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Maria A. Kiseleva

Oscillations and Stability of Drilling Systems: Analytical and Numerical Methods

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ABSTRACT

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This work is devoted to study of electromechanical models of the drilling systems actuated by an induction motor. This subject is up-to-date due to the fact that failures of the drilling equipment cause significant time and expenditure losses for the drilling companies. Although there are many papers devoted to the investigation of the drilling systems, the equipment failures still occur in the drilling industry.

In this study, we continue investigations started by researchers from Eindhoven University of Technology who introduced an experimental model of a drilling system. The model consists of two discs connected with each other by a steel string which may experience torsional deformation only. The upper disc which represents the upper part of the drill-string is connected to the driving part. The lower disc, which represents the end of the drill-string, experiences friction torque caused mainly by interaction with a shale. The key idea of the present study, that expands and refines the experimental model, was to introduce more complex equations of the driving part and, particularly, to consider the induction motor.

Towards this end, two new mathematical models are considered. The first model is a simplified one. Its prototype is an electric hand drill. It can be assumed that the drill-string is absolutely rigid. Also, another model of a friction torque acting on the lower part of the drill-string is implemented. It is assumed that the friction torque has asymmetric characteristics of the Coulomb type. The qualitative analysis of this model made it possible to obtain conditions on permissible loads (i.e., permissible values of the friction torque) in case when the system remains in operational mode after the shale's type changes. With the help of computer modeling, a particular case when there is a sudden load appearance (i.e., when the drill was idle before the transient process) was also studied.

The second mathematical model takes into account the torsional deformation of the drill. For the friction torque with an asymmetric characteristic of the Coulomb type, local analysis of the system is provided. In the case of the friction model offered by the researchers from Eindhoven, the computer modeling of the system was carried out by the author. In the context of this modeling, an interesting effect represented by hidden oscillations of the stick-slip type was found.

The results of the study have been published in 10 papers (three of which are indexed in Scopus).

Keywords: drilling systems, induction motor, friction torque, limit load problem

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This work is dedicated to the loving memory of my parents Olga Ershova and Alexey Kiselev. I would like to extend my deepest thanks to them for making me the person I am today.

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1 INTRODUCTION

1.1 Intellectual merit

The breakdown of a drilling system (see Fig. 1[1]) is a common problem in the drilling industry. That is why the study of transient processes in the drilling equipment is very important both from the theoretical and practical perspectives.

The drilling equipment breakdown happens quite often in oil and gas industries. It leads to considerable time and expenditure losses. The discontinuation of the regular performance of a drill string due to the impact of certain loads is of the special interest. Indeed, due to the high cost of each failure, in order to reduce the number of drill string elements breakdowns, the study of drilling rigs is important. According to the statistics provided in (Horbeek et al., 1995; Shokir, 2004; Vaisberg et al., 2002), in 1985, 45 % of all rigs failures were directly related to the drill string. Nowadays, losses associated with each such failure of a drill-string are very expensive. Approximately one out of seven rigs experience such drill string failure. Notwithstanding, a lot of research devoted to lessening the failures of drill strings (Besselink et al., 2011; Mihajlovic et al., 2006; Mihajlović et al., 2007; Germay et al., 2009; Viguié et al., 2009), the data mentioned above shows that drill strings still break. Thus, the study of transient processes appearing in drilling rigs is relevant to the nowadays needs of theory and practice.

The main concern of this work is the study of the dynamics of more thorough models of drilling systems actuated by an induction electrical motor.

The stability criterion developed in this work allows one to obtain the range of permissible loads for the drill when the shale's type changes. In the case of double-mass model of a drilling system, the so-called hidden oscillations have been found. These oscillations cannot be detected after the transient process which begins in the neighborhood of a stable equilibrium. Thus, the breakdowns

[1] Dwight Burdette (Own work) [CC-BY-3.0 (http://creativecommons.org/licenses/by/3.0)], via Wikimedia Commons. 2012. Oil Drilling Rig Saline Township Michigan. URL:http://commons.wikimedia.org/wiki/File%3AOil_Drilling_Rig_Saline_Township_Michigan.JPG. [Online; accessed 11-May-2013]

FIGURE 1 Photo of the drilling system

of the drilling equipment could be caused by the presence of such oscillations.

1.2 Goal of the work

The goal of this work is to create and to study mathematical models of the drilling systems which contain a more comprehensive description of the electrical drive operation in comparison with earlier investigations. The study of the new refined models includes the investigation of the influence of different loads on those models with the help of analytical and numerical methods of the investigation of dynamical systems, modern computational tools, and specialized mathematical software packages.

1.3 Methods of investigation

In this work, both analytical and numerical methods of the investigation of dynamical systems have been used. All the models included have been described with the help of differential equations with discontinuous right-hand sides. To this end, the methods of investigation of differential equations with discontinuous right-hand sides were used, including Filippov definition (see Appendix 2). In order to study the global stability of such systems, it was necessary to investigate the behavior of their trajectories in the regions of continuity. This approach enabled the use of the Lyapunov functions method (see Appendix 1) suitable for the investigation of ordinary differential equations with continuous right-hand sides. In the course of computer modeling, in order to avoid computational errors, it is of critical importance to correctly define the behavior of the system in the neighborhood of the discontinuity surface. This method of correct computer modeling is based on Filippov definition and is described in Appendix 3.

1.4 The main results

- A mathematical model of a drilling system with absolutely rigid drill string actuated by an induction motor is developed (PI; PIV).
- An appropriate load characteristics represented as a non-symmetrical dry friction is introduced. For the model with such friction type a limit load problem is solved (PI; PIV).
- A double-mass mathematical model of a drilling system actuated by an induction motor is developed. Local analysis of the system is provided for the friction torque with an asymmetric characteristic of the Coulomb type (PII; PIII; PIV).

1.5 Adequacy of the results

All analytical results developed in the study are rigorously proven. In the context of computer modelling, methods specifically designed for the integration of differential equations with discontinuous right-hand sides were used in order to avoid computational errors. The obtained results correlate with the results of other researchers, in particular, from Eindhoven University of Technology.

1.6 Novelty

In this work, for the first time, three different approaches had been used jointly: the construction and analysis of models of drilling systems (Eindhoven University of Technology) and electrical drives (Saint Petersburg State University), and the methods of numerical analysis (University of Jyväskylä). This allowed the author to study complex effects appearing in the drilling systems actuated by an induction motor.

1.7 Practicability

The obtained models enable for a more effective analysis of the performance of the drilling systems. Engineers may use the regions of stability of the drilling models in order to minimize failures. Also, it is demonstrated that the models may experience hidden stable oscillations which co-exist with a stable equilibrium state for certain systems. This means that in the course of computer modeling, it is possible to miss those hidden oscillation and to make a wrong assumption about the global stability of a system. This can result in the drilling system failure. In order to avoid that, it is necessary to use special approaches of investigation of such drilling systems.

1.8 Appraisal of the work and publications

The results of this work were reported at the international conferences "4th IEEE International Conference on Nonlinear Science and Complexity" (Budapest, Hungary –2012), XII International Conference "Stability and Oscillations of Nonlinear Control Systems", Pyatnitskiy conference (Moscow, Russia – 2012), "Sixth Polyakhov Readings" (Saint Petersburg, Russia – 2012), 4th All-Russian Multi-Conference on Control Problems "MKPU–2011" (Divnomorskoe, Russia – 2011), International Workshop "Mathematical and Numerical Modeling in Science and Technology" (Finland, Jyväskylä – 2010) and at the seminars of the department of Applied Cybernetics (Saint Petersburg State University, Russia 2009 – 2013) and the department of Information Technology (University of Jyväskylä, Finland 2009–2013).

The results of this dissertation were also published in 10 articles. The main results are published in four included articles. In articles PI–PIV, problem formulation belongs to the co-authors. In article PI, the author obtained estimates of the limiting value of the permissible rapidly alternating load for the model of a drilling system with a non-symmetrical dry friction. In articles (PII; PIII), a new model of a double-mass drilling system actuated by an induction motor is introduced. In the context of numerical analysis, hidden oscillations were found by the author. In PIV, different mathematical models of the drilling systems are studied. Also, the material of the dissertation is presented in (Kiseleva, 2012) and its extended version is in preparation in (Kiseleva, 2013).

2 THE MAIN CONTENT

2.1 Real drilling systems

In order to understand mathematical models of the drilling systems, it is necessary to know how real drilling equipment used in the oil and gas industry works.

The drilling systems of a rotary type are used for drilling wells for exploration and production of oil and gas (Mihajlovic, 2005). Schematic view of the rotary drilling system is depicted in Fig. 2 (Leine, 2000). A borehole is created with the help of a rock-cutting tool called drill-bit. The bit represents a short heavy segment containing a cutting device at the free end (Tucker and Wang, 1999) and is driven by a torque created at the surface by a motor with a mechanical transmission box. The rotary table is driven by the motor (large disc) via transmission. Rotary table acts as a kinetic energy storage unit. This energy is transmitted from the surface to the bit with the help of a drill-string. The drill-string mainly consists of drill pipes and can be up to 8 km long. The lowest part of the drill-string is the Bottom-Hole-Assembly (BHA). The BHA consists of drill collars and the bit, and it can reach several hundred meters in length.

During the drilling process, a real drill-string undergoes various types of vibrations (Mihajlovic, 2005; Mihajlović et al., 2007; Jansen, 1991; Leine, 2000; Leine and van Campen, 2002; Van den Steen, 1997): torsional (rotational), bending (lateral), axial (longitudinal) and hydraulic vibrations. Torsional vibrations are caused by nonlinear interaction between the bit and the shale or the drill-string and the borehole wall. Bending vibrations are often caused by the pipe eccentricity and they lead to centripetal forces during the rotation. Axial vibrations are due to the bouncing of the drilling bit on the shale during the rotation. Hydraulic vibrations are found in the circulation system, stemming from the pump pulsations. Extensive research on the subject of friction-induced torsional vibrations in the drill-string systems has already been conducted. Much of the research considers vibrations in the drill-string systems (Brett, 1992; Germay, 2002; Jansen and Van den Steen, 1995; Kreuzer and Kust, 1996a,b; Keuzer and Kust, 1997; Kust, 1998; Kyllingstad and Halsey, 1988; Jansen, 1991; Leine, 2000; Leine

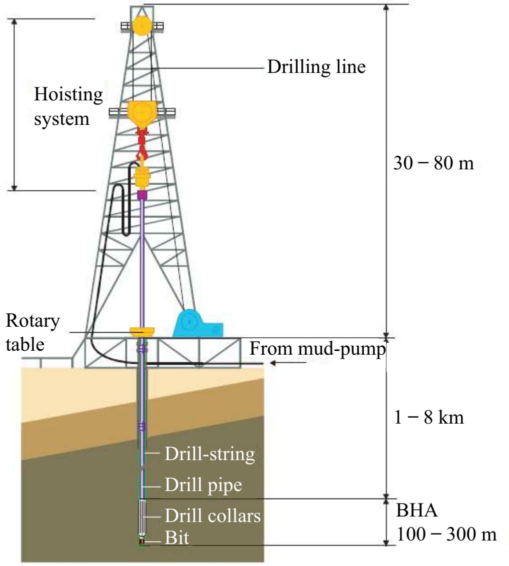

FIGURE 2 Schematic view of a real drilling rig

and van Campen, 2002; Mihajlovic et al., 2005, 2004a,b; Van den Steen, 1997). In most cases, it is concluded that torsional vibrations are caused by negative damping in the friction force present at the contact between the bit and the borehole (see, for example, (Brett, 1992; Kreuzer and Kust, 1996a; Mihajlovic et al., 2004a)). A number of experimental results provide additional evidence for such a conclusion (Brett, 1992; Leine, 2000; Leine and van Campen, 2002; Mihajlovic et al., 2005, 2004a,b; Van den Steen, 1997). Based on that conclusion, a control strategy is suggested in (Jansen and Van den Steen, 1995) to avoid torsional vibrations the in drill-string systems. It should be noted that Germay (Germay, 2002) and Richard et al. (Richard et al., 2004) have concluded that torsional vibrations in the drilling systems can appear due to the interaction between torsional and axial dynamics of the system. Moreover, according to these authors, such interaction effectively leads to the Stribeck effect.

In this work, only one type of vibrations is considered – torsional vibrations, since they are often regarded as one of the most damaging types of vibration (Omojuwa et al., 2011; Rajnauth, 2003).

