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On practical application of Zubov's optimal damping concept*

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This article presents some new ideas connected to nonlinear and nonautonomous control laws based on the application of an optimization approach. There is an essential connection between practical demands and the functionals to be minimized. This connection is at the heart of the proposed methods. The discussion is focused on the optimal damping concept first proposed by V. I. Zubov in the early 1960's. Significant attention is paid to various modern aspects of the optimal damping theory's practical implementation. Emphasis is given to the specific choice of the functional to be damped to provide the desirable stability and performance features of a closed-loop system. The applicability and effectiveness of the proposed approach are confirmed by an illustrative numerical example.

Keywords: feedback, stability, damping control, functional, optimization.

1. Introduction. At present, the intensive development of the world economy constantly generates many problems connected to the performance, safety, and reliability of various automatic control systems, which provide effective operation for different control plants in all areas of human activity.

The various approaches associated with the design of feedback control laws have already been extensively researched and reflected in numerous publications ([1–6] and many others). However, the complexity of this problem is vast until now because of the many dynamical requirements, restrictions, and conditions that must be satisfied by the control actions.

It seems to be quite evident for today that one of the most effective analytical and numerical tools for feedback connections design is the optimization approach. This point of view is supported by the flexibility and convenience of modern optimization methods with respect to the relevant practical demands for control theory implementation.

Several aspects of optimization ideology's applications for control systems design are presented in multitudinous scientific publications, including such popular monographs as [4–6]. Various analytical methods are presently used to compute the optimal control actions for linear and nonlinear systems subject to given performance indices. Importantly, optimality is not the end itself for most practical situations, as a rule. This means that the optimization approach should be rather treated as an instrument to achieve the desirable features of the system to be designed.

Nevertheless, the optimization approach is not recognized overall as a universal instrument to be practically implemented. This can be explained by the presents of some disadvantages connected to computational troubles. Therefore, there is a need to develop

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persistently analytical and numerical methods of control laws design based on optimization ideology.

Various problems in this area comprise an essential part of many scientific publications devoted to control theory and its applications. Special attention is focused on control laws synthesis for nonlinear and non-autonomous controlled plants, whose corresponding problems are the most complicated and practically significant.

At present, numerous approaches are used to practically solve these problems [1–11]. These approaches are based on Pontryagin's Maximum Principle, Bellman's Dynamic Programming Principle (using HJB equations), finite-dimensional approximation in the range of the model predictive control (MPC) technique, etc. However, all these approaches are connected to many calculations, which fundamentally impede their implementation in both laboratory design activities and real time control regimes.

This work is focused on a different concept that can be used to design stabilizing controllers based on the theory of transient processes optimal damping (OD). This theory, which was first proposed and developed by V. I. Zubov [9–11], provides effective analytical and numerical methods for control calculations with essentially reduced computational consumptions.

In modern interpretations, OD theory is closely connected to the Control Lyapunov Function (CLF) concept [12, 13]. The essence of this connection is reflected by the various constructive methods using the inverse optimal control principle [14, 15]. The initial concept was earlier proposed by Zubov, who suggested using Lyapunov constructions to provide stability and meet performance requirements.

In this paper, efforts are made to combine the modern CLF concept with the optimal damping approach. Attention is paid to various aspects of OD theory's practical implementation. This study focuses on the specific choice of the functional to be damped to provide the desirable stability and performance features of the closed-loop connection.

This paper is organized as follows. In Section 2, two feasible approaches are presented to formalize the practical requirements for the closed-loop system's dynamic properties. Here, Zubov's optimal damping problem is mathematically posed. Section 3 is devoted to the specific features of this problem, which can be used as a basis for practical feedback control laws synthesis. In Section 4, methods are proposed for the approximate minimization of the integral functionals based on OD theory. Section 5 is devoted to new practical choices of the integral items of the functional to be damped, thus providing desirable performance features. In Section 6, the proposed approach is illustrated by a simple numerical example of the approximate optimal controller design. Section 7 concludes the paper by discussing the overall results of this research.

2. About two approaches to control laws design. Let us consider a commonly used mathematical model for a nonlinear and non-autonomous control plant, presented by the following system of ordinary differential equations:

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}),\ \mathbf{x} \in E^n,\ \mathbf{u} \in E^m,\ t \in [t_0, \infty), \tag{1}$$

where, $\mathbf{x}$ is the state vector, and vector $\mathbf{u}$ implies a control action. The function $\mathbf{f} : E^{n+m+1} \to E^n$ is continuous with respect to all its arguments in the space E^{n+m+1}. Let us suppose that the system (1) has zero equilibrium, i. e.,

$$\mathbf{f}(t, \mathbf{0}, \mathbf{0}) = \mathbf{0} \quad \forall t \geqslant t_0. \tag{2}$$

The essence of the feedback design problem is to synthesize a nonlinear and non-autonomous controller of the form

$$\mathbf{u} = \mathbf{u}(t, \mathbf{x}), \tag{3}$$

such that the following requirements fulfilled:

a) the function $\mathbf{u}(t, \mathbf{x})$ is piecewise continuous in its arguments;

b) the closed-loop connection (1), (3), like (2), must have zero equilibrium

$$\mathbf{f}(t, \mathbf{0}, \mathbf{u}(t, \mathbf{0})) = \mathbf{0} \quad \forall t \geqslant t_0; \tag{4}$$

c) the aforementioned equilibrium point must be locally (globally) uniformly asymptotically stable (UAS or UGAS).

For the local variants, let us suppose that all admissible controls are limited by the condition $\mathbf{u} \in U \subset E^m$, where the set U is a metric compact set in the space E^m. For one turn, all admissible states of plant (1) are limited by belonging to the r-neighborhood $\mathbf{x} \in B_r$ of the origin.

If there is freedom in the choice of control laws in the range of the requirements to be satisfied, it is suitable to pose questions related to the performance of the control processes.

The practical problem statements are usually formulated as certain additional requirements to be undeviatingly satisfied with the help of the obtained feedback control laws of the form (3). In most cases, the aforementioned requirements can be presented as follows:

$$\mathbf{x}(t, \mathbf{x}^0, \mathbf{u}(\cdot)) \in X \; \forall t \geqslant t_0 \quad \forall \mathbf{x}^0 \in B_r, \quad \forall \mathbf{u} \in U, \tag{5}$$

where the vector function $\mathbf{x}(t, \mathbf{x}^0, \mathbf{u}(\cdot))$ is the motion of plant (1) closed by controller (3) under the initial condition $\mathbf{x}(\mathbf{t_0}) = \mathbf{x}^0$.

Herein, an admissible set determines the aforementioned complex of requirements to be satisfied and corresponds to desirable performance features. This set, in particular, can be determined by some constraints of the system's characteristics (transient time, overshoot, etc.).

Notably, numerous well-known scientific publications ([5, 6, 10] and most others) flatly connect formalized expression of the processes' performance, except for (5), which only presents the values of certain integral functionals of the form

$$J = J(\mathbf{u}(\cdot)) = \int_{t_0}^{\infty} F_0(t, \mathbf{x}, \mathbf{u}) dt. \tag{6}$$

It is supposed that the subintegral function F_0 is positively definite, i. e.,

$$F_0(t, \mathbf{x}, \mathbf{u}) > 0 \quad \forall t \geqslant t_0, \quad \forall \mathbf{x} \in B_r, \quad \forall \mathbf{u} \in U, \tag{7}$$

excluding the points $(t, \mathbf{0}, \mathbf{0})$ for any time t. For these points, $F_0 = 0$.

Notably, the choice of function F_0 is generally made outside of the range of formalized approaches for the solution of various practical problems. Usually, this question is considered based on the informal opinions of experts with a connection to the relevant requirements (5).

If the function F_0 is given, this process is much better when the value of functional (6) is less.

In this connection, the following optimization problem is of primary importance:

$$J(\mathbf{u}(\cdot)) \to \min_{\mathbf{u}\in U_c}, \quad \mathbf{u}_{c0}(t,\mathbf{x}) = \arg\min_{\mathbf{u}\in U_c} J(\mathbf{u}(\cdot)), \quad J_0 := J(\mathbf{u}_{c0}(\cdot)). \tag{8}$$

This is the problem of the integral functional minimization (MIF) on the admissible set U_c of stabilizing controllers (3). Further, it is assumed that the lower exact bound for functional J on set U_c is reached within the context of the present situation.

Currently, numerous well-known approaches are widely used to practically solve problem (8). These approaches are based on Pontryagin's Maximum Principle, Bellman's Dynamic Programming ideas, finite-dimensional approximation in the range of the MPC technique, etc.

In particular, let us consider certain specialties of the Dynamic Programming theory application [4, 5, 10]. For the feedback control design, it is necessary to carry out the following actions.

1. Given a system (1), a performance index (6), and an admissible set U, the Hamilton—Jacobi—Bellman (HJB) equation can be constructed as

$$\frac{\partial V(t,\mathbf{x})}{\partial t} + \min_{\mathbf{u}\in U}\left\{\frac{\partial V(t,\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}(t,\mathbf{x},\mathbf{u}) + F_0(t,\mathbf{x},\mathbf{u})\right\} = 0, \tag{9}$$

where the Bellman function $V(t,\mathbf{x})$ is initially unknown.

