Marat V. Yuldashev

Nonlinear Mathematical Models of Costas Loops
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ABSTRACT

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This work is devoted to the development of nonlinear mathematical models of Costas loops. A Costas loop was invented in 1956 by John P. Costas of General Electric. Nowadays, a Costas loop is widely used in many applications including telecommunication devices, global positioning systems (GPS, GLONASS), medical implants, mobile phones, and other gadgets.

In contrast to the phase-locked loop (PLL) based circuit, the Costas loop is designed to simultaneously perform two tasks — carrier recovery and data demodulation. The direct application of a PLL to these tasks is possible, but it is not effective, because after superimposing the transmitted data and carrier signal, frequent changes of transmitted data require that a PLL constantly adjusts itself. A Costas loop is designed in such a way that the transmitted data doesn’t affect transient processes and does not require frequent tuning. The requirement for simultaneous data demodulation and carrier recovery makes the Costas loop-based devices multi-loop, multi-channel circuits with multiple outputs. Also, in contrast to a PLL, the Costas loop has three non-linear elements. All this makes the development of non-linear models of Costas loops a difficult task. High-frequency signals, used in the modern devices, further complicate the application of analytical methods and numerical simulation. This is due to the fact that the transient time is greater than the signal’s periods by several orders of magnitude. Furthermore, the behaviour of Costas circuits greatly depends on the classes of signals involved. So, the development of non-linear mathematical models of Costas loops that allow one to facilitate the application of analytical methods and reduce the numerical simulation time is a relevant problem of the practical significance. It is this problem that is considered and solved in the present study.

In this work, nonlinear mathematical models of the classic Costas loop and the Quadrature Phase Shift Keying (QPSK) Costas loop have been developed. All theoretical results are rigorously proved. An effective numerical procedure for the simulation of Costas loops based on the phase-detector characteristics is proposed.

The results of the study have been published in 22 papers (8 of which are indexed in Scopus).

Keywords: Costas loop, carrier tracking, GPS, GLONASS, PLL, BPSK, QPSK Costas
Supervisors

Dr. Nikolay V. Kuznetsov
Department of Applied Cybernetics
Faculty of Mathematics and Mechanics
Saint Petersburg State University, Russia,
Faculty of Information Technology
University of Jyväskylä, Finland

Professor Gennady A. Leonov
Member (corr.) of Russian Academy of Science,
Head of Department of Applied Cybernetics,
Dean of Faculty of Mathematics and Mechanics
Saint Petersburg State University, Russia

Professor Pekka Neittaanmäki
Department of Mathematical Information Technology,
Dean of Faculty of Information Technology
University of Jyväskylä, Finland,
Honorary Doctor of Saint Petersburg State University, Russia
Opponents

Professor Alexey S. Matveev (Chairman)
Faculty of Mathematics and Mechanics
St. Petersburg State University, Russia,
Electrical & Electronic Engineering
and Telecommunications School
University of New South Wales, Australia

Professor Boris R. Andrievsky
Faculty of Mathematics and Mechanics
St. Petersburg State University, Russia,
Faculty of Information and Control Systems
Baltic State Technical University “VOENMEH”, Russia

Professor Alexander K. Belyaev
Director of Institute of Applied Mathematics & Mechanics
St. Petersburg State Polytechnical University, Russia,
Vice-Director of Institute for Problems in Mechanical
Engineering Russian Academy of Sciences, Russia,
Honorary Doctor of University of Johannes Kepler, Austria

Professor Vladimir I. Nekorkin
Faculty of Radiophysics,
Lobachevsky State University of Nizhni Novgorod, Russia,
Head of Department of Nonlinear Dynamics
Institute of Applied Physics
Russian Academy of Sciences, Russia

Professor Sergei Yu. Pilyugin
Faculty of Mathematics and Mechanics
St. Petersburg State University, Russia

Professor Vladimir Rasvan
Faculty of Automatics, Computers and Electronics,
Director of Research Center
“Nonlinear control. Stability and oscillations”
University of Craiova, Romania

