ST. PETERSBURG UNIVERSITY
Graduate School of Management
Faculty of Applied Mathematics & Control Processes
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(Russian Chapter)

CONTRIBUTIONS TO GAME THEORY AND MANAGEMENT

Volume IV

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Collected papers
Edited by Leon A. Petrosyan and Nikolay A. Zenkevich

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The collection contains papers accepted for the Fourth International Conference Game Theory and Management (June 28–30, 2010, St. Petersburg University, St. Petersburg, Russia). The presented papers belong to the field of game theory and its applications to management.

The volume may be recommended for researches and post-graduate students of management, economic and applied mathematics departments.

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Успехи теории игр и менеджмента. Вып. 4. Сб. статей четвертой международной конференции по теории игр и менеджменту / Под ред. Л.А. Петросяна, Н.А. Зенкевича. – СПб.: Высшая школа менеджмента СПбГУ, 2011. – 514 с.

Сборник статей содержит работы участников четвертой международной конференции «Теория игр и менеджмент» (28–30 июня 2010 года, Высшая школа менеджмента, Санкт-Петербургский государственный университет, Санкт-Петербург, Россия). Представленные статьи относятся к теории игр и ее приложениям в менеджменте.

Издание представляет интерес для научных работников, аспирантов и студентов старших курсов университетов, специализирующихся по менеджменту, экономике и прикладной математике.

Contents

Preface

This edited volume contains a selection of papers that are an outgrowth of the Fourth International Conference on Game Theory and Management with a few additional contributed papers. These papers present an outlook of the current development of the theory of games and its applications to management and various domains, in particular, energy, the environment and economics.

The International Conference on Game Theory and Management, a three day conference, was held in St. Petersburg, Russia in June 28-30, 2010. It was the initiative of St. Petersburg University carried out by the Graduate School of Management SPbU and the Department of Applied Mathematics Control Processes SPbU in collaboration with The International Society of Dynamic Games (Russian Chapter). More than 100 participants from 25 countries had an opportunity to hear state-of-the-art presentations on a wide range of game-theoretic models, both theory and management applications.

Plenary lectures covered different areas of games and management applications. They had been delivered by Professor Alain Haurie, University of Geneva (Switzerland); Professor Arkady Kryazhimskiy International Institute for Applied Systems Analysis (Austria) and Steklov Mathematical Institute RAS (Russia; Professor Herve Moulin, Rice University (USA) and Professor Ralph Tyrrell Rockafellar, University of Washington (USA).

The importance of strategic behavior in the human and social world is increasingly recognized in theory and practice. As a result, game theory has emerged as a fundamental instrument in pure and applied research. The discipline of game theory studies decision making in an interactive environment. It draws on mathematics, statistics, operations research, engineering, biology, economics, political science and other subjects. In canonical form, a game obtains when an individual pursues an objective(s) in a situation in which other individuals concurrently pursue other (possibly conflicting, possibly overlapping) objectives and in the same time the objectives cannot be reached by individual actions of one decision maker. The problem is then to determine each individual's optimal decision, how these decisions interact to produce equilibria, and the properties of such outcomes. The foundations of game theory were laid more than sixty years ago by von Neumann and Morgenstern (1944).

Theoretical research and applications in games are proceeding apace, in areas ranging from aircraft and missile control to inventory management, market development, natural resources extraction, competition policy, negotiation techniques, macroeconomic and environmental planning, capital accumulation and investment.

In all these areas, game theory is perhaps the most sophisticated and fertile paradigm applied mathematics can offer to study and analyze decision making under real world conditions. The papers presented at this Fourth International Conference on Game Theory and Management certainly reflect both the maturity and the vitality of modern day game theory and management science in general, and of dynamic games, in particular. The maturity can be seen from the sophistication of the theorems, proofs, methods and numerical algorithms contained in the most of the papers in these contributions. The vitality is manifested by the range of new ideas, new applications, the growing number of young researchers and the expanding world wide coverage of research centers and institutes from whence the contributions originated.

The contributions demonstrate that GTM2010 offers an interactive program on wide range of latest developments in game theory and management. It includes recent advances in topics with high future potential and exiting developments in classical fields.

We thank Anna Tur from the Faculty of Applied Mathematics (SPbU) for displaying extreme patience and typesetting the manuscript.

Editors, Leon A. Petrosyan and Nikolay A. Zenkevich

Graph Searching Games with a Radius of Capture ⋆

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Abstract. The problem of guaranteed search on graphs, the so–called ε–search problem, is considered. The properties of the Golovach function which is the ε–search number as the function of the radius of capture ε are studied. In the present work, we show that the jumps of the Golovach function for trees may have an arbitrarily large height. We describe some classes of trees with the non–degenerate Golovach function, and construct the minimal tree with a non–unit jump.

Keywords: guaranteed search, team of pursuers, evader, ε–capture, search numbers, Golovach function, unit jumps.

1. Introduction

Here, we deal with a problem of guaranteed search on graphs "with a radius of capture". As was mentioned by Parsons (1976) — one of the pioneers of the theory of the guaranteed search — the main features of the guaranteed search in general can be studied from the article (Breisch, 1967) by speleologist Richard Breisch. Breisch considered the following problem. A person is lost in a cave, which is in total darkness, and wandering aimlessly. We are looking for an efficient way for rescue party to search the lost person: what is the minimum number of searchers required to explore a cave so that it is impossible to miss finding the victim if he is in the cave. The article of Breisch doesn't contain strict statements of this problem but provides a lot of examples.

Now, let the cave be represented by the finite connected graph G so that the rooms are described by vertices and the passages — by edges. We may assume that G is embedded in $\mathbb{R}^3$ so that its vertices are points in $\mathbb{R}^3$ and its edges are represented by closed line segments which intersect only at vertices of G. The searchers must proceed according to a predetermined plan which will find the lost man even if he was that sort of victim who knows the searcher's every move, is arbitrarily fast and invisible for rescuers, and tries to avoid meeting doing his best (in such assumption it is more correct to call the lost man as *evader* and the searchers as *pursuers*). A family $\Pi = \{x_1, \ldots, x_n\}$ of continuous functions is called a *search plan* (*program*). A search plan Π is called *winning* if for every continuous function there exist $i \in \overline{1,n}$ and $t \in [0, T]$ such that $x_i(t) = y(t)$ holds. The *search number* of the graph G is the minimum cardinality of all winning search plans for G.

In this way the problem was posed by Parsons (1976). Later but independently and in slightly different terms the similar problem of guaranteed search was posed

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by the second author of the present work (Petrov, 1982). As was proven by Golovach both problems are equivalent to a certain discrete one (Golovach, 1989).

Later many other "search numbers" appear in different variations of the described problem depending on the possible behavior of the players and other constraints. So that it becomes necessary to distinguish corresponding search numbers from each other. Thus search number which is considered in the Parsons-Petrov problem is called the *edge–search number*.

There are different connections and applications of the guaranteed graph searching. Search numbers are closely connected with some parameters of linear layouts (*cut–width*, *topological bandwidth*), which play significant role in extra-large-scale integrated circuits (ELSI). There are notable relations between the graph searching problems and the theory of graph minors by Robertson and Seymour. Also search numbers appear in pebble games, which model the rational computer memory usage. There are connections between the graph searching problem and different topics of Information Security, Biology, Linguistics, Robotics and chess composition (see (Fomin and Thilikos, 2008) for more details).

2. The search problem with a radius of capture and the Golovach function

The problem of guaranteed search on graphs with a radius of capture was posed by Golovach (1990).

Let a topological graph G be embedded in three dimensional Euclidean space so as it was described in Introduction. From this time only finite connected graphs without loops and multiple edges are considered. And for further simplicity, we shall assume that the edges of the graph are polygonal lines with finite number of line segments. By ρ we denote the inner metric of the graph, i.e. $\rho(x, y)$ is the shortest path (in the Euclidean norm) with ends in x and y and contained entirely in G.

A team of pursuers $\mathcal{P} = (P_1, \ldots, P_k)$ and an evader E are bounded within the graph and possess the simple motions:

$$\begin{aligned} &(P_i)\colon \dot{x}_i = u_i, \quad \|u_i\| \leq 1, \quad i \in \overline{1, k}, \\ &(E)\colon \dot{y} = u_0, \end{aligned}$$

where the $\| \ldots \|$ is the Euclidean norm and admissible controls are piecewise constant functions defined on arbitrary closed "time" segments. The trajectories of the players are piecewise affine vector–functions with values in G. The team of pursuers tries to catch the invisible evader. It is supposed that the evader is caught by the pursuer if they are on distance less than or equal to a given non–negative number ε which is a *radius of capture*.

The *ε–search number*, denoted by $s_G(\varepsilon)$, is the minimum number of pursuers which can catch the evader with the radius of capture equal to ε. The problem is to find the ε–search number for each topological graph and given radius of capture ε. The given problem is also called the *ε–search problem* or the *Golovach problem*.

The function assigning the ε–search number to each $\varepsilon \geq 0$ is called the *Golovach function*. Let us notice the Golovach function is piecewise constant non-increasing right continuous function.

The rest of this article deals with the problem mentioned above for the case of trees.

2.1. Some properties of the Golovach function for trees

Let us remind that the program of the team of pursuers is the collection Π of the trajectories $\{x_1(t), \dots, x_k(t), t \in [0, T]\}$. The program is winning if for every trajectory y of the evader defined on $[0, T]$ there are $t \in [0, T]$ and $i \in \overline{1, k}$ such that $\rho(x_i(t), y(t)) \leq \varepsilon$.

The minimum number of pursuers required for zero–radius capture on G is denoted by $s(G)$.

Given a positive integer $k \leq s(G)$, let $\varepsilon_G(k)$ denote the minimum capture radius for which k pursuers surely catch the evader on G (it exists in virtue of the right continuity of the Golovach function).

If v is the vertex of tree T, a *branch* of T at v is a maximal subtree B of T subject to the condition that v be of degree 1 in B.

We say that a pursuer P is ε*–close* to (ε*–far* from) a point a of the tree at some moment t if $\rho(a, x(t)) \leq \varepsilon$ (respectively, $\rho(a, x(t)) > \varepsilon$).

The next two lemmas and the theorem were proved in (Abramovskaya and Petrov, 2010).

Lemma 1. *For any subtree T' of an arbitrary tree T, $\varepsilon_{T'}(k) \leq \varepsilon_T(k)$.*

Lemma 2. *Let B_1, B_2 and B_3 are distinct branches of T at a. Suppose that for each branch B_i with $i = 1, 2, 3$ the following condition holds: for any program of a team $\mathcal{P}$ winning in the ε–capture problem on B_i, there exists a moment of time at which all of the pursuers are ε–far from a. Then, team $\mathcal{P}$ cannot successfully complete the ε–search on T.*

For any $\varepsilon > 0$, let $\mathcal{T}(\varepsilon)$ denote the set of all trees T in which a distance between arbitrary vertices of degree at least 3 is not equal to 2ε.

Theorem 1. *Suppose that $T \in \mathcal{T}(\varepsilon)$ and k pursuers catch an evader with capture radius ε on T. Then, the group of $k + 1$ pursuers can surely perform the δ–capture on T for some $\delta < \varepsilon$.*

According to the hypothesis of Theorem 1, the Golovach function for a tree T has unit jump (or is continuous) at ε subject to the condition that $T \in \mathcal{T}(\varepsilon)$. (Note that this condition is sufficient but is not necessary.) Thus if for each ε at which the Golovach function for tree T may have a jump it is known that $T \in \mathcal{T}(\varepsilon)$, one can conclude that the Golovach function for T has only unit jumps. Then the following assertion holds: *let a team $\mathcal{P}$ can catch the evader on T with a capture radius $\varepsilon > 0$, then the team with more number of pursuers than in $\mathcal{P}$ can catch the evader on T with the capture radius less than ε.*

Let us consider the following problem. Suppose that a number k of the pursuers is fixed. The problem is for each topological graph G to find the minimum radius of capture $\varepsilon \geq 0$ so that the team of k pursuers can catch the evader on G with capture radius ε.

For the case of trees and one pursuer this problem was solved in (Abramovskaya, 2010). Let T be a tree. Let $Z = (a_1, \dots, a_n)$ is a chain of maximal length in T, subtrees $A_1, \dots, A_m$ are branches of T at the vertices of Z subject to the condition that each A_i, $i = 1, \dots, m$, contains only one vertex of Z. By l_i, $i = 1, \dots, m$, we denote the maximal length of a chain in A_i that starts at the vertex of Z.

Lemma 3. *The following equality holds:*

$$\varepsilon_T(1) = \frac{1}{2} \min_{i \in \overline{1,m}} l_i.$$

This assertion has a simple corollary.

Corollary 1. *Let T be a tree, if $s(T) > 1$, then $\varepsilon_T(2) < \varepsilon_T(1)$.*

Further, we study the problem of the existence of non–unit jumps of the Golovach function for trees. It was proved in (Golovach et al., 2000) that non–unit jumps appear in the case of the complete graphs for which the jump may have an arbitrarily large height. The case of trees seems greatly simpler but has the same *degeneration* (the existence of the non–unit jumps) of the Golovach function.

2.2. The main results

In the rest of the present article we denote by $\mathcal{T}$ the tree which contains one edge of length 3 and two edges of length 1, and one vertex of degree 3.

The tree $\mathcal{T}_0$ is constructed using three copies of tree $\mathcal{T}$. For each such copy the pendant vertex which is incident to the edge of length 3 identified with a certain vertex c_0 (see Fig.1, the digits indicate edge lengths).

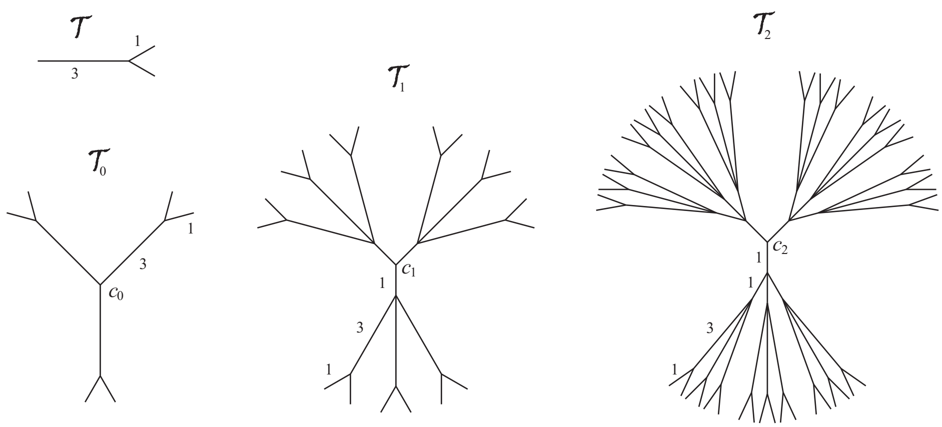

Fig. 1.

If a tree $\mathcal{T}_{n-1}$, $n \geq 1$, has been defined, let $\mathcal{T}_n$ be constructed as follows. Let's consider three disjoint copies of tree $\mathcal{T}_{n-1}$ and a new vertex c_n. Let's connect the vertex c_n with the vertex of each branch which is respective with c_{n-1} by edges of length 1.

Theorem 2. *For each $n \geq 0$ and $0 \leq \varepsilon < 0.5$ the equality $s_{\mathcal{T}_n}(\varepsilon) = n + 3$ holds.*

Proof. Let's consider an arbitrary $0 \leq \varepsilon < 0.5$ and prove the statement by induction.

Let $n = 0$, three pursuers capture the evader on $\mathcal{T}_0$ by means of the following program with zero radius of capture. P_1 is standing in vertex c_0 during the whole program. Other two pursuers are searching on the branches of $\mathcal{T}_0$ at c_0. Further, each branch of $\mathcal{T}_0$ at c_0 coincides with $\mathcal{T}$ and requires at least 2 ε–far from c_0 pursuers to succeed in ε–capture. Thus, two pursuers can't catch the evader on $\mathcal{T}_0$ with capture radius ε.

The winning program on $\mathcal{T}_1$ for the team of four pursuers with zero radius of capture is as follows. P_1 occupies c_1 and stays there in what follows. Sequentially for each branch of $\mathcal{T}_1$ at c_1, pursuers P_2, P_3, P_4 slide along the edge incident to c_1 into a vertex c' respective with c_0 in $\mathcal{T}_0$. While P_2 stays at vertex c', pursuers P_3 and P_4 are searching on the remained branches at c'. Further, three pursuers can't catch the evader on $\mathcal{T}_1$ with capture radius ε in virtue of Lemma 2, because each branch of $\mathcal{T}_1$ at c_1 requires at least 3 ε–far from c_1 pursuers to succeed in ε–capture.

Let $n-1>1$, suppose that the team of $n+2$ pursuers can catch the evader on $\mathcal{T}_{n-1}$ with zero radius of capture, and a certain winning program of $n+2$ pursuers on $\mathcal{T}_{n-1}$ allows that a single pursuer is motionless in the vertex c_{n-1} during the whole program. Furthermore, suppose that the team of $n+1$ pursuers has failed in pointwise capturing on T.

Let's proceed by induction from $n-1$ to n. The winning program of $n+3$ pursuers on $\mathcal{T}_n$ is as follows. Pursuer P_1 occupies vertex c_n and stays there in what follows. Sequentially for each branch of $\mathcal{T}_n$ at c_n, pursuers $P_2,\ldots,P_{n+3}$ slide along the edge incident to c_n to the other endpoint of the edge, say, c', and then, while P_2 stays at vertex c', pursuers $P_3,\ldots,P_{n+3}$ are searching on the remained branches at c'.

To finish the proof it is sufficient to show that the team of $n+2$ pursuers are unable to catch the evader on $\mathcal{T}_n$ with capture radius ε. Indeed, by the induction hypothesis it is necessary to put at least $n+2$ pursuers on tree $\mathcal{T}_{n-1}$ to catch the evader with capture radius ε. According to the construction of $\mathcal{T}_n$, the three copies of $\mathcal{T}_{n-1}$ connect to vertex c_n by the unit length edge. As 2ε is less than 1, the ε–catching on each branch of $\mathcal{T}_n$ at c_n requires $n+2$ pursuers ε–far from c_n, and Lemma 2 takes effect. □

Theorem 3. *Let $n\geq 1$, the Golovach function for tree $\mathcal{T}_n$ has the jump of height $\lceil \frac{n}{2}\rceil+1$.*

Proof. First we prove that $n+2-\lceil \frac{n}{2}\rceil$ pursuers can catch the evader on $\mathcal{T}_n$ with capture radius 0.5. We consider separately the cases of odd and even n.

Let us consider an arbitrary path in $\mathcal{T}_n$ from c_n to the vertex incident to the edge of length 3. Let us denote the vertices of this path so that the vertex incident to c_n is denoted by c_{n-1}, the vertex incident to c_i is denoted by c_{i-1}, $i=1,\ldots,n-1$ (see Fig.2).

Let $n=2l+1$, thus we deal with the team of $l+2$ pursuers. Initially all pursuers are located in c_n. Then sequentially for each branch of $\mathcal{T}_n$ at c_n and for each path from c_n to the vertex incident to the edge of length 3, the pursuers move as follows. We will explain the common behavior of the pursuers by the example of path $(c_n,c_{n-1},\ldots,c_0)$. First P_1 slides from c_n to the midpoint of edge (c_n,c_{n-1}). Then P_2 slides from c_n to the midpoint of edge (c_{n-2},c_{n-3}), e.t.c. P_{l+1} slides from c_n to the midpoint of edge (c_1,c_0). Thus pursuers $P_1,\ldots,P_{l+1}$ control the whole path since they possess the capture radius 0.5. The branches at c_0 which contain the edges of length 3 coincide with the tree $\mathcal{T}$. So the single pursuer P_{l+2} can check this branches with capture radius 0.5. Then P_{l+1} slides via vertex c_1 to the midpoint of the another edge incident to c_1 and adjacent to the edge of length 3. Thus, a new path from c_n is under control of pursuers $P_1,\ldots,P_{l+1}$, and so on.

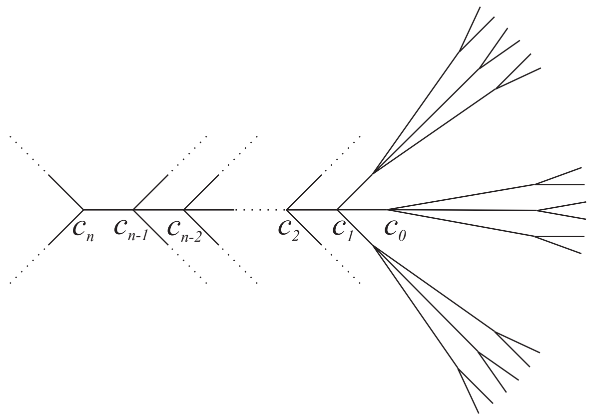

Fig. 2.

Let $n = 2l$, we deal with the team of $l + 2$ pursuers again. Unlike the even n case, pursuer P_1 stays in c_n during the whole program, P_2 slides from c_n to the midpoint of edge (c_{n-1}, c_{n-2}). Then P_3 slides from c_n to the midpoint of edge (c_{n-3}, c_{n-4}), e.t.c. P_{l+1} slides from c_n to the midpoint of edge (c_1, c_0). The whole path $(c_n, c_{n-1}, \ldots, c_0)$ is under control of pursuers $P_1, \ldots, P_{l+1}$, thus the rest of movements is similar to the previous case.

Now, let us show that the team of $n + 1 - \lceil \frac{n}{2} \rceil$ pursuers can't catch the evader on $\mathcal{T}_n$ with capture radius 0.5.

Let us prove this statement by induction. The base is trivial: single pursuer can't succeed in 0.5–catching either on $\mathcal{T}_0$ or on $\mathcal{T}_1$. Let $n \geq 1$, suppose that the required statement holds for $n - 1$ so that the team of $(n-1) + 1 - \lceil \frac{n-1}{2} \rceil$ pursuers can't succeed in 0.5–catching on tree $\mathcal{T}_{n-1}$. Consider separately the cases of odd and even n again.

Let $n = 2l$, thus to succeed in 0.5–catching on $\mathcal{T}_{n-1}$ we have to put at least

$$\left(2l - \left\lceil \frac{2l-1}{2} \right\rceil\right) + 1 = l + 1$$

pursuers on it. Tree $\mathcal{T}_n$ has three branches at c_n. By the induction hypothesis, to succeed in 0.5–catching on each branch we need at least $l + 1$ pursuers. Suppose that the team of $l + 1$ pursuers can catch the evader on at least one branch, say, B without loosing the control over vertex c_n (otherwise, the statement is obvious by Lemma 2). Denote the vertex adjacent with c_n in B by c_{n-1}, the length of (c_n, c_{n-1}) equals to 1, so one of the pursuers has to stay in the midpoint of edge (c_n, c_{n-1}). Three branches of B at c_{n-1} contain the subtrees which coincide with tree $\mathcal{T}_{n-2}$, and by the induction hypothesis we have to put at least

$$\left((2l-2) + 1 - \left\lceil \frac{2l-2}{2} \right\rceil\right) + 1 = l + 1$$

pursuers on each branch, but we have only l moving pursuers.

Let $n = 2l + 1$, then

$$\left((n-1) + 1 - \left\lceil \frac{n-1}{2} \right\rceil\right) + 1 = \left(2l + 1 + \left\lceil \frac{2l}{2} \right\rceil\right) + 1 = l + 2$$

pursuers are required to succeed in 0.5-catching on $\mathcal{T}_{n-1}$, but

$$n + 1 - \left\lceil \frac{n}{2} \right\rceil = l + 1,$$

and the statement has been proven by Lemma 1.

Thus, in virtue of Theorem 2,

$$\varepsilon_{\mathcal{T}_n}(k) \begin{cases} = 0, & k = n + 3, \\ = 0.5, & k = n + 2 - \left\lceil \frac{n}{2} \right\rceil, \ldots, n + 2, \\ > 0.5, & k \le n + 1 - \left\lceil \frac{n}{2} \right\rceil \end{cases}$$

that was to be proved. □

3. The minimal trees with the degenerate Golovach function

Consider tree $\mathcal{B}$ in Fig.3. We prove that it is the minimal (with respect to the number of the edges) tree with the degenerate Golovach function. It is rather clear that the Golovach function for $\mathcal{B}$ has the following form.

$$s_{\mathcal{B}}(\varepsilon) = \begin{cases} 4, & 0 \le \varepsilon < 0.5, \\ 2, & 0.5 \le \varepsilon < 2.5, \\ 1, & 2.5 \le \varepsilon. \end{cases}$$

Let us prove only the fact of the jump of height 2 at point 0.5.

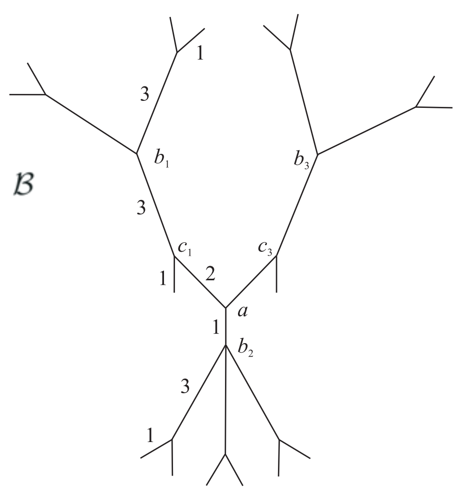

Fig. 3.

The winning program on $\mathcal{B}$ for two pursuers with capture radius 0.5 looks as follows. Pursuer P_1 occupies vertex b_1, and P_2 clears two branches of $\mathcal{B}$ at b_1 which coincide with tree $\mathcal{T}$ (see Fig.1). Then P_1 moves to the midpoint of the edge of unit length incident to c_1. After that P_1 moves to the midpoint of edge (a, b_2), and P_2 clears three branches of $\mathcal{B}$ at b_2 which coincide with tree $\mathcal{T}$. Next, P_1 moves to the midpoint of the edge of unit length incident to c_2, then to the vertex b_3, and

P_2 clears the branches at b_3 which match up with tree $\mathcal{T}$. After that, the search procedure ends.

Suppose $0 < \varepsilon' < 0.5$, then three pursuers can't catch the evader on $\mathcal{B}$ with capture radius ε. Indeed, in virtue of Theorem 2, the three pursuers are necessary to succeed in capturing on $\mathcal{T}_0$ with capture radius ε'.

Consider a tree T, $a \in T$, and $\delta > 0$, let $N_\delta(a) = \{x \in T | \rho(a,x) \leq 2\delta\}$ — a set of all 2δ–close to a points of T.

Then trees K_1 and K_3 (which contain vertices b_1 and b_3 respectively) obtained by the closure of set $\mathcal{B} \backslash N_{0.5}(a)$ have form $\mathcal{T}_0$. Further, the subtree K_2 obtained by removal edge (a, b_2) of the branch of $\mathcal{B}$ at a which contains vertex b_2 has form $\mathcal{T}_0$, too. It is clear now that we have to put at least three ε'–far from a pursuers on each subtree K_i, $i \in \overline{1,n}$ to success in ε'–catching on K_i. According to the hypothesis of Lemma 2, three pursuers can'n catch the evader on $\mathcal{B}$ with capture radius less than 0.5.

We have shown that the Golovach function for tree $\mathcal{B}$ has a jump of height 2. Actually, the Golovach function for trees with lower number of edges have no non–unit jumps. First we give several definitions and proposition, and then will prove this statement.

Let us define so–called *Parsons series*. A sequence of sets of trees $T_1, T_2, \ldots$ is constructed as follows: T_1 is the complete graph on two vertices, and if T_k $(k \geq 1)$ has been defined, let T_{k+1} contain all pairwise nonisomorphic trees T obtained as follows. Take trees $B_1, B_2, B_3 \in T_k$ (possibly isomorphic) and vertices $a_i \in VB_i$ not adjacent to vertices of degree 1 in B_i for $i = 1, 2, 3$. We choose an additional vertex a, join it by edges with a_1, a_2, and a_3, and remove the vertex of degree two from the tree thus obtained. This series of trees was constructed in (Parsons, 1976), the first four sets are shown in Fig.4.

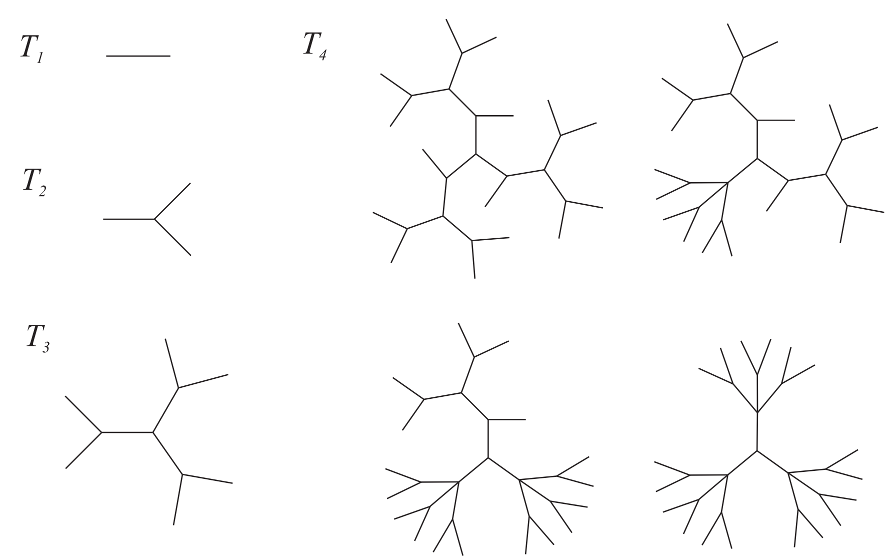

Fig. 4.

The *contraction of an edge* (a, b) in a graph G is the procedure for obtaining a new graph G' from graph G. Vertices a and b and edge (a, b) are removed, an additional vertex c is inserted so that the edges incident to c in graph G' each correspond to an edge incident to either a or b except edge (a, b).

We say that a graph H is *contained* in a graph G if G has a subgraph G' such that either this subgraph is isomorphic to H or a graph isomorphic to H can be obtained from G' by contracting some of its edges. A tree T is called a *minimal tree with search number* k if $s(T) = k$ and $s(T') < k$ for any tree T' not isomorphic to T and contained in T.

As was proven in (Golovach, 1992) the Parsons series represent the minimal trees with the given search number.

Proposition 1. *For a tree T, $s(T) = k$ if and only if there exists a $T' \in T_k$ contained in T and any $T'' \in T_{k+1}$ is not contained in T.*

We are ready to prove the main result of this section.

Theorem 4. *If a tree T has at most 27 edges, then the Golovach function for T has only unit jumps.*

Proof. If $s(T) = 1$, then the Golovach function for tree T is constant. If $s(T) = 2$, then, by definition, $\varepsilon_T(1) > 0$, whereas $\varepsilon_T(2) = 0$. It follows that the Golovach function has one unit jump. If $s(T) = 3$, then $\varepsilon_T(3) = 0$, $\varepsilon_T(2) > \varepsilon_T(3)$, and $\varepsilon_T(1) > 0$. By Corollary 1, $\varepsilon_T(2) < \varepsilon_T(1)$. Thus, $0 = \varepsilon_T(3) < \varepsilon_T(2) < \varepsilon_T(1)$.

There is the only tree, say, T_4^1 from the set T_4 with 27 edges, thus if $s(T) = 4$ then T is isomorphic to T_4^1. As above, the following inequalities are obvious:

$$\varepsilon_T(4) = 0, \varepsilon_T(4) < \varepsilon_T(3), \varepsilon_T(2) < \varepsilon_T(1), \varepsilon_T(1) > 0.$$

Now we have to check the jump at the point $\varepsilon_T(2)$ is unit.

We denote the vertices of tree T as in Fig.5. Let B, C, and D denote the branches of T at a and containing the vertices b_1, c_1, and d_1, respectively. By Lemma 1, the inequality

$$\varepsilon_T(2) \geq \max\{\varepsilon_B(2), \varepsilon_C(2), \varepsilon_D(2)\} \tag{1}$$

holds. Let

$$\varepsilon := \max\{\varepsilon_B(2), \varepsilon_C(2), \varepsilon_D(2)\}.$$

Suppose that the inequality (1) is strict, then we can instantly construct the winning program for the team of tree pursuers with capture radius ε: one pursuer occupies vertex a, and other two pursuers check branches B, C, and D.

Suppose that the inequality (1) is an equality. If the numbers $\varepsilon_B(2), \varepsilon_C(2), \varepsilon_D(2)$ are different, say, $\varepsilon_B(2) \geq \varepsilon_C(2) > \varepsilon_D(2)$, then we can again specify a program $\tilde{\Pi}$ on T for three pursuers with capture radius $\varepsilon_D(2)$. This program looks as follows. Pursuer P_1 occupies vertex b_1, and other two pursuers check the branches at b_1 and containing b_3 and b_4 (two pursuers can clear them with zero capture radius). Then P_1 slides to vertex b_2, and P_2 clears the pendant edge incident to b_2. After that, P_1 moves to a, and other two pursuers clear tree D. Then P_1 moves to c_2, P_2 clears the pendant edge incident to c_2, P_1 moves to c_1, and other two pursuers clear the branches at c_1 which containing c_3 and c_4. After this step, the program ends.

Consider the case $\varepsilon_T(2) = \varepsilon_B(2) = \varepsilon_C(2) = \varepsilon_D(2) = \varepsilon$. By Lemma 2, for at least one of branches B, C, D there exists a winning program for two pursuers with

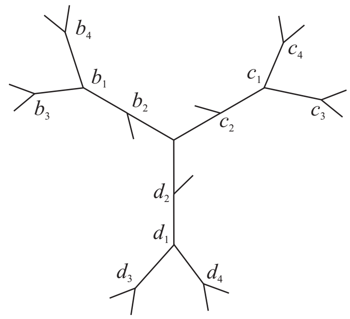

Fig. 5.

capture radius ε in which one of the pursuers is ε–close to a during all the program. Suppose that this holds for branch D.

Let the set of trees K_i, $i \in \overline{1,s}$, denote the closures of the connected components of the set $D \backslash N_\varepsilon(a)$, note that s can take the values 1, 2, 3, 4, and 5. For each $i \in \overline{1,s}$ tree K_i is isomorphic to one of trees I_1, I_2, I_4, and I_5 shown in Fig.6.

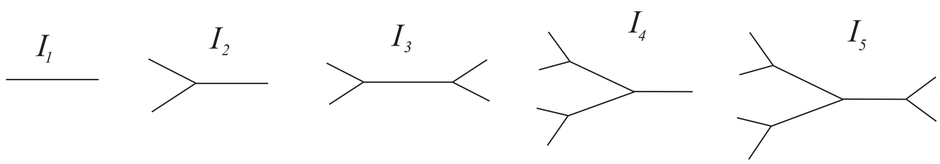

Fig. 6.

If $\rho(a, d_1) < 2\varepsilon$, then the $\delta < \varepsilon$ can be chosen so that the each tree from the set of the closures of the connected components of the set $D \backslash N_\delta(a)$ is isomorphic to one of I_1, I_2, I_3 or I_4, and the team of two pursuers can clear it with capture radius 0. In this case, we modify program $\tilde{\Pi}$ as follows. Pursuer P_1 after arrival to vertex a traverses all paths to those points of branch D which are at distance δ from a. Meanwhile, other two pursuers can clear the subtrees containing the components of $D \backslash N_\delta(a)$ with zero radius of capture.

If $\rho(a, d_1) > 2\varepsilon$, then $s = 1$ and the only tree K_1 which is obtained by the closure of the set $D \backslash N_\varepsilon(a)$ is isomorphic to I_5.

Let us make the following observation. Consider any tree K and the tree K' obtained from K by adding an arbitrary pendant edge of length α. If a team of pursuers can catch the evader on K using program $\tilde{\Pi}$ with capture radius ε, then the same team of pursuers can capture the evader on K' with capture radius $\varepsilon + \alpha$ by using the same program.

Let $\alpha = \frac{1}{2}\left(\varepsilon_{K_1}(1) - \varepsilon_{K_1}(2)\right)$ (obviously, $\varepsilon_{K_1}(1) > \varepsilon_{K_1}(2)$). Thus the tree K_1' which is obtained by the closure of the set $D \backslash N_\delta(a)$ differs from the tree K_1 by the length of one pendant edge. So that the following expressions hold:

$$\varepsilon_{K_1'}(2) \leq \varepsilon_{K_1}(2) + \alpha = \frac{1}{2}\left(\varepsilon_{K_1}(1) + \varepsilon_{K_1}(2)\right) < \varepsilon_{K_1}(1).$$

As was mentioned, there exists a winning program for the team of two pursuers with capture radius ε on D such that one of the pursuers is ε–close to a during the whole program, which means that $\varepsilon_{K_1}(1) \leq \varepsilon$. Let us modify program $\tilde{\Pi}$ again. Pursuer P_1 after arrival to vertex a moves toward vertex d_1 for a distance $\delta = \varepsilon - \frac{1}{2}\alpha$. Other two pursuers are able to clear tree K_1' with capture radius $\varepsilon_{K_1'}(2) < \varepsilon$. The rest of the program repeats $\tilde{\Pi}$.

If $\rho(a, d_1) = 2\varepsilon$, then $s = 2$, let $\delta = \frac{\varepsilon}{2}$. In this case pursuer P_1 after arrival to the vertex a passes distant δ toward the vertex d_1. Pursuers P_2 and P_3 act as follows: P_2 occupies d_3 and P_3 clears the pendant edges incident to d_3; then P_2 and P_3 occupy d_4, P_3 goes to vertex d_1, passes distance δ from d_1 along the edge (a, d_1), returns to d_1, clears the pendant edge incident to d_1, and returns to d_4; next, P_2 and P_3 move to d_5, and at the final step, P_3 clears the pendant edge incident to d_5. Program $\tilde{\Pi}$ thus modified is winning for three pursuers on T with capture radius δ.

If $s(T) \geq 5$, then T contains a tree from T_5, while the minimum number of edges for a tree from the set T_5 is 81. □

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Non-Cooperative Games with Chained Confirmed Proposals*

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Abstract. We propose a bargaining process with alternating proposals as a way of solving non-cooperative games, giving rise to Pareto efficient agreements which will, in general, differ from the Nash equilibrium of the original games.

Keywords: Bargaining; Confirmed Proposals; Confirmed Agreement.
JEL classification: C72; C73; C78.

1. Introduction

Since the seminal contributions by Nash (1950, 1953), bargaining models play a central role in the analysis of situations in which economic agents try to reach an agreement on the split of a certain asset. A plethora of approaches has resulted in a variety of bargaining mechanisms[1] keeping fixed the objective of splitting the pie. In a parallel and mostly independent effort, non-cooperative game theory undertook the task of determining the actions of individual agents in interactive strategic situations. While the efficiency of outcomes has been a central issue in both game theoretic paradigms[2], the role of bargaining as a determinant of individual actions in non-cooperative games has not been systematically explored.

In this paper, we illustrate the consequences of applying alternating proposal protocols as a way of solving non-cooperative games. From a technical point of view, the basic difference between our framework and that of bargaining over the split of a pie is that, in ours, two agents bargain about which strategy profile they will play in a non-cooperative game. Apart from the obvious departure from Rubinstein's (1982) model in that the set of possible agreements is finite,[3] in our setup a confirmed agreement between bargaining agents concerns the pair of strategies that will be actually played in the original non-cooperative game. This fact increases by one the degrees of freedom and, thus, the dimension of the outcome space, allowing the use

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[1] While an exhaustive list of the relevant references is beyond the scope of this paper, it is worth mentioning Harsanyi (1956, 1962), Sutton (1986) and Binmore (1987).

[2] Relevant references are Harsanyi (1961), Friedman (1971), Smale (1980), and Cubitt and Sugden (1994).

[3] For a formal treatment of this issue, see the insightful analysis by Muthoo (1991).

of bargaining with alternating proposals as a method of solving non-cooperative games.

Assume that two players bargain over the strategy profile to play, given that each player knows the opponent's set of possible strategies. Then, there is a non-cooperative *game* whose execution leads to the two players' final payoffs and a *supergame* whose actions in each bargaining period are *proposals* of strategies for the original non-cooperative game. Games with *confirmed* proposals are interactive strategic situations in which a player, in order to give official acceptance of a contract, must confirm his/her proposed strategy combined with the strategy chosen by his/her opponent.[4] Here we focus on non-cooperative games with complete information and finite strategy spaces. The bargaining supergame built on them is an infinite horizon game with perfect information. We show that the equilibrium outcome of the supergame can be unique even though each player's strategy space in the supergame and its number of stages are infinite. We call *equilibrium confirmed agreement* the corresponding equilibrium contract between players in the supergame, leading to a strategy profile to be played in the original non-cooperative game.

2. The Bargaining Supergame

Throughout the paper, we consider only two-players non-cooperative games; the two players alternate proposals in the bargaining supergame. The supergame ends when a player confirms the proposal he/she made the previous stage in which he/she was active. In modelling the supergame, we focus on a specific family of games with confirmed proposals, those with *chained* proposals. That is, in the case of no confirmation by one player, the non confirmed strategy profile is taken as the new starting point for the subsequent negotiation. Except for the selection of the *first* mover at the beginning of the supergame, the rules of the game are symmetric.

Let us denote by S_h the finite strategy space for player h (with $h = i, j$) in the original non-cooperative *game*. The (super-)game with confirmed proposals is built such that the set of possible proposals of player h in the supergame coincides with the set of his/her strategies S_h in the original game. The set of possible agreements of the game with confirmed proposals coincides with the set of outcomes of the original game, i.e. the product set $S_i \times S_j$ contains all possible agreements of the game with confirmed proposals built on the original game.

Denote by s_h^t the strategy proposed by player h in stage t of the supergame, with $t = 1, 2, ..., +\infty$. Suppose that player i ("she") starts the supergame with player j ("he").

The sequence of alternating proposals is as follows:

Stage 1. Player i proposes a certain strategy $s_i^1 \in S_i$ to player j. Player i would actually play s_i^1 if (and only if) she would confirm this strategy after the counterproposal of player j.

Stage 2. Player j proposes strategy $s_j^2 \in S_j$ to player i. This strategy would actually be played if (and only if) either i will confirm her previous strategy s_i^1 or j will confirm his proposal s_j^2 after the counterproposal of player i.

[4] The original non-cooperative game can be a game with perfect or imperfect information and/or with complete or incomplete information. When information is incomplete, players can exploit the bargaining process to extract information on their opponent's type through their proposals.

Stage 3. Player i chooses whether or not to confirm her previous strategy s_i^1. If she confirms s_i^1, i.e. $s_i^3 = s_i^1$, then the bargaining process ends, through the sequence (s_i^1, s_j^2, s_i^1), with the confirmed agreement (s_i^1, s_j^2) and the two players receive the payoffs corresponding to the strategy profile (s_i^1, s_j^2) in the original game. If she does not confirm, i.e. she proposes a new strategy $s_i^3 \neq s_i^1$, the bargaining process continues with s_j^2 being player j's proposal and s_i^3 being player i's counterproposal to j's proposal.

Stage 4. Player j chooses whether or not to confirm his previous strategy s_j^2. If he confirms s_j^2, i.e. $s_j^4 = s_j^2$, then the bargaining process ends, through the sequence (s_j^2, s_i^3, s_j^2), with the confirmed agreement (s_j^2, s_i^3) and the two players receive the payoffs corresponding to the strategy profile (s_i^3, s_j^2) in the original game. If he does not confirm, i.e. he proposes a new strategy $s_j^4 \neq s_j^2$, the bargaining process continues with s_i^3 being player i's proposal and s_j^4 being player j's counterproposal to i's proposal.

And so on and so forth.

If no strategy profile is ever confirmed by either player, then the outcome is the disagreement event Ω. Define with $f(s_h^{t-2}, s_{-h}^{t-1})$ the outcome of the game with confirmed proposals in case the agreement $(s_h^{t-2}, s_{-h}^{t-1})$ would be confirmed in stage t, with $t = 3, ..., +\infty$. We assume that each player h's preference relation $\succsim_h$ satisfies the following conditions:[5]

(a) *Disagreement is not better than any agreement*:
$\Omega \precsim_h f(s_h^{t-2}, s_{-h}^{t-1})$ for all $(s_h^{t-2}, s_{-h}^{t-1}) \in S_h \times S_{-h}$ and for all $t = 3, ..., +\infty$.

(b) *Patience*, i.e. the time of the agreement is irrelevant: if $s_h^{t-2} = s_h^{t'-2}$ and $s_{-h}^{t-1} = s_{-h}^{t'-1}$, then $f(s_h^{t-2}, s_{-h}^{t-1}) \sim_h f(s_h^{t'-2}, s_{-h}^{t'-1})$ for all $t \neq t'$, with $t, t' = 3, ..., +\infty$.

(c) *Stationarity*, i.e. the preference between two agreements does not depend on time: if $s_h^{t-2} = s_h^{t'-2}$, $s_{-h}^{t-1} = s_{-h}^{t'-1}$, $\tilde{s}_h^{t-2} = \tilde{s}_h^{t'-2}$ and $\tilde{s}_{-h}^{t-1} = \tilde{s}_{-h}^{t'-1}$, then $f(s_h^{t-2}, s_{-h}^{t-1}) \succsim_h f(\tilde{s}_h^{t-2}, \tilde{s}_{-h}^{t-1})$ if and only if $f(s_h^{t'-2}, s_{-h}^{t'-1}) \succsim_h f(\tilde{s}_h^{t'-2}, \tilde{s}_{-h}^{t'-1})$, for all $t \neq t'$, with $t, t' = 3, ..., +\infty$.

We refer to the extensive (super-)game with perfect information thus defined as the *game with chained confirmed proposals* (henceforth GCCP).

In the next section, we analyze the GCCP version of some well-known interactive strategic situations, extensively studied both in the theoretical and in the experimental literature. First, we analyze two examples in which the original game is a 2x2 static game. Then, by maintaining the assumption of two players only, we concentrate on three cases where the original game is a two-stage dynamic game with perfect information.

In each of the GCCP discussed in the paper, we basically look for the set of subgame perfect equilibrium outcomes (*equilibrium confirmed agreements*). Now, the backward induction solution procedure is well-defined for all *generic finite* games with *perfect* information. Games with confirmed proposals are games with *perfect* information if the non-cooperative game on which they rely is with perfect information (as in the cases we analyze here). However, they are by construction neither *finite* nor *generic*, given that different terminal histories can yield the same payoff

[5] Conditions (a) and (c) characterize also Rubinstein (1982), while in Rubinstein's model time is valuable and a discount factor is introduced accordingly.

vector. In fact, the same agreement can be obtained through different combinations of proposals and counterproposals. Nonetheless, the backward induction procedure can be easily applied to some non-generic games of perfect information, specifically to those with *no-relevant ties*.[6] Games with confirmed proposals do belong to this category, if the non-cooperative game on which they rely is static and with no relevant ties (as in the two examples in section 3.1). If the original game is dynamic (as in the three examples in section 3.2), the fact that it has no relevant ties is not enough: given that the payoff function of the game with confirmed proposals is defined over the strategic form of this game, at least one player [7] each time where he/she active, he/she will have at his/her disposal (at least) two proposals (corresponding to payoff-equivalent strategies in the original game) leading to (at least) two payoff-equivalent confirmed agreements.

In all the games with confirmed proposals analyzed in the paper, we obtain the equilibrium outcome(s) through *iterated weak dominance*, that in *generic* games of *perfect* information yields the backward induction outcome. In section 3.1 we analyze the GCCP version of the Prisoner's Dilemma and of the Battle of Sexes. In section 3.2 we focus on the Trust Game, the Entry Game and the Ultimatum Game. Here we show that iterated weak dominance yields the backward induction outcome also for the GCCP built on these specific dynamic games with no relevant ties.

3. Confirmed Agreements in Standard Two-Player Games

3.1. Bargaining over Static Games

Consider the GCCP version of the *Prisoner's Dilemma* (PD). The original game is a standard static PD, and the bargaining supergame built on it is an infinite horizon game with perfect and complete information. The sets of players' feasible proposals in the GCCP coincide with their sets of actions in the original game: $S_i = S_j = \{Cooperate, Defect\}$, henceforth $\{C, D\}$. Figure 1, with $a > c > d > z$ shows the simultaneous-move original game and all possible agreements of the bargaining supergame with chained confirmed proposals built on it.

$i \backslash j$	D	C
D	(d,d)	(a,z)
C	(z,a)	(c,c)

Fig. 1. Payoff matrix of the PD game

[6] A game with perfect information has no relevant ties if, for every pair of distinct terminal histories, the player who is decisive for one vs the other (i.e. the player who is active at the last common predecessor of the two terminal histories) is not indifferent between them.

[7] This applies to all players having the possibility to observe their opponent's action in the constituent game. Only a first mover who is active once does not belong to this category.

The original game has the profile (D, D) as equilibrium in dominant actions. The same equilibrium outcome would be found in the standard two-stage game (without bargaining and without confirmation), where one of the two players moves first and the other observes his/her "proposal" before choosing his/her own.

Let us now calculate the subgame perfect equilibrium outcome of the GCCP version of the PD game.

Observe Figure 2. The set of feasible payoffs of the bargaining game is the same as in the PD in Figure 1. The first of the two payoffs always refers to player i, as in the original game. In every decision node, the active player's weakly dominant action at that node is marked in bold. An *action* is *weakly dominant at node* ξ if every other action at ξ belongs to a weakly dominated strategy in the subgame with root ξ. Whenever there are two or more dominant actions at a given node, these actions are marked with dotted bold lines.

