DISCRETE MORSE THEORY FOR MODULI SPACES OF FLEXIBLE POLYGONS, OR SOLITAIR GAME ON THE CIRCLE

GAIANE PANINA AND ALENA ZHUKOVA

Abstract. We introduce a perfect discrete Morse function on the moduli space of a polygonal linkage.

The ingredients of the construction are: (1) the cell structure on the moduli space, and (2) the discrete Morse theory approach, which allows to reduce the number of cells to the minimal possible.

1. Introduction

The moduli space of a planar polygonal n-linkage has a cell decomposition in which cells are labeled by some cyclically ordered partitions of the set $[n] = \{1, ..., n\}$. The number of cells is big: it exceeds the sum of Betti numbers very much. Following R. Forman, we introduce a discrete Morse function on the cell complex. It turns to be a perfect Morse function. According to the discrete Morse theory, this gives a way of contracting some of the cells such that the number of remaining cells is the minimal possible. The rules of manipulating with the cells, and the rules describing gradient paths resemble the solitair game. However, this analogy should not be taken too seriously: it is a mere metaphor, not a mathematical statement.

The perfect Morse function is constructed in two steps. On the first step, we introduce some natural pairing on the cell complex which substantially reduces the number of critical cells. However, this number is not yet minimal possible.

On the second step we (following once again R. Forman) apply *path reversing technique*, which gives a perfect Morse function.

Using our approach, it is possible to compute homology groups of the configuration space of a polygonal linkage independently on the proof of M. Farber and D. Schütz [2]. However, such a proof does not seem to be a short one, so we do not give the details here.

To the best of our knowledge, no smooth perfect Morse function on the moduli space of a polygonal linkage is known. This motivates us to formulate the following open problem:

What is the smooth counterpart of the proposed discrete Morse function?

Key words and phrases. Polygonal linkage, cell complex, CW-complex, configuration space, moduli space, discrete vector field, discrete Morse theory, perfect Morse function
.

Acknowledgements. The present research is supported by RFBR project No. 15-01-02021. We also express our gratitude to all the participants of the seminar "Geometry and combinatorics" of Chebyshev Laboratory of St. Petersburg State University for inspiring discussions. In particular, we are much indebted to Pavel Galashin for producing the starting idea of the pairing.

2. Preliminaries

We start with two necessary remindings.

Cell complex on the moduli space [5]**.** A *polygonal n-linkage* is a sequence of positive numbers $L = (l_1, \dots, l_n)$. It should be interpreted as a collection of rigid bars of lengths l_i joined consecutively in a chain by revolving joints. We always assume that the triangle inequality holds, that is,

$$\forall j, \quad l_j < \frac{1}{2}\sum_{i=1}^{n} l_i$$

which guarantees that the chain of bars can close.

A planar configuration of L is a sequence of points

$$P = (p_1, \dots, p_n), \ p_i \in \mathbb{R}^2$$

with $l_i = |p_i, p_{i+1}|$, and $l_n = |p_n, p_1|$. We also call P a *polygon.*

As follows from the definition, a configuration may have self-intersections and/or self-overlappings.

Definition 2.1. *The moduli space, or the configuration space* $M(L)$ is the set of all configurations modulo orientation preserving isometries of $\mathbb{R}^2$.

Equivalently, we can define $M(L)$ as

$$M(L) = \{(u_1, ..., u_n) \in (S^1)^n : \sum_{i=1}^{n} l_i u_i = 0\}/SO(3).$$

The latter definition shows that $M(L)$ does not depend on the ordering of $\{l_1, ..., l_n\}$; however, it does depend on the values of l_i.

Throughout the paper we assume that no configuration of L fits a straight line. This assumption implies that the moduli space $M(L)$ is a closed $(n-3)$-dimensional manifold.

We remind the reader the explicit combinatorial description of $M(L)$ as a regular cell complex $\mathcal{K}(L)$.

A subset I of $[n] = \{1, 2, ..., n\}$ is *short* if

$$\sum_{I} l_i < \frac{1}{2}\sum_{1}^{n} l_i.$$

A partition of $[n] = \{1, 2, ..., n\}$ is called *admissible* if all the parts are short.

The set containing the entry "n" is called *the n-set.*

A *singleton* is a set containing exactly one entry.

A remark on notation. We write a cyclically partition as a (linearly ordered) string of sets where the n-set stands on the last position.