2.2 A simple mathematical model of a drilling system actuated by induction motor. Limit load problem.

The aim of this work was to develop mathematical models of the drilling rigs, using an induction motor as the drive (see, e.g. (Hild, 1934; Staege, 1936; Hall and Shumway, 2009)), and to study the effect of different loads on a number of

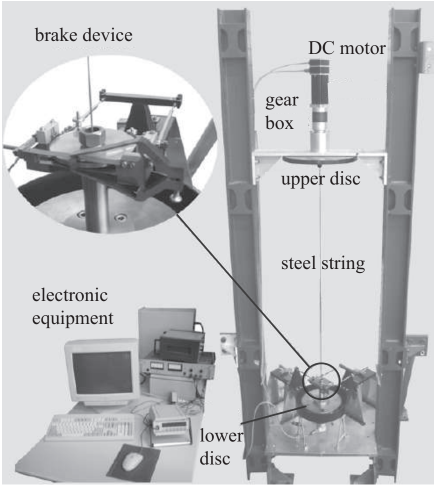

FIGURE 3 Photo of an experimental set-up of a drilling system

models. First, the model which motivated the research carried out in this work will be considered.

In the articles by Nijmeijer, van de Wouw, Mihailovi´c (De Bruin et al., 2009; Mihajlovic et al., 2006; Mihajlovic, 2005) an experimental rotor dynamic set-up was studied (see Fig. 3, (De Bruin et al., 2009)). The configuration of this set-up can be recognized in the structure of the drilling systems. The set-up mainly consists of the upper disc actuated by the driving part, flexible steel string, lower disc, and brake device. The upper and lower discs are connected with the steel string and may revolve on their axes (see Fig. 4 (Mihajlovic et al., 2006)). The brake device is used for modeling of the friction force acting on the lower disc.

Differential equations of rotation of the upper and lower discs are as follows[1]:

$$\begin{aligned} &J_u\ddot{\theta}_u(t) + k_\theta(\theta_u(t) - \theta_l(t)) + b(\dot{\theta}_u(t) - \dot{\theta}_l(t)) + T_{fu}(\dot{\theta}_u(t)) - k_m u = 0, \\ &J_l\ddot{\theta}_l(t) - k_\theta(\theta_u(t) - \theta_l(t)) - b(\dot{\theta}_u(t) - \dot{\theta}_l(t)) + T_{fl}(\dot{\theta}_l(t)) = 0. \end{aligned} \tag{1}$$

Here θ_u and θ_l are angular displacements of the upper and lower discs, respec-

[1] The derivation of this model can be explained in the following way (Kiseleva, 2013; Leonov et al., 2013a). In order to derive the model of the system we use the equations of the rotation of the upper and lower discs in the following form:

$$\begin{aligned} J_u\ddot{\theta}_u &= M_u - M_{ru}, \\ J_l\ddot{\theta}_u &= M_l - M_{rl}, \end{aligned}$$

where M_u, M_l are rotation torques, M_{ru}, M_{rl} are resistance torques. Here

$$\begin{aligned} M_u &= k_m u - T_{fu}(\dot{\theta}_u), \\ M_l &= M_{ru} = k_\theta(\theta_u - \theta_l) + b(\dot{\theta}_u - \dot{\theta}_l), \\ M_{rl} &= T_{fl}(\dot{\theta}_l), \end{aligned}$$

and M_u is a drive part torque. Later on, when considering the models actuated by an induction motor, this torque will undergo changes.

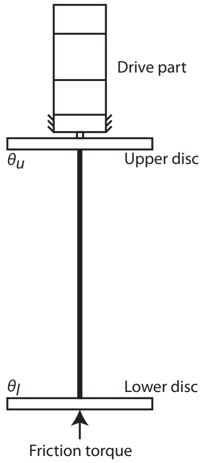

FIGURE 4 Scheme of mathematical model of drilling system

tively, J_u and J_l are inertia torques, k_θ, b, k_m are non-negative coefficients, u is a constant input voltage, $T_{fu}(\dot{\theta}_u)$ and $T_{fl}(\dot{\theta}_l)$ are friction torques acting on the upper and lower discs.

This double-mass model is convenient for the analysis of the drilling system model. However, this system lacks full dynamics consideration of the electric motor which actuates the upper disc.

The group from Saint Petersburg State University lead by Prof. Leonov has achieved significant results in the study of mathematical models of electrical machines. For example, in (Leonov and Kondrat'eva, 2009; Solovyeva, 2011; Zaretskiy, 2011) where new mathematical models of electrical machines have been developed, a new method of nonlocal reduction is suggested enabling for the improved estimates of the limit loads for the models considered. These models take into account the dynamics of the rotor of electrical machines and can be reduced to some other common models, such as a motor with squirrel-cage rotor (Leonov et al., 2013a). The derived models are described by rather simple differential equations which allow for the in-depth qualitative study of such models. This ensures that the implementation of this equations in the mechanical model of a drilling system may allow one to conduct more precise analysis (both analytical and numerical).

Here, for the both types of new models of the drilling systems, we are going to use the model of induction motor offered in (Leonov et al., 2013a).

In PI, a simple electro-mechanical model of the drilling system actuated by an induction motor is described, and the limit load problem is solved for that model. The behavior of the model is similar to the behavior of an ordinary drill (see Fig. 5[2]).

[2] By Kszapsza (Own work) [CC-BY-SA-3.0 (http://creativecommons.org/licenses/by-sa/3.0)], via Wikimedia Commons. 2013. Bosch power hammer drill model PSB 550 RE. URL:http://commons.wikimedia.org/wiki/File%3ABosch_PSB_550_RE_drill.JPG . [Online; accessed 11-May-2013]

FIGURE 5 Photo of hand electric drill

For such a drill, the torsional deformation is extremely small relatively to the rotation angle since the drill-string length is small enough. So, it can be assumed that the drill-string is absolutely rigid. Assume that it is stiffly connected to the rotor which rotates under the influence of magnetic field created by the stator of the induction motor. Interaction of the drill with the shale is defined by the resistance torque which occurs during the drilling process. Such system experiences rapidly alternating loads at the moment when the drill enters the bedrock. Thus, it is necessary to study the induction motor behavior during the load jump, i.e., when the friction toque suddenly experiences an abrupt change.

As the equations of the electromechanical model of the drilling system, let us consider the equations of the induction motor (see Appendix 4, (Leonov et al., 2013a); similarly some other types of induction motors (Leonhard, 2001; Khalil and Grizzle, 2002; PI; PIV) may also be considered) supplemented by the resistance torque M_f:

$$\begin{aligned}
&L\dot{i_1} + Ri_1 = -nBS\dot{\theta}\cos\left(\tfrac{\pi}{2} - \theta\right),\\
&L\dot{i_2} + Ri_2 = -nBS\dot{\theta}\cos\left(\tfrac{\pi}{2} - \theta - \tfrac{2\pi}{3}\right),\\
&L\dot{i_3} + Ri_3 = -nBS\dot{\theta}\cos\left(\tfrac{\pi}{2} - \theta - \tfrac{4\pi}{3}\right),\\
&I\ddot{\theta} = nBS\sum_{k=1}^{3} i_k \cos\left(\tfrac{\pi}{2} - \theta - \tfrac{2(k-1)\pi}{3}\right) + M_f(\omega_{mf} + \dot{\theta}).
\end{aligned} \tag{2}$$

Here θ is a rotation angle of the drill about the magnetic field which rotates with the constant angular speed ω_{mf}, i_1, i_2, i_3 are currents in the rotor coils, R is the resistance of the coils, L is the inductance of the coils, B is the induction of magnetic field, n is the amount of winds in every coil, S is the area of one coil wind, I is the inertia torque of the drill, $\omega = \dot{\theta} + \omega_{mf}$ is the angular velocity of the drill rotation with respect to a fixed coordinate system. Assume that the resistance

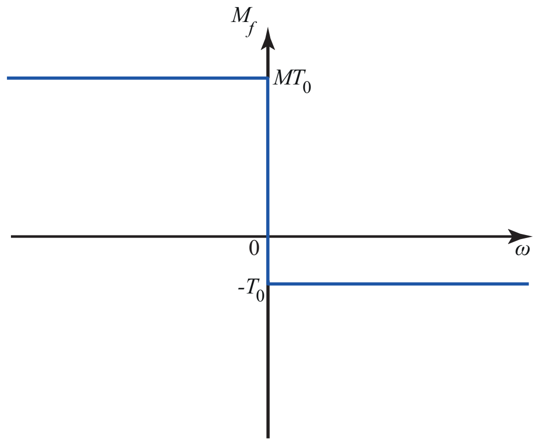

FIGURE 6 Friction torque M_f

torque M_f is of the Coulomb type. But in contrast to the classic Coulomb friction law with symmetrical discontinuous characteristics, friction torque M_f has non-symmetrical discontinuous characteristics depicted on Fig. 6 (see PI)

$$M_f \in \begin{cases} -T_0, & \text{if } \omega > 0 \\ [-T_0, MT_0], & \text{if } \omega = 0 \\ MT_0, & \text{if } \omega < 0. \end{cases} \tag{3}$$

Where $M, T_0 > 0$, number M is assumed to be large enough. This condition reflects the fact that the drilling process only takes place when $\omega > 0$. In real systems, during the transient process, such characteristic doesn't allow for switching from positive to negative ω. In this case, the system may only get stuck when $\omega = 0$ for some period of time. Such effects frequently happen during the drilling process.

Note that in (3) we used a sign $\in$ of differential inclusion. The notion of differential inclusion is directly connected to the notion of differential equations with a discontinuous right-hand side. Many papers such as (Andronov et al., 1981; Barbashin, 1967; Gelig et al., 1978; Neimark, 1972) and other articles are devoted to this subject. A detailed description of this theory can be found in the works by Filippov (Filippov, 1988, 1985). In (Filippov, 1985), Filippov considered differential equations with single-valued discontinuous right-hand sides, introduced a concept of solution and proved the basic results of the qualitative theory (see the main notions and approaches in Appendix 2).

Performing the nonsingular change of variables (Leonov et al., 2013a)

$$
\begin{aligned}
s &= -\dot{\theta}, \\
x &= -\tfrac{2}{3}\tfrac{L}{nSB}\sum_{k=1}^{3} i_k \sin\left(\tfrac{\pi}{2}-\theta-\tfrac{2(k-1)\pi}{3}\right), \\
y &= -\tfrac{2}{3}\tfrac{L}{nSB}\sum_{k=1}^{3} i_k \cos\left(\tfrac{\pi}{2}-\theta-\tfrac{2(k-1)\pi}{3}\right)
\end{aligned}
\tag{4}
$$

we reduce system (2) to the following system (PI; PIV):

$$
\begin{aligned}
\dot{s} &= ay + \xi(s,y), \\
\dot{y} &= -cy - s - xs, \\
\dot{x} &= -cx + ys,
\end{aligned}
\tag{5}
$$

where

$$
a = \frac{3(nSB)^2}{2IL}, \quad c = \frac{R}{L}.
$$

Here variables x, y define electric values in the rotor windings and the variable s defines the sliding of the rotor. For $\xi(s,y)$, the following Filippov definition is valid:

$$
\xi(s,y) = \begin{cases} \gamma, & \text{if } s = \omega_{mf}, y < -\dfrac{\gamma}{a} \text{ or } s < \omega_{mf} \\ -\gamma M, & \text{if } s = \omega_{mf}, y > \dfrac{M\gamma}{a} \text{ or } s > \omega_{mf} \\ -ay, & \text{if } s = \omega_{mf}, -\dfrac{\gamma}{a} \le y \le \dfrac{M\gamma}{a}, \end{cases}
$$

where $\gamma = \dfrac{T_0}{I}$.

Let us introduce the parameter

$$
\gamma_{max} = \frac{ac\omega_{mf}}{c^2 + {\omega_{mf}}^2}.
$$

The local analysis of equilibrium states of system (5) shows that for $0 \le \gamma < \gamma_{max}$ it has the unique asymptotically stable state of equilibrium.

Indeed, in the case $\gamma = 0$, system (5) has one asymptotically stable equilibrium state $s = 0, y = 0, x = 0$, which corresponds to the rotation of the drill with constant angular speed being congruent to the rotation speed of the magnetic field (idle speed operation).

For $\gamma \in (0, \gamma_{max})$ system (5) has one equilibrium state:

$$
s_0 = \frac{c(a - \sqrt{a^2 - 4\gamma^2})}{2\gamma}, \qquad y_0 = -\frac{\gamma}{a}, \qquad x_0 = -\frac{\gamma s_0}{ac},
$$

where s_0 is the smallest root of the equation

$$
\frac{acs}{c^2 + s^2} = \gamma.
$$

In this case, the drill rotates in the same direction as the magnetic field does but with a lower angular speed $s_0 < \omega_{mf}$.