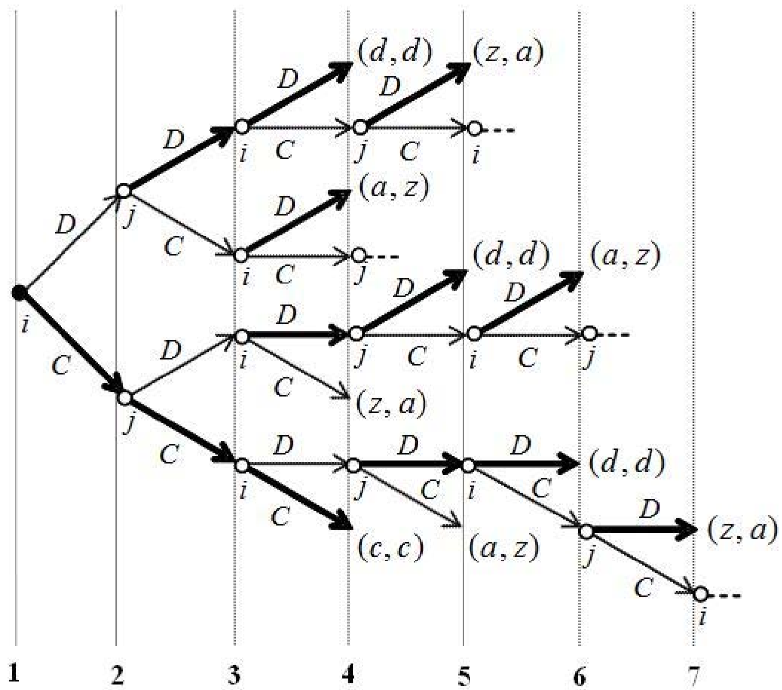

Fig. 2. PD with confirmed proposals

One main result is:

Proposition 1. *The PD with chained confirmed proposals has a unique subgame perfect equilibrium, inducing the cooperative confirmed agreement in the first stage in which a player is allowed to confirm his/her proposal.*

Proof. Let us consider the infinite game in Figure 2. Each player has only one weakly dominant action at each node in which he/she is active. Each player's equilibrium strategy leads to the following result: the payoff obtained by the player through confirming at a stage t equals the highest payoff he can get by continuing the game. Moreover, in this game, for any $\hat{t} > t$ this highest payoff can be obtained only by confirming the same agreement confirmed at t. The subgame perfect equilibrium is found through iterated weakly dominance in three steps. Observe that in Figure 2 (first 7 stages of the game) there are four decision nodes where a player can confirm the agreement yielding him/her a, his/her highest payoff possible. At stage t=4, after the non-terminal history (D, D, C), player j can get a by choosing D, hence confirming his most preferred agreement. If, instead of confirming, player j chooses to continue the game, he can get, in any subgame in the continuation game, at most a payoff of a, by confirming the same agreement he could already confirm at stage t=4. Therefore, for player j confirming (D, C) at stage t=4 weakly dominates continuing the game. The same holds for player i at stage t=3, after history (D, C), and at stage t=5 after history (C, D, D, C); and for player j at stage t=6 after history (C, C, D, D, C). Therefore, at the first step the action prescribing not to confirm the favourable asymmetric agreement is eliminated at each node in which it is available. Given that, in order to prevent the opponent from confirming his/her favorable asymmetric agreement, the agreement (D, D) should be confirmed whenever possible. Therefore, it is weakly dominant to propose D in each stage t when at least one of the two players has proposed D in at least a $\tilde{t} < t$ and to propose C otherwise. By eliminating all other actions at the second step, we are left with the two terminal histories (C, C, C) and (D, D, D). At the third step, the action D in t=1 is eliminated. □

Thus, in the unique subgame perfect equilibrium of the game, player i starts by proposing strategy C to player j, who counter-proposes strategy C. Then, player i confirms her strategy C, such that the original game strategy profile (C,C) is the (unique) confirmed agreement. This is reached already in stage $t = 3$, after the first interaction among players takes place.

Consider now the GCCP version of the *Battle of Sexes* (BS). The set of players' feasible proposals, which coincides with the set of players' actions in the original game, is $S_i = S_j = \{Opera, Football\}$, henceforth $\{O, F\}$. Figure 3 shows the one-shot original game and, also, all possible agreements of the GCCP built on it. Parameters are such that $a > b > \max\{c_1, c_2, c_3, c_4\}$.

$i \backslash j$	O	F
O	(a,b)	(c_1,c_2)
F	(c_3,c_4)	(b,a)

Fig. 3. Payoff matrix of the BS game

The original game has two Nash equilibria in pure strategies: (O, O) and (F, F). In the standard two-stage dynamic version of the game, the player moving first[8] has an advantage. Consider now the BS with confirmed proposals for the case in which player i is the first mover (Figure 4). Observe that, surprisingly, there is a *first-mover disadvantage*: the unique equilibrium confirmed agreement coincides with the Nash equilibrium of the original game which is preferred by player j.

[8] In fact, this is equivalent to a commitment.

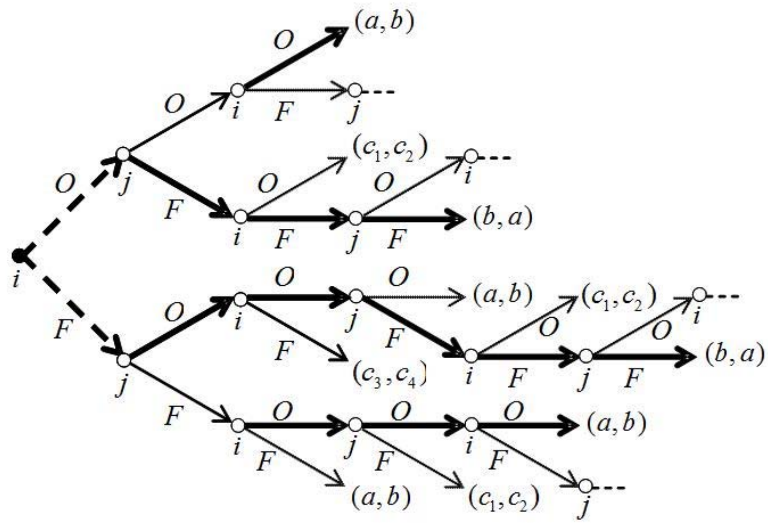

Fig. 4. BS with confirmed proposals

Proposition 2. *The BS with chained confirmed proposals has a unique equilibrium confirmed agreement, involving players' coordination on the constituent game equilibrium favourable to the second mover.*

Proof. The game ends with the confirmation of the strategy profile (F, F), whatever is player i's initial proposal. In each of the two subgame perfect equilibria in pure strategies, j replies to i's first proposal by indicating the opposite proposal (F if O and O if F). By doing that, j obliges player i to propose the same action already proposed by him (otherwise, i would confirm her initial action and would get $c_1 < b$ or $c_3 < b$). If this action is F, then j confirms F and gets his highest payoff possible. If instead this action is O, then j proposes F and i finds convenient to propose F, since, otherwise, she would get $c_1 < b$; then j confirms F and gets his highest payoff possible. Therefore, in the first stage player i is indifferent between her two possible proposals. □

Thus, in the BS with chained confirmed proposals the second mover is able to confirm the coordinating equilibrium outcome of the original game more favorable to him.

3.2. Bargaining over Dynamic Games

Let us now focus on the case where the bargaining supergame concerns a dynamic original game. Here the confirmed proposals structure is built on the strategic form of the original game. Therefore, a player's set of possible proposals in the GCCP corresponds to his/her set of strategies in the original game.

Let us begin by considering as original game the *Trust Game* (TG). In the original game, player i (the *truster*) chooses whether to *Trust* (T) or to *Not trust* (N) player j (the *trustee*). In case i trusts j, total profits are higher. In that case, j would decide whether to *Grab* (G) or to *Share* (S) the higher profits. The strategic form of the game in Figure 5, where $\underline{x}$:="x if T", with $x = G, S$, and $c_i > d_i >$

$z \geq 0$, $a > c_j > d_j > 0$, $a + z = c_i + c_j$, represents all possible agreements of the GCCP built on this dynamic game.[9]

$i \backslash j$	$\underline{G}$	$\underline{S}$
N	(d_i, d_j)	(d_i, d_j)
T	(z, a)	(c_i, c_j)

Fig. 5. Payoff matrix of the TG

In the unique subgame perfect equilibrium of the original game, i does not trust j, while the latter would choose to grab if i had trusted him in the first place.

Given players' role asymmetries in the original game, the resulting GCCP involves two possible versions: one in which the *truster* in the original game (i) is the first mover of the bargaining sequence, and one in which the *trustee* in the original game (j) is the first mover of the bargaining sequence. In the latter case, j begins the GCCP by announcing his intention to grab or to share the higher total profits in case i would trust him. The two versions of the TG with chained confirmed proposals are represented in Figure 6.a and 6.b respectively. Recall that the first of the two payoffs always refers to player i, as in the original game.

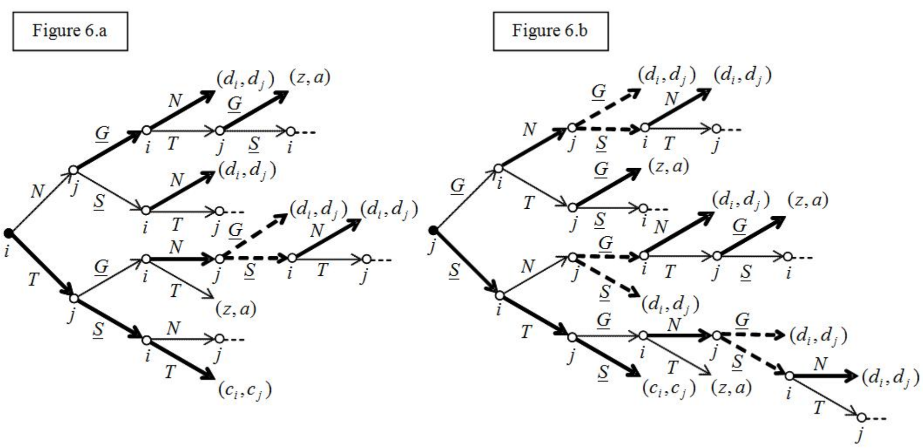

Fig. 6. TG with confirmed proposals, with i (Figure 6.a) or j (Figure 6.b) as first mover

For the two GCCP in Figure 6, the following result holds.

[9] Notice that, in order for j to confirm an agreement, he has to re-propose the same strategy in two subsequent stages in which he is active. According to this rule, for example $(\underline{S}, N, \underline{G})$ is not a terminal history of the GCCP, even though both strategy profiles $(N, \underline{S})$ and $(N, \underline{G})$ induce the same terminal history in the original game.

Proposition 3. *The TG with chained confirmed proposals has a unique equilibrium confirmed agreement, the cooperative one. This agreement is immediately confirmed by the first mover in the GCCP.*

Proof. We impose that when a player is active at a decision node, he/she plays a weakly dominant action at that node. Because of that, in both GCCP in Figure 6, at each stage t each player would: (1) confirm his/her most preferred agreement if he/she is given the possibility in that stage; (2) confirm agreements other than his/her most preferred if: (2.1) in some stage $\hat{t} > t$ (with $\hat{t} < \infty$) of the equilibrium continuation path, his/her opponent would confirm an agreement not better for him/her than the one he/she could confirm in t; (2.2) by not confirming in t, neither (1) nor (2.1) applies to any stage $t+k$, with $k = 1, ..., +\infty$, and the best agreement he/she could confirm when he/she is active in the continuation subgame is payoff-equivalent to the one he/she could confirm in stage t. When the first mover is player i, in stage 3 she would confirm $(T, \underline{S})$ because of (1). She would confirm also $(N, \underline{G})$ because of (2.1), and $(N, \underline{S})$ because of (2.2). Instead, she would not confirm $(T, \underline{G})$, given that none of the above mentioned cases applies. Hence, she would propose N after history $(T, \underline{G})$. In stage 4, after $(T, \underline{G}, N)$, player j is indifferent between confirming the agreement $(N, \underline{G})$ and proposing $\underline{S}$. In both cases the payoffs are (d_i, d_j), since, if he proposes $\underline{S}$, in the subsequent stage, player i would confirm $(N, \underline{S})$ because of (2.2) (as we have previously seen after history $(N, \underline{S})$). Thus, the subgame perfect equilibrium path is $(T, \underline{S}, T)$, with i confirming the agreement $(T, \underline{S})$ in stage 3. When the first mover is player j, in equilibrium he confirms the same agreement in stage 3. This follows from the fact that in stage 3 he would confirm the agreement $(T, \underline{G})$ because of (1), $(T, \underline{S})$ because of (2.1) and he is indifferent between confirming or not the agreements $(N, \underline{G})$ and $(N, \underline{S})$ because of (2.2) (as we have already seen when the first mover is the player i, after history $(T, \underline{G}, N)$). □

Notice that, as in the example of the BS, in both versions of the TG with chained confirmed proposals, the second mover reciprocates in stage 2 the first-mover's proposal: he/she cooperates if the first-mover's proposal is cooperative ($\underline{S}$ if T and T if $\underline{S}$, respectively) and does not cooperate otherwise ($\underline{G}$ if N and N if $\underline{G}$, respectively).

Consider now as original game the dynamic *Entry Game* (EG). In the original game i (the potential *entrant*) chooses whether to *Enter* (E) or to *Stay Out* (S) of the market, with j (the *incumbent*) deciding whether to *Accommodate (A)* or *Fight (F)* if the entrant decides to enter. The strategic form of the game in Figure 7, where $\underline{x}$:="x if E", with $x = A, F$, and $a > d > 0 > z$, represents all possible agreements of the bargaining GCCP built on this dynamic game. In the unique subgame perfect equilibrium of the original game, i's entry takes place, with j accommodating it.

$i \backslash j$	$\underline{A}$	$\underline{F}$
E	(d,d)	(z,z)
S	$(0,a)$	$(0,a)$

Fig. 7. Payoff matrix of the EG

Instead, in all subgame perfect equilibria of the correspondent GCCP, the entrant stays out. The two possible versions of the EG with chained confirmed proposals are represented in Figure 8. The first version - Figure 8.a - represents the case in which player i, the potential *entrant* in the original game, moves first in the corresponding GCCP. In the second version - Figure 8.b - player j, the *incumbent* in the original game, is the first mover.

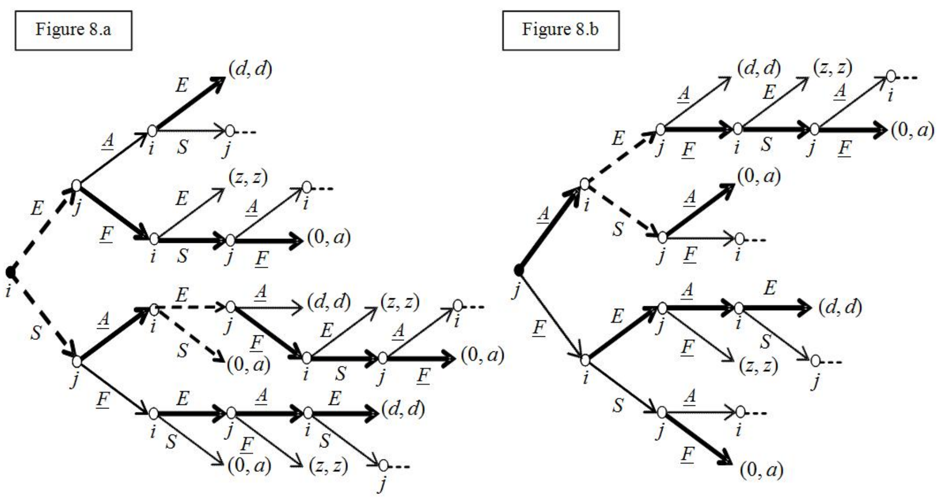

Fig. 8. EG with confirmed proposals, with i (Figure 8.a) or j (Figure 8.b) as first mover

For both GCCP in Figure 8, the following result holds.

Proposition 4. *The EG with chained confirmed proposals has two payoff-equivalent confirmed agreements, which involve the entrant to stay out.*

Proof. For the version of the game in Figure 8.a, where the first mover is player i, the proposition can be proved using the same reasoning as in the proof of Proposition 2. Notice that in the two GCCP (Figure 4 and Figure 8.a) a player h (with $h = i, j$) has a "similar" weakly dominant action in stage 2: j replies to i's "kind" ("unkind") proposal by indicating the "unkind" ("kind") proposal. By doing that, j obliges player i to propose in stage 3 the action she did not propose in stage 1. And so on and so forth. The only difference with respect to the BS is that in the EG in Figure 8.a, after history $(S, \underline{A})$, player i is indifferent between confirming this agreement and proposing E, because condition (2.1) (see proof of Proposition 3) applies. Thus, besides the agreement $(S, \underline{F})$, also $(S, \underline{A})$ is an equilibrium agreement, equivalent in payoff to $(S, \underline{F})$. More precisely, if player i starts the bargaining process, there are three equilibrium terminal histories: $(E, \underline{F}, S, \underline{F})$, $(S, \underline{A}, E, \underline{F}, S, \underline{F})$ and $(S, \underline{A}, S)$. When the first mover is player j (Figure 8.b), there are two equilibrium terminal histories: $(\underline{A}, E, \underline{F}, S, \underline{F})$ and $(\underline{A}, S, \underline{A})$. This follows from the fact that: each player always confirms agreements leading to his/her highest payoff possible, i.e. (d, d) for player i and $(0, a)$ for player j; players' optimal behavior after history $(S, \underline{A}, E)$ in Figure 8.b is the same as after history $(\underline{A}, E)$ in Figure 8.a; after history $(\underline{A})$, player i is indifferent between E and S because condition (2.1) (see proof of Proposition 3) applies. □

Therefore, in both GCCP in Figure 8, there is an equilibrium confirmed agreement in which the incumbent threatens to fight, $(S, \underline{F})$, and an additional one in which he would accommodate in case his opponent would enter, $(S, \underline{A})$. In both agreements the potential entrant accepts to stay out. If player i is the first mover (Figure 8.a), in equilibrium she will either immediately confirms the agreement $(S, \underline{A})$ or she will no longer be able to confirm any agreement at all: only player j would confirm from stage 4 onwards. If player i is not the first mover (Figure 8.b), in equilibrium she will never be able to confirm any agreement: only player j can confirm (in stage 3 or 5). Notice that, in the GCCP version of the EG, the following properties hold:

(i) only two agreements can be confirmed in equilibrium;
(ii) the two equilibrium confirmed agreements are payoff-equivalent;
(iii) both of them are Pareto efficient;
(iv) none of them is a subgame perfect equilibrium of the original game;
(v) in one of the two equilibrium agreements player i's strategy is not even a best reply strategy in the original game;
(vi) the second mover in the original game (j) is able to benefit from the confirmed proposals structure, getting his highest payoff possible;
(vii) in the equilibrium path, player i is indifferent between her two possible proposals in the first stage in which she is active;
(viii) properties (i)–(vii) hold independently of whether the player is assigned the role of first mover in the GCCP.

We show below that the same features emerge when analyzing the (super-)game with confirmed proposals version of a totally different strategic interaction setting: the *Ultimatum Game* (UG). In the original game, i (*proposer*) can offer a fair (F) or unfair (U) division to j (*respondent*); the latter, after having received i's offer, may either accept (A) or reject (R). In the (super-)game with confirmed proposals version of the original game, the set of i's possible proposals coincides with her actions in the original game, while the set of j's possible proposals coincides with his strategies in the original game, i.e. $S_j = \{\underline{AA}, \underline{AR}, \underline{RA}, \underline{RR}\}$, with $\underline{xy} :=$"x if F and y if U", with $x, y = A, R$.

The strategic form of the UG in Figure 9 (with $a > f > b > 0$) represents all possible agreements of the GCCP built on this dynamic game.[10]

[10] Recall that confirmation is achieved through re-proposal of the same strategy, thus a history like $(\underline{AR}, F, \underline{AA})$ is not terminal for the GCCP, even though both strategy profiles $(F, \underline{AR})$ and $(F, \underline{AA})$ induce the same terminal history in the original game.

$i \backslash j$	$\underline{A}\underline{A}$	$\underline{A}\underline{R}$	$\underline{R}\underline{A}$	$\underline{R}\underline{R}$
F	(f,f)	(f,f)	$(0,0)$	$(0,0)$
U	(a,b)	$(0,0)$	(a,b)	$(0,0)$

Fig. 9. Payoff matrix of the UG

In the unique subgame perfect equilibrium of the original game, unfair division takes place, with j accepting both i's offers.

Figure 10 shows the two possible versions of the UG with chained confirmed proposals. Figures 10.a refers to the case in which the proposer in the original game moves first in the supergame. Figure 10.b shows the version with the responder in the original game being the first mover in the supergame.

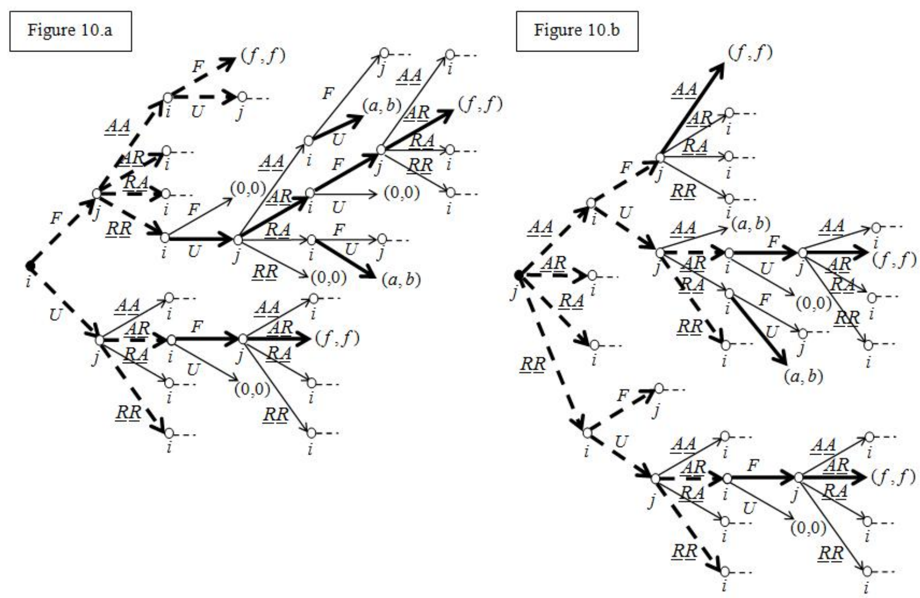

Fig. 10. UG with confirmed proposals, with i (Figure 10.a) or j (Figure 10.b) as first mover

For both GCCP in Figure 10, the following result holds.

Proposition 5. *The UG with chained confirmed proposals has an infinite number of subgame perfect equilibria, all leading to two payoff-equivalent confirmed agreements, which involve the egalitarian outcome.*

Proof. The proof is similar for the two versions of the GCCP. Consider the first version of the game, where i is the first mover (Figure 10.a). By imposing that each player chooses a weakly dominant action at each node where he/she is active, player i is never able to confirm her most preferred agreement $(U, \underline{AA})$ or $(U, \underline{RA})$. In fact, if she proposed an unfair offer, j would reply with a proposal not incorporating the acceptance of U ($\underline{AR}$ or $\underline{RR}$), otherwise i would confirm U in the subsequent stage. Therefore, in every subgame after i has proposed U, j would never propose $\underline{AA}$ or $\underline{RA}$. Then, both after each history of the type $(..., \hat{s}_j^t, U, \underline{AR})$ with $\hat{s}_j^t \neq \underline{AR}$ and after each $(..., \tilde{s}_j^t, U, \underline{RR})$ with $\tilde{s}_j^t \neq \underline{RR}$, i would propose F, otherwise she would confirm j's rejection of her unfair offer, hence getting 0. Then, after each $(..., \hat{s}_j^t, U, \underline{AR}, F)$ with $\hat{s}_j^t \neq \underline{AR}$, j would propose $\underline{AR}$, thus confirming his most preferred agreement $(F, \underline{AR})$; after each $(..., \tilde{s}_j^t, U, \underline{RR}, F)$ with $\tilde{s}_j^t \neq \underline{RR}$, j would never propose $\underline{RR}$, because that would lead to confirm an agreement involving the rejection of i's fair offer, hence getting 0. Moreover, player j is always able to avoid confirming the agreements $(U, \underline{RR})$ and $(U, \underline{AR})$. In fact, after each history $(..., \underline{RR}, U)$, he would propose the strategy $\underline{AR}$, thus avoiding to confirm the zero-payoff agreement $(U, \underline{RR})$ or to give player i the possibility to confirm $(U, \underline{AA})$ or $(U, \underline{RA})$ in the subsequent stage; after each history $(..., \underline{AR}, U)$, he would propose $\underline{RR}$, thus avoiding to confirm the zero-payoff agreement $(U, \underline{AR})$, or to give player i the possibility to confirm $(U, \underline{AA})$ or $(U, \underline{RA})$ in the subsequent stage. This explains why whenever an agreement is confirmed by j in a stage $t > 3$, this agreement is $(F, \underline{AR})$ and each terminal history is of the type $(..., U, \underline{AR}, F, \underline{AR})$. All previous considerations about player j's weakly dominant proposals in stage t conditional on i's possible proposals in stage $t-1$ explain why player i could never obtain the confirmation of an agreement which would impose player j to accept an unfair offer. Therefore, whenever in a stage $t > 3$ she has to choose between confirming an agreement $(F, \underline{Ax})$, with $x = A, R$ and entering a subgame leading to a terminal sub-history of the type $(U, \underline{AR}, F, \underline{AR})$, she is indifferent according to condition (2.2) (see proof of Proposition 3): both choices allow player i to get her second highest payoff possible and player j to get his highest payoff possible. Moreover, given that i would always propose F in each decision node where she could confirm $(U, \underline{RR})$, then, after every sequence of proposals $(F, \bar{s}_j^t, U, \underline{RR})$ with $\bar{s}_j^t = \underline{AA}, \underline{AR}, \underline{RA}$, some equilibrium loops could emerge, thus leading to an infinite number of equilibrium terminal histories, all ending with player j's confirmation of the agreement $(F, \underline{AR})$, or with player i's confirmation of the agreement $(F, \underline{Ax})$, with $x = A, R$. The only case in which player j can confirm, in equilibrium, an agreement different from $(F, \underline{AR})$ is in stage 3 of the GCP (Figure 10.b), where he confirms the payoff-equivalent agreement $(F, \underline{AA})$. □

Therefore, in every subgame perfect equilibrium of the UG with chained confirmed proposals, i offers a fair division to j, and j accepts. In each GCCP in Figure 10 we indicated the unique equilibrium terminal history leading to confirm the agreement $(F, \underline{AA})$ in stage 3, and two among the infinite possible equilibrium terminal histories leading to confirm the agreement $(F, \underline{AR})$ in a stage $t > 3$. Both kind of agreements lead to a fair division.

Quite surprisingly, the equilibrium agreements of the chained confirmed proposals version of the UG satisfy the same features (i)–(viii) characterizing the equilibria in the EG with chained confirmed proposals. In this regard, note also the strong similarity between the subgame perfect equilibrium paths of TG with chained confirmed proposals in Figure 6.a and those of the PD with chained confirmed proposals in Figure 2. The same holds for the EG with chained confirmed proposals in Figure 8.a and the BS with chained confirmed proposals in Figure 4. Finally, for the three GCCP in Figure 4 (BS), 8.a (EG) and 10.a (UG) it is common that a first-mover disadvantage exists, while instead the relative original games are all characterized by a first-mover advantage.

All these results suggest that the bargaining mechanism proposed in this paper as a supergame works in the same way for original games with different strategic structures.

4. Conclusions

Throughout the paper, we have defined *games with confirmed proposals* as a way of representing bargaining over the strategies of a non-cooperative game. We have shown the effect of this mechanism on players' ability to coordinate on Pareto efficient outcomes even in cases in which they are not equilibrium outcomes of the original non-cooperative game. Our focus was on a confirmed proposal mechanism with a *chain,* requiring that in the bargaining supergame each non-confirmed strategy profile becomes the starting point for the next negotiation round. One could discuss the implications of breaking this chain on the main features of the confirmed proposal mechanism. We leave this for future research.

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The Π-strategy: Analogies and Applications

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Abstract. The notion of the strategy of parallel pursuit (briefly Π-strategy) was introduced and used to solve the quality problem in "the game with a survival zone" by L.A.Petrosyan. Further it was found other applications of Π-strategy. In the present work Π-strategy will be constructed in the cases when 1) a control function of Pursuer should be chosen from the space L_2 and that for Evader should be chosen from L_∞; 2) control functions both of players should be chosen from the space L_2; 3) a control function of Pursuer should be chosen from intersection of spaces L_2 and L_∞ while that for Evader should belong to L_∞.

Keywords: differential game, Pursuer, Evader, strategy, parallel pursuit, domain of attainability, survival zone

AMS classification numbers: 91A23, 49N70

1. Formulation of the problem

Consider the differential game when Pursuer P and Evader E having radius vectors x and y correspondingly move in the space $\mathbb{R}^n$. If their velocity vectors are u and v then the game will be described by the equations:

$$\begin{aligned} \dot{x} = u,\ x(0) = x_0, \\ \dot{y} = v,\ y(0) = y_0, \end{aligned} \tag{1}$$

where $x, y, u, v \in \mathbb{R}^n$, $n > 1$, x_0 and y_0 are initial points of x and y correspondingly.

The family of all measurable functions $u(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ (controls of P) satisfying the following condition

$$\int_0^\infty |u(t)|^2 dt \le \rho^2, \quad \rho \ge 0, \tag{2}$$

is denoted by U^ρ, $U^\rho \subset L_2[0, \infty)$. The family of all measurable functions $u(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ satisfying the next condition

$$|u(t)| \le \alpha \ \text{ almost everywhere (a.e.) } \ t \ge 0, \tag{3}$$

is denoted by U_α, $U_\alpha \subset L_\infty[0, \infty)$. Further the family of all measurable functions satisfying both of conditions (2) and (3) will be denoted U^ρ_α.

Analogously the family of all measurable functions $v(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ satisfying

$$\int_0^\infty |v(t)|^2 dt \le \sigma^2, \quad \sigma \ge 0, \tag{4}$$

is denoted by V^σ, $V^\sigma \subset L_2[0,\infty)$ and satisfying the following condition

$$|v(t)| \le \alpha \quad \text{a. e. } \ t \ge 0 \tag{5}$$

is denoted by V_β, $V_\beta \subset L_\infty[0,\infty)$. Besides the family of all measurable functions satisfying both of conditions (4) and (5) also will be denoted V_β^σ.

In the theory of Differential Games an inequality of the forms (2) and (4) is usually called *an integral constraint for control function* (we will briefly say I-constraint). Analogously an inequality of the forms (3) and (5) is called *a geometric constraint* (briefly G-constraint).

The condition consisting of both inequalities (2)-(3) (or (4)-(5)) will be called *a complex constraint* (briefly C-constraint).

If U (correspondingly V) is one the introduced classes then there would be 9 possibilities of pairs (U, V) each of them generates the appropriate variant of simple motioned pursuit-evasion game. For brevity we will indicate this adding corresponding abbreviations to the word "game" as a prefix. For example the pair (U_α, V_β) defines G-game, (U^ρ, V_β) does IG-game and e. c.

We are going to study mainly the game with phase constraints for Evader being given by a subset L of $\mathbb{R}^n$ which is called "a Survival Zone" (for Evader naturally). (Notice that in the case $L = \emptyset$ we have a simple pursuit-evasion game.)

Each pair $(x_0, u(\cdot))$ consisting of an initial position x_0 and a control function $u(\cdot) \in U$ (correspondingly $(y_0, v(\cdot))$, $v(\cdot) \in V$) generates the path by the formula

$$x(t) = x_0 + \int_0^t u(s)ds, \quad (y(t) = y_0 + \int_0^t v(s)ds).$$

Further we will assume initial positions x_0 and y_0 are given such that $x_0 \ne y_0$ and $y_0 \notin L$.

In the game 'with the Survival Zone L' Pursuer P aims to catch Evader E, i.e. to realize the equality $x(t) = y(t)$ for some $t > 0$, while E stays in the zone $\mathbb{R}^n \setminus L$. The aim of E is to reach the zone L before being caught by Pursuer or to keep the relation $x(t) \ne y(t)$ for all $t, t \ge 0$. Notice that L doesn't anyway restrict motion of P.

The Differential Game with 'a Survival Zone' was suggested by R. Isaacs in (Isaacs, 1965) for G-game. He solved it when L was a half-plane and formulated the Problem 5.9.1 about the case when L is a disk. The game solved by L.A.Petrosyan for an arbitrary convex survival zone introducing the strategy of parallel pursuit (briefly *Π*-strategy; the prefix *Π* can be interpreted as meaning the word 'parallel' and the initial letter of the family name of its author as well). About further studies, see (Azamov, 1986) - (Azamov and Samatov, 2000), (Petrosyan, 1977) - (Petrosyan and Dutkevich, 1969), (Satimov, 2003)- (Samatov, 2008).

Well known that control functions for P are not sufficient to solve a pursuit problem as they depend only on time-parameter t, $t \ge 0$ so the suitable types of controls should be strategies. There are different ways of defining such a notion. For us it is enough the following conception.

Definition 1. The map $\mathbf{u} : V \to U$ is called *the strategy* for P if the following properties are held:

1^o (admissibility). For every $v(\cdot) \in V$ the inclusion $u(\cdot) = \mathbf{u}[v(\cdot)] \in U$ is true.

2^o (volterranianity). For every $v_1(\cdot), v_2(\cdot) \in V$ and t, $t \geq 0$, the equality $v_1(s) = v_2(s)$ a.e. on $[0,t]$ implies $u_1(s) = u_2(s)$ a.e. on $[0,t]$ with $u_i(\cdot) = \mathbf{u}[v_i(\cdot)]$, $i = 1,2$.

Definition 2. A strategy $\mathbf{u}(v)$ is called *winning* for P on the interval $[0,T]$ in the simple game if for every $v(\cdot) \in V$ there exists a moment $t^* \in [0,T]$ that is to reach the equality $x(t^*) = y(t^*)$.

Definition 3. A strategy $\mathbf{u}(v)$ for the player P is called *winning* on the interval $[0,T]$ *in the game with "the survival zone"* L if for every $v(\cdot) \in V$ there exists some moment $t^* \in [0,T]$ that
1) $x(t^*) = y(t^*)$;
2) $y(t) \notin L$ while $t \in [0,t^*]$.

Definition 4. A control function $v^*(\cdot) \in V$ for the player E is called *winning* in the game with "the survival zone" L if for every $u(\cdot) \in U$: 1) there exists some moment $\overline{t}$, $\overline{t} > 0$, that $y(\overline{t}) \in L$ and $x(t) \neq y(t)$ while $t \in [0,\overline{t}]$; or 2)$x(t) \neq y(t)$ for all $t \geq 0$.

2. The Π_G-strategy

2.1. Definition of the Π_G-strategy

The main aim of the present paper is to construct analogies of *Π*-strategy for $IG-$, $I-$ and CG-games. In this section the definition and some properties of *Π*-strategy settled by L.A.Petrosyan will be given for fullness of the exposition using analytical approach (see (Petrosyan, 1977)).

The G-game is a problem on simple pursuit with geometric constraints $|u| \leq \alpha$ and $|v| \leq \beta$ for the control functions of the players P and E respectively .

Let $z = x - y$. In order to define the Π_G-strategy consider a state vector $\zeta = (x,y)$ such that $x \neq y$ and suppose E holds the constant control $v, v \in S_\beta$ where $S_\beta = \{v : |v| \leq \beta\}$ is a ball. If P also applies the constant vector $u, |u| = \alpha$, helping P to meet E at some moment $T, T > 0$, then $Tu = Tv - z$ or

$$u = v - \lambda\xi \tag{6}$$

where $\lambda = |z|/T$ and $\xi = z/|z|$. From (6) the following equation for λ will be obtained: $\lambda^2 - 2\langle\xi, v\rangle\lambda - \alpha^2 + v^2 = 0$, where $< v, \xi >$ is the scalar production of the vectors v and ξ in $\mathbb{R}^n$. It has the positive root $\lambda_G(z,v) = \langle\xi, v\rangle + \sqrt{D_G(z,v)}$ if and only if

$$D_G(z,v) := \langle\xi, v\rangle^2 + \alpha^2 - |v|^2 \geq 0. \tag{7}$$

Substituting this value into (6) adduces to the formula

$$u = v - \lambda_G(z,v)\xi =: u_G(z,v). \tag{8}$$

Definition 5. The function $u_G(z,v)$ defined on the region (7) is called *the strategy of parallel pursuit* in the G-game or briefly *the Π_G-strategy.*

Let the initial state ζ_0 be fixed, $z_0 = x_0 - y_0$ and E applies the arbitrary control $v(\cdot) \in V_\beta$ while P runs the Π_G-strategy. Then the dynamics of the vector z will be described by the Cauchy problem:

$$\dot{z} = \dot{x} - \dot{y} = -\lambda_G(z, v(t))\xi, \;\; z(0) = z_0. \tag{9}$$

Obviously, the conditions of the existence theorem of Caratheodory (Alekseev et al., 1979) for the problem (9) are valid therefore that has the unique solution. Denote it $z(t, z_0, v(\cdot))$ or simply $z(t)$ and call *a trajectory.*

The following statement explains the term "parallel pursuit" for the Π_G-strategy (Petrosyan, 1965, 1977).

Theorem 1. *For every z_0, $z_0 \neq 0$, and $v(\cdot) \in V_\beta$ the formula $z(t) = \Lambda_G(t)z_0$ is true where $\Lambda_G(t) = 1 - \frac{1}{|z_0|}\int\limits_0^t \lambda_G(z_0, v(s))ds$.*

Corollary.
$$u_G(z(t), v(t)) = u_G(z_0, v(t)), t \geq 0. \tag{10}$$

Indeed, $u_G(z, v)$ is homogeneous on the variable z.

Henceforth $\Lambda_G(\cdot)$ will be called *the approach function in the G-game* while $\lambda_G(z, v)$ can be interpreted as *the speed of approach.*

Theorem 2. *In the G-game Π_G-strategy is wining for P on the interval $[0, T_G]$, where $T_G = |z_0|/(\alpha - \beta)$.*

2.2. Π_G-strategy and Apollonian sphere

The first essential application of Π_G-strategy was brought out to the game with survival zone (Petrosyan, 1965, 1977) basing on an interesting link between Π_G-strategy and Apollonian sphere.

To expose this link let us consider the region

$$A_G(x, y) = \left\{ w : \;|w - x| \geq \frac{\alpha}{\beta}|w - y| \right\};$$

what's boundary is just Apollonian sphere. Bringing its equation to the form $|w - c_G| = R_G$ one easily can find the center $c_G(x, y)$ and the radius of Apollonian sphere (Azamov, 1986): $c_G(x, y) = (\alpha^2 y - \beta^2 x)/(\alpha^2 - \beta^2)$, $R_G(x, y) = \alpha\beta|z|/|\alpha^2 - \beta^2|$.

It's easy to proof using (10) the following property

Theorem 3. $A_G(x(t), y(t)) = x(t) + \Lambda_G(t)[A_G(x_0, y_0) - x_0]$ *for* $t \in [0, t^*]$, *where* $t^* = \min\{t : z(t) = 0\}$.

Theorem 4 (monotonicity of Apollonian Sphere (Petrosyan, 1965, 1977)). *If* $0 \leq t_1 \leq t_2$ *then* $A_G(x(t_1), y(t_1)) \supset A_G(x(t_2), y(t_2))$.

Below in the section 6 the direct analytical proof of these assertions will be represented for more general situation.

2.3. G-game with "a Survival Zone"

It has been noted above that Isaacs' game with "line of survival" was solved by L.A.Petrosyan using the method of approximating measurable controls by piecewise constant.

Theorem 5 (L.A.Petrosyan's theorem). *If* $\alpha > \beta$ *and* $L \bigcap A_G(x_0, y_0) = \emptyset$ *then the* Π_G*-strategy is winning in G-game with the Survival zone L.*

P r o o f follows from Theorem 3.

C o m m e n t. The condition of Theorem 4 is necessary the player P to win. This is obvious for the condition $\alpha > \beta$. If $L \bigcap A_G(x_0, y_0) \neq \emptyset$, for example the intersection contains a point w, then E is able reaches w by the constant control $v = \beta(w - y_0)/|w - y_0|$ and wins the game.

3. Π_{IG}-strategy

Now let us consider IG-game where controls of P should satisfy integral constraints while controls of E do geometric constraints. Our aim is to transfer constructions of L.A.Petrosyan to IG-game.

3.1. Definition of Π_{IG}-strategy

First of all we are to notice in IG-game as distinct from the G-game the current phase state ζ should also include besides geometrical positions x, y (or reduced phase parameter $z = x - y$ if this is enough) the current resource $\overline{\rho}$ of the player P defining by the formula

$$\overline{\rho}(t) = \rho^2 - \int_0^t |u(s)|^2 ds.$$

Therefore in IG-game *a state vector* must be the triple $\zeta = (x, y, \overline{\rho})$. Choosing the vector u and the number T from the conditions: $Tu = Tv - z,\ T|u|^2 = \overline{\rho}$ one gets the following quadratic equation for the variable $\lambda = |z|/T$:

$$\lambda^2 - 2\lambda(\mu/2 + \langle \xi, v \rangle) + |v|^2 = 0, \tag{11}$$

where $\mu = \overline{\rho}/|z|,\ \xi = z/|z|$.

Further in this section we will suppose $\mu \geq 4\beta$. This implies the discriminant $D(z, \overline{\rho}, v) := (\mu/2 + \langle \xi, v \rangle)^2 - |v|^2$ is nonnegative and the equation (11) has the positive root $\lambda_{IG}(z, \overline{\rho}, v) = \mu/2 + \langle \xi, v \rangle + \sqrt{D(z, \overline{\rho}, v)}$.

Though the obtained formula for λ_{IG} is some bulky it takes part in the following

Definition 6. The function $u_{IG}(z, \overline{\rho}, v) := v - \lambda_{IG}(z, \overline{\rho}, v)\xi$ defined in the region $\mu \geq 4\beta$ is called the *Π_{IG}-strategy in the IG-game* for the player P.

Let us suppose that a perty in IG-game starts from a initial state $\zeta_0 = (x_0, y_0, \rho^2)$ and P applies Π_{IG}-strategy while E may choose any control function $v(\cdot) \in V_\beta$. These data naturally generate the augmented *trajectory* $x(t),\ y(t),\ \overline{\rho}(t)$ describing the party in IG-game. Now put $z(t) = x(t) - y(t)$. Then the path $(z(t), \overline{\rho}(t))$ will be found as the solution of Cauchy problem

$$\begin{cases} dz/dt = -\lambda_{IG}(z, \overline{\rho}, v(t))z/|z|, & z(0) = z_0, \\ d\overline{\rho}/dt = -\lambda_{IG}(z, \overline{\rho}, v(t))\overline{\rho}/|z|, & \overline{\rho}(0) = \rho^2, \end{cases} \tag{12}$$

which is valid while $\mu \geq 4\beta$ and $z \neq 0$.

Theorem 6. *If* $D(z(t), \overline{\rho}(t), v(t)) \geq 0$ *a.e. on some interval* $[0, t^*), t^* > 0$, *then*
a) $\mu(t) = \mu_0$, *where* $\mu_0 = \rho^2/|z_0|$;
b) $u_{IG}(z(t), \overline{\rho}(t), v(t),) = u_{IG}(z_0, \rho^2, v(t))$;
c) $|u_{IG}(z_0, \rho^2, v(t))|^2 = \mu_0 \lambda_{IG}(z_0, \rho^2, v(t))$.

P r o o f is immediately followed from the equation (12).

This statement allows to redefine Π_{IG}-strategy in more simply way putting

$$u_{IG}(v) = v - \lambda_{IG}(z_0, \rho^2, v)\xi_0.$$

3.2. IG-game without phase constraints

Let an initial position and control $v(\cdot) \in V_\beta$ of E be given and P run Π_{IG}-strategy. Then one has the generated trajectory $z(t)$ and can define the approach function in IG-game by the following way

$$\Lambda_{IG}(t) = 1 - \frac{1}{|z_0|} \int_0^t \lambda_{IG}(z_0, \rho^2, v(s)) ds.$$

Denote $T_{IG} = |z_0|/(\frac{\mu_0}{2} - \beta + \sqrt{\frac{\mu_0^2}{4} - \mu_0 \beta})$.

The following statements expresses an analogy of the properties of Π_G-strategy for IG-game.

Theorem 7. *If* $4\beta \leq \mu_0$ *then for every* $v(\cdot) \in V_\beta$ *the following properties are true*
a) $z(t) = \Lambda_{IG}(t) z_0$; $\overline{\rho}(t) = \Lambda_{IG}(t)\rho^2$;
b) there is $t^* \in [0, T_{IG}]$ *such that* $\Lambda_{IG}(t^*) = 0$ *and* $\Lambda_{IG}(t) > 0$ *while* $t < t^*$.

Theorem 8. *If* $4\beta \leq \mu_0$ *then in the simple motion IG-game the* Π_{IG}*-strategy is winning on* $[0, T_{IG}]$.

P r o o f of these theorems in more general situation will be given in the section 5.

3.3. Pascal's snail and Cartesian's oval

As in G-game in order to solve IG-game with a survival zone following L.A.Petrosyan we will try to find the set of all points where P is able to reach earlier than E supposing they start their motions from some current positions x, y respectively.

Let E moves holding a constant vector v and P aims to catch it by a constant control u as well spending over his resource $\overline{\rho}$. Let w be a point where P meets E. The set of all such points will be described by the following relations: $|w - x| = T|u|$, $|w - y| = T|v|$, $T|u|^2 = \overline{\rho}$. Excluding T and $|u|$ one gets the surface $|v||w - x|^2 = \overline{\rho}|w - y|$ that bounders the searching region

$$A_{IG}(x, y, \overline{\rho}) = \{w : \ |w - x|^2 \geq (\overline{\rho}/\beta)|w - y|\}$$

For $n = 2$ the boundary of the region A_{IG} will be Cartesian oval (if $4\beta < \mu$) or the inner loop of Pascal's snail (if $4\beta = \mu$). It is interesting to remember here that such curves appeared in G-game with a survival zone considered under terminal condition of l-capture (Petrosyan and Dutkevich, 1969).

3.4. IG-game with "a Survival Zone"

Now let P hold the Π_{IG}-strategy while E can apply any control $v(\cdot) \in V_\beta$. Naturally an initial position is being considered.

Theorem 9. *The following properties take a place:*
A. $A_{IG}(x(t), y(t), \overline{\rho}(t)) = x(t) + \Lambda_{IG}(t)[A_{IG}(x_0, y_0, \rho^2) - x_0]$.
B. If $0 \le t_1 \le t_2$ *then* $A_{IG}(x(t_1), y(t_1), \overline{\rho}(t_1)) \supset A_{IG}(x(t_2), y(t_2), \overline{\rho}(t_2))$.

The statement will be also proven below in the section 5. It implies

Theorem 10 (Petrosyanian type theorem). *If* $4\beta \le \mu_0$ *and* $L \bigcap A_{IG}(x_0, y_0, \rho^2) = \emptyset$ *then* Π_{IG}*-strategy is winning in IG-game with "the Survival Zone" L (for E).*

C o m m e n t. The condition of the last theorem is necessary, too, because one can easily notes that if a) $4\beta \le \mu_0$ and $L \bigcap A_{IG}(x_0, y_0, \rho^2) \neq \emptyset$ or b) $4\beta > \mu_0$ then the Player E would win the game holding simply some constant control v^*, $|v^*| = \beta$.

4. Π_I-strategy

4.1. Definition of Π_I-strategy

Now we consider I-game where admissible controls of both players should satisfy integral constraints as $u(\cdot) \in U^\rho$, $v(\cdot) \in V^\sigma$ (see (Ushakov, 1972)). Since considerations principally repeat those provided for G- and IG-games here they will expose somewhat shorter. In I-game a state vector should be a quadruple $\zeta = (x, y, \overline{\rho}, \overline{\sigma})$ including current values of the rest of resources of the players P and E respectively, i.e.

$$\overline{\rho}(t) = \rho^2 - \int_0^t |u(s)|^2 ds, \quad \overline{\sigma}(t) = \sigma^2 - \int_0^t |v(s)|^2 ds.$$

Arguments, like above, lead to the following system to build the Π-strategy for the I-Game: $Tu = Tv - z$ $\overline{\rho} - T|u|^2 = \overline{\sigma} - T|v|^2$, $T > 0$ or $u = v - \lambda\xi$, $|u|^2 - |v|^2 = \lambda\delta$ where $\lambda = |z|/T$, $\delta = (\overline{\rho} - \overline{\sigma})/|z|$. Hence

$$\lambda = \delta + 2\langle \xi, v \rangle. \tag{13}$$

Obviously, the formula (13) has sense when $\delta + 2\langle \xi, v \rangle > 0$. If $\delta + 2\langle \xi, v \rangle \le 0$ we put $\lambda = 0$. Unifying two cases gives the next formula

$$\lambda_I(z, \delta, v) := max\{0, \delta + 2\langle \xi, v \rangle\}.$$

Definition 7. The function $u_I(z, \delta, v) := v - \lambda_I(z, \delta, v)\xi$ is called *Π_I-strategy in the I-game.*

Let the initial state $\zeta = (x_0, y_0, \rho^2, \sigma^2)$ be given and $z_0 = x_0 - y_0$. Suppose P applies Π_I-strategy while E may choose any control function $v(\cdot) \in V^\sigma$. These data naturally generate the augmented trajectory $x(t), y(t), \overline{\rho}(t), \overline{\sigma}(t)$ describing the party in I-game.

