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GENERALIZED EXHAUSTERS: EXISTENCE, CONSTRUCTION, OPTIMALITY CONDITIONS

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Abstract. In this work a generalization of the notion of exhauster is considered. Exhausters are new tools in nonsmooth analysis introduced in works of Demyanov V.F., Rubinov A.M., Pshenichny B.N. In essence, exhausters are families of convex compact sets, allowing to represent the increments of a function at a considered point in an inf max or sup min form, the upper exhausters used for the first representation, and the lower one for the second representation. Using this objects one can get new optimality conditions, find descent and ascent directions and thus construct new optimization algorithms. Rubinov A.M. showed that an arbitrary upper or lower semicontinuous positively homogenous function bounded on the unit ball has an upper or lower exhausters respectively. One of the aims of the work is to obtain the similar result under weaker conditions on the function under study, but for this it is necessary to use generalized exhausters - a family of convex (but not compact!) sets, allowing to represent the increments of the function at a considered point in the form of inf sup or sup inf. The resulting existence theorem is constructive and gives a theoretical possibility of constructing these families. Also in terms of these objects optimality conditions that generalize the conditions obtained by Demyanov V.F., Abbasov M.E. are stated and proved. As an illustration of obtained results, an example of n-dimensional function, that has a non-strict minimum at the origin, is demonstrated. A generalized upper and lower exhausters for this function at the origin are constructed, the necessary optimality conditions are obtained and discussed.

1. **Introduction.**

Generalized exhausters, studied in this work, are further development of the notions of exhausters. These objects allow to consider more wide class of functions. The purpose of this work is to determine conditions of existence of generalized exhausters, describe new optimality conditions in their terms and obtain directions of ascent and descent when these conditions do not hold.

Let $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$ is an open set and let the following expansion hold

$$f(x+g) = f(x) + h_x(g) + o_x(g), \tag{1}$$

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where $o_x(g)$ satisfies one of the conditions:

$$\lim_{\alpha \downarrow 0} \frac{o_x(\alpha g)}{\alpha} = 0 \quad \forall\, g \in \mathbb{R}^n \tag{2}$$

or

$$\lim_{||g|| \to 0} \frac{o_x(g)}{||g||} = 0 \quad \forall\, g \in \mathbb{R}^n. \tag{3}$$

In the case $h_x(g) = \inf\limits_{C \in E^*(x)} \sup\limits_{v \in C} (v, g)$, where $E^*(x)$ is a family of convex sets in $\mathbb{R}^n$, $o_x(g)$ satisfies (2), we say, that $E^*(x)$ is a generalized upper exhauster of the function f at the point x in Dini's sense. If (3) is true, one speaks of a generalized upper exhauster in Hadamard's sense.

In the case $h_x(g) = \sup\limits_{C \in E_*(x)} \inf\limits_{v \in C} (v, g)$, where $E_*(x)$ is a family of convex sets in $\mathbb{R}^n$, and $o_x(g)$ satisfies (2), then we say, that $E_*(x)$ is a generalized lower exhauster of the function f at the point x in Dini's sense. If (3) is true, one speaks of a generalized lower exhauster in Hadamard's sense.

If $h_x(g)$ can be represented in the form $\min\limits_{C \in E^*(x)} \max\limits_{v \in C} (v, g)$ or $\max\limits_{C \in E_*(x)} \min\limits_{v \in C} (v, g)$, where $E^*(x)$ and $E_*(x)$ are convex compact sets, then one speaks of upper and lower exhausters respectively. If (2) is true, one speaks of an exhauster in Dini's sense, whereas when (3) hold, – of an exhauster in Hadamard's sense.

Remark 1. Note that the (generalized) exhauster of f at x coincides with the (generalized) exhauster of the functions h_x at 0_n. So unless otherwise stated, we will consider the function h_x separately and use the notation $h(g)$.

Remark 2. Note that unless otherwise stated, (generalized) exhausters will be considered in Dini's sense. As it is clear from the definition, the (generalized) exhauster in Hadamard's sense is also the (generalized) exhauster in Dini's sense.

The concept of exhauster has arised from the works of Demyanov V.F., Rubinov A.M., Pshenichny B.N. devoted to the study of nonconvex functions.

Pshenichny B.N. [12] introduced the concept of upper convex (u.c.a.) and lower concave approximations (l.c.a.). Let a positively homogeneous (p.h.) function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ be given. A convex p.h. function $\overline{h} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called an upper convex approximation of the function h, if

$$h(g) \leq \overline{h}(g) \text{ for all } g \in \mathbb{R}^n.$$

A concave p.h. function $\underline{h} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called a lower concave approximation of the function h, if

$$h(g) \geq \overline{h}(g) \text{ for all } g \in \mathbb{R}^n.$$

Rubinov A.M. introduced the concepts of exhaustive families of approximations [10]. A set Λ^* of upper convex approximations of the function h is called exhaustive, if

$$h(g) = \inf_{\overline{h} \in \Lambda^*} \overline{h}(g) \quad \forall g \in \mathbb{R}^n. \tag{4}$$

A set Λ_* of lower concave approximations of the function h is called exhaustive, if

$$h(g) = \sup_{\underline{h} \in \Lambda_*} \underline{h}(g) \quad \forall g \in \mathbb{R}^n. \tag{5}$$

Demyanov V.F. introduced the concept of exhauster [6]. Since any p.h. convex function $\overline{h}$ can be represented in the form $\overline{h}(g) = \max\limits_{v\in A(\overline{h})}(v,g)$, where $A(\overline{h})$ is subdifferential $\overline{h}$ at the null point, then (4) can be rewritten in the form

$$h(g) = \inf_{\overline{h}\in\Lambda^*}\max_{v\in A(\overline{h})}(v,g) = \inf_{C\in E^*}\max_{v\in C}(v,g), \quad \forall g\in\mathbb{R}^n,$$

where $E^* = \{A(\overline{h}) |\ \overline{h}\in\Lambda^*\}$.

