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The global stability of the Schumpeterian dynamical system*

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In this article, we present the studies that develop Schumpeter's theory of endogenous evolution of economic systems. An approach to modeling the limitation of economic growth due to the limitation of markets, resource bases and other factors is proposed. For this purpose, the concept of economic niche volume is introduced. The global stability of the equilibrium of the dynamical system with the Jacobi matrix having, at the equilibrium, all eigenvalues equal to zero, except one being negative, is proved. The proposed model makes it possible to evaluate and predict the dynamics of the development of firms in the economic sector.

Keywords: dynamical systems, Schumpeterian dynamics, global stability.

1. Introduction. The theory of endogenous economic growth, proposed by J. Schumpeter [1], is one of the actively developing branches of modern economic science [2–4]. The main idea of this theory lies in that the economic growth is caused by two interacting processes: innovations and imitations. In order to apply the results of this theory to real economic systems, there were proposed several mathematical models, beginning with K. Iwai's evolutionary model [5, 6], which was developed by V. M. Polterovich and G. M. Henkin [7] on the basis of the Burger's equation from fluid mechanics. Later on, developing the results from [8], they proposed the model of the firm capacity dynamics [9]. Let $M_i(t)$ be the integrated firm capacity at the i-th level, λ_i is the profit per capacity unit at the i-th level, $\varphi_i(t)$ is the share of the investments of the firms at the i-th level for the creation of the capacities at the next $(i+1)$-th level, $0 \leqslant \varphi_i(t) \leqslant 1$. Then the capacity dynamics equation is as follows:

$$\dot{M}_i = (1-\varphi_i)\lambda_i M_i + \varphi_{i-1}\lambda_{i-1}M_{i-1}, \quad i = 1, 2, \ldots, \tag{1}$$

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under the boundary and initial conditions

$$M_0(t) \equiv 0, \quad M_i(0) \geqslant 0, \quad \sum_{i=1}^{N} M_i(0) > 0, \quad M_i(0) = 0 \quad \text{if } i > N. \tag{2}$$

In formulas (1), (2) N is the initial number of levels, $F_i(t) = \frac{\sum_{k=0}^{i} M_k}{\sum_{k=0}^{\infty} M_k}$, $\varphi_i = \alpha + \beta(1 - F_i(T))$, $\alpha > 0$, $\beta > 0$ are constants.

In [10] the authors have shown that $M_i(t) \to \infty$ as $t \to \infty$, which is impossible for economic systems.

In this article, we, developing the approach of V. M. Polterovich and G. M. Henkin, construct the mathematical models of the capital distribution dynamics over efficiency levels in which the boundedness of the economic growth is taken into account. For that purpose the notion of the economic niche volume which is analogous to the ecological niche volume is introduced. The economic niche volume is a limit integrated capital value, for which the growth rate is so low that there is no capital growth. The equilibrium of the constructed model is found and its global stability is proved.

This article develops the research presented in [10, 11]. The main difference between models studied in these papers is the following. While in [10, 11] the considered models have distinct economic niches for efficiency levels, in this paper the unique shared economic niche for all levels is considered. This difference implies the essential difficulties while proving the stability results. Particularly, the equilibrium is not a hyperbolic point. Besides, the method, by which the global stability for n-dimensional dynamical system was proved in [10, 11], is unsuitable.

2. The model of the capital distribution dynamics over efficiency levels. Let N be the number of efficiency levels. Assume that the greater i corresponds to the higher level. Consider the following system of differential equations:

$$\begin{cases} \dot{C}_1 = \frac{1-\varphi_1}{\lambda_1} C_1 (V - C_1 - \ldots - C_N) = f_1(C_1, \ldots, C_N), \\ \dot{C}_i = \frac{1-\varphi_i}{\lambda_i} C_i (V - C_1 - \ldots - C_N) + \varphi_{i-1} C_{i-1} = f_i(C_1, \ldots, C_N), \quad i = 2, \ldots, N. \end{cases} \tag{3}$$

Here, C_i is the integrated capital of all firms at the i-th level (one firm can have the capital at different levels); V is the economic niche volume, φ_i is the share of capital of the firms at the i-th level intended to the developing of the production at the next, $(i+1)$-th, level; λ_i is the unit prime cost at the i-th level (i. e. the unit goods production cost per unit time), $i = 1, \ldots, N$, $V > 0$, $0 < \varphi_i < 1$, $\lambda_i > 0$ are constants.

