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Extended Class of Dubrovin's Equations Related to the One-Dimensional Quantum Three-Body Problem

L. A. DMITRIEVA
International Solvay Institutes for Physics and Chemistry, CP 231
Boulevard du Triomphe, B-1050 Brussels, Belgium

YU. A. KUPERIN
The Laboratory of Complex Systems Theory, St. Petersburg State University
Ul'janovskaja 1, Stary Petrgof, 198904, St. Petersburg, Russia
kuperin@cuper.niif.spb.su

G. E. RUDIN
The Laboratory of Complex Systems Theory, St. Petersburg State University
St. Petersburg, 198904, Russia

Abstract—The relation of the quantum 1D three-body problems with zero-range interaction to the matrix Riemann-Hilbert problem with meromorphic coefficient is shown. The solution of this problem is discussed using the exact analytic diagonalization of the coefficient. The problem is reduced to the boundary value problem on the Riemann surface. The solution of this problem is expressed in terms of the Riemann theta-functions. An extended class of integrable Dubrovin's type ordinary differential equations related to the one-dimensional quantum three-body problem is derived.

Keywords—Three-body problem, Riemann-Hilbert problem, Dubrovin's type equation.

1. INTRODUCTION

In this paper, we study the analytic structure of the scattering problem for the quantum system of three one-dimensional particles [1–9]. We restrict our consideration to the systems with point interaction and equal particle masses. Since in the center of mass frame this system is two-dimensional and can be reduced to the boundary value problem in a sector, it allows the application of the Sommerfeld-Maluzhinetz (SM) technique [10–17]. However, the matrix functional equations which arise for the kernel of the SM transformation were studied for some degenerated cases only [9]. To the best of our knowledge, there is no analysis similar to the one carried out in [18,19] for canonical Maluzhinetz functional equations. So we present some new results on the solutions of the matrix Maluzhinetz functional equations.

We show the relation between the three-body problem and Maluzhinetz functional equations (MFE) on one quite simple but physically very important model, namely, the system of three anyons with δ-interaction in 1D [20,21]. The main reason for this is that the resulting functional equations have the same general form for a wide range of pointwise interactions. The equations derived in [9] were studied for the case of *bosonic* and *fermionic* statistics only. Being used as

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX the kernel of SM transformation, the meromorphic vector functions which satisfy MFE provide one with solutions of the boundary value problem for the Schrödinger equation. The asymptotic behavior of the Schrödinger equation solutions is controlled by the poles of the corresponding kernel [9–17]. In order to find the kernel with given poles, one can use a *generating solution*. We call a meromorphic matrix-function, which satisfies MFE and has a meromorphic inverse, the generating solution since any meromorphic vector solution can be generated by its action on a rational function of periodic exponentials (e^{it}). In order to obtain the kernel with given poles, one needs just to solve a finite system of linear equations for the coefficients in the decomposition of this rational function into the sum of simplest rationals.

Using the iterations technique, we reduce the MFE to the matrix functional equations with the periodic coefficient. We show how the generating solution of MFE can be obtained from the generating solution of the iterated equation. We reduce the construction of the later solution to the boundary value problem on a Riemann surface using the diagonalization of the periodic coefficient. The solution of this boundary value problem is given in terms of a Baker-Akhiezer (BA) function. The BA function can be expressed in a closed form through the theta function of the corresponding Riemann surface. However, in this form the zeroes of the BA function remain unknown. These zeroes play an important role for consequent use of the corresponding generating solution. Namely, these zeroes can compensate the poles of periodic vector functions after the action of the generating solution.

Thus we use another representation for the BA function which exactly includes its zeroes. This representation allows us to derive an extended class of Dubrovin's type equations. In order to find the "zeroes" of the generating solution, one needs to solve some Cauchy problem on the interval for the system of ordinary differential equations.

Let us briefly summarize the milestones of the approach. In order to solve the scattering problem for the system of three 1D particles, one should apply the SM transform, derive the MFE for the kernel of the SM transform, iterate the MFE to get the periodic ones, diagonalize the matrix coefficient, and solve the boundary value problem on the Riemann surface of its eigenvalue. In this way one obtains a generating solution of the iterated equation. In order to come back to the scattering problem, one should use the following procedure. The generating solution of the iterated equation gives one the generating solution of the MFE; solving a linear system, one finds the kernel of SM transformation with given singularities; the steepest descent method allows one to express the scattering data through the values of the kernel at the saddle points [3,4,9–16].

The paper is organized in the following way. In Section 2, we recall the notion of the three-anyon system in 1D and fix some notations. We derive the MFE for this system with δ-interaction. General algebraic results for the iterated equation are given in Section 3. In Section 4, we rewrite the iterated equation as the matrix Riemann problem with meromorphic coefficient on the open contour and start to study its generating solutions. The construction of a generating solution through a BA function is described in Section 5. In the last section, we present the derivation of Dubrovin's type equations and discuss their relation to both BA function zeroes problem and integrable systems.

2. THREE-BODY PROBLEM AND MALUZHINETZ EQUATIONS

In this section, we shall apply the Sommerfeld-Maluzhinetz approach to the one-dimensional three-anyon system with δ-interaction. Let us briefly recall the notion of anyons. The gauge transformations which arise due to the Aharonov-Bohm (AB) type interaction [22] investigated by authors in [23,24] lead to the multivalued wave functions or the boundary conditions with the constant phase jump. This situation can be interpreted from the viewpoint of statistics for the systems of quantum particles in one and two spatial dimensions [20,25]. The particles with intermediate statistics, between *fermions* and *bosons*, are called *anyons*. Actually the one-body state loses its meaning for anyons that can be also formulated as: *there are no free* anyons. Hence, the few-anyon system can be considered as an example of the system with *persistent interaction* in the sense of Prigogine [26].