Since any p.h. concave function $\underline{h}$ can be represented in the form $\underline{h}(g) = \min\limits_{v\in B(\underline{h})}(v,g)$, where $B(\underline{h})$ is superdifferential $\underline{h}$ at the zero point, then (5) be rewritten in the form

$$h(g) = \sup_{\underline{h}\in\Lambda_*}\min_{v\in B(\underline{h})}(v,g) = \sup_{C\in E_*}\min_{v\in C}(v,g), \quad \forall g\in\mathbb{R}^n,$$

$E_* = \{B(\underline{h}) |\ \underline{h}\in\Lambda_*\}$.

In [4] M.Castellani showed that if the function h is Lipschitz then it can be represented in the form

$$h(g) = h_1(g) = \min_{C\in E^*}\max_{v\in C}(v,g) \quad \forall g\in\mathbb{R}^n, \tag{6}$$

as well as in the form

$$h(g) = h_2(g) = \max_{C\in E_*}\min_{w\in C}(w,g) \quad \forall g\in\mathbb{R}^n, \tag{7}$$

where families of sets E^* and E_* are totally bounded. Recall that a family of sets E is totally bounded if there is a ball B from $\mathbb{R}^n$, such that $C\subset B$ for all $C\in E$. Functions h_1 and h_2 are p.h. (as functions of direction g) and therefore they are p. h. approximation of the increments of f in a neighborhood of the point x.

2. **Existence theorem.**

Theorem 2.1 (Rubinov [10])**.** *Let a function $h(g):\mathbb{R}^n\to\mathbb{R}$ be p.h. and bounded on the unit ball $\sup\limits_{||g||\le 1} h(g) < \infty$. Then, if h is upper semicontinuous, an upper exhauster of the function h at the point 0_n exists, i.e. there exists a family of convex compact sets E^* such that*

$$h(g) = \inf_{C\in E^*}\max_{v\in C}(v,g) \quad \forall g\in\mathbb{R}^n.$$

If h is lower semicontinuous, then a lower exhauster of the function h at the point 0_n exists, i.e. there exists a family of convex compact sets E_* such that

$$h(g) = \sup_{C\in E_*}\min_{v\in C}(v,g) \quad \forall g\in\mathbb{R}^n.$$

The following theorem extends Rubinov's theorem to the case of generalized exhausters.

Theorem 2.2. *Let a function $h(g):\mathbb{S}\to\mathbb{R}$ be given and bounded on the unit sphere $\mathbb{S} = \{g\in\mathbb{R}^n \big| ||g|| = 1\}$. For all $p\in\mathbb{S}$ construct the set*

$$C(p) = \{v\in\mathbb{R}^n \big| (v,p) < h(p)\}.$$

Then the following holds

$$h(g) = \inf_{p\in\mathbb{S}}\sup_{v\in C(p)}(v,g) = \inf_{C\in E^*}\sup_{v\in C}(v,g),$$

where

$$E^* = \{C(p) \big| p \in \mathbb{S}\}.$$

Proof. The representation

$$\inf_{p\in\mathbb{S}} \sup_{v\in C(p)} (v,g) \le \sup_{v\in C(g)} (v,g) = h(g) \tag{8}$$

is true. Indeed,

$$(v,g) < h(g) \quad \forall v \in C(g),$$

in addition, for any $\varepsilon > 0$ there exists $v_\varepsilon = \frac{g}{||g||^2}(h(g) - \frac{\varepsilon}{2}) \in C(g)$, for which

$$(v_\varepsilon, g) = h(g) - \frac{\varepsilon}{2} > h(g) - \varepsilon,$$

hence, from the definition of the least upper bound

$$\sup_{v\in C(g)} (v,g) = h(g).$$

Let us prove that in fact in (8) the equality holds. Assume the contrary. Let

$$\inf_{p\in\mathbb{S}} \sup_{v\in C(p)} (v,g) < h(g).$$

Denote

$$\inf_{p\in\mathbb{S}} \sup_{v\in C(p)} (v,g) - h(g) = -a < 0.$$

By the definition of infimum, there exists $\overline{p} \in \mathbb{S}$ such that

$$\sup_{v\in C(\overline{p})} (v,g) < \inf_{p\in\mathbb{S}} \sup_{v\in C(p)} (v,g) + \frac{a}{2} < h(g). \tag{9}$$

Note that $\overline{p} \ne g$. Each of sets $C(p)$, $p \in \mathbb{S}$, is an open half-space. The following situation can take place:

1. $[\mathbb{R}^n \backslash C(g)] \bigcap C(\overline{p}) \ne \emptyset$.

Then there exists a vector $u \in \mathbb{R}^n$ such that

$$\begin{cases} (u,g) \ge h(g), \\ u \in C(\overline{p}). \end{cases}$$

Thus $\sup\limits_{v\in C(\overline{p})} (v,g) \ge h(g)$, which contradicts (9).