Now put $z(t) = x(t) - y(t)$ and $\delta(t) = \theta(t)/|z(t)|$, where $\theta(t) = \overline{\rho}(t) - \overline{\sigma}(t)$, $\theta(0) = \theta_0 = \rho^2 - \sigma^2$. Notice $\delta_0 = \delta(0) = \theta_0/|z_0|$. Then the path $(z(t), \theta(t))$ will be defined as the solution of Cauchy problem

$$\begin{cases} dz/dt = -\lambda_I(z, \delta, v(t))z/|z|, \; z(0) = z_0, \\ d\theta/dt = -\lambda_I(z, \delta, v(t))\theta/|z|, \; \theta(0) = \theta_0. \end{cases} \tag{14}$$

The problem (14) has the unique solution while $z \neq 0$.

Theorem 11. *If $\delta(t) > 0$ a.e. on some interval $[0, t^*), t^* > 0$, then*
a) $\delta(t) = \delta_0$;
b) $u_I(z(t), \delta(t), v(t)) = u_I(z_0, \delta_0, v(t))$;
c) $|u_I(z_0, \delta_0, v(t))|^2 = |v(t)|^2 + \lambda_I(z_0, \delta_0, v(t))\delta_0$.

Proof is easily follows from the equation (14)

Thanks to the property b) the function

$$u_I(v) = u_I(z_0, \delta_0, v) = v - \lambda_I(z_0, \delta_0, v)\xi_0$$

can be used as Π_I-strategy instead of $u_I(z, \delta, v)$.

4.2. I-game

Here the following function

$$\Lambda_I(t) = 1 - \frac{1}{|z_0|}\int_0^t \lambda_I(z_0, \delta_0, v(s))ds$$

services as the measure of approach of P to E.

Theorem 12. $z(t) = \Lambda_I(t)z_0, \quad \theta(t) = \Lambda_I(t)\theta_0.$

Theorem 13. *If $\rho > \sigma$ then Π_I-strategy is winning in the interval $[0, T_I]$ where $T_I = |z_0|^2/(\rho - \sigma)^2$.*

Proof. Applying the inequality of Cauchy to the scalar production in (13) gives the estimation $\Lambda_I(t) \leq 1 - max\{0, \delta_0 t - 2\sigma\sqrt{t}\}/|z_0|$. That's why there exists t^*, $\left(\frac{2\sigma}{\delta_0}\right)^2 < t^* \leq T_I$ such $\Lambda_I(t^*) = 0$.

Further we are to verify admissibility of the corresponding realization of Π_I-strategy when E runs his admissible control $v(\cdot) \in V^\sigma$. Thus

$$\int_0^{t_*} |u_I(v(s))|^2 ds = \int_0^{t_*} |v(s)|^2 ds + \delta_0 \int_0^{t_*} \lambda_I(v(s))ds \leq \sigma^2 + \delta_0|z_0| = \rho^2,$$

where $t_* = min\{t : \; \Lambda_I(t) = 0\}$ Q.E.D.

4.3. I-game with a "Survival Zone"

Let the current state $\zeta = (x, y, \overline{\rho}, \overline{\sigma}),\ \ x \neq y$, be considered and E be supposed moving by a constant velocity v. Consider the situation when P tries to catch E also by a constant control u unless the players spend their resource fully. If w is the point where P meets E and T is the time of occurrence of such event then $|w-x| = T|u|,\ \ |w-y| = T|v|,\ \ T|u|^2 = \overline{\rho},\ \ T|v|^2 = \overline{\sigma}$.

In the case $\overline{\rho} > \overline{\sigma}$ the exclusion of $T,\ |u|$ and $|v|$ from the relations just written gives the equation of *the Apollonian sphere* $|w - c_I| = R_I$ where $c_I = y - \overline{\sigma}\xi_0/\delta_0$ and $R_I = \sqrt{\overline{\rho}\,\overline{\sigma}}/\delta_0$.

Denote

$$A_I(\zeta) = \begin{cases} \{w:\ |w - c_I| \leq R_I\} & if \quad \overline{\rho} > \overline{\sigma}, \\ \{w: \langle 2w - x - y, z\rangle \leq 0\} & if\ \overline{\rho} = \overline{\sigma}. \end{cases}$$

And now consider the party in the I-game when P holds Π_I-strategy while E applies some his admissible control $v(\cdot) \in V_\beta$ accounting some initial position is fixed.

Theorem 14. $A_I(\zeta(t_1)) \supset A_I(\zeta(t_2))$ *for* $0 \leq t_1 \leq t_2$.

The *Proof* is elementary and differs in the cases $\rho > \sigma$ and $\rho = \sigma$ but requires long calculations that's why will be omitted here.

Theorem 15 (Petrosyanian type theorem). *If* $\rho \geq \sigma$ *and* $L \cap A_I(\zeta_0) = \emptyset$ *then the* Π_I*-strategy is winning for P in the I-Game with "the Survival Zone L."*

C o m m e n t. If either a) $\rho \geq \sigma$ and $L \cap A_I(\zeta_0) \neq \emptyset$ or b) $\rho < \sigma$ then there exists a control function $v^*(t) \in V^\sigma$ being considered as a particular sort of strategies of E is winning for E in the I-Game with "the Survival Zone L".

5. Π_{CG}-strategy

Further it will be considered the game (1) under the following conditions

$$u(\cdot) \in U_\alpha^\rho, \quad v(\cdot) \in V_\beta \tag{15}$$

called *complex-geometric constraints.* As we have come to an agreement the game (1), (15) will be called CG-game and studied from the view of the theory of L.S. Petrosyan.

Let $p = (\alpha, \beta, \mu) \in \mathbf{P}$, where

$$\mathbf{P} = \{p \in \mathbb{R}^3_+ : f(\beta, \mu) \leq \alpha, \mu \geq 4\beta, \alpha > 0, \mu > 0\},$$

$$f(\beta, \mu) = \frac{\mu}{2} - \sqrt{\frac{\mu^2}{4} - \mu\beta}, \quad \mu = \frac{\rho^2}{|z_0|}, \quad z_0 = x_0 - y_0 \neq 0.$$

Then for all $v, |v| \leq \beta$, the following functions are defined correctly
$\lambda_G(v) =< \xi_0,\ v > + \sqrt{< \xi_0,\ v >^2 + \alpha^2 - |v|^2}, \quad \xi_0 = \frac{z_0}{|z_0|}$,

$\lambda_{IG}(v) = \frac{\mu}{2} + < \xi_0,\ v > + \sqrt{(\frac{\mu}{2} + < \xi_0,\ v >)^2 - |v|^2}$,

$\lambda_{CG}(v) = \min\{\lambda_G(v), \lambda_{IG}(v)\}$, (see sections 2.1 and 3.1).

Definition 8. The function

$$\mathbf{u}_{CG}(v) = v - \lambda_{CG}(v)\xi_0 \tag{16}$$

is called *the strategy of parallel pursuit in CG-game* (briefly, Π_{CG}*-strategy*) .

The following theorem reinforces theorems 2 and 8.

Theorem 16. *If* $p \in \mathbf{P}$ *then the* Π_{CG}*-strategy is winning for* P *on* $[0, T_{CG}]$ *where* $T_{CG} = \max\{T_G, T_{IG}\}$.

Proof. An initial position takes part in the condition $p \in \mathbf{P}$. Suppose that the player P runs Π_{CG}-strategy while E may apply ani his admissible control $v(\cdot) \in V_\beta$. They generate appropriate paths $P : x(t)$ and $E : y(t)$. Putting $z(t) = x(t) - y(t)$ the definition of the Π_{CG}-strategy (see (16)) implies

$$z(t) = \Lambda_{CG}(t)z_0, \tag{17}$$

where $\Lambda_{CG}(t) = 1 - \frac{1}{|z_0|}\int\limits_0^t \lambda_{CG}(v(s))ds$. The formula (17) explains the term "parallel pursuit" for the Π_{CG}-strategy. One can write the inequality

$$\Lambda_{CG}(t) \le 1 - \frac{1}{|z_0|} \min_{v(\cdot)\in V_\beta} \int\limits_0^t \lambda_{CG}(v(s))ds$$

on the right side of that it is sitting the problem of minimizing called *an elementary control problem* (Alekseev et al., 1979, p.360). Solving it, one gets the following estimation

$$\Lambda_{CG}(t) \le 1 - \frac{t}{|z_0|} \min_{|v|\le\beta} \lambda_{CG}(v).$$

Because of the condition $p \in \mathbf{P}$ it is obvious

$$\min_{|v|\le\beta} \lambda_{CG}(v) = \min\left\{\alpha - \beta,\ \frac{\mu}{2} - \beta + \sqrt{\frac{\mu^2}{4} - \mu\beta}\right\}.$$

Henceforth $\Lambda_{CG}(t) \le 1 - t/T_{CG}$. As $\Lambda_{CG}(t)$ is continuous in $[0, T_{CG}]$ and $\Lambda_{CG}(0) = 1$ there exists $t^* \in [0, T_{CG}]$ such that $\Lambda_{CG}(t^*) = 0$ or $x(t^*) = y(t^*)$. Let t_* be the minimal value of such t^*. Notice $u = 0$ for $t \ge t_*$.

Now we are to prove admissibility of Π_{CG}-strategy on $[0, t^*]$. For that notice

$$|\mathbf{u}_{CG}(v)|^2 = \begin{cases} |v - \lambda_G(v)\xi_0|^2 & \text{if } \ \lambda_G(v) \le \lambda_{IG}(v), \\ |v - \lambda_{IG}(v)\xi_0|^2 & \text{if } \ \lambda_{IG}(v) < \lambda_G(v), \end{cases} \tag{18}$$

die to the definition of Π_{CG}-strategy. Revealing squares in (18) and due to the definitions of $\lambda_G(v)$ and $\lambda_{IG}(v)$ one can rewrite (18) in the next form

$$|\mathbf{u}_{CG}(v)|^2 = \begin{cases} \alpha^2 & \text{if } \ \lambda_G(v) \le \lambda_{IG}(v), \\ \mu\lambda_{IG}(v) & \text{if } \ \lambda_{IG}(v) < \lambda_G(v). \end{cases} \tag{19}$$

But here the inequality $\lambda_{IG}(v) < \lambda_G(v)$ implies the estimation $\mu\lambda_{IG}(v) < \alpha^2$. Thus, $|\mathbf{u}_{CG}(v)|^2 \le \alpha^2$, i.e. $u_{CG}(v(\cdot)) \in U_\alpha$.

We should also show that $u_{CG}(v(\cdot)) \in U_\rho$ on $[0, t_*]$. For that let us divide the interval $[0, t_*]$ into two parts by following way

$$E_{\leq} = \{s : \ \lambda_G(v(s)) \leq \lambda_{IG}(v(s))\}, E_{>} = \{s : \ \lambda_G(v(s)) > \lambda_{IG}(v(s))\}.$$

It is clear that this sets are measurable. We have

$$\int_0^{t_*} |\mathbf{u}_{CG}(v(s))|^2 ds = \int_{E_{\leq}} \alpha^2 ds + \mu \int_{E_{>}} \lambda_{IG}(v(s)) ds.$$

Taking into account $\alpha^2 \leq \mu \lambda_G(v(s))$ for $s \in E_{\leq}$ and $p \in \mathbf{P}$ we get

$$\int_0^{t_*} |\mathbf{u}_{CG}(v(s))|^2 ds \leq \mu \int_0^{t_*} \min\{\lambda_G(v(s)), \lambda_{IG}(v(s))\} ds =$$

$$= \rho^2 \left(1 - \Lambda_{CG}(t_*)\right) = \rho^2$$

Q.E.D.

6. CG-game

We continue to suppose that the conditions of Theorem 1 take place and consider the situation when the player E moves from a position y with a constant velocity v, $|v| \leq \beta$ while P holding Π_{CG}-strategy from a position x having some current quality of his resource $\overline{\rho}$. Let w be a point where P would meet E. The set $A_{CG}(x, y, \overline{\rho})$ of all such points can be described by the following manner

a) if $\lambda_G(v) \leq \lambda_{IG}(v)$ then $A_{CG}(x, y, \overline{\rho}) = A_G(x, y)$;

b) if $\lambda_{IG}(v) \leq \lambda_G(v)$ then $A_{CG}(x, y, \overline{\rho}) = A_{IG}(x, y, \overline{\rho})$.

We denote $A^*_{CG}(x, y, \overline{\rho})$ the union of sets $A_G(x, y)$ and $A_{IG}(x, y, \overline{\rho})$.

Now consider the party when P applies Π_{CG}-strategy (16)and some control $v(\cdot) \in V_\beta$, on the interval $[0, t^*]$ is chosen by E till the time-moment $t^* = \min\{t : \Lambda_{CG}(t) = 0\}$

Let $x(t)$, $y(t)$ are the current positions of the players and $\overline{\rho}(t)$ is the current resource of the pursuer P defining as

$$\overline{\rho}(t) = \rho^2 - \int_0^t |\mathbf{u}_{CG}(v(s))|^2 ds, \quad \overline{\rho}(0) = \rho^2, \quad t \geq 0. \tag{20}$$

We are going to study the dynamics of the attainable domains

$$A_{CG}(t) = x(t) + \Lambda_{CG}(t)(A_{CG}(0) - x_0), \quad A^*_{CG}(t) = A_G(t) \cup A_{IG}(t),$$

where

$$A_G(t) = A_G(x(t), y(t)), \quad A_{IG}(t) = A_{IG}(x(t), y(t), \overline{\rho}(t)),$$

$$A_G(0) = A_G(x_0, y_0), \quad A_{IG}(0) = A_{IG}(x_0, y_0, \rho^2),$$

$$A_{CG}(0) = A_G(0) \cup A_{IG}(0), \quad A_{CG}(0) = A^*_{CG}(0).$$

Theorem 17. $A^*_{CG}(t) \subset A_{CG}(t)$ *while* $t \in [0, t^*]$.

Proof. Obviously

$$A^*_{CG}(t) - x(t) = (A_G(t) - x(t)) \cup (A_{IG}(t) - x(t)).$$

Using formulas (17), (20) and the inequality $\overline{\rho}(t) \geq \rho^2 \Lambda_{CG}(t)$ we have

$$A_G(t) - x(t) = \{w : \ |w| \geq (\alpha/\beta)|w + \Lambda_{CG}(t) z_0|\} = \Lambda_{CG}(t)(A_G(0) - x_0),$$

$$A_{IG}(t) - x(t) = \{w : \ |w|^2 \geq (\overline{\rho}(t)/\beta)|w + \Lambda_{CG}(t) z_0|\} \subset$$

$$\subset \{w : \ |w|^2 \geq (\rho^2 \Lambda_{CG}(t)/\beta)|w + \Lambda_{CG}(t) z_0|\} = \Lambda_{CG}(t)(A_{IG}(0) - x_0),$$

those imply the desired result. Q.E.D.

Theorem 18. *The set-valued function* $coA_{CG}(t)$ *is monotonically nondecreasing on the interval* $[0, t^*]$ *in the sense of order by inclusion. (coA denotes the convex hull of the set A.)*

Proof. Using the condition $|v| \leq \beta$ and relations (18)-(19) we obtain

$$|v - \lambda_G(v)\xi_0| \geq \alpha|v|/\beta \ \text{ if } \ \lambda_G(v) \leq \lambda_{IG}(v)$$

and

$$|v - \lambda_{IG}(v)\xi_0| \geq \sqrt{\mu \lambda_{IG}(v)}|v|/\beta \ \text{ if } \ \lambda_G(v) > \lambda_{IG}(v).$$

In its turn this assertions are equivalent to the following ones

$$|z_0|v + \lambda_G(v) y_0 \in \lambda_G(v) A_G(0) \ \text{ if } \ \lambda_G(v) \leq \lambda_{IG}(v)$$

and

$$|z_0|v + \lambda_{IG}(v) y_0 \in \lambda_{IG}(v) A_{IG}(0) \ \text{ if } \ \lambda_G(v) > \lambda_{IG}(v).$$

Hence $|z_0|v + \lambda_{CG}(v) y_0 \in \lambda_{CG}(v) A_{CG}(0)$. Now using the support function $C(A, p) = \sup_{w \in A} < w, p >$ (Blagodatskiy, 1979) it easy to see

$$< |z_0|v, p > - \lambda_{CG}(v) C(A_{CG}(0) - y_0, p) \leq 0$$

for all p, $|p| = 1$. Hence

$$< v - \lambda_{CG}(v)\xi_0, p > - \frac{1}{|z_0|}\lambda_{CG}(v) C(A_{CG}(0) - x_0, p) = \frac{d}{dt} C(A_{CG}(t), p) \leq 0$$

Q.E.D.

Theorem 19 follows the main result of the paper concerning to CG-game with a Survival Zone, being denoted L as before.

Theorem 19 (Petrosyanian type theorem). *If* $p \in \mathbf{P}$ *and* $coA_{CG}(0) \cap L = \emptyset$ *then a strategy* $\mathbf{u}_{CG}(v)$ *for the player* P *is winning on the interval* $[0, T_{CG}]$ *in the game with "the survival zone"* L.

C o m m e n t. Here the property of necessity of the condition '$p \in \mathbf{P}$ and $coA_{CG}(0) \cap L = \emptyset$' is true as well. Moreover as in the case $p \in \mathbf{P}$ and $A_{CG}(0) \cap L \neq \emptyset$ and so in the case $p \in \mathbf{E} = \mathbb{R}^3_+ \setminus \mathbf{P}$ the player E is able to solve the evasion problem using some constant control v^*, $|v^*| = \beta$.

The conclusion. It has been considered the pursuit-evasion game with different constraints for both Pursuer and Evader described by the simplest differential equation (1) only. The circumstance allowed us to construct winning strategies explicitly and to solve nontrivial problem of pursuit when a Survival Zone for Evader is present. Even for the simplest dynamics (1) there are problems staying open. For example we don't know how to solve pursuit problem for GC-,CI-,IC-,GI- and C-games with a Survival Zone.

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About Some Non-Stationary Problems of Group Pursuit with the Simple Matrix*

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Abstract. We consider two linear non-stationary problems of evasion of one evader from group of pursuers provided that players possess equal dynamic possibilities and the evader does not leave limits of some set. It is proved that if the number of pursuers is less than dimension of space the evader evades from a meeting on an interval $[t_0, +\infty)$.

Keywords: differential pursuit-evasion games.

1. Introduction

The important direction of the modern theory of differential games is linked with working out of solution methods of pursuit - evasion game problems with several players. Deriving both necessary and sufficient conditions of solvability of problems of evasion and pursuit under the initial data and game parameters thus is of interest. In the given research two non-stationary linear differential games with a simple matrix are considered.

2. The Non-stationary Problem with Simple Motion

In the space $\mathbb{R}^k (k \geq 2)$ a differential game of $n+1$ persons: n pursuers $P_1, \ldots, P_n$ and a single evader E, is considered.

The motion law of each pursuer P_i has the form

$$\dot{x}_i = b(t)u_i, \ \|u_i\| \leq 1. \tag{1}$$

The motion law of the evader has the form

$$\dot{y} = b(t)v, \ \|v\| \leq 1. \tag{2}$$

At $t = t_0$ initial positions of pursuers $x_1^0, \ldots, x_n^0$ and initial position of evader y^0, are set and $x_i^0 \neq y^0$, $i = 1, \ldots, n$.

Here $b : [t_0, \infty) \to \mathbb{R}^1$ — measurable function.

It is supposed that evader E in the course of game does not leave convex set D ($D \subset \mathbb{R}^k$) with a nonempty interior.

At $b(t) \equiv 1$ and with the lack of phase restrictions the problem was considered in Petrosyan L. A., 1966, Pshenichnii B. N., 1976, at $b(t) \equiv 1$ with phase restrictions the problem was considered in Ivanov R. P., 1978, Petrov N. N., 1984,

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Kovshov A. M., 2009, Satimov N. Yu. and Kuchkarov A. Sh., 2001. The non-stationary case under a condition $n \geq k$ was considered in
Bannikov A. S. and Petrov N. N., 2010.

Let σ be some partition $t_0 < t_1 < \dots < t_s < \dots$ of interval $[t_0, \infty)$, not having finite condensation points.

Definition 1. The piecewise–program strategy V of the evader E which is defined on $[0, \infty)$ that correspond to a partition σ is the family of mappings $\{c^l\}_{l=0}^{\infty}$ which are putting in correspondence to magnitudes

$$(t_l, x_1(t_l), \dots, x_n(t_l), y(t_l))$$

measurable function $v = v_l(t)$ which is defined for $t \in [t_l, t_{l+1})$ and such that $||v_l(t)|| \leq 1,\ y(t) \in D,\ t \in [t_l, t_{l+1})$.

Let's designate the given game through Γ.

Definition 2. We say that in game Γ an evasion from a meeting occurs if a partition σ of interval $[t_0, +\infty)$ exists without having finite condensation points, strategy V of the evader E corresponding to partition σ, such that for any trajectories $x_1(t), \dots, x_n(t)$ of pursuers $P_1, \dots, P_n$ takes place

$$x_i(t) \neq y(t),\ t \geq t_0, \quad i = 1, \dots, n,$$

where $y(t)$ is the trajectory of the evader E realized in the present state of affairs.

Theorem 1. *If $y^0 \in \operatorname{Int} D$, b is the function bounded on any compact set then the evasion from meeting is happening in the game Γ.*

Proof. Because $y^0 \in \operatorname{Int} D$, $D_r(q)$ — full-sphere with the radius r with the center in the point q exists and $y^0 \in \operatorname{Int} D_r(q) \subset D$.

Let ε — distance from y^0 to boundary $D_r(q)$, $I_l = [t_0 + l - 1, t_0 + l], b_l > 0$ is such that $|b(t)| \leq b_l$ for all $t \in I_l$,

$$\Omega_j(\tau) = \Big\{ t > \tau \ : \ \int\limits_{\tau}^{t} |b(s)| ds = \frac{\varepsilon}{j+1} \Big\}.$$

Note that if $t \in \Omega(\tau) \ \ \tau, t \in I_l$ for some l, then

$$\frac{\varepsilon}{j+1} = \int\limits_{\tau}^{t} |b(s)| ds \leq b_l (t - \tau).$$

Therefore

$$t - \tau \geq \frac{\varepsilon}{b_l (j+1)}. \tag{3}$$

Let us define partition σ_l of the segment for all segments I_l and the number m_l in the this way. Let us consider the segment I_1. Let $\tau_0^1 = t_0$, $j = 1, 2 \dots$,

$$\tau_j^1 = \begin{cases} \inf\{t > \tau_{j-1}^1,\ t \in \Omega_j(\tau_{j-1}^1)\},\ \text{if } \tau_j^1 < t_0 + 1 \text{ and } \Omega_j(\tau_{j-1}^1) \neq \emptyset, \\ t_0 + 1,\ \text{otherwise.} \end{cases}$$

Then we assume that $m_1 = \min\{j : \tau_j^1 = t_0 + 1\}, \sigma_1 = \{\tau_0^1, \dots, \tau_{m_1}^1\}$.

Let us consider the segment I_2. Let $\tau_0^2 = t_0 + 1$. For all $j = 1, 2 \dots,$

$$\tau_j^2 = \inf\{t > \tau_{j-1}^2,\ t \in \Omega_{j+m_1}(\tau_{j-1}^2)\},$$

if

$$\tau_j^2 < t_0 + 2 \quad \Omega_{j+m_1}(\tau_{j-1}^2) \neq \emptyset$$

and $\tau_j^2 = t_0 + 2$, if appropriate conditions aren't fulfilled.

Then we assume that

$$m_2 = m_1 + \min\{j : \tau_j^2 = t_0 + 2\},\ \sigma_2 = \{\tau_0^2, \dots, \tau_{m_2 - m_1}^2\}.$$

We assume that partition σ_{l-1} of the segment I_{l-1} and the number m_{l-1} are defined. Let us consider the segment I_l. Let $\tau_0^l = t_0 + l - 1$. For all $j = 1, 2 \dots,$

$$\tau_j^l = \inf\{t > \tau_{j-1}^2,\ t \in \Omega_{j+m_{l-1}}(\tau_{j-1}^l)\},$$

if

$$\tau_j^l < t_0 + l \text{ and } \Omega_{j+m_{l-1}}(\tau_{j-1}^l) \neq \emptyset$$

and $\tau_j^l = t_0 + l$, if appropriate conditions aren't fulfilled.

Then we assume that

$$m_l = m_{l-1} + \min\{j : \tau_j^l = t_0 + l\},\ \sigma_l = \{\tau_0^l, \dots, \tau_{m_l - m_{l-1}}^2\}.$$

Note that in view of (3) the numbers m_l exist for all l. As the partition σ of the interval $[t_0, \infty)$, we take such partition that contraction σ on all segment I_l matches with σ_l. Let $\sigma = \{\tau_0 = t_0 < \tau_1 < \dots < \tau_r < \dots\}$. We define strategy V of the evader in follows. $v(t) = v_j \operatorname{sign} b(t), t \in [\tau_j, \tau_{j+1})$, where v_j is defined from the following conditions

$$(v_j, x_i(\tau_j) - y(\tau_j)) = 0, i = 1, \dots, n, (v_j, y(\tau_j) - q) \leq 0, \|v_j\| = 1.$$

As $n < k$ system has the solution.

Let us show that V is the evasion strategy. Let us consider the segment $[\tau_j, \tau_{j+1}]$. Then from systems (1), (2) we have

$$y(t) = y(\tau_j) + \int\limits_{\tau_j}^{t} |b(s)| ds \cdot v_j,$$

$$x_i(t) = x_i(\tau_j) + \int\limits_{\tau_j}^{t} b(s) u_i(s) ds.$$

Therefore

$$\begin{gathered} \|x_i(t) - y(t)\| = \|x_i(\tau_j) - y(\tau_j) - M_j(t) v_j + M_j(t) \hat{u}_i(t)\| \geq \\ \geq \|x_i(\tau_j) - y(\tau_j) - M_j(t) v_j\| - M_j(t) = \\ = \sqrt{\|a_i(\tau_j)\|^2 - 2M_j(t)(a_i(\tau_j), v_j) + M_j^2(t)} - M_j(t) = \\ = \sqrt{\|a_i(\tau_j)\|^2 + M_j^2(t)} - M_j(t), \end{gathered} \tag{4}$$

where $a_i(\tau_j) = x_i(\tau_j) - y(\tau_j)$, $M_j(t) = \int\limits_{\tau_j}^{t} |b(s)|ds$,

$$\hat{u}_i(t) = \begin{cases} \frac{1}{M_j(t)} \int\limits_{\tau_j}^{t} b(s)u_i(s)ds, \text{ if } M_j(t) \neq 0 \\ 0, \text{ if } M_j(t) = 0 \end{cases},$$

and $\|\hat{u}_i(t)\| \leq 1$ for all $t \in [\tau_j, \tau_{j+1}]$. From (4) it follows that if $x_i(\tau_j) \neq y(\tau_j)$ then $x_i(t) \neq y(t)$ for all $t \in [\tau_j, \tau_{j+1}]$. It means that if caption didn't occur at the moment τ_j then it wouldn't on the segment $[\tau_j, \tau_{j+1}]$.

As $x_i^0 \neq y^0$ for all i then $y(t) \neq x_i(t)$ for all $i, t \geq t_0$.

Let us prove now that $y(t) \in D$ for all $t \geq t_0$. Let us consider the segment $[\tau_0, \tau_1]$. Then

$$\|y(t) - q\| = \|y^0 + M_1(t)v_1 - q\| =$$
$$= \sqrt{\|y^0 - q\|^2 + 2(y^0 - q, v_1)M_1(t) + M_1^2(t)} \leq \sqrt{(r-\varepsilon)^2 + M_1^2(t)}.$$

As $M_1(t) \leq \frac{\varepsilon}{2}$ for all $t \in [\tau_0, \tau_1]$ then

$$\|y(t) - q\| \leq \sqrt{(r-\varepsilon)^2 + \left(\frac{\varepsilon}{2}\right)^2} \leq r - \frac{\varepsilon}{2}.$$

Thus, $y(t) \in D$ for all $t \in [\tau_0, \tau_1]$.

Let us assume that inequality $\|y(t) - q\| \leq r - \frac{\varepsilon}{j+1}$ is proved for all $t \in [\tau_{j-1}, \tau_j], j \leq l-1$. Let us prove that for all $t \in [\tau_{l-1}, \tau_l]$ inequality $\|y(t) - q\| \leq r - \frac{\varepsilon}{l+1}$ holds true.

$$\|y(t) - q\| = \|y(t_{l-1}) + M_l(t)v_l - q\| =$$
$$= \sqrt{\|y(t_{l-1}) - q\|^2 + 2(y(t_{l-1}) - q, v_l)M_l(t) + M_l^2(t)} \leq$$
$$\leq \sqrt{(r - \frac{\varepsilon}{l})^2 + M_l^2(t)}.$$

As $M_l(t) \leq \frac{\varepsilon}{l+1}$ for all $t \in [\tau_{l-1}, \tau_l]$ then

$$\|y(t) - q\| \leq \sqrt{(r - \frac{\varepsilon}{l})^2 + \left(\frac{\varepsilon}{l+1}\right)^2} \leq r - \frac{\varepsilon}{l+1}.$$

So we prove that V is the evasion strategy. □

3. Linear non-stationary problem of evasion in a cone

In the space $\mathbb{R}^k (k \geq 2)$, a differential game of $n+1$ persons: n pursuers $P_1, \ldots, P_n$ and a single evader E, is considered.

The motion law of each pursuer P_i has the form

$$\dot{x}_i = a(t)x_i + u_i, \ \|u_i\| \leq 1. \tag{5}$$

The motion law of the evader E has the form

$$\dot{y} = a(t)y + v, \ \|v\| \leq 1. \tag{6}$$

At $t = t_0$ we have the initial positions of the pursuers $x_1^0, \ldots, x_n^0$ and the initial position of the evader y^0 and $x_i^0 \neq y^0$, $i = 1, \ldots, n$.

Here $a : [t_0, \infty) \to \mathbb{R}^1$ is a measurable function.

It is supposed that the evader E in the course of the game does not leave convex cone

$$D = \{y : \ y \in \mathbb{R}^k, \quad (p_j, y) \leq 0, j = 1, \ldots r\},$$

where $p_1, \ldots p_r$ are the unit vectors $\mathbb{R}^k$ such that $\operatorname{Int} D \neq \emptyset$.

The evader uses piecewise-program strategies.

At $a(t) \equiv a, a < 0$ and lack of phase restrictions the problem was considered in Pshenichnii B. N. and Rappoport I. S., 1979, at $a(t) \equiv a, a < 0, n \geq k$ with phase restrictions the problem was considered in Petrov N. N., 1988; at $a(t) \equiv a, a < 0, n < k$ with phase restrictions the problem was considered in Petrov N. N., 1998, and at $a(t) \equiv a, a > 0, n < k$ with phase restrictions the problem was considered in Shuravina I. N., 2009.

Theorem 2. *Let $y^0 \in \operatorname{Int} D$, a is a bounded function on any compact and $n < k$. Then the evasion from meeting occurs in the game Γ.*

Proof. Let's consider segment $I_l, t_l = t_0 + l$. In systems (5), (6) let's make a change of variables

$$x_i = e^{\int_{t_l}^{t} a(s)ds} w_i^l, \ y = e^{\int_{t_l}^{t} a(s)ds} z^l.$$

We will receive systems

$$\dot{w}_i^l = e^{-\int_{t_l}^{t} a(s)ds} u_i, \ \dot{z}^l = e^{-\int_{t_l}^{t} a(s)ds} v. \tag{7}$$

Let's notice that $x_i(\tau) = y(\tau)$ at some $i, \tau \in I_l$ if and only if $w_i^l(\tau) = z^l(\tau)$. Besides $y(t) \in D$ if and only if $z^l(t) \in D$. Let further

$$b_l(t) = e^{-\int_{t_l}^{t} a(s)ds}, K_l = e^{-\int_{t_l}^{t_{l-1}} a(s)ds},$$

$D_r(q)$ – a full-sphere with radius r with the center in a point q $y^0 \subset D_r(q) \subset D$, ε – distance from y^0 to boundary $D_r(q)$, $q_1 = K_1 q, r_1 = K_1 r, \varepsilon_1 = K_1 \varepsilon$, $q_l = K_l q_{l-1}$, $r_l = K_l r_{l-1}$, $l \geq 2$. Let's notice that $b_l(t) > 0$ for all $t \in I_l$.

For the segment I_1 on ε_1 and function b_1 we will define number m_1 and partition σ_1 under the scheme of the previous section. For the segment I_2 on $\varepsilon_2 = \frac{K_2 \varepsilon_1}{m_1 + 2}$ and function b_2 we will define number m_2 and partition σ_2 and so on. For segment I_l on $\varepsilon_l = \frac{K_l \varepsilon_{l-1}}{m_{l-1} + 2}$ and function b_l we will define m_l and partition σ_l. As a partition σ of interval $[t_0, \infty)$ we take such partition, which contraction on any segment I_l coincides with σ_l.

Let $\tau_j, \tau_{j+1} \in \sigma_l$. We set strategy V of evader E on $[\tau_j, \tau_{j+1}]$ supposing $v(t) = v_j^l$, where v_j^l is defined from following system

$$(v_j^l, z^l(\tau_j) - w_i^l(\tau_j)) = 0, \ i = 1, \ldots, n, \ (v_j^l, q_l - z^l(\tau_j)) \geq 0, \ \|v_j^l\| = 1.$$

As $n < k$ then v_j^l is always exists. Let's show that strategy V guarantees evasion from meeting.

1. Let's show that $z^l(t) \neq w_i^l(t)$ for all $i, t \in I_l$. Let $\tau_j, \tau_{j+1} \in \sigma_l$. From systems (7) we have

$$z^l(t) = z^l(\tau_j) + \int\limits_{\tau_j}^{t} b_l(s)ds \cdot v_j^l,$$

$$w_i^l(t) = w_i^l(\tau_j) + \int\limits_{\tau_j}^{t} b_l(s)u_i(s)ds.$$

Therefore for all $t \in [\tau_j, \tau_{j+1})$

$$\begin{aligned}\|z^l(t) - w_i^l(t)\| &\geq \|z^l(\tau_j) + M_j^l(t)v_j^l - w_i^l(\tau_j)\| - M_j^l(t) = \\ &= \sqrt{(a_i^l(\tau_j))^2 + 2M_j^l(t)(v_j^l, a_i^l(\tau_j)) + (M_j^l(t))^2} - M_j^l(t) = \\ &= \sqrt{(a_i^l(\tau_j))^2 - (M_j^l(t))^2} - M_j^l(t) > 0 \text{ if } a_i^l(\tau_j) \neq 0.\end{aligned}$$

Here

$$M_j^l(t) = \int\limits_{\tau_j}^{t} b_l(s)ds, \;\; a_i^l(\tau_j) = z^l(\tau_j) - w_i^l(\tau_j).$$

Therefore if capture didn't occur at τ_j then it wouldn't occur on segment $[\tau_j, \tau_{j+1}]$. As $z^0 - w_i^0 \neq 0$ for all i then we prove that $z^l(t) \neq w_i^l(t)$ for all $i, l, t \in I_l$.

2. Let's introduce that for all natural l, $t \in I_l$ following inequalities hold

$$\|z^l(t) - q_l\| \leq r_l - \frac{\varepsilon_l}{j+1}, \;\; t \in [\tau_j, \tau_{j+1}], \;\; (\tau_j, \tau_{j+1} \in \sigma_l). \tag{8}$$

Let's consider the segment $I_1 = [t_0, t_1]$ and partition σ_1 of this segment.

$$\|z^1(t_0) - q_1\| = \|K_1 y^0 - K_1 q\| = K_1(r - \varepsilon) = r_1 - \varepsilon_1.$$

Let $t \in [\tau_0^1, \tau_1^1]$. Then

$$\begin{gathered}\|z^1(t) - q_1\| = \|z^1(t_0) + M_0^1(t)v_1^1 - q_1\| = \\ \sqrt{\|z^1(t_0) - q_1\|^2 + 2M_0^1(t)(z^1(t_0) - q_1, v_1^1) + (M_0^1(t))^2} \leq \\ \leq \sqrt{(r_1 - \varepsilon_1)^2 + \left(\frac{\varepsilon}{2}\right)^2} \leq r_1 - \frac{\varepsilon_1}{2},\end{gathered}$$

because $M_0^1(t) \leq \frac{\varepsilon_1}{2}$ for all $t \in [\tau_0^1, \tau_1^1]$ owing to a choice τ_1^1. The further proof of an inequality (8) for I_1 is similar to the proof of a corresponding inequality from the previous section.

Let us assume that equality (8) is proved for all $I_l, l \leq s$. Let us prove an inequality for I_{s+1}. Owing to the supposition inequalities holds true

$$\|z^s(t_s) - q_s\| \leq r_s - \frac{\varepsilon_s}{m_s + 2}.$$

Then

$$\|z^{s+1}(t_s) - q_{s+1}\| = \|K_{s+1}(y(t_s) - q_s)\| =$$
$$= K_{s+1}\|z^s(t_s) - q_s\| \leq K_{s+1}(r_s - \frac{\varepsilon_s}{m_s + 2}) = r_{s+1} - \varepsilon_{s+1}.$$

Therefore the proof of the inequality (8) for I_{s+1} is similar to the proof (8) for I_1. Therefore we have

$$z^l(t) \subset D_{r_l}(q_l) \subset D$$

for all $t \in I_l$ and for l. Therefore we have $y(t) \in D$ for all $t \geq t_0$. Thus specified strategy V is the evasion strategy. The theorem is proved. □

Theorem 3. *Let* $y^0 \in \operatorname{Int} D$,

$$0 \notin \{z_1^0, \ldots, z_n^0, p_1, \ldots, p_r\}.$$

Then evasion from meeting occurs in the game Γ.

The proof of this theorem is similar to the proof of the stationary case Petrov N. N., 1988.

4. Evasion problem from group inertial pursuits

In the space $\mathbb{R}^k (k \geq 2)$ a differential game of $n+1$ persons: n pursuers $P_1, \ldots, P_n$ and a single evader E, is considered.

The motion law of each pursuer P_i has the form

$$\ddot{x}_i(t) = a(t)u_i(t), \quad u_i \in U.$$

The motion law of the evader E is

$$\ddot{y}(t) = a(t)v(t), \quad v \in U,$$

and at $t = t_0$ initial positions of pursuers $x_i(t_0) = x_i^0$, $\dot{x}_i(t_0) = \dot{x}_i^0$ and initial position of evader

$$y(t_0) = y^0, \quad \dot{y}(t_0) = \dot{y}^0, \quad x_i^0 \neq y^0, \quad i = 1, \ldots, n,$$

are set.

Here x_i, y, u_i, $v \in \mathbb{R}^k$, $U \subset \mathbb{R}^k$ — convex compact, $0 \in \operatorname{Int} U$; $a(t)$ — bounded measurable function integrated on any compact subset of axis t, $a(t) \neq 0$ almost everywhere on interval $[t_0, +\infty)$. Measurable functions $u_i(t)$, $v(t)$ accept values at $t \geq t_0$ from set U.

Let's designate the given game through $\Gamma = \Gamma(n, z(t_0))$, where

$$z(t) = (z_1(t), \dot{z}_1(t), \ldots, z_n(t), \dot{z}_n(t)), \quad z_i(t) = x_i(t) - y(t), \quad i = 1, \ldots, n.$$

At $a(t) \equiv 1$ resolvability sufficient conditions of the local evasion problem have been got by Prokopovich P. V., Chikrii A. A., 1989.

Sufficient conditions of resolvability of the local evasion problem for a stationary control example of Pontryagin have been got by
Chikrii A. A.,Prokopovich, P. V., 1994.

In this section, using ideas of Chikrii and Prokopovich, the conditions of resolvability of the local evasion problem have been got in a non-stationary case.

Definition 3. *Positional counter-strategy* V of evader E is the measurable mapping

$$[t_0, +\infty) \times \mathbb{R}^{2nk} \times U^n \to U.$$

Then at set controls $u_i(t)$ of pursuers P_i, $i = 1, \dots, n$ strategy V defines control $v(t) = V(t, z(t), u_1(t), \dots, u_n(t))$, which will measurable function.

Definition 4. We say that in differential game Γ from initial position $z(t_0)$ *local evasion problem is solvable*, if for all measurable functions

$$u_i(t), \quad t \geq t_0, \quad u_i \in U, \quad i = 1, \dots, n,$$

strategy V of evader E exists such that $z_i(t) \neq 0$ for all $t \geq t_0$, $i = 1, \dots, n$.

We consider that controls of pursuers are formed on the basis information about position $z(t)$ of differential game.

Theorem 4. *If the condition* $0 \notin \operatorname{co}\{\dot{z}_1^0, \dots, \dot{z}_n^0\}$ *is satisfied then in game* Γ *local evasion problem from initial position* $z(t_0)$ *is solvable.*

Proof. Let $0 \notin \operatorname{co}\{\dot{z}_1^0, \dots, \dot{z}_n^0\}$. Based on the convex sets separation theorem unit vector p and number $\varepsilon > 0$ exist such that

$$\max_{i=1,\dots,n} (\dot{z}_i^0, p) \leq -2\varepsilon. \tag{9}$$

Let us designate

$$\eta(t) = \min_{i=1,\dots,n} \|z_i(t)\|, \tag{10}$$

$$\delta = \min\{1, \varepsilon, \sqrt{\eta(t_0)}\}. \tag{11}$$

1. Let $\max\limits_{1,\dots,n} (z_i^0, p) \leq 0$. Let us suppose

$$v(t) = v_p(t) = \begin{cases} v_p, & a(t) > 0, \\ v_{-p}, & a(t) < 0, \end{cases} \tag{12}$$

where $v_p, v_{-p} \in U$ are vectors such that $(v_p, p) = C(U, p)$, $(v_{-p}, -p) = C(U, -p)$. Then for all $i = 1, \dots, n$ we have

$$(z_i(t), p) =$$

$$= (z_i^0, p) + (t - t_0)(\dot{z}_i^0, p) + \int_{t_0}^{t} (t - s) a(s) (u_i(s) - v_p(t), p)\, ds \leq -2\varepsilon(t - t_0) < 0$$

at $t > t_0$. The local evasion problem from initial position $z(t_0)$ is solvable.

2. Let us assume that $(z_1^0, p) > 0$ and $(z_i^0, p) \leq 0$ for all $i \in \{2, \dots, n\}$. Let us describe evasion maneuver which guarantees the solvability of evasion problem for the such initial position $z(t_0)$. Let $K = \min\{1, \frac{1}{|U|_c}\}$, $\tau_1 = K\frac{\delta}{4}$, $\delta_1 = K\frac{\delta^2}{4} < \eta(t_0)$. Let's suppose

$$v(t) = v_p(t), \quad t \in [t_0, +\infty) \backslash [t_1, t_1 + \delta_1),$$

where t_1 is either first moment, when for the first time equalities are fulfilled $\|z_1(t_1)\| = \delta_1$ and $(z_1(t_1), p) > 0$, or $+\infty$, and if $t_1 < +\infty$, then on interval $[t_1, t_1 + \tau_1)$, we choose control $v(t)$ in special way.

This way chosen control of evader E the pursuer P_i, $i \in \{2, \dots, n\}$ does not influence a game course. Really, at any controls $u_i(t)$, $i = 1, \dots, n$, we have

$$(\dot{z}_i(t), p) = (\dot{z}_i(t_0), p) + \int\limits_{t_0}^{t} a(s)\big((u_i(s), p) - (v(s), p)\big) \leq$$
$$\leq -2\delta + 2c|U|\tau_1 < -\delta, \quad t \geq t_0. \quad (13)$$

Therefore

$$(z_i(t), p) = (z_i(t_0), p) + \int\limits_{t_0}^{t} (\dot{z}_i(s), p)\, ds < (z_i(t_0), p) \leq 0, \quad t \geq t_0, \quad i \neq 1. \quad (14)$$

So we have at $t \geq t_0$ $\|z_i(t)\| \neq 0$, $i \neq 1$.

Let us notice that at $t = t_1$ $\|z_1(t_1)\| = \delta_1$, therefore at any controls $u_1(t)$ and $v(t)$ on segment $[t_1, t_1 + \tau_1]$

$$(z_1(t_1 + \tau_1), p) = (z_1(t_1), p) + \int\limits_{t_1}^{t_1+\tau_1} (\dot{z}_1(s), p)\, ds < \delta_1 - \delta\tau_1 = 0. \quad (15)$$

Therefore, at $t = t_1 + \tau_1$ position $z(t_1 + \tau_1)$ of differential game corresponds to previous case 1:

$$(\dot{z}_i(t_1 + \tau_1), p) < -\delta,$$
$$(z_i(t_1 + \tau_1), p) < 0, \quad i = 1, \dots, n.$$

Thus,if on any controls $u_1(s)$ we can specify control $v(s)$, $s \in [t_1, t_1 + \tau_1)$, such that $\|z_1(t)\| \neq 0$ at $t \in [t_1, t_1 + \tau_1]$, then the solvability of local evasion problem for initial position $z(t_0)$ in the case 2 will be proved.

Let us assume that

$$(z_1(t_1), \dot{z}_1(t_1)) = -\|z_1(t_1)\| \|\dot{z}_1(t_1)\|. \quad (16)$$

Vectors $z_1(t_1)$ and $\dot{z}_1(t_1)$ are linearly dependent, therefore unit vector ψ exists such that

$$(\psi, z_1(t_1)) = (\psi, \dot{z}_1(t_1)) = 0. \quad (17)$$

Let $\varepsilon_1 \in (0, \tau_1)$ is some number such that at arbitrary control $u_1(s)$, $v(s)$, $s \in [t_1, t_1 + \varepsilon_1]$, inequality $(z_1(s), p) > 0$ is fulfilled. On the segment $[t_1, t_1 + \varepsilon_1]$ we choose control $v(s)$ so that

$$(\psi, v(s)) = \begin{cases} C(U, \psi), & (\psi, u_1(s)) \leq 0, \\ -C(U, -\psi), & (\psi, u_1(s)) > 0. \end{cases} \quad (18)$$

Let us show that the number $\gamma_1 \in (0, \varepsilon_1)$ exits such that

$$(z_1(t_1 + \gamma_1), \dot{z}_1(t_1 + \gamma_1)) \neq -\|z_1(t_1 + \gamma_1)\| \|\dot{z}_1(t_1 + \gamma_1)\|. \quad (19)$$

At $t \geq t_1$ let us consider the functions

$$\begin{aligned} f_1(t) &= (\psi, z_1(t)) = \int_{t_1}^{t} (t-s)a(s)(\psi, u_1(s) - v(s))\, ds, \\ f_2(t) &= (\psi, \dot{z}_1(t)) = \int_{t_1}^{t} a(s)(\psi, u_1(s) - v(s))\, ds. \end{aligned} \tag{20}$$

The functions $f_1(t)$, $f_2(t)$, $t_1 \leq t \leq t_1 + \varepsilon_1$, satisfy to the set of equations

$$\begin{aligned} \dot{f}_1(t) &= f_2(t), \\ \dot{f}_2(t) &= a(t)(\psi, u_1(t) - v(t)), \end{aligned} \tag{21}$$

and $f_1(t_1) = f_2(t_2) = 0$.

As $0 \in \operatorname{Int} U$ then $C(U, q) > 0$ for all $q \neq 0$. Therefore $|a(t)(\psi, u_1(t) - v(t))| > 0$ almost everywhere on $[t_1, t_1 + \varepsilon_1]$ and $f_2(t) \not\equiv 0$ on any segment $[\alpha, \beta] \subset [t_1, t_1 + \varepsilon_1]$, $t_1 < \alpha < \beta$. Therefore the set $G = \{t \in (t_1, t_1 + \varepsilon_1) | f_2(t) \neq 0\}$is the nonempty open set. Therefore $G = \bigcup\limits_j (\alpha_j, \beta_j)$, when $\{(\alpha_j, \beta_j)\}$ is no more than countable system of not intersected intervals.

Let $(\alpha_{j_0}, \beta_{j_0})$ is the some interval of this system. Then we have

$$f_2(\alpha_{j_0}) = f_2(\beta_{j_0}) = 0, \quad \dot{f}_2(t) \neq 0, \quad t \in (\alpha_{j_0}, \beta_{j_0}).$$

If $f_1(\alpha_{j_0}) \neq 0$ then the relation (15) is fulfilled at $t_1 + \gamma_1 = \alpha_{j_0}$. If $f_1(\alpha_{j_0}) = 0$ then

$$\dot{f}_1(t) = f_2(t) \neq 0, \quad t \in (\alpha_{j_0}, \beta_{j_0}).$$

Therefore $f_1(t) f_2(t) > 0$ on $(\alpha_{j_0}, \beta_{j_0})$. So at $t_1 + \gamma_1 = \tau$, when τ is some arbitrarily chosen number from an interval $(\alpha_{j_0}, \beta_{j_0})$, the relation (15) takes place.