Let us assume that at the moment $t = \tau$ there is a sudden change of the load from the value γ_0 to the value γ_1, where $0 < \gamma_0 < \gamma_1 < \gamma_{max}$. Such situation occurs when the drill reaches the bedrock. For $\gamma = \gamma_0$ the system has the unique stable equilibrium state

$$s_0 = \frac{c(a - \sqrt{a^2 - 4\gamma_0{}^2})}{2\gamma_0}, \quad y_0 = -\frac{\gamma_0}{a}, \quad x_0 = -\frac{\gamma_0 s_0}{ac}.$$

It is essential, that after the transient process, the solution $s(t), x(t), y(t)$ of system (5) with $\gamma = \gamma_1$ and the initial data

$$s(\tau) = \frac{c(a - \sqrt{a^2 - 4\gamma_0{}^2})}{2\gamma_0}, \quad y(\tau) = -\frac{\gamma_0}{a}, \quad x(\tau) = -\frac{\gamma_0 s_0}{ac}$$

tends to the equilibrium state

$$s_1 = \frac{c(a - \sqrt{a^2 - 4\gamma_1{}^2})}{2\gamma_1}, \quad y_1 = -\frac{\gamma_1}{a}, \quad x_1 = -\frac{\gamma_1 s_1}{ac}$$

when $t \to +\infty$.

The following theorem is proved in the included articles (PI; PIV).

Theorem 1 *Let the following conditions be fulfilled*

$$\gamma_0 < \gamma_{max},$$

$$\gamma_1 < \min\left\{\gamma_{max}, \quad 2c^2\right\},$$

$$\frac{(\gamma_1 - \gamma_0)^2}{2c^2} s_0{}^2 + \frac{(\gamma_1 - \gamma_0)^2}{2} \leq \int\limits_{s_0}^{\omega_{mf}} \phi(s) ds + \frac{(1 + M)^2}{2} \gamma_1{}^2.$$

Then the solution of system (5) with $\gamma = \gamma_1$ and the initial data

$$s(\tau) = \frac{c(a - \sqrt{a^2 - 4\gamma_0{}^2})}{2\gamma_0}, \quad y(\tau) = -\frac{\gamma_0}{a}, \quad x(\tau) = -\frac{\gamma_0 s_0}{ac}$$

tends to an equilibrium state of this system when $t \to +\infty$.

The main idea of proof is to build the following Lyapunov function (see Appendix 1) in the region of continuity:

$$V(x, y, s) = \frac{a^2}{2}(x + \frac{\gamma_1}{ac} s)^2 + \frac{1}{2}(ay + \gamma_1)^2 + \int\limits_{s_1}^{s} \left(-\frac{\gamma_1}{c} s^2 + as - c\gamma_1\right) ds.$$

The corollaries formulated below follow naturally from Theorem 1.

Corollary 1 *Let the following conditions be fulfilled*

$$\gamma_0 < \gamma_{max},$$

$$\gamma_1 < \min\left\{\gamma_{max}, \quad 2c^2\right\},$$

$$3(M^2+2M)\gamma_1^2 - 8c^2\gamma_1 + 3ac^2 \geq 0.$$

Then the solution of system (5) with $\omega_{mf} = c$, $\gamma = \gamma_1$ *and the initial data*

$$s(\tau) = \frac{c(a - \sqrt{a^2 - 4\gamma_0{}^2})}{2\gamma_0}, \quad y(\tau) = -\frac{\gamma_0}{a}, \quad x(\tau) = -\frac{\gamma_0 s_0}{ac}$$

tends to the equilibrium state of this system when $t \to +\infty$.

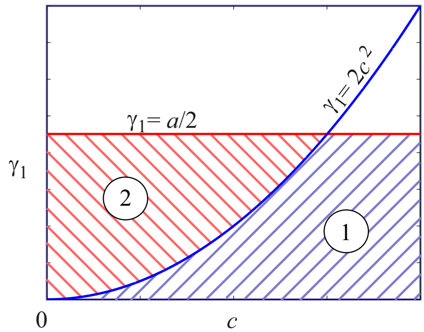

FIGURE 7 Regions of acceptable load: 1 – due to the theorem conditions, 2 – due to computer modeling results

Corollary 2 *Let* M *be sufficiently large positive number,* $\omega_{mf} = c$, $\gamma_0 = 0$ *and*

$$\gamma_1 < \min\left\{\frac{a}{2}, \quad 2c^2\right\}. \tag{6}$$

Then the solution of system (5) with $\gamma = \gamma_1$ *and the initial data*

$$s(\tau) = 0, \quad y(\tau) = 0, \quad x(\tau) = 0$$

tends to the equilibrium state of this system when $t \to +\infty$.

In the case when

$$2c^2 < \frac{a}{2} \quad \text{for} \quad \gamma_1 \in \left(2c^2, \frac{a}{2}\right)$$

(i.e., condition (6) is not fulfilled), the computer modeling of system (5) was carried out (area 2 on Fig. 7, see PI), which showed that the corollary statement remains valid. The computer modeling is based on methods applicable to the systems with discontinuous right-hand sides (see Appendix 3).

2.3 A double-mass mathematical model of a drilling system actuated by an induction motor

In PII-PIV, electromechanical model of a drilling system with induction motor, which takes into account torsional deformation of the drill string is studied. Let us extend the double-mass model of a drilling system considered above by adding the equations of an induction motor (see Appendix 4, (Leonov et al., 2013a)):

$$
\begin{aligned}
&L\dot{i_1} + Ri_1 = -nBS\dot{\theta_u}\cos\left(\frac{\pi}{2} - \theta_u\right),\\
&L\dot{i_2} + Ri_2 = -nBS\dot{\theta_u}\cos\left(\frac{\pi}{2} - \theta_u - \frac{2\pi}{3}\right),\\
&L\dot{i_3} + Ri_3 = -nBS\dot{\theta_u}\cos\left(\frac{\pi}{2} - \theta_u - \frac{4\pi}{3}\right),\\
&J_u\ddot{\theta}_u + k_\theta(\theta_u - \theta_l) + b(\dot{\theta}_u - \dot{\theta}_l) - nBS\sum_{k=1}^{3} i_k \cos\left(\frac{\pi}{2} - \theta_u - \frac{2(k-1)\pi}{3}\right) = 0,\\
&J_l\ddot{\theta}_l - k_\theta(\theta_u - \theta_l) - b(\dot{\theta}_u - \dot{\theta}_l) + T_{fl}(\omega_{mf} + \dot{\theta}_l) = 0.
\end{aligned}
$$

Here θ_u, θ_l are angular displacements of the rotor and the lower disc relatively to the rotating magnetic field, ω_{mf} is a rotation speed of the magnetic field, $T_{fl}(\omega_{mf} + \dot{\theta}_l)$ is a friction torque.

Performing the nonsingular change of variables

$$
\begin{aligned}
&s = -\dot{\theta}_u,\\
&x = -\frac{2}{3}\frac{L}{nSB}\sum_{k=1}^{3} i_k \sin\left(\frac{\pi}{2} - \theta_u - \frac{2(k-1)\pi}{3}\right),\\
&y = -\frac{2}{3}\frac{L}{nSB}\sum_{k=1}^{3} i_k \cos\left(\frac{\pi}{2} - \theta_u - \frac{2(k-1)\pi}{3}\right),\\
&u = -\dot{\theta}_l,\\
&\theta_{rel} = \theta_u - \theta_l,
\end{aligned}
$$

we obtain the system

$$
\begin{aligned}
&\dot{x} = -cx + ys,\\
&\dot{y} = -cy - s - xs,\\
&\dot{\theta}_{rel} = u - s,\\
&\dot{s} = \frac{k_\theta}{J_u}\theta_{rel} + \frac{b}{J_u}(u - s) + \frac{a}{J_u}y,\\
&\dot{u} = -\frac{k_\theta}{J_l}\theta_{rel} - \frac{b}{J_l}(u - s) + \frac{1}{J_l}T_{fl}(\omega_{mf} - u).
\end{aligned}
\tag{7}
$$

Here $a = \dfrac{3(nSB)^2}{2L}$, $c = \dfrac{R}{L}$.

Consider the case when the friction force has non-symmetrical characteristics considered above:

$$T_{fl}(\omega_l) \in \begin{cases} \gamma, & \text{if } \omega_l > 0 \\ [-M\gamma, \gamma], & \text{if } \omega_l = 0 \\ -M\gamma, & \text{if } \omega_l < 0, \end{cases}$$

where $\omega_l = \omega_{mf} - u$. Here $M, \gamma > 0$.

The results of the local analysis of the system are formulated in the following theorem:

Theorem 2 *For*

$$b = 0, \quad 0 < \gamma < \gamma_{max} = \frac{ac\omega_{mf}}{c^2 + \omega_{mf}{}^2}$$

the system (7) has one asymptotically stable equilibrium state:

$$s_0 = u_0, \quad x_0 = -\frac{\gamma s_0}{ac}, \quad y_0 = -\frac{\gamma}{a}, \quad \theta_{rel0} = \frac{\gamma}{k_\theta},$$

where s_0 is the smallest root of the equation

$$\frac{acs}{c^2 + s^2} = \gamma.$$

The proof of this theorem can be found in Appendix 5. It is similar to the proof of the theorem described in AI.

In the course of the numerical analysis of the system, stable operation modes of the drilling system as well as the modes when the drill gets stuck were found. Consider more complex model of the friction. Assume that the friction torque is as follows (see Fig. 8, (De Bruin et al., 2009))

$$T_{fl}(\omega_l) \in \begin{cases} T_{cl}(\omega_l)\,\mathrm{sign}(\omega_l), & \text{if } \omega_l \neq 0 \\ [-T_{sl}, T_{sl}], & \text{if } \omega_l = 0, \end{cases} \tag{8}$$

$$T_{cl}(\omega_l) = T_{fl} + (T_{sl} - T_{fl})e^{-|\frac{\omega_l}{\omega_{sl}}|^{\delta_{sl}}} + b_l|\omega_l|, \tag{9}$$

where T_{sl}, T_{fl}, ω_{sl}, δ_{sl} and b_l are non-negative coefficients.

Qualitative analysis of such systems is a complex task due to the friction type.

Following the works (Leonov et al., 2012; Bragin et al., 2011; Leonov and Kuznetsov, 2011b,a; Kuznetsov et al., 2011b; Leonov et al., 2010c; Bragin et al., 2010; Leonov et al., 2010b), let us describe some aspects of the numerical modeling of oscillations of continuous dynamical systems which turned out to be extremely important for engineers as a practical matter. Computer modeling of the system was carried out. Here again, for modeling a system with a discontinuous right-hand side a special numerical method is required (see Appendix 3). For such a system, frictional oscillations are of the special interest. There is a number of papers devoted to frictional oscillations (Hensen, 2002; Hensen and van de Molengraft, 2002; Juloski et al., 2005; Mallon, 2003; Mallon et al., 2006; Olsson, 1996; Olsson and Astrom, 1996, 2001; Putra, 2004; Putra et al., 2004; Putra and

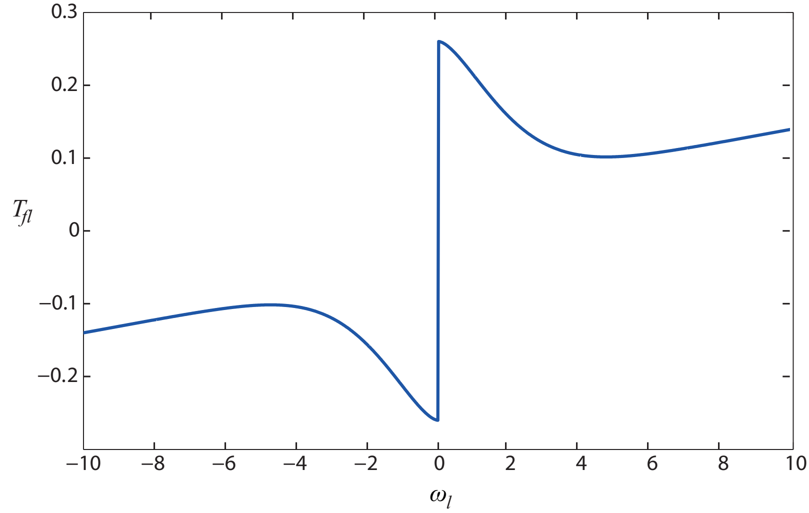

FIGURE 8 Friction torque T_{fl}

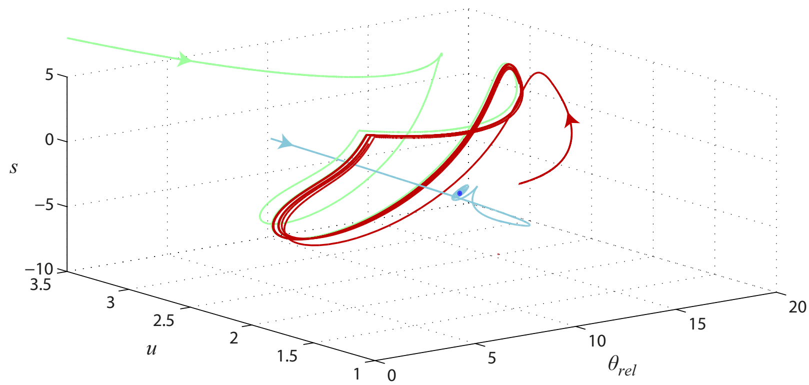

FIGURE 9 Hidden oscillations in space (θ_{rel}, s, u)

Nijmeijer, 2003, 2004; van de Wouw et al., 2005; Al-Bender et al., 2004; Batista and Carlson, 1998) due to the fact that these oscillations may cause wear or damage of various mechanical systems. Under the certain values of parameters, the so-called *hidden oscillations* (Leonov et al., 2011c; Bragin et al., 2011; Leonov et al., 2012; Leonov and Kuznetsov, 2013) may emerge – oscillations which basin of attraction does not intersect with small neighborhoods of equilibrium states.