2. $[\mathbb{R}^n \backslash C(g)] \bigcap C(\overline{p}) = \emptyset$.

Then $\overline{p} = -g$ (i.e. $[\mathbb{R}^n \backslash C(g)]$ and $C(\overline{p})$ are half-spaces, generated by the planes with normals $g, \overline{p}$ and if these half-spaces don't have common points then, obviously, planes are parallel to each other).

In this case, due to the boundedness of h on $\mathbb{S}$ there are finite λ, satisfying an inequality $\lambda \ge \max\{h(g); -h(\overline{p})\}$. At these λ vector $u = \lambda g \in C(\overline{p})$. Indeed, we have:

$$(u, \overline{p}) = (\lambda g, \overline{p}) = (\lambda g, -g) = -\lambda ||g||^2 = -\lambda \le h(\overline{p}).$$

Therefore if such a λ is chosen then

$$\sup_{v\in C(\overline{p})} (v,g) \ge (u,g) = (\lambda g, g) = \lambda ||g||^2 = \lambda \ge h(g),$$

which also contradicts (9).

So, finally we obtain

$$h(g) = \inf_{p \in \mathbb{S}} \sup_{v \in C(p)} (v, g) = \inf_{C \in E^*} \sup_{v \in C} (v, g).$$

$\square$

Similarly one can state and prove

Theorem 2.3. *Let a function $h(g)$ be given and bounded on the unit sphere $\mathbb{S} = \{g \in \mathbb{R}^n \big| ||g|| = 1\}$. For every $p \in \mathbb{S}$ construct the set*

$$C(p) = \{v \in \mathbb{R}^n \big| (v, p) > h(p)\}.$$

Then the following holds

$$h(g) = \sup_{p \in \mathbb{S}} \inf_{v \in C(p)} (v, g) = \sup_{C \in E_*} \inf_{v \in C} (v, g),$$

where

$$E_* = \{C(p) \big| p \in \mathbb{S}\}.$$

3. **Optimality conditions.** Optimality conditions in terms of exhausters were first obtained in works of Demyanov [2, 6–9, 11]. It turned out that the minimum condition are expresses by means of upper exhausters and maximum conditions - by means of lower ones, so upper exhausters are called proper for a minimization problems, and lower one - proper for a maximization problems. Correspondingly lower exhausters are called adjoint for minimization problems, and the upper adjoint for maximization problems. Expressions for the optimality conditions involving adjoint exhausters were obtained by V.A. Roshchina [14], but these conditions do not allow one to obtain directions of descent and ascent. In works [1, 3] new optimality conditions in terms of adjoint exhausters devoid of this shortcoming were presented. In this section we make an attempt to generalize the conditions of optimality in terms of generalized exhausters.

Theorem 3.1 (see [5])**.** *If in (1) $o_x(g)$ satisfies (2) and $h(x)$ is p.h., then the inequality*

$$h_x(g) \geq 0 \ \forall g \in \mathbb{S}$$

is a necessary condition for a minimum of f on X.

If it turns out that $o_x(g)$ satisfies (3), then the relation

$$h_x(g) > 0 \ \forall g \in \mathbb{S}$$

is a sufficient condition for a strict local minimum of f on X.

Theorem 3.2 (see [5])**.** *If in (1) $o_x(g)$ satisfies (2) and $h(x)$ is p.h., then the condition*

$$h_x(g) \leq 0 \ \forall g \in \mathbb{S}$$

is necessary for a maximum of f on X.

If it turns out that $o_x(g)$ satisfies (3), then the condition

$$h_x(g) < 0 \ \forall g \in \mathbb{S}$$

is sufficient for a strict local maximum of f on X.

Remark 3. From Theorems 3.1 and 3.2 it is clear, that if we deal with exhausters in Dini's sense, then the necessary condition for a local minimum of f at a point x can be stated in the form

$$\min_{C\in E^*(x)} \max_{v\in C}(v,g) \geq 0 \quad \text{or} \quad \max_{C\in E_*(x)} \min_{v\in C}(v,g) \geq 0 \quad \forall g \in \mathbb{S}$$

and conditions for a maximum – in the form

$$\min_{C\in E^*(x)} \max_{v\in C}(v,g) \leq 0 \quad \text{or} \quad \max_{C\in E_*(x)} \min_{v\in C}(v,g) \leq 0 \quad \forall g \in \mathbb{S}.$$

Remark 4. From Theorems 3.1 and 3.2 it is clear, that if we deal with generalized exhausters in Dini's sense, then the necessary condition for a local minimum of f at a point x can be stated in the form

$$\inf_{C\in E^*(x)} \sup_{v\in C}(v,g) \geq 0 \quad \text{or} \quad \sup_{C\in E_*(x)} \inf_{v\in C}(v,g) \geq 0 \quad \forall g \in \mathbb{S}$$

and conditions for a maximum – in the form

$$\inf_{C\in E^*(x)} \sup_{v\in C}(v,g) \leq 0 \quad \text{or} \quad \sup_{C\in E_*(x)} \inf_{v\in C}(v,g) \leq 0 \quad \forall g \in \mathbb{S}.$$

3.1 Optimality conditions in terms of proper generalized exhausters. As it was mentioned, first this conditions were presented in works [2, 6, 7, 9].