As one can see, the right-hand sides of the equations (3) of the capital growth rates, except for the first one, contain two addends: the first addend describes the capital growth due to the production at the level i, the second one — due to the investments from the previous level $i-1$. The first addend is positive, i. e. the capital increases, if the amount of the capitals is not greater than the economic niche volume, otherwise, the first addend is negative, i. e. the production runs down. However, the investments from the previous level provide the capital growth during some time, while the value of the positive second addend is greater than the absolute value of the negative first addend, i. e. while the investments from the previous, $(i-1)$-th, level are greater than the firm losses at the i-th level induced by exceeding of the economic niche volume.

Equating the right-hand sides of the equations to zero we obtain the equilibria: $O = (0, ..., 0)$ and $P = (0, ..., 0, V)$.

Remark 1. It is worth to note that the Jacobi matrix at the point P has all the eigenvalues equal to zero, except one — negative, which does not allow to establish the local stability using linear approximation. The Jacobi matrix is

$$f' = \{f_{ij}\}, \quad i, j = 1, \dots, N,$$

where

$$f_{ij} = \begin{cases} a_i(V - 2C_i - \sum\limits_{k \neq i} C_k), \quad i = j, \\ -a_i C_i + \varphi_{i-1}, \quad i = j + 1, \\ -a_i C_i, \quad \text{otherwise}. \end{cases}$$

Determine the eigenvalues for the equilibria $P = (0, \dots, V)$ as the roots of the characteristic polynomial

$$\det f'(0, \dots 0, V) - \lambda E = (-\lambda)^{N-1}(-a_N V - \lambda) = 0.$$

We obtain $\lambda_j = 0$, $j = 1, \dots, N-1$, $\lambda_N = -a_N V < 0$.

Proposition 1. The set $\mathbb{R}^N_+ = \{(C_1, \dots, C_N) \in \mathbb{R}^N : C_i \geqslant 0, i = 1, \dots, N\}$ is invariant.

P r o o f. If $C_1 = 0$, then from (3) we obtain $\dot{C}_1 = 0$. It means that the trajectories do not leave $\mathbb{R}^N_+$ through the boundary hyperplane $C_1 = 0$. If $C_i = 0$, then from (3) we obtain $\dot{C}_i = \varphi_{i-1} C_{i-1} \geqslant 0$. It means that the trajectories do not leave $\mathbb{R}^N_+$ through the boundary hyperplane $C_i = 0$. Thus, the trajectories do not leave $\mathbb{R}^N_+$, therefore, $\mathbb{R}^N_+$ is invariant.

Let us denote $a_i = \frac{1 - \varphi_i}{\lambda_i}$.

Proposition 2. All trajectories enter the parallelepiped $K = \{(C_1, \dots, C_N) \in \mathbb{R}^N : 0 \leqslant C_1 \leqslant V,\ 0 \leqslant C_i \leqslant \max(\frac{\varphi_{i-1}}{a_i}, V),\ i = 2, \dots, N\}$ and do not leave it.

P r o o f. If $\frac{\varphi_{i-1}}{a_i} \geqslant V$, then for $C_i = \frac{\varphi_{i-1}}{a_i}$:

$$\dot{C}_i = \varphi_{i-1}(V - C_1 - \ldots - C_{i-1} - \frac{\varphi_{i-1}}{a_i} - C_{i+1} \ldots - C_N) + \varphi_{i-1} C_{i-1} =$$
$$= \varphi_{i-1}(V - C_1 - \ldots - C_{i-2} - \frac{\varphi_{i-1}}{a_i} - C_{i+1} - \ldots - C_N) < 0.$$