Professor Timo Tiihonen
Department of Mathematical Information Technology,
Vice-Dean of Faculty of Information Technology,
University of Jyväskylä, Finland
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I would like to extend my deepest thanks to my parents Prof. Dilara Kalimullina and Prof. Vladimir Yuldashev.
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1 INTRODUCTION

1.1 Intellectual merit

The classic Costas loop was invented in 1956 by famous American electrical engineer John P. Costas of General Electric (Costas, 1956, 1962). A Costas loop is a carrier tracking and Binary Phase Shift Keying (BPSK) data demodulation device (Tomasi, 2001; Young, 2004; Couch, 2007; Proakis and Salehi, 2007; Best, 2007; Chen et al., 2010). Nowadays, a Costas loop and its modifications are widely used in telecommunication devices (Viterbi, 1983; Malyon, 1984; Hodgkinson, 1986; Stephens, 2001; Yu et al., 2011; Abe et al., 2012), Global Positioning Systems (GPS) (Jasper, 1987; Beier, 1987; Mileant and Hinedi, 1994; Braasch and Van Dierendonck, 1999; Tanaka et al., 2001; Humphreys et al., 2005; An’an and Du Yong, 2006; Kaplan and Hegarty, 2006; Tang et al., 2010; Misra and Palod, 2011; Hegarty, 2012), medical implants (Hu and Sawan, 2005; Luo and Sonkusale, 2008; Xu et al., 2009), mobile phones (Kobayashi et al., 1992; Gao and Feher, 1996; Lin et al., 2004; Shah and Sinha, 2009), and other important technological applications (Wang and Leeb, 1987; Miyazaki et al., 1991; Djordjievic et al., 1998; Djordjevic and Stefanovic, 1999; Cho, 2006; Hayami et al., 2008; Nowsheen et al., 2010).

A mathematical description and the investigation of mathematical models of Costas loops is a very difficult task (Abramovitch, 2002). The direct description of these circuits leads to the analysis of nonlinear non-autonomous differential equations with high-frequency and low-frequency components in the right-hand sides of the equations (Leonov and Seledzhi, 2002; Leonov, 2006; Kudrewicz and Wasowicz, 2007; Leonov et al., 2009; Leonov, 2010). Because in the modern applications not only sinusoidal but many other classes of signal have been used (Henning, 1981; Wang and Emura, 1998; Sutterlin and Downey, 1999; Wang and Emura, 2001; Chang and Chen, 2008; Sarkar and Sengupta, 2010), it further complicates the study of the corresponding differential equations. However, it is possible to overcome these difficulties through the development of the high-frequency asymptotic analysis methods (see (Leonov, 2008; Kuznetsov et al., 2010a) and [PI—PV]). These methods allow one to “split” high-frequency and
low-frequency parts in the mathematical models of Costas loops.

According to one of the largest publication database (www.sciencedirect.com), there exist a high interest in the investigation of Costas loops:

- 2008 — 503 publications
- 2009 — 500 publications
- 2010 — 512 publications
- 2011 — 607 publications
- 2012 — 680 publications

This work further contributes to the body of knowledge about Costas loops and it is devoted to the development and analysis of their mathematical models using the high-frequency asymptotic analysis approach.

1.2 Goal of the work

The goals of this work include: a rigorous mathematical derivation of models of the classic Costas loop for general signal waveforms, a modification of the QPSK Costas loop and their numerical simulation.

1.3 Methods of investigation

Many methods of analysis of Costas loops are considered in various publications. However, the problems of the development of adequate nonlinear models and rigorous analysis of such models are still far from being resolved. However, a simple linear analysis can lead to incorrect conclusions and, thereby, it requires rigorous justification. Numerical simulation is not a trivial task also due to the high-frequency properties of the signals involved. Therefore the development of nonlinear mathematical models of Costas loops is a must for the analysis of such systems.

