Noname manuscript No.
(will be inserted by the editor)

Constructive description of Hardy-Sobolev spaces in $\mathbb{C}^n$

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Received: date / Accepted: date

Abstract In this paper we study the polynomial approximations in Hardy-Sobolev spaces on for convex domains. We use the method of pseudoanalytical continuation to obtain the characterization of these spaces in terms of polynomial approximations.

Keywords Hardy-Sobolev Spaces · polynomial approximations, pseudoanalytical continuation · Cauchy-Leray-Fantappiè integral.

Mathematics Subject Classification (2010) 32E30 · 41A10

1 Introduction

The purpose of this paper is to give an alternative characterizations of Hardy-Sobolev (see. [1]) spaces

$$H_p^l(\Omega) = \{f \in H(\Omega) : \|f\|_{H^p(\Omega)} + \sum_{|\alpha|\leq l} \|\partial^\alpha f\|_{H^p(\Omega)} < \infty\} \tag{1}$$

on strongly convex domain $\Omega \subset \mathbb{C}^n$.

We continue the research started in [14] and devoted to description of basic spaces of holomorphic functions of several variables in terms of polynomial approximations and pseudoanalytical continuation. In particular, we show that for $1 < p < \infty$ and $l \geq 1$ a holomorphic on a strongly convex domain Ω function f is in the Hardy-Sobolev space $H_p^l(\Omega)$ if and only if there exist a polynomial sequence P_{2^k} such that

$$\int\limits_{\partial\Omega} d\sigma(z) \left(\sum_{k=1}^{\infty} |f(z) - P_{2^k}(z)|^2 \, 2^{2lk} \right)^{p/2} < \infty. \tag{2}$$

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In the one variable case this condition follows from the characterization obtained by E.M. Dynkin [5] for Radon domains.

The paper is divided into five sections with one appendix. In section 2 we give main definitions and preliminaries of this work. Section 3 is devoted to the Cauchy-Leray-Fantappiè integral formula, the polynomial approximations and estimates of its kernel. We also define internal and external Korányi regions, the multidimensional analog of Lusin regions. In section 4 we introduce the method of pseudoanalytical continuation and three constructions of the continuation with different estimates. We use these constructions to obtain the characterization of Hardy-Sobolev spaces in terms of estimates of the pseudoanalytical continuation. To prove this result we use the special analog of the Krantz-Li area-integral inequality [7] for external Korányi regions established in appendix A. Finally, section 5 contains the proof of characteristics (2).

2 Main notations and definitions

Let $\mathbb{C}^n$ be the space of n complex variables, $n \geq 2$, $z = (z_1, \ldots, z_n)$, $z_j = x_j + iy_j$;

$$\partial_j f = \frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right), \quad \bar{\partial}_j f = \frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right),$$

$$\partial f = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k} dz_k, \quad \bar{\partial} f = \sum_{k=1}^{n} \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k, \quad df = \partial f + \bar{\partial} f.$$

The notation

$$\langle \partial f(z),\ w\rangle = \sum_{k=1}^{n} \frac{\partial f(z)}{\partial z_k} w_k.$$

is used to indicate the action of ∂f on the vector $w \in \mathbb{C}^n$, and

$$|\bar{\partial} f| = \left|\frac{\partial f}{\partial z_1}\right| + \ldots + \left|\frac{\partial f}{\partial z_n}\right|.$$

The euclidian distance form the point $z \in \mathbb{C}^n$ to the set $D \subset \mathbb{C}^n$ we denote as $\operatorname{dist}(z,\ D) = \inf\{|z - w| : w \in D\}$. Lebesgue measure in $\mathbb{C}^n$ we denote as $d\mu$.

For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we set $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\alpha! = \alpha_1! \ldots \alpha_2!$, also $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ and $\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial \bar{z}_1^{\alpha_1} \ldots \partial \bar{z}_n^{\alpha_n}}$.

Let $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ be a strongly convex domain with a C^3-smooth defining function. We need to consider a family of domains $\Omega_t = \{z \in \mathbb{C}^n : \rho(z) < t\}$ that are also strongly convex for each $|t| < \varepsilon$, where $\varepsilon > 0$ is small enough. The projection of $z \in \Omega_\varepsilon \setminus \Omega_{-\varepsilon}$ to the $\partial\Omega$ we define as $\pi(z)$.

For $\xi \in \partial\Omega_t$ we define the complex tangent space

$$T_\xi = \{z \in \mathbb{C}^n : \langle \partial\rho(\xi),\ \xi - z\rangle = 0\}.$$

The space of holomorphic functions we denote as $H(\Omega)$ and consider the Hardy space (see [17], [6])

$$H^p(\Omega) := \left\{ f \in H(\Omega) : \sup_{-\varepsilon<t<0} \int\limits_{\partial\Omega_t} |f(z)|^p d\sigma_t(z) < \infty \right\},$$

where $d\sigma_t$ is induced Lebesgue measure on the boundary of Ω_t. We also denote $d\sigma = d\sigma_0$. Hardy-Sobolev spaces $H^l_p(\Omega)$ are defined by (1).

Throughout this paper we use notations $\lesssim$, $\asymp$. We let $f \lesssim g$ if $f \leq cg$ for some constant $c > 0$, that doesn't depend on main arguments of functions f and g and usually depend only on dimension n and domain Ω. Also $f \asymp g$ if $c^{-1}g \leq f \leq cg$ for some $c > 1$.

3 Cauchy-Leray-Fantappiè formula

In the context of theory of several complex variables there is no unique reproducing formula formula, however we could use the Leray theorem, that allows us to construct holomorphic reproducing kernels ([2], [11], [12]). For convex domain $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ this theorem brings us Cauchy-Leray-Fantappiè formula, and for $f \in H^1(\Omega)$ and $z \in \Omega$ we have

$$f(z) = K_\Omega f(z) = \frac{1}{(2\pi i)^n} \int\limits_{\partial\Omega} \frac{f(\xi)\partial\rho(\xi) \wedge (\bar\partial\partial\rho(\xi))^{n-1}}{\langle \partial\rho(\xi),\ \xi - z\rangle^n} = \int\limits_{\partial\Omega} f(\xi)K(\xi, z)\omega(\xi), \tag{3}$$

where $\omega(\xi) = \frac{1}{(2\pi i)^n}\partial\rho(\xi) \wedge (\bar\partial\partial\rho(\xi))^{n-1}$, and $K(\xi, z) = \langle \partial\rho(\xi),\ \xi - z\rangle^{-n}$.

The $(2n-1)$-form ω defines on $\partial\Omega_t$ Leray-Levy measure dS, that is equivalent to Lebesgue surface measure $d\sigma_t$ (for details see [2], [9], [10]). This allows us to identify Lebesgue, Hardy and Hardy-Sobolev spaces defined with respect to measures $d\sigma_t$ and dS. Also note, that measure dV defined by the $2n$-form $d\omega = (\partial\bar\partial\rho)^n$ is equivalent to Lebesgue measure $d\mu$ in $\mathbb{C}^n$.

By [13] the integral operator K_Ω defines a bounded mapping on $L^p(\partial\Omega)$ to $H^p(\Omega)$ for $1 < p < \infty$.

The function $d(w, z) = |\langle \partial\rho(w),\ w - z\rangle|$ defines on $\partial\Omega$ quasimetric, and if $B(z, \delta) = \{w \in \partial\Omega : d(w, z) < \delta\}$ is a quasiball with respect to d then $\sigma(B(z, \delta)) \asymp \delta^n$, see for example [13]. Therefore $\{\partial\Omega, d, \sigma\}$ is a space of homogeneous type.

Note also the crucial role in the forthcoming considerations of the following estimate that is proved in [14].

Lemma 3.1 *Let Ω be strongly convex, then*

$$d(w, z) \asymp \rho(w) + d(\pi(w), z),\ \ w \in \mathbb{C}^n \setminus \Omega,\ \ z \in \partial\Omega.$$

3.1 The polynomial approximation of Cauchy-Leray-Fantappié kernel

In lemma 3.3 here we construct a polynomial approximations of Cauchy-Leray-Fantappié kernel based on theorem by V.K. Dzyadyk about estimates of Cauchy kernel on domains on complex plane (theorem 1 in part 1 of section 7 in [3]). The approximation is choosed similarly to [16] and [15]. This construction allows us in theorem 5.1 to get polynomials that approximate holomorphic function with desired speed.

Lemma 3.2 *Let Ω be a strongly convex domain with $0 \in \Omega$, then for every $\xi \in \Omega_\varepsilon \setminus \Omega$ the value of $\lambda = \frac{\langle \partial\rho(\xi),\ z\rangle}{\langle \partial\rho(\xi),\ \xi\rangle}$ for $z \in \Omega$ lies in domain $L(t)$, bounded by the bigger arc of the circle $|\lambda| = R = R(\Omega)$ and the chord $\{\lambda \in \mathbb{C} : \lambda = 1 + e^{it}s,\ s \in \mathbb{R},\ |\lambda| \le R\}$, where $t = \frac{\pi}{2} - \arg(\langle \partial\rho(\xi),\ \xi\rangle)$.*

Proof For $\xi \in \partial\Omega$ define

$$\Lambda(\xi) = \left\{\lambda \in \mathbb{C} : \lambda = \frac{\langle \partial\rho(\xi),\ z\rangle}{\langle \partial\rho(\xi),\ \xi\rangle},\ z \in \Omega\right\}.$$

The convexity of Ω with $0 \in \Omega$ implies that

$$|\langle \partial\rho(\xi),\ \xi\rangle| \gtrsim |\partial\rho(\xi)|\,|\xi| \gtrsim 1, \tag{4}$$

$$\operatorname{Re}\langle \partial\rho(\xi),\ z - \xi\rangle \le 0, \quad z \in \bar{\Omega},\ \xi \in \Omega_\varepsilon \setminus \Omega. \tag{5}$$

The domain $L(\xi) \subset \mathbb{C}$ is also convex and contains 0, thus the equality

$$\frac{\langle \partial\rho(\xi),\ z\rangle}{\langle \partial\rho(\xi),\ \xi\rangle} = 1 + \frac{\langle \partial\rho(\xi),\ z - \xi\rangle}{\langle \partial\rho(\xi),\ \xi\rangle}$$

with estimates (4), (5) completes the proof of the lemma.

