Preprints, 12th IFAC Workshop on Time Delay Systems
June 28-30, 2015. Ann Arbor, MI, USA

A Novel Approach to Robust Stability Analysis of Linear Time-Delay Systems ⋆

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Abstract: In this paper, we demonstrate that the Lyapunov–Krasovskii functional, which does not admit a quadratic lower bound, can be applied for the robust exponential stability analysis as well as for obtaining the exponential estimates for the solutions of linear time-invariant differential-difference systems with multiple delays.

Keywords: time-delay systems, Lyapunov–Krasovskii functionals, exponential stability, robust stability, exponential estimates, decay rate

1. INTRODUCTION

The Lyapunov–Krasovskii approach to stability analysis of time-delay systems is based on the well-known Krasovskii theorem (Krasovskii, 1956). Roughly speaking, it states that if there is a positive definite functional, whose time-derivative along the solutions of the system is negative definite, then the system is asymptotically stable. One of the possible ways to use this theorem is to prescribe the (negative definite) time-derivative and then to construct the functional that has the same derivative along the solutions. For linear time-invariant delay systems, the functionals with a prescribed derivative have been developed in the works of Repin (1965), Infante & Castelan (1978), Huang (1989) and Kharitonov & Zhabko (2003). In these papers, the structure, the existence issue, and further the explicit form of the functionals and their positive definiteness have been studied. As a result, two functionals satisfying the Krasovskii theorem were constructed. The first one was proposed in Huang (1989), let us denote it by v_0; its time-derivative along the solutions of the system is the quadratic form of the current state $x(t)$. In its turn, the second one's derivative is the functional depending on the whole delay system's state x_t. The second functional was introduced in Kharitonov & Zhabko (2003) and was called the functional of the complete type. The important point is that the functional v_0 does not admit a quadratic lower bound and admits only the local cubic one if the system is exponentially stable (see Huang, 1989), whereas the complete-type functional admits the quadratic bound and, therefore, is effective in applications. There are many contributions addressing the applications of the complete-type functionals, see, for instance, Egorov & Mondiè (2014), Jarlebring et al. (2011), Ochoa et. al (2013), and Kharitonov (2013). The applications important for us in this paper are the robustness analysis (Kharitonov & Zhabko, 2003) and the construction of the exponential estimates for the solutions (Kharitonov & Hinrichsen, 2004), as for linear time-invariant delay systems the asymptotic stability is equivalent to the exponential one. On the contrary, the functional v_0 is considered to be not suitable for solving the problems of this kind.

However, in the works Zhabko & Medvedeva (2011), Medvedeva & Zhabko (2013; 2015) the following has been shown. In spite of the fact that the functional v_0 does not admit a quadratic lower bound on the set of arbitrary continuous functions, it admits such a bound on the set of functions satisfying the condition $\|\varphi(\theta)\| \leqslant \|\varphi(0)\|$, $\theta \in [-h, 0]$, where h is the maximal delay, if the system is exponentially stable. In terms of such bound, the exponential stability criterion was established, and the constructive approach for the stability analysis was developed.

The aim of the present paper is to demonstrate that the functional v_0 can be effective not only in the stability but also in the robust stability analysis as well as in the construction of the exponential estimates for the solutions of the exponentially stable systems. In other words, we are going to show the possibility to analyze the robustness and to estimate the decay rate and the γ-factor (see Definition 1) without making use of the complete-type functionals. Our approach is based on the abovementioned exponential stability criterion. The special integral estimate for the derivative of the functional plays a key role as well.

It is worth pointing out that there is a great variety of works where the problems we address are treated on the basis of the LMI approach, see, for instance, Mondiè & Kharitonov (2005) or the survey papers Kharitonov (1999) and Niculescu et al. (1997). In Bellman & Cooke (1963) the exponential estimates for the solutions are constructed directly in terms of the Laplace transform.

The paper is organized as follows. Section 2 contains the preliminaries. Then, Section 3 is devoted to the robustness analysis whilst the exponential estimates for the solutions are provided in Section 4. In Section 5, we illustrate the work with examples comparing our results with those obtained in Kharitonov & Zhabko (2003) and Kharitonov & Hinrichsen (2004) by use of the complete-type functionals.

⋆ The authors acknowledge Saint Petersburg State University for the research grant 9.37.157.2014.

2. PRELIMINARY RESULTS

In this paper, we consider a time-delay system of the form

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t-h_j), \tag{1}$$

where $A_j \in \mathbb{R}^{n\times n}$, $j=\overline{0,m}$, are the constant matrices, and $0 = h_0 < h_1 < \ldots < h_m = h$ are the constant delays. We use the standard notation: $x(t,\varphi)$, or briefly $x(t)$, denotes the solution of system (1) with the piecewise continuous initial function φ, i.e. $\varphi \in PC\big([-h,0],\mathbb{R}^n\big)$; then, $x_t(\varphi)$, or briefly x_t, stands for the segment of the solution

$$x_t(\varphi): \;\theta \to x(t+\theta,\varphi),\; \theta\in[-h,0].$$

On the space of the piecewise continuous functions the uniform norm

$$\|\varphi\|_h = \sup_{\theta\in[-h,0]} \|\varphi(\theta)\|$$

is defined, where $\|\cdot\|$ is the euclidian norm.