As in the center of mass frame, the configuration space of the three-body one-dimensional system is of dimension two, it is possible to introduce parastatistics for this system. Let $2m_\alpha = \hbar = 1$, $\alpha = 1, 2, 3$. Using the Jacobi coordinates [27]

$$\begin{aligned} R &= \frac{1}{3}(r_1 + r_2 + r_3), \\ x_\gamma &= r_\alpha - r_\beta, \\ y_\gamma &= \sqrt{\frac{4}{3}}\left(\frac{1}{2}(r_\alpha + r_\beta) - r_\gamma\right), \\ \mathbf{x} &= (x_\gamma, y_\gamma), \end{aligned} \tag{1}$$

where r_α is the coordinate of α^{th} particle in a laboratory system, $\alpha, \beta, \gamma \in \{1, 2, 3\}$, the Hamiltonian of the system can be expressed as

$$H = -\frac{\partial^2}{\partial R^2} \times I - I \times H_c, \tag{2}$$

where the operator H_c represents the system in the center of mass frame and reads as $H_c = -\Delta_{\mathbf{x}} = -\partial^2_{x_\alpha} - \partial^2_{y_\alpha}$ for the case of no interaction. The plane $\mathbf{R}^2_{\mathbf{x}}$ can be separated into six equal sectors with open angle equal $\frac{\pi}{3}$ which corresponds to different permutations of particles [1]. Let us define the operators of permutations Z_α and coordinate sign inversion C which act on the wave function $\psi(x_\alpha, y_\alpha)$ as

$$\begin{aligned} (Z_\alpha\psi)(x_\alpha, y_\alpha) &= \psi(-x_\alpha, y_\alpha), \\ (C\psi)(\mathbf{x}) &= \psi(-\mathbf{x}). \end{aligned} \tag{3}$$

The Hamiltonian of a three-body system with pairwise interaction and no statistics is invariant under the action of Z_α and C. Since Z_α is just a one-dimensional reflection, such system allows no parastatistics. However, if one supposes the "pure" inversions Z_α to be excluded from the symmetry group and investigates the symmetries which correspond to their combinations $\{(CZ_2Z_1)^m\}_{m=1}^6$ only, the parastatistics happen to be possible. Really, the operator CZ_2Z_1 corresponds to the rotation of configuration space by the angle $\frac{\pi}{3}$ around the three-body collision point $\mathbf{x} = 0$. So, similarly to the case of two-dimensional anyons, one can suppose the wave function to accumulate a constant phase factor $e^{i\nu}$ under the action of CZ_2Z_1. So, the Hamiltonian restricted to one sector would be supplied with corresponding boundary conditions as well as for the AB effect. From what was said above, we can see that the anyonic statistics for the one-dimensional three-body system can appear if we abandon not only the entity of one-particle states but also two-body ones and shall speak about the *three-anyon complex*. The system with such properties can appear only as a result of a special kind of interaction, namely, the three-body AB gauge interaction.

Let us briefly discuss the rigorous way to pose the problem. Let us introduce the Hamiltonian H_c with some pairwise interaction between the particles and the three-body gauge field of AB type. Namely, let us write the Hamiltonian in the form

$$H_c = \left(i\nabla_{\mathbf{x}} + \vec{A}(\mathbf{x})\right)^2 + \sum_{\gamma=1}^{3} V(x_\gamma),$$

where $V(x)$ denotes the pairwise interaction potential and

$$\vec{A}(\mathbf{x}) = \frac{B}{2\pi|\mathbf{x}|}\begin{pmatrix} -y_\alpha \\ x_\alpha \end{pmatrix} \tag{4}$$

is just the AB gauge vector potential [22]. The gauge field A in H_c can be eliminated by noncontinuous gauge transformation only. We choose this transformation in the form which keeps the rotational symmetry of the three-body system. In polar coordinates, $\mathbf{x} = (r\cos\phi, r\sin\phi)$, this gauge transformation looks like

$$\mathcal{G} : \varphi \mapsto u = e^{i(\beta\pi/3)\{3\phi/\pi\}}\varphi,$$

where $\{\ \}$ denotes the fractional part and $\beta = \left\{\frac{B}{2\pi}\right\}$. The transformation $\mathcal{G}$ being applied, one gets the Hamiltonian $H_{cg} = \mathcal{G}H_c\mathcal{G}^{-1}$ in the following form:

$$H_{cg} = -\Delta + \sum_{\gamma=1}^{3} V(x_\gamma),$$

defined on the domain which is specified by the boundary conditions

$$\left.\begin{pmatrix} u \\ u_\phi \end{pmatrix}\right|_{\phi=m\pi/3+0} = e^{i(\beta\pi/3)} \left.\begin{pmatrix} u \\ u_\phi \end{pmatrix}\right|_{\phi=m\pi/3-0}, \qquad m = 0,\dots,5,$$

for smooth potentials V. Here u stands for the derivative w.r.t. ϕ. If the pairwise interaction $V(x)$ includes some zero-range potential [9,28,29], then the boundary conditions change in the way corresponding to the singular part of interaction. In this work, we shall consider the zero-range interaction $V(x)$ only.

Let us consider the simplest case of zero-range two body interaction, so-called δ-interaction. The Hamiltonian $H_c = (i\nabla_\mathbf{x} + A(\mathbf{x}))^2 + \Gamma\sum_{\gamma=1}^{3}\delta(x_\gamma)$, with real constant $\Gamma \in \mathbf{R}$, allows the strict definition as self-adjoint operator [29]

$$H_c = (i\nabla_\mathbf{x} + A(\mathbf{x}))^2,$$

defined on the domain

$$\mathcal{D}(H_c) = \left\{ u \;\middle|\; \begin{array}{c} u \in W_2^2(\mathbf{R}^2 \setminus \{x_\alpha = 0\}); \\ u|_{\phi=m\pi/3+0} = u|_{\phi=m\pi/3-0}, \\ u_\phi|_{\phi=m\pi/3+0} - u_\phi|_{\phi=m\pi/3-0} = r\Gamma u|_{\phi=m\pi/3+0}, \\ m = 0,\dots,5 \end{array} \right\}.$$

In this case, the operator H_c after gauge transformation

$$H_{cg} = -\Delta$$

is defined on the domain

$$\mathcal{D}(H_{cg}) = \left\{ u \;\middle|\; \begin{array}{c} u \in W_2^2(\mathbf{R}^2 \setminus \{x_\alpha = 0\}); \\ u|_{\phi=m\pi/3+0} = e^{i(\beta\pi/3)} u|_{\phi=m\pi/3-0}, \\ u_\phi|_{\phi=m\pi/3+0} - e^{i(\beta\pi/3)} u_\phi|_{\phi=m\pi/3-0} = r\Gamma u|_{\phi=m\pi/3+0}, \\ m = 0,\dots,5 \end{array} \right\}.$$

This operator commutes with rotations $(CZ_2Z_1)^n$ and so allows the decomposition into the orthogonal sum of six operators corresponding to different symmetries. The spectral problem for these operators looks like the boundary value problem:

$$\begin{gathered} -\Delta u = \lambda u, \\ e^{i\nu}\frac{\partial}{\partial_\phi} u(r,\phi)\Big|_{\phi=\pi/3} - \frac{\partial}{\partial_\phi} u(r,\phi)\Big|_{\phi=0} - r\Gamma u(r,\phi)|_{\phi=\pi/3} = 0, \\ u(r,\phi)|_{\phi=\pi/3} - e^{i\nu} u(r,\phi)|_{\phi=0} = 0, \\ u(0,\phi) < \infty, \end{gathered} \tag{5}$$