Thus, control $v(s)$ at $s \in [t_1, t_1 + \varepsilon_1]$ is defined according to a rule (14). Then we have

$$\|z_1(t)\| \neq 0, \quad t \in [t_1, t_1 + \varepsilon_1],$$

and at some moment $t = t_1 + \gamma_1$ the relation (15) will be fulfilled.

If

$$(z_1(t_1), \dot{z}_1(t_1)) \neq -\|z_1(t_1)\| \|\dot{z}_1(t_1)\|, \tag{22}$$

then we suppose $\gamma_1 = 0$. Let's notice that $(z_1(t_1 + \gamma_1), p) > 0$ and $(\dot{z}_1(t_1 + \gamma_1), p) < -\delta$, therefore from the inequality (15) linear independence of the vectors $z_1(t_1 + \gamma_1)$ and $\dot{z}_1(t_1 + \gamma_1)$ follows. Further for all $s \in [t_1 + \gamma_1, t_1 + \tau_1)$ we will suppose $v(s) = u_1(s)$. Then we have

$$z_1(t) = z_1(t_1 + \gamma_1) + (t - t_1 - \gamma_1)\dot{z}_1(t_1 + \gamma_1) \tag{23}$$

at $t_1 + \gamma_1 \leq t \leq t_1 + \tau_1$, therefore $\|z_1(t)\| \neq 0$ at $t \in [t_1 + \gamma_1, t_1 + \tau_1]$.

Thus on any measurable function $u_1(s) \in U$ it is possible to construct such measurable function

$$v(s) = V(s, z(s), u_1(s), \ldots, u_n(s)), \quad v(s) \in U, \quad s \in [t_1, t_1 + \tau_1),$$

that $\|z_1(t)\| \neq 0$ at $t \in [t_1, t_1+\tau_1]$. The solvability of local evasion problem is proved in the case 2.

3. Let $(z_i^0, p) > 0$ for all $i \in I' = \{1, \dots, s\}$, $s \le n$, and $(z_i^0, p) \le 0$ for $i \in \{1, \dots, n\} \backslash I'$. We define such numbers τ_j, δ_j, $j = 1, \dots, N_1$, $N_1 \le s$, that

$$\tau_j > \tau_{j+1}, \quad \delta_j > \delta_{j+1}, \quad j = 1, \dots, N_1 - 1,$$

and if at $t' > t_0$ for some $i \in \{1, \dots, s\}$ equality $\|z_i(t')\| = \delta_j$ hold and $(z_i(t'), p) >$ then $(z_i(t' + \tau_j), p) < 0$ for any controls $u_i(s)$, $v(s)$ defined on $[t', t' + \tau_j]$.

Time moment $t_i > t_0$ when equality $\eta(t) = \delta_i$ the first time is carried out and number $\ell \in \{1, \dots, s\}$, exists such that $\|z_\ell(t_i)\| = \delta_i$, $(z_\ell(t_i), p) > 0$, we name the i-th meeting moment. Without reducing a generality, we suppose that at $t = t_i$ we have $\|z_i(t_i)\| = \delta_i$ and $(z_i(t_i), p) > 0$. it means that pursuers are numbered in that order in what their meeting with the evader happens.

Let us suppose

$$v(t) = v_p(t), \quad t \in [t_0, +\infty) \backslash T, \quad T = \bigcup_{i=1,\dots,N_1} [t_i, t_i + \tau_i), \tag{24}$$

At $t = t_0$ we construct sequences $\{\tau_1^i\}_{i=1}^{\infty}$, $\{\delta_1^i\}_{i=1}^{\infty}$ in follows:

$$\tau_1^i = K \frac{\delta}{2^{i+1}}, \quad \delta_1^i = \delta \tau_1^i = K \frac{\delta^2}{2^{i+1}}. \tag{25}$$

Numbers τ_i, $i = 1, \dots, N_1$, will be defined so that $\tau_i \le \tau_1^i$, $i = 1, \dots, N_1$, therefore

$$\sum_{i=1,\dots,N_1} \tau_i < \xi_1 = K \frac{\delta}{2} \le \frac{\delta}{2|U|c}.$$

Then for any control $u_i(s)$, $i = 1, \dots, n$, defined on $[t_0, t]$ and $v(s)$ defined on $[t_0, t] \cap T$ we have inequalities

$$\begin{aligned}(\dot z_i(t), p) = (\dot z_i^0, p) + \int\limits_{[t_0,t]\cap T} a(s)(u_i(s) - v(s), p)\, ds + \\ + \int\limits_{[t_0,t]\backslash T} a(s)(u_i(s) - v_p(t), p)\, ds \le -2\delta + 2|U|c\mu(T) < \\ < -2\delta + 2|U|c\xi_1 \le -\delta, \quad t \in [t_0, +\infty), \quad i = 1, \dots, n. \end{aligned} \tag{26}$$

Therefore $(z_i(t' + \tau), p) = (z_i(t'), p) + \int\limits_{t'}^{t'+\tau} (\dot z_i(s), p)\, ds < (z_i(t'), p)$, $t' \ge t_0$, $\tau > 0$. And it means that approaching with the pursuer P_i, $i \in \{1, \dots, n\} \backslash I'$ will not occur. Without limiting a generality, we consider further that $N_1 = n$, i.e. approaching occurs with each pursuer.

Note that, if at the time instant $t = t'$ for some $i \in \{1, \dots, n\}$ we have the relations

$$\|z_i(t')\| = \delta_1^i, \quad (z_i(t'), p) > 0,$$

then for any controls $u_i(s)$, $v(s)$ defined on segment $[t', t' + \tau_1^i]$ we have

$$(z_i(t' + \tau_i^i), p) = (z_i(t'), p) + \int_{t'}^{t'+\tau_1^i} (\dot{z}_i(s), p)\, ds < \delta_1^i - \delta\tau_1^i = 0. \tag{27}$$

Let $\tau_1 = \tau_1^1$, $\delta_1 = \delta_1^1$. Then $t_1 > t_0$. Let us define the evasion maneuver recurrently. Let us have at the time instant $t = t_i$ we have the relations $\|z_i(t_i)\| = \delta_i$, $(z_i(t_i), p) > 0$, with defined number τ_i and monotonically decreasing sequences of positive numbers $\{\tau_i^\ell\}_{\ell=i}^\infty$, $\{\delta_i^\ell\}_{\ell=i}^\infty$.

Let us suppose $(z_i(t_i), \dot{z}_i(t_i)) = -\|z_i(t_i)\|\|\dot{z}_i(t_i)\|$. The number $\varepsilon_i \in (0, \tau_i)$ exists so that at arbitrary $u_\ell(s)$, $\ell = 1, \ldots, n$, $v(s)$, $s \in [t_i, t_i + \varepsilon_i]$, we have inequalities

$$\min_{\tau \in [0, \varepsilon_i]} \|z_\ell(t_i + \tau)\| > \delta_i^{i+1}, \quad (z_i(s), p) > 0. \tag{28}$$

The vectors $z_i(t_i)$, $\dot{z}_i(t_i)$ are linear dependent, therefore there is an unit vector ψ_i that

$$(\psi_i, z_i(t_i)) = (\psi_i, \dot{z}_i(t_i)) = 0.$$

We choose control $v_(s)$ on segment $[t_i, t_i + \varepsilon_i]$ so that

$$(\psi_i, v(s)) = \begin{cases} C(U, \psi_i), & (\psi_i, u_i(s)) \leq 0, \\ -C(U, -\psi_i), & (\psi_i, u_i(s)) > 0. \end{cases} \tag{29}$$

Then we have $\gamma_i \in (0, \varepsilon_i)$ so that

$$(z_i(t_i + \gamma_i), \dot{z}_i(t_i + \gamma_i)) \neq -\|z_i(t_i + \gamma_i)\|\|\dot{z}_i(t_i + \gamma_i)\|.$$

From (28) and inequality $(\dot{z}_i(t_i + \gamma_i), p) < -\delta$ we have linear independence of vectors $z_i(t_i+\gamma_i)$ and $\dot{z}_i(t_i+\gamma_i)$. If $(z_i(t_i), \dot{z}_i(t_i)) \neq -\|z_i(t_i)\|\|\dot{z}_i(t_i)\|$ then we assume that $\gamma_i = 0$.

According to the reasonings in the case 2, the evader control $v(s)$ on interval $[t_i + \gamma_i, t_i + \tau_i)$ it is necessary to consider $u_i(s)$ as equal.However if $i < n$ then on $[t_i + \gamma_i, t_i + \tau_i)$ approaching with pursuers P_i, $i + 1, \ldots, n$, might occur. Therefore

$$v(s) = u_i(s), \quad s \in [t_i + \gamma_i, t_i + \tau_i) \setminus \bigcup_{j=i+1,\ldots,n} [t_j, t_j + \tau_j),$$

if $i < n$ and

$$v(s) = u_n(s), \quad [t_n + \gamma_n, t_n + \tau_n).$$

Let $i < n$ and $t_\ell \in [t_i + \gamma_i, t_i + \tau_i)$, $\ell = i + 1, \ldots, n$. The evader will approach with pursuers $P_{i+1}, \ldots, P_n$ so close and bypass them for such a short period of time, that on the trajectory $z_i(t)$ on segment $[t_i + \gamma_i, t_i + \tau_i]$ at any controls $u_i(s)$, $s \in [t_i + \gamma_i, t_i + \tau_i]$ next relations holds true:

$$(z_i(t_i + \tau), \dot{z}_i(t_i + \tau)) \neq -\|z_i(t_i + \tau)\|\|\dot{z}_i(t_i + \tau)\|, \quad \tau \in [\gamma_i, \tau_i], \tag{30}$$

$$\min_{t \in [t_i + \gamma_i, t_i + \tau_i]} \|z_i(t)\| \geq \alpha_i, \tag{31}$$

$$\alpha_i > \delta_{i+1}. \tag{32}$$

From (32) we have, that approach of evader with each pursuer can occur no more than once.

Let $H_i(t_i+\gamma_i+\tau)$, $\tau \in \mathbb{R}^1$, be the line which is passing through points $z_i(t_i+\gamma_i)+\tau\dot{z}_i(t_i+\gamma_i)$ and $\dot{z}_i(t_i+\gamma_i)$. On the basis of linear independence of vectors $z_i(t_i+\gamma_i)$ and $\dot{z}_i(t_i+\gamma_i)$ we have at any τ vectors $z_i(t_i+\gamma_i)+\tau\dot{z}_i(t_i+\gamma_i)$ and $\dot{z}_i(t_i+\gamma_i)$ linearly are independent. Therefore at any τ we have

$$f(\tau) = \min_{x \in H_i(t_i+\gamma_i+\tau)} \|x\| > 0. \tag{33}$$

$$\Big(f(\tau) = \frac{\|\big((1-\tau)\|b\|^2 - (a,b)\big)\big(a + (\tau-1)b\big) + \|a+b\|^2 b\|}{\|a+b\|^2},$$
$$a = z_i(t_i+\gamma_i), \quad b = \dot{z}_i(t_i+\gamma_i)\Big)$$

What is more the function $f(\tau)$ is continuous.Let us define the number at the time instant $t = t_i + \gamma_i$

$$\beta_i = \min_{\tau \in [0, \tau_i - \gamma_i]} f(\tau). \tag{34}$$

If $v(s) = u_i(s)$ at $s \in [t_i+\gamma_i, t_i+\tau_i]$ then the corresponding trajectory is

$$z_i^\circ t = z_i(t_i+\gamma_i) + (t - t_i - \gamma_i)\dot{z}_i(t_i+\gamma_i)$$

and

$$\dot{z}_i^\circ(t) = \dot{z}_i(t_i+\gamma_i), \quad t \in [t_i+\gamma_i, t_i+\tau_i].$$

It is clear that at any $t \in [t_i+\gamma_i, t_i+\tau_i]$

$$\|z_i^\circ(t)\| \geq \beta_i, \quad \|\dot{z}_i^\circ(t)\| \geq \beta_i. \tag{35}$$

Let us assume that on segment $[t_i+\gamma_i, t_i+\gamma_i)$ we have the final system of intervals $[t^r, t^r+\tau^r)$, $r = 1, \ldots, \ell$, $\ell \geq 1$, such that

$$\mu(\bigcup_{r=1,\ldots,l} [t^r, t^r+\tau^r)) < \xi_{i+1}, \tag{36}$$

$$\xi_{i+1} = \min\left\{\sqrt{\tau_i^2 + \frac{\beta_i}{2|U|c}} - \tau_i, K\frac{\beta_i}{4}\right\}. \tag{37}$$

Let us show if

$$v(s) = u_i(s), \quad s \in [t_i+\gamma_i, t_i+\tau_i) \backslash \bigcup_{r=1,\ldots,\ell} [t^r, t^r+\tau^r),$$

control $v(s)$ is defined on the set $\bigcup_{r=1,\ldots,\ell} [t^r, t^r+\tau^r)$ and control $u_i(s)$ is defined on segment $[t_i+\gamma_i, t_i+\tau_i]$ are chosen arbitrarily, so corresponding trajectory $z_i^l(t)$, $t \in [t_i+\gamma_i, t_i+\tau_i]$, such that

$$\|z_i^\ell(t) - z_i^\circ(t)\| < \frac{\beta_i}{2}, \tag{38}$$

$$\|\dot{z}_i^\ell(t) - \dot{z}_i^\circ(t)\| \leq \frac{\beta_i}{2}, \tag{39}$$

Let $\ell = 1$. It is clear that

$$z_i^1(t^1) = z_i^\circ(t^1), \quad \|z_i^1(t^1+\tau^1) - z_i^\circ(t^1+\tau^1)\| \le |U|c(\tau^1)^2$$

and at $t^1 + \tau^1 < t \le t_i + \tau_i$

$$\|z_i^1(t) - z_i^\circ(t)\| \le |U|c((\tau^1)^2 + 2\tau^1(\tau_i - \tau^1)) < |U|c((\tau^1)^2 + 2\tau^1\tau_i).$$

Therefore at any $t \in [t_i + \gamma_i, t_i + \tau_i]$

$$\|z_i^1(t) - z_i^\circ(t)\| < |U|c((\tau^1)^2 + 2\tau^1\tau_i) < |U|c(\xi_{i+1}^2 + 2\xi_{i+1}\tau_i).$$

Analogically, for $\ell \in \mathbb{N}$, $t \in [t_i + \gamma_i, t_i + \tau_i]$ we have

$$\|z_i^\ell(t) - z_i^\circ(t)\| < |U|c(\xi_{i+1}^2 + 2\xi_{i+1}\tau_i) \le \frac{\beta_i}{2} \tag{40}$$

owing to relations (36), (37).

Thus the inequality (38) is proved. The inequality (39) implies from the definition of the number ξ_{i+1}.

Let us show for any $\tau \in [\gamma_i, \tau_i]$

$$(z_i^\ell(t_i+\tau), \dot z_i^\ell(t_i+\tau)) \ne -\|z_i^\ell(t_i+\tau)\|\|\dot z_i^\ell(t_i+\tau)\|. \tag{41}$$

Let us assume the opposite, there are

$$\tau_0 \in [\gamma_i, \tau_i], \quad q \in \mathbb{R}^1, \quad q > 0,$$

such that $z_i^\ell(t_i+\tau_0) = -q\dot z_i^\ell(t_i+\tau_0)$. Owing to inequalities (38), (39) we have

$$\min_{t_i+\gamma_i, t_i+\tau_i} \|z_i^\ell(t)\| \ge \frac{\beta_i}{2}, \tag{42}$$

$$z_i^\circ(t_i+\tau_0) \in z_i^\ell(t_i+\tau_0) + \frac{\beta_i}{2}S, \quad \dot z_i^\circ(t_i+\tau_0) \in \dot z_i^\ell(t_i+\tau_0) + \frac{\beta_i}{2}S, \tag{43}$$

i.e. vectors $z_i^\circ(t_i+\tau_0)$, $\dot z_i^\circ(t_i+\tau_0)$ can be shown as

$$z_i^\circ(t_i+\tau_0) = z_i^\ell(t_i+\tau_0) + x, \quad \dot z_i^\circ(t_i+\tau_0) = \dot z_i^\ell(t_i+\tau_0) + y,$$

where x, $y \in \frac{\beta_i}{2}S$, $S \subset \mathbb{R}^n$ full-sphere with the radius 1 and with center in the origin. Let

$$\Sigma = \{\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 | \alpha_1 + \alpha_2 = 1\}.$$

According to (34),

$$\min_{\alpha\in\Sigma} \|\alpha_1(z_i^\ell(t_i+\tau_0) + x) + \alpha_2(\dot z_i^\ell(t_i+\tau_0) + y)\| \ge \beta_i.$$

On the other side for $\alpha_1^* = \frac{1}{1+q}$, $\alpha_2^* = \frac{q}{1+q}$

$$\|\alpha_1^*(z_i^\ell(t_i+\tau_0) + x) + \alpha_2^*(\dot z_i^\ell(t_i+\tau_0) + y)\| =$$
$$= \|\alpha_1^* x + \alpha_2^* y\| \le \frac{\beta_i}{2}.$$

Therefore the inequality (41) holds true for any $\tau \in [\gamma_i, \tau_i]$.

Let us define sequence $\{\tau_{i+1}^{\ell}\}_{\ell=i+1}^{\infty}$ at the instant moment $t = t_i + \gamma_i$ on ξ_{i+1} in follows:

$$\tau_{i+1}^{i+1} = \min\{\tau_i^{i+1}, \frac{\xi_{i+1}}{2}\}, \quad \tau_{i+1}^{i+k} = \frac{\tau_{i+1}^{i+1}}{2^{k-1}}, \quad k \geq 2.$$

It is clear that

$$\sum_{k=1}^{n-i} \tau_{i+1}^{i+k} < \xi_{i+1}.$$

We construct new sequence on the basis of the previous sequence

$$\{\delta_{i+1}^{i+k}\}_{k=1}^{\infty}, \quad \delta_{i+1}^{i+k} = \delta\tau_{i+1}^{i+k}.$$

Let $\delta_{i+1} = \delta_{i+1}^{i+1}$, $\tau_{i+1} = \tau_{i+1}^{i+1}$. Then we have $\delta_{i+1} \leq \tau_{i+1}^{i+1} \leq \frac{\beta_i}{8}$. In respect that (42) we have inequalities (31), (32).

Thus we choose the control $v(s)$ which is defined at $s \in [t_i, t_i + \gamma_i]$ from relations (24) and

$$v(s) = u_i(s), \quad s \in [t_i + \gamma_i, t_i + \tau_i) \setminus \bigcup_{p=i+1,\dots,n} [t_p, t_p + \tau_p),$$

if $i < n$,

$$v(s) = u_n(s), s \in [t_n + \gamma_n, t_n + \tau_n).$$

Then we have $\|z_i(t)\| \neq 0$ at $t \in [t_i, t_i + \tau_i]$ and $(z_i(t), p) < 0$ at $t \geq t_i + \tau_i$, $i = 1, \dots, n$.

Note that the approaching of evader with pursuers can occur not later than

$$t_0 + \frac{\max\limits_{i=1,\dots,n} (z_i^0, p)}{\delta}.$$

The theorem is proved. $\square$

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Mathematical Model of Diffusion in Social Networks

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Abstract. Network of interacting agents whose actions are influenced by the actions of their neighbors according to the simple diffusion rule is considered. The paper examines some special types of the diffusion function for which the conditions of the network equilibrium are derived. Network equilibrium supposes that the network dynamics is in the stationary state, i.e. relative density of active agents is not changed. The numerical experiment for social network "vkontakte.ru" was made. Some statistical hypotheses about the type of connectivity distribution were checked.

Keywords: social networks, diffusion function, diffusion mechanism.

1. Introduction

Social network is a social structure characterized by the set of individuals and the set of relationships among these individuals through which social influence operates. Many social phenomena can be simulated with the help of social networks. For example, it could be diffusion of epidemics, innovations or products at the market. At present social networks uniting people either in real or virtual communication are widespread. Most people are users of the Internet as well as special social Internet projects such as www.facebook.com, www.vkontakte.ru (popular in Russia), www.livejournal.com, etc. Social networks attract great attention of market analysts whose main task is to find new effective strategies to have a market for goods and services. It is social networks that let spread information quickly from one network user to another, i.e. if a network user possesses some product or service, those network users who communicate with him can also have the same things with a high probability.

In Gubanov et al., 2009 the main idea is the dynamics of agents' opinions takes place in the social network. Authors model this dynamics by Markov chain. Also they represent the game-theoretical models of informational confrontation in social networks.

The present work is based on the paper López-Pintado, 2008. The dynamic model of social network represented in this work is supposed to know the diffusion mechanism namely the effective spreading rate of the product and the diffusion function. Stochastic dynamics of the social network can be determined as a continuous Markov chain but analytical results are too complicated to be represented. That is why we can use an approximation (with large population) of the dynamics described in Benaim and Weibull, 2003. So we apply the mean-field dynamics to model the diffusion dynamics in social networks.

2. Mathematical Model

The matematical model of the social network is graph $\Gamma = \langle N,L \rangle$, where $N = \{1, 2, \ldots, n\}$ is the set of agents and $L \subseteq N \times N$ is the set of links between agents. We assume that network Γ is undirected and doesn't contain reflexive links, i.e. if $(i, j) \in L$ then $(j, i) \in L$ for any $i, j \in L$, and $(i, i) \notin L$ for any $i \in L$.

Let $N_i = \{j \in N : (i, j) \in L\}$ be the set of individuals with whom agent i is linked. And $k_i = |N_i|$ is the number of neighbors of agent i which is also called agent's connectivity.

One of the main parameters of the network is its connectivity distribution.

Definition 1 (López-Pintado, 2008). The connectivity distribution $P(k)$ displays for each $k \geq 1$ the fraction of agents with connectivity k in the population:

$$\sum_{k=1}^{\infty} P(k) = 1. \tag{1}$$

We assume that all agents have at least one neighbor, i.e. $P(0) = 0$.

The following types of networks have great importance:

– Homogeneous networks, where all agents have the same connectivity:

$$P(\bar{k}) = 1 \tag{2}$$

for some $\bar{k} \geq 1$;

– Exponential networks, where the connectivity distribution is an exponential function:

$$P(k) \sim e^{-k}; \tag{3}$$

– Scale-free networks, where the connectivity distribution is a power function:

$$P(k) \sim k^{-\gamma}, \tag{4}$$

where $2 \leq \gamma \leq 3$.

The majority of complex social networks, such as the Internet, is characterized by scale-free connectivity distributions.

Consider the dynamic model of diffusion of a new product on the market. At time t the state of the system is described by n–dimensional vector $s_t = (s_{1t}, \ldots, s_{it}, \ldots, s_{nt}) \in S^n = \{0, 1\}^n$, consisting of zeroes and ones, where $s_{it} = 0$ if i is a susceptible agent at time t and $s_{it} = 1$ if i is an active agent at time t. A susceptible agent doesn't have the product but is capable of obtaining it if exposed to someone who does. An active agent already has the product and can influence neighbors in favor of adopting it.

Let an agent i be susceptible at time t. This agent becomes active at a rate $F(\nu, k_i, a_i)$, which depends on the spreading rate of the product $\nu \geq 0$, the connectivity k_i and the number of active neighbors $a_i = \sum\limits_{j \in N_i} s_j$.

Assume that the rate $F(\nu, k_i, a_i)$ can be represented as follows:

$$F(\nu, k_i, a_i) = \nu f(k_i, a_i), \tag{5}$$

where $f(k_i, a_i) \geq 0$, $(k_i, a_i) \in N \times N$, $0 \leq a_i \leq k_i$, $f(k_i, a_i)$ is non-decreasing function of a_i, which is called the diffusion function.

We assume that the reverse transition is also possible. An active agent becomes susceptible at some constant positive rate δ.

Define the effective spreading rate of the product by λ:

$$\lambda = \frac{\nu}{\delta}. \tag{6}$$

So we can determine the diffusion mechanism.

Definition 2 (López-Pintado, 2008). The diffusion mechanism is a pair $m = (\lambda, f(\cdot))$ where λ is the effective spreading rate and $f(\cdot)$ is the diffusion function.

Denote some new parameters of the network. Denote the relative density of active agents with connectivity k at time t as $\rho_k(t)$. And then $\rho(t)$ is the relative density of active agents at time t:

$$\rho(t) = \sum_k P(k)\rho_k(t). \tag{7}$$

Denote $\theta(t)$ as the probability that any given link points to an active agent at time t. Therefore, the probability that a susceptible agent with k neighbors has exactly a active neighbors at time t is determined in the following way:

$$p\{\xi = a\} = C_k^a \theta(t)^a (1 - \theta(t))^{k-a}. \tag{8}$$

And the transition rate from susceptible to active for an agent with connectivity k is the following:

$$\tilde{g}_{\nu,k}(\theta(t)) = \sum_{a=0}^{k} \nu f(k,a) C_k^a \theta^a(t)(1 - \theta(t))^{k-a}. \tag{9}$$

So the dynamic equation can be written as follows:

$$\frac{d\rho_k(t)}{dt} = -\rho_k(t)\delta + (1 - \rho_k(t))\tilde{g}_{\nu,k}(\theta(t)). \tag{10}$$

Equation (10) says that the variation of the relative density of active agents with k links at time t equals the proportion of susceptible agents with k neighbors at time t who becomes active minus the proportion of active agents with k neighbors at time t who becomes susceptible.

Further we will only consider systems in stationary state, that is ρ does not change in the course of time:

$$\frac{d\rho_k(t)}{dt} = 0. \tag{11}$$

3. Some Special Types of Diffusion Function

We have examined two different diffusion functions and obtained values of some parameters for these functions.

3.1. The Piece Linear Diffusion Function

We consider the piece linear diffusion function of the following form:

$$f(k,a) = \begin{cases} 0, & a \in [0; \frac{k}{3}) \\ 3\frac{a}{k} - 1, & a \in [\frac{k}{3}; \frac{2k}{3}) \\ 1, & a \in [\frac{2k}{3}; k] \end{cases} \tag{12}$$

Suppose that if the number of active agents is less than the third part of the total number of active agents then this part of agents can't influence the transition from susceptible to active agents. Similarly if 2/3 of total number of neighbors are active then the transition occurs with the rate ν. If the number of active agents varies between 1/3 and 2/3 of the total number of neighbors then $f(k,a)$ is a linear function.

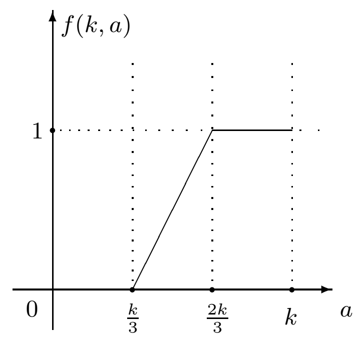

Fig. 1. Graph of the piece linear diffusion function.

To obtain the values of the main network parameters for diffusion function (12) we should consider three cases for each part of the function. We consider stationary state, so from equation (11) we can get the following equation for $\rho_k(\theta)$:

$$\rho_k(\theta) = \frac{g_{\lambda,k}(\theta)}{1 + g_{\lambda,k}(\theta)}, \tag{13}$$

where

$$g_{\lambda,k}(\theta) = \frac{\tilde{g}_{\nu,k}}{\delta}. \tag{14}$$

Calculate the values of $g_{\lambda,k}(\theta)$ for each part of function $f(k,a)$:

1. If $f(k,a) = 0$, then $g_{\lambda,k}(\theta) = 0$;
2. If $f(k,a) = 3\frac{a}{k} - 1$, then $g_{\lambda,k}(\theta) = \lambda(3\theta - 1 - (1-\theta)^k)$;
3. If $f(k,a) = 1$, then $g_{\lambda,k}(\theta) = \lambda(1 + (1-\theta)^k)$.

We use these values to find the value of $\rho_k(\theta)$. To find $\rho(\theta)$ we should use formula (7).

So we obtain the values of the main network parameters.

Proposition 1. *The relative density of active agents with connectivity k in the network with diffusion function (12) at the stationary state is*

$$\rho_k(\theta) = \begin{cases} 0, & a \in [0; \frac{k}{3}) \\ \frac{\lambda(3\theta-1-(1-\theta)^k)}{1+\lambda(3\theta-1-(1-\theta)^k)}, & a \in [\frac{k}{3}; \frac{2k}{3}) \\ \frac{\lambda(1+(1-\theta)^k)}{1+\lambda(1+(1-\theta)^k)}, & a \in [\frac{2k}{3}; k] \end{cases}; \tag{15}$$

and the relative density of active agents at the stationary state in this network is

$$\rho = \begin{cases} 0, & a \in [0; \frac{k}{3}) \\ \sum_k P(k)\rho_k = \sum_k P(k)\frac{\lambda(3\theta-1-(1-\theta)^k)}{1+\lambda(3\theta-1-(1-\theta)^k)}, & a \in [\frac{k}{3}; \frac{2k}{3}) \\ \sum_k P(k)\rho_k = \sum_k P(k)\frac{\lambda(1+(1-\theta)^k)}{1+\lambda(1+(1-\theta)^k)}, & a \in [\frac{2k}{3}; k] \end{cases}. \tag{16}$$

3.2. Diffusion Function with Exponential Part

Consider the diffusion function with exponential central part of the following form:

$$f(k,a) = \begin{cases} 0, & a \in [0; \frac{k}{3}) \\ \frac{1}{e^{\frac{2}{3}}-e^{\frac{1}{3}}}e^{\frac{a}{k}} - \frac{1}{e^{\frac{1}{3}}-1}, & a \in [\frac{k}{3}; \frac{2k}{3}) \\ 1, & a \in [\frac{2k}{3}; k] \end{cases} \tag{17}$$

This diffusion function is constructed similarly to the previous one. But if the number of active agents varies between 1/3 and 2/3 of the total number of agents then $f(k,a)$ is an exponential function.

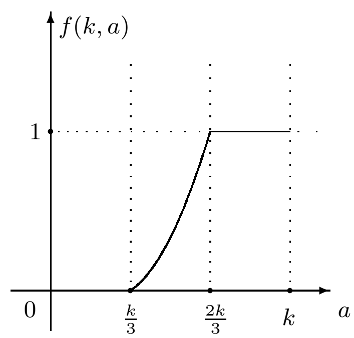

Fig. 2. Graph of diffusion function (17) with exponential central part.

For this function we obtain the values of the main network parameters.

Proposition 2. *The relative density of active agents with connectivity k in the network with diffusion function (17) at the stationary state is as follows:*

$$\rho_k(\theta) = \begin{cases} 0, & a \in [0; \frac{k}{3}) \\ \frac{x}{1+x}, & a \in [\frac{k}{3}; \frac{2k}{3}) \\ \frac{\lambda(1+(1-\theta)^k)}{1+\lambda(1+(1-\theta)^k)}, & a \in [\frac{2k}{3}; k] \end{cases}, \tag{18}$$

where $x = \frac{\lambda}{e^{\frac{2}{3}} - e^{\frac{1}{3}}}((1 - e^{\frac{1}{3}})(1-\theta)^k + \frac{1}{2}(1 - \frac{1}{k})\theta^2 + (1 + \frac{1}{2k})\theta + 1 - e^{\frac{1}{3}})$*;*
and the relative density of active agents at the stationary state of the network is

$$\rho = \begin{cases} 0, & a \in [0; \frac{k}{3}) \\ \sum_k P(k)\rho_k = \sum_k P(k)\frac{x}{1+x}, & a \in [\frac{k}{3}; \frac{2k}{3}) \\ \sum_k P(k)\rho_k = \sum_k P(k)\frac{\lambda(1+(1-\theta)^k)}{1+\lambda(1+(1-\theta)^k)}, & a \in [\frac{2k}{3}; k] \end{cases}. \quad (19)$$

4. Empirical Research of Social Network "vkontakte.ru"

We've made an empirical study of social network "vkontakte.ru". The aim of our research was to determine the connectivity distribution of the social network. For this purpose 306 agents of this network were interviewed. The number of friends of each interviewee in the network was found out.

As a result of the survey the following variational series were obtained:

$P(k)$	$[1;75)$	$[75;150)$	$[150;225)$	$[225;300)$
k	40	77	96	43

$P(k)$	$[300;375)$	$[375;450)$	$[450;525)$	$[525;600)$
k	20	18	6	6

(20)

After the analysis of the histogram (see Fig.3) of the obtained data two complex hypotheses about the type of connectivity distribution were suggested:

– The connectivity distribution is Poisson with probability of k occurrences

$$f(k, \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad (21)$$

where parameter λ is a positive real number;

– The connectivity distribution is Erlang with probability of k occurrences

$$f(k, A, B) = \begin{cases} \frac{1}{\Gamma(A)} x^{A-1} B^{-A} e^{-\frac{x}{B}}, & x \geq 0 \\ 0, & x \leq 0 \end{cases}, \quad (22)$$

where $A > 2$, $B > 0$, $\Gamma(A) = \int_0^{+\infty} t^{A-1} e^{-t}\, dt$.

We use Pearson criterion to check the first hypothesis and find the statistics by the following way:

$$\chi^2_{obs} = \sum_{i=1}^{\infty} \frac{(n_i - nf(i,\lambda))^2}{nf(i,\lambda)}. \quad (23)$$

For the considering network we estimate parameter

$$\lambda = \frac{589}{3}, \quad (24)$$

and get the following value of the statistics

$$\chi^2_{obs} = 3,10626 \cdot 10^{80}. \quad (25)$$

For the significance level $0,01$ we get the critical region $[\chi^2_{cr}; \infty) \approx [16,773; \infty)$. Since χ^2_{obs} is in the critical region we have to reject the hypothesis about Poisson distribution of the obtained data.

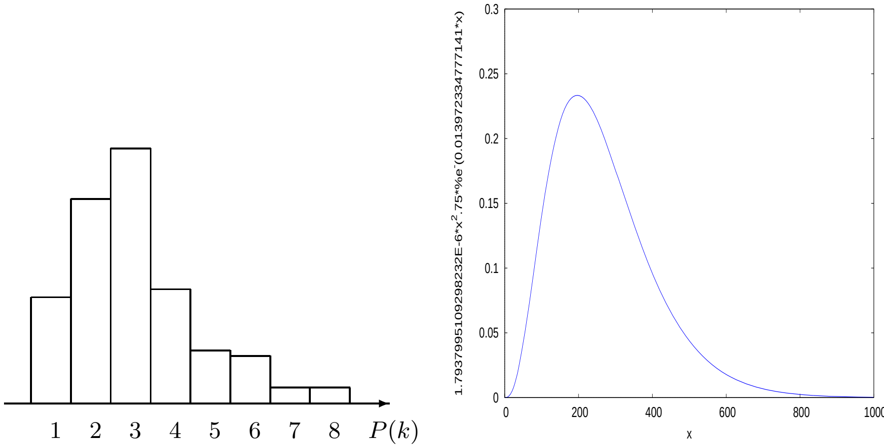

Fig. 3. The sample (20) histogram and graph of Erlang distribution with parameters $A = 3$ and $B = 72$.

We use Pearson criterion to check the second hypothesis using the method of moments to estimate parameters.

We consider $\eta = [\xi]$ as random variable, i.e. $\eta = l$ is equivalent to $l \leq \xi < l + 1$, $l = 1, 2, \ldots$ Then $P\{\eta = l\} = P\{l \leq \xi < l + 1\} = F_\xi(l + 1) - F_\xi(l)$, where $F_\xi(\cdot)$ is Erlang distribution function with parameters A, B.

Observed value of the statistics will be calculated by the following formula:

$$Z_{\text{obs}} = \frac{\bar{x} m_3}{2s^4} - 1,$$

where $m_3 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^3$ is the third central moment of the sample.

Critical region is defined as follows:

$$\left\{ Z : \quad Z > u(1-\alpha)\frac{3(A^*)^2 + 13A^* + 10}{2A^*\sqrt{n}} \right\},$$

where $u(1 - \alpha)$ is a quantile of order $(1 - \alpha)$ of the standard normal distribution, A^* is an estimation of Erlang distribution parameter A by the method of moments.

In this case we have the following value of statistics:

$$Z_{obs} = -0,29546. \tag{26}$$

The critical region is $[2,6412907; \infty)$, so we don't have any reasons to reject the hypothesis of Erlang distribution of the observed data.

Thereby the hypothesis of Poisson distribution was rejected and the hypothesis of Erlang distribution was accepted.

For this network we've considered the diffusion function $f(k, a) = (\frac{a}{k})^2$. As mentioned earlier, we can find $\rho_k(\theta)$ by using equation (13). If we consider diffusion function $f(k, a) = (\frac{a}{k})^2$, then

$$g_{\lambda,k}(\theta) = \frac{\lambda}{k}((k-1)\theta^2 + \theta). \tag{27}$$

Using formulas (28), (29) and (3) we can obtain the following results.

Proposition 3. *The relative density of active agents with connectivity k for the data* (20) *obtained from the network "vkontakte.ru" with diffusion function $f(k,a) = (\frac{a}{k})^2$ at the stationary state is the following:*

$$\rho_k(\theta) = \frac{\frac{\lambda}{k}((k-1)\theta^2 + \theta)}{1 + \frac{\lambda}{k}((k-1)\theta^2 + \theta)}, \tag{28}$$

and the relative density of active agents at the stationary state of the same network is

$$\rho(\theta) = \sum_k P(k) \frac{\frac{\lambda}{k}((k-1)\theta^2 + \theta)}{1 + \frac{\lambda}{k}((k-1)\theta^2 + \theta)}. \tag{29}$$

5. Conclusion

There are some open questions in our work. To determine the diffusion mechanism we have to know the effective spreading rate λ. It is suggested that this parameter is known to the researcher. But how will you determine this parameter? If you want to examine some social network you should make series of the empirical experiments to get more faithful type of connectivity distribution.

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Product Diversity in a Vertical Distribution Channel under Monopolistic Competition *

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Abstract. In Russia the chain-stores gained a considerable market power. In the paper we combine a Dixit-Stiglitz industry with a monopolistic retailer. The questions addressed are: Does the retailer always deteriorate welfare, prices and variety of goods? Which market structure is worse: Nash or Stackelberg behavior? What should be the public policy in this area?

Keywords: monopolistic competition, Dixit-Stiglitz model, retailer, Nash equilibrium, Stackelberg equilibrium, social welfare, Pigouvian taxation.

1. Introduction

In 2000-s, Russia and other developing markets of FSU have shown dramatic growth of chain-stores and similar retailing firms. Inspired by Wal-Mart and other successful giants abroad, Russian food traders like Perekrestok and Pyaterochka has gained considerable shares of the market and noticeable market power, both in Moscow and in province. This shift in the market organization was suspected by newspapers for negative welfare effects, accompanied by upward pressure on prices and inflation. Public interest to this question is highlighted by anti-chain-stores bill currently passed in Russian Parliament (Duma).

Leaving aside the empirical side of this question, this project focuses on constructing and analyzing the adequate model of vertical market interaction, suitable for Russian retailing markets of food, clothes and durables. We step aside from the traditional models of monopolistic or oligopolistic vertical interaction or franchising (see, e.g. review in (Tirole, 1990)), and stick instead to more modern monopolistic-competition representation of an industry in the Dixit-Stiglitz spirit (Dixit and Stiglitz, 1977), but combined with vertical interaction. This combination is rather new, being pioneered by Chen (Chen, 2004) and Hamilton and Richards (Hamilton and Richards, 2007), as described below in the literature review.

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Our departure from the former two papers is that it is the retailer who is exercising the monopsony power after market concentration, while the production is organized as monopolistic-competition industry with free entry. This hypothesis seems rather realistic, at least for developing markets. Indeed, there are numerous evidences in economic newspapers (see (Nikitina, 2006), (Sagdiev et al., 2006), (Slovak Republic, 2007), (FAS, 2007) cited in our literature review) that each of several big retailers has much stronger bargaining power than quite numerous manufacturers and importers of consumer goods like sausages, shirts, etc. (even such big international companies as Coca-Cola are not strong enough to enforce their terms of trade to Russian retailers).

To reflect such relations, our stylized model of market concentration considers a monopsonistic and monopolistic retailer (as a proxy of an oligopsonistic/oligopolistic retailer). This monopolistic intermediary deals with a continuum $[0, N]$ of Dixit-Stiglitz manufacturers on one side of two-sided market and a representative consumer on another. Each manufacturer has a fixed cost and a variable cost, he produces a single variety of the "commodity" and sets the price for this variety. The most natural timing of the model is the retailer's leadership, i.e., the retailer starts with announcing her markup policy correctly anticipating the subsequent manufacturers' responses, and simultaneously chooses the scope of varieties/firms to buy from. At the next stage the manufacturers come up with their prices and then the market clears the quantities. Both sides take into account the demand profile generated by a consumer's quasi-linear utility function. Another situation is when the retailer can also impose entrance fee on producers or/and consumers.

One or another monopolistic organization of the industry is compared to the pre-concentration situation where multiple shops operate. For simplicity, each variety is assumed to be sold through one shop. There can be three kinds of situations, ordered by increasing market power of the producers.

(1) The myopic (Nash) behavior of both manufacturers and retailer(s) who are unable to predict and influence the market. Which model is closer to reality, we tried to find out from empirical market papers (see literature review below), but did not come to a definite conclusion. So, both pre-concentration versions remain discussed.

(2) Leadership of manufacturers, who choose their wholesale prices correctly anticipating the best-response retailer's markup added later on.[1]

(3) Vertical integration from the side of producer, who owns the shop selling its variety or, equivalently, dictates the terms of trade and uses entrance fee or other tools to appropriate whole profit.[2]

The questions addressed in our paper are: Does the emergence of the monopsonistic retailer enhance or deteriorate welfare, and how much? Which retailer behavior is worse: usual monopoly or price discrimination with entrance fees? What should be the guidelines for public regulation (if any) in this area?

[1] Such interaction is somewhat similar to the concept of "common agency" considered in contract theory (see e.g. (Bernheim and Whinston, 1986)), but our leaders are in different position because each manufacturer is dealing with one or more small shops, knowing their costs. Instead of common agent (hypermarket) for all producers which should be assumed in "common agency" approach.

[2] This situation seems less realistic than other two, but used to compare with vertical integration from retailer's side.

In general the question is cumbersome, but definite results of this kind were obtained for the case of quasi-linear quadratic utility like in Ottaviano, Tabuchi and Thisse (Ottaviano et al., 2002), that means linear demands. For this model (in contrast with common wisdoms of politicians) it turns out under rather realistic assumptions that:

(i) Market concentration always *enhances* social welfare through softening the "double marginalization effect", this enhancement can work through lower prices and bigger consumption, or through adjusting the inefficiently low or inefficiently high number of varieties.

(ii) Under market concentration, further enforcement of the monopolist's market power by allowing for price discrimination (entrance fees) further *enhances* social welfare. Even the first-best (Pareto) optimum is guaranteed if the monopolist is able to use entrance fees on both sides: for producers and consumers. The latter situation turns out *equivalent* in welfare and (socially optimal) consumptions to the integrated monopoly, which means that one firm owns both production and retailing of the whole industry.

(iii) The governmental regulation of a simple monopolistic intermediary through capping the markup – *enhances* welfare. However, such regulation is *not needed* for price discriminating monopolist (even under entrance fees on manufacturing side only). If the government uses Pigouvian stimulation, then not taxes but *subsidies* to the monopsonistic retailer enhance welfare.

Additionally, in comparisons of various situations we tell what happens to prices, quantities and diversity under changing market organization.

To explain surprisingly positive influence of certain steps in increasing the retailer's market power, we can mention two general ideas. First, more market power softens the effects of double marginalization. Second, when the industry involves monopolistic competition, some sort of monopolistic behavior is present anyway. Therefore, the retailer's market power do not aggravate welfare losses of this kind, but instead internalize the externalities on the supply side. When the main decision-maker in the industry is powerful enough, she internalize them better, sometimes as good as the social planner would.

In the sections that follow we first describe more extensively the literature and motivation for our approach, then introduce the model and finally describe the results.

2. Literature Review

There is a broad literature on vertical interaction between a producer and a retailer, see reviews in (Perry, 1989) and (Rey, 2003); they study various economic consequences of such interaction. The early classical paper of Spengler (Spengler, 1950) explored the simplest case of Stackelberg game between two monopolists, one above the other, that entails "double marginalization." In essence, the second monopolist adds her own markup to the monopolistic price of the first one and further deteriorates social welfare, see (Tirole, 1990). A broad range of papers relaxed the Spengler's assumptions of homogeneous commodity, single producer and single retailer, which is natural and realistic. The difference among them lies mainly in various models of the oligopolists interaction. First of all, there is a strand of spatial Hotelling-type models (Hotelling, 1929) and another class of "representative consumer" models like Dixit-Stiglitz one (Dixit and Stiglitz, 1977). Among the spatial models, Salop (Salop, 2006) started with the circular city model with one producer and several retailers distributing themselves around this city to meet the demand of the continuous population. The main result is that under reasonable assumptions all consumers are served and the inefficiency found by Spengler disappears. Therefore there is no welfare motive for integration between the producer and the retailers. In contrast, Dixit (Dixit, 1983) modified this model to include production activity of "retailers" who used also other production factors. Then there is a welfare reason for integration, because it reducers the inefficiently-big number of retailers-producers and increases welfare. Further these ideas were developed in (Mathewson and Winter, 1983).

Another approach explores the idea of representative consumer in Dixit-Stiglitz manner. In particular, Perry and Groff (Perry and Groff, 1989) use constant-elasticity-of-substitution (CES) utility function of the consumer, defined on many discrete varieties, each produced by a single retailer. The main result is that such competition brings both distorted prices and distorted number of retailers, and interestingly, integration of two stages of production turns out welfare-deteriorating because the decrease in the varieties outweighs the decreasing prices. Here, like in Dixit's circular city, the retailers are the low-level producers also, modifying the commodity, otherwise a consumer buying from *all* retailers would look strange. Another step in this direction is Chen (Chen, 2004) where a multi-product monopolistic producer at the first stage chooses the number of varieties produced and therefore the number of retailers, because one-to-one correspondence remains. Afterwards he performs a bargaining procedure with each of them, and finally the markets clear taking into account the substitutability among the varieties. The number of differentiated goods turns out smaller than the constrained social optimum; the retailer's countervailing power lowers consumer prices but exacerbates the distortion in product diversity.

Finally, Hamilton and Richards (Hamilton and Richards, 2007) synthesize Hotelling-type and Dixit-Stiglitz-type models, i.e., the spatial-market and the differentiated-goods approaches in modeling the oligopoly competition among multi-product firms – "supermarkets," who face a competitive manufacturers. Two kinds of diversities are involved. There is a potentially large number of spatially-diversified supermarkets to choose from, but a consumer enjoys also a product-variety from the product line designed in a (single) supermarket chosen. It is found that increase in product differentiation on the manufacturers' side need not increase the equilibrium length of the product lines when retailers are specializing. Besides, under both no-entry and free-entry oligopoly conditions for retailers, product variety is undersupplied; several other effects are also found and the impact of excise taxation studied.

In contrast, in our setting, as we have mentioned, the monopolist is the retailer, but these are the producers who are organized as a Dixit-Stiglitz industry. To motivate such unusual approach, we can note that such or similar situation is rather typical for today Russian market. At least, there are many observations that modern supermarkets and hypermarkets have dominant bargaining power in relations with more numerous producers. For example, in (Nikitina, 2006) we read:

"... chain-stores ... capture the substantial part of profit of the small suppliers... These chains force suppliers to participate in various promotional actions, no matter whether such advertising or discounts are useful for suppliers or not... Overwhelming majority of the retailers require payments from suppliers, not only "payment for the shelf," but also so-called retro-bonus, i.e. "sale percentage." According to the agreements between suppliers and retailers, the latter receive 5 % on average ... Another "trap" for a supplier is the obligation to guarantee some sales volume. If the quantity sold turns out lower for some reasons, the supplier has to pay the difference between the planned and the actual sales... As a result, the total markup appropriated by a retailer can be from 30 to 50 % of the cost."

Similarly, in (Sagdiev et al., 2006) we can read:

"Retailers require from suppliers dozens thousands dollars only to start selling their goods... As the suppliers confess, the price of the "entrance ticket" depends on the producer's reputation and his spending for advertising. A retailer can make a discount when the manufacturer agrees to spend a lot on promotion of his commodities . . . The entrance ticket is not the only payment by a supplier wishing to enter a chain-store network. The total share going to the retailers can be about 35% of the consumer price."[3]

Another important citation can be found, e.g. in the website of the Federal Antimonopoly Service of Russian Federation (FAS, 2007): "Transformation of retail trade sector into big trade networks (retailers) allowed the latter, despite their seemingly small market share, dictate the rules of the game and determine the network entry conditions for suppliers and producers."

Such market power of the retailer in Russia is not surprising. The deficit of trading facilities is artificially aggravated by considerable corruption in trading land and in licensing trade in cities. As mentioned in McKinsey Global Institute survey of Russian retailing (Kaloshkina et al., 2009), during 2002-07 the sales of such big chain-stores as Eldorado, X5, Magnit, Metro Group, Auchan were growing by approximately 50% annually. However, the share of the market in the hands of all chain-stores still remains smaller than 50% in 2010.