Definition 1. (Bellman & Cooke, 1963) System (1) is called *exponentially stable*, if there exist $\gamma \geqslant 1$ and $\sigma > 0$ such that

$$\|x(t,\varphi)\| \leqslant \gamma e^{-\sigma t}\|\varphi\|_h, \quad t\geqslant 0,$$

for every solution of system (1).

Given a positive definite matrix W, the functional satisfying the condition

$$\frac{dv_0(x_t)}{dt} = -x^T(t)Wx(t), \quad t\geqslant 0,$$

along the solutions of system (1) is of the form

$$\begin{aligned} v_0(\varphi) &= \varphi^T(0)U(0)\varphi(0) \\ &+ 2\varphi^T(0)\sum_{j=1}^{m}\int_{-h_j}^{0} U(-\theta-h_j)A_j\varphi(\theta)d\theta \\ &+ \sum_{k=1}^{m}\sum_{j=1}^{m}\int_{-h_k}^{0}\varphi^T(\theta_1)A_k^T \\ &\times\Big(\int_{-h_j}^{0}U(\theta_1+h_k-\theta_2-h_j)A_j\varphi(\theta_2)d\theta_2\Big)d\theta_1, \end{aligned} \tag{2}$$

see Huang (1989); Kharitonov & Zhabko (2003). Here $U(\tau)$ is the Lyapunov matrix, associated with W, i.e. the solution of the following set of the matrix equations

$$U'(\tau) = \sum_{j=0}^{m} U(\tau-h_j)A_j,\; \tau\geqslant 0;$$

$$U(-\tau) = U^T(\tau),\; \tau\geqslant 0;$$

$$\sum_{j=0}^{m}\big[U(-h_j)A_j + A_j^T U(h_j)\big] = -W.$$

The Lyapunov matrix and, therefore, functional (2) exists for any symmetric matrix W, if and only if the so-called Lyapunov condition holds: the system does not have an eigenvalue s such that $-s$ is also an eigenvalue, see Kharitonov (2013). The Lyapunov matrix is continuous.

Functional (2) admits the following upper bound:

Lemma 2. (Kharitonov, 2013) If the Lyapunov condition holds, then

$$\big|v_0(\varphi)\big| \leqslant \eta\|\varphi\|_h^2,\;\; \varphi\in PC\big([-h,0],\mathbb{R}^n\big),$$

where

$$\eta = M\alpha^2,\;\; M = \max_{\tau\in[0,h]}\|U(\tau)\|,\;\; \alpha = 1+\sum_{j=1}^{m}\|A_j\|h_j.$$

As for a lower bound, there is only the local cubic one: If system (1) is exponentially stable, then for every H there exists $\kappa > 0$ such that

$$v_0(\varphi)\geqslant \kappa\|\varphi(0)\|^3, \quad \|\varphi\|_h\leqslant H,$$

here φ is a continuous function, see Huang (1989). Nevertheless, on the special set of functions

$$S = \big\{\varphi\in PC\big([-h,0],\mathbb{R}^n\big)\,\big|\,\|\varphi(\theta)\|\leqslant\|\varphi(0)\|,\,\theta\in[-h,0]\big\}$$

functional (2) admits a quadratic lower bound, as the following criterion states.

Theorem 3. (Zhabko & Medvedeva, 2011; 2015) Given a positive definite matrix W, system (1) is exponentially stable, if and only if there exists a functional $v_0(\varphi)$ such that the following conditions hold:

1. $\dfrac{dv_0(x_t)}{dt} = -x^T(t)Wx(t)$;
2. there exists $\mu > 0$ such that
$$v_0(\varphi)\geqslant\mu\|\varphi(0)\|^2, \quad \varphi\in S.$$

Note that in the necessity part of Theorem 3 the constant μ is obtained constructively:

$$\mu = \frac{\lambda_{\min}(W)\delta}{4},$$

where $\lambda_{\min}(W)$ is the minimal eigenvalue of W, and $\delta > 0$ is the solution of the equation

$$\alpha K e^{K\delta} = \frac{1}{2\delta},$$

here $K = \sum_{j=0}^{m}\|A_j\|$, and α is defined in Lemma 2.

3. ROBUST STABILITY ANALYSIS

In this section, we consider the same problem statement as in Kharitonov & Zhabko (2003), see also Kharitonov (2013). Assume that system (1) is exponentially stable and define the following perturbed system

$$\dot{y}(t) = \sum_{j=0}^{m}(A_j+\Delta_j)y(t-h_j). \tag{3}$$

Here the constant matrices Δ_j are such that

$$\|\Delta_j\|\leqslant\rho_j, \quad j=\overline{0,m}, \tag{4}$$

where ρ_j are the constant values. Our aim is to find the conditions on these values under which system (3) remains exponentially stable.