The development of modern computer technology allows for the numerical simulation of complex nonlinear dynamical systems. Consequently, new information about the behavior of their trajectories can be obtained. In the well-known Duffing system (Duffing, 1918), Van der Pol system (van der Pol, 1927) Belousov-Zhabotinsky system (Belousov, 1959), Lorenz system (Lorenz, 1963), Roessler sys-

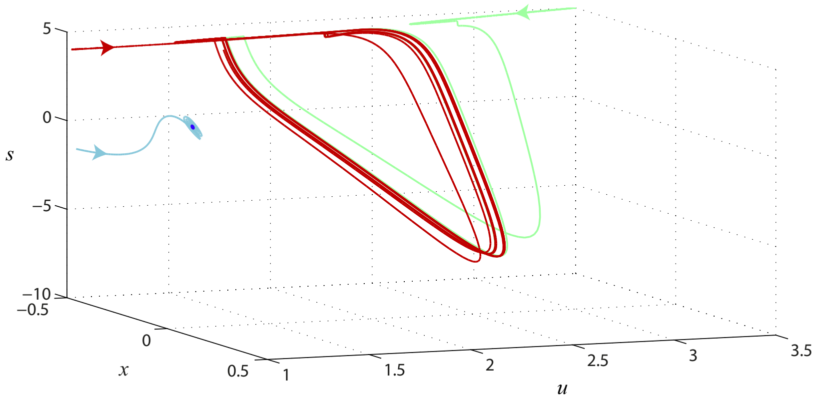

FIGURE 10 Hidden oscillations in space $\{x, s, u\}$

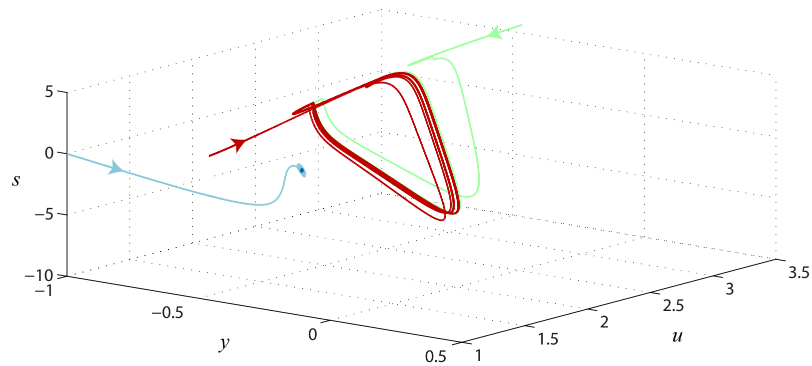

FIGURE 11 Hidden oscillations in space $\{y, s, u\}$

tem (Rössler, 1976) and other systems, classic *self-exciting* oscillations and the attractors can be obtained numerically by using *the standard computational procedure*: after the transient process, a trajectory, which begins in the neighborhood of the unstable state of equilibrium, reaches an oscillation and defines it.

However, the possibilities of this approach are limited. In the middle of the last century, in the systems with a scalar nonlinearity, oscillations of another type were obtained. These are hidden oscillations and they can not be calculated in the way described above. In this case, the simulation of trajectories with random initial data will unlikely give the desired result (e.g., the description of the experiment by Kolmogorov related to the search of limit cycles (Arnol'd, 2005; Kuznetsov, 2008; Leonov et al., 2008; Kuznetsov and Leonov, 2008; Leonov, 2010; Leonov and Kuznetsov, 2010; Bragin et al., 2011; Kuznetsov et al., 2013b)), as the domain of attraction can be very small and the dimension of the attractor can be significantly smaller than the dimension of the system.

In 1961, Gubar (Gubar', 1961; Leonov and Kuznetsov, 2013) demonstrated analytically the possibility of hidden oscillations in a two-dimensional phase-locked loop with a piecewise constant pulse nonlinearity. In the 50-60s of the last century, the study of known hypotheses (Markus and Yamabe, 1960; Aizerman, 1949; Kalman, 1957) about absolute stability have led to finding hidden oscillations in automatic control systems with piecewise-linear non-linearity, which belongs to the linear stability region (see (Pliss, 1958; Bernat and Llibre, 1996; Leonov et al., 2010a; Leonov and Kuznetsov, 2013; Bragin et al., 2011), etc.).

Recently, chaotic hidden oscillations (hidden attractors) were found in the Chua system (Kuznetsov et al., 2010, 2011c; Leonov et al., 2011a; Bragin et al., 2011; Leonov et al., 2011c; Kuznetsov et al., 2011a; Leonov et al., 2011b; Vagaitsev, 2012; Leonov et al., 2012; Kuznetsov et al., 2013a; Leonov and Kuznetsov, 2013).

In PII-PIV, hidden oscillations for a system describing electromechanical double-mass model of the drilling system driven by an induction motor were found.

In Fig. 9–11 (see PII) stable equilibrium state and stable limit cycle are depicted. According to the above definition, this fact implies that the system has hidden frictional oscillations.

This result shows that such complex phenomena as hidden oscillations appear even in rather simple models. Because the possibility of finding hidden oscillations while modeling a system with random data is low due to the small region of attraction, the detection of hidden oscillations is a difficult task. Therefore, there is a need to develop new approaches for the study of such systems. Note that in our case hidden oscillations are of the stick-slip type – they pass through the sliding region which appears due to the existence of the discontinuity. Thus, it is recommended to carry out the modelling in the neighborhood of the discontinuity surface.

REFERENCES

Adkins, B. (1957). *The General Theory of Electrical Machines*. John Wiley and Sons Inc.

Aizerman, M.A. (1949). The problem of the stability in the "large" of dynamical systems. *Russian Mathematical Surveys*, 4(4), 187–188.

Al-Bender, F., Lampaert, V., and Swevers, J. (2004). Modeling of dry sliding friction dynamics: From heuristic models to physically motivated models and back. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 14(2), 446–460.

Andronov, A.A., Vitt, A.A., and Khaikin, S.E. (1966). *Theory of oscillations*. Fizmatgiz, Moscow, 1959; English transl. Pergamon Press, Oxford, and Addison-Wesley, Reading, Mass.

Andronov, A., Vitt, A., and Haikin, S. (1981). *Theory of oscillations*. Nauka, Glavnaya redaktsiya fiziko-matematicheskoy literaturi [in Russian].

Arnol'd, V. (2005). *Experimental Mathematics*. Fazis, Moscow[in Russian].

Ashino, R., Nagase, M., and Vaillancourt, R. (2000). Behind and beyond the MATLAB ODE suite. *Computers & Mathematics with Applications*, 40(4), 491–512.

Barbashin, E. (1967). *Introduction to stability theory*. Nauka, Glavnaya redaktsiya fiziko-matematicheskoy literaturi [in Russian].

Barbashin, E. and Krasovsky, N. (1952). Stability of oscillations in general. *Doklady Akademii Nauk USSR*, 86(3), 453–456 [in Russian].

Batista, A. and Carlson, J. (1998). Bifurcations from steady sliding to stick slip in boundary lubrication. *Physical Review E*, 57(5), 4986.

Belousov, B. (1959). Collection of short papers on radiation medicine. *Medgiz, Moskow*.

Bernat, J. and Llibre, J. (1996). Counterexample to Kalman and Markus-Yamabe conjectures in dimension larger than 3. *Dynamics of Continuous, Discrete and Impulsive Systems*, 2(3), 337–279.

Bespalov, V. (1992). *Induction motors for dynamical operation modes*. Moscow Power Engineering Institute [in Russian].

Bespalov, V., Moshinsky, Y., and Petrov, A. (2002). Mathematical model of an induction motor in generalized orthogonal coordinate system. *Electrichestvo*, (8), 33–39 [in Russian].

Besselink, B., Van De Wouw, N., and Nijmeijer, H. (2011). A semi-analytical study of stick-slip oscillations in drilling systems. *Journal of Computational and Nonlinear Dynamics*, 6(2).

Bothe, D. (1999). Periodic solutions of non-smooth friction oscillators. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 50(5), 779–808.

Bragin, V.O., Kuznetsov, N.V., and Leonov, G.A. (2010). Algorithm for counterexamples construction for Aizerman's and Kalman's conjectures. *IFAC Proceedings Volumes (IFAC-PapersOnline)*, 4(1), 24–28. doi:10.3182/20100826-3-TR-4016.00008.

Bragin, V.O., Vagaitsev, V.I., Kuznetsov, N.V., and Leonov, G.A. (2011). Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits. *Journal of Computer and Systems Sciences International*, 50(4), 511–543. doi:10.1134/S106423071104006X.

Brett, J.F. (1992). The genesis of torsional drillstring vibrations. *SPE Drilling Engineering*, 7(3), 168–174.

De Bruin, J.C.A., Doris, A., van de Wouw, N., Heemels, W.P.M.H., and Nijmeijer, H. (2009). Control of mechanical motion systems with non-collocation of actuation and friction: A popov criterion approach for input-to-state stability and set-valued nonlinearities. *Automatica*, 45(2), 405–415.

Drury, B. (2001). *Control Techniques Drives and Controls Handbook*. The Institution of Electrical Engineers.

Duffing, G. (1918). *Erzwungene Schwingungen bei Veranderlicher Eigenfrequenz*. Friedrich Vieweg & Sohn Verlag, Braunschweig.

Fečkan, M. (1997). Bifurcation of periodic solutions in differential inclusions. *Applications of Mathematics*, 42(5), 369–393.

Fečkan, M. (1999). Chaotic solutions in differential inclusions: chaos in dry friction problems. *Transactions of the American Mathematical Society*, 351(7), 2861–2873.

Filippov, A. (1960). Differential equations with discontinuous righthand side. *Matematichesky Sbornik*, 51(1), 99–128 [in Russian].

Filippov, A. (1985). *Differential Equations with Discontinuous Righthand Side*. Nauka, Moscow [in Russian].

Filippov, A. (1988). *Differential Equations with Discontinuous Righthand Sides*, volume 18 of *Mathematics and its Applications*. Springer.

Gelig, A., Leonov, G., and Yakubovish, V. (1978). *Stability of nonlinear systems with non-unique equilibrium state*. Nauka [in Russian].

Germay, C. (2002). Self-excited oscillations of drag bits. Master Thesis. *The University of Minnesota, The Faculty of the Graduate School*.

Germay, C., Van de Wouw, N., Nijmeijer, H., and Sepulchre, R. (2009). Nonlinear drillstring dynamics analysis. *SIAM Journal on Applied Dynamical Systems*, 8(2), 527–553.

Golubev, A. and Zikov, B. (2003). Mathematical model of an induction motor with multi-phase windings of stator and rotor. *Electrotehnika*, (7), 35–40 [in Russian].

Gruzov, L. (1953). *Methods of mathematical investigation of electrical machines*. Gosenergoizdat M.- L. [in Russian].

Gubar′, N. (1961). Investigation of a piecewise linear dynamical system with three parameters. *Journal of Applied Mathematics and Mechanics*, (25), 1519–1535.

Hall, D.R. and Shumway, J. (2009). *System for steering a drill string*. US Patent 7,600,586.

Hensen, R. (2002). Controlled mechanical systems with friction. *Technische Universiteit Eindhoven*.

Hensen, R. and van de Molengraft, M. (2002). Friction induced hunting limit cycles: an event mapping approach. In *American Control Conference, 2002. Proceedings of the 2002*, volume 3, 2267–2272. IEEE.

Higham, D.J. and Higham, N.J. (2005). *MATLAB guide*. SIAM, Philadelphia.

Hild, F.W. (1934). *Earth drilling system*. US Patent 1,962,346.