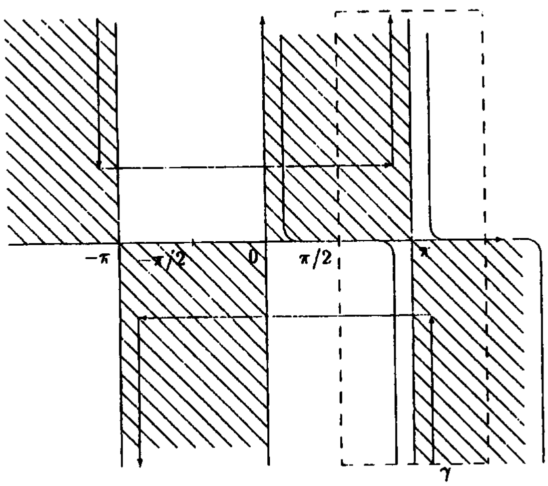

Figure 1.

where $k^2 = \lambda$ is the energy variable and $\nu = (\beta + n)\pi/3$, $n = 0, 1, \ldots, 5$. Let us consider the SM transform of a solution for (5)

$$u(r, \phi) = \int_\gamma e^{-ikr\cos\alpha} g(\alpha, \phi)\, d\alpha, \tag{6}$$

where the contour γ consists of two symmetric parts as shown in Figure 1 in which we dashed the domains $\Im(k\cos\alpha) < 0$ for real k and show their shift for $\Im k > 0$; the dashed box shows one of these domains for $\Re k = 0$.

The integration by parts in (6) being applied, the Helmholtz equation in (5) for u requires

$$\frac{\partial^2}{\partial\alpha^2} g(\alpha, \phi) = \frac{\partial^2}{\partial\phi^2} g(\alpha, \phi), \tag{7}$$

and so $g(\alpha, \phi) = g_1(\alpha + \phi) + g_2(\alpha - \phi)$.

For convenience we shall represent the solution of (5) as:

$$u(r, \phi) = \int_\gamma e^{-ikr\cos\alpha} \left(g_+(\alpha + \phi) + g_-\left(\alpha + \frac{\pi}{3} - \phi\right)\right) d\alpha. \tag{8}$$

Now the boundary conditions have the form of homogeneous integral equations with the kernel which consists of even in α functions only [10–16]. We use the freedom in even part of $g(\alpha, \phi)$ in order to set both integrals over γ_1 and γ_2 in the boundary conditions equal to zero. Namely, using the representation (8), the boundary conditions in (5) look like

$$\left\{g_+(\alpha) + g_-\left(\alpha + \frac{\pi}{3}\right) - e^{i\nu}\left(g_+\left(\alpha + \frac{\pi}{3}\right) + g_-(\alpha)\right)\right\} \equiv 0,$$

$$ik\sin\alpha\left[g_+(\alpha) - g_-\left(\alpha + \frac{\pi}{3}\right) - e^{i\nu}\left(g_+\left(\alpha + \frac{\pi}{3}\right) - g_-(\alpha)\right)\right] - \Gamma\left(g_+(\alpha) + g_-\left(\alpha + \frac{\pi}{3}\right)\right) \equiv 0.$$

If we introduce the notations $\mathbf{g}(\alpha) \equiv (g_+(\alpha), g_-(\alpha))^t$, the boundary conditions (5) can be rewritten as

$$\mathbf{g}\left(\alpha + \frac{\pi}{3}\right) = M(\alpha)\mathbf{g}(\alpha), \tag{9}$$

where the meromorphic matrix function M can be written in terms of the transition and reflection coefficients of the corresponding two-body one-dimensional problem. Namely, if one considers the two-body system with δ-interaction described by the Hamiltonian $H_2 = -\frac{d^2}{dx_\alpha^2} + \Gamma\delta(x_\alpha)$ and compare it with the system with no interaction, then the scattering matrix would be expressed through the transition and reflection coefficients $T(k) = 2ik/(2ik - \Gamma)$; $R(k) = \Gamma/(2ik - \Gamma)$, as

$$S(k) = \begin{pmatrix} T(k) & R(k) \\ R(k) & T(k) \end{pmatrix}. \tag{10}$$

The matrix M looks in these terms as

$$M(\alpha) = \begin{pmatrix} e^{i\nu}T(k\sin\alpha) & R(k\sin\alpha) \\ R(k\sin\alpha) & e^{-i\nu}T(k\sin\alpha) \end{pmatrix}. \tag{11}$$

This form of the matrix coefficient $M(\alpha)$ holds for the general case of zero-range interaction [9]. However, in the case of the interaction with internal structure, the nontrivial inhomogeneous term appears due to the kernel of the SM transformation in internal channel. These terms allow one to satisfy additional boundary conditions at $r = 0$. Thus, the general form of functional equations at hand is

$$g\left(\alpha + \frac{\pi}{3}\right) = M(\alpha)g(\alpha) + F(\alpha). \tag{12}$$

These equations have been investigated for the cases when the matrix $M(\alpha)$ has the constant basis, i.e., $\nu = 0, \pi$ only [9].

3. GENERAL SOLUTION FOR THE MATRIX MALUZHINETZ FUNCTIONAL EQUATION

Let us discuss the general case of the matrix functional equation (12). For convenience we study the equations with the shift of argument equal to 1 but not $\frac{\pi}{3}$ and suppose the matrix coefficients to have the period n instead of 2π. Namely, let $M(x)$ and $F(x)$ be meromorphic matrices depending on the complex variable x and such that for some $n \in \mathbf{N}$

$$M(x+n) = M(x); \qquad F(x+n) = F(x). \tag{13}$$

The properties of matrix coefficient $M(\alpha)$ from (11) can be reconstructed if one poses $n = 6$, $x = 3\alpha/\pi$. So, we shall consider the matrix finite-difference equation

$$g(x+1) = M(x)g(x) + F(x). \tag{14}$$

Let us introduce the notation for ordered multiplication of some matrices $\{M_i\}$, namely $\uparrow\prod_{i=l}^{k} M_i \equiv M_k M_{k-1} \dots M_l$ for $k \geq l$ and $\uparrow\prod_{i=l}^{k} M_i = I$ for $k < l$. The n^{th} iteration of the equation (14) reads as

$$g(x+n) = N(x)g(x) + \Xi(x), \tag{15}$$

where we denoted

$$N(x) \equiv \uparrow\prod_{i=0}^{n-1} M(x+i)$$

and

$$\Xi(x) \equiv \sum_{i=0}^{n-1} \uparrow\prod_{j=i+1}^{n-1} M(x+j)F(x+i).$$

Due to the definition of N, one has

$$N(x+n) = N(x); \qquad \Xi(x+n) = \Xi(x), \tag{16}$$

$$N(x+1) = M(x)N(x)M^{-1}(x). \tag{17}$$

So, the eigenvalues of $N(x)$ are of the period 1. It can be shown that two special cases:

(a) $N(x) \equiv I$,

(b) $\det(N(x) - I) \not\equiv 0$

cover the three-body problem with zero-range interaction. Since the determinant of meromorphic matrix is meromorphic, we can establish that it differs from zero almost everywhere except maybe for a countable number of points.

