$$\phi' \circ \theta = \mathrm{id}_{Y'}. \qquad (1.1.1)$$

Thus the first condition ($\mathcal{A}(\phi)$ surjective) implies that ϕ' is splits. To prove that ϕ splits as well we consider $t = \operatorname{Coker} \mathcal{Y}_{\underline{H}}(\phi) : \mathcal{Y}_{\underline{H}}(Y) \to T$. Here T is a presheaf on $\underline{H}$ and t is a natural transformation. The restriction functor $\mathcal{R}$ respects cokernels; thus $T|_{\underline{N}} = \mathcal{R}(T) = \mathcal{R}(\operatorname{Coker} \mathcal{Y}_{\underline{H}}(\phi)) = \operatorname{Coker}(\mathcal{R} \circ \mathcal{Y}_{\underline{H}}(\phi)) = \operatorname{Coker}(\mathcal{B}(\phi)) = 0$, since $\mathcal{B}$ is a surjection by our assumptions. By Proposition 1.1.1, T factors through $\underline{H}'$, i.e., there exists $S \in Pshv(\underline{H}')$ such that $T = S \circ F$.

The image of ϕ under the quotient functor splits, and its image under T also does. A bit more precisely, $\mathrm{id}_{T(Y)} = T(\mathrm{id}_Y) = S(F(\mathrm{id}_Y)) = S(\mathrm{id}_{F(Y)}) = S(\mathrm{id}_{Y'})$. Hence by (1.1.1), $\mathrm{id}_{T(Y)} = S(\mathrm{id}_{Y'}) = S(\phi' \circ \theta) = S(\theta) \circ S(\phi') = S(\theta) \circ T(\phi)$. Thus $T(\phi)$ is injective. Since the morphisms in $Pshv(\underline{H})$ are natural transformations, we have the following commutative diagram:

$$\begin{array}{ccccc}
\underline{H}(Y,X) & \xrightarrow{\mathcal{Y}_{\underline{H}}(\phi)_Y} & \underline{H}(Y,Y) & \xrightarrow{t_Y} & T(Y) \\
\downarrow & & \downarrow \mathcal{Y}_{\underline{H}}(Y)(\phi) & & \downarrow T(\phi) \\
\underline{H}(X,X) & \xrightarrow{\mathcal{Y}_{\underline{H}}(\phi)_X} & \underline{H}(X,Y) & \xrightarrow{t_X} & T(X)
\end{array} \qquad (1.1.2)$$

Now our statement can be easily verified by diagram chasing.

We want to prove that there exists $\xi \in \underline{H}(Y,X)$ such that $\phi \circ \xi = \mathrm{id}_Y$, or equivalently $\mathrm{id}_Y = \mathcal{Y}_{\underline{H}}(\phi)_Y(\xi)$. It means just that $\mathrm{id}_Y \in \operatorname{Im} \mathcal{Y}_{\underline{H}}(\phi)_Y = \operatorname{Ker} t_Y$, since t_Y is the cokernel of $\mathcal{Y}_{\underline{H}}(\phi)_Y$. Since $T(\phi)$ is injective, $\operatorname{Ker} t_Y = \operatorname{Ker} T(\phi) \circ t_Y = \operatorname{Ker} t_X \circ \mathcal{Y}_{\underline{H}}(Y)(\phi)$. Now we have $\mathrm{id}_Y \in \operatorname{Ker} t_X \circ \mathcal{Y}_{\underline{H}}(Y)(\phi) \iff \mathcal{Y}_{\underline{H}}(Y)(\phi)(\mathrm{id}_Y) \in \operatorname{Ker} t_X$ and $\mathcal{Y}_{\underline{H}}(Y)(\phi)(\mathrm{id}_Y) = \phi = \mathcal{Y}_{\underline{H}}(\phi)_X(\mathrm{id}_X) \in \operatorname{Im} \mathcal{Y}_{\underline{H}}(\phi)_X = \operatorname{Ker} t_X$, and we are done. □

Below we will also apply the following simple abstract nonsense lemma.

Lemma 1.1.3. Let $\underline{A}$ be an additive category and $F : \underline{A} \to \underline{A}$ be an additive functor. Suppose we have an adjunction $(\epsilon, \eta) : F \vdash F$. Then for any objects $X, Y \in \underline{A}$ we have $\frac{\underline{A}}{F\underline{A}}(X,Y) = \frac{\underline{A}(X,Y)}{\epsilon_Y F\underline{A}(FX,FY)\eta_X}$.

Proof. Obviously,

$$\epsilon_Y F\underline{A}(FX,FY)\eta_X \subset \sum_{Z \in \operatorname{Obj} F\underline{A}} \underline{A}(Z,Y) \circ \underline{A}(X,Z).$$

Now consider the composition $X \xrightarrow{\phi} FZ \xrightarrow{\psi} Y$. The counit-unit equation says that $id_{FZ} = \epsilon_{FZ} \circ F\eta_Z = F\epsilon_Z \circ \eta_{FZ}$. Hence $\psi \circ \phi = \psi \circ \epsilon_{FZ} \circ F\eta_Z \circ F\epsilon_Z \circ \eta_{FZ} \circ \phi$. Since ϵ and η are natural transformations, we have $\psi \circ \epsilon_{FZ} = \epsilon_Y \circ F^2\psi$ and $\eta_{FZ} \circ \phi = F^2\phi \circ \eta_X$. Thus $\psi \circ \phi = \epsilon_Y \circ F^2\psi \circ F\eta_Z \circ F\epsilon_Z \circ F^2\phi \circ \eta_X \in \epsilon_Y F\underline{A}(FX,FY)\eta_X$. Hence

$$\sum_{Z \in \operatorname{Obj} F\underline{A}} \underline{A}(Z,Y) \circ \underline{A}(X,Z) \subset \epsilon_Y F\underline{A}(FX,FY)\eta_X,$$

and we are done. □

Corollary 1.1.4. In the notation of previous lemma we have $\frac{\underline{A}}{F\underline{A}}(X,Y) = \frac{\underline{A}(X,Y)}{\epsilon_Y A(X,F^2Y)}$

1.2 Weight structures: basics

Definition 1.2.1. A pair of subclasses $\underline{C}_{w\le 0}, \underline{C}_{w\ge 0} \subset \operatorname{Obj}\underline{C}$ will be said to define a weight structure w on a triangulated category $\underline{C}$ if they satisfy the following conditions.

(i) $\underline{C}_{w\ge 0}$ and $\underline{C}_{w\le 0}$ are retraction-closed in $\underline{C}$ (i.e., contain all $\underline{C}$-retracts of their objects).

(ii) **Semi-invariance with respect to translations.**
$\underline{C}_{w\le 0} \subset \underline{C}_{w\le 0}[1]$ and $\underline{C}_{w\ge 0}[1] \subset \underline{C}_{w\ge 0}$.