A Costas loop is a PLL-based circuit and, thereby, methods similar to those used in the context of investigation of any PLL may be adapted here. The first mathematical description of physical models was proposed by Gardner and Viterbi (Gardner, 1966; Viterbi, 1966). These authors described a mathematical model of the classic Costas loop in the signal space and, without a rigorous mathematical justification, proposed a mathematical model in the phase space. Although PLL-based circuits are essentially nonlinear control systems, in the modern engineering literature devoted to the analysis of PLL-based circuits (Lindsey, 1972; Lindsey and Simon, 1973; Djordjevic et al., 1998; Djordjevic and Stefanovic, 1999; Fiocchi et al., 1992; Miyazaki et al., 1991; Cho, 2006; Wang and Leeb, 1987; Wang and Emura, 2001, 1998; Hayami et al., 2008; Young et al., 1992; Gardner et al., 1993; Gardner, 1993; Fines and Aghvami, 1991; Margaris, 2004; Kroupa, 2003;
Razavi, 2003; Shu and Sanchez-Sinencio, 2005; Manassewitsch, 2005; Egan, 2000; Suarez and Quere, 2003; Tretter, 2007; Emura, 1982; Benarjee, 2006; Demir et al., 2000a; Stephens, 2001; Xanthopoulos et al., 2001; Demir et al., 2000b; Roberts et al., 2009; Kim et al., 2010; Tomkins et al., 2009; Proakis and Salehi, 2007), the main means of investigation include the use of simplified linear models, methods of linear analysis, empirical rules, and numerical simulation (see a plenary lecture of D. Abramovitch at the 2002 American Control Conference (Abramovitch, 2002)). While the analysis of linearized models of control systems may lead to incorrect conclusions\(^1\), attempts to justify the reliability of conclusions, based on the application of such simplified approaches, are quite rare (see, e.g., (Suarez and Quere, 2003; Margaris, 2004; Banerjee and Sarkar, 2008a; Chicone and Heitzman, 2013; Best, 2007; Suarez et al., 2012; Feely, 2007; Feely et al., 2012; Kudrewicz and Wasowicz, 2007; Sarkar and Sengupta, 2010; Banerjee and Sarkar, 2008b)). A rigorous nonlinear analysis of a PLL-based circuit models is often a very difficult task (Leonov and Seledzhi, 2002; Kuznetsov, 2008; Kudrewicz and Wasowicz, 2007); therefore, for the analysis of nonlinear PLL models numerical simulations are widely used (Troedsson, 2009; Best, 2007; Bouaricha et al., 2012). However, for the high-frequency signals, complete numerical simulation of the physical model of a PLL-based circuit in signal/time space, described by nonlinear non-autonomous system of differential equations, is highly complicated since it is necessary to simultaneously observe “very fast time scale of the input signals” and “slow time scale of signal’s phases” (Abramovitch, 2008a,b). Here, a relatively small discretization step in a numerical procedure does not allow one to consider transition processes for the high-frequency signals in a reasonable time period.

To overcome these difficulties, it was suggested (Gardner, 1966; Viterbi, 1966) to construct a dynamical model of a Costas loop circuit in the space of signal phases. As noted in [PIV], this approach considers only a slow time scale of the signals phases. This requires the computation of the phase detector (PD) characteristic, which depends on waveforms of circuit signals (Leonov, 2008; Kuznetsov et al., 2009b,a, 2008; Leonov et al., 2006, 2009; Kuznetsov, 2008). However, in order to use the results of such analysis of a mathematical model describing the behaviour of the corresponding physical model, a rigorous justification is needed [PI — PV]. To this end, it is essential, in turn, to apply also the averaging methods (Krylov and Bogolyubov, 1947; Mitropolsky and Bogolubov, 1961).

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\(^1\) see, e.g. counterexamples to the filter hypothesis and to Aizerman’s and Kalman’s conjectures regarding absolute stability (Kuznetsov and Leonov, 2001; Leonov et al., 2010c,b,a,b; Bragin et al., 2010; Leonov et al., 2011c; Leonov and Kuznetsov, 2011; Kuznetsov et al., 2011b; Bragin et al., 2011b; Leonov et al., 2011b; Leonov and Kuznetsov, 2012, 2013b,c,d; Kuznetsov and Leonov, 2008; Kuznetsov et al., 2010b; Leonov et al., 2010c; Leonov and Kuznetsov, 2010; Leonov et al., 2011a; Kuznetsov et al., 2011a,c; Leonov et al., 2011d, 2012,b;a; Kiseleva et al., 2012; Andrievsky et al., 2012; Kuznetsov et al., 2013a,b; Leonov and Kuznetsov, 2013a) and Perron effects of sign inversions of Lyapunov exponents for time varying linearizations (Kuznetsov and Leonov, 2005; Leonov and Kuznetsov, 2007) etc.
1.4 The main results