2. In accordance with (9), assign the connection between a control and the Bellman function $V(t,\mathbf{x})$, providing the minimum of the expression in braces:

$$\mathbf{u} = \tilde{\mathbf{u}}\left[t,\mathbf{x},V(t,\mathbf{x})\right] = \arg\min_{\mathbf{u}\in U}\left\{\frac{\partial V(t,\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}(t,\mathbf{x},\mathbf{u}) + F_0(t,\mathbf{x},\mathbf{u})\right\}. \tag{10}$$

Here set U can be used instead of set U_c.

3. Substitute the found function $\tilde{\mathbf{u}}$ into (9), thereby obtaining the HJB equation, which is not weighed down by the minimum search operation:

$$\begin{aligned}\frac{\partial V(t,\mathbf{x})}{\partial t} + \frac{\partial V(t,\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}\{t,\mathbf{x},\tilde{\mathbf{u}}\left[t,\mathbf{x},V(t,\mathbf{x})\right]\} &+ \\ + F_0\{t,\mathbf{x},\tilde{\mathbf{u}}\left[t,\mathbf{x},V(t,\mathbf{x})\right]\} &= 0.\end{aligned} \tag{11}$$

One can easily see that (11) is a routine PDE with respect to the initially unknown function $V(t,\mathbf{x})$.

4. If the solution $V = \tilde{V}(t,\mathbf{x})$ of this equation is computed, and if the function $\tilde{V}$ is continuously differentiable and satisfies the conditions $\tilde{V}(t,\mathbf{0}) = 0 \quad \forall t \geqslant t_0$, $\tilde{V}(\infty,\mathbf{x}) = 0$ $\forall \mathbf{x} \in B_r$, then, after substituting $V = \tilde{V}(t,\mathbf{x})$ into (10), the desired solution of the MIF problem can be obtained as follows:

$$\mathbf{u} = \mathbf{u}_{c0}(t,\mathbf{x}) = \tilde{\mathbf{u}}\left[t,\mathbf{x},\tilde{V}(t,\mathbf{x})\right] \in U_c. \tag{12}$$

Here, function $V = \tilde{V}(t,\mathbf{x})$, which satisfies HJB equation (11), is called *a value function* cosidering the equality $\tilde{V}(t_0,\mathbf{x}_0) = \min_{\mathbf{u}\in U_c} J(\mathbf{u}(\cdot))$: i. e., its value determines a minimum of the functional J based on the motion of the closed-loop system with the initial condition $\mathbf{x}(t_0) = \mathbf{x_0}$.

As is well known, an application of Bellman's theory to solve the MIF problem is significantly hampered by a number of difficulties.

First of all, the aforementioned scheme for the problem's solution is notably only based on the sufficient conditions of the extreme. Actually, the function $V = \tilde{V}(t, \mathbf{x})$ by no means is always continuously differentiable or able to satisfy the desirable conditions. In addition, a search of this function can be implemented numerically with no trouble only if the halfway problem (10) admits an analytical solution. Under this condition, subsequent computing obstacles are connected only to PDE (11).

Otherwise, the computational consumption increases like an avalanche due to the so-called "curse of dimensionality".

Considering the presence of the obstacles mentioned above, let us address an alternative approach to formalize the practical judgments for dynamical processes quality. This approach is based on the concept of optimal transient process damping, which was first proposed by V. I. Zubov in [9–11].

This concept is built upon the following functional:

$$L = L(t, \mathbf{x}, \mathbf{u}) = V(t, \mathbf{x}) + \int_{t_0}^{t} F(\tau, \mathbf{x}, \mathbf{u}) d\tau, \tag{13}$$

which is introduced to check the performance of a closed-loop connection (1), (3).

Here, various scalar functions $V = V(t, \mathbf{x})$ can be used to define a distance from the current state $\mathbf{x}$ of the plant (1) to the zero equilibrium. Let us assume that these functions are continuously differentiable and satisfy the following conditions:

$$\alpha_1(\|\mathbf{x}\|) \leqslant V(t, \mathbf{x}) \leqslant \alpha_2(\|\mathbf{x}\|) \quad \forall \mathbf{x} \in E^n, \ \forall t \in [t_0, \infty), \tag{14}$$

and for some functions $\alpha_1, \alpha_2 \in K$ (or $\alpha_1, \alpha_2 \in K_\infty$) (Hahn's comparison functions, which are determined in [2, 3, 16]).

Note that the integral item in (13) inherently determines a penalty for a closed-loop system with the help of the additionally given function F connected to the performance of the motion. Let us accept that this function is positively definite in the same way as the function F_0 in (7). The problem of optimal damping (OD) with respect to functional (13) can be posed in the form

$$W = W(t, \mathbf{x}, \mathbf{u}) \to \min_{\mathbf{u} \in U}, \quad \mathbf{u} = \mathbf{u}_d(t, \mathbf{x}) := \arg \min_{\mathbf{u} \in U} W(t, \mathbf{x}, \mathbf{u}), \tag{15}$$

where the function W determines the rate of changes in functional L due to the motions of the plant (1), as follows:

$$\begin{aligned} W(t, \mathbf{x}, \mathbf{u}) := \left.\frac{dL}{dt}\right|_{(1)} = \left.\frac{dV}{dt}\right|_{(1)} + F(t, \mathbf{x}, \mathbf{u}) = \\ = \frac{\partial V(t, \mathbf{x})}{\partial t} + \frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}, \mathbf{u}) + F(t, \mathbf{x}, \mathbf{u}). \end{aligned} \tag{16}$$

Clearly, the solution

$$\mathbf{u} = \mathbf{u}_d(t, \mathbf{x}) \tag{17}$$

of the OD problem (15) determines feedback control (OD controller) for plant (1). The corresponding closed-loop system (1), (17), which has zero equilibrium, is a closed-loop OD system.

The optimal damping concept is based on the following simple idea: the process improves significantly the more rapidly the functional (13) decreases based on the motions of the closed-loop connection.

Let us consider a circumstance where the computational scheme for the OD problem solution is considerably simpler than for the MIF one. Actually, as it follows from relationships (13)–(17), it is not necessary (though, it is desirable) to obtain an analytical representation of the function $\tilde{\mathbf{u}}\,[t,\mathbf{x},V(t,\mathbf{x})]$. This is determined by the possibility to calculate the values of $\mathbf{u}=\mathbf{u}_d(t,\mathbf{x})$ numerically, using a pointwise minimization of the function $W(t,\mathbf{x},\mathbf{u})$ according to the choice of $\mathbf{u}\in U$ for the current values of the variables $t,\mathbf{x}$.

Note that the OD mathematical formalization of the exacting practical demands on process performance is reduced to the choice of the functions $V=V(t,\mathbf{x})$ and $F=F(t,\mathbf{x},\mathbf{u})$ for functional (13) to be damped. Since a direct connection is not evident between the aforementioned functions and the requirements in (5), this choice can be realized informally based on experts' opinions. Naturally, this is also true for the MIF problem.

However, because the numerical solution of the OD problem is considerably simpler than the MIF solution, it is possible to use this advantage to formalize the choice of functions V and F in the range of the optimal damping concept. This is one of the main issues discussed below. This idea was partially implemented for damping stabilization in [17, 18], but was not connected to optimality issues.

3. Basic features of optimal damping control. Problem (15) for optimal damping has certain features that should be used as a basis for practical control laws synthesis issues. We will next consider some of these principals.

First, let us introduce the concept of *the control Lyapunov function* [1, 12, 13] for plant (1).

Definition 1. Continuously differentiable function $V(t,\mathbf{x})$ such that

$$\alpha_1(\|\mathbf{x}\|)\leqslant V(t,\mathbf{x})\leqslant\alpha_2(\|\mathbf{x}\|)\quad\forall\mathbf{x}\in E^n,\ \forall t\geqslant t_0, \tag{18}$$

$\alpha_1,\alpha_2\in K_\infty$, is said to be *global Control Lyapunov Function* (*global CLF*) for plant (1) if there exists a function $\alpha_3\in K_\infty$ such that the inequality

$$\inf_{\mathbf{u}\in E^m}\left[\frac{\partial V(t,\mathbf{x})}{\partial t}+\frac{\partial V(t,\mathbf{x})}{\partial\mathbf{x}}\mathbf{f}(t,\mathbf{x},\mathbf{u})\right]+\alpha_3(\|\mathbf{x}\|)\leqslant 0\quad\forall t\geqslant t_0,\ \forall\mathbf{x}\in E^n, \tag{19}$$

holds. If conditions (18), (19) are satisfied for $\alpha_1,\alpha_2,\alpha_3\in K$, $\forall\mathbf{x}\in B_r$, then V is said to be *local CLF*.

It the CLF for system (1) (global or local) exists, then this system is globally (or locally) *uniformly asymptotically stabilizable* (UGAS or UAS) [3].

Notably, the properties of stability and performance for the motions of the closed-loop OD system, transferring from some initial point $\mathbf{x}_0=\mathbf{x}(t_0)\neq\mathbf{0}$, vary based on the choice of the functions $V=V(t,\mathbf{x})$ and $F=F(t,\mathbf{x},\mathbf{u})$ in (13). Here, the main role of V is to support the stability properties, and the purpose of F is to provide the desirable performance features.

Evidently, any choice of function V for the damping functional (13) should be treated as the choice of a Lyapunov function candidate. In particular, these functions can play a role of CLF for plant (1).

The main purpose of controller (17) is to provide the stability properties for the zero-equilibrium position of the closed-loop system. This requirement is connected to the following statement.