Horbeek, J.H., Birch, W., McMahon, M.J., and Fearnley, K. (1995). Successful reduction of north sea drillstring failures. *Offshore Europe*, 43–51.

Hrisanov, V. and Brxhezinsky, R. (2003). Adequacy questions of mathematical models of an asynchronous motor in context of analysis of the launch transient processes. *Electrotechnica*, (10), 20–25 [in Russian].

Hurwitz, A. (1964). On the conditions under which an equation has only roots with negative real parts. *Selected Papers on Mathematical Trends in Control Theory (R. Bellman and R. Kalaba, eds.)*, 65.

Jansen, J.D. (1991). Non-linear rotor dynamics as applied to oilwell drillstring vibrations. *Journal of Sound and Vibration*, 147(1), 115–135.

Jansen, J.D. and Van den Steen, L. (1995). Active damping of self-excited torsional vibrations in oil well drillstrings. *Journal of Sound and Vibration*, 179(4), 647–668.

Juloski, A.L., Mihajlović, N., Heemels, W.P.M.H., Van de Wouw, N., and Nijmeijer, H. (2005). Friction induced hunting limit cycles: an event mapping approach. In *Proceeding of the 44th IEEE Conference on Decision and Control, Seville, Spain*. IEEE.

Kalman, R.E. (1957). Physical and mathematical mechanisms of instability in nonlinear automatic control systems. *Transactions of ASME*, 79(3), 553–566.

Keuzer, E. and Kust, O. (1997). Controlling torsional vibrations through proper orthogonal decomposition. In *IUTAM Symposium on Interaction between Dynamics and Control in Advanced Mechanical Systems*, 207–214. Springer.

Khalil, H.K. and Grizzle, J.W. (2002). *Nonlinear systems*. Prentice Hall, Upper Saddle River, 3rd edition.

Kiseleva, M. (2012). *Stability and Oscillations of Drilling Systems. Thesis*. Saint Petersburg State University [in Russian].

Kiseleva, M. (2013). *Oscillations of Dynamical Systems Applied in Drilling: Analytical and Numerical Methods. PhD Thesis*. Jyväskylä University Printing House [in preparation].

Kopilov, I. (2001a). *Mathematical modeling of electrical machines*. Visshaya shkola, Moscow [in Russian].

Kopilov, I. (2001b). *Mathematical modeling of electrical machines: college textbook*. Visshaya shkola, Moscow [in Russian].

Kopilov, I., Morozkin, V., and Klokov, B. (2002). *Engineering of electrical machines*. Visshaya shkola, Moscow [in Russian].

Kreuzer, E. and Kust, O. (1996a). Analyse selbsterregter drehschwingungen in torsionsstäben. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, 76(10), 547–557.

Kreuzer, E. and Kust, O. (1996b). Analysis of long torsional strings by proper orthogonal decomposition. *Archive of Applied Mechanics*, 67(1-2), 68–80.

Krishenko, A.P. (2007). *Stability of motion of autonomous systems*. Bauman Moscow State Technical University, Moscow [in Russian].

Kunze, M. (2000). *Non-smooth dynamical systems*, volume 1744. Springer Verlag.

Kunze, M. and Küpper, T. (1997). Qualitative bifurcation analysis of a non-smooth friction-oscillator model. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, 48(1), 87–101.

Kust, O. (1998). Selbsterregte drehschwingungen in schlanken torsionssträngen. *Nichtlineare Dynamik und Regelung, Dissertation, VDI Verlag, Reihe*, 11.

Kuznetsov, N., Kuznetsova, O., Leonov, G., and Vagaitsev, V. (2013a). *Informatics in Control, Automation and Robotics, Lecture Notes in Electrical Engineering, Vol. 174, Part 4*, chapter Analytical-numerical localization of hidden attractor in electrical Chua's circuit, 149–158. Springer. doi:10.1007/978-3-642-31353-0_11.

Kuznetsov, N.V. (2008). *Stability and Oscillations of Dynamical Systems: Theory and Applications*. Jyväskylä University Printing House.

Kuznetsov, N.V., Kuznetsova, O.A., and Leonov, G.A. (2013b). Visualization of four normal size limit cycles in two-dimensional polynomial quadratic system. *Differential equations and dynamical systems*, 21(1-2), 29–34. doi:10.1007/s12591-012-0118-6.

Kuznetsov, N.V., Kuznetsova, O.A., Leonov, G.A., and Vagaytsev, V.I. (2011a). Hidden attractor in Chua's circuits. *ICINCO 2011 - Proceedings of the 8th International Conference on Informatics in Control, Automation and Robotics*, 1, 279–283. doi:10.5220/0003530702790283.

Kuznetsov, N.V. and Leonov, G.A. (2008). Lyapunov quantities, limit cycles and strange behavior of trajectories in two-dimensional quadratic systems. *Journal of Vibroengineering*, 10(4), 460–467.

Kuznetsov, N.V., Leonov, G.A., and Seledzhi, S.M. (2011b). Hidden oscillations in nonlinear control systems. *IFAC Proceedings Volumes (IFAC-PapersOnline)*, 18(1), 2506–2510. doi:10.3182/20110828-6-IT-1002.03316.

Kuznetsov, N.V., Leonov, G.A., and Vagaitsev, V.I. (2010). Analytical-numerical method for attractor localization of generalized Chua's system. *IFAC Proceedings Volumes (IFAC-PapersOnline)*, 4(1), 29–33. doi:10.3182/20100826-3-TR-4016.00009.

Kuznetsov, N.V., Vagaytsev, V.I., Leonov, G.A., and Seledzhi, S.M. (2011c). Localization of hidden attractors in smooth Chua's systems. *International Conference on Applied and Computational Mathematics*, 26–33.

Kyllingstad, A. and Halsey, G.W. (1988). A study of slip/stick motion of the bit. *SPE Drilling Engineering*, 3(4), 369–373.

LaSalle, J.P. (1968). Stability theory for ordinary differential equations. *Journal of Differential Equations*, 4, 57–65.

Leine, R.I. (2000). Bifurcations in discontinuous mechanical systems of the fillippov-type. *Technische Universiteit Eindhoven*.

Leine, R.I. and van Campen, D.H. (2002). Stick-slip whirl interaction in drillstring dynamics. *Transactions of the ASME, 220Vol*, 124, 209–220.

Leonhard, W. (2001). *Control of electrical drives*. Springer.

Leonov, G.A. (2001). *Mathematical Problems of Control Theory: An Introduction*. World Scientific Publishing Company.

Leonov, G.A. (2006). Phase synchronization: Theory and application. *Automation and Remote Control*, 1573–1609.

Leonov, G.A., Bragin, V.O., and Kuznetsov, N.V. (2010a). Algorithm for constructing counterexamples to the Kalman problem. *Doklady Mathematics*, 82(1), 540–542. doi:10.1134/S1064562410040101.

Leonov, G.A., Bragin, V.O., and Kuznetsov, N.V. (2010b). On problems of Aizerman and Kalman. *Vestnik St. Petersburg University. Mathematics*, 43(3), 148–162. doi:10.3103/S1063454110030052.

Leonov, G.A. and Kuznetsov, N.V. (2010). Limit cycles of quadratic systems with a perturbed weak focus of order 3 and a saddle equilibrium at infinity. *Doklady Mathematics*, 82(2), 693–696. doi:10.1134/S1064562410050042.

Leonov, G.A. and Kuznetsov, N.V. (2011a). Analytical-numerical methods for investigation of hidden oscillations in nonlinear control systems. *IFAC Proceedings Volumes (IFAC-PapersOnline)*, 18(1), 2494–2505.

Leonov, G.A. and Kuznetsov, N.V. (2011b). Algorithms for searching for hidden oscillations in the Aizerman and Kalman problems. *Doklady Mathematics*, 84(1), 475–481. doi:10.1134/S1064562411040120.

Leonov, G.A. and Kuznetsov, N.V. (2013). Hidden attractors in dynamical systems. From hidden oscillations in Hilbert-Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractors in Chua circuits. *International Journal of Bifurcation and Chaos*, 23(1). doi:10.1142/S0218127413300024. art. no. 1330002.

Leonov, G.A., Kuznetsov, N.V., Kiseleva, M.A., Solovyeva, E.P., and Zaretskiy, A.M. (2013a). Mathematical model of a drilling system actuated by induction motor with phase rotor [in preparation]. *Nonlinear Dynamics*.

Leonov, G.A., Kuznetsov, N.V., and Kudryashova, E.V. (2008). Cycles of two-dimensional systems: Computer calculations, proofs, and experiments. *Vestnik St.Petersburg University. Mathematics*, 41(3), 216–250. doi:10.3103/S1063454108030047.

Leonov, G.A., Kuznetsov, N.V., Kuznetsova, O.A., Seledzhi, S.M., and Vagaitsev, V.I. (2011a). Hidden oscillations in dynamical systems. *Transaction on Systems and Control*, 6(2), 54–67.

Leonov, G.A., Kuznetsov, N.V., and Seledzhi, S.M. (2011b). Hidden oscillations in dynamical systems. *Recent researches in System Science*, 292–297.

Leonov, G.A., Kuznetsov, N.V., and Vagaitsev, V.I. (2011c). Localization of hidden Chua's attractors. *Physics Letters A*, 375(23), 2230–2233. doi:10.1016/j.physleta.2011.04.037.

Leonov, G.A., Kuznetsov, N.V., and Vagaitsev, V.I. (2012). Hidden attractor in smooth Chua systems. *Physica D: Nonlinear Phenomena*, 241(18), 1482–1486.

Leonov, G.A. and Solovyeva, E.P. (2012). On a special type of stability of differential equations for induction machines with double squirrel-cage rotor. *Vestnik St. Petersburg University. Mathematics*, 45(3), 128–135.

Leonov, G.A., Solovyeva, E.P., and Zaretskiy, A.M. (2013b). An estimation method of transient processes of induction machines. *Vestnik St. Petersburg University*, 1–21 [in print].

Leonov, G.A., Vagaitsev, V.I., and Kuznetsov, N.V. (2010c). Algorithm for localizing Chua attractors based on the harmonic linearization method. *Doklady Mathematics*, 82(1), 693–696. doi:10.1134/S1064562410040411.

Leonov, G. (2010). Efficient methods in the search for periodic oscillations in dynamical systems. *Journal of Applied Mathematics and Mechanics*, 74(1), 24–50.

Leonov, G. and Kondrat'eva, N. (2009). *Stability analisys of AC electrical machines*. Unipress, Saint Petersburg [in Russian].

Lorenz, E. (1963). Deterministic nonperiodic flow. *Journal of the Atmospheric Sciences*, 20(2), 130–141.

Lyapunov, A.M. (1950). *The general problem of the stability of motion*. Gostechizdat, Moscow [in Russian].

Mallon, N. (2003). Reduced-order observer based friction compensation for a controlled one-link robot. *Eindhoven University of Technology, Department of Mechanical Engineering*.

Mallon, N., van de Wouw, N., Putra, D., and Nijmeijer, H. (2006). Friction compensation in a controlled one-link robot using a reduced-order observer. *Control Systems Technology, IEEE Transactions on*, 14(2), 374–383.

Marchaud, A. (1936). Sur les champs continus de demi-cônes convexes et leurs intégrales. *Compositio Mathematica*, 3, 89–127.

Markus, L. and Yamabe, H. (1960). Global stability criteria for differential systems. *Osaka Journal of Mathematics*, 12(2), 305–317.

Mihajlovic, N. (2005). Torsional and lateral vibrations in flexible rotor systems with friction. *Technische Universiteit Eindhoven*.

Mihajlovic, N., Van de Wouw, N., Hendriks, M.P.M., and Nijmeijer, H. (2004a). Analysis of friction-induced limit cycling in an experimental drill-string set-up. *ASME Journal of Dynamic Systems, Measurements and Control*, 126(4), 709–720.

Mihajlovic, N., Van de Wouw, N., Hendriks, M.P.M., and Nijmeijer, H. (2005). Friction-induced vibrations in an experimental drill-string system for various friction situations. *Proceedings of the ENOC-2005*.

Mihajlovic, N., Van de Wouw, N., Hendriks, M.P.M., and Nijmeijer, H. (2006). Friction-induced limit cycling in flexible rotor systems: An experimental drill-string set-up. *Nonlinear Dynamics*, 46(3), 273–291.

Mihajlović, N., Van de Wouw, N., Rosielle, P.C.J.N., and Nijmeijer, H. (2007). Interaction between torsional and lateral vibrations in flexible rotor systems with discontinuous friction. *Nonlinear Dynamics*, 50(3), 679–699.