This situation is not unique only for Russia. It is also, for instance, in some countries of East Europe, as shows the following citation from the site of Antimonopoly Office of Slovak Republic (Slovak Republic, 2007): "In view of the existing structure of the individual local markets, high barriers to entry (considerable direct and forced investments, sunk costs related to the required advertising and marketing support when entering the market, administrative barriers to entry, time necessary for entry into the market, and so forth), saturation of the individual relevant markets, and the nonexistence of potential competitors, if the concentration were carried out, the undertaking Tesco plc would establish or strengthen its dominant position. Consequently, the undertaking Tesco plc. would not be subject to substantial competition and, given its economic strength, it could act independently with respect to its suppliers, consumers, and competitors."

Besides, the Slovak case is not unique, see, e.g. (Lira et al., 2008) for the case of Chile.

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[3] The same article note that "...in international practice, a bonus payed for the entrance into the trading network is known, ... but in less scale... In Germany only small suppliers pay for starting to distribute goods through the supermarkets... In Great Britain the scheme "money for distribution" is not common, but retailer and supplier can promote the goods together..."

3. Model

We consider a monopolistic competition model modified to include the two-level interaction "manufacturer – retailer – consumer."

3.1. Demand

On consumers' side, there is a representative consumer, endowed with L units of labor supplied to the market inelastically, and labor is the only production factor. There are two types of goods in the economy. The first "commodity" consist of many varieties, for instance, milk of different brands. The second one is the numéraire representing other (perfectly competitive) goods. The income effect is neglected. A general-type quasi-linear utility function of any consumer reflects preferences over two kinds of goods:

$$U(\mathbf{q}, N, A) = V(\mathbf{q}, N) + A.$$

Here N is the length of the product line, reflecting the scope (the interval) of varieties; $q(i) \geq 0$ is "quantity" or the consumption of i-th variety chosen by any agent (consumer) and $\mathbf{q} = (q(i))_{i \in [0,N]}$ is the infinite-dimensional vector or function $q(\cdot) : [0, N] \to R$ describing the whole **profile** of varieties, all profiles keeping bold notation hereafter. Variable $A \geq 0$ is the consumption of the numéraire good. However, we use this general function V only to formulate the concepts of equilibria. Throughout we study a tractable special case, namely, the quadratic class of utility function formulated below. It was introduced by G.I.P. Ottaviano, T. Tabuchi and J.-F. Thisse (Ottaviano et al., 2002) (see also (Combes et al., 2008)) and became quite popular for modeling monopolistic competition:

$$U(\mathbf{q}, N, A) = \alpha \int_0^N q(i)di - \frac{\beta - \gamma}{2} \int_0^N [q(i)]^2 \, di - \frac{\gamma}{2} \left[\int_0^N q(i)di \right]^2 + A.$$

Here α, β and γ are some positive parameters, satisfying $\beta > \gamma > 0$, to have U quasi-concave. This condition ensures also that our consumer prefers larger diversity. In contrast, under $\beta = \gamma$, only the total quantity of consumption $Q = qN$ but not the diversity per se influences utility. Two quadratic terms here ensure strict concavity in two dimensions, i.e., definite consumer's choice among varieties and between the two sectors.

The main feature achieved by this three-term construction is that this utility generates the system of linear demands for each variety and linear demand for the whole differentiated sector.

To formulate the budget constraint, $\breve{p}(i)$ denotes the price of variety i for the consumer (which equals the wholesale price $p(i)$ when there is no retailer), $w \equiv 1$ is the wage rate in the economy, P_A is the price of the numéraire, also becoming 1 at the equilibrium.

Then the utility-maximization problem of the representative consumer takes the form

$$U(q, N, A) = V(q, N) + A \to \max_{(q,A)}$$

$$\int_0^N \breve{p}(i) q(i) di + P_A A \leq wL + \int_0^N \pi_{\mathcal{M}}(i) di + \pi_{\mathcal{R}},$$

where $\pi_{\mathcal{M}}(i)$ is the profit of i-th manufacturer, while $\pi_{\mathcal{R}}$ is the profit of retailer.

The budget constraint has a natural interpretation. Its right-hand side is Gross Domestic Product (GDP) of the economy with respect to income, while the left-hand side represents expenditures.

As to the solution to the consumer's problem, in the case when the income is sufficiently large, it does not influence the demand, due to quasi-linear utility, standardly.[4]

Therefore, for any price-profile $\mathbf{p} : [0, N] \mapsto R_+$, the individual demand-profile function $\mathbf{q}^*$ for all varieties is defined as follows (any "profile" is a function w.r.t. varieties' names $i \in [0, N]$ and bold letters $\mathbf{p}$, $\mathbf{q}$ denote such profiles, in contrast with certain points $\breve{p}(i), q(i)$ of the profiles)

$$\mathbf{q}^*(N, \mathbf{p}) = \arg\max_{\mathbf{q}} \left[V(q, N) - \int_0^N \breve{p}(i) q(i) di \right]. \tag{1}$$

Solving FOC of this problem in another direction for each price-profile, in Section 4.1 we derive the inverse demand function $p(i, q(i), N, \mathbf{p}_{-i})$ of any variety i for our quadratic utility. This function p describes how i-th consumer's price depends upon i-th quantity $q(i)$ and upon exogenous parameters: number N of competitors and the price profile $\mathbf{p}_{-i}$ including all prices but for i. This inverse-demand function is taken into account by producers, to be described now.

3.2. Supply

On the supply side there can be several different situations, modeled differently. Anyway manufacturers sell the goods only indirectly, through some retailer, but either they can interact myopically (Nash equilibrium) or one or another side of vertical relations may be the leader.

In all situations we denote by $p(i)$ wholesale price and by $r(i)$ the retailer's *markup* or price margin (we use these terms interchangeably). Therefore the ultimate i-th retailing price amounts to the sum $\breve{p}(i) = p(i) + r(i)$. Each manufacturer's cost function takes the following form: $C(q) = cq + F$. So, i-th manufacturer's profit-maximization problem takes the form

$$\pi_{\mathcal{M}}(i) = p(i) q(i, \mathbf{p} + \mathbf{r}) - C(q(i, \mathbf{p} + \mathbf{r})) \ \rightarrow \ \max_{p(i)},$$

where $\mathbf{p}$ is the wholesale price-profile for all producers and $\mathbf{p} + \mathbf{r}$ are the retail (consumer) prices; therefore, $\mathbf{r}$ is the retail markup profile. In the long run, free entry reduces profits to zero but at this stage profit maximizing determines price and quantity.

Now let us turn to the monopsonistic retailer (emerging after market concentration) or many retailers (existing before market concentration). Their number makes difference only when the retailer is big enough to control the market. In multi-retailer situation, we assume that one retailer may sell one variety or several, but each variety is sold through one retailer. As soon as markup optimization per each variety is similar, it is sufficient to describe the behavior of the "gross retailer"

[4] In the opposite case when the income is small, there can be a boundary solution when whole consumer's income is spent only for the diversified commodity. We, traditionally, ignore this case but for specifying some restrictions on parameters ensuring absence of income effect.

representing the whole population of them. Gross retailer's cost function is similar to that of the manufacturers:

$$C_{\mathcal{R}}(q) = \int_0^N p(i)q(i)di + \int_0^N c_{\mathcal{R}} q(i)di + \int_0^N F_{\mathcal{R}} di \ .$$

Here the first integral shows the expenditures to buy from the manufacturers, while the second and the third item show the retail costs: $c_{\mathcal{R}}$ is the number of labor units required from retailer to sell a unit of any differentiated good, $F_{\mathcal{R}}$ is the retailer's fixed cost (also measured in labor) required to start selling any differentiated good. Then the retailer's profit-maximization problem is

$$\pi_{\mathcal{R}} = \int_0^N [r(i) - c_{\mathcal{R}}] q(i, \mathbf{p} + \mathbf{r}) di - \int_0^N F_{\mathcal{R}} di \ \to \ \max_{\mathbf{r}},$$

where $r(i) = p_{\mathcal{R}}(i) - p(i)$ is the markup of the i-th goods variety.

We consider the following types of timing, or, rather, types of leader-follower relationship between the manufacturers and the retailer, ordered by decreasing retailer's bargaining power:

- Leading retailer, i.e., monopolistic competition with strategic behavior of retailer[5]:
  - at first the retailer chooses markups and the scope of product varieties, correctly anticipating the subsequent response of manufacturers;
  - then each manufacturer chooses, to enter the market or not and the wholesale price;
- Nash equilibrium, when manufacturers and retailer choose $\mathbf{p}$ and $\mathbf{r}$ simultaneously and myopically[6].
- Leading manufacturer, i.e., monopolistic competition with strategic behavior of manufacturers[7]:
  - at first all manufacturers simultaneously choose, to enter the market or not, and the wholesale prices (the number of firms is determined by the zero-profit condition), correctly anticipating the individually adjusted markup function;
  - then the retailer chooses the markup for each commodity (each manufacturer);

[5] Note that, in addition to the retailer presence, another distinction here from the standard monopolistic competition is that the equilibrium number of firms is *chosen* by the retailer, not by the free entry condition. This assumption is rather plausible, as shown by the following saying (Sokolov, 2006) of an owner of a retail supermarket about their policy concerning the diversity: "... for dried crusts ... we deal with a limited number of suppliers, since increasing the number of brands in such commodity groups does not lead to the increase in total sales... "

[6] The Nash variant of interaction seems less realistic than two leadership cases of interaction with more far-sighted behavior. But Nash interaction can be interesting as starting point to compare and estimate everybody's gains and losses from the strategic behavior.

[7] Though not as realistic as the previous concept, wise producers still can be a plausible approximation of reality. As we have mentioned in Introduction, this concept is, essentially, the solution concept in Common Agency models (the principals are manufacturers while the agent is retailer). The only distinction is the absence of information asymmetry among the participants.

– "No-retailer" equilibrium, when manufacturers enforce the terms of trade quite strongly, or each retailer of a variety just belongs to the manufacturer of related variety.

We formally express these four concepts in (most natural) symmetric case as follows.

Definitions. 1) Symmetric Nash-equilibrium is a quadruple $(p^{Nash}, r^{Nash}, q^{Nash}, N^{Nash}) \in R^4_+$, such that related price profile $\mathbf{p} = p(i) \equiv p^{Nash}$ solves each manufacturer's problem under external parameters $N^{Nash}, \mathbf{p}_{-i} \equiv p^{Nash}$, markup r^{Nash} maximizes the manufacturer profit under p^{Nash}, N^{Nash}, while $q(i) \equiv q^{Nash}$ solves the consumer's problem under $N^{Nash}, \mathbf{p} \equiv p^{Nash}$ and N^{Nash} satisfies the zero-profit condition.

2) Symmetric RL-equilibrium is a quadruple $(p^{RL}, r^{RL}, q^{RL}, N^{RL}) \in R^4_+$, such that related price function $\rho(i, r, \mathbf{p}_{-i}, N^{RL})$ describes the optimal response of each manufacturer to external markup r and parameters $N^{RL}, \mathbf{p}_{-i} \equiv p^{RL}$, markup r^{RL} and N^{RL} maximize the retailer's profit under this function ρ, $p^{RL} = \rho(i, r^{RL}, p^{RL}, N^{RL})$, while $q(i) \equiv q^{RL}$ solves the consumer's problem under $N^{RL}, \mathbf{p} \equiv p^{RL}$.

3) Symmetric ML-equilibrium is a quadruple $(p^{ML}, r^{ML}, q^{ML}, N^{ML}) \in R^4_+$, such that related markup function $\mu(i, p(i), \mathbf{p}_{-i}, N^{ML})$ describes the optimal response of the retailer to i-th price $p(i)$ and parameters $N^{ML}, \mathbf{p}_{-i} \equiv p^{ML}$, price p^{ML} maximize the manufacturer profit under this function μ, $r^{ML} = \mu(i, p^{ML}, p^{ML}, N^{ML})$, while $q(i) \equiv q^{ML}$ solves the consumer's problem under $N^{ML}, \mathbf{p} \equiv p^{ML}$ and N^{ML} satisfies the zero-profit condition.

4) NR-equilibrium. Each manufacturer owns the shop, bears joint fixed and variable costs of production and trade, $C(q) = (c + c_{\mathcal{R}})q + F + F_{\mathcal{R}}$, and maximizes the profit function

$$\pi_{\mathcal{NR}}(i) = p(i, q(i), N, \mathbf{p}_{-i})q(i) - C(q(i)) \ \rightarrow \ \max_{q(i)}.$$

To compare the social outcomes of these concepts of equilibria, we formulate the welfare function analogous to the previous one, but for modified retailing cost:

$$W = V\,(q, N) - \int_0^N (c + c_{\mathcal{R}})q(i)di - \int_0^N (F + F_{\mathcal{R}})di.$$

The symmetric solution q^{MaxW}, N^{MaxW} to this optimization program is called socially-optimal quantity and socially-optimal diversity.

Besides, we formulate the consumer surplus as

$$CS = V\,(q, N) - \int_0^N (p(i) + r(i))q(i)di.$$

Thus we have introduced the model and now start analyzing it.

4. Market Concentration

In this section we approach the question: Does market concentration enhance or deteriorate welfare? We consider two models of pre-concentration market: Nash behavior or strategic behavior of producers, leaving to the reader the choice of more realistic one and discussing other relevant models in the end of section. Both are compared to stylized model of extreme concentration: one retailer owns all shops and behaves as a leader, and to social optimum. Thus, we study:

– "no retailer" equilibrium (*NR*);
– strategic behavior of manufacturers-leaders (*ML*);
– Nash equilibrium (*Nash*);
– strategic behavior of retailer-leader (*RL*);
– socially-optimal quantity and socially-optimal length of the product line (*MaxW*).

4.1. Equilibria Characterization

Demand. Under this utility function, the consumer's problem can be written as following:

$$\alpha \int_0^N q(i)di - \frac{\beta-\gamma}{2} \int_0^N [q(i)]^2\, di - \frac{\gamma}{2} \left[\int_0^N q(i)di\right]^2 + A \to \max_{q,A}$$

$$\int_0^N p_{\mathcal{R}}(i)q(i)di + A \le L + \int_0^N \pi_{\mathcal{M}}(i)di + \pi_{\mathcal{R}}\ .$$

Here we analyse only the case when the consumer's income is sufficiently large and does not influence the consumption of the diversified products.

By standard technique (using the first order conditions, cf. (Ottaviano et al., 2002)) we can express the equilibrium retail price through the parameters as

$$p(i) + r(i) = \alpha - (\beta-\gamma)q(i) - \gamma \int_0^N q(j)dj\ , \quad i \in [0,N]. \tag{2}$$

Moreover, solving FOC in the opposite direction, we can obtain the linear demand function for any variety $i \in [0,N]$:

$$q(i) = a - (b+gN)[p(i)+r(i)] + gP, \tag{3}$$

where coefficients are

$$a = \frac{\alpha}{\beta+(N-1)\gamma}\ , \quad b = \frac{1}{\beta+(N-1)\gamma}\ , \quad g = \frac{\gamma}{(\beta-\gamma)[\beta+(N-1)\gamma]} \tag{4}$$

and price index

$$P = \int_0^N [p(j)+r(j)]dj$$

expresses the aggregate pricing behavior of all firms, negligibly influenced by firm i. In symmetric equilibrium $\bar{p}, \bar{q}, \bar{r}$ both above integrals can be simplified as

$$\int_0^N q(j)dj = N\bar{q},\ \ P = N(\bar{p}+\bar{r}).$$

Method of finding (*Nash*), (*ML*) equilibria and (*MaxW*). Using the demands system obtained, we calculate all kinds of market outcomes defined: (*Nash*), (*ML*) and socially-optimal (*MaxW*), and then (*RL*). All these are found in closed form in Result 2 below, proceeding as follows.

Solution *MaxW* is found directly from FOC. The method of calculating *Nash* is also rather usual: system of FOC for all participants is combined with the zero-profit condition, resulting in unique solution. For finding equilibrium *ML*, we first derive the markup function $r_i(p(i), P, N)$ which is the best-response of the retailer (a small shop) to any price $p(i)$ (P, N given). Then we optimize prices of the manufacturers having this markup function, and adding the zero-profit condition get the unique equilibrium.

Method of finding ($\boldsymbol{RL}$) equilibrium. More complicated is calculating equilibrium RL. Here, having in mind demands $q(i, r(i), N, P)$ and optimal price policies $p^*(i, r(i), N, P)$ as functions of markup $r(i)$ and diversity N, the retailer *jointly* chooses profile $\mathbf{r} = (r(i))_{i \in N}$ and N as the solution to her program in the form

$$\pi_{\mathcal{R}} = \int_0^N [r(i) - c_{\mathcal{R}}] q(i, \mathbf{p}^*(\mathbf{r}, N) + \mathbf{r}) di - \int_0^N F_{\mathcal{R}} di \ \rightarrow \ \max_{\mathbf{r}, N},$$

$$\pi_{\mathcal{M}}(p^*(i, r(i), N, P), r(i), N) \geq 0 \ .$$

If we, reasonably, assume only symmetric variables $\bar{p}, \bar{q}, \bar{r}$ (bar accent denoting symmetrization, in particular $\bar{p}^*(\bar{r}, N) = p^*(i, \bar{r}, N, N\bar{p}) \ \forall i$) this retailer's problem is simplified as

$$\pi_{\mathcal{R}} = N[\bar{r} - c_{\mathcal{R}}]\bar{q}(\bar{p}^*(\bar{r}, N) + \bar{r}) - NF_{\mathcal{R}} \ \rightarrow \ \max_{\bar{r}, N},$$

$$\pi_{\mathcal{M}}(\bar{p}^*(\bar{r}, N), \bar{r}, N) \geq 0 \ .$$

Remark that in symmetric RL case one has the following expressions q and p as functions of r and N:

$$q = q(r, N) = \frac{(b + gN)[a - b(c + r)]}{2b + gN} \tag{5}$$

$$p = p(r, N) = \frac{a + (b + gN)c - br}{2b + gN} \tag{6}$$

Strictly speaking, now we have stepped aside from the monopolistic competition model, since N need not be necessarily determined by the free entry or zero-profit condition. More specifically, there can be two types of solution, or regimes that we call *artificially restricted market* or *un-restricted* market. The first case means non-negative-profit constraint occurring non-binding, i.e., optimization of the unconstrained function $\pi_{\mathcal{R}}$ resulting in positive manufacturers' profit $\pi_{\mathcal{M}} > 0$. In this case the retailer really ignores "free entry" condition imposing instead her own restriction on entry (so, we also ignore free entry within calculations). Otherwise retailer first use "free entry" condition to calculate $N(\bar{r})$ as a function of $\bar{r}$, then maximize her profit with respect to $\bar{r}$. As we have found by direct calculations, the *artificially restricted* solution happens if and only if some crucial constant exceeds one:

$$\mathcal{F} = \frac{F_{\mathcal{R}}}{2F} \geq 1.$$

Thus, postponing its further economic discussion, we can formulate now

Result 1 *It is profitable for the retailer-leader to artificially restrict the entry, when her fixed cost exceeds doubled manufacturer's fixed cost.*

This constant $\mathcal{F}$ denoting *relation* of retailing fixed costs over manufacturing fixed cost plays hereafter an important role. To concisely formulate the characterization of all our solutions, we introduce the following auxiliary notations (some of them being interpreted economically in special section):

$$\beta_{-\gamma} = \beta - \gamma, \;\; q_{NE} = \sqrt{\frac{F}{\beta_{-\gamma}}} \;, \quad \mathcal{F} = \frac{F_{\mathcal{R}}}{2F} \;, \;\; \tilde{D} = \sqrt{\beta_{-\gamma}} \cdot \frac{\alpha - c - c_{\mathcal{R}}}{\sqrt{F}} = \frac{\alpha - c - c_{\mathcal{R}}}{q_{NE}} \;.$$

With these notations, omitting the intermediate calculations, the formulae characterizing equilibria through exogenous parameters are summarized in the next table. The types of competition (equilibria concepts) are shown in rows, while the equilibrium values of the variables are in columns, the separate column showing the optimal value of welfare function W.

Result 2 *The market outcomes under several market organizations and parameters are characterized as*

	quantity q	*price p*	*markup r*
$Nash$	q_{NE}	$c + q_{NE} \cdot \beta_{-\gamma}$	$c_{\mathcal{R}} + q_{NE}\frac{\tilde{D} - \beta_{-\gamma}}{2}$
ML	$\frac{\sqrt{2}}{2} \cdot q_{NE}$	$c + q_{NE} \cdot \sqrt{2} \cdot \beta_{-\gamma}$	$c_{\mathcal{R}} + q_{NE}\frac{\tilde{D} - \beta_{-\gamma}\sqrt{2}}{2}$
$RL, \mathcal{F} \geq 1$	$q_{NE} \cdot \sqrt{\mathcal{F}}$	$c + q_{NE} \cdot \beta_{-\gamma}\sqrt{\mathcal{F}}$	$c_{\mathcal{R}} + q_{NE}\frac{\tilde{D}}{2}$
$RL, \mathcal{F} \leq 1$	q_{NE}	$c + q_{NE} \cdot \beta_{-\gamma}$	$c_{\mathcal{R}} + q_{NE}\left(\frac{\tilde{D}}{2} + \beta_{-\gamma}(\mathcal{F} - 1)\right)$
$MaxW$	$q_{NE}\sqrt{2 + 4\mathcal{F}}$	$-$	$-$

	diversity N	*consumer surplus CS*
$Nash$	$\frac{1}{2\gamma} \cdot \left(\tilde{D} - 3\beta_{-\gamma}\right)$	$\left(\tilde{D} - 3\beta_{-\gamma}\right) \cdot \left(\tilde{D} - \beta_{-\gamma}\right) \cdot \frac{F}{8 \cdot \gamma\beta_{-\gamma}}$
ML	$\frac{\tilde{D} - 2\sqrt{2}\beta_{-\gamma}}{\sqrt{2}\gamma}$	$\left(\tilde{D} - 2\sqrt{2}\beta_{-\gamma}\right) \cdot \left(\tilde{D} - \sqrt{2}\beta_{-\gamma}\right) \cdot \frac{F}{8 \cdot \gamma\beta_{-\gamma}}$
$RL, \mathcal{F} \geq 1$	$\frac{\frac{\tilde{D}}{\sqrt{\mathcal{F}}} - 4\beta_{-\gamma}}{2\gamma}$	$\left(\tilde{D} - 4\sqrt{\mathcal{F}}\beta_{-\gamma}\right) \cdot \left(\tilde{D} - 2\sqrt{\mathcal{F}}\beta_{-\gamma}\right) \cdot \frac{F}{8 \cdot \gamma\beta_{-\gamma}}$
$RL, \mathcal{F} \leq 1$	$\frac{\tilde{D} - 2\beta_{-\gamma}(\mathcal{F} + 1)}{2\gamma}$	$\left(\tilde{D} - 2(\mathcal{F} + 1)\beta_{-\gamma}\right) \cdot \left(\tilde{D} - 2\mathcal{F}\beta_{-\gamma}\right) \cdot \frac{F}{8 \cdot \gamma\beta_{-\gamma}}$
$MaxW$	$\frac{1}{\gamma}\left(\frac{\tilde{D}}{\sqrt{2+4\mathcal{F}}} - \beta_{-\gamma}\right)$	$-$

	welfare W
$Nash$	$\left(\tilde{D} - 3\beta_{-\gamma}\right) \cdot \left(\tilde{D} - \frac{3+8\mathcal{F}}{3} \cdot \beta_{-\gamma}\right) \cdot \frac{3F}{8 \cdot \gamma\beta_{-\gamma}}$
ML	$\left(\tilde{D} - 2\sqrt{2}\beta_{-\gamma}\right) \cdot \left(\tilde{D} - \frac{\sqrt{2} \cdot (3+8\mathcal{F})}{3} \cdot \beta_{-\gamma}\right) \cdot \frac{3F}{8 \cdot \gamma\beta_{-\gamma}}$
$RL, \mathcal{F} \geq 1$	$\left(\tilde{D} - 4\sqrt{\mathcal{F}}\beta_{-\gamma}\right) \cdot \left(\tilde{D} - \frac{4+6\mathcal{F}}{3\sqrt{\mathcal{F}}} \cdot \beta_{-\gamma}\right) \cdot \frac{3F}{8 \cdot \gamma\beta_{-\gamma}}$
$RL, \mathcal{F} \leq 1$	$\left(\tilde{D} - 2\beta_{-\gamma}(\mathcal{F} + 1)\right) \cdot \left(\tilde{D} - \frac{4+6\mathcal{F}}{3} \cdot \beta_{-\gamma}\right) \cdot \frac{3F}{8 \cdot \gamma\beta_{-\gamma}}$
$MaxW$	$\left(\tilde{D} - \sqrt{2 + 4\mathcal{F}}\beta_{-\gamma}\right)^2 \cdot \frac{F}{2 \cdot \gamma\beta_{-\gamma}}$

	retailer's profit $\pi_{\mathcal{R}}$	*manufact. profit* $\pi_{\mathcal{M}}$
$Nash$	$\left(\tilde{D} - 3\beta_{-\gamma}\right)\left(\tilde{D} - \beta_{-\gamma}(1+4\mathcal{F})\right) \cdot \frac{F}{4\cdot\gamma\beta_{-\gamma}}$	0
ML	$\left(\tilde{D} - 2\sqrt{2}\beta_{-\gamma}\right)\left(\tilde{D} - 2\sqrt{2}\beta_{-\gamma}\cdot\frac{1+4\mathcal{F}}{2}\right) \cdot \frac{F}{4\cdot\gamma\beta_{-\gamma}}$	0
$RL, \mathcal{F} \geq 1$	$\left(\tilde{D} - 4\sqrt{\mathcal{F}}\beta_{-\gamma}\right)^2 \cdot \frac{F}{4\cdot\gamma\beta_{-\gamma}}$	$\frac{F_{\mathcal{R}}}{2} - F$
$RL, \mathcal{F} \leq 1$	$\left(\tilde{D} - 2\beta_{-\gamma}(\mathcal{F}+1)\right)^2 \cdot \frac{F}{4\cdot\gamma\beta_{-\gamma}}$	0

Here in every equilibria cases one has $W = CS + N \cdot \pi_{\mathcal{M}} + \pi_{\mathcal{R}}$.

Moreover, one has for NR *case:*

	quantity q	*price* p	*markup* r	*diversity* N	W
NR	$q_{\mathcal{NR}}$	$c + c_{\mathcal{M}} + f_{\mathcal{M}}$	–	$(D_{\mathcal{M}} - 2)\,\frac{\beta_{-\gamma}}{\gamma}$	$\left(D_{\mathcal{M}}^2 - 3 \cdot D_{\mathcal{M}} + 2\right) \cdot H_{\mathcal{M}}$

where

$$q_{\mathcal{NR}} = \sqrt{\frac{F + F_{\mathcal{R}}}{\beta_{-\gamma}}}\ ,\ f_{\mathcal{M}} = \sqrt{(F + F_{\mathcal{R}}) \cdot (\beta_{-\gamma})}\ ,$$

$$D_{\mathcal{M}} = \frac{\alpha - c - c_{\mathcal{R}}}{\sqrt{(F + F_{\mathcal{R}}) \cdot (\beta_{-\gamma})}}\ ,\ H_{\mathcal{M}} = \frac{(F + F_{\mathcal{R}}) \cdot (\beta_{-\gamma})}{2 \cdot \gamma}\ .$$

Remark that although prices and profits are not relevant for $MaxW$ case, we can calculate these values "artificially". Indeed, as in the case of perfect competition, "assigned consumer price" $(p + r)$ equals total marginal costs $(c + c_{\mathcal{R}})$. Therefore, total "assigned profit" of retailer and all producers is negative (due to fixed costs), i.e. always less than in all another market structures (ML, $Nash$, RL). Hence, obviously, "assigned consumer surplus" is bigger than welfare ($CS^{MaxW} > W^{MaxW}$). So this consumer surplus is bigger than consumer surplus in all another market structures.

Total quantity of consumption $Q = qN$ also can be easily derived from q, N.

Before comparing these equilibria we should discuss the interpretation of the parameters.

4.2. Interpretation of Parameters $\mathcal{F}$, q_{NE} and $\tilde{D}$

First, let us interpret the important parameter $\mathcal{F} = \dfrac{F_{\mathcal{R}}}{2F}$, named so far as "prevalence of retailing fixed cost over the manufacturing fixed cost." It would be helpful if we intuitively located this parameter in some area greater than 1 or smaller, not to study all mathematically feasible cases, but unfortunately, it is not easy.

What could $\mathcal{F}$ mean in reality? In the model, both constants $F_{\mathcal{R}}, F$ relate to whole production of any variety. So, their relation can be perceived as relation of shares of fixed cost within the price of a single variety. Up to our sketch data about the retail prices of Russian food items, the retailer's markup is usually between 20% and 40% with average about 25% of the ultimate price, the rest 75% going to the manufacturer. Thus, if we believe in similar (about 15-20%) profitability in manufacturing and retailing, total cost-share $\tilde{c}$ of manufacturer in the price of a commodity is about 4 times bigger than the retailer's cost-share $\tilde{c}_R$. Both costs can be divided into a fixed part modeled here as $F/q(i)$ and a variable cost, i.e., $\tilde{c} = c + F/q(i)$, $\tilde{c}_R = c_R + F_R/q(i)$. Can F_R be bigger than F when $\tilde{c}/\tilde{c}_R \approx 4$?

At the first glance, it is quite improbable, because on the manufacturing side the fixed cost includes essential capital expenses (like renting buildings), which can be about 1/2 of total cost. However, in the long run capital becomes a variable cost. Then the "fixed cost," more or less independent of the production volume, amounts mainly to advertising and intellectual capital of a company: analytics and main specialists with their accumulated knowledge. This cost can be below 10% of total cost. It is unclear, is such share similar or much bigger in retailing. Here "fixed cost," spent on each variety, more or less independently of the retailing volume, amounts to maintaining the shell devoted to this variety, costs of bargaining with the manufacturer, of book-keeping this item, and certain advertising efforts of the shop. Naturally, selling a bigger diversity is more costly, under the same volume of sells. We would say, that if it is more costly almost proportionally to N, then the share of fixed cost $F_R/q(i)$ in retailer's cost $\tilde{c}_R$ is close to 1. Elaborating this idea further, we can note that our fixed-plus-linear cost model $F + cq$ is only a linearization of some real cost functions $C(q)$ and $C_R(q)$. To make a good approximation, we need not ask a manufacturer about her capital stock and advertising expenses. Instead, we should ask about her current production $\bar{q}$ and total cost (the point $(\bar{q}, C(\bar{q}))$ of linearization) and about the derivative $C^{'}(\bar{q}) = c$: "How costly it could be to increase production for 10-15% or how much you can economize from a decrease." Then our "fixed cost" parameter F could be calculated as $F = C(\bar{q}) - \bar{q}C^{'}(\bar{q})$. The same for retailer. In this view, concavity or convexity of real cost functions can make F, F_R bigger or smaller, positive or negative, with small respect to expenses like advertising cost. Summarizing this discussion, we understand what kind of real data would be helpful to calibrate our stylized model and reduce the area of search for realistic effects. But we are unable to calibrate it now even approximately, so, all values of $\mathcal{F}$ are analyzed below. It turns out that this relation between manufacturing and retailing fixed cost essentially change the character of behavior in the industry.

Having explained this, another parameter $q_{NE} = \sqrt{F/\beta_{-\gamma}}$ is explained as the quantity of individual variety emerging at Nash equilibrium (Nash serves as a reference point for other equilibria). We observe that quantity q_{NE} depends only on the manufacturer's fixed cost and consumer's variety-loving, being independent of the retailer's parameters. Observe that the bigger is the entry barrier of fixed cost, the larger market space is captured by a single producer, that looks reasonable. Besides, the stronger is variety-loving $\beta_{-\gamma}$, the smaller is each manufacturer's share of the market at the Nash equilibrium, that also looks reasonable. The table shows that similar reactions to these parameters $F, \beta_{-\gamma}$ are demonstrated also by equilibria types ML and RL, only the individual quantity itself can be bigger or smaller than at Nash.

We observe also that wholesale prices at various regimes of competition include production cost and some additional term proportional to the individual quantity, which depends positively upon the fixed cost. Analogously is expressed the retailer's markup.

Different is the formula for the diversity N, which positively (and linearly) correlates with parameter $\tilde{D}$, named so especially to highlight this correlation.

$$\tilde{D} = \sqrt{\beta_{-\gamma}} \frac{\alpha - c - c_{\mathcal{R}}}{\sqrt{F}} = \frac{\alpha - c - c_{\mathcal{R}}}{q_{NE}}$$

In the numerator there is the "interval for feasible prices" in this industry, i.e., the chocking price α (upper limit on the willingness to pay) minus the marginal cost of production and retailing $c + c_{\mathcal{R}}$. If this magnitude is zero, the industry cannot survive. The greater it is, the bigger is the pie of potential wealth that can be split among the players in the game and deadweight loss. Besides there is the "preference for diversity" parameter $\beta_{-\gamma}$. Naturally, $\tilde{D}$ and the diversity both increase when the numerator increases. The increasing denominator which is our "fixed cost of manufacturing" F brings the opposite natural effect. The smaller is F, the more competitive is production in the industry (entry is cheaper). Thereby, $\tilde{D}$ describes the welfare potential of the industry multiplied by some specific measures of manufacturing competitiveness and love for diversity.

On the other hand, looking on expression $\frac{\alpha-c-c_{\mathcal{R}}}{q_{NE}}$ we see that a market where individual quantity is bigger would have smaller diversity than another one, that means total quantity $Q = Nq$ being more rigid with respect to changing parameters than individual quantity and diversity.

There can be also interpretation of $\tilde{D}$ in terms of socially-efficient quantity $q^{MaxW} = q_{NE}\sqrt{2+4\mathcal{F}}$ obtained already and *the minimal-profit quantity* q^{Min} that guarantees nonnegative total profit to retailer and any manufacturer, found from non-negative profit condition $(\alpha - c - c_{\mathcal{R}})q^{Min} - (F + F_{\mathcal{R}}) \geq 0$.

Combining these, we find that $\tilde{D} = \beta_{-\gamma}\sqrt{\mathcal{F}+\frac{1}{2}} \cdot \frac{q^{MaxW}}{q^{Min}}$. Now we see that the potential of the industry to generate profit also positively affect $\tilde{D}$ and, respectively, the diversity, but instead of price-type expression $\alpha - c - c_{\mathcal{R}}$ of this potential we have the quantity-type expression $\frac{q^{MaxW}}{q^{Min}}$. Again, preference for diversity $\beta_{-\gamma}$ affects $\tilde{D}$ and the diversity positively.

In addition, the above formula allows to linearly express the socially-optimal diversity N^{MaxW} through the potential of the industry: $\frac{\tilde{D}}{\beta_{-\gamma}\sqrt{\mathcal{F}+\frac{1}{2}}} = \frac{2\gamma}{\beta_{-\gamma}} N^{MaxW} + 2 = \frac{q^{MaxW}}{q^{Min}}$, that once again highlights tight connection of parameter $\tilde{D}$ with diversity.

4.3. Market Concentration Impact on Welfare, Quantities, Prices and Diversity

Now we are ready to compare the outcomes of equilibrium types. Here we are mostly interested in the welfare effects caused by market "monopolization" by big chain-stores, after situation modeled as RL emerged.

Which model or equilibrium concept better describes situation in Russia before the concentration that occurred in 2000-s: NR, $Nash$ or ML? Definitely, retailers did exist before chain-stores and each had certain degree of *local* market power in a city district. So, for our comparison, we reject NR model which assumes monopolistic competitive retailers[8]. To choose between $Nash$ and ML models, we guess, did manufacturers exercise some market power in bargaining with each retailer, or did they react myopically to given markup r in the market? The first hypothesis seems most plausible, and occupy our main attention, but the second hypothesis is also analyzed, with similar outcomes.

A puzzle to be resolved when interpreting these comparisons, is identifying our *single* retailer in both models ($Nash$ or ML) with *multiple* retailers in reality. Whenever we believe in local market power remaining approximately the same af-

[8] This case can also be interpreted as "manufacturer's outlet", i.e each producer has the own "manufacturer's outlet" store. Of course, this interpretation is valid under proper definition of marginal $(c + c_{\mathcal{R}})$ and fixed $(F + F_{\mathcal{R}})$ costs.

ter retailing property concentration, the main change could be in costs. Probably, in Russia the costs of selling did not change too dramatically because of ownership concentration itself (leaving aside other causes). Of course, chain-stores economized somewhat on unified book-keeping, unified marketing specialists and, notably, on unified logistics. Still, we prefer to ignore this change and do model the selling cost function $F_R + cq_R$ as remaining the same during concentration, for two reasons. First, we want to analytically separate the market-concentration effect from economies of scale in retailing. Second, since the welfare effect of concentration is found being positive, our conclusion should be only *enforced* if we take into account these economies of scale also.

Effects of Switching from $\boldsymbol{ML}$ to $\boldsymbol{RL}$. In addition to these precautions, we also should combine inequalities $N^{ML} \geq 0, N^{RL} \geq 0$ with equilibria characterization and get the following conditions on combinations of parameters $\mathcal{F}$ and $\tilde{D}/\beta_{-\gamma}$ ensuring non-empty market:

$$\begin{array}{rl} \tilde{D}/\beta_{-\gamma} \geq 4 \cdot \sqrt{\mathcal{F}} & \text{if } \mathcal{F} \geq 1, \\ \tilde{D}/\beta_{-\gamma} \geq \max\{2 \cdot \sqrt{2},\ 2 \cdot (\mathcal{F}+1)\} & \text{if } \mathcal{F} < 1. \end{array} \tag{7}$$

Now, under these restrictions, we can formulate the main results on the market concentration modeled as switching from ML to RL regimes.

Welfare, Consumer Surplus, Retailer surplus. Algebraic manipulations with the equilibria formulae obtained yield the unambiguous welfare conclusion:[9]

Restricted market: $\mathcal{F}= \frac{F_{\mathcal{R}}}{2F} > 1 \ \Rightarrow$	$W^{ML} < W^{RL}$
Unrestricted market: $\mathcal{F}= \frac{F_{\mathcal{R}}}{2F} \leq 1 \Rightarrow$	$W^{ML} \leq W^{RL}$

Result 3 *Generally there are positive social welfare gains from market concentration, similar in restricted and non-restricted markets.*

To comment on this result, the welfare increase may seem quite counter-intuitive from the first glance. Normally one expects social losses from market concentration, and exactly such anti-trust reasons stay behind recent proposal by Russian Duma of the new law restricting chain-stores. Why any gains may occur instead of losses expected?

To answer, we first give general reasons, and then look carefully on quantity and diversity changes during the concentration. Generally, it is a textbook result in IO that a two-tier monopoly causes higher deadweight loss than a simple monopoly. So, when a two-tier monopoly becomes vertically integrated through ownership concentration, it can be quite beneficial for society. In our setting something similar occurs under horizontal concentration, only the changes in the diversity complicate the picture. Indeed, when a leader-manufacturer monopolistically optimizes her price and has in mind the optimal response of the (locally) monopolistic retailer, this situation is rather similar to the two-tier monopoly. In contrast, when the monopsonistic (and monopolistic also) retailer exercise essential market power over the manufacturers, it is somewhat similar to vertical concentration, bringing the decision-making into essentially *one* hands. In this general view, the welfare gains are not too surprising.

As to consumer surplus CS, the situation is not so simple. The complete solution is given below.

[9] Equality $W^{ML} = W^{RL}$ in case $\mathcal{F} \leq 1$ occurs as a degenerate case, only under specific parameters $\frac{\tilde{D}}{\beta_{-\gamma}} = 2\sqrt{2}$ and $\mathcal{F} = \sqrt{2} - 1$.

Result 4 *During switching from ML to RL regimes, welfare is changing, depending on three regions of parameters, as follows:*

Parameter regions	*consumer surplus SC*
$D > \frac{4\cdot(\mathcal{F}^2+\mathcal{F}-1)}{4\mathcal{F}+2-3\sqrt{2}}, \mathcal{F} \leq \frac{3\sqrt{2}-2}{4}$	$CS^{RL} > CS^{ML}$
$D < \frac{4\cdot(\mathcal{F}^2+\mathcal{F}-1)}{4\mathcal{F}+2-3\sqrt{2}}$	$CS^{RL} < CS^{ML}$

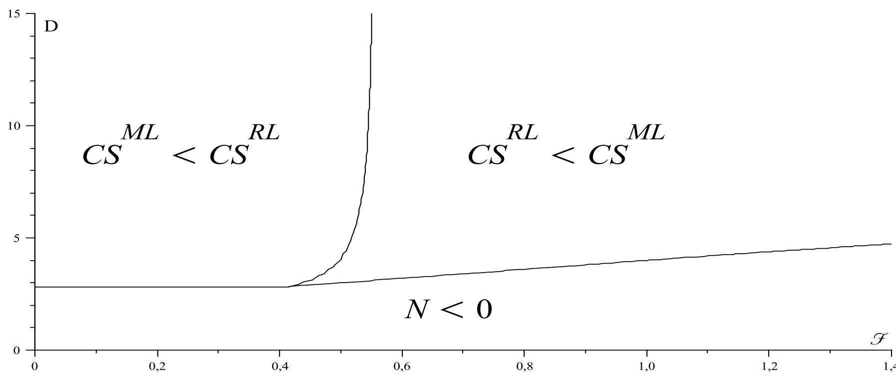

Fig. 1. Comparison of Consumer Surplus between ML and RL regimes.

Figure 1 illustrates these two cases of market-concentration effects in consumer surplus[10]. Below these regions either N^{ML} or N^{RL} is negative, so the model is inappropriate.

Finally, comparing the retailer's surplus (i.e. retailer' profit), we can conclude that the situation is obvious and similar to the one for welfare (see Result 2): if the market exists (i.e. N^{ML} and N^{RL} are non-negative) then the retailer's profit is bigger when retailer is leader then when she is follower: $\pi_{\mathcal{R}}^{RL} > \pi_{\mathcal{R}}^{ML}$.

Result 5 *Generally there are positive profit gains from market concentration, similar in restricted and non-restricted markets.*

Now we should look on market concentration more specifically: are the gains in consumption volume, or in the diversity, or in profits (recall that our representative consumer is the owner of all firms) responsible for benefits to the society?

Quantities, prices and diversity. We again perform algebraic manipulations with equilibria formulae to find regions of parameters for several inequations of interest that could explain the welfare gains: $q^{ML} <_? q^{RL}$ (increase in the consumption of a single variety), $Q^{ML} <_? Q^{RL}$ (increase in the total quantity of consumption), $p^{ML} + r^{ML} >_? p^{RL} + r^{RL}$ (decrease in the retail price), $N^{ML} <_? N^{RL}$ (increase in the diversity). Using again non-empty market condition (7), we get

Result 6 *During switching from ML to RL regimes, prices, quantities and diversity are changing, depending on four regions of parameters, as follows* [11]*:*

[10] In the figures we use the notation $D = \tilde{D}/\beta_{-\gamma}$.

[11] Note that the last region of parameters includes both restricted ($1 < \mathcal{F}$) and unrestricted markets.

Parameter regions	Quantities	Retail prices	Diversity
$\mathcal{F} \le \min\{\sqrt{2}-1, 1-\frac{\tilde{D}/\beta_{-\gamma}}{2+2\sqrt{2}}\}$	$q^{ML} < q^{RL}$ $Q^{ML} < Q^{RL}$	$p^{ML}+r^{ML} \ge p^{RL}+r^{RL}$	$N^{ML} \le N^{RL}$
$1-\frac{\tilde{D}/\beta_{-\gamma}}{2+2\sqrt{2}} < \mathcal{F} \le \sqrt{2}-1$	$q^{ML} < q^{RL}$ $Q^{ML} < Q^{RL}$	$p^{ML}+r^{ML} \ge p^{RL}+r^{RL}$	$N^{ML} > N^{RL}$
$\sqrt{2}-1 < \mathcal{F} \le \frac{\sqrt{2}}{2}$	$q^{ML} < q^{RL}$ $Q^{RL} < Q^{ML}$	$p^{ML}+r^{ML} \ge p^{RL}+r^{RL}$	$N^{ML} > N^{RL}$
$\mathcal{F} > \frac{\sqrt{2}}{2}$	$q^{ML} < q^{RL}$ $Q^{RL} < Q^{ML}$	$p^{ML}+r^{ML} < p^{RL}+r^{RL}$	$N^{ML} > N^{RL}$

Moreover, in comparison with $MaxW$, during switching from ML to RL regimes, diversity is changing, depending on four regions of parameters, as follows:

Parameter regions	Diversity
1. $2\sqrt{2} < \tilde{D}/\beta_{-\gamma} < D_1^{ML}$	$N^{ML} < N^{RL} < N^{MaxW}$
2. $D_1^{ML} < \tilde{D}/\beta_{-\gamma} < D_2^{ML}$	$N^{RL} < N^{ML} < N^{MaxW}$
3.1. $\frac{1}{2} < \mathcal{F} < 1,\quad D_2^{ML} < \tilde{D}/\beta_{-\gamma} < D_3^{ML}$	$N^{RL} < N^{MaxW} < N^{ML}$
3.2. $\mathcal{F} > 1,\quad \tilde{D}/\beta_{-\gamma} < D_4^{ML}$	$N^{RL} < N^{MaxW} < N^{ML}$
4.1. $\frac{1}{2} < \mathcal{F} < 1,\quad \tilde{D}/\beta_{-\gamma} > D_3^{ML}$	$N^{MaxW} < N^{RL} < N^{ML}$
4.2. $\mathcal{F} > 1,\quad \tilde{D}/\beta_{-\gamma} > D_4^{ML}$	$N^{MaxW} < N^{RL} < N^{ML}$

where $D_1^{ML} = 2\cdot(1+\sqrt{2})(1-\mathcal{F}),\quad D_2^{ML} = \sqrt{2}\left(1+\frac{1}{\sqrt{1+2\mathcal{F}}-1}\right),$

$$D_3^{ML} = 2\mathcal{F}\left(1+\frac{\sqrt{2}}{\sqrt{1+2\mathcal{F}}-\sqrt{2}}\right),\quad D_4^{ML} = 2\sqrt{\mathcal{F}(1+2\mathcal{F})}\left(\sqrt{1+2\mathcal{F}}+\sqrt{2\mathcal{F}}\right).$$

Figure 2 and Figure 3 illustrate these cases of market-concentration effects in prices, quantities and diversities described in the both tables of the Result 6.

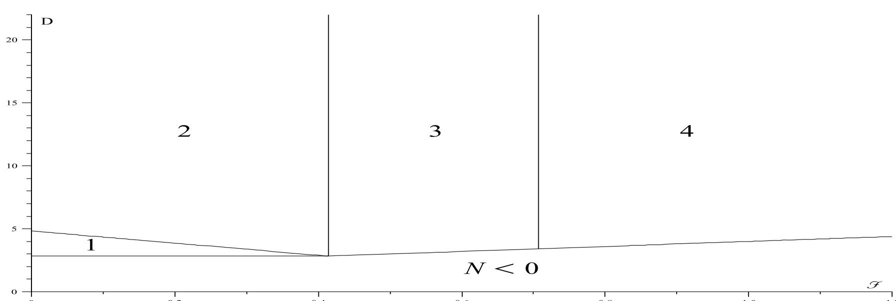

Fig. 2. Comparison between ML and RL regimes in quantities, prices and diversity:

Region 1:	$Q^{ML} < Q^{RL}$	$p^{RL}+r^{RL} < p^{ML}+r^{ML}$	$N^{ML} < N^{RL}$
Region 2:	$Q^{ML} < Q^{RL}$	$p^{RL}+r^{RL} < p^{ML}+r^{ML}$	$N^{RL} < N^{ML}$
Region 3:	$Q^{RL} < Q^{ML}$	$p^{RL}+r^{RL} < p^{ML}+r^{ML}$	$N^{RL} < N^{ML}$
Region 4:	$Q^{RL} < Q^{ML}$	$p^{ML}+r^{ML} < p^{RL}+r^{RL}$	$N^{RL} < N^{ML}$

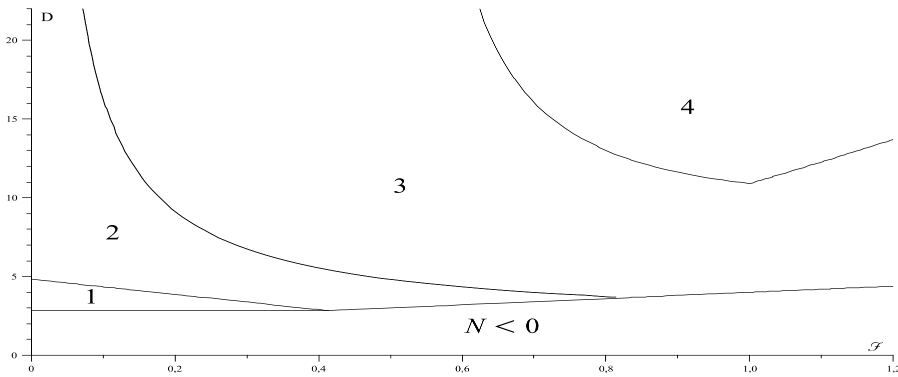

Fig. 3. Comparison between ML and RL regimes and $MaxW$ in diversity:

Region 1:	$N^{ML} < N^{RL} < N^{MaxW}$
Region 2:	$N^{RL} < N^{ML} < N^{MaxW}$
Region 3:	$N^{RL} < N^{MaxW} < N^{ML}$
Region 4:	$N^{MaxW} < N^{RL} < N^{ML}$

To discuss these results, the first question of interpretation is: Why, depending upon parameters $\mathcal{F}$ and $\tilde{D}$, it turns out more profitable for the retailer to shift the equilibrium in one or another direction when she obtains the monopsonistic power? Before obtaining it, each retailer takes the diversity and wholesale price as given when choosing the markup, now the unique retailer forecasts her influence on these variables. The more is relation $\mathcal{F}$ of retailer's fixed cost to manufacturer's cost, the more is their foregone joint profit when the free entry determines the excessive (from the profit viewpoint) diversity. This the explanation why fraction N^{RL}/N^{ML} becomes smaller and smaller with increasing $\mathcal{F}$ in Fig.2. Prices and quantities adjust to this general tendency.