If $\frac{\varphi_{i-1}}{a_i} \geqslant V$, then for $C_i = R \geqslant \frac{\varphi_{i-1}}{a_i}$:

$$\dot{C}_i = a_i R(V - C_1 - \ldots - C_{i-1} - R - C_{i+1} - \ldots - C_N) + \varphi_{i-1} C_{i-1} \leqslant$$
$$\leqslant a_i R(V - C_1 - \ldots - C_{i-1} - \frac{\varphi_{i-1}}{a_i} - C_{i+1} \ldots - C_N) + \varphi_{i-1} C_{i-1} \leqslant$$
$$\leqslant \varphi_{i-1}(V - C_1 - \ldots - C_{i-1} - \frac{\varphi_{i-1}}{a_i} - C_{i+1} \ldots - C_N) + \varphi_{i-1} C_{i-1} =$$
$$= \varphi_{i-1}(V - C_1 - \ldots - C_{i-2} - \frac{\varphi_{i-1}}{a_i} - C_{i+1} - \ldots - C_N) < 0.$$

If $\frac{\varphi_{i-1}}{a_i} < V$, then for $C_i = V$:

$$\dot{C}_i = a_i V(V - C_1 - \ldots - C_{i-1} - V - C_{i+1} - \ldots - C_N) + \varphi_{i-1} C_{i-1} =$$
$$= a_i C_{i-1}(\frac{\varphi_{i-1}}{a_i} - V) + a_i V(-C_1 - \ldots - C_{i-2} - C_{i+1} - \ldots - C_N) < 0.$$

If $\frac{\varphi_{i-1}}{a_i} < V$, then for $C_i = R \geqslant V$:

$$\dot{C}_i = a_i R(V - C_1 - \ldots - C_{i-1} - R - C_{i+1} - \ldots - C_N) + \varphi_{i-1}C_{i-1} =$$
$$= a_i C_{i-1}(\frac{\varphi_{i-1}}{a_i} - V) + a_i R(V - R - C_1 - \ldots - C_{i-2} - C_{i+1} - \ldots - C_N) < 0.$$

So, all trajectories enter any parallelepiped $K_R = \{(C_1, \ldots, C_N) \in \mathbb{R}^N : 0 \leqslant C_1 \leqslant V,$ $0 \leqslant C_i \leqslant R,\ i = 2, \ldots, N\}$, where $R \geqslant \max(\frac{\varphi_{i-1}}{a_i}, V)$ and do not leave it. Therefore, all trajectories enter the smallest parallelepiped K.

In the rest of the paper we obtain results dealing with the problem of the global stability of the equilibrium P. Let us introduce the following notions and designations. Denote by $C(t, C^0) \in \mathbb{R}^n_+$ the solution of (3) such that $C(0, C^0) = C^0 \in \mathbb{R}^n_+$, $C = (C_1, ..., C_n)$, $C^0 = (C^0_1, ..., C^0_n)$. Let $\rho(x, y) = ||x - y||$ be the Euclidean metric in $\mathbb{R}^N$.

Definition. *The equilibrium P of the system* (3) *is globally stable in $\mathbb{R}^n_+ \setminus \{O\}$ if $C(t, C^0) \to P$, as $t \to \infty$, for every $C^0 \in \mathbb{R}^n_+ \setminus \{O\}$.*

The main result of the paper is as follows: the equilibrium $P = (0, ..., 0, V) \in \mathbb{R}^n_+$ is globally stable in $\mathbb{R}^n_+ \setminus \{O\}$.

2.1. Two efficiency levels. Let us consider the model with two efficiency levels:

$$\begin{cases} \dot{C}_1 = \frac{1-\varphi_1}{\lambda_1} C_1(V - C_1 - C_2) = f_1(C_1, C_2), \\ \dot{C}_2 = \frac{1-\varphi_2}{\lambda_2} C_2(V - C_1 - C_2) + \varphi_1 C_1 = f_i(C_1, C_2). \end{cases} \tag{4}$$

Theorem 1. *The equilibrium $P = (0, V)$ of the system* (4) *is globally stable in $\mathbb{R}^2_+ \setminus \{O\}$.*