- Nonlinear mathematical models of the classic Costas loop for various signal waveforms (sinusoidal, impulse, polyharmonic, piecewise-differentiable) have been developed [see articles PI—PIV].
- A nonlinear mathematical model of the QPSK Costas loop is justified [see article PV].
- Effective numerical procedure and software tool for the simulation and analysis of Costas loops have been proposed [see articles PIII—PIV].

1.5 Adequacy of the results

All theoretical results are rigorously proved and, in special cases, coincide with the known classic results. Comparative numerical simulation of the proposed mathematical model and the corresponding physical model gave similar results.

1.6 Novelty

In this work, for the first time, a comprehensive rigorous mathematical method of constructing mathematical models of Costas loops for general signal waveforms is proposed. This method is based on the combination of engineering approaches to the investigation of PLL systems, the high-frequency analysis, and Fourier series application.

1.7 Practicability

The obtained results can be used for the analysis of the stability of Costas loops. The proposed method allows one to significantly reduce computation time spent on the numerical simulation of Costas loops (see patent application [AXVII]). It has become possible to obtain important characteristics of Costas circuits such as pull-in time, pull-in range, etc. Also, the models developed facilitate further analysis and synthesis of Costas loops.

1.8 Appraisal of the work and publications

The results of this work were presented at the following international conferences: International Congress on Ultra Modern Telecommunications and Control...

The results of this work also appeared in 21 publications: 8 publications in Scopus database, one Finnish patent application, 2 Russian patents, 2 certificates for computer program. The main results of this work are included in (Yuldashev, 2012, 2013a,b).

In the included papers [PI–PV] co-authors formulated the problems and estimated integrals, the author formulated and proved theorems.
2 THE MAIN CONTENT

2.1 Nonlinear models of the classic Costas loop

Consider an operation of the classic Costas loop (see Fig. 1) with the sinusoidal carrier and sinusoidal VCO (Voltage-Controlled Oscillator) signals after transient processes. The input signal is a BPSK signal, which is a product of the transmitted data

\[ m(t) \sin(\omega t) \]

and the high frequency \( \omega \). Because here the in-lock state of the Costas loop is considered, VCO signal is synchronized with the carrier (i.e., there is no frequency and phase difference between VCO signal and input carrier). On the lower branch (Q), after multiplying the VCO signal, shifted by \( -\frac{\pi}{2} \), by the input signal one has

\[
Q = \frac{1}{2} \left( m(t) \sin(0) - m(t) \sin(2\omega t) \right) = -\frac{1}{2} m(t) \sin(2\omega t).
\]

From an engineering point of view, the high-frequency part \( \sin(2\omega t) \) in (1) is erased by a low-pass filter on Q the branch. Since \( \sin(0) = 0 \), the signal on the
lower branch after the filtration is equal to zero, which indicates the in-lock state of the Costas loop.

On the upper branch (I), the input signal is multiplied by the output signal of VCO:

\[ I = \frac{1}{2} (m(t) \cos(0) - m(t) \cos(2\omega t)) = \frac{1}{2} (m(t) + m(t) \cos(2\omega t)). \]  

(2)

The high frequency term \( \cos(2\omega t) \) is filtered by the low-pass filter. Since \( \cos(0) = 1 \), on the upper branch, after filtration, one can obtain demodulated data \( m(t) \). Then both branches are multiplied together and, after an additional low-pass filtration, one gets the signal \( g(t) \) to adjust VCO frequency to the frequency of the input signal carrier. After the transient processes, the carrier and the VCO frequencies are equal to each other and the control input of VCO is equal to zero:

\[ g(t) = 0. \]

These results, lacking rigorous mathematical justification, were well-known to engineers (Gardner, 1966; Viterbi, 1966) in the case of sinusoidal signals. The first effective mathematical model of high-frequency signals for PLL-based circuits was proposed by Leonov (Leonov, 2008). The included papers [PI—PV] generalize this approach to the classic Costas loop with a sinusoidal VCO signal and various types of input signals. Here, we will describe the general approach to the investigation of Costas loops.