Lemma 3.3 *Let Ω be a strongly convex domain and $r > 0$. Then for every $k \in \mathbb{N}$ there exist function $K_k(\xi, z)$ defined for $\xi \in \Omega_\varepsilon \setminus \Omega$ and polynomial in $z \in \Omega$ with $\deg K_k(\xi, \cdot) \le k$ and following properties:*

$$\left|K(\xi, z) - K_k^{glob}(\xi, z)\right| \lesssim \frac{1}{k^r}\frac{1}{d(\xi, z)^{d+r}}, \quad d(\xi, z) \ge \frac{1}{k}; \tag{6}$$

$$\left|K_k^{glob}(\xi, z)\right| \lesssim k^n, \quad d(\xi, z) \le \frac{1}{k}. \tag{7}$$

Proof Due to [3] and [16] for any $j \in \mathbb{N}$ there exists function $T_j(t, \lambda)$ polynomial in λ with $\deg T_j(t, \cdot) \le j$ such that

$$\left|\frac{1}{1-\lambda} - T_j(t, \lambda)\right| \lesssim \frac{1}{j^r}\frac{1}{|1-\lambda|^{1+r}} \tag{8}$$

for $\lambda \in L(t) \setminus \left\{\lambda : |1-\lambda| < \frac{1}{j}\right\}$ and coefficients of polynomials $T_j(t, \lambda)$ continuously depend on t. Note also that by maximum principle

$$T_j(t, \lambda) \lesssim j^{-1}, \quad \lambda \in L(t) \bigcap \left\{\lambda : |1-\lambda| < \frac{1}{j}\right\}. \tag{9}$$

For $j \in \mathbb{N}$ and $(j-1) < k \le jn$ define

$$K_k^{glob}(\xi, z) = K_{jn}^{glob}(\xi, z) = \frac{1}{\langle \partial\rho(\xi),\ \xi\rangle^n} T_j^n \left(t(\xi), \frac{\langle \partial\rho(\xi),\ z\rangle}{\langle \partial\rho(\xi),\ \xi\rangle} \right).$$

Due to definition of T_j polynomials $K_k^{glob}(\xi, \cdot)$ satisfy relations (6) - (7).

3.2 Korányi regions

For $\xi \in \partial\Omega$ and $\varepsilon > 0$ we define the *inner Korányi region* as

$$D^i(\xi, \eta) = \{\tau \in \Omega : \pi(\tau) \in B(\xi, -\eta\rho(\tau)),\ \rho(\tau) > -\varepsilon\}.$$

The strong convexity of Ω implies that area-integral inequality by S. Krantz and S.Y. Li [7] for $f \in H^p(\Omega),\ 0 < p < \infty$, could be expressed as

$$\int\limits_{\partial\Omega} d\sigma(z) \left(\int\limits_{D^i(z,\eta)} |\partial f(\tau)|^2 \frac{d\mu(\tau)}{(-\rho(\tau))^{n-1}} \right)^{p/2} \le c(\Omega, p) \int\limits_{\partial\Omega} |f|^p \, d\sigma. \tag{10}$$

Consider the decomposition of vector $\tau - \xi \in \mathbb{C}^n$ as $\tau = w + tn(\xi)$, where $w \in T_\xi,\ t \in \mathbb{C}$, and $n(z) = \frac{\partial\rho(\xi)}{|\partial\rho(\xi)|}$ is a complex normal vector at z. We define the *external Korányi region* as

$$\begin{aligned} D^e(\xi, \eta) = \{\tau \in \mathbb{C}^n \setminus \Omega : \tau = w + tn(z), \\ w \in T_z,\ t \in \mathbb{C},\ |w| < \eta\rho(\tau),\ |\mathrm{Re}(t)| < \eta\rho(\tau),\ \rho(\tau) < \varepsilon\}. \end{aligned}$$

In appendix A we will proof the area-integral inequality similar to 10 for external regions D^e.

We point out two rules for integration over regions $D^e(\xi, \eta)$. First, for every function F we have

$$\int\limits_{\Omega_\varepsilon \setminus \Omega} |F(z)| \, d\mu(z) \asymp \int\limits_{\partial\Omega} d\sigma(\xi) \int\limits_{D^e(\xi,\eta)} |F(\tau)| \frac{d\mu(\tau)}{\rho(\tau)^n}.$$

Second, if $F(w) = \tilde{F}(\rho(w))$ then

$$\int\limits_{D^e(\xi,\eta)} |F(\tau)| \, d\mu(\tau) \asymp \int\limits_0^\varepsilon \left| \tilde{F}(t) \right| t^n dt.$$

Similar rules are valid for regions D^i.

We could clarify the estimate of $d(\tau, w)$ in lemma 3.1 for $\tau \in D^e(z, \eta)$.

Lemma 3.4 *Let Ω be a strongly convex domain and $\eta > 0$, then*

$$d(\tau, w) \asymp \rho(\tau) + d(z, w), \quad z, w \in \partial\Omega,\ \tau \in D^e(z, \eta). \tag{11}$$

Proof For $\tau \in D^e(z,\eta)$ we have $d(\pi(\tau),z) \lesssim \eta\rho(\tau)$ and

$$d(\tau,w) \lesssim \rho(\tau) + d(\pi(\tau),w) \lesssim \rho(\tau) + d(\pi(\tau),z) + d(z,w) \lesssim \rho(\tau) + d(z,w).$$

On the other hand,

$$\rho(\tau) + d(z,w) \lesssim \rho(\tau) + (d(z,\pi(\tau)) + d(\pi(\tau),w)) \lesssim (1+\eta)\rho(\tau) + d(\pi(\tau),w) \\ \lesssim \rho(\tau) + d(\pi(\tau),w) \lesssim d(\tau,w).$$

4 The method of pseudoanalytical continuation

4.1 Definition of pseudoanalytical continuation

The main tool of this paper is the method of continuation of function $f \in H(\Omega)$ outside the domain Ω. Let $f \in H^1(\Omega)$ and let the boundary values of f almost everywhere coincide with the boundary values of some function $\mathbf{f} \in C^1_{loc}(\mathbb{C}^n \setminus \Omega)$ such that $\left|\bar\partial \mathbf{f}\right| \in L^1(\mathbb{C}^n \setminus \Omega)$. Then by Stokes formula for $z \in \Omega$ we have

$$f(z) = \lim_{r\to 0+} \frac{1}{(2\pi i)^n} \int\limits_{\partial\Omega_r} \frac{\mathbf{f}(\xi)\partial\rho(\xi) \wedge (\bar\partial\partial\rho(\xi))^{n-1}}{\langle \partial\rho(\xi),\ \xi - z\rangle^n} = \\ \lim_{r\to 0+} \frac{1}{(2\pi i)^n} \int\limits_{\mathbb{C}^n\setminus\Omega_r} \frac{\bar\partial\mathbf{f}(\xi) \wedge \partial\rho(\xi) \wedge (\bar\partial\partial\rho(\xi))^{n-1}}{\langle \partial\rho(\xi),\ \xi - z\rangle^n} \\ = \frac{1}{(2\pi i)^n} \int\limits_{\mathbb{C}^n\setminus\Omega} \frac{\bar\partial\mathbf{f}(\xi) \wedge \partial\rho(\xi) \wedge (\bar\partial\partial\rho(\xi))^{n-1}}{\langle \partial\rho(\xi),\ \xi - z\rangle^n}, \quad (12)$$

since (for details see [12])

$$d_\xi \left(\frac{\partial\rho(\xi) \wedge (\bar\partial\partial\rho(\xi))^{n-1}}{\langle \partial\rho(\xi),\ \xi - z\rangle^n} \right) = 0, \quad z \in \Omega,\ \xi \in \mathbb{C}^n \setminus \Omega.$$

This formula allows us to study properties of function $f \in H(\Omega)$ relying on estimates of its continuation. Note that it is not necessary for the function $\mathbf{f}$ to be a continuation in terms of coincidence of boundary values. We call the function $\mathbf{f} \in C^1_{loc}(\mathbb{C}^n \setminus \Omega)$ satisfying (12) *the pseudoanalytic continuation of the function* f.

4.2 Continuation by symmetry

Theorem 4.1 *Let $f \in H^1_p(\Omega)$ and $1 < p < \infty$, $m \in \mathbb{N}$. There exist a pseudoanalytical continuation $\mathbf{f} \in C^1_{loc}(\mathbb{C}^n \setminus \Omega)$ of function f such that* $\operatorname{supp} \mathbf{f} \subset \Omega_\varepsilon$, $\left|\bar\partial\mathbf{f}(z)\right| \in L^p(\Omega_\varepsilon \setminus \Omega)$ *and*

$$\left|\bar\partial\mathbf{f}(z)\right| \lesssim \max_{|\alpha|=m} |\partial^\alpha f(z^*)|\, \rho(z)^{m-1}, \quad z \in \Omega_\varepsilon \setminus \Omega, \quad (13)$$

where z^* is a symmetric to z with respect to $\partial\Omega$.

Proof Define

$$\mathbf{f}_0(z) = \sum_{|\alpha|\le m-1} \partial^\alpha f(z^*) \frac{(z-z^*)^\alpha}{\alpha!}, \ z \in \Omega_\varepsilon \setminus \Omega. \tag{14}$$

Let $\alpha \pm e_k = (\alpha_1, \dots, \alpha_k \pm 1, \alpha_n)$ and define $(z-z^*)^{\alpha - e_k} = 0$ if $\alpha_k = 0$. In these notations we have

$$\begin{aligned}\bar\partial_j \mathbf{f}_0 &= \sum_{k=1}^\infty \sum_{|\alpha|\le m-1} \left(\partial^{\alpha+e_k} f(z^*) \frac{(z-z^*)^\alpha}{\alpha!} - \partial^\alpha f(z^*) \frac{(z-z^*)^{\alpha-e_k}}{(\alpha-e_k)!} \right) \bar\partial_j z_k^* \\ &= \sum_{k=1}^\infty \sum_{|\alpha|=m-1} \partial^{\alpha+e_k} f(z^*) \frac{(z-z^*)^\alpha}{\alpha!} \bar\partial_j z_k^*, \end{aligned} \tag{15}$$

hence,

$$\left|\bar\partial \mathbf{f}_0(z)\right| \lesssim \max_{|\alpha|=m} |\partial^\alpha f(z^*)| \, \rho(z)^{m-1}, \quad z \in \mathbb{C}^n \setminus \Omega.$$

Consider function $\chi \in C^\infty(0,\infty)$ such that $\chi(t) = 1$ for $t \le \varepsilon/2$ and $\chi(t) = 0$ for $t \ge \varepsilon$. The function $\mathbf{f}(z) = \mathbf{f}_0(z)\chi(\rho(z))$ satisfies the condition (14) and $\operatorname{supp} \mathbf{f} \subset \Omega_\varepsilon$.