Second question is: How, by which logical ties, the social welfare increases through these changes in quantities, prices and the diversity? Generally there are three variables pleasing the consumer: total quantity Q, diversity N and profit of the retailer, owned by the gross consumer (plus zero profit from manufacturers). To this end, in this quasi-linear setting we can reason in terms of usual Marshallian diagram. After symmetrization of each equilibrium, the maximal possible utility gains can be described by the simplified inverse-demand function:

$$V(Q,N) = \alpha - \left(\gamma + \frac{\beta - \gamma}{N}\right) Q.$$

We can see that the chocking price α remains the same, but increasing diversity N can stretch the demand triangle rightwards, the right corner approaching the limiting value γ/α. In this respect, increasing N enables to increase the common pie of welfare, divided among the consumer, the deadweight-loss and profit ultimately also belonging to consumer. On the other hand, increasing N increases average cost $\bar{c} = c + c_R + (F + F_R)N$. Geometrically, it shifts up the effective cost line and thus reduces the common pie of welfare. The socially optimal diversity N^{MaxW} expresses the good balance between these two forces. In contrast, any market equilibrium brings distortion in two respects: (1) non-optimal diversity N and (2) monopolistically reduced total quantity, which brings a triangle of deadweight loss.

Trying to apply this logic, first we observe in Table 3 that at all equilibria the total quantity consumed is strictly less than the socially-optimal quantity, so social loss is *always* present, and the found positive impact on welfare can be understood as minimizing this loss. Decrease in the diversity is also *always* present, which can be beneficial when the diversity is excessive.

Besides, *generally* shift in prices and total quantity during the market concentration have the opposite sign. Only in case $\sqrt{2}-1 < \mathcal{F} \leq 1/\sqrt{2}$ it seems unnatural: the quantity is growing, when the price is growing, resembling the Giffen effect, probably this effect could be explained through the equilibrium number of manufactures, which decreases.

Further, individual quantities *always* increase under market concentration, confirming our general reasoning about concentration impact on a single variety: the strategic behavior of the manufacturers, similar to two-tier monopoly, yields lower total quantity than the strategic behavior of the retailer.

Other equilibrium variables including total quantities behave more complicatedly. Under small $\mathcal{F}$ (relation of retailing fixed cost to manufacturing one) total quantity Q grows, alike individual quantities. Here the welfare gains can be explained by growing quantity. But it is not the case under large immoderate $\mathcal{F}$. Here both total quantity and diversity go down during market concentration, especially when the retailer starts restricting the entry. So, here the welfare gains can be either because of reducing too big initial diversity N^{ML} (of excessive number of producers) or because of growth of profits, ultimately going to the consumer.The retail prices $(p + r)$ grow during market concentration under large $\mathcal{F}$, but decrease under moderate or small $\mathcal{F}$, but they have only indirect impact utility, through transferring the consumer surplus into profits.

As to the diversity, or the number of manufactures, under large and moderate relation $\mathcal{F}$, it decreases. Excessive diversity can be harmful, because requiring too many producers and excessive total fixed costs. In particular, at ML regime the diversity can be excessive, so, reducing the diversity during market concentration can add to positive quantity effect under small $\mathcal{F}$. In the case of very big $\mathcal{F}$ such reduction is intentionally forced by the restrictive policy of the manufacturer and can be stronger.

We conclude from this subsection that changes in equilibrium variables explain some mechanisms of welfare gains during market concentration.

Effects of Switching from *Nash* to *RL*. For more complete discussion, we perform now the $Nash \to RL$ version of modeling the market concentration, to show that the effects are somewhat different.

Taking again the precautions $N^{Nash} \geq 0, N^{RL} \geq 0$ we get the conditions on combinations of parameters $\mathcal{F}$ and $\tilde{D}$ ensuring non-empty markets:

$$\begin{aligned} \tilde{D}/\beta_{-\gamma} &\geq 4 \cdot \sqrt{\mathcal{F}} && \text{if } \mathcal{F} \geq 1, \\ \tilde{D}/\beta_{-\gamma} &\geq \max\{3,\ 2 \cdot (\mathcal{F}+1)\} && \text{if } \mathcal{F} < 1. \end{aligned} \tag{8}$$

Now, under these restrictions, we can formulate the main results on concentration modeled as switching from $Nash$ to RL regimes.

Welfare. Algebraic manipulations with the equilibria formulae obtained yield, using again non-empty market condition (8), the unambiguous welfare conclusion:

Result 7 *During switching from Nash to RL regimes, welfare is changing, depending on three regions of parameters, as follows:*

Parameter regions	*Welfare*
$\mathcal{F} < \frac{1}{2}$	$W^{Nash} < W^{RL}$
$\left(\frac{9\cdot\sqrt{\mathcal{F}}}{2} + \frac{1}{\sqrt{\mathcal{F}}-3-2\cdot\mathcal{F}}\right)\cdot\tilde{D}/\beta_{-\gamma} > \frac{7}{4}$, $\mathcal{F} > \frac{1}{2}$	$W^{Nash} > W^{RL}$
$\left(\frac{9\cdot\sqrt{\mathcal{F}}}{2} + \frac{1}{\sqrt{\mathcal{F}}-3-2\cdot\mathcal{F}}\right)\cdot\tilde{D}/\beta_{-\gamma} < \frac{7}{4}$	$W^{Nash} < W^{RL}$

For the consumer surplus CS, the situation is more simple[12]*:*

Parameter regions	*consumer surplus SC*
1. Small: $\mathcal{F} < \dfrac{1}{2}$	$CS^{Nash} < CS^{RL}$
2. Large: $\mathcal{F} > \dfrac{1}{2}$	$CS^{RL} < CS^{Nash}$

Figure 4 illustrates these cases of market-concentration effects in welfare and consumer surplus.

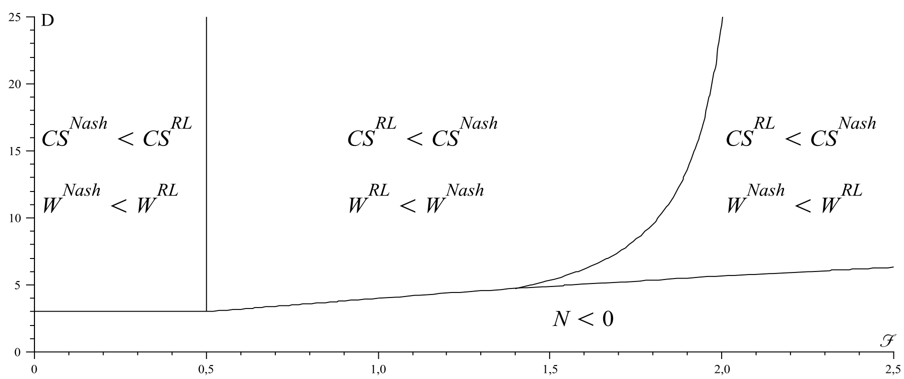

Fig. 4. Comparison between $Nash$ and RL regimes in welfare and consumer surplus.

We see that welfare increases during switching from Nash regime to RL regime only under big or small relative fixed cost $\mathcal{F}$, but decreases under moderate one; while consumer surplus increases during the switching only under small relative fixed cost $\mathcal{F}$. As an interpretation of the Figure 4, remark that, since in the middle region $W^{\mathcal{N}\dashv\int\langle} > W^{\mathcal{RL}}$, $Nash$ outcome here is more "competitive" than outcome under the leadership of retailer.

Quantities, prices and diversity. We again perform algebraic manipulations with equilibria formulae to find regions of parameters for several inequations of interest: $q^{Nash} <_? q^{RL}$ (increase in the consumption of a single variety), $Q^{Nash} <_? Q^{RL}$ (increase in the total quantity of consumption), $p^{Nash} + r^{Nash} <_? p^{RL} + r^{RL}$ (increase in the retail price), $N^{Nash} <_? N^{RL}$ (increase in the diversity).

Using again non-empty market condition (8), we get

[12] Cf. the case of switching from ML to RL, see the Table in Result 2 and Figure 1.

Result 8 *During switching from $Nash$ to RL regimes, prices, quantities and diversity are changing, depending on two regions of parameters, as follows:*

	Quantities	*Retail prices*	*Diversity*
$\mathcal{F} \leq \frac{1}{2}$	$q^{Nash} = q^{RL},\ Q^{Nash} < Q^{RL}$	$p^{Nash} + r^{Nash} > p^{RL} + r^{RL}$	$N^{Nash} < N^{RL}$
$\mathcal{F} > \frac{1}{2}$	$q^{Nash} < q^{RL},\ Q^{RL} < Q^{Nash}$	$p^{Nash} + r^{Nash} < p^{RL} + r^{RL}$	$N^{Nash} > N^{RL}$

Moreover, in comparison with $MaxW$, during switching from $Nash$ to RL regimes, diversity is changing, depending on four regions of parameters, as follows:

Parameter regions	*Diversity*
1. $\mathcal{F} < \frac{1}{2},\quad \tilde{D}/\beta_{-\gamma} > 3$	$N^{Nash} < N^{RL} < N^{MaxW}$
2. $\mathcal{F} > \frac{1}{2},\quad \tilde{D}/\beta_{-\gamma} < D_1^{RL}$	$N^{RL} < N^{Nash} < N^{MaxW}$
3.1. $\frac{1}{2} < \mathcal{F} < 1,\ D_1^{RL} < \tilde{D}/\beta_{-\gamma} < D_2^{RL}$	$N^{RL} < N^{MaxW} < N^{Nash}$
3.2. $\mathcal{F} > 1,\ \tilde{D}/\beta_{-\gamma} < D_3^{RL}$	$N^{RL} < N^{MaxW} < N^{Nash}$
4.1. $\frac{1}{2} < \mathcal{F} < 1,\ \tilde{D}/\beta_{-\gamma} > D_2^{RL}$	$N^{MaxW} < N^{RL} < N^{Nash}$
4.2. $\mathcal{F} > 1,\ \tilde{D}/\beta_{-\gamma} > D_3^{RL}$	$N^{MaxWL} < N^{RL} < N^{Nash}$

where $D_1^{RL} = 1 + \frac{1}{\sqrt{\mathcal{F}+\frac{1}{2}}-1},\ D_2^{RL} = 2\mathcal{F}\left(1 + \frac{\sqrt{2}}{\sqrt{1+2\mathcal{F}}-\sqrt{2}}\right),$

$$D_3^{RL} = 2\sqrt{\mathcal{F}\cdot(1+2\mathcal{F})}\left(\sqrt{1+2\mathcal{F}} + \sqrt{2\mathcal{F}}\right).$$

Figure 5 illustrates these four cases of market-concentration effects on quantities, prices and diversity. Remark that in Figure 3 and Figure 5 the fourth regions are the same.

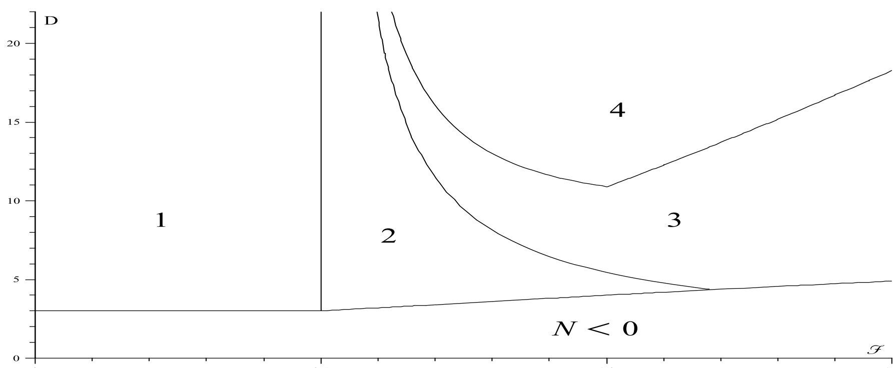

Fig. 5. Comparison between $Nash$ and RL regimes and $MaxW$ in quantities, prices and diversity:

Region 1:	$Q^{Nash} < Q^{RL}$	$p^{RL} + r^{RL} < p^{Nash} + r^{Nash}$	$N^{Nash} < N^{RL} < N^{MaxW}$
Region 2:	$Q^{RL} < Q^{Nash}$	$p^{Nash} + r^{Nash} < p^{RL} + r^{RL}$	$N^{RL} < N^{Nash} < N^{MaxW}$
Region 3:	$Q^{RL} < Q^{Nash}$	$p^{Nash} + r^{Nash} < p^{RL} + r^{RL}$	$N^{RL} < N^{MaxW} < N^{Nash}$
Region 4:	$Q^{RL} < Q^{Nash}$	$p^{Nash} + r^{Nash} < p^{RL} + r^{RL}$	$N^{MaxW} < N^{RL} < N^{Nash}$

To interpret these regions, we should say that, like in previous figures, the more is fraction $\mathcal{F} = F_R/F$ between the retailer's and the manufacturers cost, the *more is* *the need to diminish the excessive variety*, determined by the free entry without the respect for the retailer's losses from the excessive diversity (free entry pays respect only to F). When switching to RL regime, the retailer becomes capable to influence this excess (from her viewpoint) diversity directly or indirectly.

We did not compare No-retailer equilibrium with RL-equilibrium because we do not suppose NR case realistic enough. Never substantial share of the retailing market was covered by the shops belonging to producers. Instead we use this model as theoretical reference point: vertical integration of the industry performed from above, from the producer's side. It is compared in the next section to vertical integration from below, from the retailer's side.

Generally, in this section we have seen that a conclusion about welfare benefits of market concentration, switching from one or another previous industry organization may depends upon demand functions and technologies, but welfare gains from market concentration is quite plausible. Therefore, in spite of noticeable public complaints about the retailer's bargaining power (mentioned in Intro), this power and leadership in relations with manufacturers can occur socially desirable. In this case an economist should not advice a governmental intervention against chain-stores and similar practices, recently suggested by Duma. Section 6 devoted to regulation adds arguments in the same direction.

Possible extensions of the setting. We conclude this section devoted to market concentration effects by the question: Which is the most realistic model of pre-concentration retailing? Instead of two alternative models that we could use several other versions. First, we could study a market with direct selling by manufacturers without any retailing shops, but it seems less realistic (as suggested by discussants in our presentations). Second, the case when manufacturers have bigger market power than retailing sector, seems irrelevant to pre-concentration regime in Russia and similar countries, because several manufacturers jointly controlling common retailer seems unrealistic.

It means that manufacturers behave simultaneously but strategically towards the retailing sector and consumers, percept as their common agent in principal-agent relations, and considering externalities onto each other. It seems not a good model for our question. Indeed, before market concentration, typical manufacturer of consumer goods in Russia sold to several shops shared with several other suppliers. For big manufacturers the markup was the result of individual bargaining with a shop (like in ML equilibrium), for small ones it was a market constant (like in $Nash$ equilibrium). Anyway, a common-agency story seems irrelevant. Instead, a more sophisticated and realistic description of concentration would like to undertake two detailizations of the model.

(1) We can consider big and small manufacturers or/and retailers, describing concentration as appearance of more and more big chain-stores among the competitive fringe of small shops.

(2) In addition, spatial structure of the market should be described somehow to explain why in reality the *same* varieties are sold for much smaller prices in chain-stores outside the city center than in small shops in the center.

However, both extensions wait for another paper.

5. Entrance Fees

Now we describe an extension of our setting motivated as follows. In Intro we have mentioned that trade relations between the manufacturers and the retailers often involves more complex terms than just a markup. Rather typically, to start and maintain selling anything through a chain-store, a manufacturer is forced to pay an entrance fee to the retailer, annually or monthly. Denoting this per-period fee as F_E we can look on it as an addition to the manufacturer's fixed cost, that becomes now $\breve{F} = F + F_E$. This amount is subtracted from the retailers fixed cost that becomes $\breve{F}_R = F_R - F_E$. This fee F_E is as a new pricing tool optimized by the retailer simultaneously with optimizing the markup and the diversity. The regime studied is when the retailer is leading the game.

We want to know: What happens to price and variety because of the two-part tariff practice? Is this practice really harmful for society, as manufacturers and many journalists and politicians suggest in the cited papers (Nikitina, 2006) and (Sagdiev et al., 2006), or not? We are going to show positive gains of the entrance fee. So, the government and legislature should not do what they do now, prohibiting the fee.[13]

Equilibrium with entrance fee. To support our hypothesis, we modify the retailer's optimization program and the equilibrium formulae for the case "RL". The profit function of i-th manufacturer becomes

$$\pi_{\mathcal{M}}(i) = [p(i) - c]q(i) - (F + F_E)$$

and the profit of the retailer is now

$$\pi_{\mathcal{R}} = \int_0^N [r(i) - c_{\mathcal{R}}]q(i)di - \int_0^N (F_{\mathcal{R}} - F_E)di.$$

Since the retailer is the leader, in <u>**symmetric**</u> equilibria we have the same expressions as before (see (5)-(6)) for the wholesale price p and quantity q of each variety as functions of markup r and diversity N:

$$q = q(r,N) = \frac{(b+gN)[a - b(c+r)]}{2b+gN}\ , \quad p = p(r,N) = \frac{a + (b+gN)c - br}{2b+gN}\ .$$

Therefore, the new profit-maximization problem of the retailer is

$$\pi_{\mathcal{R}} = N[(r - c_{\mathcal{R}})q(r,N) - F_{\mathcal{R}} + F_E] \ \to\ \max_{r,N,F_E},$$

$$\pi_{\mathcal{M}} = [p(r,N) - c]q(r,N) - (F + F_E) \geq 0\ .$$

Since

$$\frac{\partial \pi_{\mathcal{R}}}{\partial F_E} = N > 0, \tag{9}$$

the objective function should attain a *boundary* maximum, so restriction on non-negative profit $\pi_{\mathcal{M}}$ is active in the equilibria, i.e. free entry condition $\pi_{\mathcal{M}} = 0$ holds.

[13] Notably, Russian parliament has just adopted a bill ("Law on trade...") that prohibits the entrance fees in retailing, because they are supposed harmful for competition and welfare.

Hence

$$F_E = -F + [p(r,N) - c]q(r,N) = -F + \frac{q^2(r,N)}{b+gN}. \tag{10}$$

Substituting this into $\pi_{\mathcal{R}}$ we obtain the unconstrained optimization problem

$$\pi_{\mathcal{R}} = N[\frac{q^2(r,N)}{b+gN} + (r - c_{\mathcal{R}})q(r,N) - (F_{\mathcal{R}} + F)] \ \rightarrow \ \max_{r,N}.$$

Its solution is

$$N^E = \frac{\dfrac{\tilde{D}}{\sqrt{1+2\mathcal{F}}} - 2\beta_{-\gamma}}{2\gamma}, \qquad r^E = c_{\mathcal{R}} + q_{NE}\left(\frac{\tilde{D}}{2} - \beta_{-\gamma}\sqrt{1+2\mathcal{F}}\right).$$

Surprisingly, for each diversity the optimal **entrance fee equals** exactly to the **fixed selling cost** $F_E = F_{\mathcal{R}}$. The quantity is $q^E = q_{NE}\sqrt{1+2\mathcal{F}}$ and the wholesale price is $p^E = c + q_{NE}\beta_{-\gamma}\sqrt{1+2\mathcal{F}}$.

Finally, we calculate profit, social welfare and consumer surplus under two-part tariff:

$$\pi_R^E = \left(\tilde{D} - 2\sqrt{1+2\mathcal{F}}\cdot\beta_{-\gamma}\right)^2 \cdot \frac{F}{4\gamma\beta_{-\gamma}}, \quad CS^E = \frac{F}{8\gamma\beta_{-\gamma}} \cdot \left(\tilde{D} - 2\sqrt{1+2\mathcal{F}}\cdot\beta_{-\gamma}\right)\tilde{D},$$

$$W^E = \pi_R^E + CS^E = \frac{3F}{8\gamma\beta_{-\gamma}} \cdot \left(\tilde{D} - 2\sqrt{1+2\mathcal{F}}\cdot\beta_{-\gamma}\right) \cdot \left(\tilde{D} - \frac{4}{3}\cdot\sqrt{1+2\mathcal{F}}\cdot\beta_{-\gamma}\right).$$

Comparing all the above formulae with the ones in Result 2 (see page 11), it is easy to derive the following consequences of introducing the two-part tariff.

Result 9 *When the leader-retailer introduces the entrance fee,*

- $p^E > p^{RL}$, i.e. entrance fee always **pushes up the wholesale price**, thus compensating the fee to manufacturers,
- $r^E < r^{RL}$, *i.e. entrance fee* ***decreases the markup****, and, surprisingly,*
- $p^E + r^E < p^{RL} + r^{RL}$, *i.e. entrance fee always* ***pushes down the retail price!***
- $q^E > q^{RL}$, *i.e. the entrance fee always* ***increases quantity of each variety,***
- $Q^E > Q^{RL}$, *i.e. the entrance fee* ***increases total quantity of the industry****,*
- $W^E > W^{RL}, CS^E > CS^{RL}, \pi_R^E > \pi_R^{RL}$, *i.e. entrance fee* ***always increases social welfare and consumer surplus****.*

As to the diversity N, under the fee N^E can be less or more than N^{RL}, depending on two regions of parameters $\tilde{D}$ and $\mathcal{F}$ as follows:

Parameter regions	*Diversity*
$\frac{\tilde{D}}{\beta_{-\gamma}} > \overline{D}^{RL}$	$N^E < N^{RL}$
$\overline{D}^{RL} > \frac{\tilde{D}}{\beta_{-\gamma}} > \underline{D}^{RL}$	$N^E > N^{RL}$

where bounds of these regions are the following continuous functions of $\mathcal{F}$:

$$\underline{D}^{RL} = \begin{cases} 4\sqrt{\mathcal{F}}, & \mathcal{F} \geq 1; \\ 2(\mathcal{F}+1), & \mathcal{F} \leq 1; \end{cases} \qquad \overline{D}^{RL} = \begin{cases} \left(1 + \sqrt{\frac{1}{\mathcal{F}} + 2}\right)\cdot\sqrt{1+2\mathcal{F}}\cdot\frac{2\mathcal{F}}{1+\mathcal{F}}, & \mathcal{F} \geq 1; \\ \left(1 + \sqrt{1+2\mathcal{F}}\right)\cdot\sqrt{1+2\mathcal{F}}, & \mathcal{F} \leq 1. \end{cases}$$

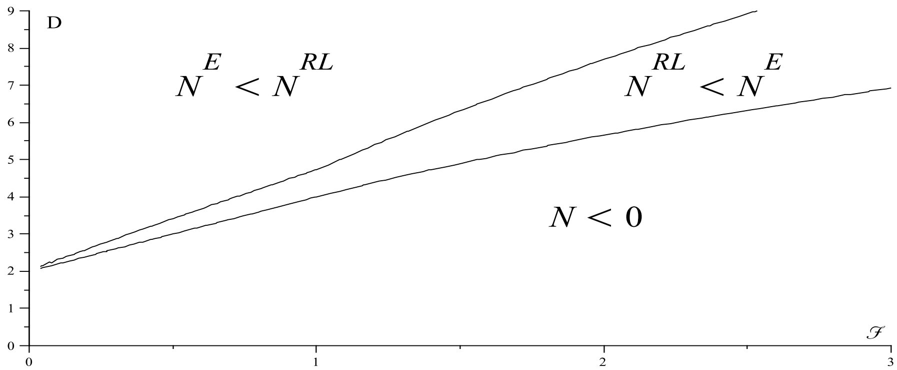

Fig. 6. Comparison between RL and $Entrance\ fee$ regimes in diversity.

Figure 6 explains the comparison in diversity: when $N^E < N^{RL}$ and when $N^E > N^{RL}$. One can see that when the retailer's fixed cost is sufficiently bigger than manufacturer's one and the slack between the chocking price and costs is small, the entrance fee increases the diversity.

Why the optimal entrance (shelf) fee becomes equal to the retailer's fixed costs? To understand the result obtained, let us note that an increase in entrance fee (in first approximation) leads to the growth of retailer's profit (it is easy to see, for instance, from (9)). On the other hand, when entrance fee increases, the fixed costs of producers also increase. These additional costs must be covered by higher wholesale price or/and by bigger per-firm output. Therefore, generally $\dfrac{\partial N}{\partial F_E} < 0$, i.e, the number of varieties (i.e. the number of firms in the industry) decreases[14]. This tendency hampers the increase in the retailer's profit. The interaction of these two forces determines the equilibrium entrance fee. For the linear functional form of demand, it turns out that resulting entrance fee equals to the retailer's fixed costs. Probably (in our opinion), for other functional forms the optimal entrance fee may be bigger or less than the retailer's fixed costs, depending on the value of these opposite forces.

Another way to understand this effect is the comparison of entrance fee practice with the vertical integration of the industry. We use such comparison below to explain why the entrance fee **increases** social welfare and even the consumer surplus (in contrast to what Russian legislature thinks about it).

This increase becomes not so surprising, if we recall that generally in many market situations it is not the monopolism *per se* responsible for the welfare losses, but the inability of the monopolist to use the efficient tools of pricing like perfect price discrimination, etc. In particular, the entrance fee works here exactly like in typical situations where two-part tariff compensates the fixed cost and increases the efficiency of the market through making prices closer to costs. Here the markup works like a price for manufacturers to get the service of retailer. Thus, the welfare increase under entrance fee has the common nature with a two-part tariff.

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[14] If we substitute in (10) the expression (see (5)) $q = q(r, N) = \frac{(b+gN)[a-b(c+r)]}{2b+gN}$, then $F_E + F = \frac{(\alpha-c-r)^2 \cdot \beta_{-\gamma}}{\left(2\beta_{-\gamma}+\gamma N\right)^2}$. So, $\frac{\partial N}{\partial F_E} < 0$.

More light on increasing welfare is shaded by the following argument. Suppose that starting from situation with entrance fees the next step in retailer's enforcement is undertaken: vertical integration of the industry. It means that the monopolistic retailer becomes the *owner* of the whole industry with its manufacturing and retailing. She chooses the retail price and the number of varieties (the product line), bearing all manufacturing and retailing costs. Does this integration change the market outcome and her profit? The surprising answer is "no".

To show this, we maximize the owner's profit $\pi = ((p_R - c - c_R)q(p_R, N) - F - F_R)N$ with respect to the length N of the product line and the retail price p_R equal to previous $p + r$. Here, as previously, the demand for a variety i is $q_i(p_R, N) = a - (b + gN)p_{iR} + g\int_0^N p_{jR}dj$ where a, b, g are dependent on N as in (4). Taking into account that the solution should be symmetric, we replace p_i and p_j by p_R and get

$$q(p_R, N) = \frac{\alpha - p_R}{\beta + \gamma(-1 + N)} \tag{11}$$

Maximizing profit and simplifying the result we get the market outcome $(q^{Mon}, p_R^{Mon}, N^{Mon})$ of such integrated monopoly:

$$q^{Mon} = \sqrt{\frac{F + F_R}{\beta_{-\gamma}}} = q^E, \quad p_R^{Mon} = \frac{\alpha + c + c_R}{2} = p^E + r^E,$$

$$N^{Mon} = \frac{\beta_{-\gamma}}{2\gamma}\left(-2 + \frac{\alpha - c - c_R}{\sqrt{(F + F_R)\beta_{-\gamma}}}\right) = N^E,$$

which turns out exactly the same as under retailer leadership with entrance fee. Naturally, profit and welfare also coincide. Thus, we have proved that combination of two pricing tools: **entrance fee plus markup is equivalent to integrated monopoly**. With their help the retailer fully controls the industry, to the benefit of consumers, neutralizing the shortcomings of two-tier monopoly and resolving the negative externalities among manufacturers.

The next improvement in the same direction, aiming to get first-best Pareto efficiency would be exercising the entrance fee F_C for consumers (using two-part tariff instead of linear pricing). To show this, use (5)-(6) under symmetry assumption and maximize such price-discriminating monopolist's profit

$$\pi_{DMon} = ((p_{DM} - c - c_R)q(p_{DM}, N) - F - F_R)N + F_C \to \max_{p_{DM}, N, F_C}$$

$$F_C \leq [\alpha - \frac{\beta_{-\gamma}}{2}q(p_{DM}, N) - \frac{\gamma}{2}Nq(p_{DM}, N) - p_{DM}]Nq(p_{DM}, N).$$

Here the right-hand side describes the potential consumer surplus from using the diversified industry. This CS is completely captured by the monopolist through the consumer's entrance fee (nothing remains actually), but it returns to consumer in the form of monopolist's profit and spent on other goods. It can be observed from the above program that the monopolist just maximizes the consumer's welfare, and the outcome will be price equal to marginal cost, efficient quantity, see Result 2, and efficient diversity N^{MaxW} :

$$p_R^{DMon} = c + c_R, \quad q^{DMon} = \sqrt{2 \cdot \frac{F + F_R}{\beta_{-\gamma}}},$$

$$F_C = \frac{\alpha - c - c_R}{2\gamma} \cdot \left(\alpha - c - c_R - \sqrt{2 \cdot (F + F_R)\beta_{-\gamma}}\right),$$

$$N^{DMon} = \frac{\beta_{-\gamma}}{2\gamma} \cdot \left(-2 + \frac{\alpha - c - c_R}{\sqrt{(F + F_R)\beta_{-\gamma}}}\sqrt{2}\right),$$

$$\pi_R^{DMon} = W^{DMon} = \left(\tilde{D} - \sqrt{2(1 + 2\mathcal{F})(\beta - \gamma)}\right)^2 \cdot \frac{F + F_R}{2 \cdot \gamma\beta_{-\gamma}}.$$

Comparing with previous results, observe that $q^{DMon} = q^{MaxW} > q^{Mon} = q^E$, $N^{DMon} = N^{MaxW} > N^{Mon} = N^E$, $W^{DMon} = W^{MaxW} > W^{Mon}$; therefore allowing for more powerful and sophisticated monopolist increases both consumption and diversity, thereby increasing welfare (as sophistication typically does in IO).

Our conjecture is that this market outcome can be obtained indirectly, namely a retailer-leader can use both entrance fees (F_E for manufacturers and F_C for consumers) instead of integrating the industry to arrive at the *same* first-best outcome as we have just found for integrated monopoly. Then, usage of discount cards payed by consumers in some shops (entrance fees) can be welfare improving.

Finally, let us compare *Entrance fee* equilibrium with *NR* equilibrium. It turns out that $q^{NR} = q^E$, $N^{NR} = 2 \cdot N^E$, i.e., ***under NR regime, the quantity equals to the quantity in the situation with Entrance fee, while the diversity is two times more then the diversity in the situation with Entrance fee.***

So, we find the correspondence between NR and $Entrance\ fee$ regimes. Therefore, it seems interesting to compare consumer surplus and welfare for these regimes. One has

$$CS^E = \frac{F}{8\gamma\beta_{-\gamma}} \cdot \left(\tilde{D} - 2\sqrt{1 + 2\mathcal{F}} \cdot \beta_{-\gamma}\right)\tilde{D},$$

$$W^E = \frac{3F}{8\gamma\beta_{-\gamma}} \cdot \left(\tilde{D} - 2\sqrt{1 + 2\mathcal{F}} \cdot \beta_{-\gamma}\right) \cdot \left(\tilde{D} - \frac{4}{3} \cdot \sqrt{1 + 2\mathcal{F}} \cdot \beta_{-\gamma}\right),$$

$$CS^{NR} = W^{NR} = \frac{3F}{8\gamma\beta_{-\gamma}} \cdot \left(\tilde{D} - 2\sqrt{1 + 2\mathcal{F}} \cdot \beta_{-\gamma}\right) \cdot \left(\frac{4}{3} \cdot \tilde{D} - \frac{4}{3} \cdot \sqrt{1 + 2\mathcal{F}} \cdot \beta_{-\gamma}\right).$$

Hence

$$CS^{NR} - CS^E = \frac{3F}{8\gamma\beta_{-\gamma}} \cdot \left(\tilde{D} - 2\sqrt{1 + 2\mathcal{F}} \cdot \beta_{-\gamma}\right) \cdot \left(\tilde{D} - \frac{2}{3} \cdot \sqrt{1 + 2\mathcal{F}} \cdot \beta_{-\gamma}\right) > 0,$$

$$W^{NR} - W^E = \frac{F}{8\gamma\beta_{-\gamma}} \cdot \left(\tilde{D} - 2\sqrt{1 + 2\mathcal{F}} \cdot \beta_{-\gamma}\right) \cdot \tilde{D} > 0,$$

i.e., ***under NR regime, welfare and consumer surplus are bigger then the correspondence ones under Entrance fee regime.***

For the government, the ideas of this section suggest *refraining from any regulation, when profits are transferred* to consumers and redistribution needs does not force the regulation. The next section expands these ideas.

6. Governmental Regulation of Retailing

Based on the above analysis, this section considers the outcomes of several possible measures that the state may wish to use, under one or another market situation. Namely, when the retailer's leadership after market concentration arise, it may tempt the state (the government or legislature) to restrict the market size of each firm, or/and the markup or/and restrict entrance fee when it is used (exactly like the Russian legislature did).

Section 4.3 about concentration showed that restriction of concentration by the state can be detrimental, at least under our assumptions.

As we have seen in the previous section, *prohibition of entrance fee for manufacturers is detrimental* for welfare. Now we study alternative measures.

6.1. Government Intervention through Capping Markup

Consider the specific situation when the government starts regulating the markup of the retailer-leader who does not use any entrance fees. So, regulated markup r maximizes welfare, being chosen by the government who correctly anticipates the subsequent choices of retailer, producers and consumers. The retailer-leader chooses only variety N within the constraint of non-negative producers' profits, which is typically binding.

To solve the equilibrium by backward induction, we have seen in (5)–(6) that the output and wholesale price of a firm in the long-run is always

$$q = q(r, N) = \frac{(b + gN)[a - b(c + r)]}{2b + gN} \;, \qquad p = p(r, N) = \frac{a + (b + gN)c - br}{2b + gN} \;.$$

Having this in mind, the retailer solves the problem

$$\begin{aligned} \pi_{\mathcal{R}} &= N[(r - c_{\mathcal{R}})q(r, N) - F_{\mathcal{R}}] \;\to\; \max_{N}, \\ \pi_{\mathcal{M}} &= [p(r, N) - c]q(r, N) - F \geq 0 \end{aligned} \tag{12}$$

obtaining her best response $N = N(r)$ to the markup regulated by the government. Anticipating this $N(r)$, the government solves the welfare maximization problem

$$W = W(r) \;\to\; \max_{r}.$$

We restrict our study only to the important case when the parameters of the model make "the non-negative manufacturer's profit" condition $\pi_{\mathcal{M}} \geq 0$ active, i.e. the retailer becomes a dummy in the game whereas free entry determines $N(r)$ [15]. Then, it is easy to see that under **direct** intervention of the government into pricing, quantity q^d, wholesale price p^d, markup r^d, diversity N^d and social welfare W^d become

$$q^d = q_{NE} = \sqrt{\frac{F}{\beta_{-\gamma}}}, \quad p^d = c + q_{NE} \cdot \beta_{-\gamma} \;, \quad r^d = c_{\mathcal{R}} + q_{NE} \cdot \tilde{D} \cdot \frac{\mathcal{F}}{\mathcal{F} + 1} \;,$$

[15] The class of parameters generating the opposite type of equilibrium (restricted entry) is not so easy to analyse. Besides, the regulating measures hardly can be confined to capping markup in this case, so a richer model is needed.

$$N^d = \frac{\dfrac{\tilde{D}}{1+\mathcal{F}} - 2\beta_{-\gamma}}{\gamma}\ , \quad \pi^d_{\mathcal{R}} = \left(\tilde{D} - 2\beta_{-\gamma}\cdot(1+\mathcal{F})\right)^2 \cdot \frac{\mathcal{F}}{(1+\mathcal{F})^2}\cdot\frac{F}{\gamma\beta_{-\gamma}}\ ,$$

$$CS^d = \left(\tilde{D} - 2\beta_{-\gamma}\cdot(1+\mathcal{F})\right)\cdot\left(\tilde{D} - \beta_{-\gamma}\cdot(1+\mathcal{F})\right)\cdot\frac{1}{2\cdot(1+\mathcal{F})^2}\cdot\frac{F}{\gamma\beta_{-\gamma}}\ ,$$

$$W^d = \left(\tilde{D} - 2\beta_{-\gamma}\cdot(1+\mathcal{F})\right)\cdot\left(\tilde{D} - \frac{1+4\mathcal{F}}{1+2\mathcal{F}}\cdot\beta_{-\gamma}\cdot(1+\mathcal{F})\right)\cdot\frac{1+2\mathcal{F}}{2\cdot(1+\mathcal{F})^2}\cdot\frac{F}{\gamma\beta_{-\gamma}}\ .$$

Comparing this outcome with the previous results we get the following comparison of the direct intervention of the government with the "RL" (free entry!) case [16].

Result 10 *One has*

- $q^d = q^{RL}$, *i.e direct intervention of the government* ***does not change the quantity of each variety***
- $p^d = p^{RL}$, *i.e direct intervention of the government* ***does not change the wholesale price***
- $r^d < r^{RL}$, *i.e direct intervention of the government* ***decreases the markup***
- $N^d > N^{RL}$, *i.e direct intervention of the government* ***increases the diversity***
- $W^d > W^{RL}$, *i.e direct intervention of the government* ***increases the social welfare***

Now, using the above formulae, we can compare diversities N^d N^{RL} with the social optimal diversity N^{MaxW}. A natural conjecture is that socially efficient diversity is the largest among N^{RL}, N^d, N^{MaxW}. Surprisingly, it is not true in general. The table below shows that all three cases $N^{RL} < N^d < N^{MaxW}$, $\quad N^{RL} < N^{MaxW} < N^d$, $\quad N^{MaxW} < N^{RL} < N^d$ are possible:

Parameter regions	*Diversity*
1. $2(1+\mathcal{F}) < \frac{\tilde{D}}{\beta_{-\gamma}} < \underline{D}^{MaxW},\ \mathcal{F} \le 1$	$N^{RL} < N^d < N^{MaxW}$
2.1. $\underline{D}^{MaxW} < \frac{\tilde{D}}{\beta_{-\gamma}}$, $\mathcal{F} < 0.5$	$N^{RL} < N^{MaxW} < N^d$
2.2. $\underline{D}^{MaxW} < \frac{\tilde{D}}{\beta_{-\gamma}}$, $\mathcal{F} > 0.5$, $\frac{\tilde{D}}{\beta_{-\gamma}} < \overline{D}^{MaxW}$	$N^{RL} < N^{MaxW} < N^d$
3. $\overline{D}^{MaxW} < \frac{\tilde{D}}{\beta_{-\gamma}}$, $0.5 < \mathcal{F} \le 1$	$N^{MaxW} < N^{RL} < N^d$

where

$$\underline{D}^{MaxW} = -4 - \sqrt{2+4\mathcal{F}} + \frac{14\mathcal{F} + 6 + (4\mathcal{F}+2)^{3/2}}{-\mathcal{F}^2 + 2\mathcal{F} + 1}\ ,$$

$$\overline{D}^{MaxW} = 3 + \sqrt{2+4\mathcal{F}} + \frac{3+\sqrt{4\mathcal{F}+2}}{2\mathcal{F}-1}\ .$$

Moreover (see Figure 7), region 1 where $N^{RL} < N^d < N^{MaxW}$ is rather small, while the other two zones are large.

Thus, we conclude that direct intervention of the government may **lead to "over-production of diversity"** and in the case when the diversity is already too large, the intervention **aggravates "over-production of diversity"**.

Further, we can ask: Does the government enhance social welfare when it prohibits entrance fee (previously existing), and simultaneously restricts markup from above? Some ideas for the answer we can find in the table below

[16] Let us recall that we consider here only free entry case, hence $\mathcal{F} < 1$.

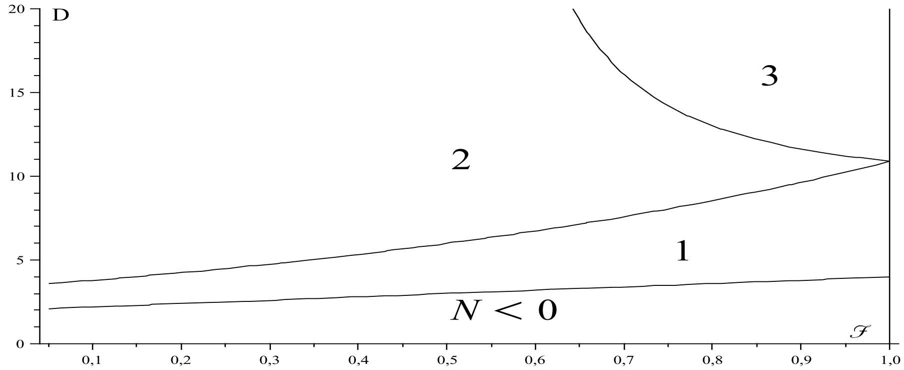

Fig. 7. Comparison between RL, $Capping\ markup$ and $MaxW$ regimes in diversity:

Region 1:	$N^{RL} < N^d < N^{MaxW}$
Region 2:	$N^{RL} < N^{MaxW} < N^d$
Region 3:	$N^{MaxW} < N^{RL} < N^d$

Parameter regions	*Welfare*
$\frac{\tilde{D}}{\beta_{-\gamma}} > D^{Ed}$	$W^E < W^d$
$2(1+\mathcal{F}) < \frac{\tilde{D}}{\beta_{-\gamma}} < D^{Ed}$, $\mathcal{F} \leq 1$	$W^E > W^d$

where $D^{Ed} = \frac{-8\mathcal{F}}{d_1+\sqrt{d_1^2+d_2}}$, $d_1 = \frac{5\sqrt{1+2\mathcal{F}}}{2} + \frac{5}{1+\mathcal{F}} - 8 < 0$, $\quad d_2 = \frac{8\mathcal{F}(\mathcal{F}-1)(3\mathcal{F}+1)}{(1+\mathcal{F})^2} \leq 0 \ \ \forall\ \mathcal{F} \in [0;1]$.

Comparing such intervention with entrance fee situation, we see on Figure 8 that capping markup is better for welfare than entrance fee where $\tilde{D} = (\alpha - c - c_R)\sqrt{\frac{\beta-\gamma}{F}}$ is big and $\mathcal{F} = \frac{F_R}{2F}$ small, but the opposite is true in another region. So, when retailer's fixed cost are sufficiently bigger than manufacturer's one and the slack between the chocking price and costs is small, the government should not intervene in this fashion.

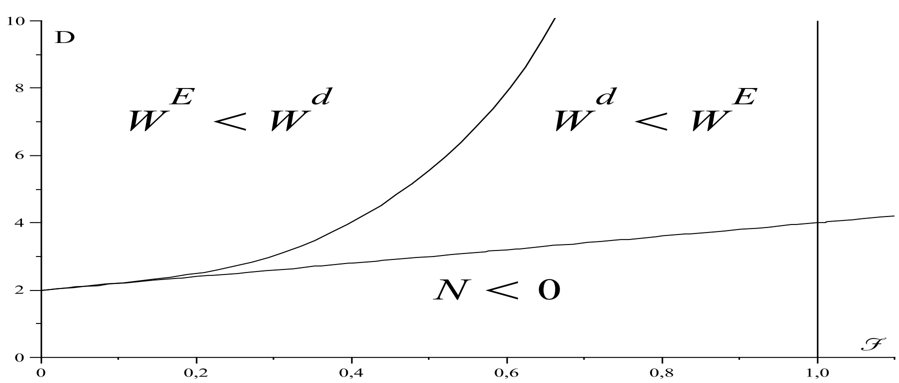

Fig. 8. Comparison between $Entrance\ fee$ and $Capping\ markup$ regimes in welfare.

6.2. Pigouvian Regulation: to Tax or to Subside the Retailer?

Suppose that the government stimulates the market in Pigouvian way. The retailer pays the tax τ per every unit of goods sold. Then the retailer's profit takes the form

$$\int_0^N [r(i) - (c_{\mathcal{R}} + \tau)]q(i)di - \int_0^N F_{\mathcal{R}} di.$$

The collected taxes are redistributed among the consumers in a lump-sum manner. In the case of negative τ it means a Pigouvian subsidy, funded from a lump-sum taxes on consumers in amount $N\overline{F} = \tau \int_0^N q(i)di$.

The welfare function $W^{RL}(\bar{q}(\tau), N(\tau))$ has the previous form, only equilibrium variables $\bar{q}(\tau), N(\tau)$ become the functions of tax τ. They, and welfare, can be found from the previous formulae. Knowing these equilibrium expressions, the government can maximize the welfare function with respect to τ, and find the taxation policy that maximizes the social welfare. We did not solve this problem here (though it is possible), finding instead only the sign: Should the government start taxing or subsidizing the retailer for a small amount? In other words, the question is: When it is beneficial for the government to tax the retailer ($\tau > 0$), or to subside her ($\tau < 0$)?

Result 11 *Under the strategic behavior of the retailer, it is beneficial for the government to subside her, i.e.* $\tau^* < 0$.

This fact seems unnatural in anti-trust logics: the state should subsidize a monopolist/monopsonist. However, such effects are rather typical in IO literature and the proofs are simple. We just have ensured that our more complicated situation with varieties keeps the same effect.

7. Conclusion

We study the model where the monopolistic-competition industry is selling many varieties through the monopsonistic retailer. The emergence of such retailer with a two-sided market power is compared to initial situation, when the manufacturers were selling through many retailers having a local selling market power but no buying power. Thereby we model the welfare and marketing effects of the market concentration after the emergence of big chain-stores in Russia and other CIS countries.

In the special case of quadratic valuations (linear demands) it turns out that generally the concentration enhances social welfare. This effect, unexpected for non-economists, actually is not too surprising, because it reminds the switch from a two-tier monopoly to a simple monopoly. The welfare increase here, however, can go not only through the growth of the total quantity, but in some cases through growing total profits and, notably, through decreasing excessive diversity. The retailer is shown to be generally interested in restricting the diversity, but it is not harmful itself.

In the case of the governmental regulation of the monopsonistic retailer through Pigouvian taxes, it turns out that subsidies instead of taxes are needed, that also reminds similar effect known for a simple monopoly/monopsony.

The anti-chain-stores bill recently suggested by Russian Parliament (Duma) shows the public interest in this type of questions.

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Strong Strategic Support of Cooperative Solutions in Differential Games

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Abstract. The problem of strategically provided cooperation in n-persons differential games with integral payoffs is considered. Based on initial differential game the new associated differential game (CD-game) is designed. In addition to the initial game it models the players actions connected with transition from the strategic form of the game to cooperative with in advance chosen principle of optimality. The model provides possibility of refusal from cooperation at any time instant t for each player. As cooperative principle of optimality the core operator is considered. It is supposed that components of an imputation form the core along any admissible trajectory are absolutely continuous functions of time. In the bases of CD-game construction lies the so-called imputation distribution procedure described earlier in (Petrosjan and Zenkevich, 2009). The theorem established by authors says that if at each instant of time along the conditionally optimal (cooperative) trajectory the future payments to each coalition of players according to the imputation distribution procedure exceed the maximal guaranteed value which this coalition can achieve in CD-game, then there exist a strong Nash equilibrium in the class of recursive strategies first introduced in (Chistyakov, 1999). The proof of this theorem uses results and methods published in (Chistyakov, 1999, Chentsov, 1976).

Keywords: strong Nash equilibrium, time-consistency, core, cooperative trajectory.

1. Introduction

Similar to (Petrosjan and Zenkevich, 2009) in this paper the problem of strategically support of the cooperation in differential m-person game with prescribed duration T and independent motions is considered.

$$\frac{dx^{(i)}}{dt} = f^{(i)}(t, x^{(i)}, u^{(i)}),\ i \in I = [1; m], \tag{1}$$

$$x^{(i)} \in R^{n(i)}, u^{(i)} \in P^{(i)} \in CompR^{k(i)}$$

$$x^{(i)}(t_0) = x_0^{(i)}, \quad i \in I. \tag{2}$$

The payoffs of players $i \in I = [1 : m]$ have integral form

$$H^{(i)}_{t_0,x_0}(u(\cdot)) = \int_{t_0}^{T} h^{(i)}(t, x(t, t_0, x_0, u(\cdot)))dt. \tag{3}$$

Here $u(\cdot) = (u^{(1)}(\cdot), \ldots, u^{(m)}(\cdot))$ is a given m-vector of open loop controls,

$$x(t, t_0, x_0, u(\cdot)) = \left(x^{(1)}(t, t_0, x_0, u^{(1)}(\cdot)), \ldots, x^{(m)}(t, t_0, x_0, u^{(m)}(\cdot))\right),$$

where $x^{(i)}(\cdot) = x(\cdot, t_0, x_0^{(i)}, u^{(i)}(\cdot))$ is the solution of the Cauchy problem for i-th subsystem of (1) with corresponding initial conditions (2) and admissible open loop control $u^{(i)}(\cdot)$ of player i.