Now, consider the operation of a Costas loop before synchronization (Fig. 2), i.e., when the carrier phase \( \theta_1(t) \) and the VCO phase \( \theta_2(t) \) are different.

![Diagram of the classic Costas loop](image)

FIGURE 2 The classic Costas loop with the phase difference \( \theta_2(t) - \theta_1(t) \)

Let us formulate the high-frequency property of signals \( f^{1,2}(t) = f^{1,2}(\theta^{1,2}(t)) \) (here \( f^{1,2}(\theta) \) are waveforms) in the following way: suppose that for the frequencies

\[ \omega^{1,2}(t) = \dot{\theta}^{1,2}(t) \]

there exist a large number \( \omega_{\min} \) such that within a fixed time interval \([0, T]\) the following relation holds true:

\[ \omega^{1,2}(t) \geq \omega_{\min} > 0, \]  

(3)
where $T$ doesn’t depend on $\omega_{min}$.

The frequency difference is supposed to be uniformly bounded, i.e.,

$$\left| \omega^1(t) - \omega^2(t) \right| \leq \Delta \omega, \ \forall t \in [0, T]. \quad (4)$$

Denote $\delta = \omega_{min}^{-1}$, then

$$\left| \omega^p(t) - \omega^p(\tau) \right| \leq \Delta \Omega, \ p = 1, 2,$$

$$|t - \tau| \leq \delta, \ \forall t, \tau \in [0, T], \quad (5)$$

where $\Delta \Omega$ doesn’t depend on $\delta$.

Following the application of a Costas loop to GPS (Kaplan and Hegarty, 2006), let us consider a simplified loop shown in Fig. 3. It is the same loop as in Fig. 2, yet without Filter 1 and Filter 3, $m(t) \equiv 1$.

For the piecewise differentiable signal waveforms $f^{1,2}(\theta)$, which can be presented in the form of Fourier series

$$f^p(\theta) = \frac{a_0^p}{2} + \sum_{i=1}^{\infty} \left( a_i^p \cos(i\theta) + b_i^p \sin(i\theta) \right), \ \theta \geq 0$$

$$a_0^p = \frac{1}{\pi} \int_{-\pi}^{\pi} f^p(\theta) d\theta, \quad a_i^p = \frac{1}{\pi} \int_{-\pi}^{\pi} f^p(\theta) \cos(i\theta) d\theta,$$

$$b_i^p = \frac{1}{\pi} \int_{-\pi}^{\pi} f^p(\theta) \sin(i\theta) d\theta, \quad p = 1, 2,$$

it is possible to obtain the phase detector characteristics. Let us assume, that the linear filter satisfies the relation

$$\sigma(t) = a_0(t) + \int_0^t \gamma(t - \tau) \xi(\tau) d\tau,$$

$$|\gamma(\tau) - \gamma(t)| = O(\delta), \quad |t - \tau| \leq \delta, \ \forall \tau, t \in [0, T], \quad (6)$$

where $\xi(t)$ and $\sigma(t)$ are filter’s input and output respectively, $\gamma(t)$ is the impulse transient function with bounded derivative $a_0(t)$ is exponentially damped function depending on the initial conditions of the filter.

Using relation (6) we get

$$g(t) = a_0(t) + \int_0^t \gamma(t - \tau) f^1(\theta^1(\tau)) f^2(\theta^2(\tau)) \left( f^1(\theta^1(\tau)) \right) \left( f^2(\theta^2(\tau) - \frac{\pi}{2} \right) d\tau. \quad (7)$$
Consider the block-scheme shown in Fig. 4. Here, the phase detector (PD) is a nonlinear element with the output $\varphi(\theta_1(t) - \theta_2(t))$, which represents all intermediate multipliers; $G(t)$ is the output of the filter.