Following Kharitonov & Zhabko (2003), for the stability analysis of system (3) we will use functional (2), corresponding to system (1). The time-derivative of this functional along the solutions of system (3) is of the form

$$\frac{dv_0(y_t)}{dt} = -y^T(t)Wy(t) + l(y_t), \tag{5}$$

where

$$\begin{aligned} l(y_t) &= 2\Big[\sum_{j=0}^{m}\Delta_j y(t-h_j)\Big]^T\times \\ &\times\Big[U(0)y(t) + \sum_{k=1}^{m}\int_{-h_k}^{0}U(-\theta-h_k)A_k y(t+\theta)d\theta\Big], \end{aligned}$$

see Kharitonov & Zhabko (2003). Let us estimate each term of functional $l(y_t)$. First, for every $j=\overline{0,m}$ we have

$$2y^T(t-h_j)\Delta_j^T U(0)y(t) \leqslant M\rho_j\big(\|y(t)\|^2+\|y(t-h_j)\|^2\big).$$

Then, for every $j=\overline{0,m}$ and $k=\overline{1,m}$ we obtain

$$2y^T(t-h_j)\Delta_j^T \int_{-h_k}^{0} U(-\theta-h_k)A_k y(t+\theta)d\theta$$

$$\leqslant M\rho_j\|A_k\|\Big(h_k\|y(t-h_j)\|^2+\int_{-h_k}^{0}\|y(t+\theta)\|^2 d\theta\Big).$$

Combining all the estimates together, we arrive at the following lemma. The similar one, for the one-delay case, can be found in Kharitonov (2013).

Lemma 4. Functional $l(y_t)$ admits the upper bound

$$l(y_t) \leqslant \sum_{j=0}^{m} l_j\|y(t-h_j)\|^2+\sum_{j=1}^{m} l_{m+j}\int_{-h_j}^{0}\|y(t+\theta)\|^2 d\theta.$$

Here $l_0 = M\Big(\alpha\rho_0+\sum_{k=0}^{m}\rho_k\Big), \quad l_j=\alpha M\rho_j,$

$$l_{m+j}=M\|A_j\|\sum_{k=0}^{m}\rho_k, \quad j=\overline{1,m}.$$

The structure of the functional $l(y_t)$ shows that time-derivative (5) is not negative definite for any ρ_j. For this reason, the functional v_0 was considered not to be suitable for the robustness analysis. The key idea, that allows us to avoid this difficulty, is to use the following integral estimate.

Lemma 5. The following estimate holds

$$\int_0^t l(y_s)ds \leqslant \Big(l_0+\sum_{j=1}^{m}\big(l_j+h_j l_{m+j}\big)\Big)\int_0^t\|y(s)\|^2 ds$$

$$+\sum_{j=1}^{m}\big(l_j+h_j l_{m+j}\big)\int_{-h_j}^{0}\|\varphi(s)\|^2 ds.$$

Proof. The assertion of the lemma follows directly from Lemma 4 and the estimations

$$\int_0^t\|y(s-h_j)\|^2 ds \leqslant \int_{-h_j}^{0}\|\varphi(s)\|^2 ds+\int_0^t\|y(s)\|^2 ds,$$

$$\int_{-h_j}^{0}\int_0^t\|y(s+\theta)\|^2 ds d\theta$$

$$\leqslant h_j\int_{-h_j}^{0}\|\varphi(s)\|^2 ds+h_j\int_0^t\|y(s)\|^2 ds. \qquad \square$$

Observe that

$$l_0+\sum_{j=1}^{m}\big(l_j+h_j l_{m+j}\big)=2\alpha M\sum_{j=0}^{m}\rho_j.$$

We are now ready to present the main result of this section whose proof is based on the proof of the sufficiency part of Theorem 3, see Medvedeva & Zhabko (2015).

Theorem 6. Let system (1) be exponentially stable. If

$$\sum_{j=0}^{m}\rho_j < \frac{\lambda_{\min}(W)}{2\alpha M}, \tag{6}$$

where M and α are defined in Lemma 2, then system (3) remains exponentially stable.

Proof. Suppose that the matrices Δ_j, $j=\overline{0,m}$, are such that inequalities (4) and (6) hold but system (3) is not exponentially stable. It means that there exists a sequence

$$\{t_k\}_{k=1}^{+\infty}, \quad t_k \xrightarrow[k\to+\infty]{} +\infty,$$

such that $\|y(t_k)\|\geqslant\beta=\text{const}>0$ for every k, here $y(t)$ is the solution of system (3). Without loss of generality we can assume that $t_1\geqslant h$ and $t_{k+1}-t_k\geqslant h$ for every k. Consider two cases. In the first case the solution $y(t)$ is uniformly bounded whereas in the second one it is not.

Case 1. Suppose that there exists $G>0$ such that $\|y(t)\|\leqslant G,\ t\geqslant 0$. Then,

$$\dot y(t)\leqslant K_\Delta G, \quad t\geqslant h, \quad \text{where } K_\Delta=\sum_{j=0}^{m}\|A_j+\Delta_j\|.$$

We will now show that there exists $\tau>0$ such that

- $\|y(t)\|\geqslant\dfrac{\beta}{2}$, $t\in[t_k,t_k+\tau]$, for every k;
- the intervals $[t_k,t_k+\tau]$ do not intersect with each other for different values of k.