(iii) **Orthogonality.**
$\underline{C}_{w\le 0} \perp \underline{C}_{w\ge 0}[1]$.

(iv) **Weight decompositions**.
For any $M \in \operatorname{Obj}\underline{C}$ there exists a distinguished triangle

$$LM \to M \to RM \to LM[1]$$

such that $LM \in \underline{C}_{w\le 0}$ and $RM \in \underline{C}_{w\ge 0}[1]$.

If $\underline{C}$ is endowed with a weight structure then we will say that $\underline{C}$ is a *weighted* category.

We will also need the following definitions.

Definition 1.2.2. Let $i, j \in \mathbb{Z}$; assume that a triangulated category $\underline{C}$ is endowed with a weight structure w.

1. The full subcategory $\underline{Hw}$ of $\underline{C}$ whose objects are $\underline{C}_{w=0} = \underline{C}_{w\ge 0} \cap \underline{C}_{w\le 0}$ is called the *heart* of w.

2. $\underline{C}_{w\ge i}$ (resp. $\underline{C}_{w\le i}$, resp. $\underline{C}_{w=i}$) will denote the class $\underline{C}_{w\ge 0}[i]$ (resp. $\underline{C}_{w\le 0}[i]$, resp. $\underline{C}_{w=0}[i]$).

3. $\underline{C}^b \subset \underline{C}$ will be the category whose object class is $\cup_{i,j\in\mathbb{Z}}\underline{C}_{[i,j]}$.

   We will say that its objects are the *w-bounded* objects of $\underline{C}$.

4. Moreover, the elements of $\cup_{i\in\mathbb{Z}}\underline{C}_{w\ge i}$ (resp. of $\cup_{i\in\mathbb{Z}}\underline{C}_{w\le i}$) will be said to be *$w$-bounded below* (resp. above).

   We will say that $(\underline{C}, w)$ is *bounded* (resp. bounded below, resp. above) if all objects of $\underline{C}$ are w-bounded (resp. bounded below, resp. above).

5. Let $\underline{C}'$ be a triangulated category endowed with a weight structure w'; let $F : \underline{C} \to \underline{C}'$ be an exact functor.

   Then F is said to be *weight-exact* (with respect to w, w') if it maps $\underline{C}_{w\le 0}$ into $\underline{C}'_{w'\le 0}$ and sends $\underline{C}_{w\ge 0}$ into $\underline{C}'_{w'\ge 0}$.

6. Let $\underline{D}$ be a full triangulated subcategory of $\underline{C}$.

   We will say that w *restricts* to $\underline{D}$ whenever the couple $(\underline{C}_{w\le 0} \cap \operatorname{Obj}\underline{D}$, $\underline{C}_{w\ge 0} \cap \operatorname{Obj}\underline{D})$ is a weight structure on $\underline{D}$.

Remark 1.2.3. It is easily seen that the notion of a weight structure is self-dual, i.e., for w as above and $\underline{D} = \underline{C}^{op}$ (so, $\operatorname{Obj}\underline{D} = \operatorname{Obj}\underline{C}$) there exists an (opposite) weight structure w' on $\underline{D}$ such that $\underline{D}_{w'\le 0} = \underline{C}_{w\ge 0}$ and $\underline{D}_{w'\ge 0} = \underline{C}_{w\le 0}$.

Lemma 1.2.4. Let $\underline{C}$ be a weighted category with heart $\underline{H}$, and $\underline{H}' \subset \underline{H}$ an additive subcategory. Let $\underline{C}'$ be the thick triangulated category *generated* by $\underline{H}'$ in $\underline{C}$ (that is, the smallest full thick triangulated subcategory of $\underline{C}$ containing $\underline{H}'$).

Then the weight structure of $\underline{C}$ restricts to $\underline{C}'$.

Proof. The statement easily follows from Corollary 2.1.2 of [BoS18]. □

1.3 Pure functors

Let us define an important class of (co)homological functors from categories endowed with weight structures.

Definition 1.3.1. Assume that $\underline{C}$ is endowed with a weight structure w.

We will say that a (co)homological functor H from $\underline{C}$ into an abelian category $\underline{A}$ is *w-pure* (or just pure if the choice of w is clear) if H kills both $\underline{C}_{w\geq 1}$ and $\underline{C}_{w\leq -1}$.

To ensure that the pure functors we need exist, we recall the following statement.

Proposition 1.3.2. The correspondence sending a pure functor $H : \underline{C} \to \underline{A}$ into its restriction to $\underline{Hw}$ gives an equivalence of the (not necessarily small) category of pure functors of this sort with the category of additive functors $\underline{Hw} \to \underline{A}$.

Proof. Immediate from Theorem 2.1.2 of [Bon18]. □

Inverting this correspondence (cf. Proposition 1.5.1(3) below), for an additive functor $F : \underline{Hw} \to \underline{A}$ we will denote by $H^F : \underline{C} \to \underline{A}$ the corresponding pure (homological) functor.

1.4 Weight structures in localizations and detecting weights

Proposition 1.4.1. 1. Let $\underline{D} \subset \underline{C}$ be a triangulated subcategory of $\underline{C}$; suppose that w induces a weight structure on $\underline{D}$ (i.e., $\operatorname{Obj} \underline{D} \cap \underline{C}_{w\leq 0}$ and $\operatorname{Obj} \underline{D} \cap \underline{C}_{w\geq 0}$ give a weight structure on $\underline{D}$). We denote the heart of the latter weight structure by HD.

Then w induces a weight structure on $\underline{C}/\underline{D}$ (the localization, i.e., on the Verdier quotient of $\underline{C}$ by $\underline{D}$). Being more precise, the retraction-closures of $\underline{C}_{w\leq 0}$ and $\underline{C}_{w\geq 0}$ in $\underline{C}/\underline{D}$ give a weight structure on $\underline{C}/\underline{D}$ (note that $\operatorname{Obj} \underline{C} = \operatorname{Obj} \underline{C}/\underline{D}$).

2. The heart $H(\underline{C}/\underline{D})$ of this weight structure is the retraction-closure of $\frac{\underline{Hw}}{\underline{HD}}$ in $\underline{C}/\underline{D}$.

3. If $\underline{C}, w$ is bounded above, below, or both, then $\underline{C}/\underline{D}$ is so as well.

Proof. See Proposition 8.1.1 and Remark 8.1.3(5) of [Bon10]. □

Definition 1.4.2. Let $F : \underline{C} \to \underline{C}'$ be a weight-exact functor, where $\underline{C}'$ is a triangulated category endowed with a weight structure w'.

We will say that F is right (resp. left) weight-conservative if given $X \in \underline{C}$ with $F(X) \in \underline{C}'_{w'\geqslant 0}$ (resp. $F(X) \in \underline{C}'_{w'\leqslant 0}$) we have $X \in \underline{C}_{w\geqslant 0}$ (resp. $X \in$

$\underline{C}_{w\leqslant 0}$). A family of functors will be said to be right (left) weight-conservative if its product is.