Let $d = \operatorname{dist}(z^*, \partial\Omega)/10$, then for every mutiindex α such that $|\alpha| = m$ by Cauchy maximal inequality we have

$$|\partial^\alpha f(z^*)| \lesssim d^{-m+1} \sup_{|\tau - z^*| < d} |\partial f(\tau)| \lesssim \rho(z)^{-m+1} \sup_{\tau \in D^i(\pi(z), c_0 d)} |\partial f(\tau)|,$$

for some $c_0 > 0$. Finally, by theorem 2.1 from [7] we get

$$\int_{\Omega_\varepsilon \setminus \Omega} \left|\bar\partial \mathbf{f}(z)\right|^p d\mu(z) \lesssim \int_{\Omega \setminus \Omega_{-\varepsilon}} d\mu(z) \left(\sup_{\tau \in D^i(\pi(z), c_0 d)} |\partial f(\tau)| \right)^p \lesssim \|\partial f\|^p_{H^p(\Omega)}.$$

4.3 Continuation by global approximations.

Let $f \in H^1(\Omega)$ and consider a polynomial sequence $P_1, P_2, \dots$ converging to f in $L^1(\partial\Omega)$. Define

$$\lambda(z) = \rho(z)^{-1} \left| P_{2^{k+1}}(z) - P_{2^k}(z) \right|, \quad 2^{-k} < \rho(z) < 2^{-k+1}.$$

Theorem 4.2 *Assume that $\lambda \in L^p(\mathbb{C}^n \setminus \Omega)$ for some $p \ge 1$. Then there exist a pseudoanalytical continuation $\mathbf{f}$ of function f such that*

$$\left|\bar\partial \mathbf{f}(z)\right| \lesssim \lambda(z), \quad z \in \mathbb{C}^n \setminus \Omega. \tag{16}$$

Proof Consider function $\chi \in C^\infty(0,\infty)$ such that $\chi(t) = 1$ for $t \le 1$ and $\chi(t) = 0$ for $t \ge 2$. We define continuation of function f by formula

$$\mathbf{f}(z) = P_{2^k}(z) + \chi(2^k\rho(z))(P_{2^{k+1}}(z) - P_{2^k}(z)), \quad 2^{-k} < \rho(z) < 2^{-k+1}.$$

Now $\mathbf{f}$ is C^1-function on $\mathbb{C}^n \setminus \bar{\Omega}$ and $\left|\bar{\partial}\mathbf{f}(z)\right| \lesssim \lambda(z)$. We define a function $F_k(z)$ as $F_k(z) = \mathbf{f}(z)$ for $\rho(z) > 2^{-k}$ and as $F_k(z) = P_{2^{k+1}}(z)$ for $\rho(z) < 2^{-k}$. The the function F_k is smooth and holomorphic in $\Omega_{2^{-k}}$, and $\left|\bar{\partial}F_k(z)\right| \lesssim \lambda(z)$ for $z \in \mathbb{C}^n \setminus \Omega_{2^{-k}}$. Thus similarly to 12 we get

$$P_{2^{k+1}}(z) = F_k(z) = \frac{1}{(2\pi i)^n} \int\limits_{\mathbb{C}^n\setminus\Omega} \frac{\bar{\partial}F_k(\xi) \wedge \partial\rho(\xi) \wedge (\bar{\partial}\partial\rho(\xi))^{n-1}}{\langle \partial\rho(\xi),\ \xi - z\rangle^n}, \ z \in \Omega,$$

We can pass to the limit in this formula by the dominated convergence theorem; hence, function $\mathbf{f}$ satisfies the formula (12) and is a pseudoanalytical continuation of function f.

4.4 Pseudoanalytical continuation of Hardy-Sobolev spaces

Theorem 4.3 *Let Ω be a strongly convex domain, $1 < p < \infty$, $l \in \mathbb{N}$ and $f \in H^p(\Omega)$. Then $f \in H^l_p(\Omega)$ if and only if there exists such pseudoanalytical continuation $\mathbf{f}$ that*

$$\int\limits_{\partial\Omega} d\sigma(z) \left(\int\limits_{D^e(z,\alpha)} \left|\bar{\partial}\mathbf{f}(\tau)\rho(\tau)^{-l}\right|^2 d\nu(\tau) \right)^{p/2} < \infty, \tag{17}$$

where $d\nu(\tau) = \frac{d\mu(\tau)}{\rho(\tau)^{n-1}}$.

Proof Let $f \in H^l_p(\Omega)$. By theorem 4.3 we could construct pseudoanalytical continuation $\mathbf{f}$ such that

$$\left|\bar{\partial}\mathbf{f}(z)\right| \lesssim \max_{|\alpha|=m+1} \left|\partial^\alpha f(z^*)\right| \rho(z)^m, \quad z \in \mathbb{C}^n \setminus \Omega.$$

Note that the symmetry with respect to $\partial\Omega$ maps the external sector $D^e(z,\eta)$ into some internal Korányi sector. Indeed, for every $\alpha > 0$ there exists $\eta_1 = \eta_1(\eta) > 0$ such that

$$\{\tau^* : \tau \in D^e(z,\eta)\} \subseteq D^i(z,\eta_1).$$

Applying area-integral inequality (10) we obtain

$$\int\limits_{\partial\Omega} d\sigma(z) \left(\int\limits_{D^e(z,\eta)} \left|\bar\partial \mathbf{f}(\tau)\rho(\tau)^{-l}\right|^2 d\nu(\tau) \right)^{p/2}$$
$$\lesssim \max_{|\alpha|=m} \int\limits_{\partial\Omega} d\sigma(z) \left(\int\limits_{D^e(z,\eta)} \left|\partial^\alpha f(\tau^*)\right|^2 d\nu(\tau) \right)^{p/2}$$
$$\lesssim \max_{|\alpha|=m} \int\limits_{\partial\Omega} d\sigma(z) \left(\int\limits_{D^i(z,\eta_1)} \left|\partial^\alpha f(\tau)\right|^2 \frac{d\mu(\tau)}{(-\rho(\tau))^{n-1}} \right)^{p/2} < \infty$$

To prove the sufficiency, assume that function $f \in H^1(\Omega)$ admits the pseudoanalytical continuation $\mathbf{f}$ with estimate (17.) We will prove that for every function $g \in L^{p'}(\partial\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, and every multiindex α, $|\alpha| \le l$,

$$\left| \int\limits_{\partial\Omega} g(z)\partial^\alpha f(z) dS(z) \right| \le c(f) \left\| g \right\|_{L^{p'}(\partial\Omega)}.$$

Assume, without loss of generality, that $\alpha = (l, 0, \ldots, 0)$. By representation (12) we have

$$f(z) = \int\limits_{\mathbb{C}^n \setminus \Omega} \frac{\bar\partial \mathbf{f}(\xi) \wedge \omega(\xi)}{\langle \partial\rho(\xi),\ \xi - z\rangle^n}$$

and with $C_{nl} = \frac{(n+l-1)!}{(n-1)!}$

$$\int\limits_{\partial\Omega} g(z)\partial^\alpha f(z) dS(z) = C_{nl} \int\limits_{\partial\Omega} g(z) \left(\int\limits_{\mathbb{C}^n \setminus \Omega} \left(\frac{\partial\rho(\xi)}{\partial\xi_1}\right)^l \frac{\bar\partial \mathbf{f}(\xi) \wedge \omega(\xi)}{\langle \partial\rho(\xi),\ \xi - z\rangle^{n+l}} \right) dS(z)$$
$$= C_{nl} \int\limits_{\mathbb{C}^n \setminus \Omega} \left(\frac{\partial\rho(\xi)}{\partial\xi_1}\right)^l \bar\partial \mathbf{f}(\xi) \wedge \omega(\xi) \int\limits_{\partial\Omega} \frac{g(z) dS(z)}{\langle \partial\rho(\xi),\ \xi - z\rangle^{n+l}}.$$

Define $\Phi_l(\xi) = \int\limits_{\partial\Omega} \frac{g(z)dS(z)}{\langle \partial\rho(\xi),\ \xi - z\rangle^{n+l}}$. Applying Hölder inequality twice we have

$$\begin{aligned}
\left|\int\limits_{\partial\Omega} g(z)\partial^\alpha f(z)dS(z)\right| &\lesssim \int\limits_{\mathbb{C}^n\setminus\Omega} \left|\bar\partial \mathbf{f}(\xi)\right| |\Phi_l(\xi)|\, d\mu(\xi) \\
&\lesssim \int\limits_{\partial\Omega} dS(\xi) \int\limits_{D^e(\xi,\alpha)} \left|\bar\partial F(\tau)\right| |\Phi_l(\tau)| \frac{d\mu(\tau)}{\rho(\tau)^n} \\
&\lesssim \int\limits_{\partial\Omega} d\sigma(\xi) \left(\int\limits_{D^e(\xi,\alpha)} \left|\bar\partial F(\tau)\right|^2 \rho(\tau)^{-2l} \frac{d\mu(\tau)}{\rho(\tau)^{n-1}}\right)^{1/2} \times \\
&\quad \times \left(\int\limits_{D^e(\xi,\alpha)} |\Phi_l(\tau)|^2 \rho(\tau)^{2l-2} \frac{d\mu(\tau)}{\rho(\tau)^{n-1}}\right)^{1/2} \\
&\lesssim \left(\int\limits_{\partial\Omega} dS(\xi) \left(\int\limits_{D^e(\xi,\alpha)} \left|\bar\partial F(\tau)\right|^2 \rho(\tau)^{-2l} d\nu(\tau)\right)^{p/2}\right)^{1/p} \times \\
&\quad \times \left(\int\limits_{\partial\Omega} dS(\xi) \left(\int\limits_{D^e(\xi,\alpha)} |\Phi_l(\tau)|^2 \rho(\tau)^{2l-2} d\nu(\tau)\right)^{p'/2}\right)^{1/p'}.
\end{aligned}$$

The first product term is bounded by (17), and the second one by the area-integral inequality (30), that we will prove in the appendix A in theorem A.1.

5 Constructive description of Hardy-Sobolev spaces

Theorem 5.1 *Let $f \in H^1(\Omega)$ and $1 < p < \infty$, $l \in \mathbb{N}$. Then $f \in H^l_p(\Omega)$ iff there exists a polynomial sequence P_{2^k} such that*

$$\int\limits_{\partial\Omega} d\sigma(z) \left(\sum_{k=1}^{\infty} |f(z) - P_{2^k}(z)|^2\, 2^{2lk}\right)^{p/2} < \infty. \tag{18}$$

Proof Assume that condition (18) holds, then polynomials P_{2^k} converge to function f in $L^p(\partial\Omega)$ and by the theorem 4.2 we could construct pseudoanalytical continuation $\mathbf{f}$ such that

$$\left|\bar\partial \mathbf{f}(z)\right| \lesssim |P_{2^{k+1}}(z) - P_{2^k}(z)|\, \rho(z)^{-1}, \quad z \in \mathbb{C}^n \setminus \Omega,\ 2^{-k} \le \rho(z) < 2^{-k+1}.$$

Consider the decomposition of region $D^e(z,\eta)$ to sets $D_k(z) = \{\tau \in D^e(z,\eta) : \ 2^{-k} \le \rho(\tau) < 2^{-k+1}\}$, and define functions

$$a_k(z) = |P_{2^{k+1}}(z) - P_{2^k}(z)|\, 2^{-kl},$$
$$b_k(z) = \left(\int\limits_{D_k(z)} \left|\bar\partial \mathbf{f}(\tau)\rho(\tau)^{-l}\right|^2 d\nu(\tau) \right)^{1/2}.$$

Lemma 5.1 $b_k(z) \lesssim M a_k(z)$

Assume, that this lemma holds, then by Fefferman-Stein maximal theorem we have

$$\int\limits_{\partial\Omega} \left(\sum_{k=1}^{\infty} b_k(z)^2 \right)^{p/2} d\sigma(z) \lesssim \int\limits_{\partial\Omega} \left(\sum_{k=1}^{\infty} a_k(z)^2 \right)^{p/2} d\sigma(z).$$

The right-hand side of this inequality is finite by the condition 18, also we have

$$\sum_{k=1}^{\infty} b_k(z)^2 \asymp \int\limits_{S(z)} \left|\bar\partial \mathbf{f}(\xi)\rho(\xi)^{-l}\right|^2 d\nu(\xi),$$

which completes the proof of the sufficiency in the theorem..