P r o o f . The interior of the set $\mathbb{R}^2_+ \setminus \{O\}$ is divided by the $\dot{C}_1 = 0$ isocline: $C_1 + C_2 = V$ and $\dot{C}_2 = 0$ isocline: $C_2 = \tilde{C}_2$, where $\tilde{C}_2 = \frac{V - C_1 + \sqrt{(V - C_1)^2 + \frac{4\varphi_1 C_1}{a_2}}}{2}$, into 3 domains D_i as follows (Figure):

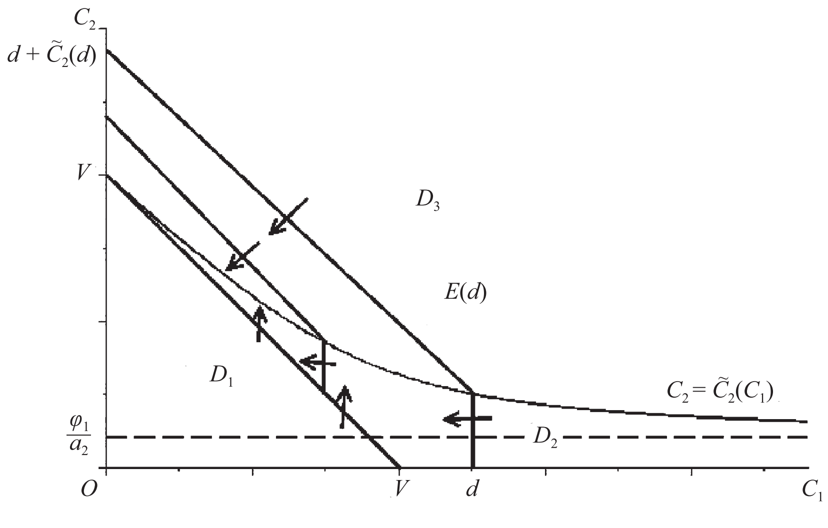

Figure. $E(d)$, $\frac{\varphi_1}{a_2} < V$

- $D_1 = \{(C_1, C_2) : C_1 + C_2 < V\}$, in which $f_1 > 0, f_2 > 0$;
- $D_2 = \{(C_1, C_2) : C_1 + C_2 > V, C_2 < \tilde{C}_2\}$, in which $f_1 < 0, f_2 > 0$;
- $D_3 = \{(C_1, C_2) : C_1 + C_2 > V, C_2 > \tilde{C}_2\}$, in which $f_1 < 0, f_2 < 0$.

To prove the global stability of the equilibrium $P = (0, V)$ we construct the family of nested sets $\{E(d), d > 0\}$ containing P, diameters of which tend to 0 as $d \to 0$, such that all trajectories enter and do not leave each of these sets. Taking into account Proposition 2, consider trajectories $C(t, C^0) = (C_1(t), C_2(t))$, where $C(0, C^0) = C^0$, for which $C^0 \in K \setminus \{O\}$. Since $\dot{C}_1(t) + \dot{C}_2(t) > 0$, if $0 < C_1(t) + C_2(t) < V$, then all trajectories leave the triangle $\{(C_1, C_2) : 0 < C_1 + C_2 \leqslant V\}$. Consider the family of sets:

$$E(d) = \{(C_1, C_2) : V \leqslant C_1 + C_2 \leqslant d + \tilde{C}_2(d), 0 \leqslant C_1 \leqslant d, C_2 \geqslant 0\}.$$

Obviously, $E(d_2) \subset E(d_1)$, if $d_2 < d_1$, and $(0, V) \in E(d)$ for any $d \geqslant 0$ (see Figure).

All trajectories enter $E(d)$ for $d > 0$. Really, trajectories intersect any straight line $C_1 = d$ from right to left (so that C_1 is decreasing) for $C_1 + C_2 > V$. As shown above, the straight line $C_1 + C_2 = V$ is intersected by trajectories from below to upward. Clearly, the inner product of the vector $\bar{f} = (f_1, f_2)$ and the normal vector $\bar{n} = (-1, -1)$ to any straight line $C_1 + C_2 = d + \tilde{C}_2(d)$ is positive, $(\bar{f}, \bar{n}) > 0$, for all points belonging to a straight line $C_1 + C_2 = d + \tilde{C}_2(d)$ in $\mathbb{R}^2_+$. Hence, trajectories intersect any straight line $C_1 + C_2 = d + \tilde{C}_2(d)$ so that C_1, C_2 are decreasing.