LEMMA 1. *The general solution of inhomogeneous functional equation (15) corresponding to cases (a) and (b) looks like:*

$$\begin{aligned} &\text{(a)} \qquad g(x) = \frac{x}{n}\Xi(x) + g_0(x); \\ &\text{(b)} \qquad g(x) = (I - N(x))^{-1}\Xi(x) + g_0(x), \end{aligned} \tag{18}$$

where $g_0(x)$ is a solution of homogeneous functional equation

$$g_0(x+n) = N(x)g_0(x). \tag{19}$$

PROOF. One needs to show only that $(x/n)\Xi(x)$ and $(I - N(x))^{-1}\Xi(x)$ are particular solutions of inhomogeneous equation (15) for cases (a) and (b). This can be checked by direct calculation. Due to (16) one has

$$\frac{x+n}{n}\Xi(x+n) = \frac{x}{n}\Xi(x) + \Xi(x).$$

In the case $N(x) \equiv I$, this is equivalent to (15) for $g(x) = (x/n)\Xi(x)$. In case (b), one has

$$\begin{aligned} (I - N(x+n))^{-1}\Xi(x+n) &= (I - N(x))^{-1}\Xi(x) \\ &= N(I-N)^{-1}\Xi + (I-N)(I-N)^{-1}\Xi \\ &= N(x)(I - N(x))^{-1}\Xi(x), \end{aligned}$$

which is equivalent to (15) with $g(x) = (I - N(x))^{-1}\Xi(x)$. ∎

Using this statement we are able to prove the following theorem.

THEOREM 1. *The general solution of inhomogeneous functional equation (14) for cases (a) and (b) looks like:*

$$\text{(a)} \qquad g(x) = \sum_{i=0}^{n-1} \uparrow\prod_{j=i+1}^{n-1} M(x+j)\left(\frac{x+i}{n}\right)F(x+i) + \sum_{i=0}^{n-1}\uparrow\prod_{j=i}^{n-1} M(x+j)g_0(x+i), \tag{20}$$

$$\text{(b)} \qquad g(x) = (I - N(x))^{-1}\Xi(x) + \sum_{i=0}^{n-1}\uparrow\prod_{j=1}^{n-1} M(x+j)g_0(x+i), \tag{21}$$

where $g_0(x)$ is a solution of homogeneous functional equation (19). For the case $\det N(x) \not\equiv 0$, the solution can be also expressed as

$$g(x) = \sum_{i=0}^{n-1}\uparrow\prod_{j=i}^{n-1} M(x+j)(I - N(x+i))^{-1}F(x+i) + \sum_{i=0}^{n-1}\uparrow\prod_{j=i}^{n-1} M(x+j)g_0(x+i). \tag{22}$$

PROOF. The case $N(x) \equiv I$ has been considered for scalar function M and $n = 6$ in [9]. The proof for the matrix case is similar to the scalar one and given by direct calculation. Let us substitute the expression (20) into the equation (14)

$$g(x+1) - M(x)g(x) - F(x) = 0. \tag{23}$$

Then, using (13) one gets

$$\begin{aligned} g(x+1) &= \sum_{i=0}^{n-1} \uparrow \prod_{j=i+1}^{n-1} M(x+j+1) \left(\frac{x+i+1}{n}\right) F(x+i+1) \\ &\quad + \sum_{i=0}^{n-1} \uparrow \prod_{j=i}^{n-1} M(x+j+1) g_0(x+i+1) \\ &= \sum_{i=1}^{n} \uparrow \prod_{j=i+1}^{n} M(x+j) \left(\frac{x+i}{n}\right) F(x+i) \\ &\quad + \sum_{i=1}^{n} \uparrow \prod_{j=i}^{n} M(x+j) g_0(x+i) \\ &= M(x)g(x) + F(x) + g_0(x+n) - N(x)g_0(x). \end{aligned}$$

It means that the equation (23) is equivalent to (19).

The proof of this statement for $\det(N(x)-I) \not\equiv 0$ is based on the following property of $\Xi(x)$:

$$\Xi(x+1) = M(x)\Xi(x) + (I - N(x+1))F(x). \tag{24}$$

From this equation together with (17) one has

$$(I - N((x+1))^{-1}\Xi(x+1) = M(x)(I - N(x))^{-1}M^{-1}(x)M(x)\Xi(x) + F(x). \tag{25}$$

Thus, the first term in (21) is a particular solution of inhomogeneous equation (14). For the second term, it can be verified by direct calculation that it satisfies the homogeneous equation (14) if and only if the function g_0 satisfies the homogeneous functional equation (19).

The representation (22) of the general solution can be proven by the substitution into (23). Namely, the second term is the general solution of homogeneous equation and for the inhomogeneous part ($g_0 = 0$), one has

$$\begin{aligned} g(x+1) &= \sum_{i=1}^{n} \uparrow \prod_{j=1}^{n} M(x+j)(I - N(x+i))^{-1} F(x+i) \\ &= M(x)g(x) + F(x). \end{aligned}$$

∎

4. FUNCTIONAL EQUATIONS AND THE RIEMANN-HILBERT PROBLEM

For the application to the solution of scattering problem, one needs to find analytic solutions of functional equations (14) with given singularities. In order to do this, it is useful to find a generating solution, i.e., a meromorphic matrix function having meromorphic inverse one with finite number of singularities in the band $\Re x \in [0,1)$. Then, the solution with given properties can be obtained through the action of a generating solution on some periodic vector function which is simply a rational function of exponentials. A generating solution for the functional equations (14) can be obtained from the one for iterated equations (15) using the results of Theorem 1. So, in the rest of the paper we shall concentrate on the functional equations (15) only.