Indeed, let $t\in[t_k,t_k+\tau]$, then

$$\|y(t)-y(t_k)\|\leqslant K_\Delta G(t-t_k)\leqslant K_\Delta G\tau,$$

and, therefore,

$$\|y(t)\|\geqslant\|y(t_k)\|-K_\Delta G\tau\geqslant\beta-K_\Delta G\tau.$$

It is clear that

$$\tau=\min\Big\{\frac{\beta}{2K_\Delta G};h\Big\}$$

satisfies the required conditions. Next, equality (5) leads to

$$v_0(\varphi)=v_0(y_t)+\int_0^t\Big[y^T(s)Wy(s)-l(y_s)\Big]ds.$$

Applying Lemma 5 to the latter integral, we obtain

$$v_0(\varphi)\geqslant v_0(y_t)+\Big(\lambda_{\min}(W)\ -\ L\Big)\int_0^t\|y(s)\|^2 ds$$
$$-\sum_{j=1}^{m}\big(l_j+h_j l_{m+j}\big)\int_{-h_j}^{0}\|\varphi(s)\|^2 ds, \quad \text{here} \tag{7}$$

$$L=l_0+\sum_{j=1}^{m}\big(l_j+h_j l_{m+j}\big)=2\alpha M\sum_{j=0}^{m}\rho_j.$$

Note that $\lambda_{\min}(W)-L>0$ by condition (6). According to Lemma 2, the first summand in the right-hand side of (7), i.e. functional (2), is bounded:

$$v_0(y_t)\geqslant-\eta\|y_t\|_h^2\geqslant-\eta G^2, \quad \eta>0.$$

Then, the third one does not depend on t. Consider the second summand. Let $N(t)$ be the number of intervals

$$[t_k,t_k+\tau]\subset[0,t];$$

it is clear that $N(t) \xrightarrow[t\to+\infty]{} +\infty$. Since for different k the intervals $[t_k, t_k+\tau]$ do not have common points, we obtain

$$\int_0^t \|y(s)\|^2 ds \geqslant \sum_{k=1}^{N(t)} \int_{t_k}^{t_k+\tau} \|y(s)\|^2 ds \geqslant \frac{\beta^2\tau}{4} N(t) \xrightarrow[t\to+\infty]{} +\infty,$$

which gives the contradiction in inequality (7).

Case 2. We now suppose that the constant G from Case 1 does not exist. Then the sequence $\{t_k\}_{k=1}^{+\infty}$ can be chosen as follows:

$$\|y(t_k)\| = \max_{-h\leqslant t\leqslant t_k} \|y(t)\| \xrightarrow[k\to+\infty]{} +\infty.$$

Such a choice gives $y_{t_k} \in S$ for every k. Since system (1) is exponentially stable, we can apply the necessity part of Theorem 3: there exists $\mu > 0$ such that

$$v_0(\varphi) \geqslant \mu\|\varphi(0)\|^2, \quad \varphi \in S.$$

Hence, $v_0(y_{t_k}) \geqslant \mu\|y(t_k)\|^2$ for every k, and, therefore, $v_0(y_{t_k}) \xrightarrow[k\to+\infty]{} +\infty$. Consider inequality (7) for $t = t_k$, $k = 1, 2, \dots$ Now the second summand in its right-hand side is nonnegative, the third summand does not depend on k, and we again obtain the contradiction: $v_0(\varphi) \xrightarrow[k\to+\infty]{} +\infty$. The theorem is proved. □

Remark 7. If we use the complete-type functional (see Kharitonov & Zhabko, 2003)

$$v(\varphi) = v_0(\varphi) + \sum_{j=1}^m \int_{-h_j}^0 \varphi^T(\theta)\big[W_j + (h_j+\theta)W_{m+j}\big]\varphi(\theta)d\theta,$$

where the Lyapunov matrix in v_0 is associated with

$$W = W_0 + \sum\nolimits_{j=1}^m \big[W_j + h_j W_{m+j}\big],$$

and W_j, $j = \overline{0, 2m}$, are the positive definite matrices, to analyze the robust stability of system (1), we obtain the conditions

$$l_0 < \lambda_{\min}(W_0), \quad l_j \leqslant \lambda_{\min}(W_j), \quad j = \overline{1, 2m}. \tag{8}$$

This is because the time-derivative $dv(x_t)/dt$ along the solutions of system (1) is equal to $-w(x_t)$, where

$$w(\varphi) = \varphi^T(0)W_0\varphi(0) + \sum_{j=1}^m \varphi^T(-h_j)W_j\varphi(-h_j) + \sum_{j=1}^m \int_{-h_j}^0 \varphi^T(\theta)W_{m+j}\varphi(\theta)d\theta,$$

hence, inequalities (8) ensure the negative definiteness of the derivative of the functional v along the solutions of system (3). Inequalities (8) are similar to those obtained in Kharitonov & Zhabko (2003); Kharitonov (2013).