Theorem 1. *Let the condition*

$$W_{d0}(t,\mathbf{x}) := W(t,\mathbf{x},\mathbf{u}_d(t,\mathbf{x})) \leqslant -\alpha_4(\|\mathbf{x}\|) \quad \forall t \geqslant t_0,\ \forall \mathbf{x} \in B_r, \tag{20}$$

holds for feedback control (17)*, where* $\alpha_4 \in K$*. Then the function* $V(t,\mathbf{x})$ *is a CLF for plant* (1)*, and zero equilibrium for the closed-loop system* (1)*,* (17) *is locally uniformly asymptotically stable, i. e., the feedback* (17) *serves as a stabilizing controller for plant* (1)*.*

P r o o f. Thus, let condition (20) holds for the controller (17), which is the solution of OD problem (15), i. e., the following relationships are correct:

$$\min_{\mathbf{u}\in U} W(t,\mathbf{x},\mathbf{u}) = \min_{\mathbf{u}\in U}\left[\left.\frac{dV}{dt}\right|_{(1)}(t,\mathbf{x},\mathbf{u}) + F(t,\mathbf{x},\mathbf{u})\right] \geqslant \tag{21}$$
$$\geqslant \min_{\mathbf{u}\in U}\left.\frac{dV}{dt}\right|_{(1)}(t,\mathbf{x},\mathbf{u}) + \min_{\mathbf{u}\in U} F(t,\mathbf{x},\mathbf{u}) \leqslant -\alpha_4(\|\mathbf{x}\|).$$

However, the function F satisfies the condition $F(t,\mathbf{x},\mathbf{u}) \geqslant 0$ for any arguments that provides $-\min\limits_{\mathbf{u}\in U} F(t,\mathbf{x},\mathbf{u}) \leqslant 0$. Substituting the last relation into (21), we can obtain

$$\min_{\mathbf{u}\in U}\left.\frac{dV}{dt}\right|_{(1)}(t,\mathbf{x},\mathbf{u}) \leqslant -\alpha_4(\|\mathbf{x}\|) - \min_{\mathbf{u}\in U} F(t,\mathbf{x},\mathbf{u}) \leqslant -\alpha_4(\|\mathbf{x}\|),$$

which is equivalent to

$$\min_{\mathbf{u}\in U}\left[\frac{\partial V(t,\mathbf{x})}{\partial t} + \frac{\partial V(t,\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}(t,\mathbf{x},\mathbf{u})\right] \leqslant -\alpha_4(\|\mathbf{x}\|),$$

i. e., the function $V(t,\mathbf{x})$ is, by definition, the local CLF for the system (1).

Now, in accordance with the equality (16) on the basis of (20), the following is true:

$$\tilde{W}_{d0} = \tilde{W}_{d0}(t,\mathbf{x}) := \frac{\partial V(t,\mathbf{x})}{\partial t} + \frac{\partial V(t,\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}(t,\mathbf{x},\mathbf{u}_d(t,\mathbf{x})) \leqslant$$
$$\leqslant -\alpha_4(\|\mathbf{x}\|) - F(t,\mathbf{x},\mathbf{u}_d(t,\mathbf{x})) \leqslant -\alpha_4(\|\mathbf{x}\|),$$

i. e.,

$$\tilde{W}_{d0} = \tilde{W}_{d0}(t,\mathbf{x}) := \left.\frac{dV}{dt}\right|_{(1),\mathbf{u}=\mathbf{u}_d(t,\mathbf{x})} \leqslant -\alpha_4(\|\mathbf{x}\|),$$

where $\alpha_4 \in K$.

It follows from this ([2, 8] and others) that the zero equilibrium of the closed-loop system (1), (17) is locally uniformly asymptotically stable, i. e., the feedback (17) is a stabilizing controller for plant (1). ■

Remark 1. If all the aforementioned conditions of Theorem 1 are fulfilled for the whole space, i. e., if $B_r = E^n$, $U = E^n$, and if all the aforementioned functions $\alpha_i,\ i = \overline{1,4}$ belong to class K_∞, then the zero equilibrium point for the closed-loop system is globally uniformly asymptotically stable (UGAS) [2, 8].

Let us specify one of the most important features for the solution (17) of OD problem (15), which was first developed and investigated by V. I. Zubov [9–11].

Theorem 2. *Let MIF problem* (8) *have a unique solution, and let the control law* (17) *be a solution of OD problem* (15) *with respect to functional* (13) *with the subintegral* *function* $F(t, \mathbf{x}, \mathbf{u}) \equiv F_0(t, \mathbf{x}, \mathbf{u})$ *and with function* V*, which coincides with the solution* $V(t, \mathbf{x}) \equiv \tilde{V}(t, \mathbf{x})$ *of HJB equation* (11)*.*

Then the controller $\mathbf{u} = \mathbf{u}_d(t, \mathbf{x})$ *is simultaneously a solution for the MIF problem* (8)*, i. e.,* $\mathbf{u}_{c0}(t, \mathbf{x}) \equiv \mathbf{u}_d(t, \mathbf{x})$*, where* $\mathbf{u}_{c0}$ *is determined by* (12)*.*

If the mentioned solution is not unique, then any OD controller can be taken as a MIF optimal feedback.

Proof. This statement can be proven based on the scheme proposed by V. I. Zubov with respect to integral functionals with finite limits.

Given a control law $\mathbf{u} = \mathbf{u}_d(t, \mathbf{x})$ and initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$, let us integrate the equations of the closed-loop system

$$\dot{\mathbf{x}} = \mathbf{f}[t, \mathbf{x}, \mathbf{u}_d(t, \mathbf{x})] \Leftrightarrow \dot{\mathbf{x}} = \mathbf{f}_d(t, \mathbf{x}); \tag{22}$$

as a result, we can obtain the corresponding motion $\mathbf{x} = \mathbf{x}_d(t)$ and the control $\mathbf{u} = \mathbf{u}_d(t)$ as functions of $t \in [t_0, \infty)$. Let us suppose that the zero equilibrium of system (22) is asymptotically stable, i. e., for any $\mathbf{x}_0 \in B_r$ $\lim\limits_{t\to\infty} \mathbf{x}_d(t) = \mathbf{0}$.

Based on (9) and (11), the following identity is valid for these functions (see f in (2), (4)):

$$\left[\frac{\partial V(t, \mathbf{x})}{\partial t} + \frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}, \mathbf{u}) + F(t, \mathbf{x}, \mathbf{u})\right]_{\mathbf{x}=\mathbf{x}_d(t), \mathbf{u}=\mathbf{u}_d(t)} \equiv 0,$$

i. e.,

$$\left[\left.\frac{dV(t, \mathbf{x})}{dt}\right|_{(1)} + F(t, \mathbf{x}, \mathbf{u})\right]_{\mathbf{x}=\mathbf{x}_d(t), \mathbf{u}=\mathbf{u}_d(t)} \equiv 0,$$

which is equivalent to the identity (by time)

$$dV(t, \mathbf{x}) \equiv -F(t, \mathbf{x}_d(t), \mathbf{u}_d(t))dt \tag{23}$$

for the OD motion $\mathbf{x} = \mathbf{x}_d(t)$.

Both parts of identity (23) can be integrated by a curvilinear integral from the initial position $[t_0, \mathbf{x}_d(t_0)]$ to the endpoint $\lim\limits_{\tau\to\infty} [\tau, \mathbf{x}_d(\tau)]$ along the motion $\mathbf{x}_d(t)$:

$$\int\limits_{[t_0, \mathbf{x}_d(t_0)]}^{\lim\limits_{\tau\to\infty}[\tau, \mathbf{x}_d(\tau)]} dV(t, \mathbf{x}) = -\int\limits_{t_0}^{\infty} F(t, \mathbf{x}_d(t), \mathbf{u}_d(t))dt,$$

which leads to the equality

$$\lim_{\tau\to\infty} V[\tau, \mathbf{x}_d(\tau)] - V[t_0, \mathbf{x}_d(t_0)] = -\int\limits_{t_0}^{\infty} F(t, \mathbf{x}_d(t), \mathbf{u}_d(t))dt. \tag{24}$$

However, since the optimal motion passes through the given initial point $A(t_0, \mathbf{x}^0)$, we obtain

$$V[t_0, \mathbf{x}_d(t_0)] = V(t_0, \mathbf{x}^0), \tag{25}$$

and, according to the condition $\lim\limits_{t\to\infty, \mathbf{x}\to\mathbf{0}} V(t, \mathbf{x}) = 0$ and considering the property of asymptotic stability, the equality

$$\lim_{\tau\to\infty} V[\tau, \mathbf{x}_d(\tau)] = 0 \tag{26}$$

holds, because $\lim\limits_{\tau\to\infty} \mathbf{x}_d(\tau) = \mathbf{0}$.