We say that F (or a family of functors) is weight-conservative if it both right and left weight-conservative.

Note that a functor F is right (left) weight-conservative if and only if the corresponding functor $F^{op}: \underline{C}^{op} \to \underline{D}^{op}$ is left (right) weight-conservative (with respect to the opposite weight structures; see Remark 1.2.3).

Definition 1.4.3. We will say that a functor $F: C \to D$ detects sections if a C-morphism $f: X \to Y$ admits a section whenever $F(f)$ does.

We say that a family of functors detects sections if its product does.

We need the following lemma.

Lemma 1.4.4. Let $F_i: C \to D_i$ be a set of weight-exact triangulated functors, and that the weight structure w on C is bounded below.

Then $\{F_i\}$ is right weight-conservative if and only if (the family of) the restrictions of F_i to $\underline{Hw}$ is detects sections.

Proof. This is Corollary 5.15 in [Bac16]; One can also prove this statement similarly to Proposition 2.2.3(1) of [Bon18]. □

Theorem 1.4.5. Let $\underline{C}$ be a small triangulated category equipped with a weight structure w, $\underline{N} \subset \underline{Hw}$ is an additive subcategory. Take $\underline{Hw}' = \underline{Hw}/\underline{N}$ and denote by $F: \underline{Hw} \to \underline{Hw}'$ the obvious factorization functor. Consider the following Yoneda-type functors:

$$\mathcal{A}: \underline{Hw} \to Pshv(\underline{N})$$

$$A \to (B \to \underline{Hw}(B, A))$$

and

$$\mathcal{B}: \underline{Hw} \to Pshv(\underline{Hw}')$$

$$A \to (B \to \underline{Hw}'(B, F(A))).$$

Then for any $M \in Obj(\underline{C})$ and $n \in \mathbb{Z}$ we have the following: M is w-bounded below and $H_i^{\mathcal{A}}(M) = H_i^{\mathcal{B}}(M) = 0$ for all $i < n$ if and only if $M \in \underline{C}_{w\geqslant n}$.

Proof. This is an easy combination of Lemmas 1.1.2 and 1.4.4. □

Now we study a setting where these pure functors come from certain weight-exact ones.

Theorem 1.4.6. Let w be a bounded below (resp. above) weight structure on $\underline{C}$ and let $f_s^*: \underline{C} \to \underline{D}_s$ be a family of weight-exact functors. Suppose that $f_{!s}: \underline{D}_s \to \underline{C}$ (resp. $f_{*s}: \underline{D}_s \to \underline{C}$) are weight-exact functors left (resp. right) adjoint to f_s^* for all s. Denote by $\underline{C}'$ the thick triangulated subcategory of $\underline{C}$ generated by $\cup_s f_{!s} f_s^* \underline{C}_{w=0}$ (resp. by $\cup_s f_{*s} f_s^* \underline{C}_{w=0}$), and let $F: \underline{C} \to \underline{C}/\underline{C}'$ be the Verdier localization functor.

Suppose also that w is bounded below (resp. above). Then the family $\{f_s^*\} \cup \{F\}$ is right (resp. left) weight-conservative.

Proof. It is easily seen that the left weight-conservative version of our statement (stated in brackets) is the dual to the right weight-conservative one (see Remark 1.2.3).

Thus it suffices to treat the functors $f_{!s}$. Let us apply Lemma 1.4.4. We set $\underline{N} = \oplus_s f_{!s} f_s^* \underline{Hw}$. Let $\phi : X \to Y$ be an $\underline{Hw}$-morphism. Suppose that the morphisms $F(\phi)$ and $f_s^*(\phi)$ admit sections.

Assume first that the category $\underline{Hw}$ is small. Consider the functors $\mathcal{A}$ and $\mathcal{B}$ from Lemma 1.1.2 (for $\underline{H} = \underline{Hw}$). By this lemma it suffices to verify that both $\mathcal{A}(\phi)$ and $\mathcal{B}(\phi)$ are epimorphic. Now, $\mathcal{B}(\phi)$ epimorphic since $F(\phi)$ is. Next, N generated by $\cup_s f_{!s} f_s^* \underline{H}$; hence to prove that $\mathcal{A}(\phi)$ epimorphic it suffices to check that $\mathcal{A}(\phi)(T)$ epimorphic for $T \in \cup_s f_{!s} f_s^* \underline{H}$. Let $T = f_{s!} f_s^*(M)$ for some $M \in \underline{H}$. By adjunction we have the following commutative diagram:

$$\begin{array}{ccc} \underline{H}(T, X) & \xrightarrow{\mathcal{A}(\phi)(T)} & \underline{H}(T, Y) \\ \downarrow & & \downarrow \\ \underline{H}(f_s^*(M), f_s^*(X)) & \longrightarrow & \underline{H}(f_s^*(M), f_s^*(Y)) \end{array} \tag{1.4.1}$$

Here the vertical arrows are isomorphisms and the bottom arrow is epimorphic since $f_s^*(\phi)$ is. Hence the top arrow is epimorphic as well.

Lastly, if $\underline{Hw}$ is not small then it is easily seen that we can replace the presheaf functors $\mathcal{A}$ and $\mathcal{B}$ by the corresponding collections of functors $\mathcal{A}_X = \underline{H}'(X, F(-))$ and $\mathcal{B}_Y = \underline{Hw}(Y, -)$ for X running through $\underline{C}_{w=0} = \operatorname{Obj} \underline{H}'$ and Y running through $\operatorname{Obj} \underline{N}$. Note however that we will not apply this case of our theorem in the next section.

□

The following statement follows from our theorem immediately.

Corollary 1.4.7. Let the assumptions of Theorem 1.4.6 holds; assume that $\underline{Hw} \subset \operatorname{Kar}_{\underline{C}} N$.

Then the family $\{f_s^*\}$ is right (left) weight-conservative.

1.5 Weight complexes

Now we recall (a little of) the theory of so-called "strong" weight complex functors.

Recall that the category of $K^b(\underline{Hw})$ endowed with a stupid weight-structure w^{st}. A bit more precise we take $K^b(\underline{Hw})_{w^{st} \le 0}$ (resp. $K^b(\underline{Hw})_{w^{st} \ge 0}$) to be the class of objects in $K^b(\underline{Hw})$ that are homotopy equivalent to those complexes that are concentrated in degrees ≥ 0 (resp. ≤ 0). See also Remark 1.2.3.(1) in [BoS18].

Proposition 1.5.1. Assume that $\underline{C}$ possesses an ∞-enhancement (see §1.1 of [Sos19] for the corresponding references), and is endowed with a bounded weight structure w.