Prove the necessity. Now $f \in H^l_p(\Omega)$ with $1 < p < \infty$ and $l \in \mathbb{N}$. By theorem 4.3 we could construct continuation $\mathbf{f}$ of function f with estimate (17). Applying the approximation of Cauchy-Leray-Fantappiè kernel from lemma 3.3 to function $\mathbf{f}$ we define polynomials

$$P_{2^k}(z) = \int_{\mathbb{C}^n \setminus \Omega} \bar\partial \mathbf{f}(\xi) \wedge \omega(\xi) K^{glob}_{2^k}(\xi, z).$$

We will prove that these polynomials satisfy the condition (18). From lemma 3.3 we obtain

$$|f(z) - P_{2^k}(z)| \lesssim \int_{\mathbb{C}^n \setminus \Omega} \left|\bar\partial \mathbf{f}(\xi)\right| \left| \frac{1}{\langle \partial\rho(\xi),\ \xi - z\rangle^d} - K^{glob}_{2^k}(\xi, z) \right| d\mu(\xi)$$
$$= U(z) + V(z) + W_1(z) + W_2(z),$$

where

$$U(z) = \int\limits_{d(\tau,z)<2^{-k}} \frac{\left|\bar\partial \mathbf{f}(\tau)\right|}{\left|\langle \partial\rho(\tau),\ \tau - z\rangle\right|^n} d\mu(\tau),$$

$$V(z) = 2^{kn} \int\limits_{d(\tau,z)<2^{-k}} \left|\bar\partial \mathbf{f}(\tau)\right| d\mu(\tau),$$

$$W_1(z) = 2^{-kr} \int\limits_{\substack{d(\tau,z)>2^{-k}\\ \rho(\tau)<2^{-k}}} \frac{\left|\bar\partial \mathbf{f}(\tau)\right|}{\left|\langle \partial\rho(\tau),\ \tau - z\rangle\right|^{n+r}} d\mu(\tau),$$

$$W_2(z) = 2^{-kr} \int\limits_{\rho(\tau)>2^{-k}} \frac{\left|\bar\partial \mathbf{f}(\tau)\right|}{\left|\langle \partial\rho(\tau),\ \tau - z\rangle\right|^{n+r}} d\mu(\tau).$$

The parameter $r > 0$ will be chosen later.

Note that $V(z) \lesssim cU(z)$ and estimate the contribution of $U(z)$ to the sum. For some $c_1, c_2 > 0$ we have

$$U(z) \le \int\limits_{\substack{d(w,z)<c_1 2^{-k}\\ w\in\partial\Omega}} d\sigma(w) \sum_{j>c_2 k} \int\limits_{D_j(w)} \frac{\left|\bar\partial \mathbf{f}(\tau)\right|}{\left|\langle \partial\rho(\tau),\ \tau - z\rangle\right|^n} \frac{d\nu(\tau)}{\rho(\tau)}$$

$$\le \int\limits_{\substack{d(w,z)<c_1 2^{-k}\\ w\in\partial\Omega}} d\sigma(w) \sum_{j>c_2 k} \left(\int\limits_{D_j(w)} \left|\bar\partial \mathbf{f}(\tau)\rho(\tau)^{-l}\right|^2 d\nu(\tau) \right)^{1/2} \times$$

$$\times \left(\int\limits_{D_j(w)} \frac{\rho(\tau)^{2(l-1)} d\nu(\tau)}{\left|\langle \partial\rho(\tau),\ \tau - z\rangle\right|^n} \right)^{1/2} = \sum_{j>c_2 k} \int\limits_{d(w,z)<c_1 2^{-j}} b_j(w) m_j(w) d\sigma(w)$$

Consider the integral $m_j(w)$. Since $\tau \in D_j(w)$ then by estimates from lemma 3.4 $d(\tau, z) \asymp \rho(\tau) + d(w, z) > 2^{-j}$ and

$$m_j(w) = \left(\int\limits_{D_j(w)} \frac{\rho(\tau)^{2(l-1)} d\nu(\tau)}{\left|\langle \partial\rho(\tau),\ \tau - z\rangle\right|^n} \right)^{1/2} \lesssim \frac{2^{-j(l-1)}}{2^{-jn}} 2^{-j} = 2^{jn-jl}. \tag{19}$$

Thus

$$2^{kl} U(z) \lesssim \sum_{j>c_1 k} 2^{-(j-k)l} 2^{jn} \int\limits_{d(w,z)<c_2 2^j} b_j(w) d\sigma(w) \lesssim \sum_{j>c_1 k} 2^{-(j-k)l} M b_j(z) \tag{20}$$

Now estimate the value $W_1(z)$. Similarly to the previous we have

$$W_1(z) \le 2^{-kr} \sum_{j>k} \int\limits_{d(w,z)\ge c_1 2^{-k}} b_j(w) m_j^r(w) d\sigma(w),$$

where

$$m_j^r(w) = \left(\int\limits_{D_j(w)} \frac{\rho(\tau)^{2(l-1)} d\nu(\tau)}{|\langle \partial\rho(\tau),\ \tau - z\rangle|^{2(n+r)}} \right)^{1/2}. \tag{21}$$

Applying the estimate $d(\tau, z) \asymp \rho(\tau) + d(w, z) \gtrsim 2^{-j}$, we obtain

$$m_j^r(w) \lesssim 2^{j(n+r-l)}. \tag{22}$$

Finally

$$2^{kl} W_1(z) \lesssim 2^{kl-kr} \sum_{j>k} 2^{j(r-l)+jn} \int\limits_{d(w,z)\le 2^{-j}} b_j(w) d\sigma(w)$$
$$\lesssim \sum_{j>k} 2^{-(j-k)(r-l)} M b_j(z). \tag{23}$$

Similarly, estimating the contribution of $W_2(z)$, we obtain

$$2^{kl} W_2(z) \lesssim 2^{-k(r-l)} \sum_{j=0}^{k} \int\limits_{\partial\Omega} b_j(w) m_j^r(w) d\sigma(w). \tag{24}$$

Since $d(\tau, z) \gtrsim 2^{-j} + d(w, z)$ for $\tau \in \partial\Omega,\ \tau \in D_j(z)$ then

$$m_j^r(w) \lesssim \frac{2^{-jl}}{(2^{-j} + d(w, z))^{n+r}} \le \min\left(2^{j(n+r-l)}, 2^{-jl} d(w, z)^{-n-r}\right).$$

Thus

$$\int\limits_{\partial\Omega} b_j(w) m_j^r(w) d\sigma(w) \lesssim \int\limits_{d(w,z)\le 2^{-j}} \frac{2^{-jl}}{2^{-j(n+r)}} b_j(w) d\sigma(w)$$
$$+ \sum_{t=1}^{j-1} \int\limits_{2^{-t-1}\le d(w,z)\le 2^{-t}} \frac{2^{-jl}}{2^{-t(n+r)}} b_j(w) d\sigma(w)$$
$$\lesssim \sum_{t=1}^{j} 2^{-jl} 2^{tr} M b_j(z) \lesssim 2^{-jl} 2^{jr} M b_j(z).$$

Choosing $r = l + 1$, we have

$$W_2(z) 2^{kl} \lesssim \sum_{j=1}^{k} 2^{-(k-j)(r-l)} M b_j(z) \le \sum_{j=1}^{k} 2^{-(k-j)} M b_j(z). \tag{25}$$

Combining the estimates (20, 23, 25) we finally obtain

$$|f(z)-P_{2^k}(z)| \lesssim \sum_{j=1}^{k} 2^{-(k-j)} Mb_j(z) + \sum_{j>k} 2^{-(j-k)l} Mb_j(z),$$

which similarly to [5] implies

$$\sum_{k=1}^{\infty} |f(z)-P_{2^k}(z)|^2 \lesssim \sum_{k=1}^{\infty} (Mb_k(z))^2 = \int\limits_{D^e(z,\eta)} \left|\bar{\partial}\mathbf{f}(\tau)\right|^2 \rho(\tau)^{-2l} d\mu(\tau) < \infty.$$

This completes the proof of the theorem and it remains to prove lemma 5.1.

Proof (of the lemma 5.1) . Define $g_k(z) := 2^{-kl}(P_{2^{k+1}}(z) - P_{2^k}(z))$.

Let $z \in \partial\Omega$ and $\tau \in S_k(z)$. Consider complex normal vector $n(z) = \frac{\partial\rho(z)}{|\partial\rho(z)|}$ at z, complex tangent hyperplane $T_z = \{w \in \mathbb{C}^n : \langle \partial\rho(z),\ w-z\rangle = 0\}$ and complex plane $T^{\perp}_{z,\tau}$, orthogonal to T_z and containing the point τ

$$T^{\perp}_{z,\tau} := \{\tau + sn(z) : s \in \mathbb{C}\}.$$

Projection of vector $\tau \in \mathbb{C}$ to $\partial\Omega \bigcap T^{\perp}_{z,\tau}$ we will denote as $\pi_z(\tau)$.

Define $\Omega_{z,\tau} = \Omega \bigcap T^{\perp}_{z,\tau}$ and $\gamma_{z,\tau} = \partial\Omega_{z,\tau}$. There exist a conformal map $\varphi_{z,\tau} : T^{\perp}_{z,\tau} \setminus \Omega_{z,\tau} \to \mathbb{C} \setminus \{w \in \mathbb{C} : |w| = 1\}$ such that $\varphi_{z,\tau}(\infty) = \infty,\ \varphi'_{z,\tau}(\infty) > 0$, and we could consider analytical in $T^{\perp}_{z,\tau} \setminus \Omega_{z,\tau}$ function $G_k(s) := \frac{g_k(s)}{\varphi^{2k+1}_{z,\tau}(s)}$.

Applying to function G_k Dyn'kin maximal estimate from [4] for domain $T^{\perp}_{z,\tau} \setminus \Omega(z,\tau)$ we obtain the estimate

$$|G_k(\tau)| \lesssim \frac{1}{\rho(\tau)} \int\limits_{s\in I_{z,\tau}} |G_k(s)|\,|ds| + \int\limits_{\partial\Omega_{z,\tau}\setminus I_{z,\tau}} |G_k(s)| \frac{\rho(\tau)^m}{|s-\pi_z(\tau)|^{m+1}}\,|ds|\,,$$

where $I_{z,\tau} = \{s \in \gamma_{z,\tau} : |s-\pi_z(\tau)| < \mathrm{dist}(\tau,\ \partial\Omega_{z,\tau})/2\}$, and $m > 0$ could be chosen arbitrary large.

Note that $|\varphi_{z,\tau}(s)| - 1 \asymp \mathrm{dist}(s,\ \partial\Omega_{z,\tau}) \asymp 2^{-k}$, thus $|g_k(s)| \asymp |G_k(s)|$ for $s \in D_k(z) \bigcap T^{\perp}_{z,\tau}$. Hence,

$$|g_k(\tau)| \lesssim \sum_{j=1}^{\infty} 2^{-jm} \frac{1}{2^j\rho(\tau)} \int\limits_{\substack{s\in\partial\Omega_{z,\tau} \\ |s-\pi_z(\tau)|<2^j\rho(\tau)}} |g_k(s)|\,|ds|\,. \tag{26}$$

Since the boundary of the domain Ω is C^3-smooth, we can assume that the constant in this inequality (26) does not depend on $z \in \partial\Omega$ and $\tau \in \Omega_\varepsilon \setminus \Omega$.

