Multiprogram Stabilization Problem for the Mathematical Pendulum

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***Abstract*—In this paper, the model of mathematical pendulum is formulated as a non-linear dynamic system. The equilibrium positions of the dynamic system are obtained as a solution of corresponding problem of multiprogram stabilization. This solution is eventually formalized in a form of Hermit's polynomial.**

I. Introduction

Numerous applications (for example, [1], [2], [3]) require realization of a given set of motions. In such a case *the problem of synthesis of multiprogram stable controls* could be formulated [1], [3], [5]. In the present paper, this problem is solved for the quasi-linear time-invariant controlled system [3], [4]:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mu\mathbf{G}(\mathbf{x}, \mathbf{u}, \mu), \tag{1}$$

where $\mathbf{x}$ – n-dimensional state vector; $\mathbf{u}$ – r-dimensional control vector; $\mathbf{A} = \{a_{ij}\}$, $\mathbf{B} = \{b_{ik}\}$, $i = \overline{1,n}$, $j = \overline{1,n}$, $k = \overline{1,r}$, – constant matrices, $\mathbf{G}(\mathbf{x}, \mathbf{u}, \mu)$ – real continuously differentiable by $\mathbf{x}$ and $\mathbf{u}$; $\mu \geq 0$ – small parameter.

Suppose that a set of controls $\mathbf{u}_1(t), \ldots, \mathbf{u}_N(t)$ and a set of program motions $\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t)$ are constructed for the system (1). Note, that each program control $\mathbf{u}_j(t)$ and each program motion $\mathbf{x}_j(t)$, $j = \overline{1,N}$ are designed as the decision of certain boundary problem.

The **problem 1** of multiprogram stabilization for the system (1) consists of a controls $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, that realize the program motions $\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t)$ and guarantee the asymptotic stability.

II. Control synthesis

Consider the *multiprogram control*

$$\mathbf{u}(\mathbf{x}, t) = \sum_{j=1}^{N}\left(\mathbf{u}_j + \mathbf{C}(\mathbf{x}-\mathbf{x}_j) - 2\mathbf{u}_j \sum_{i=1, i\neq j}^{N} \frac{(\mathbf{x}_j - \mathbf{x}_i)(\mathbf{x} - \mathbf{x}_j)}{(\mathbf{x}_j - \mathbf{x}_i)^2}\right) p_j(\mathbf{x}), \tag{2}$$

where

$$p_j(\mathbf{x}) = \prod_{i=1, i\neq j}^{N} \frac{(\mathbf{x} - \mathbf{x}_i)^2}{(\mathbf{x}_j - \mathbf{x}_i)^2}. \tag{3}$$

Here the expressions $(\mathbf{x}_j - \mathbf{x}_i)(\mathbf{x} - \mathbf{x}_j)$, e $(\mathbf{x}_j - \mathbf{x}_i)^2$ are the scalar multiplication of corresponding vectors. Note that the function (2) is the Hermit's interpolating polynomial. The nodal location of the polynomial is a control motion $\mathbf{x}_j(t)$ and polynomial value is a program control $\mathbf{u}_j(t)$, $t = \overline{1,N}$. Actually, for the function (3) it is true, that $\mathbf{u}(\mathbf{x}_j(t), t) \equiv \mathbf{u}_j(t)$. Consider the following theorem.

Theorem 1. *Let the following conditions hold*

1) the homogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ *under control* $\mathbf{u} = \mathbf{C}\mathbf{x}$ *has the major stability margin;*

2) $\inf\limits_{t\geq 0} \parallel \mathbf{x}_i(t) - \mathbf{x}_j(t) \parallel > 0,\ i \neq j;$

3) $\frac{\mu\|\mathbf{G}(\mathbf{x},\mathbf{u},\mu)\|}{\|\mathbf{x}\|} \to 0$ *with* $\|\mathbf{x}\| \to 0$ *uniformly by t.*

Then the control (2), (3) exists and realize the given set of program motions $\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t)$.