4. EXPONENTIAL ESTIMATES

In this section, we show that the approach based on Theorem 3 makes it possible to find a lower bound for the exponential decay rate σ and an upper bound for the γ-factor from Definition 1.

4.1 Estimation of Decay Rate

Assume that system (1) is exponentially stable, and denote the absolute value of the real part of its rightmost eigenvalue by $\bar{\sigma}$. In this subsection, our aim is to find a lower bound for $\bar{\sigma}$ without any information about the system's spectrum. Making the change of variable $y(t) = e^{\sigma t}x(t)$, where $\sigma > 0$, we obtain the following system

$$\dot{y}(t) = (A_0 + \sigma I)y(t) + \sum_{j=1}^m e^{\sigma h_j} A_j y(t-h_j). \tag{9}$$

It is clear that if, for some σ, system (9) is exponentially stable then $\bar{\sigma} > \sigma$. Since system (9) can be considered as a particular case of system (3) with

$$\Delta_0 = \sigma I, \quad \Delta_j = (e^{\sigma h_j} - 1)A_j, \quad j = \overline{1, m},$$

we can directly apply the approach of Section 3. In this case, the constants l_j, $j = \overline{0, 2m}$, depend on σ. They can be written in the form

$$l_0(\sigma) = M\big(\sigma\alpha + R_\sigma\big), \quad l_j(\sigma) = \alpha M(e^{\sigma h_j} - 1)\|A_j\|,$$
$$l_{m+j}(\sigma) = M\|A_j\|R_\sigma, \quad j = \overline{1, m},$$

where α and M are defined in Lemma 2, and

$$R_\sigma = \sigma + \sum_{k=1}^m (e^{\sigma h_k} - 1)\|A_k\|.$$

The next statement is a direct consequence of Theorem 6.

Theorem 8. Let system (1) be exponentially stable. If

$$R_\sigma < \frac{\lambda_{\min}(W)}{2\alpha M}, \tag{10}$$

then system (9) remains exponentially stable.

Theorem 8 provides a lower bound for the decay rate σ from Definition 1. The interesting point is that it is possible to construct the sequence containing the bounds from Theorem 8 that converges to the exact decay rate $\bar{\sigma}$. To do this, at each step we should find, with a given accuracy, the maximal σ satisfying (10), then use system (9) with this σ as the original system and repeat the procedure. To prove this fact, let us first suppose that system (9) with $\sigma = \xi$ is exponentially stable and denote by $U_\xi(\tau)$ its Lyapunov matrix associated with W. Then, introduce the function

$$F(\xi, \sigma) = 2M_\xi\left(1 + \sum_{j=1}^m h_j e^{\xi h_j}\|A_j\|\right) \times \left(\sigma + \sum_{k=1}^m (e^{\sigma h_k} - 1)e^{\xi h_k}\|A_k\|\right),$$

here $M_\xi = \max\limits_{\tau\in[0,h]} \|U_\xi(\tau)\|$. The inequality

$$F(\xi, \omega) < \lambda_{\min}(W) \tag{11}$$

represents condition (10), where system (9) with $\sigma = \xi$ plays the role of system (1), and system (9) with $\sigma = \xi + \omega$ plays the role of system (9). In other words, (11) means that system (9) with $\sigma = \xi + \omega$ is exponentially stable. Observe that $F(\xi, 0) = 0$ for every ξ, and $F(\xi, \omega)$ is a continuous function for $\xi \in [0, \bar{\sigma})$, $\omega \geqslant 0$.

Set $\sigma_0 = 0$ and consider the sequence

$$\sigma_k = \sigma_{k-1} + S_k - \varepsilon_k, \quad \text{where} \tag{12}$$
$$S_k = \sup\big\{\sigma \,\big|\, \sigma > 0,\ F(\sigma_{k-1}, \sigma) < \lambda_{\min}(W)\big\},$$

the numbers ε_k are such that

$$\varepsilon_k \xrightarrow[k\to+\infty]{} 0,$$

$0 \leqslant \varepsilon_k \leqslant S_k$, and $\varepsilon_k > 0$, if $S_k > 0$. Then, $\{\sigma_k\}_{k=0}^{+\infty}$ is the nondecreasing sequence, and

$$F(\sigma_{k-1}, \sigma_k - \sigma_{k-1}) < \lambda_{\min}(W).$$

For every k system (9) with $\sigma = \sigma_k$ is exponentially stable.

Theorem 9. The sequence $\{\sigma_k\}_{k=0}^{+\infty}$ converges, and
$$\lim_{k\to+\infty} \sigma_k = \bar{\sigma}.$$

Proof. First, $\{\sigma_k\}$ is the nondecreasing sequence, and $\sigma_k < \bar{\sigma}$ for every k. Therefore, the sequence has the limit $\tilde{\sigma} \leqslant \bar{\sigma}$. Suppose, by contradiction, that
$$\lim_{k\to+\infty} \sigma_k = \tilde{\sigma} < \bar{\sigma}.$$
Then, system (9) with $\sigma = \tilde{\sigma}$ is exponentially stable, and, of course, $F(\tilde{\sigma}, 0) = 0$.