Note that function $g_k(\tau+z-w)$ is holomorphic in $w \in T_z$, then estimating the mean we obtain

$$|g_k(\tau)| \leq \frac{1}{\rho(\tau)^{n-1}} \int\limits_{|w-z|<\sqrt{\rho(\tau)}} |g_k(\tau+z-w)|\, d\mu_{2n-2}(w)$$

$$\lesssim \sum_{j=1}^{\infty} 2^{-jm} \frac{1}{\rho(\tau)^{n-1}} \int\limits_{|w-z|<\sqrt{\rho(\tau)}} \frac{d\mu_{2n-2}(w)}{2^j \rho(\tau)} \int\limits_{\substack{s\in\partial\Omega_{z,\tau} \\ |s-\pi_z(\tau+z-w)|<2^j\rho(\tau)}} |g_k(s)|\,|ds|$$

$$\lesssim \sum_{j=1}^{\infty} 2^{-j(m-n+1)} \int\limits_{B(z,2^j\rho(\tau))} |g_k(w)|\, d\sigma(w), \quad (27)$$

where $d\mu_{2n-2}$ is Lebesgue measure in T_z

Assume that $m > n-1$, then $|g_k(\tau)| \lesssim M g_k(z),\ z \in \partial\Omega,\ \tau \in D_k(z)$. Finally,

$$b_k(z) = \left(\int\limits_{D_k(z)} \left|\bar\partial \mathbf{f}(\tau)\rho(\tau)^{-l}\right|^2 d\nu(\tau) \right)^{1/2} \lesssim \left(\int\limits_{D_k(z)} \left|g_k(\tau)\rho(\tau)^{-l-1}\right|^2 d\nu(\tau) \right)^{1/2}$$

$$\lesssim M a_k(z) \left(\int\limits_{D_k(z)} \frac{d\nu(\tau)}{\rho(\tau)^2} \right)^{1/2} \lesssim M a_k(z) \quad (28)$$

and the lemma is proved. □

A Area-integral inequality for external Korányi region

Let $\Omega \subset \mathbb{C}^n$ be a strongly convex domain and $\eta > 0$. For function $g \in L^1(\partial\Omega)$ and $l \in \mathbb{N}$ we define a function

$$I_l(g,z) = \left(\int\limits_{D^e(z,\eta)} \left| \int\limits_{\partial\Omega} \frac{g(w)dS(w)}{\langle \partial\rho(\tau),\ \tau-w\rangle^{n+l}} \right|^2 d\nu_l(\tau) \right)^{1/2}, \quad (29)$$

where $d\nu_l(\tau) = \frac{d\mu_{2n}(\tau)}{\rho(\tau)^{n-2l-1}}$.

Theorem A.1 *Let Ω be strongly convex domain and $g \in L^p(\partial\Omega),\ 1<p<\infty$, Then*

$$\int\limits_{\partial\Omega} I_l(g,z)^p d\sigma(z) \lesssim \int\limits_{\partial\Omega} |g(z)|^p\, d\sigma(z). \quad (30)$$

Note that in the one-variable case the integral 29 gives the holomorphic function and the result of the theorem follows from [5].

Definition 1 Assume, that defining function ρ for strongly convex domain Ω has the following form near $z_0 \in \partial\Omega$

$$\rho(z) = \mathrm{Im}(z_n) + \sum_{j,k=1}^{n} A_{jk} z_j \bar{z}_k + O(|z|^3). \tag{31}$$

Define a set

$$D_0(\eta, \varepsilon) = \{\tau \in \mathbb{C}^n \setminus \Omega : |\tau_1|^2 + \ldots + |\tau_{n-1}|^2 < \eta \mathrm{Im}(\tau_n), \\ |\mathrm{Re}(\tau_n)| < \eta \mathrm{Im}(\tau_n),\ |\mathrm{Im}(\tau_n)| < \varepsilon_1\}. \tag{32}$$

Theorem A.2 *There exists such covering of the set $\overline{\Omega}_\varepsilon \setminus \Omega_{-\varepsilon}$ by open sets Γ_j such that for every $\xi \in \Gamma_j$ we can find a holomorphic change of coordinates $\varphi_j(\xi, \cdot) : \mathbb{C}^n \to \mathbb{C}^n$ such that*

1. *The mapping $\varphi_j(\xi, \cdot)$ transforms function ρ to the type (31) and could be expressed as follows*
$$\varphi_j(\xi, z) = \Phi_j(\xi)(z - \xi) + i(z - \xi)^\perp B_j(\xi)(z - \xi)e_n, \tag{33}$$
where matrices $\Phi_j(\xi), B_j(\xi)$ depend smooth on $\xi \in \Gamma_j$, and $e_n = (0, \ldots, 0, 1)$.
2. *Let $\psi_j(\xi, \cdot)$ be an inverse map of $\varphi_j(\xi, \cdot)$, and let $J_j(\xi, \cdot)$ be a complex Jacobian of ψ. Note that real Jacobian is then equal to $|J_j(\xi, \cdot)|^2 = J_j(\xi, \cdot)\overline{J_j(\xi, \cdot)}$, Then*
$$\sup_{\tau \in \Omega_\varepsilon \setminus \overline{\Omega}_\varepsilon} \left|J_j(\xi, \cdot) - J_j(\xi', \cdot)\right| \lesssim \left|\xi - \xi'\right|, \tag{34}$$
$$\sup_{\tau \in \Omega_\varepsilon \setminus \overline{\Omega}_\varepsilon} \left|\psi_j(\xi, \cdot) - \psi_j(\xi', \cdot)\right| \lesssim \left|\xi - \xi'\right|. \tag{35}$$
3. *There exist constants $c > 0,\ 0 < \eta_0, \varepsilon_1 < 1$ such that*
$$\varphi_j(\xi, D^e(\xi, \eta)) \subseteq D_0(c\eta, c\varepsilon), \quad \psi_j(\xi, D^0_\eta) \subseteq D^e(\xi, c\eta),\ \textit{for } 0 < \eta < \eta_0. \tag{36}$$

Proof Let $\xi \in \partial\Omega$, by linear change of coordinates $z' = (z - \xi)\Phi(\xi)$ we could obtain the following form for function ρ

$$\rho(z) = \rho(\xi + \Phi^{-1}(\xi)z') = \mathrm{Im}(z'_n) + \sum_{j,k=1}^{n} A^1_{jk}(\xi) z'_j \bar{z}'_k + \mathrm{Re} \sum_{j,k=1}^{n} A^2_{jk}(\xi) z'_j z'_k + O(\left|z'\right|^3).$$

Setting $z''_n = z'_n + iA^2_{jk} z'_j z'_k$ and $z''_j = z'_j,\ 1 \le j \le n - 1$, we have

$$\rho(z'') = \mathrm{Im}(z'_n) + \sum_{j,k=1}^{n} A^1_{jk}(\xi) z''_j \bar{z}''_k + O(\left|z''\right|^3).$$

Denote $B(\xi) = \Phi(\xi)^\perp A^1(\xi)\Phi(\xi)$, then

$$\varphi(\xi, z) = \Phi(\xi)(z - \xi) + i(z - \xi)^\perp B(\xi)(z - \xi)e_n.$$

We choose Γ_j such that the matrix $\Phi(\xi)$ could be defined on Γ_j smoothly, this choice we denote as Φ_j, and the change corresponding to this matrix as φ_j

$$\varphi_j(\xi, z) = \Phi_j(\xi)(z - \xi) + i(z - \xi)^\perp B_j(\xi)(z - \xi)e_n.$$

Thus mappings φ_j satisfy the first condition. Easily, the second condition also holds.

To prove the last condition (36) it is sufficient to show, that if function ρ has the form (31), then for some $c, \eta_0 > 0$ we have

$$D^e(0, \eta) \subseteq D_0(c\eta), \quad D_0(\eta) \subseteq D^e(0, c\eta), \mathrm{for} 0 < \eta < \eta_0.$$

Indeed, for the function ρ of the form (31) the Korányi sector could be expressed as follows

$$D^e(0,\eta) = \{\tau \in \mathbb{C}^n \setminus \Omega : |\tau_1|^2 + \ldots + |\tau_{n-1}|^2 \le (\eta\rho(\tau))^2,\ |\mathrm{Re}(\tau_n)| \le \eta\rho(\tau),\ \rho(\tau) < \varepsilon\}$$

Assume $\tau \in D^e(0,\eta)$, then

$$\rho(\tau) = \mathrm{Im}(\tau_n) + \sum_{j,k=1}^{n} A_{jk}\tau_j\bar{\tau}_k \le \mathrm{Im}(\tau_n) + c_0\,|\mathrm{Im}(\tau_n)|^2$$
$$+ c_0(|\tau_1|^2 + \ldots + |\tau_{n-1}| + |\mathrm{Re}(\tau_n)|^2) \le \mathrm{Im}(\tau_n) + c_0\,|\mathrm{Im}(\tau_n)|^2 + c_0\eta\rho(\tau) + c_0(\eta\rho(\tau))^2. \tag{37}$$

By choosing $\eta_0 < 1/(4c_0)$ we have $\rho(\tau) \le c\mathrm{Im}(\tau_n)$ for $\tau \in D^e(0,\eta)$ and $0 < \eta < \eta_0$. Hence, $D^e(0,\eta) \subseteq D_0(c\eta)$ for $0 < \eta < \eta_0$ and $\varepsilon_1 = c\varepsilon$. Note, that

$$\rho(\tau) = \mathrm{Im}(\tau_n) + \sum_{j,k=1}^{n} A_{jk}\tau_j\bar{\tau}_k > \mathrm{Im}(\tau_n) \text{ for } \tau \in D_0(\eta),$$

since matrix (A_{jk}) is strictly positive definite. Hence, $D_0(\eta) \subseteq D^e(0,\eta)$. This ends the proof of the theorem.