Then there exists an exact functor $t^{st} : \underline{C} \to K^b(\underline{Hw})$, $M \mapsto (M^i)$, that enjoys the following properties.

1. The composition of the embedding $\underline{Hw} \to \underline{C}$ with t^{st} is isomorphic to the obvious embedding $\underline{Hw} \to K^b(\underline{Hw})$.

2. The functor t^{st} is weight-exact and weight-conservative, i.e., an object M of $\underline{C}$ belongs to $\underline{C}_{w\leq n}$ (resp. to $\underline{C}_{w\geq n}$) if and only if $t^{st}(M)$ belongs to $K(\underline{Hw})_{w^{st}\leq n}$ (resp. to $K(\underline{Hw})_{w^{st}\geq n}$).

3. If $F : \underline{Hw} \to \underline{A}$ is an additive functor as in Definition 1.3.1 then the corresponding pure functor H^F can be computed as the functor that sends M into the zeroth homology of the complex $(F(M^i))$.

Proof. All the statements easily follow from Corollary 3.5 of [Sos19] combined with Proposition 1.3.4 and Theorem 2.1.2 of [Bon18]. □

Remark 1.5.2. To drop the assumption that $\underline{C}$ possesses an ∞-enhancement (and w is bounded) one can consider the so-called weak weight complex functor $t : \underline{C} \to K_{\mathfrak{w}}(\underline{Hw})$ (see [Bon18]).

2 Applications to Voevodsky motives

In this section we apply the results above to Voevodsky motives (over a field). Below we will always assume that the (coefficient) ring R is associative unital commutative, and the exponential characteristic of our fields is invertible in R.

2.1 A reminder on motives and Chow weight structures

We recall some properties of the triangulated categories $DM^{gm}(k,R)$ of Voevodsky's motives over a perfect field k, where R is as above.

For $X \in Sm(k)$ we will write $M_R(X) = M(X) \in \operatorname{Obj} DM^{gm}(k,R)$ for the motif of X.

Proposition 2.1.1. I.1. There exist an embedding $\mathrm{Chow}(k,R) \to DM^{gm}(k,R)$ of category of Chow motives to $DM^{gm}(k,R)$ such that following diagram commutes

$$\begin{array}{ccc} SmProj(k) & \longrightarrow & Sm(k) \\ \Big\downarrow{\scriptstyle M_{\mathrm{Chow},R}} & & \Big\downarrow{\scriptstyle M_R} \\ Chow(k,R) & \longrightarrow & DM^{gm}(k,R), \end{array} \tag{2.1.1}$$

where $M_{\mathrm{Chow},R}$ is the usual (covariant) Chow motif functor.

2. There exists a (unique) bounded weight structure $w = w_{Chow}$ on $DM^{gm}(k,R)$ whose heart equals $\mathrm{Chow}(k,R)$.

3. $DM^{gm}(k,R)$ is a tensor category, and for any pair X, Y of smooth schemes over k the corresponding projections give an isomorphism $M(X\times Y) \simeq M(X)\otimes M(Y)$.

II. Let $f : \mathrm{Spec}(l) \to \mathrm{Spec}(k)$ be an algebraic extension, $X \in \operatorname{Obj} DM^{gm}(l,R)$, and $Y \in \operatorname{Obj} DM^{gm}(k,R)$. Then the following statements are valid.

1. There is an exact functor $f^* : DM^{gm}(k,R) \to DM^{gm}(l,R)$ such that $f^*(M(Z)) \cong M(Z_l)$ for any smooth variety Z/l. Moreover, for any morphism $g : \mathrm{Spec}(l') \to \mathrm{Spec}(l)$ of perfect field spectra we have $(fg)^* = g^*f^*$.

2. If f is finite then there is a functor $f_* = f_! : DM^{gm}(l,R) \to DM^{gm}(k,R)$ such that $f_!(M_l(Z)) \cong M_k(Z)$; here Z is a smooth projective l-variety that we

consider as a k-scheme in the right hand side. Moreover, for any finite morphism $g : \mathrm{Spec}(l') \to \mathrm{Spec}(l)$ we have $(fg)_! = f_!g_!$.

3. Moreover, in this case the functors f^* and $f_!$ are both left and right adjoint to each other, and the projection formula $f_!(X \otimes f^*(Y)) \simeq f_!(X) \otimes Y$ holds.

In particular $f_!f^*(Y) = M(\mathrm{Spec}(l)) \otimes Y$.

III. The functors f^* and $f_!$ are weight-exact.

Proof. Assertion I.1 originates from [Voe00]; cf. [BeV08] for the general case. The existence of w_{Chow} is given by Proposition 2.3.2 of [BoI15]; see also Theorem 2.2.1 of [Bon11]. I.3 is given by Proposition 2.1.3. of [Voe00].

Assertion II easily follows from Theorem 3.1 and Corollary 3.2(2) of [CiD15].

Assertion III is given by Theorem 2.2.1(2) of [BoI15]. □

2.2 Some motivic notation

We fix some perfect field k (and suppose that is exponential characteristic is invertible in R). We will write k^s for the separable closure of k.

We choose a prime number p. k is said to be *p-special* if every finite extension l/k is a p-extension.

We recall that for any field k and prime p there exists an extension k_p/k such that k_p is p-special and every finite subextension k_p of l/k has degree $[l : k]$ coprime to p. This fact easily follows from infinite Galois theory and is given for example by Proposition 101.16. of [EKM].

Definition 2.2.1. Let $S \subset SmProj(k)$ be a set of smooth projective schemes over field k. We write $D\langle S\rangle M(k, R)$ for the thick triangulated subcategory of $DM^{gm}(k, R)$ generated by S.

In this section we will deal with the following situation. For every algebraic extension l/k we are given a set $S_l \in SmProj(l)$. We will assume it is "functorial" in the following sense: for any morphism $f : \mathrm{Spec}(l') \to \mathrm{Spec}(l)$ we have $f^*(D\langle S_l\rangle M(l, R)) \subset D\langle S_{l'}\rangle M(l', R)$. Moreover, if f is finite then we assume $f_!(D\langle S_{l'}\rangle M(l', R)) \subset D\langle S_l\rangle M(l, R)$. We write $D\langle S\rangle M(l, R)$ for $D\langle S_l\rangle M(l, R)$. Note that the Chow weight structure on $DM^{gm}(k, R)$ restricts to $D\langle S\rangle M(k, R)$ by Lemma 1.2.4. A special case of a family of categories of this sort is Artin motives; we will discuss it in Section 2.4 below. More generally, one may take the subcategory generated by motives of (smooth projective) varieties of dimension at most d (for some $d \geq 0$).