Let us reformulate the equations (15) to the form of the boundary value problem. Namely, let us introduce a new variable $z = e^{2\pi i(x/n)}$. Then, due to (16) the matrix function $N(z)$ will be meromorphic and the boundary values of solutions for (15) on the cut $z \in [0, +\infty)$ will satisfy

$$g(z - i0) = N(z)g(z + i0) + \Xi(z). \tag{26}$$

This is the special case of the problem known as the matrix Riemann-Hilbert problem [30]. Below we propose the way to solve this type of boundary value problem. But first let us describe more carefully the kind of problem we shall analyze.

Let $N(z)$ be a *rational* matrix function, i.e., $N \mid \mathbf{C} \mapsto \mathbf{C}^n \times \mathbf{C}^n$ so that all the matrix elements $N_{ij}(z)$, $i, j = 1, \ldots, n$, are *rational* functions of the complex variable z. Let $\mathcal{L}$ be some open smooth curve on the complex plane $\mathbf{C}$. Let us denote its ends by z_{l+} and z_{l-}, and fix the orientation so that the oriented curve $\mathcal{L}$ runs from the point z_{l-} to the point z_{l+}. The side of the curve which goes clockwise we shall denote by $\mathcal{L}_+$ and the other one by $\mathcal{L}_-$. Let us also suppose that the points $z_{l\pm}$ are chosen so that $N(z_{l\pm}) = I$.[1]

HOMOGENEOUS RIEMANN-HILBERT PROBLEM. Find the vector function $\mathbf{g}(z) \mid \mathbf{C} \mapsto \mathbf{C}^n$ analytic on $\bar{\mathbf{C}} \setminus \{\mathcal{L}\}$, extendable on both sides of $\mathcal{L}$ and such that its boundary values satisfy

$$\mathbf{g}(z)|_{\mathcal{L}_+} = N(z)\mathbf{g}(z)|_{\mathcal{L}_-}. \tag{27}$$

INHOMOGENEOUS RIEMANN-HILBERT PROBLEM. Find the vector function $\mathbf{g}(z) \mid \mathbf{C} \mapsto \mathbf{C}^n$ analytic on $\bar{\mathbf{C}} \setminus \{\mathcal{L}\}$, extendable on both sides of $\mathcal{L}$ and such that its boundary values satisfy

$$\mathbf{g}(z)|_{\mathcal{L}_+} = N(z)\mathbf{g}(z)|_{\mathcal{L}_-} + \mathbf{f}(z), \tag{28}$$

where $\mathbf{f}(z)$ is some smooth function defined on $\mathcal{L}$.

Let us define the function

$$\ln_{\mathcal{L}} z = \begin{cases} \ln\left\{\dfrac{(z - z_{l+})}{(z - z_{l-})}\right\}, & |z_{l\pm}| < \infty, \\ \ln(z - z_{l+}), & z_{l-} = \infty, \end{cases}$$

with the cut of the logarithm being chosen to coincide with $\mathcal{L}$.

In our consideration, we shall use the matrix coefficient $N(z)$ in the diagonal (or Jordan) form. In order to get this form one can consider the function $p(z, \lambda) = \det(N(z) - \lambda I)$. Due to the properties of $N(z)$, $p(z, \lambda)$ is a rational function in the z variable and polynomial in λ. The roots of the equation $p(z, \lambda) = 0$ determine the algebraic functions $\lambda_i(z)$, $i = 1, \ldots, N$. The functions $\lambda_i(z)$ can be grouped by their analytic properties. Namely, λ_i's which are rational correspond to rational eigenvectors; λ_i's which have a number of branching points can be grouped by the analytic function whose values on different sheets they represent, and correspond to the eigenvectors being meromorphic on the same Riemann surface. This classification should also include separation of Jordan blocks corresponding to some of everywhere (in z) degenerated eigenvalues if these blocks are kept by the analytic continuation.

The eigenvalues being classified, one can find the eigenvectors corresponding to nondegenerated (a.e.) eigenvalues λ_i using the projections $\Lambda_i(z)$ defined by [3]

$$\Lambda_i(z) = -(N(z) - \lambda I)^{-1} p(z, \lambda) \left(\frac{\partial p(z, \lambda)}{\partial \lambda}\right)^{-1}\Bigg|_{\lambda = \lambda_i(z)}. \tag{29}$$

The matrices $\Lambda_i(z)$ can be expressed as $\Lambda_i(z) = \mathbf{v}_i(z) \times \tilde{\mathbf{v}}_i^t(z)$, where $\mathbf{v}_i(z)$ is the eigenvector of $N(z)$ corresponding to the eigenvalue $\lambda_i(z)$ and $\tilde{\mathbf{v}}_i^t(z)$ is one for $N(z)^t$. Due to (29), these eigenvectors are normalized as $\tilde{\mathbf{v}}_i^t(z) \times \mathbf{v}_i(z) = 1$, so $\operatorname{Tr} \Lambda_i(z) = 1$, $\Lambda_i^2(z) = \Lambda_i(z)$, and $\Lambda_j(z)\Lambda_i(z) = 0$ for $i \neq j$.

For the general situation, the consideration of degenerated eigenvalues requires more work to be done; however, in most applications the coefficient $N(z)$ has some helpful features. For example,

[1] This requirement is made just in order to simplify further discussion. The solution algorithm given below can also be applied for the general situation; however, the functional class of solutions must be changed in this case to allow appropriate singularities at the points $z_{l\pm}$.

the complete diagonalization is always possible if $N(z)$ is unitary or self-adjoint on some curve in $\mathbf{C}$, say on $\mathbf{R}$.

We shall consider in detail the problem for 2×2 matrices only. For any given n, the construction we present can in principle be performed, but the absence of exact formulae would lead the general discussion too far from applications. Since the 2×2 matrix has just two eigenvalues, only one Jordan block can appear. We shall separate four situations.

1a. $\lambda_1(z) \neq \lambda_2(z)$, a.e., and both are rational functions. This is so iff all the poles and zeroes of the rational function

$$\operatorname{Tr}^2 N(z) - 4 \det N(z)$$

have even multiplicity. So, the corresponding eigenvectors $\mathbf{v}_1(z)$ and $\mathbf{v}_2(z)$ can be normalized so that they are rational too.

1b. $\lambda_1(z) \neq \lambda_2(z)$, a.e., and have branching points. So, they represent the values of the two-sheet analytic function

$$\lambda(z) = \frac{1}{2}\left(\operatorname{Tr} N(z) + \sqrt{\operatorname{Tr}^2 N(z) - 4\det N(z)}\right).$$

2a. $\lambda_1(z) = \lambda_2(z)$, but there is just one eigenvector $\mathbf{v}(z)$ and the adjoint vector $\mathbf{u}(z)$.