Further we will assume, that the covering $\overline{\Omega}_\varepsilon \setminus \Omega_{-\varepsilon} \subset \bigcup_{j=1}^{N} \Gamma_j$ and maps φ_j, ψ_j are chosen by the theorem A.2. For covering $\{\Gamma_j\}$ we consider a smooth decomposition of identity on $\partial\Omega$:

$$\chi_j \in C^\infty(\Gamma_j),\ 0 \le \chi_j \le 1,\ \mathrm{supp}\ \chi_j \subset \Gamma_j,\ \sum_{j=1}^{N} \chi_j(z) = 1,\ z \in \partial\Omega.$$

Fix the parameter $\eta < \eta_0$ and denote $D_0 = D_0(\eta)$. Then by (36)

$$I_l(g,z)^2$$
$$= \sum_{j=1}^{N} \chi_j(z) \int\limits_{\varphi_j(z,D^e(z,\eta/c))} \left| \int\limits_{\partial\Omega} \frac{g(w)J_j(z,\tau)dS(w)}{\langle \partial\rho(\psi_j(z,\tau)),\ \psi_j(z,\tau) - w\rangle^{n+l}} \right|^2 \frac{d\mu(\tau)}{\mathrm{Im}(\tau_n)^{n-2l+1}}$$
$$\lesssim \sum_{j=1}^{N} \int\limits_{D_0} \left| \int\limits_{\partial\Omega} \frac{g(w)\chi_j^{1/2}(z)J_j(z,\tau)dS(w)}{\langle \partial\rho(\psi_j(z,\tau)),\ \psi_j(z,\tau) - w\rangle^{n+l}} \right|^2 \frac{d\mu(\tau)}{\mathrm{Im}(\tau_n)^{n-2l+1}}. \tag{38}$$

We will consider the function

$$K_j(z,w)(\tau) = \frac{\chi_j^{1/2}(z)J_j(z,\tau)}{\langle \partial\rho(\psi_j(z,\tau)),\ \psi_j(z,\tau) - w\rangle^{n+1}} \tag{39}$$

as a map $\partial\Omega \times \partial\Omega \to \mathscr{L}(\mathbb{C}, L^2(D_0, d\nu_l))$, such that its values are operator of multiplication from $\mathbb{C}$ to $L^2(D_0, d\nu_l)$, where $d\nu_l(\tau) = \frac{d\mu(\tau)}{\mathrm{Im}(\tau_n)^{n-2l+1}}$ is a measure on the region D_0. Throughout the proof of the theorem A.1 j, l will be fixed integers and the norm of function F in the space $L^2(D_0, d\nu_l)$ will be denoted as $\|F\|$.

We will show that integral operator defined by kernel K_j is bounded on L^p. To prove this we apply $T1$-theorem for transformations with operator-valued kernels formulated by Hytönen and Weis in [8], taking in account that in our case concerned spaces are Hilbert. Some details of the proof are similar to the proof of the boundedness of operator Cauchy-Leray-Fantappiè K_Ω for lineally convex domains introduced in [13]. Below we formulate the $T1$-theorem, adapted to our context.

Definition 2 We say that the function $f \in C_0^\infty(\partial\Omega)$ is a normalized bump-function, associated with the quasiball $B(w_0, r)$ if supp $f \subset B(z, r)$, $|f| \le 1$, and

$$|f(\xi) - f(z)| \le \frac{d(\xi, z)^\gamma}{r^\gamma}.$$

The set of bump-functions associated with $B(w_0, r)$ is denoted as $A(\gamma, w_0, r)$.

Theorem A.3 *Let $K : \partial\Omega \times \partial\Omega \to \mathscr{L}(\mathbb{C}, L^2(D_0, d\nu_l))$ verify the estimates*

$$\|K(z,w)\| \lesssim \frac{1}{d(z,w)^n}; \tag{40}$$

$$\|K(z,w) - K(\xi,w)\| \lesssim \frac{d(z,\xi)^\gamma}{d(z,w)^{n+\gamma}}, \quad d(z,w) > Cd(z,\xi); \tag{41}$$

$$\left\|K(z,w) - K(z,w')\right\| \lesssim \frac{d(w,w')^\gamma}{d(z,w)^{n+\gamma}}, \quad d(z,w) > Cd(w,w'). \tag{42}$$

Assume that operator $T : \mathscr{S}(\partial\Omega) \to \mathscr{S}'(\partial\Omega, \mathscr{L}(\mathbb{C}, L^2(D_0, d\nu_l)))$ with kernel K verify the following conditions.

- *$T1,\ T'1 \in \mathrm{BMO}(\partial\Omega, L^2(D_0, d\nu_l))$, where T' is formally adjoint operator.*
- *Operator T satisfies the weak boundedness property, that is for every pair of normalized bump-functions $f, g \in A(\gamma, w_0, r)$ we have*

$$\|\langle g,\ Tf\rangle\| \le Cr^{-n}.$$

Then $T \in \mathscr{L}(L^p(\partial\Omega), L^p(\partial\Omega, L^2(D_0, d\nu_l))$ for every $p \in (1, \infty)$.

In the following three lemmas we will prove that kernels K_j and corresponding operators T_j satisfy the conditions of the $T1$-theorem.

Lemma A.1 *The kernel K_j verify estimates (40-42).*

Proof By lemma 3.4 we have $|\langle\partial\rho(\tau),\ \tau - w\rangle| \asymp \rho(\tau) + |\langle\partial\rho(z),\ z - w\rangle|,\ z, w \in \partial\Omega,\ \tau \in D^e(z, \eta)$. Thus

$$\|K_j(z,w)\|^2 = \int\limits_{D_0} |K_j(z,w)(\tau)|^2\, d\nu_l(\tau) \lesssim \int\limits_{D^e(z,c\eta)} \frac{1}{|\langle\partial\rho(\tau),\ \tau - w\rangle|^{2n+2l}} d\nu_l(\tau)$$

$$\lesssim \int\limits_{D^e(z,c\eta)} \frac{1}{(\rho(\tau) + |\langle\partial\rho(z),\ z - w\rangle|)^{2n+2l}} \frac{d\mu(\tau)}{\rho(\tau)^{n-2l+1}} \lesssim \int\limits_0^\infty \frac{t^{2l-1}dt}{(t + |\langle\partial\rho(z),\ z - w\rangle|)^{2n+2l}}$$

$$\lesssim \frac{1}{|\langle\partial\rho(z),\ z - w\rangle|^{2n}} \lesssim \frac{1}{d(z,w)^{2n}}.$$

Similarly,

$$\left\|K_j(z,w) - K_j(z,w')\right\|^2 \lesssim \int\limits_{D^e(z,c\eta)} \left|\frac{1}{\langle\partial\rho(\tau),\ \tau - w\rangle^{n+l}} - \frac{1}{\langle\partial\rho(\tau),\ \tau - w'\rangle^{n+l}}\right|^2 d\nu_l(\tau)$$

$$\lesssim \int\limits_{D^e(z,c\eta)} \left|\frac{\langle\partial\rho(\tau),\ w - w'\rangle \left(\sum\limits_{k=0}^{n+l-1} \langle\partial\rho(\tau),\ \tau - w\rangle^k \langle\partial\rho(\tau),\ \tau - w'\rangle^{n+l-1-k}\right)}{\langle\partial\rho(\tau),\ \tau - w\rangle^{n+l} \langle\partial\rho(\tau),\ \tau - w'\rangle^{n+l}}\right|^2 d\nu_l(\tau)$$

Note that

$$\begin{aligned}|\langle\partial\rho(\tau),\ \tau-w\rangle| &\lesssim \rho(\tau)+|\langle\partial\rho(\pi(\tau)),\ \pi(\tau)-w\rangle| \\ &\lesssim \rho(\tau)+|\langle\partial\rho(z),\ z-w\rangle|+|\langle\partial\rho(\pi(\tau)),\ \pi(\tau)-z\rangle| \lesssim \rho(\tau)+|\langle\partial\rho(z),\ z-w\rangle|\,,\end{aligned}$$

which combined with lemma 3.4 and condition $\langle\partial\rho(w),\ w-w'\rangle < c\,\langle\partial\rho(z),\ z-w\rangle$ implies

$$\langle\partial\rho(\tau),\ \tau-w\rangle \asymp \rho(\tau)+|\langle\partial\rho(z),\ z-w\rangle| \asymp \rho(\tau)+\left|\left\langle\partial\rho(z),\ z-w'\right\rangle\right| \asymp \left|\left\langle\partial\rho(\tau),\ \tau-w'\right\rangle\right|.$$

Next, we have

$$\begin{aligned}&\left|\left\langle\partial\rho(\tau),\ \tau-w'\right\rangle-\langle\partial\rho(\tau),\ \tau-w\rangle\right| = \left|\langle\partial\rho(\tau),\ \pi(\tau)-w\rangle-\left\langle\partial\rho(\tau),\ \pi(\tau)-w'\right\rangle\right| \\ &\le \left|\left\langle\partial\rho(\tau)-\partial\rho(\pi(\tau)),\ w-w'\right\rangle\right|+\left|\langle\partial\rho(\pi(\tau)),\ \pi(\tau)-w\rangle-\left\langle\partial\rho(\pi(\tau)),\ \pi(\tau)-w'\right\rangle\right| \\ &\lesssim \rho(\tau)\left|\left\langle\partial\rho(w),\ w-w'\right\rangle\right|^{1/2}+|\langle\partial\rho(\pi(\tau)),\ \pi(\tau)-w\rangle|^{1/2}\left|\left\langle\partial\rho(w),\ w-w'\right\rangle\right|^{1/2} \\ &\lesssim |\langle\partial\rho(\tau),\ \tau-w\rangle|^{1/2}\left|\left\langle\partial\rho(w),\ w-w'\right\rangle\right|^{1/2}\end{aligned}$$

Hence,

$$\begin{aligned}\left\|K_j(z,w)-K_j(z,w')\right\|^2 &\lesssim \int\limits_{D^e(z,c\eta)} \frac{|\langle\partial\rho(w),\ w-w'\rangle|}{|\langle\partial\rho(\tau),\ \tau-w\rangle|^{2n+2l+1}}d\nu_l(\tau) \\ &\lesssim \int\limits_0^\infty \frac{|\langle\partial\rho(w),\ w-w'\rangle|\,t^{2l-1}dt}{(t+|\langle\partial\rho(z),\ z-w\rangle|)^{2n+2l+1}} \lesssim \frac{|\langle\partial\rho(w),\ w-w'\rangle|}{|\langle\partial\rho(z),\ z-w\rangle|^{2n+1}} = \frac{d(w,w')}{d(z,w)^{n+1}}.\end{aligned}$$

The last inequality (42) is a bit harder to prove. Let $z,\xi,w\in\partial\Omega,\ d(z,\xi)<d(z,w)$, and estimate the value

$$A = |\langle\partial\rho(\psi_j(z,\tau)),\ \psi_j(z,\tau)-w\rangle-\langle\partial\rho(\psi_j(\xi,\tau)),\ \psi_j(\xi,\tau)-w\rangle|\,.$$

Denote $\tau_z=\psi_j(z,\tau),\ \tau_\xi=\psi_j(\xi,\tau)$, then

$$\tau=\Phi(z)(\tau_z-z)+i(\tau_z-z)^TB(z)(\tau_z-z)e_n=\Phi(\xi)(\tau_\xi-\xi)+i(\tau_\xi-\xi)^TB(\xi)(\tau_\xi-\xi)e_n,$$

whence we obtain denoting $\Psi(z)=\Phi(z)^{-1}$

$$\begin{aligned}\tau_z &= z+\Psi(z)\tau-i(\tau_z-z)^TB(z)(\tau_z-z)\Psi(z)e_n, \\ \tau_\xi &= \xi+\Psi(\xi)\tau-i(\tau_\xi-\xi)^TB(\xi)(\tau_\xi-\xi)\Psi(\xi)_n, \\ \tau_z-\tau_\xi &= z-\xi+(\Psi(z)-\Psi(\xi))\tau+I(z,\xi,\tau).\end{aligned}$$