Theorem 3.3. *For the relation*

$$h(g) = \min_{C\in E^*} \max_{v\in C}(v,g) \geq 0 \;\forall g \in \mathbb{S}, \tag{10}$$

where E^* is a family of convex compacts in $\mathbb{R}^n$, to hold it is necessary and sufficient that

$$0_n \in \; C \quad \forall\, C \in E^*. \tag{11}$$

Theorem 3.4. *For the relation*

$$h(g) = \max_{C\in E_*} \min_{v\in C}(v,g) \leq 0 \;\forall g \in \mathbb{S}, \tag{12}$$

where E_* is a family of convex compacts in $\mathbb{R}^n$, to hold it is necessary and sufficient that

$$0_n \in \; C \quad \forall\, C \in E_*. \tag{13}$$

Let us state and prove a similar theorem for generalized exhausters.

Theorem 3.5. *For the relation*

$$h(g) = \inf_{C\in E^*} \sup_{v\in C}(v,g) \geq 0 \;\forall g \in \mathbb{S}, \tag{14}$$

where E^* is a family of convex sets in $\mathbb{R}^n$, to hold it is necessary and sufficient that

$$0_n \in cl\; C \quad \forall\, C \in E^*. \tag{15}$$

Proof. First, prove that (14) implies (15). Assume, on the contrary, that (14) is true, but there exists $\widetilde{C} \in E^*$ for which $0_n \notin cl\; \widetilde{C}$. Then by the separation theorem there are $\varepsilon > 0$, $\widetilde{\Delta} \in \mathbb{S}$, such that

$$(v, \widetilde{\Delta}) \leq -\varepsilon \quad \forall v \in \widetilde{C}.$$

Therefore

$$\sup_{v\in\widetilde{C}}(v, \widetilde{\Delta}) \leq -\varepsilon < 0,$$

whence

$$\inf_{C\in E^*}\sup_{v\in C}(v,g)<0,$$

which contradicts (14).

Now show that the opposite is true, i.e., that (15) yields (14). Let (15) be true, then for all C in E^* there is a sequence $v_k(C)\in C$ such that $v_k(C)\longrightarrow 0_n$ when $k\longrightarrow\infty$. It means that for all C an $\varepsilon>0$ there exists K_C such that the relation $||v_k(C)||\le\varepsilon$ is valid for all $k\ge K_C$.

Suppose that there exists $\widetilde{C}\in E^*$ and $\widetilde{\Delta}\in\mathbb{S}$, $\widetilde{\varepsilon}>0$ such that $\sup\limits_{v\in\widetilde{C}}(v,\widetilde{\Delta})<0$, i.e. there exists $\widetilde{\varepsilon}>0$ such that

$$\sup_{v\in\widetilde{C}}(v,\widetilde{\Delta})=-\widetilde{\varepsilon} \tag{16}$$

Taking $\varepsilon=\widetilde{\varepsilon}/2$, find $K_{\widetilde{C}}>0$, such that for all $k\ge K_{\widetilde{C}}$ the inequality $||v_k(\widetilde{C})||\le\widetilde{\varepsilon}/2$ holds. Then

$$(v_k(\widetilde{C}),\widetilde{\Delta})\ge-||v_k(\widetilde{C})||\;||\widetilde{\Delta}||\ge-\widetilde{\varepsilon}/2>-\widetilde{\varepsilon}\text{ and, besides, }\;v_k(\widetilde{C})\in\widetilde{C}.$$

Therefore the inequality

$$\sup_{v\in\widetilde{C}}(v,\widetilde{\Delta})>-\widetilde{\varepsilon},$$

is valid, which contradicts (16).

Therefore

$$\forall C\in E^*,\;\Delta\in\mathbb{S}\text{ the relation }\sup_{v\in C}(v,\Delta)\ge 0\text{ holds}$$

which implies

$$\inf_{C\in E^*}\sup_{v\in C}(v,\Delta)\ge 0\;\forall\Delta\in\mathbb{S}.$$

□

Remark 5. Note that if condition (15) does not hold, then there exists $C\in E^*$, such that $0_n\notin cl\ C$. Denote $\widetilde{E}^*=\{C\in E^*|\ 0_n\notin cl\ C\}$. Take $g_C\in\ cl\ C$, such that $||g_C||=\inf\limits_{v\in C,\ C\in\widetilde{E}^*}||v||$. Any such direction $-g_C/||g_C||$ (by the separation theorem [13]) is a direction of descent. At the same time any direction $\widetilde{g}\in\mathbb{S}$ which satisfies the equality

$$\inf_{C\in E^*}\sup_{v\in C}(v,\widetilde{g})=\inf_{g\in\mathbb{S}}\inf_{C\in E^*}\sup_{v\in C}(v,g)$$

is a direction of steepest descent.

Similarly, one can prove the following necessary condition for a maximum.

Theorem 3.6. *For the condition*

$$h(g)=\sup_{C\in E_*}\inf_{v\in C}(v,g)\le 0\;\forall g\in\mathbb{S}, \tag{17}$$

where E_* is a family of convex sets in $\mathbb{R}^n$, to hold, it is necessary and sufficient that

$$0_n\in cl\ C\quad\forall\ C\in E_*. \tag{18}$$

Remark 6. Note that if condition (18) does not hold, then exists $C \in E_*$, such that $0_n \notin cl\ C$. Denote $\widetilde{E}_* = \{C \in E_* | \ 0_n \notin cl\ C\}$. Take $g_C \in \ cl\ C$, such that $||g_C|| = \inf\limits_{v \in C,\ C \in \widetilde{E}_*} ||v||$. Any such direction $g_C/||g_C||$ (by the separation theorem [13]) is a direction of ascent. At the same time any direction $\widetilde{g} \in \mathbb{S}$ which satisfies the relation

$$\sup_{C \in E_*} \inf_{v \in C} (v, \widetilde{g}) = \sup_{g \in \mathbb{S}} \sup_{C \in E_*} \inf_{v \in C} (v, g)$$

is a direction of steepest ascent.