Substituting relationships (25) and (26) into (24), we obtain

$$\int\limits_{t_0}^{\infty} F(t, \mathbf{x}_d(t), \mathbf{u}_d(t))dt = V(t_0, \mathbf{x}^0).$$

However, the integral on the right is equal to $J_d = J(\mathbf{u}_d)$, i. e.,

$$J_d = J(\mathbf{u}_d) = V(t_0, \mathbf{x}^0). \tag{27}$$

Next, let us consider a contrary proof: suppose that there exists an admissible control $\bar{\mathbf{u}} \in U$ such that

$$J(\bar{\mathbf{u}}) < J_d = J(\mathbf{u}_d). \tag{28}$$

Let us suppose that the controller $\mathbf{u} = \bar{\mathbf{u}}(t, \mathbf{x})$ provides the corresponding motion $\bar{\mathbf{x}}(t)$ of plant (1), satisfying the boundary conditions $\bar{\mathbf{x}}(t_0) = \mathbf{x}^0$ and $\lim\limits_{\tau\to\infty} \bar{\mathbf{x}}(\tau) = \mathbf{0}$, and providing the corresponding function $\bar{\mathbf{u}}(t)$ for the closed-loop system.

Since the control $\bar{\mathbf{u}}$ is not necessary a solution of OD problem, based on (15), we obtain

$$W(t, \mathbf{x}_d(t), \mathbf{u}_d(t)) \leqslant W(t, \bar{\mathbf{x}}(t), \bar{\mathbf{u}}(t)) \quad \forall t \geqslant t_0.$$

In accordance with (16), it follows that

$$\begin{gathered}\frac{\partial V(t, \bar{\mathbf{x}})}{\partial t} + \frac{\partial V(t, \bar{\mathbf{x}})}{\partial \mathbf{x}} \mathbf{f}(t, \bar{\mathbf{x}}, \bar{\mathbf{u}}) + F(t, \bar{\mathbf{x}}, \bar{\mathbf{u}}) \geqslant \\ \geqslant \frac{\partial V(t, \mathbf{x}_d)}{\partial t} + \frac{\partial V(t, \mathbf{x}_d)}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}_d, \mathbf{u}_d) + F(t, \mathbf{x}_d, \mathbf{u}_d) = 0 \quad \forall t \geqslant t_0,\end{gathered}$$

or

$$\begin{gathered}\frac{\partial V(t, \bar{\mathbf{x}})}{\partial t} + \frac{\partial V(t, \bar{\mathbf{x}})}{\partial \mathbf{x}} \mathbf{f}(t, \bar{\mathbf{x}}, \bar{\mathbf{u}}) + F(t, \bar{\mathbf{x}}, \bar{\mathbf{u}}) = \\ = \left[\left.\frac{dV(t, \mathbf{x})}{dt}\right|_{(1)} + F(t, \mathbf{x}, \mathbf{u}) \right]_{\mathbf{x}=\bar{\mathbf{x}}(t), \mathbf{u}=\bar{\mathbf{u}}(t)} \geqslant 0 \quad \forall t \geqslant t_0.\end{gathered}$$

The last inequality can be rewritten in the equivalent form

$$\left[\left.\frac{dV(t, \mathbf{x})}{dt}\right|_{(1)} \right]_{\mathbf{x}=\bar{\mathbf{x}}(t), \mathbf{u}=\bar{\mathbf{u}}(t)} \equiv -F(t, \bar{\mathbf{x}}, \bar{\mathbf{u}}) + \alpha(t), \tag{29}$$

where $\alpha(t)$ is a function satisfying the condition

$$\alpha(t) \geqslant 0 \quad \forall t \geqslant t_0. \tag{30}$$

Relation (29) defines the following identity:

$$dV(t, \mathbf{x}) \equiv -F(t, \bar{\mathbf{x}}(t), \bar{\mathbf{u}}(t))dt + \alpha(t)dt \tag{31}$$

for the aforementioned motion $\mathbf{x} = \bar{\mathbf{x}}(t)$.

As before, both parts of identity (31) can be integrated by a curvilinear integral from the initial position $[t_0, \bar{\mathbf{x}}(t_0)]$ to the end position $\lim\limits_{\tau\to\infty}[\tau, \bar{\mathbf{x}}(\tau)]$ along the motion $\bar{\mathbf{x}}(t)$:

$$\int\limits_{[t_0,\bar{\mathbf{x}}(t_0)]}^{\lim\limits_{\tau\to\infty}[\tau,\bar{\mathbf{x}}(\tau)]} dV(t,\mathbf{x}) = -\int\limits_{t_0}^{\infty} F(t,\bar{\mathbf{x}}(t),\bar{\mathbf{u}}(t))dt + \int\limits_{t_0}^{\infty} \alpha(t)dt,$$

which leads to the equality

$$\lim_{\tau\to\infty} V[\tau,\bar{\mathbf{x}}(\tau)] - V[t_0,\bar{\mathbf{x}}(t_0)] = -\int\limits_{t_0}^{\infty} F(t,\bar{\mathbf{x}}(t),\bar{\mathbf{u}}(t))dt + \int\limits_{t_0}^{\infty} \alpha(t)dt. \tag{32}$$

However, since the motion $\bar{\mathbf{x}}(t)$ also passes through the given starting point $A(t_0, \mathbf{x}^0)$,

$$V[t_0,\bar{\mathbf{x}}(t_0)] = V(t_0,\mathbf{x}^0). \tag{33}$$

Further, considering $\lim\limits_{\tau\to\infty} \bar{\mathbf{x}}(\tau) = 0$, we obtain

$$\lim_{\tau\to\infty} V[\tau,\bar{\mathbf{x}}(\tau)] = 0. \tag{34}$$

Substituting (33) and (34) into (32), obtain

$$\int\limits_{t_0}^{\infty} F(t,\bar{\mathbf{x}}(t),\bar{\mathbf{u}}(t))dt = V(t_0,\mathbf{x}^0) + \int\limits_{t_0}^{\infty} \alpha(t)dt.$$

The integral on the right is equal to $\bar{J} = J(\bar{\mathbf{u}})$. Considering (27), we arrive at the equality

$$\bar{J} = J(\bar{\mathbf{u}}) = J(\mathbf{u}_d) + \int\limits_{t_0}^{\infty} \alpha(t)dt. \tag{35}$$

Since function $\alpha(t)$ satisfies condition (30), it follows from equality (35) that

$$\bar{J} = J(\bar{\mathbf{u}}) \geqslant J(\mathbf{u}_d) = J_d.$$

However, this contradicts the assumption of (28), i. e., a control $\bar{\mathbf{u}}(t)$ satisfying condition (28) does not exist.

This means that the OD controller $\mathbf{u} = \mathbf{u}_d(t,\mathbf{x})$ gives the same optimal value $J(\mathbf{u}_d) = J_d = J_{c0} = J(\mathbf{u}_{c0})$ as the MIF controller $\mathbf{u} = \mathbf{u}_{c0}(t,\mathbf{x})$. Considering the uniqueness of problem (8)'s solution, the identity $\mathbf{u}_{c0}(t,\mathbf{x}) \equiv \mathbf{u}_d(t,\mathbf{x})$ is valid.

Clearly, if a mentioned solution is not unique, then any OD controller can be used for MIF optimal feedback. ■

Notably, Theorem 2 formally reduces the solution of the MIF problem to a solution of an essentially simpler OD problem. However, it is natural that the direct utilization of such a transformation has no practical sense, since one need to determine a solution $\tilde{V}(t,\mathbf{x})$ for the HJB equation (11) to state the OD problem. However, solving the HJB equation is the essence of the MIF problem.

Nevertheless, the aforementioned peculiarity can be successfully used for various theoretical constructions. For example, the conformity of these two problems was applied by Zubov for a minimum-time problem investigation presented in [9–11], which was carried out with the help of OD theory.

It directly follows from Theorem 2 that the MIF problem can be treated as a particular case of the OD problem for plant (1). Indeed, under the conditions $F_0(t, \mathbf{x}, \mathbf{u}) \equiv F(t, \mathbf{x}, \mathbf{u})$ and $V(t, \mathbf{x}) \equiv \tilde{V}(t, \mathbf{x})$, the OD controller (17) minimizes functional (6).

In this way, the OD problem has the following significant advantages over the MIF problem. First, the OD problem can be more simply numerically solved; second, the OD problem is more general because the set of its solutions for the various functionals (13) also provides solutions for the MIF problem (8).

The aforementioned advantages suggest the two following main directions for OD theory's application:

1) the choice of the approximate solution of the MIF problem, if this problem plays a self-contained role in feedback (3) synthesis;

2) the construction of the methods guaranteeing fulfillment of the practical requirements (5) to support the desirable performance of the closed-loop system.

The priority of these two directions is determined by the following circumstance: all MIF and OD problems are no more then variants of the approximate mathematical formalization for the practical requirements presented by (5). Thus, both approaches are valid. Nevertheless, their successful implementation is determined by the correct selection of the functionals under consideration. For the MIF problem (8), the function $F_0(t, \mathbf{x}, \mathbf{u})$ should be used for functional (6). On the other hand, to set the OD problem (15), functions $V(t, \mathbf{x})$ and $F(t, \mathbf{x}, \mathbf{u})$ should be selected. A choice of these functions should be made considering the initially given requirements (5).

In the end, these two functions play a central role in the process of designing the optimal controllers (12) and (17), which are the subintegral functions F and Lyapunov—Bellman functions V.

Nevertheless, there is a fundamental difference between the aforementioned approaches. For the MIF problem, the integrand $F_0(t, \mathbf{x}, \mathbf{u})$ is initially given for the functional (6), while the Lyapunov—Bellman function $V = \tilde{V}(t, \mathbf{x})$ is computed as a solution of the HJB equation in accordance with the scheme presented above, which leads to the optimal controller $\mathbf{u} = \mathbf{u}_{c0}(t, \mathbf{x})$.