Let the initial conditions of the filters in Fig. 3 and Fig. 4 be the same, then

$$G(t) = a_0(t) + \int_0^t \gamma(t - \tau) \varphi(\theta_1(\tau) - \theta_2(\tau)) d\tau. \quad (8)$$

Consider a $2\pi$-periodic function

$$\varphi(\theta) = \frac{A_1^0 A_2^0}{4} + \frac{1}{2} \sum_{l=1}^{\infty} \left( (A_1^l A_2^l + B_1^l B_2^l) \cos(l\theta) + (A_1^l B_2^l - B_1^l A_2^l) \sin(l\theta) \right), \quad (9)$$

where $A_1^p$, $B_1^p$ can be calculated from the Fourier series coefficients of $f_1^1, 2(\theta)$ as follows

$$A_1^l = \frac{a_1^1 a_1^l}{2} + \frac{1}{2} \sum_{m=1}^{\infty} [a_1^m (a_1^m a_1^{m+l} + a_1^m b_1^{m+l}) + b_1^m (b_1^m a_1^{m+l} + b_1^m)],$$

$$B_1^l = \frac{a_1^1 b_1^l}{2} + \frac{1}{2} \sum_{m=1}^{\infty} [a_1^m (b_1^m a_1^{m-l} - b_1^m a_1^{m-l}) - b_1^m (a_1^m a_1^{m-l} - a_1^m)],$$

$$A_2^l = \frac{a_2^2 a_2^l}{2} + \frac{1}{2} \sum_{m=1}^{\infty} [a_2^m (a_2^m a_2^{m+l} + a_2^m b_2^{m+l}) + b_2^m (b_2^m a_2^{m+l} + b_2^m)],$$

$$B_2^l = \frac{a_2^2 b_2^l}{2} + \frac{1}{2} \sum_{m=1}^{\infty} [a_2^m (b_2^m a_2^{m-l} - b_2^m a_2^{m-l}) - b_2^m (a_2^m a_2^{m-l} - a_2^m)],$$

where

$$\alpha_2^k = \begin{cases} a_2^k, & k = 4p, \\ b_2^k, & k = 4p + 1, \\ -a_2^{-k}, & k = 4p + 2, \\ -b_2^{-k}, & k = 4p + 3, \end{cases} \quad \beta_2^k = \begin{cases} b_2^k, & k = 4p, \\ -a_2^k, & k = 4p + 1, \\ -b_2^k, & k = 4p + 2, \\ a_2^{-k}, & k = 4p + 3, \end{cases} \quad (11)$$

where $a_{-h} = a_h$, $b_{-h} = b_h$, $h < 0$.

The following theorem allows one prove the equivalence of the models shown in Fig. 3 and Fig. 4.
Theorem 1. If (3)–(6), are satisfied then

\[ |g(t) - G(t)| = O(\delta), \quad \forall t \in [0, T]. \] (12)

The proof of this theorem with some additional clarification can be found in the included articles [PI–PV].
Phase detector characteristics examples

\[ f^{1,2}(\theta) = \sin(\theta), \quad \varphi(\theta) = -\frac{1}{8} \sin(2\theta) \]

\[ f^{1,2}(\theta) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\theta), \quad \varphi(\theta) = -\frac{1}{2\pi} + \frac{1}{\pi^2} \sum_{l=1}^{\infty} \begin{cases} \frac{16}{\pi^2 l^4} \cos(l\theta), & l = 4p, \quad p \in \mathbb{N} \\ -\frac{2(\pi l - 2)}{\pi^2 l^4} \cos(l\theta) - \frac{4(\pi l - 2)}{\pi^4 l^4} \sin(l\theta), & l = 4p + 1, \quad p \in \mathbb{N}_0 \\ -\frac{8}{\pi^2 l^4} \sin(l\theta), & l = 4p + 2, \quad p \in \mathbb{N}_0 \\ \frac{4(\pi l + 2)}{\pi^4 l^4} \cos(l\theta) - \frac{4(\pi l + 2)}{\pi^4 l^4} \sin(l\theta), & l = 4p + 3, \quad p \in \mathbb{N}_0 \end{cases} \]

\[ f^{1,2}(\theta) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)\theta), \quad \varphi(\theta) = \frac{1}{\pi^2} \sum_{l=1}^{\infty} \frac{1}{(4l+2)^5} \sin((4l+2)\theta) \]