Admissible open loop controls of players $i \in I$ are Lebesgue measurable open loop controls

$$u^{(i)}(\cdot) : t \mapsto u^{(i)}(t) \in R^{k(i)}$$

such that

$$u^{(i)}(t) \in P^{(i)} \text{ for all } t \in [t_0, T].$$

It is supposed that each of the functions

$$f^{(i)} : R \times R^{k(i)} \times P^{(i)} \to R^{k(i)}, \quad i \in I$$

are continuous, locally Lipschitz with respect to $x^{(i)}$ and satisfies the following condition: $\exists \lambda^{(i)} > 0$ such, that

$$||f^{(i)}(t, x^{(i)}, u^{(i)})|| \leq \lambda^{(i)}(1 + ||x^{(i)}||) \quad \forall x^{(i)} \in R^{k(i)}, \quad \forall u^{(i)} \in P^{(i)}.$$

Each of the functions

$$h^{(i)} : R \times R^{k(i)} \times P^{(i)} \to R, \quad i \in I$$

are also continuous.

It is supposed that at each time instant $t \in [t_0, T]$ the players have information about the trajectory (solution) $x^{(i)}(\tau) = x(\tau, t_0, x_0, u^{(i)}(\cdot))$ of the system (1), (2) on the time interval $[t_0, t]$ and use recursive strategies (Chistyakov, 1977, Chistyakov, 1999).

2. Recursive strategies

Recursive strategies were first introduced in (Chistyakov, 1977) for justification of dynamic programming approach in zero sum differential games, known as method of open loop iterations in non regular differential games with non smooth value function. The ε-optimal strategies constructed with the use of this method are universal in the sense that they remain ε-optimal in any subgame of the previously defined differential game (for every $\varepsilon > 0$). Exploiting this property it became possible to prove the existence of ε-equilibrium (Nash equilibrium) in non zero sum differential games (for every $\varepsilon > 0$) using the so called "punishment strategies" (Chistyakov, 1981).

The basic idea is that when one of the players deviates from the conditionally optimal trajectory other players after some small time delay start to play against the deviating player. As result the deviating player is not able to get much more than he could get using the conditionally optimal trajectory. The punishment of the deviating player at each time instant using one and the same strategy is possible because of the universal character of ε-optimal strategies in zero sum differential games.

In this paper the same approach is used to testify the stability of cooperative agreements in the game $\Gamma(t_0, x_0)$ and as in mentioned case the principal argument is the universal character of ε-optimal recursive strategies in specially defined zero sum games $\Gamma_S(t_0, x_0)$, $S \subset I$ associated with the non-zero sum game $\Gamma(t_0, x_0)$.

The recursive strategies lie somewhere in-between piecewise open loop strategies (Petrosjan, 1993) and ε-strategies introduced by B. N. Pshenichny (Pschenichny, 1973). The difference from piecewise open loop strategies consists in the fact that like in the case of ε-strategies of B. N. Pshenichny the moments of correction of open loop controls are not prescribed from the beginning of the game but are defined during the game process. In the same time they differ from ε-strategies of B. N. Pshenichny by the fact that the formation of open loop controls happens in finite number of steps.

Recursive strategies $U_i^{(n)}$ of player i with maximal number of control corrections n is a procedure for the admissible open loop formation by player i in the game $\Gamma(t_0, x_0), (t_0, x_0) \in D$.

At the beginning of the game $\Gamma(t_0, x_0)$ player i using the recursive strategy $U_i^{(n)}$ defines the first correction instant $t_1^{(i)} \in (t_0, T]$ and his admissible open loop control $u^{(i)} = u^{(i)}(t)$ on the time interval $[t_0, t_1^{(i)}]$. Then if $t_1^{(i)} < T$ having the information about state of the game at time instant $t_1^{(i)}$ he chooses the next moment of correction $t_2^{(i)}$ and his admissible open loop control $u^{(i)} = u^{(i)}(t)$ on the time interval $(t_1^{(i)}, t_2^{(i)}]$ and so on. Then whether on k-th step ($k \leq n-1$) the admissible control will be formed on the time interval $[t_*, T]$ or on the step n player i will end up with the process by choosing at time instant $t_{n-1}^{(i)}$ his admissible control on the remaining time interval $(t_{n-1}^{(i)}, T]$.

3. Associated zero sum games and corresponding solutions

For each given state $(t_*, x_*) \in D$ and non void coalition $S \subset I$ consider zero sum differential game $\Gamma_S(t_*, x_*)$ between coalition S and $I \backslash S$ with the same dynamics as in $\Gamma(t_*, x_*)$ and payoff of the coalition S equal to the sum of payoffs of the players $i \in S$ in the game $\Gamma(t_*, x_*)$:

$$\sum_{i \in S} H_{t_* x_*}^{(i)}(u^{(S)}(\cdot), u^{(I \backslash S)}(\cdot)) = \sum_{i \in S} H_{t_* x_*}^{(i)}(u(\cdot)) = \sum_{i \in S} \int_{t_0}^{T} h^{(i)}(t, x(t), u(t)) dt$$

here

$$u^{(S)}(\cdot) = \{u^{(i)}(\cdot)\}_{i \in S},$$
$$u^{(I \backslash S)}(\cdot) = \{u^{(j)}(\cdot)\}_{j \in I \backslash S},$$
$$u(\cdot) = (u^{(S)}(\cdot), u^{(I \backslash S)}(\cdot)) = (u^{(1)}(\cdot), \ldots, u^{(m)}(\cdot)).$$

The game $\Gamma_S(t_*, x_*)$, $S \subset I$, $(t_*, x_*) \in D$, as $\Gamma(t_*, x_*), (t_*, x_*) \in D$ we consider in the class of recursive strategies. Under the above formulated conditions each of the games $\Gamma_S(t_*, x_*)$, $S \subset I$, $(t_*, x_*) \in D$ has a value

$$val \Gamma_S(t_*, x_*).$$

If $S = I$ the game $\Gamma_S(t_*, x_*)$ became an one player optimization problem. We suppose that in this game there exist an optimal open loop solution. The corresponding trajectory — solution of (1), (2) on the time interval $[t_0, T]$ we denote by

$$x_0(\cdot) = (x_0^{(1)}(\cdot), \dots, x_0^{(m)}(\cdot))$$

and call "*conditionally optimal cooperative trajectory*". This trajectory may not be necessary unique. Thus on the set D the mapping

$$v(\cdot) : D \to R^{2^I}$$

is defined with coordinate functions

$$v_S(\cdot) : D \to R, \quad S \subset I,$$

$$v_S(t_*, x_*) = val\Gamma_S(t_*, x_*).$$

This mapping correspond to each state $(t_*, x_*) \in D$ a characteristic function $v(t_*, x_*) : 2^I \to R$ of non zero-sum game $\Gamma(t_*, x_*)$ and thus m-person classical cooperative game $(I, v(t_*, x_*))$.

Let $E(t_*, x_*)$ be the set of all imputations in the game $(I, v(t_*, x_*))$. Multivalue mapping

$$M : (t_*, x_*) \mapsto M(t_*, x_*) \subset E(t_*, x_*) \subset R^m,$$

$$M(t_*, x_*) \neq \Lambda \quad \forall (t_*, x_*) \in D,$$

is called "*optimality principle*" (defined over the family of games $\Gamma(t_*, x_*)$, $(t_*, x_*) \in D$) and the set $M(t_*, x_*)$ "*cooperative solution of the game* $\Gamma(t_*, x_*)$ *corresponding to this principle*".

As it follows from (Fridman, 1971) under the above imposed conditions the following Lemma holds.

Lemma 1. *The functions* $v_S(\cdot) : D \to R, S \in I$, *are locally Lipschitz.*

Since the solution of the Cauchy problem (1), (2) in the sense of Caratheodory is absolutely continuous, from Lemma 1 it follows.

Theorem 1. *For every solution of the Cauchy problem (1), (2) in the sense of Caratheodory*

$$x(\cdot) = (x^{(1)}(\cdot), \dots, x^{(m)}(\cdot)),$$

corresponding to the m*-system of open loop controls*

$$u(\cdot) = (u^{(1)}(\cdot), \dots, u^{(m)}(\cdot))$$

$$(x^{(i)}(\cdot) = x(\cdot, t_0, x_0^{(i)}, u^{(i)}(\cdot)), \quad i \in I),$$

the functions

$$\varphi_S : [t_0, T] \to R, \quad S \subset I, \quad \varphi_S(t) = v_S(t, x(t))$$

are absolutely continuous functions on the time interval $[t_0, T]$.

Suppose that $M(t_*, x_*)$ is the core of the game $\Gamma(t_*, x_*)$, and let the imputation

$$\xi(t_*, x_*) = \{\xi_1(t_*, x_*), \dots, \xi_m(t_*, x_*)\} \in M(t_*, x_*).$$

Then for each coalition $S \subset I$ we have

$$\sum_{i \in S} \xi_i(t_*, x_*) \geq v_S(t_*, x_*).$$

4. Realization of cooperative solutions.

The realization of the solution of the game $\Gamma(t_0, x_0)$ we shall connect with the known "*imputation distribution procedure*" (IDP) (Petrosjan and Danilov, 1979, Petrosjan, 1995).

Under IDP of the imputation $\xi(t_0, x_0)$ from the core $M(t_0, x_0)$ of the game $\Gamma(t_0, x_0)$ along conditionally optimal trajectory $x_0(\cdot)$ we understand such function

$$\beta(t) = (\beta_1(t), \ldots, \beta_m(t)), \quad t \in [t_0, T], \tag{4}$$

that

$$\xi(t_0, x_0) = \int_{t_0}^{T} \beta(t) dt \tag{5}$$

and

$$\int_{t}^{T} \beta(t) dt \in E(t, x_0(t)) \quad \forall t \in [t_0, T] \tag{6}$$

where $E(t, x_0(t))$ is the set of imputations in the game $(I, v(t, x_0(t)))$.

The IDP $\beta(t), \quad t \in [t_0, T]$ of the solution $M(t_0, x_0)$ of the game $\Gamma(t_0, x_0)$ is called *dynamically stable (time-consistent)* along the conditionally optimal trajectory $x_0(\cdot)$ if

$$\int_{t}^{T} \beta(t) dt \in M(t, x_0(t)) \quad \forall t \in [t_0, T] \tag{7}$$

The solution $M(t_0, x_0)$ of the game $\Gamma(t_0, x_0)$ is *dynamically stable (time-consistent)* if along at least one conditionally optimal trajectory the dynamically stable IDP exist.

Suppose that $M(t, x_0(t)) \neq \emptyset$, $t \in [t_0, T]$ ($M(t, x_0(t))$ is the core in the subgame $\Gamma(t, x_0(t))$ with initial conditions on conditionally optimal cooperative trajectory with duration $T - t$), and $\xi(t, x_0(t)) \in M(t, x_0(t))$ can be selected as absolutely continuous function of t. Then the following theorem holds.

Theorem 2. *For any conditionally optimal trajectory $x_0(\cdot)$ the following IDP of the solution $\xi(t_0, x_0) \in M(t_0, x_0)$ of the game $\Gamma(t_0, x_0)$*

$$\beta(t) = -\frac{d}{dt} \xi(t, x_0(t)), \quad t \in [t_0, T], \tag{8}$$

is the dynamically stable IDP along this trajectory. Therefore the solution $\xi(t_0, x_0) \in M(t_0, x_0)$ of the game $\Gamma(t_0, x_0)$ is dynamically stable.

5. About the strategically support of the imputation $\boldsymbol{\xi(t_0, x_0)}$ from the core $\boldsymbol{M(t_0, x_0)}$.

If in the game the cooperative agreement is reached and each player gets his payoff according to the IDP (8), then it is natural to suppose that those who violate this agreement are to be punished. The effectiveness of the punishment (sanctions) comes to question of the existence of strong Nash Equilibrium in the following differential game $\Gamma^{\xi}(t_0, x_0)$ which differs from $\Gamma(t_0, x_0)$ only by payoffs of players.

The payoff of player i in $\Gamma^{\xi}(t_0, x_0)$ is equal to

$$H^{(\xi, i)}_{t_0, x_0}(u(\cdot)) = -\int_{t_0}^{t(u(\cdot))} \frac{d}{dt} \xi_i(t, x_0(t)) dt + \int_{t(u(\cdot))}^{T} h^{(i)}(t, x(t, t_0, x_0, u(\cdot))) dt$$

where $t(u(\cdot))$ is the last time instant $t \in [t_0, T]$ for which

$$x_0(\tau) = x(\tau, t_0, x, u(\cdot)) \quad \forall \tau \in [t_0, t].$$

In this paper we use the following definition of strong Nash equilibrium.

Definition 1. Let $\gamma = \langle I, \{X_i\}_{i \in I}, \{K_i\}_{i \in I} \rangle$ be the m-person game in normal form, here $I = [1 : m]$ is the set of players, X_i the set of strategies of player i and

$$K_i : X = X_1 \times X_2 \times \cdots \times X_m \to R$$

the payoff function of player i. We shall say that in the game γ there exists *a strong Nash equilibrium* if

$$\forall \varepsilon > 0 \quad \exists x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, \ldots, x_m^\varepsilon) \in X$$

such that

$$\forall S \subset I, \forall x_S \in X_S = \prod_{i \in S} X_i$$

$$\sum_{i \in S} K_i(x_s, x^\varepsilon_{I \setminus S}) - \varepsilon \leq \sum_{i \in S} K_i(x^\varepsilon),$$

where

$$x^\varepsilon_{I \setminus S} = \{x_j^\varepsilon\}_{j \in I \setminus S} \quad (x^\varepsilon_{I \setminus S} \in X_{I \setminus S}).$$

Let $C(t_*, x_*)$ be the core of the game $(I, v(t_*, x_*))$.

Theorem 3. *In the game $\Gamma^\xi(t_0, x_0)$ there exist a strong Nash equilibrium with outcomes (payoffs) of players in this equilibrium equal to*

$$\xi(t_0, x_0) = \{\xi_1(t_0, x_0), \ldots, \xi_m(t_0, x_0)\} \in M(t_0, x_0).$$

The idea of the proof is following. Since $\xi(t_0, x_0)$ belongs to the core $M(t_0, x_0)$ of the game $\Gamma(t_0, x_0)$ we have

$$\sum_{i \in S} \xi_i(t, x_0(t)) \geq v_S(t, x_0(t)) \quad \forall S \subset I \quad \forall t \in [t_0, T] \tag{9}$$

This means that at each time instant $t \in [t_0, T]$ moving along conditionally optimal trajectory $x_0(\cdot)$ no coalition can guarantee himself the payoff $[t, T]$ more than according to IDP (8), i.e. more than

$$\sum_{i \in S} \int_t^T \beta(\tau) d\tau = -\sum_{i \in S} \int_t^T \frac{d}{dt} \xi(\tau, x_0(\tau)) d\tau = \sum_{i \in S} \xi_i(t, x_0(t)),$$

in the same time on the time interval $[t_0, t]$ according to the IDP she already got the payoff equal to

$$\sum_{i \in S} \int_{t_0}^t \beta_i(\tau) d\tau = -\sum_{i \in S} \int_{t_0}^t \frac{d}{dt} \xi_i(\tau, x_0(\tau)) d\tau = \sum_{i \in S} \xi_i(t_0, x_0) - \sum_{i \in S} \xi_i(t, x_0(t))$$

Consequently no coalition can guarantee in the game $\Gamma^{\xi}(t_0, x_0)$ the payoff more than

$$\sum_{i \in S} \xi_i(t_0, x_0).$$

According to the cooperative solution $x_0(\cdot)$ but moving always in the game $\Gamma^{\xi}(t_0, x_0)$ along conditionally optimal trajectory each coalition will get his payoff according to the imputation $\xi(t_0, x_0)$ from the core $M(t_0, x_0)$. Thus no coalition can benefit from the deviation from the conditionally optimal trajectory which in this case is natural to call "strongly equilibrium trajectory".

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Strategic Bargaining and Full Efficiency

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Abstract. In strategic bargaining, Bolton (1991) assumes that the decider feels envy or inequality aversion, Rabin (1997) characterized how fairness affects bargaining efficiency and distribution of bargaining outcomes, but leaves tolerance open. It is a difficult topic to incorporate tolerance to fair bargaining structure, and discuss bargaining outcomes. In this paper, we consider decider withholding cooperation from proposer even if the proposer mistreated the decider, and explore the proposer's force and the decider's deciding power. For self-interested and tolerably fair motivated preferences, we present some results predictable in classical game theoretic perspective. When full efficiency is obtained as the maximum payoffs possible are reached when parties make the type of small sacrifices people make all the time to cooperate with one another, we can characterize full efficiency. Since our bargaining structure incorporates tolerance, our results extends (Bolton, 1991, Rabin, 1997).

Keywords: Strategic Bargaining, Fairness, Tolerance, Full Efficiency
JEL classification: A12, A13, B49, C70, D63

1. Introduction

In abstract proposer-decider game, the proposer makes a take-it-or-leave-it offer to the decider on how they should split some money. The decider then chooses whether to accept or reject that split considering his tolerance. If she accepts, the two get the proposed allocation. If she rejects, each get nothing. Well, sometimes she will accept the offer even if the proposer treats her unfairly. Division of surplus will be more even than in classical game theory assuming self-interested players.

In complete information, it is expected that the actual outcome will accord with classical game-theoretic prediction in one respect. Hence, it will be efficient. If the proposer is fully aware of what the decider's threshold is, she will always make an offer sharing the entire surplus that the decider will accept.

Some proposers may be motivated by fairness, but some proposers would be tempted to make ungenerous offer, some deciders may tolerate to accept, while some may not. As a consequence, inefficient outcomes arise possibly. The inefficiency here may be surprising in a traditional economic perspective, since it is complete and perfect information with no transactions costs, and people may tolerate to cooperate with those who treat them unfairly. This phenomena has been observed, for example, by Akerlof (1982), (1984), Akerlof and Yellen (1990) in labor economics, Rudolph (2006) in financial economics, and Bolton (1991), Rabin (1997) in theoretical analysis.

In strategic bargaining, Bolton (1991) assumes that the decider feels envy or inequality aversion, Rabin (1997) characterized how fairness affects bargaining efficiency and distribution of bargaining outcomes. In this stage, as we know, tolerance is not incorporated to bargaining structure. It is a bit more difficult to incorporate tolerance to fair bargaining structure, and discuss bargaining outcomes.

In this paper, we consider decider withholding cooperation from proposer even if the proposer mistreated the decider, formalizes a simple class of bargaining structure incorporating tolerance into fair bargaining structure, and explore the proposer's force and the decider's deciding power. For self-interested and tolerably fair motivated preferences, we present some results predictable in classical game theoretic perspective. When full efficiency is obtained as the maximum payoffs possible are reached when parties make the type of small sacrifices people make all the time to cooperate with one another, we can characterize full efficiency. Since our bargaining structure incorporates tolerance, our results extends ones from (Bolton, 1991, Rabin, 1997).

The remaining is organized as follows: In Section 2, we present the class of bargaining structures and characterize the preferences we will consider in this paper; in Section 3, we formalize full efficiency; Section 4 is for some concluding remarks.

2. Proposer-Decider Games

In two-person bargaining games, one party (the proposer) makes an offer on how to split some money, and a second party (the decider) determines by her response whether and how it is divided. In this section, we define a class of Proposer-Decider Games with following structure, but different from (Rabin, 1997) in that the decider may withhold cooperation from the proposer even if mistreated.

Definition 1. A game splitting a pie of size 1 is an extended proposer-decider game (EPDG henceforth) if

The proposer can propose any $(1-x, x)$ such that $x \in [Y, 1]$, for an exogenous parameter $Y \in [0, 1]$;

After observing x, the decider chooses some $\gamma \in [0, 1]$;

The proposer and the decider respectively get monetary outcomes $g_1(1-x, \gamma)$ and $g_2(1-x, \gamma)$ with $g_1(1-x, 0) = 1-x+x^*$ and $g_2(x, 0) = x - x^*$ for all x, where for $x < Y$, $x^* > 0$ dependent on $Y - x$; for $x > Y$, g_1 and g_2 are non-increasing in γ, i.e. for all $\gamma > \gamma'$, $g_1(x, \gamma) < g_1(x, \gamma')$, $g_2(x, \gamma) < g_2(x, \gamma')$.

In the EPDG, the proposer is allowed to offer the decider an amount less than Y. The decider can choose either to accept the proposer's offer or to take some according to exogenously given rules. Here different from the PDG, $\gamma = 0$ does not still mean accepting all, and $\gamma = 1$ does not still mean accepting nothing. To a certain degree the value Y is to determine the decider's bargaining power considering the decider's withholding degree. We define the deciding power so that the decider can always guarantee herself at least proportion ρ of the pie, in convenience to clarify the role of Y.

Definition 2. A decider has deciding power ρ in a EPDG where $\rho = \sup_{x-x^*(Y-x)\geq Y}\{k \mid$ for every offer x, there exists a γ for the decider such that $g_2(x, \gamma) \geq k\}$.

In any EPDG, assuming pure self-interest classical game-theoretic prediction is for the proposer to propose $(1-Y, Y)$, and the decider to accept such an offer.

Proposition 1. If both the proposer and the decider were purely self-interested, the unique subgame-perfect Nash equilibrium (SPNE henceforth) would always be that the proposer proposes $(1-Y,Y)$, and the decider accepts this offer, i.e., chooses γ such that $(g_1(Y,\gamma),g_2(Y,\gamma))=(1-Y,Y)$.

Proof. Trivial.

From the lemma above, ρ can be taken as a measure of bargaining power for the decider. Though completely determining the outcome for pure self interest, ρ does not completely determine the outcome for the alternative preferences, since the proposer can force a particular outcome. For example, the decider has no say at all in the dictator game (Camerer and Thaler, 1995). Therefore we give a measure of force for the proposer below.

Definition 3. A proposer has force π in a EPDG where $\pi=\sup_{\gamma\in[0,1]}\{k \mid$ there exists x for the proposer such that $g_1(x,\gamma)\geq k$ for all $\gamma\}$.

π is a measure of how much the proposer can grab without the decider's cooperation, ρ is a measure of how much the decider is guaranteed possibly without the proposer's cooperation, and in all EPDGs, $\rho+\pi\geq 1$. When $\rho+\pi>1$, the decider withhold cooperation from the proposer even if he is treated unfairly. When $\rho+\pi\equiv 1$, the subgame-perfect equilibrium outcome is (ρ,π) when the two players are each trying to maximize their money outcome. When $\rho+\pi<1$, the part $1-\rho-\pi$ needs the consent of both parties before its allocation. Starkness below is taken as a measurement of the decider's ability to harm the proposer of a EPDG.

Definition 4. Let $r(x)\equiv\sup_{\gamma}\{g_1(1-x,\gamma)|g_1(1-x,\gamma)<1-x+x^*(Y-x)\}$. If $g_1(1-x,\gamma)=1-x+x^*(Y-x)$ for all γ, then $r(x)=0$. Starkness σ in a EPDG is defined as $\sigma=\min_{x-x^*(Y-x)\geq Y}\frac{1-x+x^*(Y-x)-r(x)}{1-x+x^*(Y-x)}$.

The measure $r(x)$ is the largest amount which the decider can give to the proposer who has just offered x to her without fully accepting the offer, from which we can derive a measure σ of the minimum the proposer can be punished (if he is punished at all) if he just offers x.

The EPDG above specifies payoffs in terms of money outcomes. Economical analysis specifies outcomes in terms of players' utilities, here we take the proposer's utility $U_1(g_1,g_2,x,Y)\equiv g_1$. We allow the decider to lower the proposer's payoff whenever the proposer has proposed an unfair split. The decider's payoff can depend on all the relevant variables and parameter on their monetary payoffs g_1 and g_2, the offer x, the minimum allowable offer Y, and withholding function x^* with respect to $Y-x$.

Definition 5. The decider's preferences $U_2(g_1,g_2,x,Y,x_0)$ are self-interest and tolerably fairness motivated (SITFM preferences henceforth) if there exists a function $E(Y)$ such that:

(1) U_2 is continuous in all variables;

(2) U_2 is increasing in g_2;

(3) U_2 is increasing in x_0;

(4) When $x-x^*(E(Y)-x)\geq E(Y)$, U_2 does not depend on g_1;

(5) When $x-x^*(E(Y)-x)<E(Y)$, U_2 is decreasing in g_1 for all $g_1>1-E(Y)$;

(6) For all x,Y and x^*, the set of indifference curves has slope $\frac{dg_2}{dg_1}>0$;

(7) There exists $K>0$ such that, for all x,Y,x^*,g_1, and g_2, these indifference curves have slope $\frac{dg_2}{dg_1}\leq K$;

(8) For all Y and $\epsilon > 0$, there exists $L(Y, \epsilon) > 0$ such that for all $x - x^*(E(Y) - x) < E(Y) - \epsilon$ and for all $g_1 \geq 1 - x$, the slope of these indifference curves $\frac{dg_2}{dg_1} \geq L(Y, \epsilon)$.

SITFM preferences are similar to SIFM preferences in (Rabin, 1997) in conditions (1-3) and (6-8), but condition (5-6) depends on $x^*(E(Y) - x)$, important here for a proposer's force and decider's deciding power, which is distinguished from (Rabin, 1997).

3. Full Efficiency

Given a class of EPDGs and SITM preferences, we present some general results that we can predict, and characterize full efficiency.

It is always feasible for the proposer to offer $(1 - x, x) = (1 - \max[E(Y), Y] + x^*(Y - E(Y)), \max[E(Y), Y] - x^*(Y - E(Y)))$ and the decider accepts, hence, any equilibrium in a EPDG must give at least $1 - E(Y) + x^*(Y - E(Y))$ to the Proposer.

Proposition 2. For all EPDGs, and all SITFM preferences, the proposer's equilibrium payoff must be at least $1 - E(Y) + x^*(Y - E(Y))$.

Proof. Given the assumptions, the proposer can get $1 - E(Y) + x^*(Y - E(Y))$ himself by offering $(1 - E(Y) + x^*(Y - E(Y)), E(Y) - x^*(Y - E(Y)))$, which lowers his payoff.

For any SITFM preferences and any allocation that gives the proposer at least a payoff of $1 - E(Y) + x^*(Y - E(Y))$, there exists a EPDG for which this allocation is the equilibrium.

Proposition 3. Given any Y and any SITFM preferences, for every $z_1 \in (\min[1 - E(Y) + x^*(Y - E(Y)), 1 - Y + x^*(Y - E(Y))], 1 - Y + x^*(Y - E(Y)))$ and every $z_2 \in [Y - x^*(Y - E(Y)), 1 - z_1]$, there exists a EPDG with $\rho = Y - x^*(Y - E(Y))$ such that (z_1, z_2) is the unique equilibrium of EPDG.

Proof. It can be adapted from the proof of Proposer 1 in (Rabin, 1997).

From the proposition above, for any allocation where the proposer gets more than $1 - E(Y) + x^*(Y - E(Y))$, there exists a bargaining structure yielding that outcome in the unique equilibrium, also any combination of offer by the proposer and degree of rejection by the decider is possible if the proposer gets more than $1 - E(Y) + x^*(Y - E(Y))$.

Since parties may sacrifice a little to cooperate with one another, we define full efficiency below as the maximum payoffs possible.

Definition 6. An equilibrium (z_1, z_2) is full efficient in a EPDG if it yields payoffs $g_1(z_1, \gamma^*) + g_2(z_2, \gamma^*) \equiv \max_{\gamma \in [0,1]}\{g_1(z_1, \gamma) + g_2(z_2, \gamma)\}$.

Proposition 4. For all SITFM preferences, and all Y, the outcome in every EPDG with starkness $\sigma > E(Y) - x^*(Y - E(Y))$ will be fully efficient.

Proof. Since the proposer can guarantee himself $1 - E(Y) + x^*(Y - E(Y))$ by offering $(1 - E(Y) + x^*(Y - E(Y)), E(Y) - x^*(Y - E(Y)))$, and any rejected offer will yield him $g_1 < 1 - Y - \sigma < (1 - E(Y) + x^*(Y - E(Y))$, he will never make an offer that will be rejected, hence the proposition follows.

4. Concluding Remarks

In this paper, incorporating tolerance to fair bargaining structure, we consider decider withholding cooperation from proposer even if the proposer mistreated the decider, and explore the proposer's force and the decider's deciding power. For self-interested and tolerably fair motivated preferences, we present some results predictable in classical game theoretic perspective. When full efficiency is obtained as the maximum payoffs possible are reached when parties make the type of small sacrifices people make all the time to cooperate with one another, we can characterize full efficiency.

Since our bargaining structure incorporates tolerance, our results extends (Bolton, 1991, Rabin, 1997), since Bolton (1991) assumes that the decider feels envy or inequality aversion, Rabin (1997) characterized how fairness affects bargaining efficiency and distribution of bargaining outcomes, but they all leaves tolerance aside of bargaining structure.

Two problems, as we know, are still open. In bargaining structures of more than two stages, full efficiency need be considered carefully since it is complicated to allow preferences to depend on the other player's behavior. In addition, full efficiency may be more complicated when the exact preferences of the decider are not common knowledge, since inefficiency can arise from complete information.

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Socially Acceptable Values for Cooperative TU Games

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Abstract. In the framework of the solution theory for cooperative transferable utility games, a value is called socially acceptable with reference to a certain basis of games if, for each relevant game, the payoff to any productive player covers the payoff to any non-productive player. Firstly, it is shown that two properties called desirability and monotonicity are sufficient to guarantee social acceptability of type I. Secondly, the main goal is to investigate and characterize the subclass of efficient, linear, and symmetric values that are socially acceptable for any of three types (with clear affinities to simple unanimity games).

Keywords: cooperative game, unanimity game, socially acceptable value, Shapley value, solidarity value, egalitarian value

Mathematics Subject Classification 2000: 91A12

1. Introduction and notions

Formally, a *transferable utility game* (or cooperative game or coalitional game with side payments) is a pair $\langle N, v\rangle$, where N is a finite set of at least two *players* and $v : 2^N \to \mathbb{R}$ is a *characteristic function* satisfying $v(\emptyset) = 0$. An element of N (notation: $i \in N$) and a nonempty subset S of N (notation: $S \subseteq N$ or $S \in 2^N$ with $S \neq \emptyset$) is called a *player* and *coalition* respectively, and the real number $v(S)$ is called the *worth* of coalition S. A TU game $\langle N, v\rangle$ is called *monotonic* if $v(S) \leq v(T)$ for all $S, T \subseteq N$ with $S \subseteq T$. The size (cardinality) of coalition S is denoted by $|S|$ or, if no ambiguity is possible, by s. Particularly, n denotes the size of the player set N. Let $\mathcal{G}_N$ denote the linear space consisting of all games with fixed player set N. Given two games $\langle N, v\rangle$, $\langle N, w\rangle$, and two scalars $\beta, \delta \in \mathbb{R}$, their *linear combination* $\langle N, \beta \cdot v + \delta \cdot w\rangle$ is defined by $(\beta \cdot v + \delta \cdot w)(S) = \beta \cdot v(S) + \delta \cdot w(S)$ for all $S \subseteq N$.

The solution part of cooperative game theory deals with the allocation problem of how to divide, for any game $\langle N, v\rangle$, the worth $v(N)$ of the grand coalition N among the players. The traditional one-point solution concepts associate, with every game, a single allocation called the value of the game. Formally, a *value* on $\mathcal{G}_N$ is a function ψ that assigns a single payoff vector $\psi(N, v) = (\psi_i(N, v))_{i \in N} \in \mathbb{R}^N$ to every TU game $\langle N, v\rangle$. The so-called *value* $\psi_i(N, v)$ of player i in the TU game $\langle N, v\rangle$ represents an assessment by i of his gains for participating in the game. For instance, the *egalitarian value* ψ^{EG} allocates the same payoff to every player in that $\psi_i^{EG}(N, v) = \frac{v(N)}{n}$ for all games $\langle N, v\rangle$ and all $i \in N$. Throughout the paper we restrict ourselves to the class of efficient, linear, and symmetric values.

Definition 1. A value ψ on $\mathcal{G}_N$ is said to possess

(i) *efficiency*, if $\sum_{i \in N} \psi_i(N,v) = v(N)$ for all games $\langle N,v \rangle$;

(ii) *linearity*, if $\psi(N, \beta \cdot v + \delta \cdot w) = \beta \cdot \psi(N,v) + \delta \cdot \psi(N,w)$ for all games $\langle N,v \rangle$, $\langle N,w \rangle$, and all scalars $\beta, \delta \in \mathbb{R}$;

(iii) *symmetry*, if $\psi_{\pi(i)}(N, \pi v) = \psi_i(N,v)$ for all games $\langle N,v \rangle$, all $i \in N$, and every permutation π on N. Here the game $\langle N, \pi v \rangle$ is defined by $(\pi v)(\pi S) := v(S)$ for all $S \subseteq N$.

Our main goal is to develop the notion of social acceptability on the class of efficient, linear, and symmetric values. Undoubtedly, the Shapley value (Shapley, 1953) is the most appealing value of this class, whereas the solidarity value introduced in (Nowak and Radzik, 1994) has clear affinities to the Shapley value. In fact, these clear affinities have been stressed in Calvo's approach (Calvo, 2008) to non-transferable utility (NTU) games (inclusive of TU games) by introducing the so-called "random marginal NTU value" and "random removal NTU value" as the NTU counterparts of the Shapley TU value and the solidarity TU value, respectively, in the sense that pairwise coincidence of values happens to occur on the class of TU games. Surprisingly, it turns out that the solidarity value and the various social acceptability notions are well-matched. In order to review similar axiomatizations of both the Shapley value and the solidarity value, we recall three essential properties of values for TU games.

Definition 2. A value ψ on $\mathcal{G}_N$ possesses

(i) *substitution property*, if $\psi_i(N,v) = \psi_j(N,v)$ for all games $\langle N,v \rangle$, all pairs $i,j \in N$, such that players i and j are *substitutes* in the game $\langle N,v \rangle$, i.e., $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i,j\}$;

(ii) *null player property*, if $\psi_i(N,v) = 0$ for all games $\langle N,v \rangle$, all $i \in N$, such that player i is a *null player* in the game $\langle N,v \rangle$, i.e., $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \backslash \{i\}$;

(iii) *A-null player property*, if $\psi_i(N,v) = 0$ for all games $\langle N,v \rangle$, all $i \in N$, such that player i is a *A-null player* in the game $\langle N,v \rangle$, i.e., $\sum_{k \in S} \Big[v(S) - v(S \backslash \{k\}) \Big] = 0$ for all $S \subseteq N$ with $i \in S$.

It is well-known that the symmetry property implies the substitution property. (Shapley, 1953) and (Nowak and Radzik, 1994) proved that there exists a unique value on $\mathcal{G}_N$ satisfying the following four properties: efficiency, linearity, symmetry, and either null player property or A-null player property. In fact, the explicit formulas for the *Shapley value* $\psi^{Sh}(N,v) = (\psi_i^{Sh}(N,v))_{i \in N}$ and the *solidarity value* $\psi^{Sol}(N,v) = (\psi_i^{Sol}(N,v))_{i \in N}$ are as follows (Shapley, 1953; Roth, 1988; Driessen, 1988; Nowak and Radzik, 1994): for all $i \in N$

$$\psi_i^{Sh}(N,v) = \sum_{S \subseteq N \backslash \{i\}} \frac{1}{n \cdot \binom{n-1}{s}} \cdot \Big[v(S \cup \{i\}) - v(S) \Big], \tag{1}$$

or equivalently,

$$\psi_i^{Sh}(N,v) = \sum_{\substack{T\subseteq N,\\ T\ni i}} \frac{1}{n\cdot\binom{n-1}{t-1}} \cdot \Big[v(T) - v(T\backslash\{i\})\Big];$$

$$\psi_i^{Sol}(N,v) = \sum_{\substack{T\subseteq N,\\ T\ni i}} \frac{1}{n\cdot\binom{n-1}{t-1}} \cdot \frac{1}{t} \cdot \sum_{k\in T}\Big[v(T) - v(T\backslash\{k\})\Big]. \tag{2}$$

According to the so-called "Equivalence Theorem" concerning the class of efficient, linear, and symmetric values, the following equivalent interpretations will be exploited throughout the remainder of this paper (cf. (Driessen and Radzik, 2002), (Driessen and Radzik, 2003), (Ruiz et al., 1998)).

Theorem 1. *The next four statements for a value ψ on $\mathcal{G}_N$ are equivalent.*

(i) ψ verifies efficiency, linearity, and symmetry;

(ii) There exists a unique collection of constants $\{\rho_k\}_{k=1}^n$ with $\rho_n = 1$ such that, for every n-person game $\langle N,v\rangle$ with at least two players, the value payoff vector $(\psi_i(N,v))_{i\in N}$ is of the following form (cf. (Ruiz et al., 1998), Lemma 9, page 117): for all $i\in N$

$$\psi_i(N,v) = \sum_{\substack{S\subseteq N,\\ S\ni i}} \frac{\rho_s}{s}\cdot v(S) - \sum_{\substack{S\subseteq N,\\ S\not\ni i}} \frac{\rho_s}{n-s}\cdot v(S); \tag{3}$$

(iii) There exists a unique collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$ such that, for every n-person game $\langle N,v\rangle$ with at least two players, the value payoff vector $(\psi_i(N,v))_{i\in N}$ is of the following form (cf. (Driessen and Radzik, 2002), (Driessen and Radzik, 2003)): for all $i\in N$

$$\psi_i(N,v) = \sum_{S\subseteq N\backslash\{i\}} \frac{1}{n\cdot\binom{n-1}{s}} \cdot \Big[b_{s+1}\cdot v(S\cup\{i\}) - b_s\cdot v(S)\Big]; \tag{4}$$

*(iv) There exists a unique collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$ such that $\psi(N,v) = \psi^{Sh}(N,\mathcal{B}v)$ for every n-person game $\langle N,v\rangle$ with at least two players. Here the n-person game $\langle N,\mathcal{B}v\rangle$, called $\mathcal{B}$-*scaled game*, is defined by $(\mathcal{B}v)(S) = b_s\cdot v(S)$ for all $S\subseteq N$, $S\neq\emptyset$.*

By straightforward computations, the reader may verify that the expression on the right hand of (3) agrees with the one on the right hand of (4) by choosing $b_k = \binom{n}{k}\cdot\rho_k$ for all $k = 1,2,\ldots,n$. Clearly, the expression on the right hand of (4) reduces to the Shapley value payoff (1) of player i in the n-person game $\langle N,v\rangle$ itself (denoted by $\psi = \psi^{Sh}$) whenever $b_k = 1$ for all $k = 1,2,\ldots,n$, that is $\rho_k = \binom{n}{k}^{-1} = \frac{k!\cdot(n-k)!}{n!}$.

Remark 1. Let ψ be an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to the collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$. Fix two players $i\in N$, $j\in N$, $i\neq j$. Without going into details, by distinguishing between coalitions containing none, one or both players, straightforward calculations yield the next relationship about the difference between the value payoffs of both players:

$$\psi_j(N,v) - \psi_i(N,v) = \sum_{S \subseteq N \setminus \{i,j\}} \frac{\gamma(n-1,s)}{n} \cdot b_{s+1} \cdot \Big[v(S \cup \{j\}) - v(S \cup \{i\}) \Big] \quad (5)$$

where $\gamma(n-1,s) = \frac{s! \cdot (n-2-s)!}{(n-1)!}$ for all $s = 0, 1, 2, \ldots, n-2$.

Generally speaking, in view of (1), the right hand of (4) equals the Shapley value payoff $Sh_i(N, \mathcal{B}v)$ of player i in the $\mathcal{B}$-scaled game $\langle N, \mathcal{B}v \rangle$. In summary, the Equivalence Theorem 1 states that a value ψ is efficient, linear, and symmetric if and only if the ψ-value of a game coincides with the Shapley value of the $\mathcal{B}$-scaled game (denoted by $\psi(N,v) = \psi^{Sh}(N, \mathcal{B}v)$). We call ψ the per-capita Shapley value whenever $b_s = \frac{1}{s}$ for all $s = 1, 2, \ldots, n-1$. It appears that the solidarity value $\psi^{Sol}(N,v)$ of the form (2) arises whenever $b_s = \frac{1}{s+1}$ for all $s = 1, 2, \ldots, n-1$, that is $\rho_s = \frac{s! \cdot (n-s)!}{n! \cdot (s+1)}$. As a last, but appealing example, ψ is called a *discount Shapley value* if there exists a discount factor $0 < \delta \leq 1$ such that the value payoff $\psi(N,v)$ is of the form (4) with reference to the collection of constants $b_s = \delta^{n-s}$ for all $s = 1, 2, \ldots, n$, that is the larger the coalition size, the larger the discount factor δ^{n-s} of the worth $v(S)$ of any coalition S.

Remark 2. For future purposes, we list a number of combinatorial (in)equalities.

$$\sum_{k=t}^{n-1} \binom{k}{t} \cdot \tfrac{1}{k} = t^{-1} \cdot \binom{n-1}{t} \qquad \text{for all } t = 1, 2, \ldots, n-1. \quad (6)$$

$$\sum_{k=t}^{n-1} \binom{k}{t} \cdot \tfrac{n}{k^2 \cdot (k+1)} \leq t^{-1} \cdot \binom{n-1}{t} \qquad \text{for all } t = 1, 2, \ldots, n-1. \quad (7)$$

$$\sum_{k=t}^{n-1} \binom{k}{t} \cdot \tfrac{n}{k \cdot (n-k) \cdot (n-k+1)} \geq t^{-1} \cdot \binom{n-1}{t} \qquad \text{for all } t = 1, 2, \ldots, n-1. \quad (8)$$

The proofs of both (6) and (7) proceed by induction on n and are left to the reader. In fact, (8) applied to $t = 1$ reduces to an equality since it concerns a telescoping sum. For all $t \geq 2$, the expression on the left hand of (8) applied to $k = n-1$ already covers the single term on the right hand.

2. Socially Acceptable Values of Three Types

Any linear value ψ on $\mathcal{G}_N$ is fully determined by the value payoffs of games that form a basis of $\mathcal{G}_N$. It is well-known that the collection of simple *unanimity games* $\mathcal{U} = \{\langle N, u_T \rangle \mid T \subseteq N, T \neq \emptyset\}$ forms a $(2^n - 1)$-dimensional basis of $\mathcal{G}_N$. Here the $\{0,1\}$-*unanimity game* $\langle N, u_T \rangle$ is defined by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. In order to simplify forthcoming mathematical expressions, we prefer to deal with the adapted collection of non-simple unanimity games $\langle N, u_T^t \rangle$, $T \subseteq N$, $T \neq \emptyset$, given by $u_T^t(S) = t$ if $T \subseteq S$, and $u_T^t(S) = 0$ otherwise. Throughout this paper we aim to investigate the value payoffs for *productive players* of T in comparison with *non-productive players* of $N \setminus T$.

Definition 3. A value ψ on $\mathcal{G}_N$ is called *socially acceptable of type I* if the collection of value payoffs $(\psi_k(N, u_T^t))_{k\in N}$ of any adapted unanimity game $\langle N, u_T^t\rangle$ are such that, for all $T \subseteq N$, $T \neq \emptyset$, every productive player of T receives at least as much as every non-productive player of $N\backslash T$, that is

$$\psi_i(N, u_T^t) \geq \psi_j(N, u_T^t) \geq 0 \qquad \text{for all } i \in T, \text{ and all } j \in N\backslash T. \tag{9}$$

Remark 3. Since non-productive players $j \in N\backslash T$ are null players in the adapted unanimity game $\langle N, u_T^t\rangle$, their Shapley value payoff $\psi_j^{Sh}(N, u_T^t) = 0$, whereas productive players $i \in T$ are treated as substitutes who allocate the worth $u_T(N) = t$ equally in that $\psi_i^{Sh}(N, u_T^t) = 1$. Without going into details, it is possible to derive from (2) that the solidarity value payoffs for these adapted unanimity games are bounded such that for all $T \subseteq N$, $T \neq \emptyset$,

$$0 < \psi_j^{Sol}(N, u_T^t) < \tfrac{t}{n} \qquad \text{if } j \in N\backslash T \qquad \textit{and} \qquad \tfrac{t}{n} < \psi_i^{Sol}(N, u_T^t) < 1 \qquad \text{if } i \in T.$$

In words, the egalitarian value, the Shapley value and the solidarity value are socially acceptable of type I in that these three linear values favour, in a weak or strict sense, the productive players to the non-productive players of any (adapted) unanimity game. We remark that non-linear values like the nucleolus (Schmeidler, 1969) and the τ-value (Tijs, 1981) are also socially acceptable in that, for simple unanimity games, both of them coincide with the Shapley value. As already mentioned, we restrict ourselves to the class of efficient, linear, and symmetric values.

Definition 4. Let the collection $\mathcal{W} = \{\langle N, w_T\rangle \mid T \subseteq N, T \neq \emptyset\}$ of *coalition-size dependent unanimity games* be defined by $w_T(S) = \frac{s}{t} \cdot \binom{s}{t}^{-1}$ if $T \subseteq S$, and $w_T(S) = 0$ otherwise.
A value ψ on $\mathcal{G}_N$ is called *socially acceptable of type II* if the collection of value payoffs $(\psi_k(N, w_T))_{k\in N}$ of any coalition-size dependent unanimity game $\langle N, w_T\rangle$ are such that, for all $T \subseteq N$, $T \neq \emptyset$, every productive player of T receives at least as much as every non-productive player of $N\backslash T$, that is

$$\psi_i(N, w_T) \geq \psi_j(N, w_T) \geq 0 \qquad \text{for all } i \in T, \text{ and all } j \in N\backslash T. \tag{10}$$

Remark 4. The $(2^n - 1)$-dimensional collection $\mathcal{W}$ of coalition-size dependent unanimity games forms a basis of $\mathcal{G}_N$ since, for any TU game $\langle N, v\rangle$, its game representation is given by $v = \sum_{T\subseteq N} \alpha_T^v \cdot w_T$, where $\alpha_T^v = v(T)$ if $|T| = 1$ and $\alpha_T^v = v(T) - \sum_{k\in T} \frac{v(T\backslash\{k\})}{|T|-1}$ if $|T| \geq 2$. Notice that $w_T(T) = 1$ and further, $w_T(N) < 1$ if and only if $1 < t < n$.
In this setting, a player i is called a *scale dummy* in the game $\langle N, v\rangle$ if, for all $S \subseteq N$ with $|S| \geq 2$ containing i, it holds $\sum_{k\in S} v(S\backslash\{k\}) = (|S| - 1) \cdot v(S)$. Particularly, any player $j \in N\backslash T$ is a scale dummy in the coalition-size dependent unanimity game $\langle N, w_T\rangle$.

Definition 5. Let the collection $\mathcal{Z} = \{\langle N, z_T^t\rangle \mid T \subsetneq N\}$ of *complementary unanimity games* be defined by $z_T^t(S) = t$ if $S \cap T = \emptyset$, $S \neq \emptyset$, and $z_T(S) = 0$ otherwise. Note that $z_T^t(N) = 0$ whenever $T \neq \emptyset$. In case $T = \emptyset$, then $z_\emptyset^0(S) = 1$ for all $S \subseteq N$, $S \neq \emptyset$, and all players are substitutes in the unitary game $\langle N, z_\emptyset^0\rangle$.
A value ψ on $\mathcal{G}_N$ is called *socially acceptable of type III* if the collection of value payoffs $(\psi_k(N, z_T^t))_{k\in N}$ of any complementary unanimity game $\langle N, z_T^t\rangle$ are such

that, for all $T \subsetneqq N$, $T \neq \emptyset$, every player of T (considered as an enemy) receives at most as much as every player of $N \backslash T$ (considered as a friend), of which the payoff is bounded above by the ratio of the number of enemies to the number of players, that is

$$\psi_i(N, z_T^t) \leq \psi_j(N, z_T^t) \leq \frac{t}{n} \qquad \text{for all } i \in T, \text{ and all } j \in N \backslash T. \tag{11}$$

This paper is organized as follows. In Sections 3. and 4. we investigate and characterize the class of efficient, linear, and symmetric values that verify the social acceptability. In Section 3. it is shown that two additional properties called desirability and monotonicity are sufficient to guarantee social acceptability of type I because of unitary conditions $0 \leq b_k \leq 1$ for all $k = 1, 2, \ldots, n-1$. In Section 4. the main goal is, given an efficient, linear, and symmetric value ψ, to determine the exact conditions for social acceptability of each of three types, in terms of column sums of suitably chosen $n \times n$ lower triangular matrices A^{ψ}, B^{ψ}, and C^{ψ} respectively. Section 5. contains some concluding remarks. Throughout this paper we deal with efficient, linear, and symmetric values in such a way that the value representation (4) with reference to the collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ is the most appropriate tool.

3. A Sufficient Property for Social Acceptability of Values

To start with, we list the following two properties of values that turn out to be sufficient for social acceptability of type I.