2b. $\lambda_1(z) = \lambda_2(z)$; this corresponds to the trivial case $N(z) = \lambda(z)I$.

Let us call an analytic matrix function $X(z)$ a *generating solution* of the Riemann-Hilbert problem[2] (27) if the following hold.

1. It satisfies the boundary value problem

$$X(z)|_{\mathcal{L}_+} = N(z)X(z)|_{\mathcal{L}_-}. \tag{30}$$

2. $X(z)$ and $X^{-1}(z)$ are regular on $\mathbf{C}/\{\mathcal{L}\}$ except maybe for some finite number s of points $\{z_i\}_{i=1}^{s}$ where they have poles of finite multiplicities.

Suppose we know some generating solution of the problem (27). One can show that for any regular solution $\mathbf{g}(z)$, the function $\mathbf{g}_0(z) \equiv X^{-1}(z)\mathbf{g}(z)$ should satisfy

$$\mathbf{g}_0(z)|_{\mathcal{L}_+} = \mathbf{g}_0(z)|_{\mathcal{L}_-},$$

and so would be a rational function with poles at the points $\{z_i\}$ only and of multiplicity not higher than one of $X^{-1}(z)$. Any such function can be expanded in finite linear combination of simplest rationals and the regularity conditions at the points $\{z_i\}$ would lead to the homogeneous system of linear equations for the coefficients in this decomposition.

The general solution of inhomogeneous problem (28) is given by

$$\mathbf{g}(z) = X(z)\left\{g_0(z) + \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\mathbf{f}(z')\,dz'}{X|_{\mathcal{L}_+}(z')(z-z')}\right\}, \tag{31}$$

where again the coefficients in the decomposition of $g_0(z)$ are determined by the inhomogeneous system of linear equations. Due to the Fredholm alternative, one would apply a number of solvability conditions to $\mathbf{f}(z)$ in order to find a solution with given analytic properties. The properties of these linear systems will be studied elsewhere. Here we shall concentrate on the construction of generating solutions.

[2]Note that the generating solution for the functional equation (15) can be obtained as $X(e^{2\pi i(x/n)})$.

5. CONSTRUCTION OF GENERATING SOLUTIONS

In this section, we propose the way to construct a generating solution. To do this we shall act analogously to the scalar case. Namely, the scalar Riemann-Hilbert problem

$$g(z)|_{\mathcal{L}_+} = \lambda(z)g(z)|_{\mathcal{L}_-} \tag{32}$$

with rational scalar coefficient $\lambda(z)$ can be solved in the closed form. Let us denote by $\{x^i\}_{i=1}^s$ and $\{y^i\}_{i=1}^s$ the sets of poles and zeroes (the degenerated poles and zeroes are included many times according to their multiplicity) of the function $\lambda(z)$. Then the solution of the problem (32) is given by the following formula:

$$g(z) = \exp\left\{\ln\lambda(z)\frac{\ln_{\mathcal{L}} z}{2\pi i} - \frac{1}{2\pi i}\sum_{i=1}^{s}\int_{x^i}^{y^i}\frac{\ln_{\mathcal{L}} z'}{z'-z}\,dz\right\}, \tag{33}$$

where the cuts of $\ln\lambda(z)$ are chosen to coincide with the integration paths from x^i to y^i.

In order to investigate the 2×2 matrix problem we shall consider separately the cases described above.

2b. In this case, the matrix problem does not differ from the scalar one since the generating solution is given by

$$X(z) = g(z)I,$$

with $g(z)$ defined by (33). The homogeneous problem (27) has two linearly independent solutions, and the general solution of inhomogeneous problem (28) looks like

$$\mathbf{g}(z) = g(z)\left\{g_0(z) + \frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\mathbf{f}(z')\,dz'}{g|_{\mathcal{L}_+}(z')(z-z')}\right\}.$$

1a. In this case, the construction of a generating solution (27) is reduced to the two independent scalar problems (32) with $\lambda = \lambda_1$ and $\lambda = \lambda_2$. If we denote by $g_1(z)$ and $g_2(z)$ the corresponding solutions (33), then the generating solution can be constructed as

$$X(z) = g_1(z)\Lambda_1(z) + g_2(z)\Lambda_2(z).$$

2a. In this case, the matrix coefficient can be represented as $N(z) = \lambda I + \Lambda(z)$ with $\Lambda^2(z) = 0$. So, the matrix function

$$X(z) = g(z)\left\{I + \frac{\ln_{\mathcal{L}} z}{2\pi i\lambda(z)}\Lambda(z)\right\},$$

where $g(z)$ is given by (33), is the generating solution of the problem (27). Really,

$$\begin{aligned} X(z)|_{\mathcal{L}_+} &= g(z)|_{\mathcal{L}_+}\left\{I + \frac{\ln_{\mathcal{L}} z|_{\mathcal{L}_+}}{2\pi i\lambda(z)}\Lambda(z)\right\} \\ &= \lambda(z)g(z)|_{\mathcal{L}_-}\left\{I + \frac{2\pi i + \ln_{\mathcal{L}} z|_{\mathcal{L}_-}}{2\pi i\lambda(z)}\Lambda(z)\right\} \\ &= \lambda(z)X(z)|_{\mathcal{L}_-} + g(z)|_{\mathcal{L}_-}\Lambda(z) \\ &= N(z)X(z)|_{\mathcal{L}_-}, \end{aligned}$$

and $0 < |\det X(z)| = |g(z)|^2 < \infty$.

1b. This case is the most interesting since it cannot be reduced to the scalar problem (32). We propose the following construction for a generating solution. We make use of the fact that the matrix coefficient $N(z)$ can be represented as

$$N(z) = \lambda_+(z)\Lambda_+(z) + \lambda_-(z)\Lambda_-(z),$$

where the signs $\pm$ denote the values of functions on different sheets of the Riemann surface determined by $\sqrt{\mathcal{F}(z)} = \sqrt{\operatorname{Tr}^2 N(z) - 4\det N(z)}$.