For the OD problem, both the function $V(t, \mathbf{x})$ and the function $F(t, \mathbf{x}, \mathbf{u})$ are initially given for the functional (13), and these functions directly determine the optimal controller $\mathbf{u} = \mathbf{u}_d(t, \mathbf{x})$. As observed earlier, the selection of function V is primarily done to provide stability for the closed-loop system.

Under the consideration of stability and desirable performance issues, the following variants of the functions $V(t, \mathbf{x})$ and $F(t, \mathbf{x}, \mathbf{u})$ can be chosen for the functional $L(t, \mathbf{x}, \mathbf{u})$ (13) to be damped.

1. The aforementioned functions are taken from the MIF problem (8), i. e., the identities $V(t, \mathbf{x}) \equiv \tilde{V}(t, \mathbf{x})$ and $F(t, \mathbf{x}, \mathbf{u}) \equiv F_0(t, \mathbf{x}, \mathbf{u})$ are valid. As follows from Theorem 2, the solution of the OD problem in this case is simultaneously a solution for the MIF problem: $\mathbf{u}_d(t, \mathbf{x}) \equiv \mathbf{u}_{c0}(t, \mathbf{x})$.

2. The subintegral functions F is taken as before from the MIF problem (8), i. e., $F(t, \mathbf{x}, \mathbf{u}) \equiv F_0(t, \mathbf{x}, \mathbf{u})$, while the function $V(t, \mathbf{x})$ is selected from the some given class $\Re$ to provide an approximate solution $\tilde{V}(t, \mathbf{x})$ for the HJB equation.

3. The function $V(t, \mathbf{x})$ is initially fixed in the range of the class $\Re_0$ of the CLF, while

the function $F(t, \mathbf{x}, \mathbf{u})$ is computed based on the requirements (5), thereby providing the desirable performance of the control process. This case corresponds to the concept of *inverse optimality*, first presented in [14].

4. Functions $V(t, \mathbf{x})$ and $F(t, \mathbf{x}, \mathbf{u})$ are simultaneously selected in the range of certain classes with no direct connection to the integral functional (6) and with the MIF problem (8). This selection is initially performed to provide stability and the desirable performance.

The last three variants presented here generate concrete computational methods of the stabilizing controllers (3) design based on the optimal damping theory.

4. Approximate optimal control design based on optimal damping. The following subtle issue is connected to the coincidence of the aforementioned problems. For the MIF problem, the choice of the function F_0 uniquely determines the function $V = \tilde{V}$ as a solution of the correspondent HJB equation. If this function is used together with the function $F \equiv F_0$ for the OD problem (15), then the OD controller $\mathbf{u} = \mathbf{u}_d(t, \mathbf{x})$ provides the same optimal value $J = J_0$ as the MIF controller $\mathbf{u} = \mathbf{u}_{c0}(t, \mathbf{x})$.

However, if any function $V(t, \mathbf{x})$ is used in functional (13) instead of $\tilde{V}(t, \mathbf{x})$, thereby maintaining the identity $F \equiv F_0$, then the corresponding OD controller (17) will not be a solution of the MIF problem, i. e., this controller will provide a value $J \geqslant J_0$ for the performance index (6). Retaining function F_0 means that the functional (6) has real fundamental worth for practical situation.

In that case, by solving the OD problem (15) for different functions V, one can determine which function V approximates the HJB solution $\tilde{V}(t, \mathbf{x})$ in the best way. Thus, the OD problem can be treated as an instrument for dragging of the function V to the aforementioned optimal solution $\tilde{V}$, with the trend $J \to J_0$.

It is evident that the presented idea is applicable only for a situation where a direct MIF problem solution is connected to large computational troubles. In this case, it is suitable to construct an approximate optimal controller that is similar to an optimal one, $\mathbf{u} = \mathbf{u}_{c0}(t, \mathbf{x})$, but can be designed with lower computational consumption.

Here, a specialized approach is proposed to construct an approximate optimal controller based on the optimal damping concept.

Thus, let us consider the MIF problem (8) with integral functional (6), which is given based on the motions of the closed-loop system with the controller $\mathbf{u} = \mathbf{u}_{c0}(t, \mathbf{x})$ for the plant

$$\dot{\mathbf{x}} = \mathbf{f_0}(t, \mathbf{x}, \mathbf{u}), \tag{36}$$

where the right part has the same properties as plant (1).

As mentioned above, the MIF problem is equivalent to the OD problem in the form

$$W = W(t, \mathbf{x}, \mathbf{u}) \to \min_{\mathbf{u}\in U}, \ \mathbf{u} = \mathbf{u}_d(t, \mathbf{x}) := \arg \min_{\mathbf{u}\in U} W(t, \mathbf{x}, \mathbf{u}), \tag{37}$$

$$W(t, \mathbf{x}, \mathbf{u}) := dL/dt|_{(36)},$$

$$L = L(t, \mathbf{x}, \mathbf{u}) = V(t, \mathbf{x}) + \int_{t_0}^{t} F_0(\tau, \mathbf{x}, \mathbf{u}) d\tau, \tag{38}$$

if $V(t, \mathbf{x}) \equiv \tilde{V}(t, \mathbf{x})$ for the solution $\tilde{V}$ of the HJB equation

$$\frac{\partial V(t, \mathbf{x})}{\partial t} + \min_{\mathbf{u}\in U} \left\{ \frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}} \mathbf{f}_0(t, \mathbf{x}, \mathbf{u}) + F_0(t, \mathbf{x}, \mathbf{u}) \right\} = 0.$$

There are two possible situations of solution processes for both optimization problems:

a) it is possible to analytically find the function

$$\mathbf{u} = \tilde{\mathbf{u}}\left[t, \mathbf{x}, \frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}}\right] = \arg\min_{\mathbf{u}\in U}\left\{\frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}}\mathbf{f}_0(t, \mathbf{x}, \mathbf{u}) + F_0(t, \mathbf{x}, \mathbf{u})\right\};$$

b) this function can not be found analytically.

The first situation leads to the HJB equation presented in the following form:

$$\frac{\partial V(t, \mathbf{x})}{\partial t} + \frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}}\mathbf{f}_0\left\{t, \mathbf{x}, \tilde{\mathbf{u}}\left[t, \mathbf{x}, \frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}}\right]\right\} + \qquad (39)$$

$$+ F_0\left\{t, \mathbf{x}, \tilde{\mathbf{u}}\left[t, \mathbf{x}, \frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}}\right]\right\} = 0.$$

Because function $\tilde{\mathbf{u}}$ is known, PDE equation (39) for function $V(t, \mathbf{x})$ can be solved numerically (for example, using power series [19]).

If the second situation occurs, it is impossible to transform the HJB equation into the form in (39). Thus, it is necessary to solve equation (9) directly, which usually leads to the "curse of dimensionality".

For the OD problem, the first situation is also preferable. If the function $\tilde{\mathbf{u}}\left[t, \mathbf{x}, \frac{\partial V(t,\mathbf{x})}{\partial \mathbf{x}}\right]$ is known, then it is possible to immediately obtain the OD controller

$$\mathbf{u} = \mathbf{u}_d^*(t, \mathbf{x}) := \tilde{\mathbf{u}}\left[t, \mathbf{x}, \frac{\partial V^*(t, \mathbf{x})}{\partial \mathbf{x}}\right]$$

for any specified function $V = V^*(t, \mathbf{x})$ in (38).

Nevertheless, in contrast to the MIF problem, the second situation here is not critical. Numerically realizing the pointwise minimization of the function $W(t, \mathbf{x}, \mathbf{u})$ for every fixed point $(t, \mathbf{x})$, we can obtain the OD controller

$$\mathbf{u} = \mathbf{u}_d^{**}(t, \mathbf{x}) = \arg\min_{\mathbf{u}\in U}\left\{\frac{\partial V^{**}(t, \mathbf{x})}{\partial \mathbf{x}}\mathbf{f}_0(t, \mathbf{x}, \mathbf{u}) + F_0(t, \mathbf{x}, \mathbf{u})\right\}$$

for the given partial function $V = V^{**}(t, \mathbf{x})$. Clearly, $\mathbf{u}_d^*(t, \mathbf{x}) \equiv \mathbf{u}_d^{**}(t, \mathbf{x})$ if $V^*(t, \mathbf{x}) \equiv V^{**}(t, \mathbf{x})$.

For both situations, accepting $V^* \equiv V^{**} \equiv \tilde{V}(t, \mathbf{x})$, we can obtain OD controllers such that they are simultaneously solutions of the MIF problem, i. e.,

$$\mathbf{u} = \tilde{\mathbf{u}}_d(t, \mathbf{x}) := \tilde{\mathbf{u}}\left[t, \mathbf{x}, \frac{\partial \tilde{V}(t, \mathbf{x})}{\partial \mathbf{x}}\right] \equiv \mathbf{u}_{c0}(t, \mathbf{x}).$$

The last position serves as a basis for constructing the approximate optimal solutions of the aforementioned problem. This construction demand appears either in certain situations when the choice of the optimal controller is essentially hindered or for cases when the exact solution $\mathbf{u} = \mathbf{u}_{c0}(t, \mathbf{x})$ is obtained but is practically unusable.

The choice of the aforementioned approximation can be realized as a solution of the corresponding OD problem. Let us consider the space $\Re_0$ of the CLF, which contains the function $V = \tilde{V}(t, \mathbf{x})$.

