Pareto-optimal Nash Equilibrium in Dynamic Games with Non-transferable Utility

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1. Introduction

Cooperative games suggest the possibility of enhancing the participants' well-being in situations involving strategic interactions. Various cooperative solutions have been presented, like the Nash (1950, 1953) bargaining solution, the Shapley (1953) value, and the stable set of Von Neumann and Morgenstern (1944). An essential property that a cooperative scheme has to satisfy is individual rationality which guarantees that each player's cooperative payoffs will be no less than his non-cooperative payoff. Moreover, conditional upon the fulfillment of individual rationality a desirable property of the scheme is Pareto efficiency. Pareto efficiency ensures that the cooperative gains of any player cannot be enhanced without the reduction of gains of some other players.

For games that are played over time the derivation of a cooperative solution satisfying individual rationality throughout the cooperation duration becomes extremely strenuous. In addition to individual rationality and Pareto efficiency, the sustainability of the agreed-upon solution is also of concern to the participating players. Frequently, the lack of guarantee that individual rationality during cooperation leads to break-ups of the scheme as the game evolves. Haurie (1976) pointed out that the property of dynamic consistency, which is crucial in maintaining sustainability in cooperation, is absent in the direct application of the Nash bargaining solution in differential games. Time consistent solutions for differential games under deterministic and stochastic dynamics can be found in Petrosyan and Zenkevich (1996), Petrosyan (1997), Yeung and Petrosyan (2004) and Yeung and Petrosyan (2006).

In this article we present a solution formula for the payoff distribution procedure of a cooperative differential game that would lead to a time consistent outcome.

2. Problem Formulation

Consider the cooperative differential game $\Gamma(x_0, T - t_0)$ with non-transferable payoffs in which the state dynamics is:

$$\dot{x}(t) = f[t, x(t), u_1(t), u_2(t), \cdots, u_n(t)], \quad x(t_0) = x_0 \text{ and } t \in [t_0, T].$$

(2.1)

The payoff of player i

$$K_i(x_0, T - t_0; u_1, u_2, \cdots, u_n) = \int_{t_0}^{T} g^i[\tau, x(\tau), u_1(\tau), u_2(\tau), \cdots, u_n(\tau)]d\tau, \qquad (2.2)$$

for $i \in \{1, 2, \cdots, n\} \equiv N$.

Let $V^i(t_0, x_0)$ be the Nash equilibrium payoff of player i if it exists. We side-step the multiple solution case and assume that there exist a unique Nash equilibrium or a certain equilibrium is chosen.

Consider the case when the players agree to cooperate and bargain. The agents consent to use a vector of weights $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, for $\alpha > 0$ and $\sum_{j=1}^{n} \alpha_j = 1$, on their payoffs and obtain a Pareto optimal outcome. Conditional upon the agreed-upon vector of weights α, the agents' optimal cooperative strategies can be generated by solving the following control problem (See Leitmann (1974), Dockner and Jorgensen (1984), Hamalainen et al (1986), and Yeung and Petrosyan (2005)):

$$\max_{u_1, u_2, \cdots, u_n} \int_{t_0}^{T} \sum_{j=1}^{n} \alpha_j g^j[\tau, x(\tau), u_1(\tau), u_2(\tau), \cdots, u_n(\tau)] d\tau \qquad (2.3)$$

subject to (2.1).