Mihajlovic, N., Van Veggel, A.A., Van de Wouw, N., Nijmeijer, H., and Den Dolech, P.O. (2004b). Friction-induced torsional vibrations in an experimental drill-string system. *Proceedings of the 23rd IASTED International Conference*.

Moshinsky, Y.A. and Petrov, A.P. (2001). Mathematical model of an induction capasitor-start motor according to the method of symmetrical components with the help of standart software. *Elektrichestvo*, 40–43 [in Russian].

Moshinsky, Y.A. and Petrov, A.P. (2007). Mathematical model for investigation of induction motor in the context of stopway. *Elektrotehnika*, 1, 19–22 [in Russian].

Moshinsky, Y.A. and T., A.V. (2007). Generalized mathematical model of frequency regulated induction motor taking into account steel losses. *Elektrichestvo*, 11, 60–66 [in Russian].

Neimark, Y.I. (1972). *Method of point mapping in the theory of nonlinear oscillations*. Nauka [in Russian].

Olsson, H. (1996). *Control systems with friction*. Department of Automatic Control, Lund Institute of Technology.

Olsson, H. and Astrom, K. (1996). Friction generated limit cycles. In *Control Applications, 1996., Proceedings of the 1996 IEEE International Conference on*, 798–803. IEEE.

Olsson, H. and Astrom, K. (2001). Friction generated limit cycles. *Control Systems Technology, IEEE Transactions on*, 9(4), 629–636.

Omojuwa, E., Osisanya, S., and Ahmed, R. (2011). Measuring and controlling torsional vibrations and stick-slip in a viscous-damped drillstring model. *International Petroleum Technology Conference, Bangkok, Thailand*.

Pankratov, V. and Zima, E. (2003). Mathematical modeling og induction electrical machines and double-powered machines. *Elektrotehnika*, (9), 19–25 [in Russian].

Piiroinen, P.T. and Kuznetsov, Y.A. (2008). An event-driven method to simulate filippov systems with accurate computing of sliding motions. *ACM Transactions on Mathematical Software (TOMS)*, 34(3), 13.

Pliss, V.A. (1958). *Some Problems in the Theory of the Stability of Motion*. Izd. LGU, Leningrad [in Russian].

Popescu, M. (2000). Induction motor modelling for vector control purposes. Technical report.

Putra, D. (2004). Control of limit cycling in frictional mechanical systems. *Technische Universiteit Eindhoven*.

Putra, D., Moreau, L., and Nijmeijer, H. (2004). Observer-based compensation of discontinuous friction. In *Decision and Control, 2004. CDC. 43rd IEEE Conference on*, volume 5, 4940–4945. IEEE.

Putra, D. and Nijmeijer, H. (2003). Limit cycling in observer-based controlled mechanical systems with friction. In *Proceedings of the 2003 European Control Conference (to appear). Cambridge*.

Putra, D. and Nijmeijer, H. (2004). Limit cycling in an observer-based controlled system with friction: Numerical analysis and experimental validation. *International Journal of Bifurcation and Chaos*, 14(09), 3083–3093.

Rajnauth, J. (2003). Reduce torsional vibration and improve drilling operations. *SPE Latin American and Caribbean Petroleum Engineering Conference*.

Richard, T., Germay, C., and Detournay, E. (2004). Self-excited stick–slip oscillations of drill bits. *Comptes rendus MECANIQUE*, 332(8), 619–626.

Rössler, O. (1976). An equation for continuous chaos. *Physics Letters*, 57A(5), 397–398.

Routh, E.J. (1877). *A treatise on the stability of a given state of motion: particularly steady motion*. Macmillan and co.

Shampine, L.F. and Reichelt, M.W. (1997). The MATLAB ODE suite. *SIAM journal on scientific computing*, 18(1), 1–22.

Shokir, E.M. (2004). A novel pc program for drill string failure detection and prevention before and while drilling specially in new areas. *Oil and gas business*, (1).

Sipaylov, G. and Loos, A. (1980). *Mathematical model of electrical machines (AVM): Textbook for college students*. Visshaya shkola, Moscow [in Russian].

Skubov, D. and Khodzhaev, K.S. (2008). *Non-Linear Electromechanics*. Springer.

Solovyeva, E. (2011). Mathematical models and stability analysis of induction motors. Master Thesis. *University of Jyväskylä*.

Solovyeva, E. (2013). *PhD Thesis*. Jyväskylä University Printing House [in preparation].

Staege, S.A. (1936). *Electrical differential oil well*. Westinghouse Electric & Manufacturing Company. US Patent 2,032,155.

Tucker, W.R. and Wang, C. (1999). An integrated model for drill-string dynamics. *Journal of Sound and Vibration*, 224(1), 123–165.

Vagaitsev, V. (2012). Analytical-numerical methods for finding hidden oscillations in dynamical systems. Jyväskylä Studies in Computing. *Jyväskylä University Printing House*, 158, 148.

Vaisberg, O., Vincke, O., Perrin, G., Sarda, J., and Fay, J. (2002). Fatigue of drill-string: state of the art. *Oil and Gas Science and Technology*, 57(1), 7–37.

van de Wouw, N., Nijmeijer, H., and Mihajlovic, N. (2005). Friction-induced limit cycling in flexible rotor systems: an experimental drill-string system. In *Proc. International Design Engineering Technical Conference, Long Beach, California, USA*.

Van den Steen, L. (1997). Suppressing stick-slip-induced drill-string oscillations: a hyper stability approach. *University of Twente*.

van der Pol, B. (1927). On relaxation-oscillations. *Philosophical Magazine and Journal of Science*, 2(7), 978–992.

Viguié, R., Kerschen, G., Golinval, J.C., McFarland, D.M., Bergman, L.A., Vakakis, A.F., and van de Wouw, N. (2009). Using passive nonlinear targeted energy transfer to stabilize drill-string systems. *Mechanical Systems and Signal Processing*, 23(1), 148–169.

White, D.C. and Woodson, H.H. (1968). *Electromechanical Energy Conversion*. The MIT Press.

Yakubovich, V., Leonov, G., and Gelig, A.K. (2004). *Stability of stationary sets in control systems with discontinuous nonlinearities*. World Scientific, Singapore.

Zaremba, S.K. (1936). Sur les équations au paratingent. *Bulletin des Sciences Mathématiques*, 60(2), 139–160.

Zaretskiy, A. (2011). Mathematical models and stability analysis of synchronous motors. Master Thesis. *University of Jyväskylä*.

APPENDIX 1 LYAPUNOV FUNCTIONS METHOD

Lyapunov proposed a method for the study of the stability of solutions with the help of a suitably chosen special functions, which were then called Lyapunov functions (Lyapunov, 1950). For using this method, finding the solutions of the system is not required. In what follows, the notions of Lyapunov stability, Lyapunov functions, and theorems used in the dissertation are formulated.

Following (Leonov, 2001; Krishenko, 2007), let us formulate the main notions and results. Consider the differential equation:

$$\frac{dx}{dt} = f(t, x), \quad t \in \mathbb{R}^1, \quad x \in \mathbb{R}^n, \tag{10}$$

where $f(t, x)$ is a continuous vector-function: $\mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$. Hereafter, we assume that all considered solutions $x(t, t_0, x_0)$ with initial data $x(t_0, t_0, x_0) = x_0$ are defined in the interval $(t_0, +\infty)$.

Definition 1 *Solution $x(t, t_0, x_0)$ of system (10) is Lyapunov stable, if for any number $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such as for all y_0, which satisfy the inequality $|x_0 - y_0| \le \delta(\varepsilon)$, the following relation holds true:*

$$|x(t, t_0, x_0) - x(t, t_0, y_0)| \le \varepsilon \qquad \forall t \ge t_0. \tag{11}$$

Definition 2 *If the solution $x(t, t_0, x_0)$ of system (10) is Lyapunov stable and there exists a number $\delta(\varepsilon) > 0$ such as for all y_0, which satisfy the inequality $|x_0 - y_0| \le \delta(\varepsilon)$, the following relation holds true:*

$$\lim_{t \to +\infty} |x(t, t_0, x_0) - x(t, t_0, y_0)| = 0, \tag{12}$$

then the solution $x(t, t_0, x_0)$ is asymptotically stable.

Let us consider the case of the zero solution: $x(t, t_0, x_0) \equiv 0$. The generalized case is reduced to this particular case by the following change of variables $x = y + x(t, t_0, x_0)$. However, this requires knowledge of the solution $x(t, t_0, x_0)$, something that is not always convenient.

Let us consider the function $V(x)$ differentiable in a certain neighborhood of the point $x = 0$ ($V : \mathbb{R}^n \to \mathbb{R}^1$), for which $V(0) = 0$.

It is clear that if x is replaced by the solution $x(t, t_0, x_0)$, then, according to the rules of differentiation of a composite function, we obtain the following identity:

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t, x). \tag{13}$$

Here x_i – i-th component of the vector x and f_i – i-th component of the vector function f.

Theorem 3 *(on asymptotic stability) Let there exist differentiable function* $V(x)$ *and continuous function* $W(x)$*, for which in certain neighborhood of the point* $x = 0$ *the following conditions hold true:*

1. $V(x) > 0$ *if* $x \neq 0$*,* $V(0) = 0$*,*

2. $\dot{V}(x) \leq W(x) < 0$ *if* $x \neq 0$*.*

Then the zero solution of system (10) is asymptotically stable.

Theorem 4 *(on instability) Let there exist differentiable function* $V(x)$ *and continuous function* $W(x)$*, for which in certain neighborhood of the point* $x = 0$ *the following conditions hold true:*

1. $V(0) = 0$ *and for certain sequence* $x_k \to 0$ *when* $k \to \infty$ *inequations* $V(x_k) < 0$ *are fulfilled,*

2. $\dot{V}(x) \leq W(x) < 0$ *when* $x \neq 0$*.*

Then the zero solution of system (10) is Lyapunov unstable.

Let us introduce an important notion of the global asymptotic stability.

Definition 3 *If the equilibrium state of system (10) is asymptotically stable and its region of attraction is congruent to* $\mathbb{R}^n$*, then the equilibrium state is globally asymptotically stable.*

Definition 4 *Equilibrium state of system (10) is asymptotically stable in region* D *if the region of attraction of this solution is congruent to* D*.*

There are theorems on asymptotic stability by Barbashin and Krasovskii that are special cases of the theorem by La Salle (LaSalle, 1968). Let us formulate two of them (Barbashin and Krasovsky, 1952).

Theorem 5 *(the first theorem of Barbashin-Krasovsky) Let* $x = 0$ *be an equilibrium state of system (10), defined in* $\mathbb{R}^n$*, and let there exists continuous and differentiable, positively-defined and infinite for* $||x|| \to \infty$ *function* $V : \mathbb{R}^n \to \mathbb{R}^1$*, derivative of which along the trajectories of system (10) is a negatively-defined function. Then* $x = 0$ *is a globally stable equilibrium state.*

Theorem 6 *(the second theorem of Barbashin-Krasovsky) Let* $x = 0$ *be an equilibrium state of system (10), defined in* D*, and there exists continuous and differentiable, positively-defined function* $V : D \to \mathbb{R}^1$*, for which* $\dot{V}(x) \leq 0$ *in* D*, and set* $S = \{x \in D : \dot{V}(x) = 0\}$ *doesn't contain full trajectories except of* $x = 0$*. Then* $x = 0$ *is an asymptotically stable equilibrium state.*

APPENDIX 2 DIFFERENTIAL INCLUSIONS AND FILIPPOV DEFINITION

All models of the drilling systems considered in this study are described by differential equations with discontinuous right-hand sides (due to the friction torque origin). The investigation of these equation requires the introduction of a new definition of solution which would be applicable to such systems. Following the works (Andronov et al., 1966; Yakubovich et al., 2004; Filippov, 1960) bellow, we will consider mechanical systems with friction and will try to explain why the well-known definition of solution of an ordinary differential equation doesn't work for discontinuous systems. We will introduce the notion of the differential inclusion and Filippov definition for systems of differential equations with discontinuous right-hand sides.

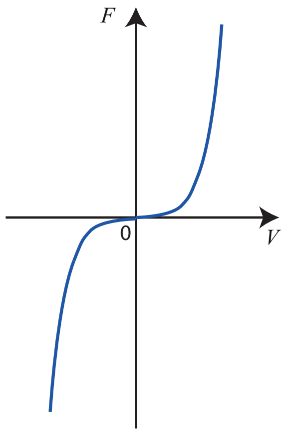

FIGURE 12 Viscous friction

The origin of the theory of differential inclusions is usually associated with the names of the French mathematician Marchaud (Marchaud, 1936) and the Polish mathematician Zaremba (Zaremba, 1936). We will introduce later the definition of a discontinuous system solution proposed by Filippov (Filippov, 1960, 1985, 1988). Other well-known definition of discontinuous systems solutions for our problem are equivalent to the definition by Filippov.