We now turn to the formulation of sufficient conditions for a strict local extremum. These conditions for exhausters for the first time appeared in [6].

Theorem 3.7. *For the condition*

$$h(g) = \min_{C \in E^*} \max_{v \in C} (v, g) > 0 \ \forall g \in \mathbb{S}, \tag{19}$$

where E^* is a family of convex compact sets in $\mathbb{R}^n$, to hold it is necessary and sufficient that

$$0_n \in \ int\ C \quad \forall\ C \in E^*. \tag{20}$$

Theorem 3.8. *For the condition*

$$h(g) = \max_{C \in E_*} \min_{v \in C} (v, g) < 0 \ \forall g \in \mathbb{S}, \tag{21}$$

where E_* is a family of convex compact sets in $\mathbb{R}^n$, to hold it is necessary and sufficient that

$$0_n \in \ int\ C \quad \forall\ C \in E_*. \tag{22}$$

Now we will derive a condition for a strict local minimum in terms of generalized exhausters.

Theorem 3.9. *To fulfill the condition*

$$h(g) = \inf_{C \in E^*} \sup_{v \in C} (v, g) > 0 \ \forall g \in \mathbb{S}, \tag{23}$$

where E^* is a family of convex sets in $\mathbb{R}^n$, it is necessary and sufficient that there exists $\delta > 0$ such that

$$B_\delta(0_n) \subset cl\ C \quad \forall\ C \in E^*, \tag{24}$$

where $B_\delta(0_n) = \big\{x \in \mathbb{R}^n \big| \ ||x|| \leq \delta\big\}$ is the ball of radius δ centered at zero.

Proof. We first prove *necessity*. Let (23) hold but (24) be not valid. Then for any $\delta > 0$ there exists $C_\delta \in E^*$, such that $B_\delta(0_n) \not\subset cl\ C_\delta$, whence we get, that there exists $g_\delta \in \mathbb{S}$, for which $g_\delta \delta \notin cl\ C_\delta$ is true. Therefore, by the separation theorem, there is a $\widehat{g}_\delta \in \mathbb{S}$ such that the inequality $(\widehat{g}_\delta, g_\delta \delta) > (\widehat{g}_\delta, v)$ holds for all $v \in C_\delta$. Thus, there exists $\widehat{g}_\delta \in \mathbb{S}$ for which

$$(\widehat{g}_\delta, v) < (\widehat{g}_\delta, g_\delta \delta) \leq ||\widehat{g}_\delta|| \ ||g_\delta \delta|| = \delta \quad \forall v \in C_\delta$$

is true. Furthermore, choosing a positive sequence $\{\delta_k\}$, that tends to zero ($\delta_k \downarrow 0$), we find corresponding sequences $\{C_{\delta_k}\}$, $\{\widehat{g}_{\delta_k}\}$, for which

$$(\widehat{g}_{\delta_k}, v) < \delta_k \forall v \in C_{\delta_k} \text{ and therefore } \sup_{v \in C_{\delta_k}} (\widehat{g}_{\delta_k}, v) \leq \delta_k.$$

Since $\{\widehat{g}_{\delta_k}\} \subset \mathbb{S}$, then without loss of generality, we can assume that $\widehat{g}_{\delta_k} \longrightarrow \widehat{g}$, where $\widehat{g} \in \mathbb{S}$. By the definition of convergence for all $\varepsilon > 0$ there is a $K > 0$ such that for every $k \geq K$ the inequality $||\widehat{g}_{\delta_k} - \widehat{g}|| < \varepsilon$ holds, i.e. $\widehat{g} \in \widehat{g}_{\delta_k} + B_\varepsilon(0_n)$. Therefore

$$(\widehat{g}, v) = (\widehat{g}_{\delta_k}, v) + (p, v) < \delta_k + \varepsilon, \quad p \in B_\varepsilon(0_n).$$

Selecting an arbitrary sequence $\{\varepsilon_l\}$, $\varepsilon_l \downarrow 0$, we find the corresponding sequence $\{K_l\}$, for which we get that for all $p \in \mathbb{N}$, $\forall v \in C_{\delta_k}$, $\forall k > K_l$, the inequality $(\widehat{g}, v) < \delta_k + \varepsilon_l$ holds. Thus,

$$\sup_{v \in C_{\delta_k}} (\widehat{g}, v) \leq \delta_k + \varepsilon_l.$$

Since $\delta_k \downarrow 0$, $\varepsilon_l \downarrow 0$ then

$$\inf_{C \in E^*} \sup_{v \in C} (\widehat{g}, v) \leq 0.$$

We now prove *sufficiency*. Since (24) is true, then

$$\sup_{v \in C} (v, g) \geq (g\delta, g) = \delta \quad \forall C \in E^*, \ \forall g \in \mathbb{S},$$

hence,

$$\inf_{C \in E^*} \sup_{v \in C} (v, g) \geq \delta > 0 \ \forall g \in \mathbb{S}.$$

□

Similarly one can prove the following condition for a strict local maximum in terms of generalized lower exhausters.