For any extension l/k we will write $f_l : \mathrm{Spec}(l) \to \mathrm{Spec}(k)$ for the corresponding morphism of spectra and C_l for the thick triangulated category of $D\langle S\rangle M(l, R)$ generated by $M \otimes \mathrm{Spec}(l')$ for $M \in D\langle S\rangle M(l, R)_{w=0}$ and l'/l finite. Denote by F_l the localization functor $D\langle S\rangle M(l, R) \to D\langle S\rangle M(l, R)/C_l = D\langle S\rangle M'(l, R)$, and let $\Phi_l = F_l \circ f_l^*$. We will prove that the family $\{\Phi_l\}$ is weight-conservative (see Theorem 2.3.5 and Theorem 2.4.4).

We will also consider a "relative" version of the functors Φ_l. For any extension K/k and its subextension l/k define C_l^K to be the thick triangulated subcategory of $D\langle S\rangle M(l, R)$ generated by $M \otimes \mathrm{Spec}(l')$ for $M \in D\langle S\rangle M(l, R)_{w=0}$ and l'/l being a finite subextension of K/l. The Chow weight structure restricts to C_l^K by Lemma 1.2.4. Let F_l^K be the localization functor $D\langle S\rangle M(l, R) \to$

$D\langle S\rangle M(l,R)/C_l^K = D\langle S\rangle M'_K(l,R)$. It is weight-exact with respect to the corresponding weight structure by Proposition 1.4.1. Denote $\Phi_l^K = F_l^K \circ f_l^*$. We will prove that for any K the family $\{\Phi_l^K\}$ is conservative (see Theorem 2.3.5).

Note that $F_l = F_l^{k^s}$ and $\Phi_l = \Phi_l^{k^s}$. Also, $\Phi_K^K = f_K^*$. Φ_l^K is weight-exact as composition of weight-exact functors. We recall that the functors F_l^K and f_l^* are monoidal; hence all Φ_l^K also are.

2.3 Motivic "weight detection" statements

Theorem 2.3.1. Let $f_s : \mathrm{Spec}(l_s) \to \mathrm{Spec}(k)$ be a family of finite separable extensions. Let $\underline{C}$ be the thick triangulated subcategory of $DM^{gm}(k,R)$ generated by $M \otimes \mathrm{Spec}(l_s)$ for all $M \in D\langle S\rangle M(k,R)_{w=0}$ and all s; consider the localization functor $F : D\langle S\rangle M(k,R) \to D\langle S\rangle M(k,R)/\underline{C}$. Then the family $\{f_s^*\} \cup \{F\}$ is weight-conservative.

Proof. Recall that the Chow weight structure restricts to $\underline{C}$ by Lemma 1.2.4. Hence the functor F is weight-exact with respect to the corresponding weight structures according to Proposition 1.4.1. We have adjunctions $f_{!s} \vdash f_s^*$ and $f_s^* \vdash f_{!s}$. Note that the functors f_s^* and $f_{s!}$ are weight-exact. The composition $f_{s!} f_s^*(M)$ equals $M \otimes M(\mathrm{Spec}(l_s))$. Thus the statement follows from Theorem 1.4.6. ☐

Corollary 2.3.2. Let $f_s : \mathrm{Spec}(l_s) \to \mathrm{Spec}(k)$ be a family of finite separable extensions. Assume that the greatest common divisor of degrees $d_s = [l_s : k]$ is invertible in R. Then the family $\{f_s^*\}$ is weight-conservative.

Proof. Consider the transpose f_s^T of f_s given by the graph $\Gamma_{f_s} \subset \mathrm{Spec}(l_s) \times \mathrm{Spec}(k) \simeq \mathrm{Spec}(k) \times \mathrm{Spec}(l_s)$. The composite $f_s f_s^T$ equals $d_s\,\mathrm{id}$. The greatest common divisor of these degrees is invertible in R; hence there exist $s_1, ..., s_n$ and $a_1, ..., a_n$ such that $t = \sum\limits_{i=1}^{n} a_i d_{s_i}$ is invertible in R. Let

$$h = \sum_{i=1}^{n} a_i f_{s_i}^T : M(\mathrm{Spec}(k)) \to \oplus_{i=1}^n M(\mathrm{Spec}(l_{s_i})).$$

The composite $\sqcup_{i=1}^n f_{s_i} \circ h$ equals $\sum\limits_{i=1}^{n} a_i f_{s_i} f_{s_i}^T = t\,\mathrm{id}$. Thus the motif $M(\mathrm{Spec}(k))$ is a retract of $\oplus_{i=1}^n M(\mathrm{Spec}(l_{s_i}))$. Next, $M = M \otimes M(\mathrm{Spec}(k))$ is a retract of $M \otimes \oplus_{i=1}^n M(\mathrm{Spec}(l_{s_i})) = \oplus_{i=1}^n f_{s!} f_s^*(M)$. Hence our statement follows from Corollary 1.4.7. ☐

Let us recall some "continuity" properties of Chow weight structure (cf. §4.3 of [CiD19]).

Lemma 2.3.3. Let $f : \mathrm{Spec}(K) \to \mathrm{Spec}(k)$ be a separable extension of k, and assume that $K = \varinjlim k_s$, that is, the direct limit of some subextensions. Then for any $N \in DM(k,R)$ we have $f^*(N) \in DM(K,R)_{w\geqslant 0}$ ($f^*(N) \in DM(K,R)_{w\leqslant 0}$) if and only if for some s and $g : \mathrm{Spec}(k_s) \to \mathrm{Spec}(k)$ we have $g^*(N) \in DM(l,R)_{w\geqslant 0}$ ($g^*(N) \in DM(l,R)_{w\leqslant 0}$).

Proof. The proof is similar to that of [Bon15, Theorem 2.3.1(V.4)]; cf. also Lemma 2.2.4 of [BoI15]. ☐

Now we can prove an "infinite" version of Theorem 2.3.1.

Theorem 2.3.4. Let $g : \mathrm{Spec}(K) \to \mathrm{Spec}(k)$ be a separable extension. For every finite non-trivial separable subextension l/k of K/k let D_l be the thick triangulated subcategory of $D\langle S\rangle M(k,R)$ generated by $M \otimes \mathrm{Spec}(l)$ for $M \in D\langle S\rangle M(k,R)_{w=0}$. Consider the localization functor $G_l : D\langle S\rangle M(k,R) \to D\langle S\rangle M(k,R)/D_l$. Then the family $\{G_l\} \cup \{g^*\}$ is weight-conservative.

Proof. Let us prove that this family is right weight-conservative. It will be left weight-conservative for the categorically dual reason.

Let $N \in D\langle S\rangle M(k,R)$. Since $g^*(N) \in D\langle S\rangle M(K,R)_{w\geqslant 0}$, by Lemma 2.3.3 for some finite subextension $f : \mathrm{Spec}(l) \to \mathrm{Spec}(k)$ we have $f^*(N) \in D\langle S\rangle M(l,R)_{w\geqslant 0}$. By Theorem 1.4.6 the family $\{G_l, f^*\}$ is right weight-conservative. Hence $N \in D\langle S\rangle M(k,R)_{w\geqslant 0}$. □

Next we will prove that the functors Φ_l from Section 2.2 form a weight-conservative family. They are monoidal and their codomains are understandable is some cases (see Corollary 2.4.2 below). Hence they can be used for calculating Picard groups (see [Bac16]).