Therefore, all trajectories enter each $E(d)$. Taking into account that d may be arbitrary small, we obtain the result: $C(t, C^0) \to P = (0, V)$ as $t \to \infty$.

Remark 2. We notice that the location of the asymptote $C_2 = \frac{\varphi_1}{a_2}$ with respect to the line $C_2 = V$ and, hence, the form of the isocline $C_2 = \tilde{C}_2$, depends on whether $\frac{\varphi_1}{a_2}$ is greater or less than V. Let us give an economic interpretation of the cases $\frac{\varphi_1}{a_2} < V$ and $\frac{\varphi_1}{a_2} \geqslant V$. In the case $\frac{\varphi_1}{a_2} < V$ the asymptote $C_2 = \frac{\varphi_1}{a_2}$ of the isocline $C_1 = \tilde{C}_1$ is lower than V. From economic point of view it can be explained as follows: if $\frac{\varphi_1}{a_2} < V$, then the contribution of the investments from the first level to the capital growth at the second level is much less than the contribution of the production at the second level. As a result these investments from the first level provide small capital growth at the second level when the production at this level runs down. If $\frac{\varphi_1}{a_2} \geqslant V$, then the contribution of the investments from the first level to the capital growth at the second level is significant in comparison with the contribution of the production at the second level. Thus, these investments provide greater capital growth at the second level when the production at this level runs down. It is clear that the first case, $\frac{\varphi_1}{a_2} < V$, is typical while the latter case, $\frac{\varphi_1}{a_2} \geqslant V$, indicates at some problems in economy.

Now, we show the existence of a separatrix in the domain D_2, if $\frac{\varphi_1}{a_2} < V$, or in the domain D_3, if $\frac{\varphi_1}{a_2} > V$. Let us prove the existence of a separatrix in the former case (the proof of the latter case is analogous). Consider the set Ω bounded by two curves, L^+ and L^-. The upper boundary L^+ is the $\dot{C}_2 = 0$ isocline, $C_2 = \tilde{C}_2$. The lower boundary L^- is the union of the asymptote $C_2 = \frac{\varphi_1}{a_2}$ of the curve $C_2 = \tilde{C}_2$ with $C_1 > V - \frac{\varphi_1}{a_2}$ and the segment of the straight line $C_1 + C_2 = V$ with $C_1 \in [V - \frac{\varphi_1}{a_2}, V]$, $C_2 \in [\frac{\varphi_1}{a_2}, V]$. Let us call the sets of trajectories entering Ω through L^+ and L^- the upper and below flows, respectively. It is obvious that Ω is invariant set. Below we show that there exists a unique trajectory in Ω which does not intersects L^+ or L^-. We call this trajectory a separatrix. We may say that the separatrix separates the upper and below flows.

Theorem 2. *There exists a unique separatrix in Ω.*

P r o o f . Denote by $[AB]$ a segment, perpendicular to the axis C_1, such that $A \in L^+$, $B \in L^-$. Denote by $C_1(A)$ the C_1 coordinate of point A, where $C_1(A) = C_1(B)$. Denote

$[A,\infty) = \{(C_1,C_2) \in L^+ : C_1 \geqslant C_1(A)\}$, $[B,\infty) = \{(C_1,C_2) \in L^- : C_1 \geqslant C_1(B)\}$. For every point $M \in [A,\infty)$ the inequality $\dot C_1 < 0$ implies that the trajectory $C(t,M)$ intersects $[AB]$ at some point $M_1 \in [AB]$. Thus, we obtain a homeomorphic map from $[A,\infty)$ onto some interval $[A,A_1) \subset [AB]$. Similarly, we obtain a homeomorphic map from $[B,\infty)$ onto $[B,B_1) \subset [AB]$. Note that $A_1 \neq B_1$ and $[A,A_1) \cap [B,B_1) = \emptyset$. Hence, in Ω there exist trajectories intersecting $[A_1,B_1]$. Usually the set of these trajectories is called a separatrix fan.