Invoking the standard dynamic programming technique an optimal solution to the control problem (2.1) and (2.3) can be characterized as follows. A set of control strategies $\{\psi_i^{(\alpha)}(\tau, x)$, for $i \in N$ and $\tau \in [t_0, T]\}$ brings about an optimal solution for the dynamic programming problem (2.1) and (2.3) if there exists a differentiable function $W^{(\alpha)}(t, x) : [t_0, T] \times R^m \to R$ satisfying the following partial differential equation:

$$-W_t^{(\alpha)}(t, x) = \max_{u_1, u_2, \cdots, u_n} \left\{ \sum_{j=1}^{n} \alpha_j g^j [t, x, u_1, u_2, \cdots, u_n] \right.$$

$$\left. + W_x^{(\alpha)}(t, x) f[t, x, u_1, u_2, \cdots, u_n] \right\}$$

$$= \sum_{j=1}^{n} \alpha_j g^j [t, x, \psi_1^{(\alpha)}(t, x), \psi_2^{(\alpha)}(t, x), \cdots, \psi_n^{(\alpha)}(t, x)]$$

$$+ W_x^{(\alpha)}(t, x) f[t, x, \psi_1^{(\alpha)}(t, x), \psi_2^{(\alpha)}(t, x), \cdots, \psi_n^{(\alpha)}(t, x)],$$

$$W^{(\alpha)}(T, x) = 0. \tag{2.4}$$

Substituting the cooperative strategies $\{\psi_i^{(\alpha)}(\tau,x)$, for $i\in N$ and $\tau\in[t_0,T]\}$ into (2.1) yields the dynamics of the cooperative state trajectory

$$x(t) = f[t,x,\psi_1^{(\alpha)}(t,x),\psi_2^{(\alpha)}(t,x),\cdots,\psi_n^{(\alpha)}(t,x)], \quad x(t_0) = x_0 . \tag{2.5}$$

We use $\{x^{(\alpha)}(t)\}_{t=t_0}^{T}$ to denote the solution to (2.5).

Note that the cooperative strategies $\{\psi_i^{(\alpha)}(\tau,x)$, for $i\in N$ and $\tau\in[t_0,T]\}$ of the dynamic programming problem generated by (2.4) are also strategies solving the optimal control problem (2.3). We can call these cooperative strategies Pareto optimal controls under cooperation.

The payoff of player i under cooperation can be obtained as:

$$\int_{t_0}^{T} g^i[\tau,x^{(\alpha)}(\tau),\psi_1^{(\alpha)}(\tau,x^{(\alpha)}(\tau)),\psi_2^{(\alpha)}(\tau,x^{(\alpha)}(\tau)),\cdots,\psi_n^{(\alpha)}(\tau,x^{(\alpha)}(\tau))]d\tau$$

$$= \int_{t_0}^{T} h_i(\tau,x^{(\alpha)}(\tau))d\tau, \qquad \text{for } i\in N .$$

(2.6)

At the start of the game for individual rationality to hold under the optimal state $\{x^{(\alpha)}(\tau)\}_{\tau=t_0}^{T}$ in the cooperative game $\Gamma(x_0, T-t_0)$ it is required that

$$\int_{t_0}^{T} h_i\,(\tau, x^{(\alpha)}(\tau))d\tau \geq V_i\ (t_0, x_0),\ \text{for}\ i \in N. \qquad (2.7)$$

It is obvious that there exists $\left\{x^{(\alpha)}(\tau)\right\}_{\tau=t_0}^{T}$ such that (2.7) is satisfied at initial time t_0.

But it may happen that as the game proceeds there exist $t \in [t_0, T]$ such that

$$\int_{t}^{T} h_i\,(\tau, x^{(\alpha)}(\tau))d\tau < V^{i}(t, x^{(\alpha)}(t)), \qquad \text{for some}\ i \in N. \qquad (2.8)$$

Time-inconsistency of the individual rationality condition appears if (2.8) happens.