Notice. The function $\mathbf{u}^k(\mathbf{x}, t)$ such as

$$\lim_{k\to+\infty} \mathbf{u}^k(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t)$$

could be used as a k-approximation of the control (2), (3), while selected regime $\mathbf{x}_j^k(t)$ ($\|\mathbf{x}_s(t) - \mathbf{x}_s^k(t)\| \leq \varepsilon_s$) could be used as a k-approximation of $\mathbf{x}_j(t)$ [3].

III. Model of Mathematical Pendulum

Consider the problem of multiprogram stabilization of some equilibrium positions for a model of the mathematical pendulum:

$$\ddot{x} + \gamma\dot{x} + \sin x = u,$$

where x is a deviation angle at vertical axis; u is a scalar control; $\gamma > 0$ is a friction coefficient.

Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{g}(\mathbf{x}) = \begin{pmatrix} 0 \\ x_1 - \sin x_1 \end{pmatrix}.$$

Then the equation of pendulum can be presented as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u + \mathbf{g}(\mathbf{x}). \tag{4}$$

Consider two defined constant vectors ($N = 2$):

$$\mathbf{x_1} = \begin{pmatrix} x_{10} \\ 0 \end{pmatrix}, \quad \mathbf{x_2} = \begin{pmatrix} x_{20} \\ 0 \end{pmatrix}.$$

To find an equilibrium positions of the system (4), it is necessary to find a control action from the equation:

$$\mathbf{A}\mathbf{x_j} + \mathbf{b}u_{pj} + \mathbf{g}(\mathbf{x_j}) = 0, \quad j = 1, 2. \tag{5}$$

From (9) we obtain $u_{p1} = \sin x_{10}$, $u_{p2} = \sin x_{20}$.

Then, the equation of the multiprogram control (2), (3) is

$$\mathbf{u}(\mathbf{x}) = ((u_{p1} + \mathbf{c}_1(\mathbf{x} - \mathbf{x}_1) -$$
$$-2u_{p1}\frac{(\mathbf{x}_1 - \mathbf{x}_2)(\mathbf{x} - \mathbf{x}_1)}{(\mathbf{x}_1 - \mathbf{x}_2)^2}))l_1(\mathbf{x}) + ((u_{p2} +$$
$$+\mathbf{c}_2(\mathbf{x} - \mathbf{x}_2) - 2u_{p2}(\frac{\mathbf{x}_2 - \mathbf{x}_1)(\mathbf{x} - \mathbf{x}_2)}{(\mathbf{x}_2 - \mathbf{x}_1)^2}))l_2(\mathbf{x}), \tag{6}$$

$$l_1(x) = \frac{(\mathbf{x} - \mathbf{x}_2)^2}{(\mathbf{x}_1 - \mathbf{x}_2)^2} = \frac{(\mathbf{x}_1 - \mathbf{x}_{20})^2 + \mathbf{x}_2^2}{(\mathbf{x}_{10} - \mathbf{x}_{20})^2},$$
$$l_2(x) = \frac{(\mathbf{x} - \mathbf{x_1})^{\mathbf{2}}}{(\mathbf{x_2} - \mathbf{x_1})^{\mathbf{2}}} = \frac{(\mathbf{x}_1 - \mathbf{x}_{10})^2 + \mathbf{x}_2^2}{(\mathbf{x}_{20} - \mathbf{x}_{10})^2}.$$

According to the Theorem 1, the closed loop system (4), (6) has the equilibrium positions $\mathbf{x_1}$, $\mathbf{x_2}$.

In the paper [4], there is an example of implementation of the multiprogram control function (6). Moreover, the process of stabilization for different deviation positions are investigated. Eventually, Matlab-code is given.

IV. Conclusion

The main result of the paper is a conceptual development of the multiprogram controls synthesis approach. For a further implementation of this technique, the more detailed algorithmic investigation and a code optimization should be reached.

References

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