On the other hand, by formula (12),
$$\sup\{\sigma > 0 \,|\, F(\sigma_{k-1}, \sigma) < \lambda_{\min}(W)\} = \sigma_k - \sigma_{k-1} + \varepsilon_k.$$
Then, for every $\delta_k > 0$ we have
$$F(\sigma_{k-1}, \sigma_k - \sigma_{k-1} + \varepsilon_k + \delta_k) \geqslant \lambda_{\min}(W).$$
Let us set $\delta_k = 1/k \xrightarrow[k\to+\infty]{} 0$. By the continuity of F, letting $k \to +\infty$, we obtain
$$F(\tilde{\sigma}, 0) \geqslant \lambda_{\min}(W) > 0.$$
The contradiction proves the theorem. $\square$

Theorem 9 shows that our approach allows to find the estimation of the decay rate which is arbitrarily close to $\bar{\sigma}$.

4.2 Estimation of γ-factor

In this subsection, we provide two alternative ways to estimate the γ-factor from Definition 1, in both cases the obtained γ depends on σ. Let M, α, η, μ, and R_σ stand for the same values as in the previous sections, and $U_\sigma(\tau)$ be the Lyapunov matrix of system (9) associated with W. Introduce the following values:
$$M_\sigma = \max_{\tau\in[0,h]} \|U_\sigma(\tau)\|, \quad \alpha_\sigma = 1 + \sum_{j=1}^{m} e^{\sigma h_j}\|A_j\|h_j,$$
$$\eta_\sigma = M_\sigma \alpha_\sigma^2, \quad K_\sigma = \|A_0 + \sigma I\| + \sum_{j=1}^{m} e^{\sigma h_j}\|A_j\|,$$
$$\beta_\sigma = M\Big(\alpha^2 + \alpha\sum_{j=1}^{m}(e^{\sigma h_j} - 1)\|A_j\|h_j + R_\sigma \sum_{j=1}^{m}\|A_j\|h_j^2\Big),$$
$$\mu_\sigma = \frac{\lambda_{\min}(W)\delta_\sigma}{4},$$
where δ_σ is the solution of the equation
$$\alpha_\sigma K_\sigma e^{K_\sigma \delta_\sigma} = \frac{1}{2\delta_\sigma}.$$
In other words, M_σ, K_σ, α_σ, η_σ, and μ_σ are the corresponding values of Section 2 constructed for system (9), where system (9) is supposed to be exponentially stable.

Theorem 10. Let system (1) be exponentially stable. Then
$$\|x(t, \varphi)\| \leqslant \sqrt{\frac{\eta_\sigma}{\mu_\sigma}} e^{-\sigma t}\|\varphi\|_h, \quad t \geqslant 0,$$
where σ is an arbitrary value such that system (9) is exponentially stable.

Proof. Take σ, and denote by $v_\sigma(\varphi)$ functional (2), corresponding to system (9). Let $y(t, \widetilde{\varphi})$ be the solution of system (9) with the initial function $\widetilde{\varphi}$. Since system (9) is exponentially stable, there exists $t^* \in \mathbb{R}$ such that
$$\|y(t, \widetilde{\varphi})\| \leqslant \|y(t^*, \widetilde{\varphi})\|, \quad t \geqslant -h.$$

If $t^* \geqslant 0$, then $y_{t^*} \in S$. Hence, by Theorem 3,
$$v_\sigma(y_{t^*}(\widetilde{\varphi})) \geqslant \mu_\sigma \|y(t^*, \widetilde{\varphi})\|^2.$$
Functional v_σ does not increase along the solutions of system (9), and the following inequalities hold
$$\begin{aligned}\|y(t, \widetilde{\varphi})\|^2 &\leqslant \|y(t^*, \widetilde{\varphi})\|^2 \leqslant \frac{1}{\mu_\sigma} v_\sigma(y_{t^*}(\widetilde{\varphi})) \\ &\leqslant \frac{1}{\mu_\sigma} v_\sigma(\widetilde{\varphi}) \leqslant \frac{\eta_\sigma}{\mu_\sigma}\|\widetilde{\varphi}\|_h^2.\end{aligned} \tag{13}$$
Hence,
$$\|y(t, \widetilde{\varphi})\| \leqslant \sqrt{\frac{\eta_\sigma}{\mu_\sigma}}\|\widetilde{\varphi}\|_h, \quad t \geqslant 0.$$
If $t^* \in [-h, 0]$, then $\|y(t, \widetilde{\varphi})\| \leqslant \|\widetilde{\varphi}\|_h$, $t \geqslant 0$, and the obtained estimate is also true (obviously, $\eta_\sigma \geqslant \mu_\sigma$).