Given a function $V^*(t, \mathbf{x}) \in \Re_0$ that is not identically equal to $\tilde{V}(t, \mathbf{x})$, let us solve the OD problem (37), thereby deriving the OD controller $\mathbf{u}_d^*(t, \mathbf{x}) := \mathbf{u}_d(t, \mathbf{x}, V^*)$. Since this controller is not MIF optimal, we obtain

$$J^* := J(V^*) := J(\mathbf{u}_d(t, \mathbf{x}, V^*)) \geqslant J(\mathbf{u}_{c0}(\cdot)) = J_0.$$

If the assessment is true

$$\Delta J = (J^* - J_0)/J_0 \leqslant \varepsilon_J \tag{40}$$

for a given value ε_J of the admissible functional J degradation, then the controller $\mathbf{u} = \mathbf{u}_d^*(t,\mathbf{x})$ can be accepted as an approximate solution for problems (6), (8), and (36).

Remark 2. The aforementioned function $V^*(t,\mathbf{x})$ can be treated as an approximate solution of the HJB equation (28). Its approximation quality is interpreted as in (40).

To choice the function $V^*(t,\mathbf{x}) \in \Re_0$ that satisfies (40), one can use an optimization approach. Next, we state a minimization problem

$$J = J(V^*) := J(\mathbf{u}_d(t,\mathbf{x},V^*)) \rightarrow \min_{V^* \in \Re_0}, \tag{41}$$

which has the obvious solution

$$V_0^*(t,\mathbf{x}) := \arg \min_{V^* \in \Re_0} J(\mathbf{u}_d(t,\mathbf{x},V^*)) \equiv \tilde{V}(t,\mathbf{x}).$$

Any numerical method for this problem solution generates the minimizing sequence $\{V_k^*(t,\mathbf{x})\} \in \Re_0$, which trends toward the function $\tilde{V}(t,\mathbf{x})$:

$$\lim_{k\rightarrow\infty} \{V_k^*(t,\mathbf{x})\} = \tilde{V}(t,\mathbf{x}) \quad \forall(t,\mathbf{x}).$$

Clearly, for any ε_J there is the function $V_{\varepsilon 0}^*(t,\mathbf{x})$ (among the items of the sequence $\{V_k^*(t,\mathbf{x})\} \in \Re_0$), such that condition (40) is valid. This function determines the approximate optimal controller $\mathbf{u} = \mathbf{u}_d^*(t,\mathbf{x}) := \mathbf{u}_d(t,\mathbf{x},V_{\varepsilon 0}^*)$.

Naturally, if the exact solution $\mathbf{u} = \mathbf{u}_{c0}(t,\mathbf{x})$ cannot be obtained simply or if this solution is known but requires an essential simplification, it is necessary to implement the problem of

$$J = J(V^*) := J(\mathbf{u}_d(t,\mathbf{x},V^*)) \rightarrow \min_{V^* \in \Re_{d0} \subset \Re_0} \tag{42}$$

instead of (41). Here, the set $\Re_{d0}$ is a contraction of the set $\Re_0$, including CLF $V(t,\mathbf{x})$.

If the set $\Re_{d0}$ does not include the optimal function, i. e., if $\tilde{V}(t,\mathbf{x}) \notin \Re_{d0}$, then the solution of problem (42),

$$V_{d0}^*(t,\mathbf{x}) := \arg \min_{V^* \in \Re_{d0} \subset \Re_0} J(\mathbf{u}_d(t,\mathbf{x},V^*)),$$

which gives an OD controller $\mathbf{u} = \mathbf{u}_{d0}^*(t,\mathbf{x}) := \mathbf{u}_d(t,\mathbf{x},V_{d0}^*)$ that is generally spiking, can interrupt requirement (40) for a given ε_J. In this case, the admissible set $\Re_{d0}$ must be changed in (42).

Note that the set $\Re_{d0}$ can be introduced in the simplest parametric way. To this end, one should fix a structure of the CLF V^* and extract the vector $\mathbf{h} \in E^p$ of its parameters to be varied: $V^* = V^*(t,\mathbf{x},\mathbf{h})$.

By analogy with (42), it is next possible to pose the optimization problem such that its solution with respect to $\mathbf{h}$ results in an approximate optimal controller.

Let us consider this question in detail, introducing the metric compact set $H_v \in E^p$. Suppose that the functions of V^* are formed as follows:

$$\mathbf{h} \in H_v \subset E^p \Rightarrow V^*(t,\mathbf{x},\mathbf{h}) \in \Re_0.$$

Given the initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0 \in B_r$ for plant (36), let us compose the series of computational procedures, which should be executed in the range of the proposed method.

1. Assign the vector $\mathbf{h} \in H_v \subset E^p$ of the tunable parameters.

2. Specify the function $V^*(t, \mathbf{x}, \mathbf{h})$.

3. Solve the OD problem with the following functional to be damped:

$$L = L(t, \mathbf{x}, \mathbf{u}, \mathbf{h}) = V(t, \mathbf{x}, \mathbf{h}) + \int_{t_0}^{t} F_0(\tau, \mathbf{x}, \mathbf{u}) d\tau,$$

thereby obtaining the OD controller $\mathbf{u} = \mathbf{u}_d^*(t, \mathbf{x}, \mathbf{h})$.

4. Compose the equations of the closed-loop system

$$\dot{\mathbf{x}} = \mathbf{f}_{0d}(t, \mathbf{x}, \mathbf{h}), \quad \mathbf{f}_{0d}(t, \mathbf{x}, \mathbf{h}) := \mathbf{f}_0(t, \mathbf{x}, \mathbf{u}_d^*(t, \mathbf{x}, \mathbf{h})). \tag{43}$$

5. Solve the Cauchy problem for system (43) with the given initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ that result in the motion $\mathbf{x}_d(t, \mathbf{h})$.

6. Specify the function $\mathbf{u}_d(t, \mathbf{h}) := \mathbf{u}_d^*(t, \mathbf{x}_d(t, \mathbf{h}), \mathbf{h})$.

7. Calculate a value of the function $J_d(\mathbf{h})$, determined by the expression

$$J_d = J_d(\mathbf{h}) = \int_{t_0}^{\infty} F_0 \left[t, \mathbf{x}_d(t, \mathbf{h}, \mathbf{x_0}), \mathbf{u}_d(t, \mathbf{h}, \mathbf{x_0}) \right] dt.$$

8. Minimize the function $J_d(\mathbf{h})$ on the set H_v, i. e., solve the problem of

$$J_d = J_d(\mathbf{h}) \to \min_{\mathbf{h} \in H_v}, \ \mathbf{h}_d := \arg \min_{\mathbf{h} \in H_v} J_d(\mathbf{h}), \ J_{d0} := J_d(\mathbf{h_d}), \tag{44}$$

repeating the steps 1–7 of this scheme.

The solution $\mathbf{h} = \mathbf{h}_d$ of the problem (44) allows us to construct an approximation of the Bellman function as follows:

$$V_{d0}^*(t, \mathbf{x}) \equiv V^*(t, \mathbf{x}, \mathbf{h}_d).$$

Correspondingly, the control law

$$\mathbf{u} = \mathbf{u}_{d0}^*(t, \mathbf{x}) := \mathbf{u}_d^*(t, \mathbf{x}, \mathbf{h}_d)$$

represents the approximate optimal controller for the initial MIF problem.

If the optimal value J_0 is known, one can estimate the following measure of the functional J degradation due to the approximate solution using

$$\Delta J = (J_{d0} - J_0)/J_0.$$

If there is a vector $\mathbf{h}^* \in H_v \subset E^p$ such that the identity is valid

$$V^*(t, \mathbf{x}, \mathbf{h}^*) \equiv \tilde{V}(t, \mathbf{x}),$$

then the following evident relationships are fulfilled:

$$\mathbf{u}_{d0}^*(t, \mathbf{x}) \equiv \mathbf{u}_{c0}(t, \mathbf{x}), \quad J_{d0} = J_0, \quad \Delta J = 0.$$

5. On practical choice of integral item. As mentioned in Zubov's works [9–11], the OD problem has obvious advantages in its implementation simplicity over the MIF problem. Consequently, there is a reason to abandon the exclusive use of functional (6) and concentrate initially on supporting practical requirements (5) using OD concept.

Under this approach, there is a reason to first assign not the integrand $F(t, \mathbf{x}, \mathbf{u})$ but the function $V(t, \mathbf{x})$ for functional (13) to be damped. The primary choice of V should be done as Lyapunov function candidate (ideally, as a CLF). At the same time, the subintegral function F should be varied to fulfill the requirements of (5).

Note that this idea originates from the following statement proven in [14]: any CLF $V(t, \mathbf{x})$ is a value function for certain performance index, i. e., this function satisfies the HJB equation associated with functional (6).

Let us next consider the suggested OD oriented approach in detail. Suppose that function $V(t, \mathbf{x})$ is assigned to the functional $L(t, \mathbf{x}, \mathbf{u})$ and that this function meets the conditions in (14).

Let us introduce a certain class $\Re_F$ of positively definite functions of type (7) and specify a functional to be damped:

$$L = L(t, \mathbf{x}, \mathbf{u}) = V(t, \mathbf{x}) + \int_{t_0}^{t} F(\tau, \mathbf{x}, \mathbf{u}) d\tau \tag{45}$$

for a given function $F \in \Re_F$.