3. Time-consistent Solution Formula

To overcome the time inconsistency problem in (2.8), we follow Petrosyan (1993 and 1997) and introduce a payoff distribution procedure (PDP) with a set of functions $\beta_i(\tau)$ for $\tau \in [t_0, T]$ such that

$$\int_{t_0}^{T} \beta_i(\tau) d\tau = \int_{t_0}^{T} h_i(\tau, x^{(\alpha)}(\tau)) d\tau, \quad \text{for } i \in N, \tag{3.1}$$

which requires the satisfaction of the condition:

$$\int_{t}^{T} \beta_i(\tau) d\tau \geq V^i(t, x^{(\alpha)}), \text{ for } i \in N \text{ and } t \in [t_0, T]. \tag{3.2}$$

If we substitute $h_i(\tau, x^{(\alpha)}(\tau))$ by $\beta_i(\tau)$, individual rationality will hold in all subgames along the cooperative trajectory $x^{(\alpha)}(\tau)$ for $\tau \in [t_0, T]$. Next we present a formula for the function $\beta_i(\tau)$ which satisfies (3.2).

Formula 3.1.

A payoff distribution procedure $\beta_i(\tau)$ with the form

$$\beta_i(\tau) = \frac{\int_{t_0}^{T} h_i(\tau, x^{(\alpha)}(\tau))d\tau - V^i(t_0, x_0)}{T - t_0} - \frac{d}{d\tau} V^i(\tau, x^{(\alpha)}(\tau)), \text{ for } \tau \in [t_0, T], \qquad (3.3)$$

would yield a time-consistent payoff which guarantees individual rationality along the cooperative trajectory $x^{(\alpha)}(\tau)$ for $\tau \in [t_0, T]$.

Proof:

Using (3.3) we obtain

$$\int_t^T \beta_i(\tau)d\tau = \int_t^T \left[\frac{\int_{t_0}^T h_i(\tau, x^{(\alpha)}(\tau))d\tau - V^i(t_0, x_0)}{T - t_0} - \frac{d}{d\tau}V^i(\tau, x^{(\alpha)}(\tau)) \right] d\tau$$

$$= \frac{T-t}{T-t_0}\left(\int_{t_0}^T h_i(\tau, x^{(\alpha)}(\tau))d\tau - V^i(t_0, x_0) \right) - \int_t^T \frac{d}{d\tau}V^i(\tau, x^{(\alpha)}(\tau))d\tau . \qquad (3.4)$$

We use F_i to denote $\frac{T-t}{T-t_0}\left(\int_{t_0}^T h_i(\tau, x^{(\alpha)}(\tau))d\tau - V^i(t_0, x_0) \right)$ and express (3.4) as:

$$\int_t^T \beta_i(\tau)d\tau = \frac{T-t}{T-t_0}F_i - \int_t^T \frac{d}{d\tau}V^i(\tau, x^{(\alpha)}(\tau))d\tau$$

$$= \frac{T-t}{T-t_0}F_i - [V^i(x^{(\alpha)}(T), T-T) - V^i(t, x^{(\alpha)}(t))]$$

$$= \frac{T-t}{T-t_0}F_i - [0 - V^i(t, x^{(\alpha)}(t))]$$

$$= \frac{T-t}{T-t_0}F_i + V^i(t, x^{(\alpha)}(t)) \geq V^i(t, x^{(\alpha)}(t))$$, because $F_i \geq 0$.

Hence individual rationality is upheld throughout $t \in [t_0, T]$ along the cooperative trajectory. □

4. Strategic Support of Pareto Optimal Solution

Consider a new game $\Gamma_\alpha(x_0, T - t_0)$ which differs from the original game $\Gamma(x_0, T - t_0)$ only with payoffs of players along the Pareto-optimal trajectory $x^{(\alpha)}(\tau)$ for $\tau \in [t_0, T]$ and connected trajectories.

Let $K_i^\alpha(x_0, T - t_0; u_1, \ldots, u_n)$ denote the payoff of player $i \in N$ in the game $\Gamma_\alpha(x_0, T - t_0)$, and let $x(\tau)$ for $\tau \in [t_0, T]$ denote the corresponding trajectory.

Then

$$K_i^\alpha(x_0, T - t_0; u_1, \ldots, u_n) = K_i(x_0, T - t_0; u_1, \ldots, u_n)$$

If do not exist such $t \in (t_0, T]$ that $x(\tau) = x^{(\alpha)}(\tau)$ for $\tau \in [t_0, T]$.