Let $\{h_i\}_{i=1}^{a}$ be the set of poles and zeroes of $\mathcal{F}(z)$ with odd multiplicity and τ be the order of this function at infinity. Two cases may happen: $a = 2m$ and $a = 2m + 1$. Let us split the set $\{h_i\}_{i=1}^{a}$ into two parts $\{h_i\}_{i=1}^{a} = \{u_i\}_{i=1}^{m} \cup \{v_i\}_{i=1}^{a-m}$ and fix the cuts δ_i which run between the points u_i and v_i. On the cut complex plane $\mathbf{C} \setminus \cup \delta_i$ define two univalued continuous branches of the functions $\sqrt{\mathcal{F}(z)}$ and also the function

$$K(z) = \begin{cases} \prod_i^m \sqrt{(z-u_i)(z-v_i)}, & a = 2m, \\ \sqrt{z - v_{m+1}} \prod_i^m \sqrt{(z-u_i)(z-v_i)}, & a = 2m+1. \end{cases}$$

Let us denote by $K_{\pm}(z)$ the branch for which

$$K_{\pm}(z) \underset{z\to\infty}{=} \pm z^m + \cdots; \qquad a = 2m,$$

$$\Im\left\{\frac{K_{\pm}(z)}{(z^m)}\right\} > 0; \qquad a = 2m+1.$$

The corresponding branches of $\sqrt{\mathcal{F}(z)}$ can be defined as

$$\left(\pm\sqrt{\mathcal{F}(z)}\right)(K_{\pm}(z)) \underset{z\to\infty}{=} z^{(\tau+a)/2} + \cdots.$$

So we have fixed the branches of $\lambda(z)$, $\lambda_{\pm}(z)$ and $\Lambda(z)$, $\Lambda_{\pm}(z)$. If we denote the points of the Riemann surface $\mathcal{R}$ of $K(z)$ by $\mathbf{z} \equiv (z, j)$; $z \in \mathbf{C}$; $j = \pm$, these branches are defined as

$$K((z,\pm)) = K_{\pm}(z); \qquad \mathcal{F}((z,\pm)) = \pm\mathcal{F}(z);$$
$$\lambda_{\pm}(z) \equiv \lambda((z,\pm)); \qquad \lambda_{\pm}(z) \equiv \lambda((z,\pm)).$$

As well as for the scalar case, let us denote by $\{\mathbf{x}_i\}_{i=1}^{s}$ and $\{\mathbf{y}_i\}_{i=1}^{s}$ the zeroes and poles of the function $\lambda(\mathbf{z})$ and fix the cuts γ_i of $\ln\lambda(\mathbf{z})$ between these points so that they do not coincide with cuts δ_i.

For the fixed values of $\{n_i\}_{i=1}^{m}$; $n_i \in \mathbf{N}$ and $\{\tilde{x}_i, \tilde{y}_i\}_{i=1}^{m'}$; $\tilde{x}_i, \tilde{y}_i \in \mathbf{C}$, let us define function $g_0(\mathbf{z})$ on $\mathcal{R}$

$$g_0(\mathbf{z}) = \exp\left\{\ln\lambda(\mathbf{z})\frac{\ln_{\mathcal{L}} z}{2\pi i} - \frac{1}{2\pi i}\sum_{i=1}^{s}\int_{\mathbf{x}^i}^{\mathbf{y}^i} \frac{[K(\mathbf{z}) + K(\mathbf{z}')]}{2K(\mathbf{z}')(z'-z)} \ln_{\mathcal{L}} z' \, d\mathbf{z}'\right\}. \tag{34}$$

The function $g_0(\mathbf{z})$ is meromorphic on $\mathcal{R} \setminus \{\infty\}$ and has the essential singularity at $z \to \infty$. As $z \to \infty$, the function $g_0(\mathbf{z})$ behaves as $e^{q(z)}$ where the coefficients of the polynomial $q(z)$ are given by the Taylor expansion of

$$-\frac{1}{2\pi i}K(\mathbf{z})\sum_{i=1}^{s}\int_{\mathbf{x}^i}^{\mathbf{y}^i} \frac{\ln_{\mathcal{L}} z' \, d\mathbf{z}'}{2K(\mathbf{z}')(z'-z)}.$$

This singularity can be canceled if one multiplies $g_0(\mathbf{z})$ by a the Baker-Akhiezer (BA) function $\psi(\mathbf{z})$ of the Riemann surface $\mathcal{R}$ corresponding to the point ∞, the polynomial $q(z)$ and some nonspecial divisor D of degree n [31]. Really, if one fixes n points on $\mathcal{R}$ and allows the function $\psi(\mathbf{z})$ to have simple poles at these points, i.e., specifies the divisor D, then the condition $\psi(\mathbf{z}) \sim e^{-q(z)}$ at $\mathbf{z} \to \infty$ will specify the function ψ uniquely up to the constant factor. In order to get the explicit expression for the BA function, one chooses the canonical sections $\{a_i\}_{i=1}^{m}$ and $\{b_i\}_{i=1}^{m}$ on the Riemann surface $\mathcal{R}$. Then one constructs the second kind differential Ω with the main part $-dq(z)$ at ∞ and normalized by the conditions

$$\oint_{a_i} \Omega = 0.$$

When one denotes by $U = \{U_1, \dots, U_m\}$ the vector of b-periods of Ω,

$$U_i = \oint_{b_i} \Omega,$$

and chooses some point $\mathbf{z}_0$ on $\mathcal{R}$, the explicit expression for ψ looks like [31]:

$$\psi(\mathbf{z}) = \exp\left(\int_{\mathbf{z}_0}^{\mathbf{z}} \Omega\right) \frac{\theta(A(\mathbf{z}) - A(D) + U - K)}{\theta(A(\mathbf{z}) - A(D) - K)}, \tag{35}$$

where K is the vector of Riemann constants, A denotes the Abel transform, and θ is the Riemann theta-function of the surface $\mathcal{R}$.

Let us introduce the function $g(\mathbf{z}) = g_0(\mathbf{z})\psi(\mathbf{z})$. The branches of $g(\mathbf{z})$ on the sheets $\mathcal{R}$ we shall denote by

$$g_+(z) \equiv g((z,+)); \qquad g_-(z) \equiv g((z,-)).$$

In this notation, the generating solution of the Riemann-Hilbert problem can be found as

$$X(z) = g_+(z)\Lambda_+(z) + g_-(z)\Lambda_-(z).$$

The regularization of $g_0(\mathbf{z})$ described above has some shortcomings. Namely, the investigation of $X^{-1}(z)$ meets the problem of theta-function zeroes, the construction includes such notions as canonical sections which require special analytic consideration for every case, and, finally, the formulae obtained are very hard to apply for numerical simulations. In the next section, we propose another way to construct the BA function which seems to be easier to realize numerically.

6. BAKER-AKHIEZER FUNCTION AND DUBROVIN'S TYPE EQUATIONS

Another way to construct the BA function can be formulated as follows.