We stress once again that the order of the sets matters, whereas there is no ordering inside a set. For example,

$$\big(\{1\}\{3\}\{4,2,5,6\}\big) \neq \big(\{3\}\{1\}\{4,2,5,6\}\big) = \big(\{3\}\{1\}\{2,4,5,6\}\big).$$

Before we describe the cell complex, remind that a CW-complex can be constructed inductively by defining its skeleta. Once the $(k-1)$-skeleton is constructed, we attach a collection of closed k-balls C_i by some continuous mappings φ_i from their boundaries ∂C_i to the $(k-1)$-skeleton. For a *regular* complex, each of the mappings φ_i is injective, and φ_i maps ∂C_i to a subcomplex of the $(k-1)$-skeleton. Regularity of a complex implies that a complex is uniquely defined by the poset of its cells. Regularity also guarantees the existence of well-defined barycentric subdivision and (for manifolds) the well-defined dual complex.

Theorem 2.2. *We have a structure of a regular CW-complex $\mathcal{K}(L)$ on the moduli space $M(L)$. Its complete combinatorial description reads as follows:*

(1) *k-dimensional cells of the complex $\mathcal{K}(L)$ are labeled by cyclically ordered admissible partitions of the set $[n]$ into $(n-k)$ non-empty parts.*
(2) *A closed cell C belongs to the boundary of some other closed cell C' iff the partition $\lambda(C)$ is finer than $\lambda(C')$* $\square$.

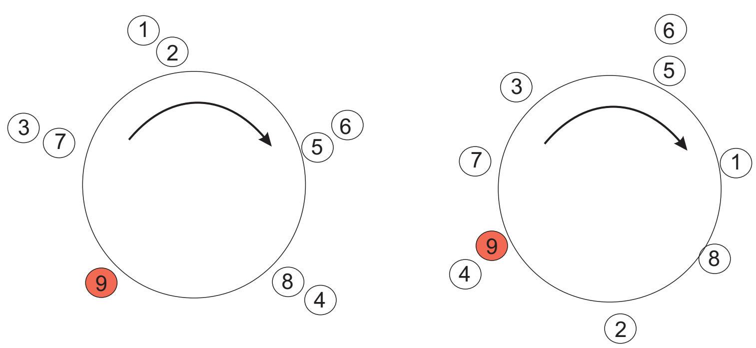

FIGURE 1. A 4-cell and a 2-cell. We write these labels as $\big(\{3,7\}\{1,2\}\{5,6\}\{4,8\}\{9\}\big)$ and $\big(\{7\}\{3\}\{5,6\}\{1\}\{8\}\{2\}\{4,9\}\big)$

In the sequel, instead of saying "the cell of the complex labeled by λ" we say for short "*the cell* λ".

Given a cell λ, its facets are obtained by splitting one of the parts of the partition λ into two non-empty parts. For example, the cells $\big(\{7\}\{3\}\{1,2\}\{5,6\}\{4,8\}\{9\}\big)$ and $\big(\{3\}\{7\}\{1,2\}\{5,6\}\{4,8\}\{9\}\big)$ are facets of the cell $\big(\{3,7\}\{1,2\}\{5,6\}\{4,8\}\{9\}\big)$

Let us explain in some more details how the cell structure appears. We put *labels* on the elements of the configuration space: according to the Definition 2.1, each configuration is a collection of unit vectors $\{u_i\}$. If the vectors are different, there is an induced cyclic ordering on $[n]$. If some of them coincide, there arises a cyclically ordered partition of $[n]$, whose parts correspond to coinciding sets of vectors. Clearly, all the labels are admissible partitions.

Two points from $M(L)$ (that is, two configurations) are *equivalent* if they have one and the same label. Equivalence classes of $M(L)$ are the *open cells*. The closure of an open cell in $M(L)$ is called a *closed cell*. For a cell C, either closed or open, its label $\lambda(C)$ is defined as the label of (any) its interior point. The collection of open cells yields a structure of a regular CW-complex which is dual to the complex $\mathcal{K}(L)$.

Discrete Morse function on a regular cell complex [3]**.** Assume we have a regular cell complex. By α^p, β^p we denote its p-dimensional cells, or p-*cells*, for short.

A *discrete vector field* is a set of pairs

$$\left(\alpha^p, \beta^{p+1}\right)$$

such that:

(1) each cell of the complex participates in at most one pair, and
(2) in each pair, the cell α^p is a facet of β^{p+1}.

Given a discrete vector field, a *path* is a sequence of cells

$$\alpha_0^p,\ \beta_0^{p+1},\ \alpha_1^p,\ \beta_1^{p+1},\ \alpha_2^p,\ \beta_2^{p+1}, ..., \alpha_m^p,\ \beta_m^{p+1},\ \alpha_{m+1}^p,$$

which satisfies the conditions:

(1) Each $\left(\alpha_i^p,\ \beta_i^{p+1}\right)$ is a pair.
(2) Whenever α and β are neighbors in the path, α is a facet of β.
(3) $\alpha_i \neq \alpha_{i+1}$.

A path is a *closed path* if $\alpha_{m+1}^p = \alpha_0^p$.