Mechanical system with a dry friction

As shown in (Andronov et al., 1966), we can establish the relationship between the work required to overcome friction and the speed. This relationship is totally different for the case of the movement of a body of mass in the fluid and the friction against any solid surface. In the first case (the case of "viscous friction"), the work essentially depends on the speed and if the speed decreases, the work decreases as well and it can be made arbitrarily small. In the second case (the case of "dry friction"), on the contrary, the work is slightly dependent

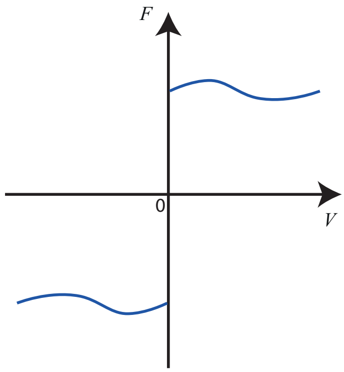

FIGURE 13 Dry friction

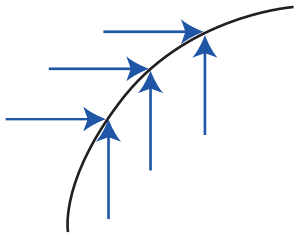

FIGURE 14 Slope field in the neighborhood of discontinuity surface: the sliding mode

on the speed. Regardless of how slow the body is moved, a certain clearly delineated work is required for that, i.e., the friction force has a finite value even when the speed is arbitrary small. In addition, the friction force is always applied in the direction opposite to the velocity, and, thereby, when passing through zero, the friction force changes its sign. In the case of the "viscous friction", it follows that the friction force passes through zero without a jump and changes the sign (Fig. 12, (Andronov et al., 1966)).

In the case of the "dry friction", as the speed tends to zero, the friction force on both sides tends to different finite limits (in particular case, it tends to the limits of opposite signs, but equal in absolute value), i.e., it is discontinuous at zero (Fig. 13, (Andronov et al., 1966)).

Thus, mathematical models of mechanical systems with the Coulomb friction, obtained from knowing the mechanics of the systems of rigid bodies, are differential equations, right-hand sides of which are functions that are discontinuous with respect to generalized velocities (the friction force changes abruptly when the direction of the motion changes) (Bothe, 1999; Fečkan, 1997, 1999; Filippov, 1988; Kunze, 2000; Kunze and Küpper, 1997).

One operation mode of such systems with the dry friction is the sliding mode. The sliding mode occurs when phase trajectories are directed towards each other in the neighborhood of the surface where the control function has discontinuities (see Fig. 14, (Gelig et al., 1978)). After a contact with the discontinuity surface, the image point can not move during any even arbitrarily small, but fi-

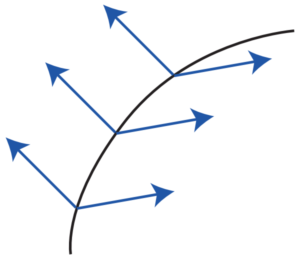

FIGURE 15 Slope field in the neighborhood of discontinuity surface: the solution moves away from the discontinuity surface

nite time interval along any trajectory adjacent to the surface (in the case of any displacement, there is always a movement that returns an affix on the discontinuity surface). Also, another case is possible: here on the contrary, the solution can not get to the corresponding part of the discontinuity surface as time increases (see Fig. 15, (Gelig et al., 1978)).

Justification of the need to generalize the notion of solution of differential equation

Here is the classic definition of solutions of differential equation (Filippov, 1985).

Definition 5 *Solution of differential equation*

$$\dot{x} = f(x, t)$$

with continuous right-hand side is a function $x(t)$, which has derivative everywhere on this interval and satisfies this equation.

Furthermore, the equation with a continuous right-hand side is equivalent to the following integral equation

$$x(t) = \int f(t, x(t))\, dt + C. \tag{14}$$

However, in the case of differential equations with discontinuous right-hand sides this definition doesn't work. In the case when $f(t, x)$ is discontinuous for t and continuous for x, we may call the functions satisfying integral equation solutions of the equation. In that case, solutions from one side of S go to S, from the other side they go off S (trajectories puncture the surface), see Fig. 16, (Gelig et al., 1978).

Solution $x(t)$, which falls for $t = t_0$ into the discontinuity surface S, extends uniquely for $t > t_0$ and close to t_0; by intersecting S the solution satisfies the equation everywhere except the point of intersection in which the solution doesn't have a derivative. In another case, when the solution approach the discontinuity surface S from both sides (trajectories merge – the sliding mode), this

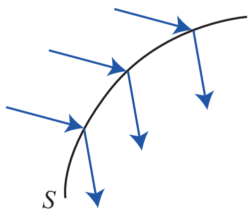

FIGURE 16 Slope field in the neighborhood of discontinuity surface: trajectories puncture the surface S

definition of solution is meaningless, because it tells us nothing about the behavior of the solution on the surface S.

It is necessary to give such definition of the solution which would embody these both cases and which formulation would not be dependent on the lines' position and discontinuity surfaces.

Definition of solution

Consider the equation of the system in the vector notation

$$\dot{x} = f(t, x), \tag{15}$$

with a piecewise continuous function f in the region G; $x \in \mathbb{R}^n$, M - set (of measure zero) of discontinuity point of function f.

Most of the well-known definitions of solution of equation (15) could be described in the following way. For each point (t, x) of the region G set $F(t, x)$ in n-dimensional space is introduced. If at the point (t, x) function f is continuous, then the set $F(t, x)$ consists of one point, which is congruent to the value of the function f at this point. If (t, x) is a discontinuity point of the function f, then the set $F(t, x)$ is defined in certain chosen way.

Definition 6 *Solution of equation (15) is called the solution of the differential inclusion*

$$\dot{x} \in F(t, x), \tag{16}$$

i.e., absolutely continuous vector function $x(t)$, defined on the interval or segment I, for which almost everywhere on I

$$\dot{x} \in F(t, x). \tag{17}$$

In other words, the solution of the differential equation (15) is defined as a function which derivative $\dot{x} = dx/dt$ may have any values from a certain set $F(t, x)$. Sometimes (16) are called differential equations with a set-valued right-hand side. A function is called a set-valued function and we emphasize that $F(t, x)$ is a set. If for all (t, x) the set $F(t, x)$ contains only one point, then (16) is an ordinary differential equation. The function $F(t, x)$ is called one-valued at the point (t_0, x_0), if the set $F(t_0, x_0)$ contains one point.

The most common definition of the solution of discontinuous system is Filippov definition. (Filippov, 1960, 1985, 1988).

Extending convex definition

This extension could also be applied to systems with a small delay of a certain type and to some systems with a dry friction including the systems considered in this study.

For every point $(t, x) \in G$ let $F(t, x)$ be the minimal convex closed set which contains all limit vector-functions $f(t, x^*)$; when $(t, x^*) \notin M$, $x^* \to x$, $t = const$. We call the solution of system (15) a solution of differential inclusion (16) with the just constructed $F(t, x)$. Since M is a set of measure zero, for almost all $t \in I$ the measure of the section of set M with the plane $t = const$ is zero. For these t, the set $F(t, x)$ is defined for all $(t, x) \in G$. At the points of continuity for function f, a set $F(t, x)$ consists of one point $f(t, x)$ and the solution satisfies equation (15) in the ordinary sense. If the point $(t, x) \in M$ lies on the boundaries of two or more regions $G_1, \ldots, G_k$ of plane $t = const$, then the set $F(t, x)$ is a segment, convex polygon or polyhedron with vertexes $f_i(t, x)$, $i \le k$, where

$$f_i(t, x) = \lim_{\substack{(t, x^*) \in G_i \\ x^* \to x}} f(t, x^*). \tag{18}$$

All the points $f_i(t, x)$ $(i = 1, \ldots, k)$ are contained in $F(t, x)$, but not necessarily all of them are vertexes.

Definition 7 *Vector-function $x(t)$, defined on the interval J, is called a solution of system (15), if it is absolutely continuous and if for almost all $t \in J$ and for all $\delta > 0$ the vector $\dot{x}(t)$ belongs to the minimal convex closed set (n-dimensional space), which contains all the values of the vector function $f(t, x^*)$, when x^* runs through almost entire δ–neighborhood of the point $x(t)$ in the space X (for a fixed t), that is, through the entire neighborhood, except of the set of measure zero.*

Such definition outlines the unique extension of the solution on the discontinuity surface. Let us consider the case when the function $f(t, x)$ is discontinuous on a smooth surface S, defined by the equation $s(x) = 0$. The surface S divides its neighborhood in the space into the regions G^- and G^+. Let for $t = const$ and for approximation x^* for $x \in S$ from regions G^- and G^+ the function has limit values

$$\begin{aligned} \lim_{\substack{x^* \in G^- \\ x^* \to x}} f(t, x^*) &= f^-(t, x), \\ \lim_{\substack{x^* \in G^+ \\ x^* \to x}} f(t, x^+) &= f^+(t, x). \end{aligned} \tag{19}$$

Then the set $F(t, x)$, mentioned in the definition of extension, is the segment which connects endings of the vectors $f^-(t, x)$ and $f^+(t, x)$, which start at x.

• If for $t \in I$ the segment lies on one side of the surface P, which is tangent to the surface S at the point, then the solutions for those t go from one side of the surface S to its other side (Fig. 17, (Filippov, 1985)).

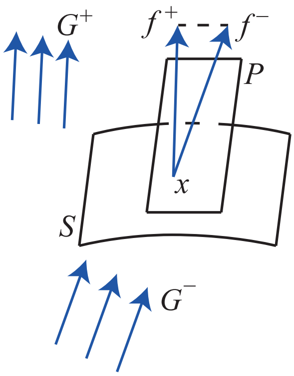

FIGURE 17 The solution goes from one side of the surface S to its other side

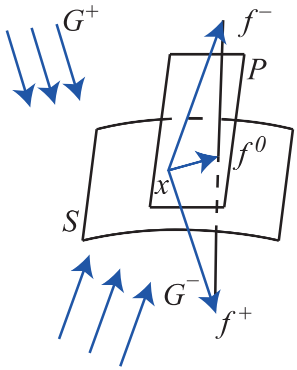

FIGURE 18 Motion of the solution along the surface S

• If this segment crosses the surface P, the crossing point is the end of the vector $f^0(t,x)$, which defines the speed of the motion

$$\dot{x} = f^0(t,x) \tag{20}$$

along the surface S in space X (Fig. 18, (Filippov, 1985)).

Note that the vector is tangent to S $f^0(t,x) \in P$, so, $f^0(t,x) \in F(t,x)$. This means that the function $x(t)$, which satisfies equation (20) due to the extension, is the solution of equation (15). This implies that $x(t)$, which on this part of the considered time interval passes in the area G^-(or in the area G^+) and satisfies equation (15); on the remaining part it goes along the surface S and satisfies equation (20), it is also considered as a solution of (15) in the sense of the definition of extension introduced above.

In equation (20) $f^0 = \alpha f^+ + (1-\alpha)f^-$,

$$\alpha = \frac{f_N^-}{f_N^- - f_N^+}, \qquad (0 \le \alpha \le 1), \tag{21}$$

f_N^+, f_N^- are vectors projections of f^+ and f^+ to the normal of the surface S at the point x (the normal is oriented towards the direction of G^+).

• If the whole segment with the endings f^- and f^+ lies on the surface P, then the movement speed f^0 on discontinuity surface S is multi-valued.

For $f^0 \neq f^-$, $f^0 \neq f^+$ there is a sliding mode, which was discussed before. Let the ideal sliding mode equation look as equation (20). Calculating α for $f^0 = \alpha f^+ + (1-\alpha)f^-$ from the condition $\operatorname{grad} S \cdot f^0 = 0$, we find the equation

$$\dot{x} = \frac{\operatorname{grad} S \cdot f^-}{\operatorname{grad} S \cdot (f^- - f^+)} f^+ - \frac{\operatorname{grad} S \cdot f^+}{\operatorname{grad} S \cdot (f^- - f^+)} f^-, \tag{22}$$

with the help of which we define the motion in the sliding mode (the initial data for (22) is chosen on the discontinuous surface, i.e., $S(x(0)) = 0$).

The connection of the theory of equations (15) with discontinuous right-hand side to the theory of differential inclusions (16) is obvious. If there is equation (17) with discontinuous function $f(t,x)$, we need to replace its value $f(t_0, x_0)$ in the discontinuity point (t_0, x_0) with a certain set. This set should be limited, convex, and self-contained. Furthermore, it should contain all limit values $f(t,x)$ when $(t,x) \to (t_0, x_0)$. After such change of variables (for any discontinuity point) instead of (15) we obtain differential inclusion (16), in which a set-valued function satisfies the above stated requirements.