Theorem 2.3.5. The family $\{\Phi_l^K\}_{K/l/k}$ is weight-conservative, where l runs through subextensions of K/k. In particular the family $\{\Phi_l\}$ is weight-conservative if l runs through all separable extensions l/k.

Proof. Let us prove that this family is right weight-conservative. It will be left weight-conservative for essentially the same reason.

Suppose that for some $X \in D\langle S\rangle M(k,R)$ we have $\Phi_l^K(X) \in D\langle S\rangle M'_K(l,R)_{w\geqslant 0}$ for any subextension $K/l/k$ but $X \notin D\langle S\rangle M(k,R)_{w\geqslant 0}$.

Let P be a set of subextensions $K/l/k$ such that $f_l^*(X) \notin D\langle S\rangle M(l,R)_{w\geqslant 0}$. Note that $k \in P$, and therefore P is non-empty. This set is partially ordered by inclusion. Let $T \subset P$ be a chain. We set $F = \varinjlim_T l = \cup_{l\in T} l$. By Lemma 2.3.3 we have $f_F^*(X) \notin D\langle S\rangle M(F,R)_{w\geqslant 0}$; thus $F \in P$ is an upper bound for T. Hence by Zorn's lemma P contains a maximal element l.

Since l is maximal, for every subextension $g_{l'} : \mathrm{Spec}(l') \to \mathrm{Spec}(l)$ of K we have $g_{l'}^*(f_l^*(X)) = f_{l'}^*(X) \in D\langle S\rangle M(l',R)_{w\geqslant 0}$. By our assumptions, $F_l^K(f_l^*(X)) = \Phi_l^K(X) \in D\langle S\rangle M'_K(l,R)_{w\geqslant 0}$. The family $\{g_{l'}\} \cup \{F_l^K\}$ is weight-conservative by Theorem 2.3.1; thus $f_l^*(X) \in D\langle S\rangle M(l,R)_{w\geqslant 0}$ and we obtain a contradiction.

□

Remark 2.3.6. By Corollary 2.3.2, the functor Φ_l^K is zero if the extension K/l contains a finite subextension whose degree invertible in R. In the general case we can compute the hearts of the corresponding localizations by means of Lemma 1.1.3 since the functors $-\otimes M(\mathrm{Spec}(l))$ are self-adjoint.

Corollary 2.3.7. Suppose R if of prime characteristic p, $f_{k_p} : \mathrm{Spec}(k_p) \to \mathrm{Spec}(k)$ is the structure morphisms. Then the functor $f_{k_p}^*$ is weight-conservative.

Proof. Let l/k be a subextension of k_p/k. Suppose $l \neq k_p$. Then there exists $\xi \in k_p \setminus l$. Let $Q(t) = t^n + t^{n-1}\beta_{n-1} + \cdots + \beta_0 \in l[t]$ be the minimal polynomial of ξ over l. Denote $k_1 = k[\beta_0, \ldots, \beta_{n-1}]$ and $k_2 = k_1[\xi]$.

k_1 is a subfield of l and $Q \in k_1[t]$; thus Q is a minimal polynomial of ξ over k_1. Hence $[l[\xi] : l] = n = [k_2 : k_1]$. Since k_p is p-special, the degree $[k_2 : k]$ is coprime to p. Next, $[k_2 : k] = [k_2 : k_1][k_1 : k] = [l[\xi] : l][k_1 : k]$; thus $[l[\xi] : l]$ is coprime to p as well. Hence by Remark 2.3.6 the functor $\Phi_l^{k_p}$ is zero. Thus the only non-zero functor in the family $\{\Phi_l^{k_p}\}$ is the functor $\Phi_{k_p}^{k_p} = f_{k_p}^*$. Thus $f_{k_p}^*$ is weight-conservative indeed.

□

Recall also that our weight-conservative functors can be combined with the corresponding weight complex functors.

Corollary 2.3.8. The family of functors

$$\tilde{\Phi}_l = t^{st} \circ \Phi_l : D\langle S\rangle M(k, R) \to K^b(D\langle S\rangle M'(l, R)_{w=0})$$

is weight-conservative.

Proof. The functor t^{st} is weight-conservative Proposition 1.5.1(2); hence the assertion follows from Theorem 2.3.4 immediately. □

2.4 The case of Artin motives

The category of mixed Artin motives $DMA(k, R)$ is defined as the thick triangulated subcategory of $DM^{gm}(k, R)$ generated by the motives of zero-dimensional smooth k-schemes. The retraction-closure of the full subcategory of $DMA(k, R)$ containing the objects $M(X)$ for smooth zero dimensional X/k is equivalent to the category of Artin motives $MA(k, R)$ (cf. §3.5 of [Voe00]), and the heart of the Chow weight structure on $DMA(k, R)$ equals $MA(k, R)$.

In the notation of section 2.2, $DMA(k, R)$ equals $D\langle S\rangle M(k, R)$, where S_l is the set of all zero-dimensional smooth schemes over l. Hence all the statements of previous subsection can be applied to this setting. Denote $DMA'(k, R) = D\langle S\rangle M'(k, R)$ and $MA'(k, R) = D\langle S\rangle M'(k, R)_{w=0}$. We also write $DMA'_K(k, R)$ for $D\langle S\rangle M'_K(k, R)$.

In this particular case it is possible to compute the codomains of the functors Φ_l very explicitly (see Corollary 2.4.3 below). More generally, one can compute explicitly the quotient categories of the form $\frac{MA(k,R)}{MA(k,R)\otimes M(\mathrm{Spec}(l)}$. The specifics of this answer allows to improve Theorem 2.3.5 in this case (see Theorem 2.4.4 below).

Proposition 2.4.1. Let $l_1/k, l_2/k, l/k$ be finite extensions (note that they are separable). Consider the decompositions $l_1 \otimes l_2 = \oplus_i F_i$ and $F_i \otimes l = \oplus_s F_{is}$. Then

$$\frac{MA(k, R)}{MA(k, R) \otimes M(\mathrm{Spec}(l)}(M(\mathrm{Spec}(l_1)),\ M(\mathrm{Spec}(l_2))) = \oplus_i \frac{R}{\sum_s [F_{is} : F_i]R}.$$

Proof. First consider the case $l_2 = k$. In this situation we have $F_i = l_1$. Let $B = MA(k, R)$, $F : B \to B$, be the functor $F : X \mapsto X \otimes \mathrm{Spec}(l)$. We apply Corollary 1.1.4 for $X = M(\mathrm{Spec}(l_1))$, $Y = M(\mathrm{Spec}(l_2)) = M(\mathrm{Spec}(k))$.