Let $q(z) = K(\mathbf{z})\left\{\sum_{i=1}^m e_i z^{-i}\right\} + O(1)$, when $z \to \dot\infty$. Let us choose k smooth oriented curves $\{\alpha_i\}_{i=1}^k$ far from the cuts δ_i parametrized by the point of the unit interval $\alpha_i = \{\alpha_i(t),\ t \in [0,1]\}$. Fix a smooth function $p_i(t)$, on α_i, such that

$$\sum_{i=1}^k \int_0^1 \frac{p_i(t)\alpha_i^l(t)\, d\alpha_i(t)}{K_+(\alpha_i(t))} = e_{l+1}; \qquad l = 0, \dots, m-1. \tag{36}$$

For example, $\alpha(t)$ can be some linear function and $p(t)$ can be a polynomial times $K_+(\alpha(t))$ whose coefficients can then be determined from the linear system (36).

We shall look for the set of m smooth functions $\{\tilde y_i(t)\}_{i=1}^m$ such that $\tilde y_i(0) = \tilde x_i$ and such that

$$\sum_{i=1}^m \int_0^1 \frac{\tilde y_i^l(t)\, d\tilde y_i(t)}{K_+(\tilde y_i(t))} = -e_{l+1}; \qquad l = 0, \dots, m-1. \tag{37}$$

In order to do this, consider the function

$$\tau(z) = \sum_{i=1}^k \int_0^1 \frac{p_i(t)\, d\alpha_i(t)}{K_+(\alpha_i(t))} \frac{\prod_{j=1}^m (\alpha(t) - \tilde y_i(t))}{(z - \alpha_i(t))\prod_{j=1}^m (z - \tilde y_i(t))}, \tag{38}$$

which behaves at $z \to \infty$ as $O(z^{-m+1})$. On the other hand,

$$\begin{aligned}\tau(z) = &\sum_{i=1}^k \int_0^1 \frac{p_i(t)\, d\alpha_i(t)}{K_+(\alpha_i(t))(z - \alpha_i(t))} \\ &- \sum_{i=1}^m \sum_{s=1}^k \int_0^1 \frac{p_s(t)\, d\alpha_s(t)}{K_+(\alpha_s(t))(z - \tilde y_i(t))} \frac{\prod_{i\neq j=1}^m (\alpha_s(t) - \tilde y_j(t))}{\prod_{i\neq j=1}^m (\tilde y_i(t) - \tilde y_j(t))}.\end{aligned} \tag{39}$$

So,

$$\tau(z) = \sum_{i=1}^{k} \int_0^1 \frac{p_i(t)\, d\alpha_i(t)}{K_+(\alpha_i(t))(z - \alpha_i(t))} - \sum_{i=1}^{m} \int_0^1 \frac{d\tilde{y}_i(t)}{K_+(\tilde{y}_i(t))(z - \tilde{y}_i(t))}, \tag{40}$$

iff

$$\tilde{y}_i'(t) = \frac{K(\tilde{y}_i(t))}{\prod_{i\neq j=1}^{m}(\tilde{y}_i(t) - \tilde{y}_j(t))} \left\{ \sum_{s=1}^{k} \frac{p_s(t)\alpha_s'(t)}{K(\alpha_s(t))} \prod_{i\neq j=1}^{m} (\alpha_s(t) - \tilde{y}_j(t)) \right\}. \tag{41}$$

The decomposition (40) provides (37) immediately when the Cauchy problem $\tilde{y}_i(0) = \tilde{x}_i$ for the system of ordinary differential equations (41) allows the direct application of numerical simulations. The functions $\tilde{y}_i(t)$ being found, one can construct the Baker-Akhiezer function $\psi(\mathbf{z})$ corresponding to $q(z)$ and $D = \tilde{x}_1^1 \tilde{x}_2^1 \dots \tilde{x}_m^1$ as

$$\psi(\mathbf{z}) = \exp\left\{ \sum_{i=1}^{m} \int_0^1 \frac{[K(\mathbf{z}) + K(\tilde{y}_i(t))]\, d\tilde{y}_i(t)}{2K(\tilde{y}_i(t))(z - \tilde{y}_i(t))} \right\}.$$

We shall say that the equations (41) belong to the extended class of Dubrovin's type equations for the two-sheet Riemann surface since they include the classical Dubrovin's equations [31]

$$\tilde{y}_i'(t) = \frac{K(\tilde{y}_i(t))}{\prod_{i\neq j=1}^{m}(\tilde{y}_i(t) - \tilde{y}_j(t))} \tag{42}$$

as a particular case and can be integrated in a similar way. Namely, let us consider the term in the big brackets in (41). The expression

$$\sum_{s=1}^{k} \frac{p_s(t)\alpha_s'(t)}{K(\alpha_s(t))} \prod_{i\neq j=1}^{m} (\alpha_s(t) - \tilde{y}_j(t))$$

with arbitrary functions $\alpha_i(t)$ and $p_i(t)$ is just a general form the polynomial $\tilde{y}_j(t)$, $i \neq j$. The choice of arbitrary functions allows one to set this expression, for example, to 1 in order to obtain the equations (42).

One can see from the way the equations (41) were constructed that the generalized Abel transformation (the basis of Abel differentials is not normalized in our case):

$$\{\tilde{y}_j(t)\}_{j=1}^m \to \{\xi_j(t)\}_{j=1}^m,$$

$$\xi_l(t) = \sum_{i=1}^{m} \int_{\tilde{x}_i}^{\tilde{y}_i(t)} \frac{z^l\, dz}{K(z)},$$

integrates the equations (41) since

$$\xi_l(t) = \sum_{i=1}^{k} \int_0^1 \frac{\alpha_i^l(t) p_i(t)\, d\alpha_i(t)}{K(\alpha_i(t))},$$

due to the properties of $\tau(z)$. The classical Dubrovin's equations cover some very important mechanical problems such as the problem of Kovalevskaja [31]. Thus, the extension of this class of equations can allow the construction of new integrable mechanical systems.

7. CONCLUSION

Let us briefly summarize results obtained in the paper. We have derived the Maluzhinetz functional equations for the three-anyon system in 1D with δ-interaction. General algebraic results we obtained for these equations allowed us to investigate the iterated equation with periodic coefficient. We reduced the iterated equation to the matrix Riemann-Hilbert problem with meromorphic coefficient on the open contour and studied its generating solutions. We described the construction of a generating solution through a Baker-Akhiezer function. We have derived the Dubrovin's type equations and discussed their relation to both Baker-Akhiezer function zeroes problem and integrable systems.

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