Let us next solve OD problem (15), thereby obtaining the OD controller

$$\mathbf{u} = \mathbf{u}_{dF}(t, \mathbf{x}) := \arg \min_{\mathbf{u} \in U} W(t, \mathbf{x}, \mathbf{u}, F), \tag{46}$$

where the rate W is defined as

$$W(t, \mathbf{x}, \mathbf{u}, F) := \frac{\partial V(t, \mathbf{x})}{\partial t} + \frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}, \mathbf{u}) + F(t, \mathbf{x}, \mathbf{u}).$$

Let us accept a comparison function $\alpha_3 \in K$ and check the condition

$$W_{F0}(t, \mathbf{x}, \mathbf{u}) := W(t, \mathbf{x}, \mathbf{u}_{dF}(t, \mathbf{x}), F) \leqslant -\alpha_3(\|\mathbf{x}\|) \quad \forall \mathbf{x} \in B_r, \ \forall t \geqslant t_0. \tag{47}$$

If this condition is valid, then it follows from Theorem 1 that the controller (46) is stabilizing controller for plant (1).

Repeating this computations using formulae (45)–(47) for various functions $F \in \Re_F$, let us introduce *a functional of stability* given on the set $\Re_F$:

$$J_c(F) := \sup_{t \in [0, \infty)} \sup_{\mathbf{x} \in B_r} \left[W(t, \mathbf{x}, \mathbf{u}_{dF}(t, \mathbf{x})), F(t, \mathbf{x}, \mathbf{u}_{dF}(t, \mathbf{x})) + \alpha_3(\|\mathbf{x}\|) \right].$$

Further, let us extract the subset $\Re_c \subset \Re_F$ of functions F such that

$$\Re_c = \left\{ F \in \Re_F : J_c(F) < 0 \right\}.$$

For these functions, all controllers (45) are stabilizing. The next step addresses the requirements (5) for the dynamics of the transient processes and introduce *a functional of performance* given on set $\Re_c$:

$$J_d(F) := \sup_{t \in [0, \infty)} \sup_{\mathbf{x}^0 \in B_r} \operatorname{dist} \left\{ \mathbf{x}(t, \mathbf{x}^0, \mathbf{u}_{dF}(t, \mathbf{x}), X) \right\},$$

where the function $\operatorname{dist}(\mathbf{x}(\cdot), X)$ determines the distance from the motion $\mathbf{x}(t, \mathbf{x}^0, \mathbf{u}_{dF})$ to the admissible set X.

The presented reasoning allows us to pose a problem of the performance functional minimization on the set $\Re_c$:

$$J_d(F) \to \inf_{F \in \Re_c}.$$

Clearly, if the function $F = \tilde{F} \in \Re_c$ is obtained in the course of this problem solution such that $J_d(\tilde{F}) = 0$, then the corresponding OD controller

$$\mathbf{u} = \mathbf{u}_{dF}(t, \mathbf{x}) := \arg \min_{\mathbf{u} \in U} W(t, \mathbf{x}, \mathbf{u}, \tilde{F})$$

is locally uniformly asymptotically stabilizing for the plant (1). In addition, the practical requirements (5) for the motion of the closed-loop connection are satisfied by this controller.

Naturally, the presented global approach determines only a general theory of the OD concept's implementation to provide stability and performance features for nonlinear and non-autonomous control plants. This theory should be reflected in various particular practically realizable methods.

The simplest specific definition of the aforementioned approach can be determined by the vector parameterization of the functions F population. Really, let us introduce p-parametrical family of the functions

$$F = F(t, \mathbf{x}, \mathbf{u}, \mathbf{h}) \tag{48}$$

with the certain given structure, where $\mathbf{h} \in E^p$ is a vector parameter.

Here, it is possible to accept the quadratic form $F = \mathbf{u}^T \mathbf{Q}(\mathbf{h})\mathbf{u}$ with positive definite symmetric matrix $\mathbf{Q}$, particularly with the form $\mathbf{Q} = \text{diag}\ \left\{(h_1^2\ h_2^2\ ...\ h_p^2)\right\}$.

For any fixed vector $\mathbf{h}$, one can specify a functional to be damped as follows:

$$L = L(t, \mathbf{x}, \mathbf{u}, \mathbf{h}) = V(t, \mathbf{x}) + \int_{t_0}^{t} F(\tau, \mathbf{x}, \mathbf{u}, \mathbf{h}) d\tau,$$

which determines a solution of OD problem (15) as

$$\mathbf{u} = \mathbf{u}_{dh}(t, \mathbf{x}) := \arg \min_{\mathbf{u} \in U} W(t, \mathbf{x}, \mathbf{u}, \mathbf{h}), \tag{49}$$

where

$$W(t, \mathbf{x}, \mathbf{u}, \mathbf{h}) := \frac{\partial V(t, \mathbf{x})}{\partial t} + \frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}, \mathbf{u}) + F(t, \mathbf{x}, \mathbf{u}, \mathbf{h}).$$

For a general case, it is possible to assign any comparison function $\alpha_3 = \alpha_3 \in K$ and check the condition

$$W_{h0}(t, \mathbf{x}, \mathbf{u}, \mathbf{h}) := W(t, \mathbf{x}, \mathbf{u}_{dh}(t, \mathbf{x}), \mathbf{h}) \leqslant -\alpha_3(\|\mathbf{x}\|) \quad \forall \mathbf{x} \in B_r,\ \forall t \geqslant t_0. \tag{50}$$

If this condition is valid, using Theorem 1, one can conclude that controller (49) stabilizes plant (1).

On this occasion, *a functional of stability* turns into the function of the p variables, which, in conformity with (48)–(50), can be presented as

$$J_c(F) := \sup_{t \in [0, \infty)} \sup_{\mathbf{x} \in B_r} \left[W(t, \mathbf{x}, \mathbf{u}_{dh}(t, \mathbf{x}), \mathbf{h}) + \alpha_3(\|\mathbf{x}\|) \right].$$

Next, let us extract the subset $H_c \subset E^p$ of vectors $\mathbf{h}$ such that

$$H_c = \{\mathbf{h} \in E^p : J_c(\mathbf{h}) < 0\}.$$

For any $\mathbf{h} \in H_c$ controller (49) is a stabilizing one. Similarly, one can determine *a function of performance* using requirements of (5):

$$J_d(\mathbf{h}) := \sup_{t\in[0,\infty)} \sup_{\mathbf{x}^0 \in B_r} \operatorname{dist}\left\{\mathbf{x}(t, \mathbf{x}^0, \mathbf{u}_{dh}(t, \mathbf{x}), X)\right\},$$

which are given on the set H_c.

Next, the finite dimensional minimization problem

$$J_d(\mathbf{h}) \to \inf_{\mathbf{h}\in H_c}$$

can be posed. If the vector $\mathbf{h} = \tilde{\mathbf{h}} \in H_c$ is obtained in the course of this problem solution such that $J_d(\tilde{\mathbf{h}}) = 0$, then the corresponding OD controller

$$\mathbf{u} = \mathbf{u}_{dh}(t, \mathbf{x}) := \arg\min_{\mathbf{u}\in U} W(t, \mathbf{x}, \mathbf{u}, \tilde{\mathbf{h}})$$

is locally uniformly asymptotically stabilizing one for plant (1). As before, practical requirements (5) for the motion of the closed-loop connection are satisfied.

6. Practical example of approximate synthesis. To illustrate the applicability of the presented approach, let us consider a numerical example [20] with the following linear plant model of the first order:

$$\dot{x} = -x + u, \tag{51}$$

where the controlled variable x and the control u are scalar values. The performance of the motion for plant (51) can be specified by the non-quadratic functional

$$J = \int_0^\infty \left(x^2 + x^4 + u^2\right) dt. \tag{52}$$

The MIF problem consists of designing the stabilizing controller $u = u_{c0}(x)$ design, thereby providing a minimum of the functional (52) on the set $U = E^1$.

It was shown in [20] that the exact solution of HJB equation (39) is the value function

$$V_t(x) = -x^2 + \frac{2}{3}\left[(2 + x^2)^{3/2} - 2\sqrt{2}\right]. \tag{53}$$

A corresponding optimal controller can be presented by the formula

$$u = u_{c0}(x) = x - x\sqrt{2 + x^2}. \tag{54}$$

This solution provides the minimal value $J_0 = 0.579$ of functional (52) for the motion of the closed-loop system (51), (54) with the initial condition $x(0) = 1$.