Let us prove the uniqueness of the separatrix. We prove this fact by contradiction using the arguments from [12]. Since $\dot C_1 < 0$ in Ω then we can represent the trajectories in Ω by the graphics, i. e. as functions of C_1. Assume, there are at least two separatrices, $C_2 = h_1(C_1)$ and $C_2 = h_2(C_1) = h_1(C_1)+\varepsilon(C_1)$, where $\varepsilon(C_1) > 0$, $\varepsilon(C_1) \to 0$ for $C_1 \to \infty$,

$$\frac{dC_2}{dC_1} = \frac{a_2C_2(V - C_1 - C_2) + \varphi_1 C_1}{a_1C_1(V - C_1 - C_2)} = G(C_1,C_2).$$

Clearly, $\varepsilon(C_1) = h_2(C_1) - h_1(C_1)$:

$$\frac{d\varepsilon}{dC_1} = G(C_1,h_2(C_1)) - G(C_1,h_1(C_1)) =$$

$$= \frac{dG(C_1,h(C_1))}{dC_2}(h_2(C_1) - h_1(C_1)) = \frac{dG(C_1,h(C_1))}{dC_2}\varepsilon,$$

where $h(C_1) \in (h_1(C_1),h_2(C_1))$,

$$\frac{dG(C_1,C_2)}{dC_2} = \frac{a_1a_2C_1(V - C_1 - C_2)^2 + \varphi_1 a_1 C_1^2}{(a_1C_1(V - C_1 - C_2))^2} > 0 \text{ for any } C_1, C_2.$$

We have $\frac{d\varepsilon}{dC_1} = \frac{dG(C_1,h(C_1))}{dC_2}\varepsilon(C_1) > 0$. Therefore, $\varepsilon(C_1)$ increases for $C_1 \to \infty$. But $\varepsilon(C_1) \to 0$ for $C_1 \to \infty$. We have reached a contradiction. Thus, there exists the unique separatrix.

In the case $\frac{\varphi_1}{a_2} = V$ the $\dot C_2$ isocline $C_2 = \tilde C_2$, being the straight line, coincides with its asymptote $C_2 = \frac{\varphi_1}{a_2}$.

2.2. *N* ***efficiency levels.*** Consider now the general case of N efficiency levels. This case is described by the system (3)

$$\begin{cases} \dot C_1 = \frac{1-\varphi_1}{\lambda_1} C_1(V - C_1 - ... - C_N) = f_1(C_1,...,C_N), \\ \dot C_i = \frac{1-\varphi_i}{\lambda_i} C_i(V - C_1 - ... - C_N) + \varphi_{i-1}C_{i-1} = f_i(C_1,...,C_N), \quad i = 2,\dots,N. \end{cases}$$

Theorem 3. *The equilibrium* $P = (0,...,0,V)$ *of the system* (3) *is globally stable in* $\mathbb{R}^N_+ \setminus \{O\}$.

P r o o f . We use the method of mathematical induction with respect to N — the dimension of system (3). For $N = 2$ the desired result is given by Theorem 1.

Denote by S_N the system of the form (3) in $\mathbb{R}^N_+$ and by $P_N = (0,\dots,0,V) \in \mathbb{R}^N_+$ the equilibrium of S_N. Assume that P_{N-1} is globally stable equilibrium of the system S_{N-1} in $\mathbb{R}^{N-1}_+ \setminus \{O\}$. Let us prove that P_N is globally stable equilibrium of S_N in $\mathbb{R}^N_+ \setminus \{O\}$.

Consider the system S_N in $\mathbb{R}^N_+$. According to Proposition 2 all trajectories enter the parallelepiped K and do not leave it. Therefore, all trajectories are positively Lagrange stable. Hence, the ω-limit set $\Omega_C \neq \varnothing$ for any trajectory $C(t,C^0) \in \mathbb{R}^N_+$ [13].