Theorem 3.10. *To fulfill the conditions*

$$h(g) = \sup_{C \in E_*} \inf_{v \in C} (v, g) < 0 \ \forall g \in \mathbb{S}, \tag{25}$$

where E_* is a family of convex sets in $\mathbb{R}^n$, it is necessary and sufficient that there exist $\delta > 0$ such that

$$\exists B_\delta(0_n) \subset cl\ C \quad \forall\ C \in E_*. \tag{26}$$

Here $B_\delta(0_n) = \left\{x \in \mathbb{R}^n \middle| \ ||x|| \leq \delta\right\}$ is the ball of radius δ centered at zero.

3.2 Optimality conditions in terms of adjoint generalized exhausters. The following results were first formulated in [1].

Theorem 3.11. *For the condition $h(g) = \max\limits_{C \in E_*} \min\limits_{v \in C} (v, g) \geq 0$ to hold for any $g \in \mathbb{S}$, where E_* is a family of convex compact sets in $\mathbb{R}^n$, it is necessary and sufficient that the closed positive half-space, generated by any hyperplane, passing through the origin, contain at least one set from the family E_*, i.e. for any $g \in \mathbb{S}$ a set $\widehat{C} \in E_*$ should exist such that for all $v \in \widehat{C}$ the inequality $(v, g) \geq 0$ holds.*

Theorem 3.12. *For the condition $h(g) = \min\limits_{C \in E^*} \max\limits_{v \in C} (v, g) \leq 0$ to hold for any $g \in \mathbb{S}$, where E_* – is a family of convex compact sets in $\mathbb{R}^n$, it is necessary and sufficient that the closed negative half-space, generated by any hyperplane, passing through the origin, contain at least one set from the family E^*, i.e. for any $g \in \mathbb{S}$ a set $\widehat{C} \in E^*$ should exists such that for all $v \in \widehat{C}$ the inequality $(v, g) \leq 0$ holds.*

Now we state and prove similar theorems for generalized exhausters.

Theorem 3.13. *For the condition*

$$h(g) = \sup_{C \in E_*} \inf_{v \in C} (v, g) \geq 0, \tag{27}$$

where E_* is a family of convex sets in $\mathbb{R}^n$, to hold for any $g \in \mathbb{S}$, it is necessary and sufficient that for any $g \in \mathbb{S}$ condition

$$\forall \varepsilon > 0 \ \exists C_\varepsilon \in E_* : \ (v, g) \geq -\varepsilon \ \forall v \in C_\varepsilon \tag{28}$$

be valid.

Proof. We first prove *necessity*. Let (27) be true. Choose and fix an arbitrary g in $\mathbb{S}$ and let

$$\sup_{C \in E_*} \inf_{v \in C} (v, g) = a, \ a \geq 0.$$

If $a > 0$, then, by the definition of least upper bound, there is a set $\widetilde{C}$ in E_*, such that $\inf_{v \in \widetilde{C}} (v, g) > a/2 > 0$, hence $(v, g) > 0$ for all v in $\widetilde{C}$, what means that (28) holds.

If $a = 0$, then directly by the definition of least upper bound we get that for any $\varepsilon > 0$ there is $C_\varepsilon \in E_*$, such that

$$\inf_{v \in C_\varepsilon} (v, g) > a - \varepsilon = -\varepsilon,$$

what implies that for any $\varepsilon > 0$ there is C_ε in E_*, for which $(v, g) > -\varepsilon$ for every v in C_ε, it means that (28) holds.

Now we prove *sufficiency*. Choose and fix an arbitrary g in $\mathbb{S}$ and let (28) hold. Take a sequence $\{\varepsilon_k\}$, such that $\varepsilon_k > 0$, $\varepsilon_k \longrightarrow 0$. Then we use it to find the corresponding sequence $\{C_{\varepsilon_k}\}$, for which $(v, g) \geq -\varepsilon_k$ for any $v \in C_{\varepsilon_k}$, i.e. $\inf_{v \in C_{\varepsilon_k}} (v, g) \geq -\varepsilon_k$. Assume that

$$\sup_{C \in E_*} \inf_{v \in C_{\varepsilon_k}} (v, g) = a, \ a < 0. \tag{29}$$

As $\varepsilon_k \longrightarrow 0$, there is $K > 0$, such that for any $k > K$ the inequality $a < -\varepsilon_k$ holds. Then

$$\inf_{v \in C_{\varepsilon_k}} (v, g) \geq -\varepsilon_k > a, \ \forall k > K,$$

whence

$$\sup_{C \in E_*} \inf_{v \in C} (v, g) > a,$$

what contradicts (29). Thus, $a \geq 0$, which means that (27) holds. □

Remark 7. If the necessary condition for a minimum from Theorem 3.13 is not satisfied, then

$$\exists \overline{g} \in \mathbb{S} \ \ \exists \overline{\varepsilon} > 0 : \ \forall C \in E_* \ \exists v_\varepsilon \in C \ \ (v_\varepsilon, g) < -\varepsilon.$$

Any such direction is a descent direction, and the direction $\widetilde{g} \in \mathbb{S}$ such that

$$\sup_{C \in E_*} \inf_{v \in C} (v, \widetilde{g}) = \inf_{g \in \mathbb{S}} \sup_{C \in E_*} \inf_{v \in C} (v, g)$$

is a direction of steepest descent.