Let us next address the OD problem for constructing the approximate solutions of the aforementioned MIF problem. To this end, as proposed in Section 4, we introduce a set $\Re_{d0} \subset \Re_d$ of the CLF V^*, which are determined by the formula

$$V^* = V^*(x, h) = h^2 x^2.$$

Introducing the metric compact set $H_v = [0, 1.2] \subset E^1$, it can be readily seen that

$$h \in H_v \Rightarrow V^*(x,h) \in \Re_0.$$

Giving the initial condition of $x(0) = 1$ for plant (51), one can solve the OD problem with respect to the functional to be damped of the form

$$L = L(x,u) = V^*(x,h) + \int_0^t \left(x^2 + x^4 + u^2\right) d\tau,$$

which leads to the relationships

$$\tilde{u}\left[x, V^*(x,h)\right] := \arg \min_{u \in E^1} \left\{ \frac{\partial V^*(x,h)}{\partial x}(-x+u) + x^2 + x^4 + u^2 \right\} = \tag{55}$$

$$= \arg \min_{u \in E^1} \left\{ \frac{\partial V^*(x,h)}{\partial x} u + u^2 \right\} = -\frac{1}{2}\frac{\partial V^*(x,h)}{\partial x}.$$

As long as $\frac{\partial V^*(x,h)}{\partial x} = 2h^2 x$, we obtain the following linear OD controller from (55):

$$u = u_d^*(x,h) = -h^2 x. \tag{56}$$

For the equation

$$\dot{x} = -\left(1 + h^2\right) x$$

of the closed-loop system (51), (56), it is possible to solve the Cauchy problem with the initial condition $x(0) = 1$, which determines the motion $x_d(t,h)$ and the corresponding control $u_d(t,h) := u_d^*(t, x_d(t,h), h)$. The value of the functional (52) for this motion can be presented as a function of h:

$$J_d = J_d(h) := \int_0^\infty \left(x_d^2(t,h) + x_d^4(t,h) + u_d^2(t,h)\right) dt.$$

Minimizing the aforementioned function $J_d(h)$ on the set H_v, i. e., considering the optimization problem as follows:

$$J_d = J_d(h) \to \min_{h \in H_v}, \ h_d := \arg \min_{h \in H_v} J_d(h), \ J_{d0} := J_d(h_d),$$

we obtain the values $h_d = 0.762$ and $J_{d0} = 0.581$.

The corresponding approximation for the value function $V_t(x)$ (53) is

$$V_{d0}^*(x) \equiv V^*(x, h_d) = h_d^2 x^2 = 0.581 x^2,$$

which leads to the approximate optimal controller

$$u = u_{d0}^*(x) := u_d^*(x, h_d) = -h_d^2 x = -0.581x. \tag{57}$$

For the optimal value J_0, the expression

$$\Delta J = (J_{d0} - J_0)/J_0 = 0.35\,\%$$

represents a relative degradation of the performance index, which is determined by a transition to the approximate optimal solution. Since the value $\Delta J = 0.35\,\%$ seems to be highly convincing, controller (57) can be practically implemented instead of the optimal solution (54).

Figure 1 illustrates a graph of the function $J_d(h)$. The optimal value of the functional (52) is also shown here.

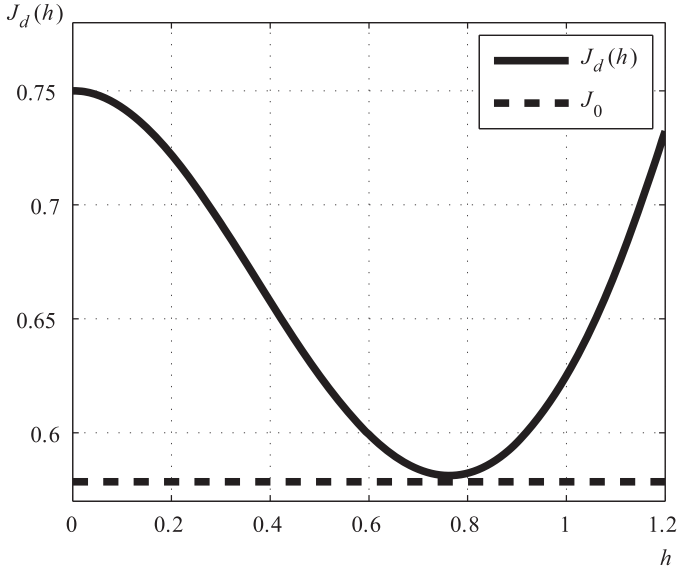

Fig. 1. The graph of the function $J_d(h)$ compared to the optimal value $J_0 = 0.579$ of the functional (52)

Figure 2 presents the graphs of the Lyapunov functions $V_t(x)$ and $V_{d0}(x)$. As mentioned above, the first is a value function with respect to functional (52), i. e., this function is a solution for the corresponding HJB equation. The next one, $V_{d0}(x) \equiv V^*(x, h_d)$, can be treated as an approximate representation of the value function. A comparison can illustrate their vicinity.

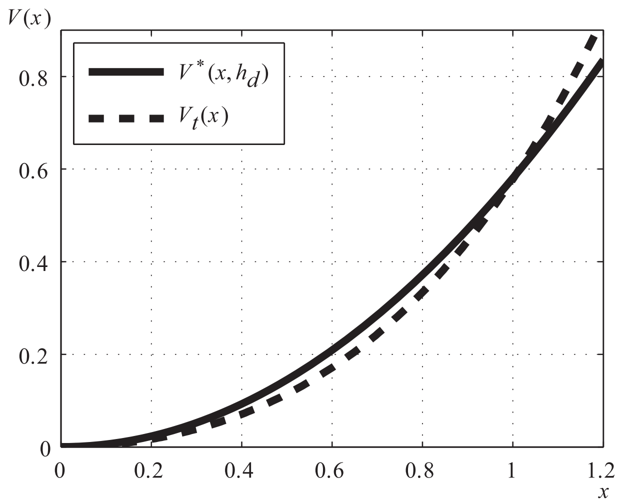

Fig. 2. The graphs of the Lyapunov functions $V_t(x)$ and $V^*_{d0}(x) \equiv V^*(x, h_d)$

The dynamics of the closed-loop system (51), (57) are illustrated by Figure 3, where the motion $x_d(t, h_d)$ and the corresponding control $u_d(t, h_d)$ are presented. A nearly identical process corresponds to the closed-loop connection (51), (54) with the optimal controller (54).

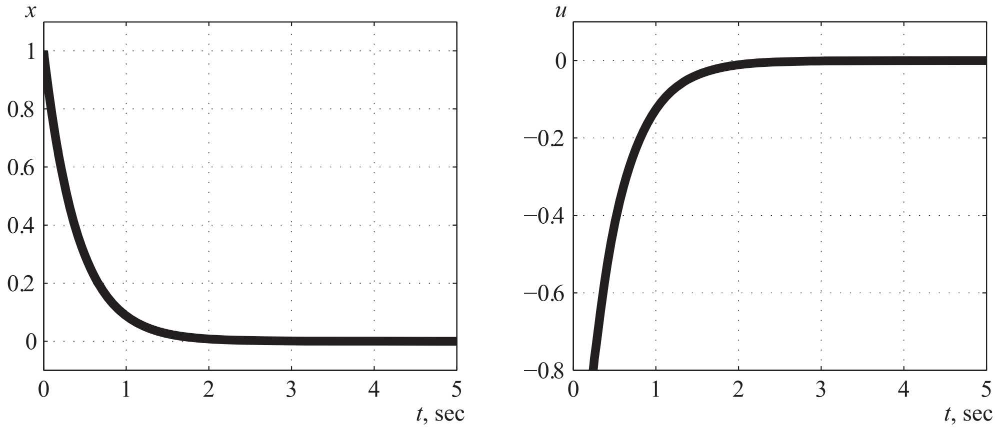

Fig. 3. The motion $x_d(t, h_d)$ and the corresponding control $u_d(t, h_d)$ for the closed-loop system (51), (57)

7. Conclusions. This work aimed to discuss some vital questions connected to various design applications of the modern optimization theory for the modeling, analysis, and synthesis of nonlinear and nonautonomous control systems. There are many practical problems to be mathematically formalized based on the optimization approach.

Nevertheless, most such problems involve providing desirable dynamic features, usually presented in the form of (5). This allows one to attract different ideas for their formalization using Bellman's theory and Zubov's optimal damping concept [1–3, 9–11]. These approaches are closely connected, but the latter has certain advantages related to the practical requirements for the dynamic features of a closed-loop connection.

First, the numerical solution of the OD problem is considerably simpler than that of the MIF problem. This factor facilitates the fair formalization of functional choice considering the optimal damping concept. This is one of the main issues discussed above, which is based on the fundamental coincidence of the mention problems' solutions under the execution of certain conditions.

This paper focused on two principal questions: the construction of an approximate solution for the MIF problem using the OD approach, and the choice of the integral items of the functional to be damped. Both the questions are oriented toward the initial requirements for the dynamic features of stability and performance. The corresponding numerical methods for controllers synthesis are proposed considering the aforementioned questions. Finally, the proposed approach was illustrated using a simple numerical example of approximate optimal controller synthesis.

The results of the above investigations could be expanded to consider the robust features of the optimal damping controller and to take into account transport delays in both the input and the output of a controlled plant. The obtained results are intended for application in studies for the multipurpose control of marine vehicles [21–24].

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О практическом применении Зубовского принципа оптимального демпфирования*

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Данная работа представляет некоторые новые идеи, связанные с синтезом нелинейных и неавтономных законов управления, базирующихся на применении оптимизационного подхода. Имеет место существенная связь между практическими требованиями и функционалом, который подлежит минимизации. Эта связь определяет основу предлагаемых методов. Обсуждение сфокусировано на принципе оптимального демпфирования, который был впервые предложен В. И. Зубовым в начале 1960-х годов. Центральное внимание уделено различным современным аспектам практического применения теории оптимального демпфирования. Ударение сделано на специальном выборе функционала, подлежащего демпфированию, для обеспечения желаемых свойств устойчивости и качества замкнутой системы. Работоспособность и эффективность предложенного подхода подтверждены иллюстративным числовым примером.

Ключевые слова: обратная связь, устойчивость, демпфирующее управление, функционал, оптимизация.

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