\[ f^1(\theta) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)\theta), \quad \varphi(\theta) = -\frac{1}{\pi^2} \sum_{l=1}^{\infty} \frac{1}{l} \sin(l\theta) \]

\[ \varphi(\theta) = -\frac{1}{\pi^2} + \frac{8}{\pi^2} \sum_{l=1}^{\infty} \begin{cases} \frac{4}{\pi^2 (l+4)^4} \cos((l+4)\theta), & l = 4p, \\ -\frac{2}{\pi^3} \sin(l\theta), & l = 4p + 2, \\ 0, & l = 2p + 1, \quad p \in \mathbb{N}_0 \end{cases} \]

\[ f^1(\theta) \quad f^2(\theta) \]

\[ \varphi(\theta) \]
2.2 QPSK Costas loop

The following description of the QPSK Costas loop follows article [PV]. Below, we provide a rigorous mathematical description of the QPSK Costas loop and formulate the corresponding theorem.

Consider the QPSK Costas loop shown in Fig. 5. The input signal has the following form

\[ m_1(t)\cos(\theta_1(t)) - m_2(t)\sin(\theta_1(t)) \]

where \( m_1,2(t) = \pm 1 \) is the transmitted data, \( \sin(\theta_1(t)) \) and \( \cos(\theta_1(t)) \) are carriers. Consider a sinusoidal output of VCO \( \cos(\theta_2(t)) \). The input signal and VCO output are high-frequency signals, i.e., for \( \theta_1(t) \) and \( \theta_2(t) \) conditions (3)–(5) are satisfied.

On the lower branch \( I \), after the multiplying input signal by the VCO signal, we get

$$ u^I(t) = (m_1(t)\cos(\theta_1(t)) - m_2(t)\sin(\theta_1(t)))\cos(\theta_2(t)), $$

Then, a low-pass filter (Filter 3) forms the signal

$$ I(t) = \int_0^t h(t - \tau)u^I(\tau)d\tau, $$

where \( h(t - \tau) \) is an impulse transient function. The signal \( I(t) \) allows one to obtain one of the carriers of the input signal.

Similarly, on the upper branch \( Q \) the product of the input signal and the VCO signal, shifted by \(-90^\circ\), forms the signal

$$ u^Q(t) = (m_1(t)\cos(\theta_1(t)) - m_2(t)\sin(\theta_1(t)))\sin(\theta_2(t)). $$
After the filtration by a low-pass filter (Filter 1) we get

\[ Q(t) = \int_0^t h(t - \tau) u^Q(\tau) d\tau. \]

This signal allows one to obtain the second carrier of the input signal. After the filtration, both signals, \( I(t) \) and \( Q(t) \), pass through the limiters. The outputs of the limiters are equal to \( \text{sign}(I(t)) \) and \( \text{sign}(Q(t)) \), respectively. Then, these signals are multiplied as shown in Fig. 5. The resulting difference, after the filtration by Filter 2, forms the control signal \( g(t) \). This signal is used as the input of the VCO for frequency and phase corrections. Similarly to the classic Costas loop, Filter 2 satisfies conditions (6).

Based on the applications of the QPSK Costas loop to GPS (Kaplan and Hegarty, 2006), we may consider a simplified loop (see Fig. 6). Here \( m_{1,2}(t) \equiv 1 \).

\[ \cos(\theta(t)) - \sin(\theta(t)) \]

\[ \cos(\theta(t)) \]

\[ \text{Filter 1} \]

\[ \text{Filter 2} \]

\[ \text{Filter 3} \]

\[ \text{g(t)} \]

**FIGURE 6** A simplified QPSK Costas loop

Suppose that Filter 1 and Filter 3 satisfy the conditions

\[
\int_0^t h(t - \tau) \sin(\omega \tau) d\tau = O\left(\frac{1}{\omega}\right), \ \forall \omega > \omega_{\text{min}},
\]

\[
\int_0^t h(t - \tau) \sin(\omega \tau) d\tau = \sin(\omega \tau) + O\left(\frac{1}{\omega}\right), \ \forall \omega < \Delta \omega.
\]

(13)

Consider the block-scheme shown in Fig. 4, where Filter 2 is the same filter as the one in Fig. 6.