The module $B(X, F^2Y)$ is generated by the classes $[L]$ of the components of $\mathrm{Spec}(l_1) \times \mathrm{Spec}(l)^{\times 2}$; hence $\epsilon_Y B(X, F^2Y)$ is generated by $\epsilon_Y \circ [L]$. The morphism ϵ_Y given by the diagonal $\Delta \subset \mathrm{Spec}(l)^{\times 2}$. Hence $\epsilon_Y \circ [L]$ is the pushforward of $\mathrm{Spec}(L) \cap \mathrm{Spec}(l_1) \times \Delta \in \mathrm{Chow}(\mathrm{Spec}(l_1) \times \mathrm{Spec}(l)^{\times 2})$ under the projection $pr_1 : \mathrm{Spec}(l_1) \times \mathrm{Spec}(l)^{\times 2} \to \mathrm{Spec}(l_1)$ by the definition of the category $\mathrm{Chow}(k, R)$

(see [Ful84], Chapter 16). Thus for an element $u \in \mathrm{Chow}(\mathrm{Spec}(l_1) \times \mathrm{Spec}(l)^{\times 2})$ given by $\mathrm{Spec}(L) \cap \mathrm{Spec}(l_1) \times \Delta$ we have $\epsilon_Y \circ [L] = pr_{1*}(u)$. Since $\mathrm{Spec}(L)$ is an irreducible component we have either $u = 0$ or $u = [L]$. If $u = 0$ then $\epsilon_Y \circ [L] = 0$.

Otherwise $\mathrm{Spec}(L)$ should be in the image of diagonal embedding $\delta : \mathrm{Spec}(l_1) \times \mathrm{Spec}(l) \to \mathrm{Spec}(l_1) \times \mathrm{Spec}(l)^{\times 2}$; thus $u = \delta_*([F_{is}])$ for some F_{is}. Hence $\epsilon_Y \circ [L] = pr_{1*}(u) = (pr_1 \circ \delta)_*([F_{is}]) = [F_i][F_{is} : F_i] = [l][F_{is} : l]$ since $pr_1 \circ \delta : \mathrm{Spec}(l_1) \times \mathrm{Spec}(l) \to \mathrm{Spec}(l_1)$ is just the projection map.

In the general case we recall that the functor $\otimes M(\mathrm{Spec}(l_2))$ is self-adjoint; thus

$$\frac{B}{FB}(M(\mathrm{Spec}(l_1)),\ M(\mathrm{Spec}(l_2))) = \frac{B}{FB}(M(\mathrm{Spec}(l_1 \otimes l_2)), M(\mathrm{Spec}(k))).$$

Next,

$$\frac{B}{FB}(M(\mathrm{Spec}(l_1 \otimes l_2)), M(\mathrm{Spec}(k))) = \oplus \frac{B}{FB}(M(\mathrm{Spec}(F_i)), M(\mathrm{Spec}(k))).$$

Hence our statement in this case follows from that in the previous one. □

Corollary 2.4.2. Suppose R is of prime characteristic p and k is p-special. Then for any two extensions $l_1/k, l_2/k$ we have

$$MA'(k,R)(M(\mathrm{Spec}(l_1)), M(\mathrm{Spec}(l_2))) = \begin{cases} R & \text{if } l_1 = l_2 = k \\ 0 & \text{otherwise} \end{cases}$$

In particular if R is a field then $MA'(k,R)$ is equivalent to the category of finite dimensional R-vector spaces.

Proof. If for some i we have $l_i \neq k$ then $\mathrm{Spec}(l_i)$ becomes zero in $MA'(k,R)$; thus $MA'(k,R)(M(\mathrm{Spec}(l_1)), M(\mathrm{Spec}(l_2))) = 0$.

If $l_1 = l_2 = k$ then by Proposition 2.4.1 we have

$$MA'(k,R)(M(\mathrm{Spec}(l_1)), M(\mathrm{Spec}(l_2))) = \frac{R}{\sum\limits_{l/k} [l:k]R} = R.$$

□

Corollary 2.4.3. Suppose R is a field of characteristic p and k is p-special. Then $DMA'(k,R) \simeq K^b(R)$. Here $K^b(R)$ is a category of bounded complexes of finite-dimensional R-vector spaces.

Proof. It is easily seen that Proposition 3.4.1 in [Voe00] yields that the weight complex functor $t^{st} : DMA'(k,R) \to K^b(MA'(k,R))$ is an equivalence. Next, the category $MA'(k,R)$ is equivalent to the category of finite-dimensional R-vector spaces by the previous corollary. □

Now we establish an improvement of Theorem 2.3.5.

Theorem 2.4.4. Suppose k is p-special and R is of characteristic p.

Then the family $\{\Phi_l\}$ is weight-conservative, where l runs through all finite extensions l/k.

Proof. We will prove that this family is right weight-conservative. It will be left weight-conservative for essentially the same reason.

Let M be an object of $DMA(k,R)$. It belongs to the subcategory of $DMA(k,R)$ generated by finitely many motives of spectra $M(\mathrm{Spec}(k_1)),...,M(\mathrm{Spec}(k_n))$. Let K be the normalization of the compositum of $k_1,...,k_n$. Note that K/k is finite and for any finite set of subfields $F_s \subset K$ we have $\otimes_s F_s \subset \otimes_s K = K^{m}$[2] since the extension K/k is Galois. Hence any component of $\otimes_s F_s$ is a subfield of K. For a separable extension l/k denote by D_l the subcategory of $DMA(l,R)$ generated by $M(\mathrm{Spec}(k_i)_l)$ and let $D'_{l,K} = F_l^K(D_l)$ and $D'_l = F_l(D_l)$. The Chow weight structure on $DMA(k,R)$ restricts to D_l by Lemma 1.2.4. The obvious localization functor $G_{l,K} : D'_{l,K} \to D'_l$ is weight-exact with respect to the corresponding weight structures by Proposition 1.4.1.