Consider the set $J(d) = \{(C_1,\dots,C_N) \in \mathbb{R}^N_+ : 0 \leqslant C_1 + \ldots + C_N \leqslant d, d \in [0,V]\}$, and the trajectory $\tilde C(t,C^0)$ belonging to $J(V) \cap \{(C_1,...,C_N) : C_j = 0, j = 1,...,N-1\}$,

i. e. $\tilde{C}(t, C^0)$ is the trajectory belonging to the interval $I_N = \{(C_1, ..., C_N) : C_j = 0, j = 1, ..., N-1, C_N \in (O, V)\}$ of the C_N axis.

Denote $g(t) = C_1(t) + \ldots + C_N(t)$, where $(C_1(t), ..., C_N(t)) = C(t, C^0)$. Then $\dot{g}(t) > 0$ for $C(t, C^0) \in J(V) \setminus \{O\}$. Hence, all trajectories, except for $\tilde{C}(t, C^0)$ and $C(t, O) = O$, leave the set $J(V)$. As for $\tilde{C}(t, C^0)$, it is easily to understand that $\tilde{C}(t, C^0) \to P$ as $t \to \infty$, where $C^0 \in I_N$.

Denote $L = K \setminus J(V)$. Obviously, all trajectories, except for $\tilde{C}(t, C^0)$ and $C(t, O) = O$, enter L and do not leave it. Therefore, if $C^0 \in \mathbb{R}^N_+$, then $C_1(t) \to 0$ as $t \to \infty$ for any trajectory. Hence, $C_1 = 0$ for any point belonging to Ω_C, or $\Omega_C \subset \mathbb{R}^{N-1} = \{(C_1, \ldots, C_N) \in \mathbb{R}^N : C_1 = 0\}$. Let us show that $\Omega_C = P = (0, ..., 0, V) \in \mathbb{R}^N_+$ for any trajectory $C(t, C^0) \in \mathbb{R}^N_+ \setminus \{O\}$. Assume, on the contrary, that there exists a trajectory $C(t, C^0)$, $C^0 \in \mathbb{R}^N_+ \setminus \{O\}$ for which there exists an ω-limit point $Q \in \Omega_C \subset \mathbb{R}^{N-1}_+$ and $Q \neq P$.

Since $\mathbb{R}^{N-1}$ is the invariant set of (3), then (3) in $\mathbb{R}^{N-1}$ has the following form:

$$\begin{cases} \dot{C}_2 = \frac{1-\varphi_2}{\lambda_2} C_2 (V - C_2 - ... - C_N) = f_2(C_2, ..., C_N), \\ \dot{C}_i = \frac{1-\varphi_i}{\lambda_i} C_i (V - C_2 - ... - C_N) + \varphi_{i-1} C_{i-1} = f_i(C_2, \ldots, C_N), \\ i = 3, \ldots, N. \end{cases} \tag{5}$$

Thus, we can assume that the system S_{N-1} has the form (5). From the induction assumption the equilibrium $P_{N-1} = (0, ..., 0, V) \in \mathbb{R}^{N-1}$ of this system is globally stable in $\mathbb{R}^{N-1}_+ \setminus \{O\}$.