A *discrete Morse function on a regular cell complex* is a discrete vector field without closed paths.

Assuming that a discrete Morse function is fixed, the *critical cells* are those cells of the complex that are not paired. Morse inequality says that we cannot avoid them completely. However, our goal is to minimize their number.

A *gradient path* of a discrete Morse function leading from one critical cell β^{p+1} to some another critical cell α^p is the sequence of cells:

$$\beta^{p+1},\ \alpha_0^p,\ \beta_0^{p+1},\ \alpha_1^p,\ \beta_1^{p+1},\ \alpha_2^p,\ \beta_2^{p+1},\ ...,\ \alpha_m^p,\ \beta_m^{p+1},\ \alpha^p$$

satisfying the three above conditions.

A discrete Morse function is a *perfect Morse function* whenever the number of critical k-cells equals the k-th Betty number of the complex. It is equivalent to the condition that the number of all critical cells equals the sum of Betty numbers.

3. Pairing on the complex $\mathcal{K}$: "rules of the game".

Assume that a linkage $L = (l_1, ..., l_n)$ is fixed. Without loss of generity we may assume that

$$l_n \geq l_{n-1} \geq ... \geq l_1.$$

First we give some **notation**:

(1) By "$\cdots$" we denote any ordered admissible collection of subsets of $[n]$, which as well can be the empty set.
(2) By "$*$" we denote any subset of $[n]$, which as well can be the empty set.
(3) A set $I \subset [n]$ is *k-prelong*, if I is short, and $I \cup \{k\}$ is long.
(4) For a set $I \subset [n]$ and an entry $k \in [n]$, we write $k < I$ whenever $\forall i \in I, \quad k < i$.
(5) Analogously, we write $k = Min(I)$ whenever k is the minimal entry of the set I.

Below we describe a Morse function. According to the definition, we introduce some pairings of the cells.

Step 1. We pair together

$$\alpha = \big(\cdots\ \{1\}\ I\ \ \cdots\big) \text{ and } \beta = \big(\cdots\ \{1\} \cup I\ \ \cdots\big)$$

iff the following holds:

(1) the set I does not contain n, and
(2) the set $\{1\} \cup I$ is short.

Before we pass to step 2, observe that the non-paired cells are labeled by one of the following types of labels:

$$\big(\cdots\ \{n, 1, *\}\big)$$
$$\big(\cdots\ \{1\}\ \{n, *\}\big))$$
$$\big(\cdots\ \{1\}\ \big(\text{a 1-prelong set}\big)\ \cdots\big)$$

Step 2. We pair together

$$\alpha = \big(\cdots\ \{2\}\ I\ \ \cdots\big) \text{ and } \beta = \big(\cdots\ \{2\} \cup I\ \ \cdots\big)$$

iff the following holds:

(1) The set I contains neither n, nor 1.
(2) The set $\{2\} \cup I$ is short.
(3) α and β were not paired at the previous step.

After this step, the non-paired cells are labeled by one of the following types of labels:

$$\big(\cdots\ \{n, 1, 2, *\}\big)$$
$$\big(\cdots\ \{1\}\ \{n, 2, *\}\big)$$
$$\big(\cdots\ \{2\}\ \{n, 1, *\}\big)$$

$$\big(\cdots\ \{2\}\ \{1\}\{n,*\}\big)$$

$$\big(\cdots\ \{2\}\ \{1\}\big(\text{ a 1-prelong set}\big)\ \cdots\big)$$

$$\big(\cdots\ \{1\}\ \big(\text{a 1-prelong set}\big)\cdots\{2\}\{n,*\}\big)$$

$$\big(\cdots\ \{2\}\ \big(\text{a 2-prelong set not containing 1}\big)\ \cdots\ \{1\}\{n,*\}\big)$$

$$\big(\cdots\ \{1\}\ \big(\text{a 1-prelong set not containing 2}\big)\ \cdots\ \{n,2,*\}\big)$$

$$\big(\cdots\ \{2\}\ \big(\text{a 2-prelong set not containing 1}\big)\ \cdots\ \{n,1,*\}\big)$$

We proceed this way for all $k < n$, assuming that the step number k looks as follows:

Step k. We pair together

$$\alpha = \big(\cdots\ \{k\}\ I\ \cdots\big) \text{ and } \beta = \big(\cdots\ \{k\} \cup I\ \cdots\big)$$

iff the following holds:

(1) The set I contains none of $n, 1, 2, ..., k-1$.
(2) The set $\{k\} \cup I$ is short.
(3) α and β were not paired at the previous steps.

We proceed pairing for all $k = 1, 2, ..., n-1$.

Pair search algorithm. Now we describe an algorithm that finds a pair (if there is one) for a given cell α. It is a necessary tool for finding gradient paths.

