A report of the scientific supervisor

on the Graduation Thesis

"On contractible subgraphs of 3-connected graphs"

by Nadezhda Vlasova

The graduation thesis "On contractible subgraphs of 3-connected graphs" by Nadezhda Vlasova is a research in Connectivity Theory — a rather complicated and abstract area of Graph Theory. Working with connectivity of graphs, we deal with abstract objects. It is rather difficult to illustrate these objects. Very often, some properties which seem to be obvious appear difficult or, sometimes, incorrect. The conjecture by McQuaig and Ota (1994) can be easily formulated, which is not often in this area of Graph Theory.

A classic fact — Györi-Lovász Theorem (1976) — tells us that the vertex set of a k-connected graph can be devided into k connected sets of any given sizes (the sum of these sizes must be equal to the number of vertices of the graph). It is not surprising that attempts of generalization of this fact for higher connectivity appear. Consider the following generalization: for a positive integer m, does any sufficiently large k-connected graph contain a connected set of m vertices such that the graph obtained by deleting this set is $(k-1)$-connected? For $k \geq 4$ and any $m \geq 2$, Mader gave a negative answer. Thus, this question remains open only for $k = 3$, and this is the conjecture formulated by McQuaig and Ota in 1994. Such vertex set is called *contractible*, since its contraction into a vertex does not break 3-connectivity.

However, this conjecture with a simple formulation appears complicated. For $m = 2$, the conjecture follows from Tutt's results (1966). For $m = 3$, the conjecture was proved by the authors. For $m = 4$, Kriesell proved it in 2000. For $m \geq 6$, no results are known. Note, that the notion of a "sufficiently large graph" in papers on this conjecture is completely different from this notion in probabilistic combinatorics. For example, Kriesell proved that every 3-connected graph on at least 8 vertices has a 4-vertex contractible set.

Kriesell tried to prove the conjecture for $m = 5$. He has proved this statement for cubic 3-connected graphs and for 3-conntected graphs with average degree close to 3. It's also known, that every 3-connected graph on at least 11 vertices has a contractible set on 5 or 6 vertices (it's my result proved in 2018).

Nadezhda Vlasova proved that every 3-connected graph on at least 11 vertex and minimal degree at least 4 has a 5-vertex contractible set. In her Thesis, both Kriesell's methods (from the paper on the case $m = 4$) and the structural block tree of a 2-connected graph are applied. The block tree of a 2-connected graph helps to analyse graphs obtained by deleting a 3 or 4-vertex contractible set which cannot be added by a vertex. Vlasova proved an interesting result which is in some sense opposite to proved by Kriesell: in his paper, vertices of degree 3 are needed and in Vlasova's Thesis they are forbidden. I hope that methods developed by Vlasova will help to prove the full version of the Conjecture for $m = 5$ (in my opinion, at the current moment Nadezhda have finished this proof, but it must be well written and checked again). Moreover, the results obtained show that a great amount of difficulties are concerned with vertices of degree 3. Hence, the ideas of this Thesis can help to prove the Conjecture for $m > 5$ in the case where the minimal degree of the graph is at least 4.

This Thesis is a serious scientific research and Nadezhda Vlasova is a self consistent reasearcher. I evaluate this Thesis by an "excellent" mark. ' It is a good beginning for the future Candidate's dissertation.

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