Consider the trajectory $C(t - t_Q, Q) \in \mathbb{R}^{N-1}$, where $C(0, Q) = Q$. Consider a point A of the intersection of the trajectory $C(t - t_Q, Q)$ with the boundary of the set L. Let us show that this point exists. Consider in $\mathbb{R}^{N-1}_+$ the set $L^{N-1} = L \cap \{(C_1, \ldots, C_N) : C_1 = 0\}$ and the system $\dot{C} = -f(C)$. Obviously, the boundary $\partial L^{N-1} = \partial L \cap \{(C_1, \ldots, C_N)$: $C_1 = 0\} \subset \mathbb{R}^{N-1}_+$ and in $\mathbb{R}^{N-1}_+$ boundaries of L and L^{N-1} coincide. Since $\dot{C}_2 = -a_2 C_2 (V - C_2 - ... - C_N) > 0$, then the trajectory $C(t - t_Q, Q)$ intersects any hyperplane $\{(C_1, \ldots, C_N) : C_2 = \text{const} > 0\}$ so that C_2 is increasing. Therefore, this trajectory exits L^{N-1} through ∂L^{N-1} and there exists the point $A = \partial L^{N-1} \cap \{C(t - t_Q, Q)\} = \partial L \cap \{C(t - t_Q, Q)\}$. Denote by $\tilde{Q}$ the point such that $C(\tilde{t} - t_Q, Q) = \tilde{Q}$ (i. e. the time of movement along the trajectory $C(t - t_Q, Q)$ from Q to $\tilde{Q}$ equals $\tilde{t}$). Denote by $U(A), U(\tilde{Q}) \subset \mathbb{R}^N$ some neighborhoods of the points A and $\tilde{Q}$. Let $\pi(A), \pi(\tilde{Q})$ be hyperplanes transversal to the trajectory $C(t - t_Q, Q)$ at the points A, $\tilde{Q}$ respectively. Consider the sets $U_\pi(A) = U(A) \cap \pi(A)$, $U_\pi(\tilde{Q}) = U(\tilde{Q}) \cap \pi(\tilde{Q})$. Though $U_\pi(A), U_\pi(\tilde{Q}) \not\subset \mathbb{R}^N_+$, but because of invariance of $\mathbb{R}^{N-1}_+$ the trajectories, intersecting the sets $U^+_\pi(A) = U_\pi(A) \cap \mathbb{R}^N_+$, $U^+_\pi(\tilde{Q}) = U_\pi(\tilde{Q}) \cap \mathbb{R}^N_+$, belong to $\mathbb{R}^N_+$. From the Theorem (17.4) in [12] there exists a closed trajectory cylinder $Z \subset \mathbb{R}^N_+$ with the trajectory $C(t - t_Q, Q)$ as its axis and $U^+_\pi(A)$, $U^+_\pi(\tilde{Q})$ as its bases. By construction, $Z \subset L$. Since Q is ω-limit point for $C(t, C_0)$, then this trajectory has to enter the cylinder $Z \subset L$ infinite number of times. The entrance to the cylinder Z occurs through $U^+_\pi(A)$ from $\mathbb{R}^N \setminus L$. Thus, the trajectory $C(t_0, C^0)$ has to leave L before entering the cylinder Z. But the trajectories cannot leave L. We have reached a contradiction. Thus, Q is not a ω-limit point of this trajectory. Thus we have, $\Omega_C = \{P_N = (0, ..., 0, V)\}$ for any trajectory $C(t)$. Since any trajectory $C(t)$ is positively Lagrange stable then, according to the Theorem (3.07) in [13], $\rho(C(t), \Omega_C = P_N) \to 0$ for $t \to \infty$ for any $C(t)$. Thus, $P_N = (0, ..., 0, V)$ is global stable in $\mathbb{R}^N_+ \setminus \{O\}$.

5. Conclusion. In this article, the mathematical models of sector capital distribution dynamics over efficiency levels with shared economic niche are proposed. The authors develop the approach of Polterovich—Henkin by taking into account the boundedness of the economic growth. The models give an opportunity for the prediction of the behavior of the economic system dynamics. The qualitative analysis of these models is presented. The equilibria of the constructed dynamical models are determined, their global stability is proved. The global stability of the equilibrium $P = (0, \ldots, 0, V)$ means that the capital at the highest level approaches the value of the economic niche volume, while the capitals at other levels tend to zero. Thus, the highest level beats the competition in the case of shared economic niche. It is worth to note that global stability of the equilibrium means the economic stagnation. This situation is extremely unfavorable and, thus, the further economic growth requires structural variations of the economic system. This is the problem for further research.

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Глобальная устойчивость шумпетеровской динамической системы*

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В статье представлены исследования, развивающие шумпетеровскую теорию эндогенной эволюции экономических систем. Предложен подход к моделированию ограниченности экономического роста, обусловленного ограниченностью рынков, ресурсной базы и другими факторами. С этой целью введено понятие объема экономической ниши. Показана глобальная устойчивость положения равновесия динамической системы с матрицей Якоби, имеющей в положении равновесия все собственные числа, равные нулю, кроме одного — отрицательного. Предложенная модель позволяет оценивать и предсказывать динамику развития фирм отрасли экономики.

Ключевые слова: динамические системы, шумпетеровская динамика, глобальная устойчивость.

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