In every pair of cells from the discrete vector field, the cells differ by moving one entry either inside or outside one of the sets. Note that no pairing changes the n-set, so we can ignore the n-set.

An entry k is *forward-movable* (with respect to the cell α), if it forms a singleton, which is followed by a set I, $n \notin I$ such that

(1) $k < i$ for every $i \in I$, and
(2) $\{k\} \cup I$ is short.

An entry k is *backward-movable* if the following holds:

(1) entry k lies in a non-singleton set J, $n \notin J$;
(2) $k = Min(J)$;
(3) one of the following conditions holds:
    (a) the set J is preceded by a non-singleton set;
    (b) the set J is preceded by a singleton $\{m\}$ with $m > k$;
    (c) the set J is preceded by the n-set.

In this notation, the **algorithm** looks as follows:

Given a cell α, take the minimal movable entry k in α. Then the cell α is paired with a cell that is formed from α by moving k either forward, or backward, according to pairing step number k.

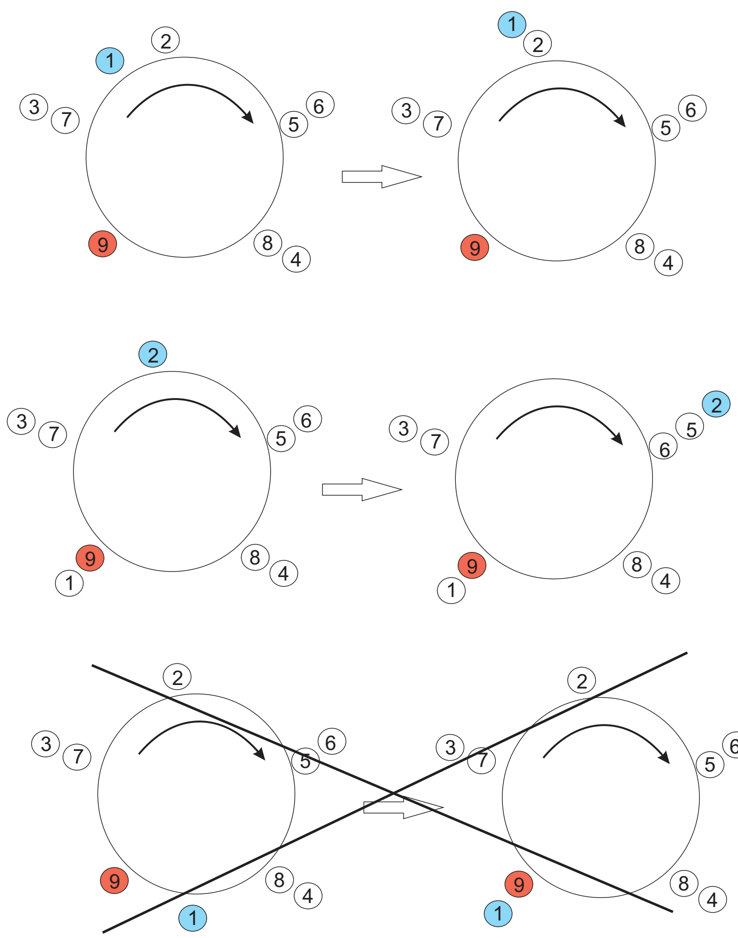

FIGURE 2. Pairing in the complex: example and a non-example.

Lemma 3.1. *Assume we have a gradient path in the complex. Assume also that for some cell α from the path and for two entries $m < k$ belonging to different sets, k is placed to the left with respect to m. That is,*

$$\alpha = \big(\cdots \ \{k, *\} \ \cdots \ \{m, *\} \ \cdots\big)$$

Then during the gradient path after the cell α, the entries k and m never get in one and the same set and never change their order. □

The lemma implies the proposition:

Proposition 3.2. *The introduced discrete vector field is a discrete Morse function.*

Proof. We have to show that there are no closed gradient paths. Indeed, the above lemma implies that in any closed path no two entries interchange their order. □

4. Critical cells of the complex $\mathcal{K}$

Let us list all the critical cells (that is, the cells that are non-paired). They are exactly those with empty set of movable entries. We classify them by types and give examples and non-examples.

Notation: unlike "$\cdots$", by "$\spadesuit$" and "$\clubsuit$" we denote a (possibly empty) string of singletons going in the decreasing order. For instance, "$\spadesuit$" can be $\big(\{7\}\{5\}\{3\}\big)$, but can be neither $\big(\{7,5,3\}\big)$ nor $\big(\{5\}\{3\}\{7\}\big)$.