Consider a \( 2\pi \)-periodic function \( \varphi(\theta) \)

\[
\varphi(\theta) = 0.5\sqrt{2} \sin(\theta(t)) \text{sign}(\sin(\theta(t))) - \sin(\theta(t)) \text{sign}(\sin(\theta(t))).
\]

(14)

The following theorem allows one to justify the transition from the block-scheme shown in Fig. 6 to that of Fig. 4.
Theorem 2. If conditions (3)–(6) and (13) are satisfied, then

\[ |g(t) - G(t)| = O(\delta), \quad \forall t \in [0, T]. \quad (15) \]

This result is discussed in details in (Yuldashev, 2013a).

2.3 Differential equations of Costas loops

A physical model of the classic Costas loop can be described by the following system of differential equations

\[
\dot{x} = Ax + b f^1(\theta^1(t)) f^2(\theta^2(t)) f^1(\theta^1(t)) f^2(\theta^2(t) - \frac{\pi}{2}),
\]

\[
\dot{\theta}^1 = \omega_{\text{free}}^2 + L c^* x,
\]

\[
\dot{\theta}^1 \equiv \omega^1.
\] (16)

Here \(A\) is the constant matrix of the filter, \(x(t)\) represents the state of the filter, \(b\) and \(c\) are constant vectors — parameters of the filter, \(L\) is a constant, which defines feedback strength of the system, \(\omega_{\text{free}}^2\) is the VCO self (free) frequency, * is the transpose operator.

The QPSK Costas loop can be described by the following differential equations:

\[
x_1 = A_1 x_1 + b_1 (\cos(\theta^2) (\cos(\theta^1) - \sin(\theta^1))),
\]

\[
x_2 = A_2 x_2 + b_2 (\text{sign}(c_1^* x_1) (c_3^* x_3) - \text{sign}(c_3^* x_3) (c_1^* x_1)),
\]

\[
x_3 = A_3 x_3 + b_3 (\sin(\theta^2) (\cos(\theta^1) - \sin(\theta^1))),
\]

\[
\dot{\theta}^1 \equiv \omega^1,
\]

\[
\dot{\theta}^2 = \omega_{\text{free}}^2 + L c^* x_2,
\] (17)

where \(A_{1,2,3}, b_{1,2,3}, c_{1,2,3}\) are parameters of the filter and \(x_{1,2,3}(t)\) is the state of the filter.

Using the phase-detector characteristics, it is possible to derive differential equations of Costas loops in the phase space as follows

\[
\dot{x} = Ax + b \varphi(\theta),
\]

\[
\dot{\theta} = \omega_{\text{free}}^2 - \theta^1 + L c^* x,
\]

\[
\theta \equiv \theta^2 - \theta^1.
\] (18)

Here \(\varphi(\theta)\) is the phase detector characteristics, which depends on the signal waveforms.

The averaging methods allow one to justify that the solutions of differential equations in the phase space are close to the solutions in the signal/time space.
2.4 Numerical simulation

The numerical simulations of the considered Costas loops with different sets of parameters in the signal space and the phase space confirms the theoretical results. In Fig. 7 one can see some results of the simulation of the classic Costas loop and that of the QPSK Costas loop. It should be noted, that the numerical simulation in the phase space is more than hundred-fold times efficient than the simulation in the signal space.

The efficiency of the proposed method is confirmed by the numerical simulation of the Costas loop for high-frequency signals. In Fig. 8, an example of modeling the classic Costas loop with 1Ghz signals is shown. Here, in the course of 10 seconds of computing in the signal space (where the process of computation is very slow), only $2.5 \times 10^{-7}$ seconds of the transient processes have been modeled. Therefore, the numerical simulation of the full transient process in the signal space is almost impossible for high-frequency signals. At the same time, the full simulation time of 20 seconds of the transient processes in the phase space took less than a second.

The derived method can be adapted for the numerical simulation of the digital Costas loops. An example of such simulation is shown in Fig. 9.
FIGURE 8 Comparison of the effectiveness of simulation in signal space and phase space

FIGURE 9 Simulation of digital Costas loop
REFERENCES