By the construction of system (9) the solutions $x(t, \varphi)$ and $y(t, \widetilde{\varphi})$ and the initial functions φ and $\widetilde{\varphi}$ are connected by the relations
$$y(t, \widetilde{\varphi}) = e^{\sigma t}x(t, \varphi), \quad t \geqslant 0,$$
$$\widetilde{\varphi}(\theta) = e^{\sigma\theta}\varphi(\theta), \quad \theta \in [-h, 0].$$
Then,
$$\|\widetilde{\varphi}\|_h = \sup_{\theta\in[-h,0]} e^{\sigma\theta}\|\varphi(\theta)\| \leqslant \sup_{\theta\in[-h,0]} \|\varphi(\theta)\| = \|\varphi\|_h,$$
and, finally,
$$\|x(t, \varphi)\| = e^{-\sigma t}\|y(t, \widetilde{\varphi})\| \leqslant \sqrt{\frac{\eta_\sigma}{\mu_\sigma}} e^{-\sigma t}\|\varphi\|_h.$$
The theorem is proved. $\square$

Theorem 11. Let system (1) be exponentially stable. Then
$$\|x(t, \varphi)\| \leqslant \sqrt{\frac{\beta_\sigma}{\mu}} e^{-\sigma t}\|\varphi\|_h, \quad t \geqslant 0,$$
where σ satisfies inequality (10).

Proof. Take σ such that inequality (10) holds, then, due to Theorem 8, system (9) is exponentially stable. In the proof of the previous theorem, instead of the inequalities (13) write the following:
$$\|y(t, \widetilde{\varphi})\|^2 \leqslant \|y(t^*, \widetilde{\varphi})\|^2 \leqslant \frac{1}{\mu} v_0(y_{t^*}(\widetilde{\varphi})) \leqslant \frac{\beta_\sigma}{\mu}\|\widetilde{\varphi}\|_h^2.$$
The second inequality here is obtained directly from Theorem 3, as $y_{t^*} \in S$. Let us prove the third one. To this end, consider formula (7) with $t = t^*$. The second term in its right-hand side is nonnegative, so it can be dropped:
$$v_0(\widetilde{\varphi}) \geqslant v_0(y_{t^*}(\widetilde{\varphi})) - \sum_{j=1}^{m}\big(l_j(\sigma) + h_j l_{m+j}(\sigma)\big)\int_{-h_j}^{0}\|\widetilde{\varphi}(s)\|^2 ds.$$
Next, by Lemma 2, $v_0(\widetilde{\varphi}) \leqslant \eta\|\widetilde{\varphi}\|_h^2$, and we obtain
$$v_0(y_{t^*}(\widetilde{\varphi})) \leqslant \Big(\eta + \sum_{j=1}^{m}\big(l_j + h_j l_{m+j}\big)h_j\Big)\|\widetilde{\varphi}\|_h^2 = \beta_\sigma\|\widetilde{\varphi}\|_h^2.$$
The rest of the proof coincides with that of Theorem 10. $\square$

Remark 12. The problem to apply inequalities (13) directly with the functional v_0 lies in the fact that its time-derivative along the solutions of system (9) is not negative definite, hence the functional can, in general, increase. Inequality (7) allows to avoid this problem.

Remark 13. Notice that Theorems 10 and 11 deal with different values of σ. Theorem 10 is true for every σ such that system (9) is exponentially stable. In practice, we can take $\sigma = \sigma_k + \omega$, where ω is such that $F(\sigma_k, \omega) < \lambda_{\min}(W)$, $\forall\, k$. As for Theorem 11, it requires inequality (10). However, applying Theorem 11, we can also make a few iterations: Find σ, consider system (9) with this σ as the original system and then repeat the procedure.

Remark 14. In Kharitonov & Hinrichsen (2004), $\gamma = \sqrt{\dfrac{\alpha_2}{\alpha_1}}$ was obtained, where $\alpha_1 > 0$ and $\alpha_2 > 0$ are such that

$$\alpha_1 \|\varphi(0)\|^2 \leqslant v(\varphi) \leqslant \alpha_2 \|\varphi\|_h^2, \quad \varphi \in PC\big([-h, 0], \mathbb{R}^n\big),$$

and v is the complete-type functional (see Remark 7). In their approach, the estimation of σ is also derived through the bounds of the functional v, and there is no explicit dependence γ of σ.

5. EXAMPLES

In this section, we illustrate our work with examples. We use the semianalytic method (see Kharitonov, 2013) to compute the Lyapunov matrix, assuming $W = I$, if functional (2) is used, and $W_j = I$, $j = \overline{0, 2m}$, for the complete-type functional.

Example 15. Consider the system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} x(t-1),$$

whose robustness was analyzed in Kharitonov & Zhabko (2003). Application of condition (6) gives

$$\rho_0 + \rho_1 < 0.0481,$$

compare with $\rho_0 < 1.3 \times 10^{-4}$ and $\rho_1 < 1.5 \times 10^{-4}$ obtained in Kharitonov & Zhabko (2003), or $\sqrt{\rho_0^2 + \rho_1^2} < 0.0112$ obtained by use of the condition from Kharitonov (2013), see p. 125. Next, conditions (8) lead to

$$3.4142\rho_0 + \rho_1 < 0.0387, \; \rho_1 \leqslant 0.0160, \; \rho_0 + \rho_1 \leqslant 0.0274.$$

These inequalities are also more restrictive than our result.