Theorem 4.1. *The critical cells of the introduced above discrete Morse function are exactly all cells of the two following types.*

Type 1.

$$\big(\spadesuit\ \{n,*\}\big).$$

Examples:

(1) $\big(\{7\}\{5\}\{3\}\{8,1,2,4,6\}\big)$,
(2) $\big(\{7\}\{6\}\{5\}\{3\}\{2\}\{1\}\{8,4\}\big)$.

Non-examples:

(1) The cell $\big(\{7,5\}\{3\}\{8,1,2,4,6\}\big)$ is non-critical because it is paired with $\big(\{5\}\{7\}\{3\}\{8,1,2,4,6\}\big)$. Here 5 is a movable entry.
(2) The cell $\big(\{7\}\{5\}\{6\}\{3\}\{2\}\{1\}\{8,4\}\big)$is non-critical because it is paired with $\big(\{7\}\{5,6\}\{3\}\{2\}\{1\}\{8,4\}\big)$. Here singletons do not come in decreasing order, 5 is a movable entry.

Type 2.

$$\big(\spadesuit\ \{k\}\ I\ \clubsuit\ \{n,*\}\big),\ \text{if the following conditions hold:}$$

(1) I is a k-prelong set not containing n.
(2) $k < I$.
(3) $k < \spadesuit$.
(In other words, $\big(\spadesuit\ \{k\}\big)$ is an ordered string of singletons.)

Example:

$\big(\{5\}\{3\}\{6,4\}\{1\}\{7,2\}\big)$ is a critical cell assuming that $\{6,4\}$ is 3-prelong.

Non-examples:

(1) The cell $\big(\{5\}\{7\}\{3\}\{6,4\}\{1\}\{8,2\}\big)$ is non-critical because it is paired with $\big(\{5,7\}\{3\}\{6,4\}\{1\}\{8,2\}\big)$.
(2) The cell $\big(\{7\}\{5\}\{3\}\{6,2\}\{1\}\{8,4\}\big)$ is also non-critical since it violates condition 2. The cell is paired with $\big(\{7\}\{5\}\{3\}\{2\}\{6\}\{1\}\{8,4\}\big)$.

Proof. Clearly, all the above cells have no movable entries and therefore are critical. To prove the converse, consider two cases for a critical cell α:

(1) The partition α consists only of singletons. Then the singletons necessarily go in decreasing order, otherwise there exists a forward-movable entry. Thus we get a critical cell of type 1.
(2) The partition α contains some non-singleton sets. Each non-singleton is either a prelong set (with respect to its preceding entry), or the n-set; otherwise a simple case analysis yields a movable entry.

□

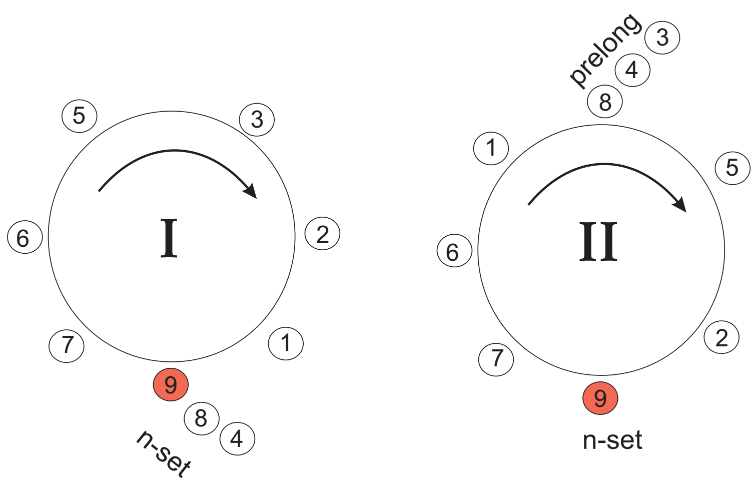

FIGURE 3. Critical cells

Example 4.2. *Assume that* $L = (1, 1, ..., 1, (n - 1 - \varepsilon))$. *In this case the configuration space* $M(L)$ *is known to be the* $(n-3)-$*sphere. The (only two) critical cells of the Morse function are*

$$\big(\{n-1\}...\{3\}\{2\}\{1\}\{n\}\big)$$

and

$$\big(\{1\}\{n-1, ..., 3, 2\}\{n\}\big),$$

that is, we have a perfect Morse function for this particular case.

Example 4.3. *Another example when we have a perfect Morse function is given by* $L = \big(\varepsilon, \varepsilon, \varepsilon, ..., \varepsilon, 1, 1, 1\big)$ *The configuration space* $M(L)$ *equals the disjoint union of two tori. The critical cells are labeled either by*

$$\big(\{n-1\}\{n-2\}\clubsuit\{n, *\}\big), \textit{(Type 1)}$$

or by

$$\big(\{n-2\}\{n-1\}\clubsuit\{n, *\}\big), \textit{(Type 2)}$$

so one easily concludes that the number of critical cells of a fixed dimension k equals the Betti number $b_k(M(L))$.