Similarly, one can prove a theorem describing the condition for a maximum in terms of generalized upper exhauster, which is an adjoint one in this case.

Theorem 3.14. *For the condition* $h(g) = \inf_{C\in E^*} \sup_{v\in C}(v, g) \le 0$ *where* E^* *is a family of convex sets in* $\mathbb{R}^n$*, to hold for any* $g \in \mathbb{S}$*, it is necessary and sufficient that for every* $g \in \mathbb{S}$ *the condition*

$$\forall\varepsilon > 0\ \exists C_\varepsilon \in E^* :\ (v, g) \le \varepsilon\ \forall v \in C_\varepsilon$$

holds.

Remark 8. If the necessary condition for a maximum from Theorem 3.14 is not satisfied, then

$$\exists\overline{g} \in \mathbb{S}\ \ \exists\overline{\varepsilon} > 0 :\ \forall C \in E^*\ \exists v_\varepsilon \in C\ \ (v_\varepsilon, g) > \varepsilon.$$

Any such direction is an ascent direction, and the direction $\widetilde{g} \in \mathbb{S}$ such that

$$\inf_{C\in E^*} \sup_{v\in C}(v, \widetilde{g}) = \sup_{g\in\mathbb{S}} \inf_{C\in E^*} \sup_{v\in C}(v, g)$$

is a direction of steepest ascent.

Conditions for a strict minimum in terms of adjoint exhausters have the following form [1].

Theorem 3.15. *For the condition* $h(g) = \max_{C\in E_*} \min_{v\in C}(v, g) > 0$ *to hold for any* $g \in \mathbb{S}$*, where* E_* *is a family of convex compact sets in* $\mathbb{R}^n$*, it is necessary and sufficient that that the open positive half-space, generated by any hyperplane, passing through the origin, contain at least one set from the family* E_**, i.e. for any* $g \in \mathbb{S}$ *there must be a set* $\widehat{C} \in E_*$ *such that for all* $v \in \widehat{C}$ *the following inequality holds* $(v, g) > 0$*.*

Theorem 3.16. *For the condition* $h(g) = \min_{C\in E^*} \max_{v\in C}(v, g) < 0$ *to hold for any* $g \in \mathbb{S}$*, where* E^* *is a family of convex compact sets in* $\mathbb{R}^n$*, it is necessary and sufficient that that the open negative half-space, generated by any hyperplane, passing through the origin, contain at least one set from the family* E^**, i.e. for any* $g \in \mathbb{S}$ *there must be a set* $\widehat{C} \in E^*$ *such that for all* $v \in \widehat{C}$ *the following inequality holds* $(v, g) < 0$*.*

Now state and prove conditions for a strict extremum in terms of generalized exhausters.

Theorem 3.17. *For the condition*

$$h(g) = \sup_{C\in E_*} \inf_{v\in C}(v, g) > 0\ \forall g \in \mathbb{S}, \tag{30}$$

to hold, where E_* *is a family of convex sets in* $\mathbb{R}^n$*, it is necessary and sufficient that* $\widetilde{\varepsilon} > 0$ *can be found, such that*

$$\exists\widetilde{\varepsilon} > 0 : \forall g \in \mathbb{S}\ \ \exists C_g \in E_*,\ (v, g) > \widetilde{\varepsilon},\ \forall v \in cl\ C_g. \tag{31}$$

Proof. We now prove *necessity.* Let (30) be true. Choose and fix an arbitrary $g \in \mathbb{S}$. Then there is $a > 0$, such that

$$\sup_{C\in E_*} \inf_{v\in C}(v, g) = a.$$

By the definition of least upper bound, for $a/2$ one can find $C \in E_*$, for which $\inf_{v\in C}(v, g) > a/2$, whence

$$(v, g) > a/2\ \ \forall v \in cl\ C.$$

Thus taking $\widetilde{\varepsilon} = a/2$, we get (31).

Sufficiency is obvious. Indeed, let (31) be true, then

$$\inf_{v\in C_g}(v,g)\geq\varepsilon\ \ \forall g\in\mathbb{S},$$

therefore, finally,

$$\sup_{v\in E_*}\inf_{v\in C_g}(v,g)\geq\varepsilon>0\ \ \forall g\in\mathbb{S}.$$

□

Similarly one can state and prove

Theorem 3.18. *For the condition*

$$h(g)=\inf_{C\in E^*}\sup_{v\in C}(v,g)<0\ \forall g\in\mathbb{S}, \tag{32}$$

to hold, where E^* is a family of convex sets in $\mathbb{R}^n$, it is necessary and sufficient that $\widetilde{\varepsilon}>0$ can be found, such that

$$\forall g\in\mathbb{S}\ \exists C_g\in E^*,\ (v,g)<-\widetilde{\varepsilon},\ \forall v\in cl\ C_g. \tag{33}$$