Example 16. Consider system (1) with $m = 2$, $h_1 = 1$, $h_2 = 2$, and the matrices

$$A_0 = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \; A_1 = \begin{pmatrix} 0 & 0.7 \\ 0.7 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} -0.49 & 0 \\ 0 & -0.49 \end{pmatrix}.$$

In Kharitonov & Hinrichsen (2004), the exponential estimate for the solutions of this system with $\sigma \approx 0.046$ and $\gamma \approx 12.96$ was constructed. Applying the results of Section 4, we obtain

$$\sigma \approx 0.12, \quad \gamma_1(\sigma) \approx 23.373, \quad \gamma_2(\sigma) \approx 19.841,$$

where γ_1 and γ_2 are from Theorems 10 and 11 respectively. Taking $\sigma = 0.046$, we have

$$\gamma_1(\sigma) \approx 19.584, \quad \gamma_2(\sigma) \approx 18.496.$$

Thus, the σ obtained by our method is better whereas the γ is worse than the values obtained by use of the complete-type functional. As was mentioned, we can increase σ making it arbitrarily close to the absolute value of the spectral abscissa of the system, however, this will lead to the increase of $\gamma(\sigma)$.

6. CONCLUSION

In this paper, functional (2) is shown to be effective in the robust stability analysis as well as in the construction of the exponential estimates. Although the example shows that we did not improve the existing estimation of the γ-factor, see Definition 1, the possibility to use functional (2) for such kind of problems is demonstrated. Another example gives the improved robustness bounds, in comparison with the ones obtained with the help of the complete-type functional.

REFERENCES

R. Bellman & K.L. Cooke (1963). Differential-difference equations. Academic Press, New York.

A.V. Egorov & S. Mondiè (2014). Necessary stability conditions for linear delay systems. *Automatica*, 50, 3204–3208.

W. Huang (1989). Generalization of Liapunov's theorem in a linear delay system. *Journal of Mathematical Analysis and Applications*, 142, 83–94.

E.F. Infante & W.B. Castelan (1978). A Liapunov functional for a matrix difference-differential equation. *Journal of Differential Equations*, 29, 439–451.

E. Jarlebring, J. Vanbiervliet & W. Michiels (2011). Characterizing and computing the $\mathcal{H}_2$ norm of time-delay systems by solving the delay Lyapunov equation. *IEEE Transactions on Automatic Control*, 56(4), 814–825.

V.L. Kharitonov (1999). Robust stability analysis of time-delay systems: A survey. *Annual Reviews in Control*, 23, 185–196.

V.L. Kharitonov (2013). Time-Delay Systems: Lyapunov Functionals and Matrices. Birkhäuser, Basel.

V.L. Kharitonov & D. Hinrichsen (2004). Exponential estimates for time-delay systems. *Systems & Control Letters*, 53, 395–405.

V.L. Kharitonov & A.P. Zhabko (2003). Lyapunov – Krasovskii approach to the robust stability analysis of time-delay systems. *Automatica*, 39, 15–20.

N.N. Krasovskii (1956). On the second Lyapunov method application to the equations with delay. *Prikl. Mat. Mekh.*, 20, 315–327 (in Russian).

I.V. Medvedeva & A.P. Zhabko (2013). Constructive method of linear systems with delay stability analysis. In *Proceedings of 11th IFAC Workshop on Time-Delay Systems*, Grenoble, France, 1–6.

I.V. Medvedeva & A.P. Zhabko (2015). Synthesis of Razumikhin and LyapunovKrasovskii approaches to stability analysis of time-delay systems. *Automatica*, 51, 372–377.

S. Mondiè & V.L. Kharitonov (2005). Exponential estimates for retarded time-delay systems: An LMI approach. *IEEE Transactions on Automatic Control*, 50(2), 268–273.

S.-I. Niculescu, E.I. Verriest, L. Dugard & J.-M. Dion (1997). Stability of linear systems with delayed state: A guided tour. In *Stability and Control of Time-Delay Systems*, L. Dugard, E.I. Verriest (eds), Lecture Notes in Control and Information Sciences, Springer-Verlag, London, 228, 1–71.

G. Ochoa, V.L. Kharitonov & S. Mondie (2013). Critical frequences and parameters for linear delay systems: A Lyapunov matrix approach. *Systems & Control Letters*, 62, 781–790.

Y.M. Repin (1965). Quadratic Lyapunov functionals for systems with delay. *J. Appl. Math. Mech.*, 29, 669–672 (transl. of Prikl. Mat. Mekh., 29, 1965, pp. 564–566).

A.P. Zhabko & I.V. Medvedeva (2011). The algebraic approach to stability analysis of differential-difference systems. *Vestnik Saint-Petersburg State University*, 1, 9–20 (in Russian).