However, the above two examples are very exceptional: in other cases the introduced Morse function is far from perfect. Rough estimates show that the number of critical cells is much bigger than the sum of Betti numbers.

5. Gradient paths between critical cells

According to the definition, a gradient path in our setting is an alternating sequence of *join-steps* (connecting α_i^p and β_i^{p+1}), and *split-steps* (choosing a facet α_{i+1}^p of β_i^{p+1}). A gradient path always starts by a split and ends also by a split.

A join-step decreases the number of sets in the partition by one, whereas a split-step increases the number of sets by one.

Each join-step is uniquely defined according to our pairing algorithm: it is performed by moving the minimal movable entry, which moves forward and joins the consequent set.

Another important remark is that if one starts a series of steps with a cell β^{p+1}, one does not necessarily arrive at some critical cell α^p. This is similar to a solitair player, who not always wins, but sometimes gets stuck. Below we exemplify "successful" solitair games. The reader can try some other types of splitting and work out some loosing examples.

So we have some freedom for a split-step, but in many cases the freedom is illusive: the split-step for a gradient path often is defined uniquely. Indeed, if after some split-step the smallest movable entry is backward-movable, there exists no consecutive join-step.

Assume we have a gradient path from a critical cell $\beta = \big(\spadesuit_1\ \{j_1\}\ I_1\ \clubsuit_1\ \{n, *_1\}\big)$ to a critical cell $\alpha = \big(\spadesuit_2\ \{j_2\}\ I_2\ \clubsuit_2\ \{n, *_2\}\big)$. We say that *the prelong set I is maintained during the gradient path* if each cell of the path has a set containing I. In other words, during the path, the set I may accept and lose new entries, but it may not lose its initial entries.

Lemma 5.1. *Assume we have a gradient path from a critical cell $\beta = \big(\spadesuit_1\ \{j_1\}\ I_1\ \clubsuit_1\ \{n, *_1\}\big)$ to a critical cell $\alpha = \big(\spadesuit_2\ \{j_2\}\ I_2\ \clubsuit_2\ \{n, *_2\}\big)$.*

If $I_1 \neq I_2$, then the entry j_2 belongs to $*_1$.

Proof. Consider the join-step after which the set I_2 appears in the path and stays maintained until the end. On this step, the entry $k = Min(I_2)$ joins the set $I_2 \setminus \{k\}$. Since k is the minimal movable entry, for $j_2 < k$ there are only two possibilities: (1) either j_2 is in the n-set, or (2) j_2 goes after I_2. The second case is excluded, since no entry can pass through the n-set. $\square$

The lemma together with a case analysis allows us to describe the gradient paths between critical cells. We do not present the complete list of all possible gradient paths, since we actually do not need all of them. The point is that in the next section we are going to reduce the number of the critical cells using path reversing, and arrive at a perfect Morse function.

Proposition 5.2. *There are no gradient paths from a critical cell of type 1 to a critical cell of type 2.*

Proof. Assume that there is a path leading from the cell $\beta = \big(\spadesuit_1 \ \{n, *_1\}\big)$ to the cell $\alpha = \big(\spadesuit_2 \ \{k\} \ I \ \clubsuit \ \{n, *_2\}\big)$.

Then by Lemma 3.1, not more than one singleton j from $\spadesuit_1$ belongs to I. Moreover, since all others entries of I eventually join it, we have $j = Max(I)$. All other entries of I and also k come from $*_1$. So we necessarily have

$$\big(I \setminus \{Max(I)\}\big) \cup \{k\} \subseteq *_1.$$

Since $n > Max(I)$, the set $\{n, *_1\}$ is longer than $Max(I) \cup \{*_1\}$, which contains the long set $I \cup \{k\}$. Then the set $\{n, *_1\}$ is long, which is impossible. □

Proposition 5.3. *For two critical cells, both of type 2, labeled by*

$$\beta = \big(\spadesuit \ \{k\} I \ \clubsuit \{n, *, j\}\big) \ \textit{and} \ \alpha = \big(\spadesuit \ \{k\} I \ \clubsuit \cup \{j\} \ \{n, *\}\big)$$

the following is true:

If I is j-prelong, the cells are connected by exactly one path. The entry j splits from the n-set backward, and joins $\clubsuit$.