APPENDIX 3 NUMERICAL METHODS OF STABILITY INVESTIGATION OF DISCONTINUOUS SYSTEMS

The models studied in this work are described by equations with discontinuous right hand-sides, thereby, a special method for numerical computation of their solutions is required. In what follows, we are going to provide a brief description of one of such methods following the work (Piiroinen and Kuznetsov, 2008).

The event-driven simulation method

In some cases it is definitely possible to find explicit expressions for the solutions of the ordinary differential equation that describes sliding if the vector fields in the non-sliding regions are given. For example, this is the case when small linear systems are considered. However, in the case of the above models of the drilling systems actuated by an induction motor, the idea is to present a numerical algorithm where the user only provides different vector fields and information about the discontinuity surface, and then the vector fields for the sliding regions are automatically computed. The method that has been chosen here for modeling the considered systems with a discontinuous right-hand side is similar to the hybrid system approach, where the integrations of smooth ordinary differential equations are mixed with discrete maps and vector field switches. In practice, this means that an initial value problem is solved for one of the possible smooth dynamical systems until the trajectory reaches one of the predefined surfaces. At such a point, depending on the state at that instance, the vector field is possibly switched. It is very important to have a reliable ODE solver that is accompanied by an accurate routine to locate discontinuity surface and the tangent surface crossings. In what follows, a surface crossing will be called an event and a scalar function defining an event surface is referred to as the event function. The existence of event detection routines will be assumed here. For instance, in MATLAB (Higham and Higham, 2005) the event detection routines are built-in and can easily be used together with the likewise built-in ODE solvers to integrate trajectories and to locate events along them as precisely as the accuracy of MATLAB permits (for more details of the MATLAB ODE routines, see (Shampine and Reichelt, 1997; Ashino et al., 2000)). However, standard methods, for example, the secant type methods, can easily be implemented and have proven to be fast and reliable. The type of events to be detected also plays an important role in how to numerically deal with them and how sensitive the event detection needs to be. The basic ideas behind the simulation algorithm for one discontinuity surface and a schematic description on how the algorithm works are given in (Piiroinen and Kuznetsov, 2008).

APPENDIX 4 MATHEMATICAL MODEL OF INDUCTION MOTOR

In what follows, we are going to provide a brief derivation of a system of equations describing a mathematical model of the induction motor with a wound rotor. The full version of this derivation can be found in (Leonov et al., 2013a; Leonov and Solovyeva, 2012; Solovyeva, 2013).

Currently, the methods of research and design of asynchronous motors based on mathematical modeling have become widespread (Bespalov et al., 2002; Bespalov, 1992; Golubev and Zikov, 2003; Gruzov, 1953; Kopilov et al., 2002; Kopilov, 2001b,a; Moshinsky and Petrov, 2001; Moshinsky and T., 2007; Moshinsky and Petrov, 2007; Pankratov and Zima, 2003; Sipaylov and Loos, 1980; Hrisanov and Brxhezinsky, 2003). Both Russian and foreign researchers have made great contribution to the establishment and development of scientific methods of calculation of induction motors, among them: Adkins, Bespalov, Blondel, Woodson, Glebov, Gorev, Danilevich, Ivanov-Smolensky, Ilinskiy, Kazovsky, Kovacs, Kononenko, Kopylov, Kostenko, Kron, Luther, Park, Petrov, Postnikov, Radin, Raz, Sipailov, Soroker, Treschev, White, Filtz and others.

The basic constructive elements of induction electrical machines are stationary stator and rotating rotor. On the stator and the rotor the windings are located. Stator winding is arranged in such a way that in the case of alternate current networking it generates the rotating magnetic field.

Consider the induction motor with a wound rotor (Drury, 2001) shown in Fig. 19. In the simplest case, a wound rotor winding consists of three coils, each consists of several turns of insulated conductor. Furthermore, one considers induction motors with wound rotor, when a rotor winding is short-circuited and no external devices are connected. A working gear is connected to the rotor shaft (in our case, it is a drill-string). Thus, the induction motor, by transforming the electric energy into the mechanical one, imparts rotational motion to the working gear via shaft.

The classic derivation of expressions for the currents in rotor winding and the electromagnetic torque of induction motor are based on the following simplifying assumptions (Popescu, 2000; Leonhard, 2001; Skubov and Khodzhaev, 2008):

I It is assumed that the magnetic permeability of the stator and rotor iron is equal to infinity. This assumption makes it possible to use the principle of superposition for the determination of the magnetic field, generated by the stator;

II one may neglect energy losses in electrical steel, i.e., the losses related to motor heating, magnetic hysteresis, and whirling currents;

III one does not take into account saturation of the rotor iron, i.e. the current of any force can run in rotor winding;

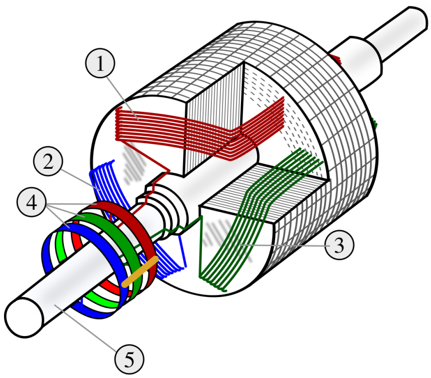

FIGURE 19 Wound rotor of induction motor: 1 – the first coil with the current i_1, 2 – the second coil with the current i_2, 3 – the third coil with the current i_3, 4 – slip rings, 5 – rotor shaft

IV one may neglect the effects arising at the ends of rotor winding and in rotor slots, i.e., one may assume that the magnetic field is distributed uniformly along the circumference of the motor;

Let us make an additional assumption:

V Stator windings are fed by a powerful outlet of sinusoidal voltage.

Then, following (Adkins, 1957; White and Woodson, 1968; Skubov and Khodzhaev, 2008), due to the last assumption, the effect of rotor currents on stator currents may be ignored. Thus, a stator generates uniformly a rotating magnetic field with a constant in magnitude induction. So, one may assume that magnetic induction vector is constant in magnitude and rotates with a constant angular velocity. This assumption is due to the classical ideas of Tesla and Ferraris and allows one to consider dynamics of induction motor from the point of view of its rotor dynamics (Leonov, 2006; Leonov and Solovyeva, 2012; Leonov et al., 2013b).

Suppose, magnetic field rotates clockwise. One introduces the uniformly rotating coordinates, rigidly connected with the vector magnetic induction, and considers the motion of wound rotor in this coordinate system. Also, suppose that the positive direction of the rotation axis of the rotor coincides with the direction of the rotation of the magnetic induction vector.

The rotating magnetic field crosses rotor winding and, by the law of electromagnetic induction, it induces EMF in it. Thus, taking into account the number of turns, EMF in coils is equal to

$$\varepsilon_k = -nSB\cos\left(\frac{\pi}{2} - \theta - \frac{2(k-1)\pi}{3}\right)\dot{\theta}, \qquad k = 1, 2, 3.$$

Here B is the inductance of the magnetic field, n is the number of turns in each coil, S is an area of one turn of coil, θ is a mechanical angle of rotation of rotor.

Acted by EMF, a variable current arises in rotor winding. According to Ampere's force law, as a result of the interaction of currents in coils with rotating magnetic field, there arise electromagnetic forces. Electromagnetic forces generate electromagnetic rotating moment, under which the rotor begins rotating with a certain frequency.

Using Ampere's force law for calculating electromagnetic forces and taking into account the number of turns in coil and positive direction of the rotor rotation axis, it follows that a generated rotating electromagnetic torque, acting on a coil with the current i_k, is equal to

$$M_k = nSB\cos\left(\frac{\pi}{2} - \left(\theta + \frac{2(k-1)\pi}{3}\right)\right) i_k, \qquad k = 1, 2, 3.$$

Thus, the electromagnetic torque of the induction motor with a wound rotor is equal to

$$M_{em} = M_1 + M_2 + M_3.$$

The dynamics of the rotating induction motor is described by the equations of electric chains (voltage equations) and the equation of moments of forces, acting on the motor rotor (equation of moments).

Using the second Kirchgoff's law and following the positive direction of the by-pass of the circuit in the clockwise order, one arrives at the following differential equations

$$\begin{aligned} L(\dot{i}_1 - \dot{i}_2) + R(i_1 - i_2) &= \varepsilon_1 - \varepsilon_2, \\ L(\dot{i}_2 - \dot{i}_3) + R(i_2 - i_3) &= \varepsilon_2 - \varepsilon_3, \end{aligned} \tag{23}$$

where R, L are active and inductive resistance of each coil; ε_k is EMF, induced in k-th coil by rotating magnetic field.

The motion of wound rotor of induction motor about shaft in the chosen coordinate system is described by the equation of torques:

$$J\ddot{\theta} = M_{em} - M_f,$$

where θ is mechanical angle of rotor rotation; J is inertia moment of the rotor relative to shaft; M_{em} is electromagnetic torque; M_l is load torque.

Thus, the system of differential equations

$$\begin{aligned} J\ddot{\theta} &= nBS\sum_{k=1}^{n} i_k \cos\left(\frac{\pi}{2} - \theta - \frac{2(k-1)\pi}{3}\right) - M_f, \\ L\dot{i}_1 + Ri_1 &= -nBS\dot{\theta}\cos\left(\frac{\pi}{2} - \theta\right), \\ L\dot{i}_2 + Ri_2 &= -nBS\dot{\theta}\cos\left(\frac{\pi}{2} - \theta - \frac{2\pi}{3}\right), \\ L\dot{i}_3 + Ri_3 &= -nBS\dot{\theta}\cos\left(\frac{\pi}{2} - \theta - \frac{4\pi}{3}\right) \end{aligned} \tag{24}$$

describes the dynamics of induction motor with a wound rotor.

Note that non-singular change of variables (4), makes it possible to reduce system (24) to the model used in (5) [1].

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1 Some other types of induction motors may also be reduced to this model under certain additional assumptions (Leonhard, 2001; Khalil and Grizzle, 2002)

APPENDIX 5 PROOF OF THE THEOREM ON LOCAL STABILITY

Proof

In case when

$$0 < \gamma < \gamma_{max} = \frac{a}{2}$$

system (7) has one equilibrium state:

$$u_0 = s_0, \quad x_0 = -\frac{\gamma}{ac}s_0, \quad y_0 = -\frac{\gamma}{a}, \quad \theta_{rel0} = \frac{\gamma}{k_\theta}. \tag{25}$$

Here s_0 is the smallest root of the equation

$$\frac{acs}{c^2 + s^2} = \gamma.$$

The characteristic polynomial of the first approximation matrix of (7) in stationary point (25) is as follows:

$$p_1(\lambda) = a_0\lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5,$$

where

$$a_0 = 1,$$

$$a_1 = 2c,$$

$$a_2 = \frac{1}{2J_u}(a + 2k_\theta + D) + \frac{1}{2J_l\gamma^2}(2\gamma^2 k_\theta + J_l a^2 c^2 - J_l ac^2 D),$$

$$a_3 = \frac{c}{J_u J_l}((2k_\theta + D)J_l + 2J_u k_\theta),$$

$$a_4 = \frac{k_\theta}{2J_u J_l \gamma^2}(\gamma^2 a + \gamma^2 D + J_l a^2 c^2 + J_u a^2 c^2 - J_l ac^2 D - J_u ac^2 D),$$

$$a_5 = \frac{cDk_\theta}{J_u J_l}.$$

Here $D = \sqrt{a^2 - 4\gamma^2}$. For defining stability, one uses Routh–Hurwitz stability criterion (Hurwitz, 1964; Routh, 1877) for a polynomial of the 5th order:

$$a_i > 0, \qquad i = 0..5, \tag{26}$$

$$a_1 a_2 - a_0 a_3 > 0, \tag{27}$$

$$(a_1 a_2 - a_0 a_3)(a_3 a_4 - a_2 a_5) - (a_1 a_4 - a_0 a_5)^2 > 0. \tag{28}$$

Thus, the conditions (27) and (28) for the local stability of equilibrium state (25) are as follows ((26) holds automatically)

$$\frac{ac}{J_u} + \frac{ac^3}{\gamma^2}(a - D) > 0,$$

$$\frac{1}{2J_u^3 J_l \gamma^4}(J_l \gamma^2 a^2 c^4 k_\theta (D(a - D) + 2k_\theta (a - D)) + J_u J_l a^2 c^6 D k_\theta (D - a)^2 + 4\gamma^4 a^2 c^2 k_\theta^2,$$

$$+4J_u \gamma^2 a^2 c^4 k_\theta^2 (a - D)) > 0.$$

Therefore, equilibrium state (25) is asymptotically stable.