Note, that norms of matrices $\|\Psi(\xi)\|$ are bounded, thus

$$\begin{aligned}|I(z,\xi,\tau)| &= \left|(\tau_z-z)^TB(z)(\tau_z-z)(\Psi(z)-\Psi(\xi))\right| \\ &\quad+\left|(\tau_z-z)^TB(z)(\tau_z-z)-(\tau_\xi-\xi)^TB(\xi)(\tau_\xi-\xi)\right|\|\Psi(\xi)\| \lesssim |z-\xi|\,|\tau_z-z|^2 \\ &\quad+\left|(\tau_z-z-\tau_\xi+\xi)^TB(z)(\tau_z-z)\right|+\left|(\tau_\xi-\xi)^TB(z)(\tau_z-z)-(\tau_\xi-\xi)B(\xi)(\tau_\xi-\xi)\right| \\ &\lesssim |z-\xi|\,|\tau_z-z|^2+|z-\xi|\,|\tau|+\left|((\Psi(z)-\Psi(\xi))\tau+I(z,\xi)^T)B(z)(\tau_z-z)\right| \\ &\quad+\left|(\tau_\xi-\xi)^T(B(z)-B(\xi))(\tau_z-z)\right|+\left|(\tau_\xi-\xi)^TB(\xi)(\tau_z-z-\tau_\xi-\xi)\right| \\ &\lesssim |z-\xi|\,|\tau_z-z|^2+|z-\xi|\,|\tau|+|\tau|\,|I(z,\xi,\tau)|+|z-\xi|\,|\tau|^2+|\tau|\,I(z,\xi,\tau).\end{aligned}$$

Choosing $\varepsilon_1>0$ small enough we get $|\tau| < \eta\ |\mathrm{Im}(\tau_n)| + (1+\eta)\,|\mathrm{Im}(\tau_n)| \le 3\varepsilon_1$ and

$$|I(z,\xi,\tau)| \lesssim d(z,\xi)^{1/2}\,|\tau|\,,\text{for } \tau\in D_0.$$

Hence,

$$A(z,\xi,\tau) \le \left|\left\langle \partial\rho(\tau_z) - \partial\rho(\tau_\xi),\ \tau_z - w\right\rangle\right| + \left|\left\langle \partial\rho(\tau_\xi),\ \tau_z - w\right\rangle\right| \lesssim \left|\tau_z - \tau_\xi\right| (\rho(\tau_z) + d(z,w)^{1/2})$$
$$+\left|\left\langle \partial\rho(\tau_z) - \partial\rho(\tau_\xi),\ z - \xi\right\rangle\right| + \left|\langle \partial\rho(z),\ z - \xi\rangle\right| + \left|\left\langle \partial\rho(\tau_\xi),\ (\Psi(z) - \Psi(\xi))\tau\right\rangle\right| + \left|\left\langle \partial\rho(\tau_\xi),\ I(z,\xi)\right\rangle\right|$$
$$\lesssim d(z,\xi)^{1/2} d(\tau_z, w) + |\tau_z - \xi|\,|z - \xi| + d(z,\xi) + |z - \xi|\,|\tau| + |I(z,\xi,\tau)|$$
$$\lesssim d(z,\xi) + d(z,\xi)^{1/2} d(z,w)^{1/2} \lesssim d(z,\xi)^{1/2} d(z,w)^{1/2}$$

Combining this estimate with inequality $|\langle \partial\rho(\tau_z),\ \tau_z - w\rangle| \asymp \left|\left\langle \partial\rho(\tau_\xi),\ \tau_\xi - w\right\rangle\right|$ we obtain

$$\|K_j(z,w) - K_j(\xi,w)\|^2 \lesssim \int\limits_{D^e(z,c\eta)} \frac{\left|\chi_j(z)^{1/2} - \chi_j(\xi)^{1/2}\right|^2}{|\langle \partial\rho(\tau),\ \tau - w\rangle|^{2n+2l}} \frac{d\mu(\tau)}{\rho(\tau)^{n-2l+1}}$$
$$+ \chi_j(\xi) \int\limits_{D_0} \frac{|\langle \partial\rho(z),\ z - \xi\rangle|\,|\langle \partial\rho(z),\ z - w\rangle|}{|\langle \partial\rho(\tau_z),\ \tau_z - w\rangle|^{2n+4}} \frac{d\mu(\tau)}{\mathrm{Im}(\tau_n)^{n-2l+1}}$$
$$\lesssim \frac{|\langle \partial\rho(z),\ z - \xi\rangle|}{|\langle \partial\rho(z),\ z - w\rangle|^{2n}} + \frac{|\langle \partial\rho(z),\ z - \xi\rangle|}{|\langle \partial\rho(z),\ z - w\rangle|^{2n+1}} \lesssim \frac{|\langle \partial\rho(z),\ z - \xi\rangle|}{|\langle \partial\rho(z),\ z - w\rangle|^{2n+1}} \lesssim \frac{d(z,\xi)}{d(z,w)^{2n+1}}.$$

Lemma A.2 *$T_j(1) = 0$ and $\left\|T_j'(1)\right\| \lesssim 1$.*

Proof Introduce the notation $\tau_z = \psi_j(z,\tau)$. The function $\langle \partial\rho(\tau_z),\ \tau_z - w\rangle$ is holomorphic in Ω with respect to w, then the form $\langle \partial\rho(\tau_z),\ \tau_z - w\rangle^{-n-l}\, dS(w)$ is closed in Ω and

$$T_j(1)(\tau) = \chi_j(z)^{1/2} J_j(z,\tau) \int\limits_{\partial\Omega} \frac{dS(w)}{\langle \partial\rho(\tau_z),\ \tau_z - w\rangle^{n+l}} = 0.$$

It remains to estimate the value of formally-adjoint operator T' on $f \equiv 1$.

$$f(w)(\tau) = T_j'(1)(w)(\tau) = \int\limits_{\partial\Omega} \frac{\chi_j(z)^{1/2} J_j(z,\tau) dS(z)}{\langle \partial\rho(\tau_z),\ \tau_z - w\rangle^{n+l}}$$
$$= \int\limits_{\partial\Omega} \frac{\chi_j(z)^{1/2} J_j(z,\tau)(dS(z) - dS(\tau_z))}{\langle \partial\rho(\tau_z),\ \tau_z - w\rangle^{n+l}} + \int\limits_{\partial\Omega} \frac{\chi_j(z)^{1/2} J_j(z,\tau) dS(\tau_z)}{\langle \partial\rho(\tau_z),\ \tau_z - w\rangle^{n+l}} = I_1 + I_2.$$

Note that $|z - \tau_z| \lesssim \mathrm{Im}(\tau_n)$, therefore $|S(z) - dS(\psi(z,\tau))| \lesssim \mathrm{Im}(\tau_n)$ and

$$|I_1| \lesssim \int\limits_{\partial\Omega} \frac{\mathrm{Im}(\tau_n) d\sigma(z)}{|\langle \partial\rho(\tau_z),\ \tau_z - w\rangle|^{n+l}} \lesssim \frac{\mathrm{Im}(\tau_n) d\sigma(z)}{(\mathrm{Im}(\tau_n) + |\langle \partial\rho(z),\ z - w\rangle|)^{n+l}}$$
$$\lesssim \int\limits_0^\infty \frac{\mathrm{Im}(\tau_n) v^{n-1} dv}{(\mathrm{Im}(\tau_n) + v)^{n+l}} \lesssim \frac{1}{\mathrm{Im}(\tau_n)^{l-1}}.$$

Thus we get

$$\int\limits_{D_0} |I_1|^2\, d\nu_l(\tau) \lesssim \int\limits_{D_0} \frac{1}{\mathrm{Im}(\tau_n)^{2l-2}} \frac{d\mu(\tau)}{\mathrm{Im}(\tau_n)^{n-2l+1}} \lesssim \int\limits_0^{\varepsilon_1} \frac{t^n dt}{t^{n-1}} \lesssim 1 \tag{43}$$

To estimate I_2 we recall that $d_\xi \frac{dS(\xi)}{\langle \partial\rho(\xi),\ \xi - z\rangle^n} = 0,\ z \in \partial\Omega,\ \xi \in \mathbb{C}^n \setminus \Omega$, and consequently

$$d \frac{dS(\xi)}{\langle \partial\rho(\xi),\ \xi - z\rangle^{n+l}} = \frac{(\bar\partial\partial\rho(\xi))^n}{\langle \partial\rho(\xi),\ \xi - z\rangle^{n+l}} - (n+l) \frac{(\bar\partial_\xi(\langle \partial\rho(\xi),\ \xi - z\rangle) \wedge \bar\partial\partial\rho(\xi))^{n-1}}{\langle \partial\rho(\xi),\ \xi - z\rangle^{n+l}}$$
$$= -\frac{l}{n} \frac{dV(\xi)}{\langle \partial\rho(\xi),\ \xi - z\rangle^{n+l}}.$$

By Stokes' theorem we obtain

$$I_2 = \int\limits_{\partial\Omega} \frac{\chi_j(z)^{1/2} J_j(z,\tau) dS(\tau_z)}{\langle \partial\rho(\tau_z),\ \tau_z - w\rangle^{n+l}}$$
$$= \int\limits_{\Omega_{\varepsilon_1}\setminus\Omega} \frac{\bar\partial_z \left(\chi_j(z)^{1/2} J_j(z,\tau)\right) \wedge dS(\tau_z)}{\langle \partial\rho(\tau_z),\ \tau_z - w\rangle^{n+l}} - \frac{l}{n} \int\limits_{\Omega_{\varepsilon_1}\setminus\Omega} \frac{\chi_j(z)^{1/2} J_j(z,\tau) dV(\tau_z)}{\langle \partial\rho(\tau_z),\ \tau_z - z\rangle^{n+l}}$$

Analogously to lemma 3.4 we have

$$|\langle \partial\rho(\tau_z),\ \tau_z - w\rangle| \asymp \mathrm{Im}(\tau_n) + \rho(z) + |\langle \partial\rho(\pi(z)),\ \pi(z) - w\rangle|\,.$$

Hence,

$$|I_2| \lesssim \int\limits_{\Omega_{\varepsilon_1}\setminus\Omega} \frac{d\mu(z)}{|\langle \partial\rho(\tau_z),\ \tau_z - w\rangle|^{n+l}}$$
$$\lesssim \int\limits_0^{\varepsilon_1} dt \int\limits_{\partial\Omega_t} \frac{d\sigma_t}{(t + \mathrm{Im}(\tau_n) + |\langle \partial\rho(\pi(z)),\ \pi(z) - w\rangle|)^{n+l}}$$
$$\lesssim \int\limits_0^{\varepsilon_1} dt \int\limits_0^{\infty} \frac{v^{n-1} dv}{(t + \mathrm{Im}(\tau_n) + v)^{n+l}} \lesssim \int\limits_0^{\varepsilon_1} \frac{dt}{(t + \mathrm{Im}(\tau_n))^l} \lesssim (\mathrm{Im}(\tau_n))^{1-l} \ln\left(1 + \frac{1}{\mathrm{Im}(\tau_n)}\right), \quad (44)$$

and

$$\int\limits_{D_0} |I_2|^2\, d\nu_l(\tau) \lesssim \int\limits_{D_0} (\mathrm{Im}(\tau_n))^{2-2l} \ln^2\left(1 + \frac{1}{\mathrm{Im}(\tau_n)}\right) d\nu_l(\tau) \lesssim \int\limits_0^{\varepsilon_1} \ln^2\left(1 + \frac{1}{s}\right) s ds \lesssim 1,$$

which with the estimate (43) completes the proof of the lemma.