Proof. We search for possible paths from β to α. By Lemma 5.1, these paths do not contain splits of the prelong set. So the path starts with the split of the n-set. We easily conclude that the entry j spits backward. □

Example. Here $k = 1$, $j = 4$.

$$\big(\{1\}\{6,5,3\}\{7\}\{2\}\{8,4\}\big)$$

$$\big(\{1\}\{6,5,3\}\{7\}\{2\}\{4\}\{8\}\big), \ \big(\{1\}\{6,5,3\}\{7\}\{2,4\}\{8\}\big)$$

$$\big(\{1\}\{6,5,3\}\{7\}\{4\}\{2\}\{8\}\big).$$

6. Path reversing: new rules of the game

Our next step is to reduce the number of critical cells using the following theorem:

Theorem 6.1. [3] *Suppose we have a discrete Morse function with critical cells α, β such that there exists exactly one gradient path from β to α. Then reversing the direction of this gradient path produces a discrete Morse function with α, β no longer critical.* □

A necessary warning is: such paths should be reversed one by one, since reversing one path may create additional paths between other pais of critical cells. Keeping this in mind, we do not reverse all the paths that are described in Proposition 5.3, but pose some extra condition on the paths to be reversed. Namely, we do the following.

We reverse the path between two critical cells labeled by
$$\beta = \big(\spadesuit\ \{k\}I\ \clubsuit\{n,*,j\}\big) \text{ and } \alpha = \big(\spadesuit\ \{k\}I\ \clubsuit \cup \{j\}\ \{n,*\}\big)$$
if and only if $j > *, \clubsuit, k$.

The **critical cells** that survive path reversing are (See Figure 4):

(1) All the cells of type 1, and

(2) All the cells $\big(\spadesuit\ \{k\}I\ \clubsuit\{n,*\}\big)$ of type 2 such that
$$k > *,\ k > \clubsuit.$$

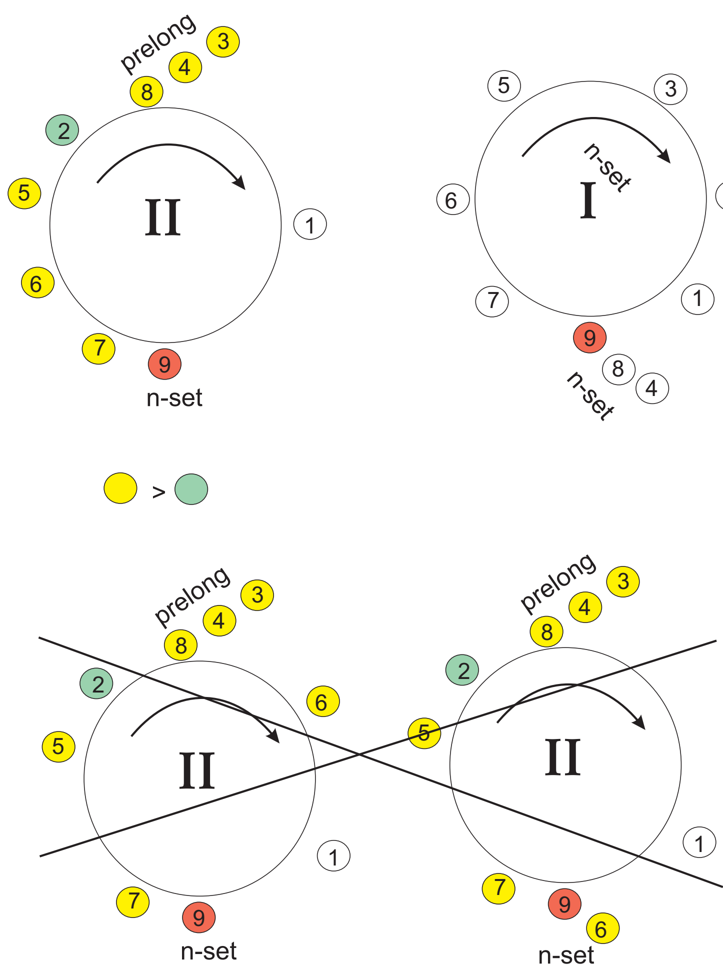

Figure 4. Critical cells that survived the path reversing: examples and non-examples

Proposition 6.2. *The above described path reversing produces no closed gradient paths.*

Proof. Assume the contrary: there exists a closed path Γ. It can be decomposed into some reversed and some unreversed gradient paths between the (former) critical cells. Since a path from type 1 to type 2 never exists, we conclude that all (former) critical cells that appear in Γ are of type 2. For them there are two possibilities: either (1) all these former critical cells have one and the same entry k preceding the prelong set, or (2) for some of these (former critical) cells the entries preceding the prelong set are different. We treat these cases separately.

(1) Lemma 5.1 implies that the prelong set is maintained during the path. Therefore, no entry greater than k passes through the prelong set. Also no entry i smaller than k passes through the n-set. So, no entry makes a full turn.