4. **Examples.**

Example 4.1. Consider the function:

$$h(g)=\begin{cases}0, & \text{if } g/||g||\in\mathbb{Q}^n \text{ or } g=0_n\\ ||g||, & \text{if } g/||g||\notin\mathbb{Q}^n\end{cases},$$

at the origin, where $\mathbb{Q}$ is the set of rational numbers. Obviously, at the point 0_n this function has a non-strict global minimum and it is not locally lipschitz in a neighborhood of the origin. Construct a family $E^*(0_n)$ for this function. According to Theorem 2.2,

$$E^*(0_n)={E_0}^*(0_n)\bigcup {E_1}^*(0_n),$$

where

$${E_0}^*(0_n)=\{C_0(\Delta)|\ \Delta\in\mathbb{Q}^n,\ \Delta\in\mathbb{S}\},\ C_0(\Delta)=\{v\in\mathbb{R}^n|\ (v,\Delta)<0\},$$
$${E_1}^*(0_n)=\{C_1(\Delta)|\ \Delta\notin\mathbb{Q}^n,\ \Delta\in\mathbb{S}\},\ C_1(\Delta)=\{v\in\mathbb{R}^n|\ (v,\Delta)<1\}.$$

According to Theorem 2.3,

$$E_*(0_n)={E_*}^0(0_n)\bigcup {E_*}^1(0_n),$$

where

$${E_*}^0(0_n)=\{C^0(\Delta)|\ \Delta\in\mathbb{Q}^n,\ \Delta\in\mathbb{S}\},\ C^0(\Delta)=\{v\in\mathbb{R}^n|\ (v,\Delta)>0\},$$
$${E_*}^1(0_n)=\{C^1(\Delta)|\ \Delta\notin\mathbb{Q}^n,\ \Delta\in\mathbb{S}\},\ C^1(\Delta)=\{v\in\mathbb{R}^n|\ (v,\Delta)>1\}.$$

For any $C\in E^*(0_n)$ the relation $0_n\in cl\ C$ holds, what by Theorem 3.5 implies, that the necessary condition for a minimum in terms of an upper (proper) exhauster holds at the point 0_n. The maximum condition (see Theorem 3.14) in terms of an upper (adjoint) exhauster does not hold: for all $g\in\mathbb{S}$, $g\notin\mathbb{Q}^n$ and any $C\in E^*(0_n)$ one can find $v\in C$ for which $(v,g)>1/2$. Therefore, all directions $g\in\mathbb{S}$, $g\notin\mathbb{Q}^n$, according to Remark 8 are ascent directions and since for every such a g it holds that

$$\inf_{C\in E^*(0_n)}\sup_{v\in C}(v,g)=\sup_{\Delta\in\mathbb{S}}\inf_{C\in E^*(0_n)}\sup_{v\in C}(v,\Delta)=1,$$

all these directions are at the same time directions of steepest ascent.

If $g\in\mathbb{S}$ and $g\in\mathbb{Q}^n$, then for all $v\in C^0(g)$ inequality $(v,g)>0$ is true, whereas if $g\in\mathbb{S}$ and $g\notin\mathbb{Q}^n$, then for all $v\in C^1(g)$ the inequality $(v,g)>1$ is true. Therefore for all g in $\mathbb{S}$ the condition (28) from Theorem 3.13 holds. Hence at the point 0_n the necessary condition for a minimum in terms of the lower (adjoint) exhausters holds. The maximum condition in terms of lower (proper) exhauster (see Theorem 3.6) does not hold at the point 0_n. Indeed for all $C(g) \in {E_*}^1(0_n)$ (remind that $g \in \mathbb{S}$, $g \notin \mathbb{Q}^n$) we have $0_n \notin C$, $\inf\limits_{v \in C(g)} ||v|| = ||g|| = 1$. But all such directions satisfy the equality

$$\sup_{C \in E_*} \inf_{v \in C} (v, g) = \sup_{\Delta \in \mathbb{S}} \sup_{C \in E_*} \inf_{v \in C} (v, \Delta) = 1.$$

That is why directions $g \in \mathbb{S}$, $g \notin \mathbb{Q}^n$, are directions of steepest ascent.

Example 4.2. Consider the function

$$h(g) = \max\{1;\ 2g + 1\},\ g \in \mathbb{R},$$

at the origin. It obvious that $E_*(0) = \{\frac{2}{n} |\ n \in \mathbb{N}\}$. According to Theorem 3.13 the non-strict minimum condition in terms of lower (adjoint) exhauster holds at the point 0 . Indeed, as $g = \pm 1$, condition (28) means that for any $\varepsilon > 0$ there is $C_\varepsilon \in E_*(0)$, such that $-\varepsilon \le v \le \varepsilon$ for all $v \in C_\varepsilon$. In our case

$$\forall \varepsilon > 0\ \exists N > 0 :\ \forall n \ge N \text{ the relation } -\varepsilon \le \frac{2}{n} \le \varepsilon \text{ holds}$$

that means therefore the validity of (28). At the same time the necessary optimality condition for a maximum at the point 0 does not hold (see theorem 3.6): $0 \ne 2/n$ for all $n \in \mathbb{N}$, hence $\widetilde{E}_*(0) = E_*(0)$ whence

$$g_C = \inf_{v \in C,\ C \in \widetilde{E}_*(0)} ||v|| = \frac{2}{n}.$$

Therefore we have only one direction of ascent $g_C/||g_C|| = +1$ which is the direction of steepest ascent. Indeed,

$$\sup_{C \in E_*} \inf_{v \in C} (v, +1) = \sup_{\Delta \in \{-1;1\}} \sup_{C \in E_*} \inf_{v \in C} (v, \Delta) = 2.$$

5. **Conclusion.** The notion of generalized exhauster allows one to extend the class of functions which can be treated by this notion.

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