Now recall that this functor is the identity on objects. We will prove that it is bijective on morphisms. Since D_l is generated by $M(\mathrm{Spec}(k_i)_l)$, it suffices to show that the homomorphism

$$g : DMA'_K(M(\mathrm{Spec}(k_i)_l), M(\mathrm{Spec}(k_j)_l)) \to DMA'(M(\mathrm{Spec}(k_i)_l), M(\mathrm{Spec}(k_j)_l))$$

induced by $G_{l,K}$ is bijective. Consider the decompositions $\mathrm{Spec}(k_i)_l = \sqcup_t \mathrm{Spec}(k_{it})$ and $\mathrm{Spec}(k_j)_l = \sqcup_s \mathrm{Spec}(k_{js})$. Then g is the direct sum of the homomorphisms

$$g^{s,t} : DMA'_K(M(\mathrm{Spec}(k_{it})), M(\mathrm{Spec}(k_{js}))) \to DMA'(M(\mathrm{Spec}(k_{it})), M(\mathrm{Spec}(k_{js}))).$$

We will prove that it is bijective. Consider the decomposition $k_{it} \otimes_l k_{js} = \oplus F_r$. Note that $\mathrm{Spec}(k_{it})$ is a component of $\mathrm{Spec}(k_i)_l$; thus $k_{it} \subset k_i \otimes_k l$. Similarly, $k_{js} \subset k_j \otimes_k l$. Hence for any r we have $F_r \subset k_{it} \otimes_l k_{js} \subset k_j \otimes_k l \otimes_l k_i \otimes_k l = k_j \otimes_k l \otimes_k k_i$. Thus F_r embeds into K. By Proposition 2.4.1, $g^{s,t}$ is a direct sum of projections of the form $g_r^{s,t} : R/I \to R/J$. Here R/I and R/J are the modules coming from the component $\mathrm{Spec}(F_r) \subset \mathrm{Spec}(k_{it}) \times \mathrm{Spec}(k_{js})$.

Let us now describe ideals I and J explicitly. For every extension l'/l consider the decomposition $F_r \otimes l' = \oplus F_{rn}$. Denote by $I_{l'}$ the ideal $\sum_n [F_{rn} : F_r]R$. By Proposition 2.4.1 we have $I = \sum I_{l'}$, where l' runs through all subextensions of K/k and $J = \sum I_{l'}$, where l' runs through all separable extensions of k.

Next let l' be an extension of l that does not embed into K. For any n the field F_{rn} an extension of l'; thus F_{rn} does not embed into K as well. Hence $F_r \neq F_{rn}$ and the degree $[F_{rn} : F_r]$ is divisible by p since k is p-special. Thus $[F_{rn} : F_r]R = 0$ and $I_{l'} = 0$. Hence $I = J$ and $g_r^{s,t}$ is bijective; thus $G_{l,K}$ is fully faithful indeed. Hence $G_{l,K}$ is an isomorphism of categories. In particular, $G_{l,K}$ is weight-conservative.

Now suppose $\Phi_l(M) \in DMA'(k,R)_{w\geqslant 0}$ for every finite l/k. Since K/k is finite, for every subextension l/k of K/k we have $\Phi_l(M) \in DMA'(k,R)_{w\geqslant 0}$. Next $\Phi_l(M) = G_{l,K}(\Phi_l^K(M))$. Since $G_{l,K}$ is weight-conservative, we have $\Phi_l^K(M) \in DMA'_K(k,R)_{w\geqslant 0}$. Thus $M \in DMA(k,R)_{w\geqslant 0}$ by Theorem 2.3.5. □

Suppose k is p-special. A certain weight-conservative family $\Phi_{Bac}^l : DMA(k,\mathbb{Z}/p) \to D(\mathbb{Z}/p)$ for l running through finite (separable) extensions was defined in [Bac16] (Section 5.2.3). Let us establish a connection between $\{\Phi_l\}$ and $\{\Phi_{Bac}^l\}$.

[2] Comparing the dimensions (over k) of left and right hand sides we obtain $m = [K : k]^{q-1}$, where q is the number of multipliers in $\otimes_s K$

Proposition 2.4.5. Let $F : K^b(\mathbb{Z}/p) \to D^b(\mathbb{Z}/p)$ be the obvious equivalence, and $t_l : DMA'(l, \mathbb{Z}/p) \to K^b(\mathbb{Z}/p)$ be the equivalence from Corollary 2.4.3. Then $\Phi^l_{Bac} \cong F \circ t_l \circ \Phi_l$.

Proof. Recall that $\Phi^k_{Bac}(M(\mathrm{Spec}(l))) = 0$ if $k \neq l$ and $\Phi^k_{Bac}(M(\mathrm{Spec}(k))) = \mathbb{Z}/p[0]$. Φ^l_{Bac} defined as the composite with base change $DMA(k, \mathbb{Z}/p) \to DMA(l, \mathbb{Z}/p) \to D(\mathbb{Z}/p)$.

Since Φ_l and Φ^l_{Bac} defined as the corresponding composites with base change functors, it sufficient to verify that $\Phi^k_{Bac} = F \circ t_k \circ \Phi_k$. For every extension l/k we have $\Phi^k_{Bac}(M(\mathrm{Spec}(l))) = 0$. Therefore for any element $X \in DMA(k, \mathbb{Z}/p)_{w=0}$ we have $\Phi^k_{Bac}(X \otimes M(\mathrm{Spec}(l))) = 0$. Thus by the universal property of localizations (we recall that Φ_k is the Verdier quotient functor) there exists an exact functor $T : DMA'(k, \mathbb{Z}/p) \to D(\mathbb{Z}/p)$ such that $\Phi^k_{Bac} = T \circ \Phi_k$. We should prove that $T = F \circ t_k$.

The equivalence $DMA'(k, R) \simeq K^b(\mathbb{Z}/p)$ implies that $DMA'(k, R)$ generated by $\mathrm{Spec}(k)[j]$ as additive category and $DMA'(k, R)(\mathrm{Spec}(k)[i], \mathrm{Spec}(k)[j]) = 0$ if $i \neq j$. Therefore it suffices to verify that $T(M(\mathrm{Spec}(k))) = F(t_k(M(\mathrm{Spec}(k))))$ (since both functors are exact) and that $T(\phi) = F(t_k(\phi))$ for any $\phi \in DMA'(k, R)(\mathrm{Spec}(k)[i], \mathrm{Spec}(k)[j])$.

Now, we have $T(M(\mathrm{Spec}(k))) = T \circ \Phi_k(M(\mathrm{Spec}(k))) = \Phi^k_{Bac}(M(\mathrm{Spec}(k))) = \mathbb{Z}/p[0]$. Next it's easily seen that $t_k(M(\mathrm{Spec}(k))) = \mathbb{Z}/p[0]$ and therefore $F(t_k(M(\mathrm{Spec}(k)))) = F(\mathbb{Z}/p[0]) = \mathbb{Z}/p[0] = T(M(\mathrm{Spec}(k)))$.

Next let $\phi \in DMA'(k, R)(\mathrm{Spec}(k)[i], \mathrm{Spec}(k)[j])$. If $i \neq j$ then $\phi = 0$ and $T(\phi) = 0 = F(t_k(\phi))$. If $i = j$ then $\phi = a\,\mathrm{id}$ since $DMA'(k, R)(\mathrm{Spec}(k)[i], \mathrm{Spec}(k)[j]) = \mathbb{Z}/p$. Therefore $T(\phi) = a\,\mathrm{id} = F(t_k(\phi))$. □

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