 The closed path Γ necessarily includes a reversed path. This means that at some moment, an entry i greater than k comes from ♣ and joins the n-set. Consider the consequent split-step.
 (a) If some entry j of the n-set moves forward, it never comes back.
 (b) If some entry j of the n-set moves backward, j is necessarily smaller than k, and the entry j never comes back.

(2) Assume there are different entries right before the prelong sets in this path. Let j be the minimal of these entries. At some step of the path, j quits the place before the prelong set. The entry j is smaller than the next entry that gets to the place before the prelong set, so it cannot stay in ♠ or in the prelong set. Therefore, j eventually joins the ♣. The only way for j to get back leads through the n-set, where it can get only via some reversed path. Since j is minimal, during that path before the prelong set stands an entry greater than j, which is impossible, according to the conditions on the paths we reverse. □

Theorem 6.3. *The number of critical cells equals the sum of Betti numbers of the manifold $M(L)$. Consequently, the above described pairing together with path reversal gives a perfect Morse function.*

Proof. We know from [2] that each short set containing the entry n contributes "2" to the sum of Betti numbers. So, to prove the theorem, we establish a bijection between the short sets containing n and pairs of critical cells.

More precisely, we will show that for every short set containing the n of cardinality $k+1$ corresponds one k-cell of type 1, and one $(n-3-k)$-cell of type 2.

(1) **Cells of type 1**. Assume I is a short set containing entry n and of cardinality $k+1$. We take it as the n-set of the critical cell of type 1. This defines the critical cell uniquely.

(2) **Cells of type 2** Assume J is a short set of cardinality $k+1$ containing entry n
 (a) **Compose a prelong set I.** The set $\overline{J} := [n] \setminus J$ is long. Take the largest entry of $\overline{J}$ and start the prelong set I with it. Keep adding entries from $\overline{J}$ to I in the decreasing order as long as I stays short.

The process stops once I becomes prelong with respect to all other entries of $\overline{J}$.

(b) **Specify an entry preceding** I**.** Let j be the largest of the $\overline{J} \setminus I$. Turn j to the singleton that precedes the prelong set I.

(c) **Compose an** n**-set.** By construction, each entry in $\overline{J} \setminus \big(I \cup \{j\}\big)$ is smaller than j. We put to the n-set the entry n and all the entries of $\overline{J} \setminus \{j\}$.

(d) **Positions of the rest of the singletons are now defined uniquely.** We turn all other entries to singletons, which are placed before $\{j\}$, if they are larger than j, and after I if they are smaller than j.

Now compute the number of the sets in the partition. All entries of J except n turn to singletons. Moreover, we have a singleton j and two non-singleton sets. That makes $k+3$ sets which gives us the dimension $(n-3-k)$

This defines a critical $(n-3-k)$-cell of type 2 uniquely. Moreover, each critical cell of type 2 (that survived the second series of pairing) arises in this way. Indeed, the inverse mapping looks even more easy:

Assume we have a critical cell of type 2. Take all the singletons except for the singleton that precedes the prelong set. Add the entry n. Altogether they give the short set associated to the cell.

□

6.1. **Two examples.** Let $L = (1, 1, 1, 1, 1, 1, 1)$ be the equilateral 7-linkage.

(1) The short set is $J = \{7\}$. Then:

(a) The corresponding cell of type 1 is:

$$\big(\{6\}\ \{5\}\{4\}\ \{3\}\ \{2\}\ \{1\}\ \{7\}\big);$$

(b) $\overline{J} = \{1, 2, 3, 4, 5, 6\}$, $I = \{4, 5, 6\}$, $j = 3$, and the corresponding cell of type 2 is:

$$\big(\{3\}\ \{4, 5, 6\}\ \{7, 1, 2\}\big)$$

(2) The short set is $J = \{5, 6, 7\}$. Then:

(a) The corresponding cell of type 1 is:

$$\big(\{4\}\ \{3\}\ \{2\}\ \{1\}\ \{7, 5, 6\}\big);$$

(b) $\overline{J} = \{1, 2, 3, 4\}$, $I = \{2, 3, 4\}$, $j = 1$ and the corresponding cell of type 2 is:

$$\big(\{1\}\ \{2, 3, 4\}\ \{6\}\ \{5\}\ \{7\}\big)$$

References

[1] M. Farber, *Invitation to topological robotics*, European Mathematical Society, 2008.

[2] M. Farber and D. Schütz, *Homology of planar polygon spaces,* Geom. Dedicata, 125 (2007), 75-92.

[3] R. Forman, *Morse Theory for Cell Complexes*, Adv. in Math. Vol. 134, 1(1998), 90-145.

[4] R. Forman, *A User's guide to discrete Morse theory*, Séminaire Lotharingien de Combinatoire 48 (2002), Article B48c
[5] G. Panina, *Moduli space of a planar polygonal linkage: a combinatorial description*, arXiv:1209.3241