Lemma A.3 *Operator T_j is weakly bounded.*

Proof Let $f, g \in A(\frac{1}{2}, w_0, r)$, denote again $\tau_z = \psi_j(z,\tau)$, then

$$\|\langle g,\ Tf\rangle\|^2 \lesssim \int\limits_{D_{c\eta}} d\nu_l(\tau) \left(\int\limits_{B(w_0,r)} |g(z)|\, dS(z) \left| \int\limits_{B(w_0,r)} \frac{f(w) dS(w)}{\langle \partial\rho(\tau_z),\ \tau_z - w\rangle^{n+1}} \right| \right)^2.$$

Denote $t := \inf\limits_{w\in\partial\Omega} |\langle \partial\rho(\tau_z),\ \tau_z - w\rangle|$ and introduce the set

$$W(z,\tau,r) := \{w \in \partial\Omega : |\langle \partial\rho(\tau_z),\ \tau_z - w\rangle| < t + r\}\,.$$

Note that $B(w_0,r) \subset W(z,\tau,cr) \subset B(z,c^2 r)$ for some $c > 0$, therefore,

$$\left| \int\limits_{B(w_0,r)} \frac{f(w) dS(w)}{\langle \partial\rho(\tau_z),\ \tau_z - w\rangle^{n+l}} \right| = \left| \int\limits_{W(z,\tau,cr)} \frac{f(w) dS(w)}{\langle \partial\rho(\tau_z),\ \tau_z - w\rangle^{n+l}} \right|$$
$$\lesssim \int\limits_{W(z,\tau,cr)} \frac{|f(z) - f(w)|\, dS(w)}{|\langle \partial\rho(\tau_z),\ \tau_z - w\rangle|^{n+l}} + |f(z)| \left| \int\limits_{\partial\Omega\setminus W(z,\tau,cr)} \frac{dS(w)}{\langle \partial\rho(\tau_z),\ \tau_z - w\rangle^{n+l}} \right|$$
$$= I_1(z,\tau) + |f(z)|\, I_2(z,\tau).$$

It follows from the estimate $|f(z)-f(w)|\le\sqrt{v(w,z)/r}$ that

$$I_1(z,\tau)\lesssim\frac{1}{\sqrt{r}}\int\limits_{B(z,c^2r)}\frac{v(w,z)^{1/2}}{(\mathrm{Im}(\tau_n)+v(w,z))^{n+l}}\lesssim\frac{1}{\sqrt{r}}\int\limits_0^{c^2r}\frac{t^{n-1/2}dt}{(\mathrm{Im}(\tau_n)+t)^{n+l}}$$
$$\lesssim\frac{1}{\sqrt{r}}\int\limits_0^{c^2r}\frac{dt}{(\mathrm{Im}(\tau_n)+t)^{l+1/2}}\lesssim\frac{1}{\sqrt{r}}\left(\frac{1}{\mathrm{Im}(\tau_n)^{l-1/2}}-\frac{1}{(\mathrm{Im}(\tau_n)+r)^{l-1/2}}\right)$$
$$=\frac{1}{\sqrt{r}}\frac{(\mathrm{Im}(\tau_n)+r)^{l-1/2}-r^{l-1/2}}{\mathrm{Im}(\tau_n)^{l-1/2}(\mathrm{Im}(\tau_n)+r)^{l-1/2}}\lesssim\frac{1}{\sqrt{r}}\frac{(\mathrm{Im}(\tau_n)+r)^{2l-1}-r^{2l-1}}{\mathrm{Im}(\tau_n)^{l-1/2}(\mathrm{Im}(\tau_n)+r)^{2l-1}}$$
$$\lesssim\frac{1}{\sqrt{r}}\frac{r\mathrm{Im}(\tau_n)^{2l-2}+r^{2l-1}}{\mathrm{Im}(\tau_n)^{l-1/2}(\mathrm{Im}(\tau_n)+r)^{2l-1}}=\frac{\sqrt{r}\mathrm{Im}(\tau_n)^{l-3/2}}{(\mathrm{Im}(\tau_n)+r)^{2l-1}}+\frac{r^{2l-3/2}}{\mathrm{Im}(\tau_n)^{l-1/2}(\mathrm{Im}(\tau_n)+r)^{2l-1}}.$$

Estimating the $L^2(D_0,d\nu_l)-$norm of the function $I_1(z,\tau)$, we obtain

$$\int\limits_{D_0(\tau)}I_1(z,\tau)^2d\nu_l(\tau)$$
$$\lesssim\int\limits_{D_0(\tau)}\left(\frac{r\mathrm{Im}(\tau_n)^{2l-3}}{(\mathrm{Im}(\tau_n)+r)^{4l-2}}+\frac{r^{4l-3}}{\mathrm{Im}(\tau_n)^{2l-1}(\mathrm{Im}(\tau_n)+r)^{4l-2}}\right)\frac{d\mu(\tau)}{\mathrm{Im}(\tau_n)^{n-2l+1}}$$
$$\lesssim r\int\limits_0^\infty\frac{s^{4l-4}}{(s+r)^{4l-2}}ds+r^{4l-3}\int\limits_0^\infty\frac{ds}{(s+r)^{4l-2}}\lesssim 1\qquad(45)$$

To estimate the second summand we apply the Stokes theorem to the domain

$$\{w\in\Omega:|\langle\partial\rho(\tau_z),\ \tau_z-w\rangle|>t+cr\}$$

and to the closed in this domain form $\frac{dS(w)}{\langle\partial\rho(\tau_z),\ \tau_z-w\rangle^{n+l}}$

$$\int\limits_{\partial\Omega\setminus W(z,\tau,cr)}\frac{dS(w)}{\langle\partial\rho(\tau_z),\ \tau_z-w\rangle^{n+l}}=-\int\limits_{\substack{w\in\Omega\\|v(\tau_z,w)|=t+cr}}\frac{dS(w)}{\langle\partial\rho(\tau_z),\ \tau_z-w\rangle^{n+l}}$$
$$=-\frac{1}{(t+cr)^{2n+2l}}\int\limits_{\substack{w\in\Omega\\|v(\tau_z,w)|=t+cr}}\overline{\langle\partial\rho(\tau_z),\ \tau_z-w\rangle}^{n+l}dS(w).$$

Applying Stokes' theorem again, now to the domain

$$\{w\in\Omega:|\langle\partial\rho(\tau_z),\ \tau_z-w\rangle|<t+cr\},$$

we obtain

$$I_3 = \int\limits_{\substack{w\in\Omega\\ |v(\tau_z,w)|=t+cr}} \overline{\langle\partial\rho(\tau_z),\ \tau_z-w\rangle}^{n+l} dS(w)$$

$$= -\int\limits_{\substack{w\in\partial\Omega\\ |v(\tau_z,w)|<t+cr}} \overline{\langle\partial\rho(\tau_z),\ \tau_z-w\rangle}^{n+l} dS(w)$$

$$+ \int\limits_{\substack{w\in\Omega\\ |v(\tau_z,w)|<t+cr}} \bar\partial_w\left(\overline{\langle\partial\rho(\tau_z),\ \tau_z-w\rangle}^{n+l}\right)\wedge dS(w)$$

$$+ \int\limits_{\substack{w\in\Omega\\ |v(\tau_z,w)|<t+cr}} \overline{\langle\partial\rho(\tau_z),\ \tau_z-w\rangle}^{n+l} dV(w).$$

Since $\left|\ \bar\partial_w\left(\overline{\langle\partial\rho(\tau_z),\ \tau_z-w\rangle}^{n+l}\right)\wedge dS(w)\right| \lesssim |\langle\partial\rho(\tau_z),\ \tau_z-w\rangle|^{n+l-1}$ we get

$$|I_3| \lesssim \int\limits_t^{t+cr} (s^{n+l}s^{n-1}+s^{n+l}s^n+s^{n+l-1}s^n)ds \lesssim \int\limits_t^{t+cr} s^{2n+l-1}ds \lesssim r(t+r)^{2n+l-1}.$$

Note that $t \asymp \rho(\tau_z) \asymp \mathrm{Im}(\tau_n)$ and consequently

$$\int\limits_{D_0(\tau)} I_2(z,\tau)^2 d\nu_l(\tau) \lesssim \int\limits_{D_0(\tau)} \left(\frac{r(\mathrm{Im}(\tau_n)+r)^{2n+l-1}}{(\mathrm{Im}(\tau_n)+r)^{2n+2l}}\right)^2 d\nu_l(\tau)$$

$$\lesssim \int\limits_0^\infty \frac{r^2}{(t+r)^{2l+2}}\frac{t^n dt}{t^{n-2l+1}} = r^2\int\limits_0^\infty \frac{t^{2l-1}}{(t+r)^{2l+2}}\frac{t^n dt}{t^{n-2l+1}} \lesssim r^2\int\limits_0^\infty \frac{dt}{(r+t)^3} \lesssim 1. \quad (46)$$

Summarizing estimates (45, 46) and condition $|f(z)|\le 1,\ z\in\partial\Omega$, we obtain

$$\|\langle g,\ Tf\rangle\|^2 \le \int\limits_{D_0} d\nu_l(\tau)\left(\int\limits_{B(w_0,r)} |g(z)|\,(I_1(z)+I_1(z)f(z))dS(z)\right)^2$$

$$\lesssim \|g\|^2_{L^1(\partial\Omega)} \sup_{z\in\partial\Omega}\int\limits_{D_0}\left(I_1(z)^2+I_1(z)^2\right)d\nu_l(\tau) \lesssim \|g\|^2_{L^1(\partial\Omega)} \lesssim |B(w_0,r)|^2\,.$$

The last estimate implies weak boundedness of operator T and completes the proof of the lemma.

Proof (of the theorem A.1) Since operators T_j with kernels K_j verify the conditions of $T1$-theorem, we have $T_j \in \mathscr{L}(L^p(\partial\Omega), L^p(\partial\Omega, L^2(D_0,d\nu_l))$ and

$$\sum_{j=1}^N \int\limits_{\partial\Omega} \|T_j g(z)\|^p\, dS(z)$$

$$= \sum_{j=1}^N \int\limits_{\partial\Omega} dS(z)\left(\int\limits_{D_0}\left|\int\limits_{\partial\Omega}\frac{g(w)\chi_j^{1/2}(z)J_j(z,\tau)dS(w)}{\langle\partial\rho(\psi_j(z,\tau)),\ \psi_j(z,\tau)-w\rangle^{n+1}}\right|^2 \frac{d\mu(\tau)}{\mathrm{Im}(\tau_n)^{n-1}}\right)^p$$

$$\lesssim \|g\|^p_{L^p(\partial\Omega)}\,.$$

Thus by decomposition (38) $\int\limits_{\partial\Omega} I(g,z)^p\, d\sigma(z) \lesssim \int\limits_{\partial\Omega} |g(z)|^p\ d\sigma(z)$, which proves the theorem.

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