(Here for $x^{(\alpha)}(\tau)$ is the Pareto-optimal trajectory).

Let $t = \sup\left\{t^1 : x(\tau) = x^{(\alpha)}(\tau), \tau \in \left[t_0, t^1\right]\right\}, t > t_0$. Then

$$K_i^{\alpha}(x_0, T - t_0; u_1, \dots, u_n) = \int_{t_0}^{t} \beta_i(\tau) dt + K_i\left(x^{\alpha}(\tau), T - t; u_1, \dots, u_n\right)$$

$$= \int_{t_0}^{t} \beta_i(\tau) dt + \int_{t}^{T} g^i[\tau, x(\tau), u_1(\tau), \dots, u_n(\tau)] d\tau .$$

In a special case when $x(\tau) = x^{(\alpha)}(\tau)$ for $\tau \in [t_0, T]$, that is $x(\tau)$ coincides with the Pareto-optimal trajectory, we obtain

$$K_i^{\alpha}(x_0, T - t_0; u_1, \dots, u_n) = \int_{t_0}^{T} \beta_i(\tau) dt .$$

Be definition of the payoff functions in the game $\Gamma_\alpha(x_0, T - t_0)$ we obtain the condition that along the Pareto-optimal trajectory the payoffs in $\Gamma_\alpha(x_0, T - t_0)$ and the in $\Gamma(x_0, T - t_0)$ coincide.

Definition 4.1. The game $\Gamma_{\alpha}(x_0, T-t_0)$ is a regularization of the game $\Gamma(x_0, T-t_0)$ if the PDP $\beta_i(\tau)$ is defined by formula (3.3).

Definition 4.2. Consider the dynamic system

$$\dot{x}(\tau) = f[\tau, x(\tau), u_1(\tau), \ldots, u_n(\tau)], x(t_0) = x_0 \text{ and } \tau \in [t_0, T]. \quad (1)$$

Let $y = x(t)$ denote the point that is reached from the solution of system (1) under a given n-tuple of controls $u(\tau) = (u_1(\tau), \ldots, u_n(\tau))$ for $\tau \in [t_0, T]$. The set of all possible $y = x(t)$ under different controls $u(\tau)$ for $\tau \in [t_0, T]$ is called the reachable set of system (1) from initial state x_0 at time instant $t \in [t_0, T]$. We denote the reachable set of system (1) by $C(x_0, T-t)$.

Consider now the problem of strategic stability of the cooperation scheme. Using (3.3) one can prove the following theorem.

Theorem 4.1. In the regularized game $\Gamma_{\alpha}(x_0, T-t_0)$ for every $\varepsilon > 0$, there exists an ε -Nash equilibrium with Pareto-optimal payoffs

$$K_i^{\alpha}(x_0, T-t_0; u_1, \ldots, u_n) = \int_{t_0}^{T} g^i[\tau, x(\tau), u_1(\tau), \ldots, u_n(\tau)]d\tau$$

$$= \int_{t_0}^{T} \beta_i(\tau)dt\,. \qquad (*1)$$

if the following condition is satisfied:

Suppose $y \in C(x_0, T-t)$ for $t \in [t_0, T]$, where $y \in C(x_0, T-t)$ is the reachable set of the dynamical system (2.1) from initial state x_0 at moment t, and $\psi = (\psi_1, \ldots, \psi_n)$ is any fixed n-tuple of feedback strategies in $\Gamma(x_0, T-t_0)$. Then the payoff function

$$K_i(y, T-t; \psi_1, \ldots, \psi_n)$$

is a continuous function of y and t, for $y \in C(x_0, T-t)$ and $t \in [t_0, T]$ for any n-tuple of feedback strategies $\psi = (\psi_1, \ldots, \psi_n)$.

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