Definition 6. Let ψ be a value on $\mathcal{G}_N$.

(i) ψ satisfies *desirability* if $\psi_j(N, v) \leq \psi_i(N, v)$ whenever player j is *less desirable* than player i in the game $\langle N, v \rangle$, that is $v(S \cup \{j\}) \leq v(S \cup \{i\})$ for all $S \subseteq N \backslash \{i, j\}$.

(ii) ψ satisfies *monotonicity* if $\psi_i(N, v) \geq 0$ for all $i \in N$ and every *monotonic game* $\langle N, v \rangle$.

Theorem 2. *If a value ψ on $\mathcal{G}_N$ verifies both desirability and monotonicity, then ψ is socially acceptable of type I.*

Proof. Suppose a value ψ on $\mathcal{G}_N$ verifies both desirability and monotonicity. Let $T \subseteq N$, $T \neq \emptyset$, $i \in T$, $j \in N \backslash T$. Since $u_T^t(S \cup \{j\}) = 0 \leq u_T^t(S \cup \{i\})$ for all $S \subseteq N \backslash \{i, j\}$, we obtain that player j is less desirable than player i in the adapted unanimity game $\langle N, u_T^t \rangle$. ¿From the desirability property of ψ, we derive $\psi_j(N, u_T^t) \leq \psi_i(N, u_T^t)$. Because the adapted unanimity game $\langle N, u_T^t \rangle$ is monotonic, it follows from the monotonicity property of ψ that $\psi_k(N, u_T^t) \geq 0$ for all $k \in N$. So, ψ is socially acceptable of type I. □

Neither the coalition-size dependent unanimity games $\langle N, w_T \rangle$ nor the complementary unanimity games $\langle N, z_T^t \rangle$ are monotonic games, so the latter proof does not apply in their context. Next we show that the two properties of desirability and monotonicity are equivalent to $[0, 1]$ boundedness for the underlying collection of constants associated with any efficient, linear, and symmetric value.

Theorem 3. *Let ψ be an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to a collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$.*

(i) ψ verifies desirability if and only if $b_k \geq 0$ for all $k = 1, 2, \ldots, n-1$.
(ii) ψ verifies desirability and monotonicity if and only if $0 \leq b_k \leq 1$ for all $k = 1, 2, \ldots, n-1$.

Proof. (i). If $b_k \geq 0$ for all $k = 1, 2, \ldots, n-1$, then the desirability property of ψ follows immediately from (5). In order to prove the converse statement, suppose ψ verifies desirability. Fix any two players $i \in N$, $j \in N$, and any coalition $T \subseteq N \backslash \{i, j\}$. Define the n-person game $\langle N, w\rangle$ by $w(T \cup \{i\}) = 1$ and $w(S) = 0$ for all $S \subseteq N$, $S \neq T \cup \{i\}$. On the one hand, from (5) we derive $\psi_j(N, w) - \psi_i(N, w) = -\frac{\gamma(n-1,t)}{n} \cdot b_{t+1}$. On the other, player j is less desirable than player i in the game $\langle N, w\rangle$, and so, the desirability property of ψ implies $\psi_j(N, w) \leq \psi_i(N, w)$. We conclude that $b_{t+1} \geq 0$ for all $t = 0, 1, \ldots, n-2$. This proves the statement in part (i).

(ii) Suppose ψ verifies monotonicity. Let $k = 1, 2, \ldots, n-1$ and fix player $i \in N$. Define the n-person game $\langle N, u\rangle$ by $u(S) = 1$ if either $i \in S$ and $s \geq k+1$ or $i \notin S$ and $s \geq k$, and $u(S) = 0$ otherwise. On the one hand, the game $\langle N, u\rangle$ is monotonic and so, the monotonicity property of ψ implies $\psi_i(N, u) \geq 0$. On the other, from (4) we derive

$$\psi_i(N, u) = \sum_{S \subseteq N \backslash \{i\}} \frac{1}{n \cdot \binom{n-1}{s}} \cdot \Big[b_{s+1} \cdot u(S \cup \{i\}) - b_s \cdot u(S) \Big]$$

$$= \sum_{\substack{S \subseteq N \backslash \{i\}, \\ s \geq k}} \frac{b_{s+1} - b_s}{n \cdot \binom{n-1}{s}} = \sum_{s=k}^{n-1} \binom{n-1}{s} \cdot \frac{b_{s+1} - b_s}{n \cdot \binom{n-1}{s}} = \frac{b_n - b_k}{n}.$$

Recall $b_n = 1$. We obtain that $\psi_i(N, u) = \frac{1-b_k}{n} \geq 0$ and hence, $b_k \leq 1$ for all $k = 1, 2, \ldots, n-1$. The technical proof of the converse statement is postponed till the end of Section 5.. □

Unfortunately, in the setting of efficient, linear, and symmetric values, it turns out that both the desirability and monotonicity conditions are not necessary for the value to be socially acceptable. That is, the class of socially acceptable values strictly contains the class of values verifying the desirability and monotonicity properties. In the next section we provide a full characterization of socially acceptable values of each of three types.

4. Characterizations of Socially Acceptable Values

In the setting of values satisfying the substitution property, it suffices to distinguish two types of players, called productive players (members of a certain coalition T) and non-productive players (nonmembers of T), respectively. For any efficient value ψ on $\mathcal{G}_N$ satisfying the substitution property, the efficiency condition applied to the adapted unanimity game $\langle N, u_T^t\rangle$ reduces to the equality $t \cdot \psi_i(N, u_T^t) + (n-t) \cdot \psi_j(N, u_T^t) = t$ for all $t = 1, 2, \ldots, n$, for all $i \in T$, $j \in N \backslash T$. Consequently, by (9), an efficient and symmetric value ψ on $\mathcal{G}_N$ is socially acceptable of type I if and only if

$$\frac{t}{n} \leq \psi_i(N, u_T^t) \leq 1 \qquad \text{for all } T \subseteq N,\ T \neq \emptyset, \text{ all } i \in T. \tag{12}$$

Theorem 4. *Let ψ be an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to the collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$. With the* *value ψ, there is associated the $n \times n$ lower triangular matrix A^{ψ} of which the rows are indexed by the coalition size k, and the columns by the number t of productive players in the adapted unanimity games, such that each matrix entry $[A^{\psi}]_{k,t}$ is given by $[A^{\psi}]_{k,t} = \binom{k}{t} \cdot \frac{b_k}{k}$ if $t \leq k \leq n$, and $[A^{\psi}]_{k,t} = 0$ otherwise. Then the value ψ is socially acceptable of type I if and only if the sum of the entries in each column (except for the entry in the last row n) of A^{ψ} is not less than zero, and not more than $t^{-1} \cdot \binom{n-1}{t}$ with reference to its t-th column. That is,*

$$0 \leq \sum_{k=t}^{n-1} [A^{\psi}]_{k,t} \leq t^{-1} \cdot \binom{n-1}{t} \qquad \text{for all } t = 1, 2, \ldots, n-1. \tag{13}$$

Proof. Fix $T \subsetneq N$, $T \neq \emptyset$, and $i \in T$. Then $u_T^t(S) = 0$ for all $S \subseteq N \backslash \{i\}$. From (4) and some combinatorial calculations, we derive

$$\begin{aligned}
\psi_i(N, u_T^t) &= \sum_{S \subseteq N \backslash \{i\}} \frac{1}{n \cdot \binom{n-1}{s}} \cdot \Big[b_{s+1} \cdot u_T^t(S \cup \{i\}) - b_s \cdot u_T^t(S) \Big] \\
&= \sum_{S \subseteq N \backslash \{i\}} \frac{b_{s+1} \cdot u_T^t(S \cup \{i\})}{n \cdot \binom{n-1}{s}} = \sum_{T \backslash \{i\} \subseteq S \subseteq N \backslash \{i\}} \frac{t \cdot b_{s+1}}{n \cdot \binom{n-1}{s}} \\
&= \sum_{s=t-1}^{n-1} \binom{n-t}{s-t+1} \cdot \frac{t \cdot b_{s+1}}{n \cdot \binom{n-1}{s}} = \sum_{k=t}^{n} \binom{n-t}{k-t} \cdot \frac{t \cdot b_k}{n \cdot \binom{n-1}{k-1}} \\
&= t \cdot \sum_{k=t}^{n} \frac{\binom{k}{t}}{\binom{n}{t}} \cdot \frac{b_k}{k} = t \cdot \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n} [A^{\psi}]_{k,t}
\end{aligned}$$

From this we conclude that (12) holds if and only if $\frac{t}{n} \leq t \cdot \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n} [A^{\psi}]_{k,t} \leq 1$ if and only if $0 \leq t \cdot \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n-1} [A^{\psi}]_{k,t} \leq 1 - \frac{t}{n}$ or equivalently, (13) holds. □

Each non-zero entry of the k-th row of matrix A^{ψ} is proportional to the average expression $\frac{b_k}{k}$. Clearly, the egalitarian value is socially acceptable of type I since it arises as one extreme case whenever the whole matrix A^{ψ}, except for its bottom row, equals zero (or equivalently, $b_k = 0$ for all $k = 1, 2, \ldots, n-1$). The Shapley value ψ^{Sh}, associated with the unitary collection $b_k = 1$ for all $k = 1, 2, \ldots, n$, arises as the second extreme case in that the inequalities on the right hand of (13) are met as combinatorial equalities (to be verified by induction on the number n of players). Any linear combination $\psi^{\beta} = (1-\beta) \cdot \psi^{EG} + \beta \cdot \psi^{Sh}$ is of the form (4) with reference to a constant collection $b_k = \beta$ for all $k = 1, 2, \ldots, n-1$, and such value ψ^{β} is socially acceptable of type I if and only if $0 \leq \beta \leq 1$. Moreover, the solidarity value ψ^{Sol} is socially acceptable of type I since its associated collection $b_k = \frac{1}{k+1} \leq 1$ for all $k = 1, 2, \ldots, n-1$.
Generally speaking, (13) applied to $t = n-1$ and $t = 1$ respectively require $0 \leq b_{n-1} \leq 1$ and $0 \leq \sum_{k=1}^{n-1} b_k \leq n-1$. In case $n = 3$, the social acceptability condition (13) reduces to both $0 \leq b_2 \leq 1$ and $0 \leq b_1 + b_2 \leq 2$, whereas, in case $n = 4$, (13) reduces to $0 \leq b_2 \leq 1$, $0 \leq b_1 + b_2 + b_3 \leq 3$, together with $b_2 + 2 \cdot b_3 \leq 3$.

Further, observe that a $n \times n$ lower triangular matrix $A = [A]_{k,t}$ induces an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to a collection of constants $\{b_k\}_{k=1}^n$ with $b_n = 1$ provided that, for all $1 \leq k \leq n-1$, $\binom{k}{t}^{-1} \cdot [A]_{k,t}$ is the same for all $1 \leq t \leq k$.

In the context of values satisfying the substitution property, the efficiency condition applied to the coalition-size dependent unanimity game $\langle N, w_T \rangle$ reduces to the equality $t \cdot \psi_i(N, w_T) + (n-t) \cdot \psi_j(N, w_T) = \frac{n}{t} \cdot \binom{n}{t}^{-1}$ for all $t = 1, 2, \ldots, n$, for all $i \in T$, $j \in N \backslash T$. Thus, by (10), an efficient and symmetric value ψ on $\mathcal{G}_N$ is socially acceptable of type II if and only if

$$\frac{1}{t} \cdot \binom{n}{t}^{-1} \leq \psi_i(N, w_T) \leq \frac{n}{t^2} \cdot \binom{n}{t}^{-1} \qquad \text{for all } T \subseteq N,\ T \neq \emptyset, \text{ all } i \in T. \tag{14}$$

Theorem 5. *Let ψ be an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to the collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$. With the value ψ, there is associated the $n \times n$ lower triangular matrix B^ψ of which the rows are indexed by the coalition size k, and the columns by the number t of productive players in the coalition-size dependent unanimity games, such that each matrix entry $[B^\psi]_{k,t}$ is given by $[B^\psi]_{k,t} = b_k$ if $t \leq k \leq n$, and $[B^\psi]_{k,t} = 0$ otherwise. Then the value ψ is socially acceptable of type II if and only if the sum of the entries in each column (except for the entry in the last row n) of B^ψ is not less than zero, and not more than $\frac{n-t}{t}$ with reference to its t-th column. That is,*

$$0 \leq \sum_{k=t}^{n-1} [B^\psi]_{k,t} \leq \frac{n-t}{t} \qquad \textit{for all } t = 1, 2, \ldots, n-1. \tag{15}$$

Proof. The same proof technique applies as before by modifying the choice of the basis of $\mathcal{G}_N$. Fix $T \subsetneq N$, $T \neq \emptyset$, and $i \in T$. Then $w_T(S) = 0$ for all $S \subseteq N \backslash \{i\}$. In the current framework, from (4) and some combinatorial calculations, we derive

$$\begin{aligned}
\psi_i(N, w_T) &= \sum_{S \subseteq N \backslash \{i\}} \frac{b_{s+1} \cdot w_T(S \cup \{i\})}{n \cdot \binom{n-1}{s}} = \sum_{T \backslash \{i\} \subseteq S \subseteq N \backslash \{i\}} \frac{s+1}{t} \cdot \binom{s+1}{t}^{-1} \cdot \frac{b_{s+1}}{n \cdot \binom{n-1}{s}} \\
&= \sum_{s=t-1}^{n-1} \binom{n-t}{s-t+1} \cdot \frac{s+1}{t} \cdot \binom{s+1}{t}^{-1} \cdot \frac{b_{s+1}}{n \cdot \binom{n-1}{s}} = \sum_{k=t}^{n} \binom{n-t}{k-t} \cdot \frac{k}{t} \cdot \binom{k}{t}^{-1} \cdot \frac{b_k}{n \cdot \binom{n-1}{k-1}} \\
&= t^{-1} \cdot \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n} b_k = t^{-1} \cdot \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n} [B^\psi]_{k,t}
\end{aligned}$$

From this we conclude that (14) holds if and only if $t^{-1} \cdot \binom{n}{t}^{-1} \leq t^{-1} \cdot \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n} [B^\psi]_{k,t} \leq \frac{n}{t^2} \cdot \binom{n}{t}^{-1}$ if and only if $1 \leq \sum_{k=t}^{n} [B^\psi]_{k,t} \leq \frac{n}{t}$ or equivalently, (15) holds. □

Clearly, the egalitarian value is socially acceptable of type II, whereas the Shapley value, associated with the unitary collection, fails to be of type II. The extreme case in that the inequalities of (15) are met as combinatorial equalities happens for the collection of constants $b_k = \frac{n}{k \cdot (k+1)}$ for all $k = 1, 2, \ldots, n-1$, because of its telescoping sum. Consequently, the solidarity value ψ^{Sol} is socially acceptable of type II

since its associated collection $b_k = \frac{1}{k+1} \leq \frac{n}{k\cdot(k+1)}$ for all $k = 1, 2, \ldots, n-1$. Due to the development of the theory about social acceptability of type II, we end up with the introduction of an appealing value on $\mathcal{G}_N$ transforming an n-person game $\langle N, v\rangle$ into its per-capita game $\langle N, v_{pc}\rangle$, applying the solidarity value and finally, repairing efficiency in a multiplicative fashion. In formula, $\psi(N, v) = n\cdot\psi^{Sol}(N, v_{pc})$ where the characteristic function of the per-capita game $\langle N, v_{pc}\rangle$ is defined by $v_{pc}(S) = \frac{v(S)}{|S|}$ for all $S \subseteq N$, $S \neq \emptyset$.

In the framework of values satisfying the substitution property, the efficiency condition applied to the complementary unanimity game $\langle N, z_T^t\rangle$ reduces to the equality $t \cdot \psi_i(N, z_T^t) + (n-t) \cdot \psi_j(N, z_T^t) = 0$ for all $t = 1, 2, \ldots, n-1$, for all $i \in T$, $j \in N\backslash T$. Thus, by (11), an efficient and symmetric value ψ on $\mathcal{G}_N$ is socially acceptable of type III if and only if

$$\frac{t}{n} - 1 \leq \psi_i(N, z_T^t) \leq 0 \qquad \text{for all } T \subsetneqq N,\ T \neq \emptyset, \text{ all } i \in T. \tag{16}$$

Theorem 6. *Let ψ be an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to the collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$. With the value ψ, there is associated the $n \times n$ lower triangular matrix C^ψ of which the rows are indexed by the coalition size k, and the columns by the number t of productive players in the complementary unanimity games, such that each matrix entry $[C^\psi]_{k,t}$ is given by $[C^\psi]_{k,t} = \binom{k}{t} \cdot \frac{b_{n-k}}{k}$ if $t \leq k \leq n-1$, and $[C^\psi]_{k,t} = 0$ otherwise. Then the value ψ is socially acceptable of type III if and only if the sum of the entries in each column (except for the entry in the last row n) of C^ψ is not less than zero, and not more than $t^{-1} \cdot \binom{n-1}{t}$ with reference to its t-th column. That is,*

$$0 \leq \sum_{k=t}^{n-1} [C^\psi]_{k,t} \leq t^{-1} \cdot \binom{n-1}{t} \qquad \text{for all } t = 1, 2, \ldots, n-1. \tag{17}$$

Proof. The same proof technique applies as before by modifying the choice of the basis of $\mathcal{G}_N$. Fix $T \subsetneqq N$, $T \neq \emptyset$, and $i \in T$. Then $z_T^t(S \cup \{i\}) = 0$ for all $S \subseteq N\backslash\{i\}$. In the current framework, from (4) and some combinatorial calculations, we derive

$$\begin{aligned}
\psi_i(N, z_T^t) &= \sum_{\substack{S \subseteq N\backslash\{i\},\\ S\neq\emptyset}} \frac{-b_s \cdot z_T^t(S)}{n\cdot\binom{n-1}{s}} = -\sum_{\substack{\emptyset\neq S \subseteq N\backslash\{i\},\\ S\cap T=\emptyset}} \frac{t\cdot b_s}{n\cdot\binom{n-1}{s}} = -\sum_{\substack{S \subseteq N\backslash T,\\ S\neq\emptyset}} \frac{t\cdot b_s}{n\cdot\binom{n-1}{s}} \\
&= -\sum_{s=1}^{n-t} \binom{n-t}{s} \cdot \frac{t\cdot b_s}{n\cdot\binom{n-1}{s}} = -\sum_{s=1}^{n-t} \frac{\binom{n-s}{t}}{\binom{n}{t}} \cdot \frac{t\cdot b_s}{n-s} \\
&= -t \cdot \sum_{k=t}^{n-1} \frac{\binom{k}{t}}{\binom{n}{t}} \frac{b_{n-k}}{k} = -t \cdot \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n-1} [C^\psi]_{k,t}
\end{aligned}$$

From this we conclude that (16) holds if and only if $\frac{t}{n} - 1 \leq -t\cdot\binom{n}{t}^{-1} \cdot \sum_{k=t}^{n-1} [C^\psi]_{k,t} \leq 0$ if and only if $0 \leq \sum_{k=t}^{n-1} [C^\psi]_{k,t} \leq t^{-1} \cdot \binom{n-1}{t}$ or equivalently, (17) holds. □

Remark 5. The well-known notion of the *dual game* $\langle N, v^* \rangle$ of a TU game $\langle N, v \rangle$ is defined by $v^*(S) = v(N) - v(N \backslash S)$ for all $S \subseteq N$. Particularly, $v^*(N) = v(N)$ as well as $(v^*)^*(S) = v(S)$ for all $S \subseteq N$. The interrelationship between any adapted unanimity game $\langle N, u_T^t \rangle$ and any complementary unanimity game $\langle N, z_T^t \rangle$ is given by $z_T^t(S) = t - (u_T^t)^*(S)$ for all $S \subseteq N$, $S \neq \emptyset$. Due to its efficiency, linearity, symmetry, and self-duality (expressing $\psi(N, v^*) = \psi(N, v)$ for all games $\langle N, v \rangle$), the Shapley values of both types of games are related by $\psi_i^{Sh}(N, z_T^t) = \frac{t}{n} - \psi_i^{Sh}(N, u_T^t)$ for all $i \in N$. Thus, $\psi_i^{Sh}(N, z_T^t) = \frac{t}{n} - 1$ if $i \in T$, whereas $\psi_j^{Sh}(N, z_T^t) = \frac{t}{n}$ if $j \in N \backslash T$.
Both the egalitarian value and the Shapley value are socially acceptable of type III as the two extreme cases in that the inequalities in (17) are met as equalities. Notice the similarity of both conditions (13) and (17), while the underlying matrix entries $[A^\psi]_{k,t}$ and $[C^\psi]_{k,t}$ only differ in the usual or reversed order of numbering concerning the collection of fundamental constants b_k, $k = 1, 2, \ldots, n-1$. Finally, we remark that each adapted unanimity game $\langle N, u_T^t \rangle$ is a so-called convex game, whereas each complementary unanimity game $\langle N, z_T^t \rangle$ is a so-called 1-concave game (Driessen et al., 2010).

5. Concluding Remarks

The social acceptability properties for the egalitarian, Shapley, and solidarity values may be summarized as follows.

Value	ψ	b_k	**Type** I	**Type** II	**Type** III
Egalitarian value	ψ^{EG}	$b_k = 0$	Yes	Yes	Yes
Shapley value	ψ^{Sh}	$b_k = 1$	Yes	No	Yes
Solidarity value	ψ^{Sol}	$b_k = \frac{1}{k+1}$	Yes	Yes	Yes
New value	ψ	$b_k = \frac{n}{k \cdot (k+1)}$	Yes	Yes	No

The efficient, linear, and symmetric value ψ, associated with the collection of constants $b_k = \frac{n}{k \cdot (k+1)}$ is indeed of type I since (13) is met because of (7), to be of type II too since (15) is met as an equality because of a telescoping sum, but this value fails to be of type III since (17) is not met because of (8). In fact, the latter value satisfies the strict reversed inequalities. Remarkably, the solidarity value is socially acceptable of each of these three types.

Corollary 1. Let ψ be an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to a collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$. For every $t = 1, 2, \ldots, n$, define the payoff p_t^ψ to productive players in the $\{0,1\}$-unanimity game $\langle N, u_T \rangle$ by

$$p_t^\psi = \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n} [A^\psi]_{k,t} \qquad \text{where} \qquad [A^\psi]_{k,t} = \binom{k}{t} \cdot \frac{b_k}{k} \qquad \text{for all } t \leq k \leq n$$

If ψ verifies desirability and monotonicity, then ψ is socially acceptable of type I such that the payoffs $(p_t^\psi)_{t=1}^n$ form a decreasing sequence (the more productive players, the less their payoffs), that is

$$\tfrac{1}{n} = p_n^\psi \leq p_{n-1}^\psi \leq p_{n-2}^\psi \leq \ldots\ldots \leq p_1^\psi \leq 1. \tag{18}$$

Proof. Let $t = 1, 2, \ldots, n-1$. Due to some combinatorial calculations, we derive

$$\begin{aligned} p_t^\psi - p_{t+1}^\psi &= \tfrac{[A^\psi]_{t,t}}{\binom{n}{t}} + \sum_{k=t+1}^{n} \left[\tfrac{[A^\psi]_{k,t}}{\binom{n}{t}} - \tfrac{[A^\psi]_{k,t+1}}{\binom{n}{t+1}} \right] \\ &= \tfrac{[A^\psi]_{t,t}}{\binom{n}{t}} + \sum_{k=t+1}^{n} \left[\frac{\binom{k}{t}}{\binom{n}{t}} - \frac{\binom{k}{t+1}}{\binom{n}{t+1}} \right] \cdot \frac{b_k}{k} \\ &= \tfrac{[A^\psi]_{t,t}}{\binom{n}{t}} + \sum_{k=t+1}^{n} \frac{n-k}{n-t} \cdot \frac{\binom{k}{t}}{\binom{n}{t}} \cdot \frac{b_k}{k} = \sum_{k=t}^{n} \frac{n-k}{n-t} \cdot \tfrac{[A^\psi]_{k,t}}{\binom{n}{t}} \end{aligned}$$

By Theorem 3 (i), $b_k \geq 0$ for all $k = 1, 2, \ldots, n-1$, and so, $[A^\psi]_{k,t} \geq 0$ for all $t \leq k \leq n$. It follows immediately that $p_t^\psi \geq p_{t+1}^\psi$ for all $t = 1, 2, \ldots, n-1$. So, (18) holds. □

Remark 6. In (Hernández-Lamoneda et al., 2007) the basic representation theory of the group of permutations $\mathcal{S}_n$ has been applied to cooperative n-person game theory. Through a specific direct sum decomposition of both the payoff space $\mathbb{R}^n$ and the space $\mathcal{G}_N$ of n-person games, it is shown that an efficient, linear, and symmetric value ψ on $\mathcal{G}_N$ is of the following form (cf. Hernández-Lamoneda et al., 2007, Theorem 2, page 411): for all $i \in N$

$$\psi_i(N, v) = \frac{v(N)}{n} + \sum_{\substack{S \subsetneq N, \\ S \ni i}} (n-s) \cdot \Big[\beta_s \cdot v(S) - \beta_{n-s} \cdot v(N \backslash S) \Big]. \tag{19}$$

Clearly, the above expression agrees with the one on the right hand of (3) by choosing $\beta_k = \frac{\rho_k}{k \cdot (n-k)}$ for all $k = 1, 2, \ldots, n-1$, and hence, (19) and (4) are equivalent by choosing $\beta_k = \frac{b_k}{k \cdot (n-k) \cdot \binom{n}{k}}$ for all $k = 1, 2, \ldots, n-1$. According to (Hernández-Lamoneda et al., 2007, Corollary 5, page 419), an efficient, linear, and symmetric value ψ verifies self-duality (i.e., $\psi(N, v^*) = \psi(N, v)$ for all games $\langle N, v \rangle$) if and only if $\beta_k = \beta_{n-k}$ for all $k = 1, 2, \ldots, , n-1$. The latter condition is equivalent to $b_k = b_{n-k}$ or $[A^\psi]_{k,t} = [C^\psi]_{k,t}$, i.e., coincidence of the two matrices A^ψ and C^ψ.

Remark 7. In (Joosten, 1994) it is shown that a value is efficient, symmetric, additive, and β-egalitarian (for some $\beta \in \mathbb{R}$) if and only if the value is the convex combination of the egalitarian value and the Shapley value in that $\psi(N, v) = \beta \cdot \psi^{EG}(N, v) + (1-\beta) \cdot \psi^{Sh}(N, v)$ for all games $\langle N, v \rangle$. Here a value ψ on $\mathcal{G}_N$ is called *β-egalitarian* if

$$\psi_i(N, v) = \frac{\beta}{n} \cdot \sum_{j \in N} \psi_j(N, v) \qquad \text{for every null player } i \text{ in the game } \langle N, v \rangle.$$

A similar result is shown in (Nowak and Radzik, 1996) concerning an axiomatization of the class of values that are convex combinations of the Shapley value and the solidarity value. Clearly, each value of this class is socially acceptable. In (Dragan et al., 1996) collinearity between the Shapley value and various types of egalitarian values has been treated for a class of zero-normalized games called proportional average worth games.

Remark 8. We conclude this paper with the proof of the "if" part of Theorem 3(ii). Let ψ be an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to a collection of constants $\mathcal{B} = \{b_s\}_{s=1}^n$ with $b_n = 1$ and $0 \le b_s \le 1$ for all $s = 1, 2, \ldots, n-1$ as well. By Theorem 3(i), ψ verifies desirability. It remains to prove that ψ verifies monotonicity too. Let $\langle N, v\rangle$ be a monotonic n-person game and $i \in N$. We show $\psi_i(N, v) \ge 0$. Write $b_0 = 0$ and as usual, $\gamma(n, s) = \frac{s!\cdot(n-1-s)!}{n!}$ for all $s = 0, 1, \ldots, n-1$. At this stage, we put forward our claim that the player's payoff satisfies, for all $k = 0, 1, \ldots, n-2$,

$$\psi_i(N, v) \ge f_k(\psi, v, \{i\}) + g_{k+1}(\psi, v, \{i\}) \quad \text{where for all } \ell = 0, 1, \ldots, n-1 \tag{20}$$

$$f_\ell(\psi, v, \{i\}) = \sum_{\substack{S \subseteq N\setminus\{i\},\\ s \le \ell}} \gamma(n, s) \cdot [b_{s+1} - b_s] \cdot v(S \cup \{i\}) \tag{21}$$

$$g_\ell(\psi, v, \{i\}) = \gamma(n, \ell) \cdot [b_n - b_\ell] \cdot \sum_{\substack{S \subseteq N\setminus\{i\},\\ s = \ell}} v(S \cup \{i\}) \tag{22}$$

The proof of the claim (20) proceeds by backwards induction on k, $k = 0, 1, \ldots, n-2$. For $k = n-2$, the claim follows immediately from the representation (4) for ψ by observing that $b_n = 1$ and $b_s \cdot v(S) \le b_s \cdot v(S \cup \{i\})$ for all $S \subseteq N\setminus\{i\}$ due to the monotonicity of the game $\langle N, v\rangle$ together with $b_s \ge 0$ for all $s = 0, 1, \ldots, n-1$. Suppose that the claim holds for some k, $k \in \{1, 2, \ldots, n-2\}$. We verify the claim for $k-1$. For that purpose, note that $s \cdot v(S \cup \{i\}) \ge \sum_{j\in S} v((S \cup \{i\})\setminus\{j\})$ for all $S \subseteq N\setminus\{i\}$ by the monotonicity of the game $\langle N, v\rangle$. By summing up over all coalitions of size $k+1$, not containing player i, we obtain

$$\sum_{\substack{S \subseteq N\setminus\{i\},\\ s = k+1}} v(S\cup\{i\}) \ge \frac{1}{k+1} \sum_{\substack{S \subseteq N\setminus\{i\},\\ s = k+1}} \sum_{j \in S} v((S\cup\{i\})\setminus\{j\}) = \frac{n-1-k}{k+1} \sum_{\substack{T \subseteq N\setminus\{i\},\\ t = k}} v(T\cup\{i\}).$$

where the last equality is due to the combinatorial argument that any $T \subseteq N\setminus\{i\}$ of size k arises from $n-k-1$ coalitions S of the form $T \cup \{j\}$, where $j \in N\setminus T$, $j \ne i$. ¿From the latter inequality, together with (22) and $b_{k+1} \le 1 = b_n$, we derive the following:

$$\begin{aligned} g_{k+1}(\psi, v, \{i\}) &= \gamma(n, k+1) \cdot [b_n - b_{k+1}] \cdot \sum_{\substack{S \subseteq N\setminus\{i\},\\ s = k+1}} v(S \cup \{i\}) \\ &\ge \gamma(n, k+1) \cdot [b_n - b_{k+1}] \cdot \frac{n-1-k}{k+1} \sum_{\substack{T \subseteq N\setminus\{i\},\\ t = k}} v(T \cup \{i\}) \\ &= \gamma(n, k) \cdot [b_n - b_{k+1}] \sum_{\substack{S \subseteq N\setminus\{i\},\\ s = k}} v(S \cup \{i\}) \end{aligned}$$

where the latter equality holds because of $\gamma(n, k+1) \cdot \frac{n-1-k}{k+1} = \gamma(n, k)$. ¿From the latter inequality, together with the induction hypothesis (20), (21), (22) respectively, it follows that

$$\begin{aligned}
\psi_i(N,v) &\geq f_k(\psi,v,\{i\}) + g_{k+1}(\psi,v,\{i\}) \\
&= f_{k-1}(\psi,v,\{i\}) + \gamma(n,k)\cdot[b_{k+1}-b_k]\cdot \sum_{\substack{S\subseteq N\setminus\{i\},\\ s=k}} v(S\cup\{i\}) + g_{k+1}(\psi,v,\{i\}) \\
&\geq f_{k-1}(\psi,v,\{i\}) + \gamma(n,k)\cdot[b_n-b_k]\cdot \sum_{\substack{S\subseteq N\setminus\{i\},\\ s=k}} v(S\cup\{i\}) \\
&= f_{k-1}(\psi,v,\{i\}) + g_k(\psi,v,\{i\})
\end{aligned}$$

This completes the backwards inductive proof of the claim (20). For $k=0$ the claim yields

$$\begin{aligned}
\psi_i(N,v) &\geq f_0(\psi,v,\{i\}) + g_1(\psi,v,\{i\}) \\
&= \gamma(n,0)\cdot[b_1-b_0]\cdot v(\{i\}) + \gamma(n,1)\cdot[b_n-b_1]\cdot \sum_{j\in N\setminus\{i\}} v(\{i,j\}) \\
&= \frac{b_1}{n}\cdot v(\{i\}) + \frac{1-b_1}{n\cdot(n-1)}\cdot \sum_{j\in N\setminus\{i\}} v(\{i,j\}).
\end{aligned}$$

Note that $v(S)\geq 0$ for all $S\subseteq N$ by monotonicity of $\langle N,v\rangle$. Together with $0\leq b_1\leq 1$, the latter inequality yields $\psi_i(N,v)\geq 0$. This completes the proof of Theorem 3(ii). □

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Auctioning Big Facilities under Financial Constraints*

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Abstract. This paper analyzes auctions for big facilities or contracts where bidders face financial constraints that may force them to resell part of the property of the good (or subcontract part of a project) at a resale market. We show that the interaction between resale and financial constraints changes previous results on auctions with financial constraints and those in auctions with resale. The reason is the link between the resale price and the auction price introduced by the presence of financial constraints. Such link induces a potential loser to modify the auction price in order to fine-tune the winner's resale offer, which may require forcing the winner to be financially constrained.

Keywords: auctions, big facilities, resale, financial constraints, subcontracting

Introduction

The allocation of a big contract or facility usually involves a small number of qualified buyers who assign a large value to the good and face financial constraints. One specific example is the allocation problem of the European Spallation Source that had to be allocated to a single country or location but whose property can be shared after the initial allocation, to alleviate the winner's financial constraints.[1] Similarly, operating licences (e.g., in the telecommunications sector) are awarded to one firm, and (some or all of) the actual services can be subcontracted.[2] Partial resale or horizontal subcontracting is also a common assumption in two-stage contract games (Kamien et al., 1989; Spiegel, 1993; Chen et al., 2004; and Meland and Straume, 2007).

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[1] On September 10, 2008, the European Strategy Forum on Research Infrastructures (ESFRI) published its report on the three candidates to host the European Spallation Source (ESS): Lund in Sweden, Bilbao in Spain, and Debrecen in Hungary. A decision was reached in May 2009 in favour of Lund. In June 2009, Spain and Sweden agreed to collaborate in order to build the European Spallation Source; a facility for manufacturing some accelerator components will be built in Bilbao.

[2] There is horizontal subcontracting or horizontal outsourcing whenever parts of the final production of a good are subcontracted to rival firms.

Previous work on auctions with resale relies mostly on the potential inefficiencies of the auction allocation mechanism to provide the basis for resale. An inefficient allocation may result from noisy signals at the time of the auction, as in Haile (2000, 2001, 2003), from asymmetries between bidders when the auction is conducted as first price, as in Gupta and Lebrun (1998) or Hafalir and Krishna (2008), or from the presence of speculators who value the object only by its resale price, as in Garratt and Tröger (2006). In contrast, in our model the auction is a second price auction and there are no pure speculators, all participants value the good, and the resale market is justified by the presence of financial constraints which may force the winner of the auction to sell part of the property of the good. That is, if bidders were wealthy enough the resale market would be inactive and it is the presence of financial constraints which makes participants aware that there will be partial resale.

The main purpose of the paper is to check the robustness of previous results of auctions with resale and auctions where bidders face financial constraints. This is particularly important when auctioning big facilities or contracts. Our results are driven by the link between auction price and resale price, which induces a potential loser to increase the auction price in order to fine-tune the winner's resale offer, which may require forcing the winner to be financially constrained. We first compare the outcomes of the auction with resale and financial constraints, to those without resale or without financial constraints, and find that the loser's incentives to modify the resale price may preclude a truth telling behavior and also bidding the minimum between valuation and wealth as part of the equilibrium. We also find that the presence of financial constraints may eliminate the speculative equilibria à la Garrat and Tröger in auctions with resale.

The paper is organized as follows. Section 1 presents the model. In Section 2 we solve the resale stage. Section 3 presents the bidding stage under private information and the main results of the paper. Section 4 concludes.

1. The model

A government wants to auction the location of a facility, or to assign a big project to one of two potential risk-neutral buyers, buyer A and buyer B. The worth of the auctioned good may be large compared to the buyers' wealth, so that default may occur. Each buyer i has a budget or wealth w_i. As in Zheng (2001), a buyer wealth represents both her liquidity constraint and her liability. Thus, w_A and w_B will set the maximum amount by which buyers can be penalized if they default. We assume that w_A and w_B are known.

Buyer i has use value v_i when i is the solo owner of the good.[3] If i obtains a fraction z of the property of the good then her use value will be zv_i. Use values v_A and v_B are private information. Each v_i has distribution F_i with associated density f_i and supports $[\underline{v}_i, \bar{v}_i]$ for v_i, with $\underline{v}_A \geq \bar{v}_B$ and $\bar{v}_i - \underline{v}_i = 1$. We will denote by $h_i(v_i)$ the hazard rate of F_i, i.e. $h_i(x) = \frac{f_i(x)}{1-F_i(x)}$ and we will assume that it is increasing. The hazard rate represents the instantaneous probability that the valuation of buyer i is v_i given that it is not smaller than v_i. Monotonicity of the hazard rate implies

[3] Due to the possibility of resale, and following Haile (2003), we will distinguish between buyers' *use value* of the object which is exogenously determined and buyers' *valuation* - the value players attach to winning the auction- which will be endogenously determined.

that the virtual use valuations ($\psi_i(v_i) = v_i - \frac{1-F_i}{f_i}$, $i = A, B$) are strictly increasing, and it is equivalent to assume log-concavity of the reliability function.[4] We further assume $\underline{v}_A h_A(\underline{v}_A) > 1$ which is the necessary condition for no buyer A exclusion through a positive reserve price being profitable for the seller.

We will define buyers ex-ante financial situation by the relationship between their use values and their wealth. We will say that buyer i is ex-ante financially constrained if $v_i > w_i$, and that i is ex-ante unconstrained otherwise.

An important assumption of the model is the inability of the initial seller to prohibit resale. Because of this, buyers participate *de facto* in a two-stage selling game; in the first stage, they compete for the object at the auction, and in the second stage the auction winner can make a take-it-or-leave-it offer to the auction loser for the entire or for part of the property of the object. Player i can always guarantee himself w_i, by not participating in the selling game.

At the first stage, the good is sold through a second-price auction and assigned to a single buyer. Bids are denoted $b = (b_A, b_B)$. We will denote by p the price to be paid by the winner, i; this payment will take place at the end of the game. The loser does not pay anything to the auctioneer. We will denote by U_i^{Wj} the utility of player i, $i = A, B$, when the auction winner is j, $j = A, B$. At the end of this first stage the auction price is announced publicly.

At the second stage, the winner of the first stage auction, i, must decide whether to keep the object or to resell it, and if so, at what price and which fraction. We will assume that the winner has all the bargaining power.[5] Thus, resale takes place via monopoly pricing - the winner of the auction makes an offer to the loser after updating her prior beliefs based on her winning and on the information revealed by the auction price. A resale offer by bidder i, O_i, is a pair $[r_i, z_i]$ which comprises a resale price r_i and a fraction of the good z_i. Keeping the object is dominated by reselling it if the auction winner does not have wealth enough to cover the auction price, i.e., if $w_i < p$.[6] We will denote the option of keeping the object by the offer $O_i = [0, 0]$. If the winner is unable to pay p after resale, she defaults and loses all her wealth. We will denote defaulting by an empty offer, i.e., by $O_i = \varnothing$. Note that if $p > w_A + w_B$ then $O_i = \varnothing$ no matter the identity of the auction winner. The auction loser must decide whether to accept or reject the resale offer. It can be easily verified that the auction loser j will accept to buy z_i at a price r_i if and only if the following two conditions simultaneously hold: 1) $r_i \leq v_j$ and 2) $r_i z_i \leq w_j$.

We search for the Perfect Bayesian Equilibria of the selling game (PBE, for short). A strategy for a player must hence specify a first round bid, a second round offer if the player is the auction winner, and a second round acceptance decision if the player is the auction loser. Furthermore, posterior beliefs are determined by Bayes rule whenever possible and the resale offer is optimal given the posterior beliefs and the first round bids. Regarding the off-equilibrium beliefs, we will assume that the

[4] See Bagnoli and Bergstrom (2005) for the class of distributions satisfying this property.

[5] Similar assumption is adopted in Zheng (2002) to characterize the optimal auction with resale, and can also be found in Hafalir and Krishna (2008). In contrast, Pagnozzi (2007) assumes that bidders bargain in the resale market, so that the outcome is given by the Nash bargaining solution.

[6] An alternative interpretation is that the winning bidder can sell equity to finance a portion of his bid as in Rhodes-Kropf and Viswanathan (2005). But here the equity provider is the losing bidder and not the equity market.

posteriors will coincide with the priors whenever the first stage bid is outside the range of equilibrium bids.[7] Finally, we will only consider rationalizable equilibria or equilibria which survive the elimination of (weakly) dominated strategies.

2. The resale stage

The optimal offers at the resale stage will depend on the winner's use value, on the first round price, on bidders' wealth and on the information revealed at the first stage. Since bidders' posteriors depend on the first round bids, different information structures can exist at the second stage. At one extreme, after the auction and before the resale stage, the loser's use value becomes common knowledge (perfectly revealing first round bids or separating equilibrium); at the other extreme, the posterior and the prior on the loser's use value coincide (perfect pooling equilibrium). The resale offers in a first-stage fully separating equilibrium coincide with those described in previous section as there will be perfect information at the resale stage. We concentrate here on the case of incomplete information at the resale stage.

To solve the resale stage, we assume that bidding strategies are non-decreasing, so that the behavior of the winner at the resale stage must take into account what she learns from the auction price.

Lemma 1. *If the auction loser, bidder j, followed a non-decreasing bidding strategy at the first stage such that $p = b_j(v_j)$ for all $v_j \in \left[v_j^L, v_j^H\right]$ and $p \neq b_j(v_j)$ for $v_j \notin \left[v_j^L, v_j^H\right]$, the winner, bidder i, has updated beliefs given by $\hat{F}_j(x) = \Pr\left(v_j \leq x \middle| \, v_j \in [v_j^L, v_j^H]\right)$, where*

$$\hat{F}_j(x) = \begin{cases} \frac{F_j(x) - F_j(v_j^L)}{F_j\left(v_j^H\right) - F_j\left(v_j^L\right)} & \text{if } x \in [v_j^L, v_j^H) \\ 1 & \text{if} \quad x \geq v_j^H \\ 0 & \text{if} \quad x < v_j^L \end{cases}$$

with $p = b_j\left(v_j^L\right) = b_j\left(v_j^H\right)$, and $\underline{v}_j \leq v_j^L < v_j^H \leq \bar{v}_j$.
If $v_j^L = \underline{v}_j$ and $v_j^H = \bar{v}_j$, then the updated distribution coincides with the prior distribution, i.e., $\hat{F}_j(x) = F_j(x)$. Conversely, if $v_j^L = v_j^H$ then the updated distribution is a point distribution.

At the resale stage, the auction winner will set the resale offer that maximizes her expected payoff given her posterior beliefs about the loser's use value. The objective function for player i is hence:

$$\max_{z_i, r_i} \left\{ [w_i - p + r_i z_i + v_i(1 - z_i)] \left(1 - \hat{F}_j(r_i)\right) + K_i(p)\hat{F}_j(r_i) \right\}$$

where

$$K_i(p) = \begin{cases} 0 & \text{if } p > w_i \\ v_i + w_i - p & \text{if } p \leq w_i \end{cases}$$

$K_i(p)$ stands for the utility when buyer i keeps the object because the resale offer is rejected. It coincides with the utility in an auction without resale and it hence

[7] We will do so whenever these off-equilibrium beliefs satisfy Cho and Kreps Intuitive Criterion.

takes on the positive value $v_i + w_i - p$ when there is no risk of default ($p \leq w_i$) and 0 when there is risk of default.

The key difference between the strong buyer and the weak buyer behavior at the resale stage lies in the shares they put up for sale. Whereas buyer A resells the minimum fraction needed to cover the auction price, i.e., $z_A = \min\left(\frac{p-w_A}{r_A}, 0\right)$, buyer B may resell the entire object if $w_A \geq v_A^L$. Resale offers will hence depend on the identity of the winner.

2.1. *Resale offers by the strong buyer*

If the winner is player A, she will not resell if $p \leq w_A$ as $v_A > v_B$. Thus, if $p \leq w_A$ then $O_A = [0, 0]$. At the other extreme, A will always default, $O_A = \varnothing$, if $p > \min(w_B, v_B^H) + w_A$.

For intermediate prices, $p \in (w_A, \min(w_B, v_B^H) + w_A]$, player A resells the minimum fraction needed to cover the first round price, i.e., $z_A \in (0, 1] : z_A r_A = p - w_A$. Note that the loss from increasing z_A out-weights its gain as $v_A \geq \overline{v}_B \geq v_B^H > r_A$, so that $z_A r_A = p - w_A$ follows. Thus, buyer A will set $r_A \in \left[v_B^L, v_B^H\right)$ to maximize her expected utility given by

$$\max_{r_A} \left\{ \left(v_A \left(1 - \left(\frac{p - w_A}{r_A} \right) \right) + r_A z_A + w_A - p \right) \left(1 - \hat{F}_B\left(r_A\right) \right) + 0 \hat{F}_B\left(r_A\right) \right\}$$

where $\hat{F}_B\left(r_A\right)$ is the probability that A's offer be rejected. Since the objective function is concave, the unconstrained optimal resale price must solve the first order condition, which after some simplifications, can be written as

$$r_A \left(\frac{r_A}{p - w_A} - 1 \right) = \frac{1}{\hat{h}_B(r_A)} \tag{1}$$

where $\hat{h}_B(r_A) = \frac{\hat{f}_B(r_A)}{1 - \hat{F}_B(r_A)}$. Let $r_A^*(p)$ be implicitly defined by $\frac{r_A^*}{p - w_A} = \frac{1}{r_A^* \hat{h}_B\left(r_A^*\right)} + 1$. Since $\frac{1}{r_A^* \hat{h}_B\left(r_A^*\right)} + 1$ is decreasing in r_A for any regular cdf, a necessary condition for the unconstrained maximum to satisfy $r_A^* \geq v_B^L$ is that $\frac{\left(v_B^L\right)^2 \hat{f}_B\left(v_B^L\right)}{1 + v_B^L \hat{f}_B\left(v_B^L\right)} \leq p - w_A$.

Denoting $M_B(v_B^L) = \frac{1 + v_B^L \hat{f}_B\left(v_B^L\right)}{\left(v_B^L\right)^2 \hat{f}_B\left(v_B^L\right)}$, buyer A's optimal resale offers when $p - w_A \leq \min(w_B, v_B^H)$ are hence:

$$O_A^*(p - w_A) = \begin{cases} [0, 0] & \text{if } p - w_A \leq 0 \\ \left[v_B^L, \; \frac{p - w_A}{v_B^L}\right] & \text{if } 0 < p - w_A \leq \frac{1}{M_B(v_B^L)} \\ \left[r_A^*, \; \frac{(p - w_A)}{r_A^*}\right] & \text{if } \frac{1}{M_B(v_B^L)} < p - w_A \leq \min(w_B, v_B^H) \end{cases}$$

Discussion above is summarized in the next lemma.

Lemma 2. *At the resale stage, player A will set $O_A = O_A^*(p - w_A)$ if $p - w_A \leq \min(w_B, v_B^H)$, and she will default, $O_A = \varnothing$, if $p - w_A > \min(w_B, v_B^H)$.*

If the first stage bids are weakly non-decreasing, so that $\hat{F}_B$ is a truncation of F_B, then $O_A^*(p - w_A)$ is non-decreasing in p, and $r_A^*(p)$ is increasing in p. Player B's behavior at the first stage auction will take this into account if he were to set p.

2.2. *Resale offers by the weak buyer*

Assume now that winner is player B and that $p < \min(w_A, v_A^H) + w_B$ so that he is not forced to default.

Denoting by (r_B^o, z_B^o) the optimal pair, some basic properties of the optimal offer are first derived.

Lemma 3. *An optimal offer (r_B^o, z_B^o) must satisfy the following properties: (i) $v_A^H \geq r_B^o \geq v_A^L$ and $r_B^o z_B^o \leq w_A$; (ii) If $r_B^o z_B^o < w_A$, then $z_B^o = 1$; (iii) If $z_B^o < 1$, then $r_B^o z_B^o = w_A$, and (iv) If $w_A < v_A^L$, then $z_B^o < 1$.*

Buyer B can either sell the entire object ($z_B = 1$) or only a part of it ($z_B < 1$). When $z_B = 1$, r_B must solve

$$\max_{r_B} \left\{ [w_B - p + r_B] \left(1 - \hat{F}_A(r_B)\right) + K_B(p)\hat{F}_A(r_B) \right\}$$

In an interior solution, the optimal resale price when $z_B = 1$, denoted $r_B^*(p)$, is implicitly defined by:

$$r_B - (p + K_B(p) - w_B) = \frac{1}{\hat{h}_A(r_B)}. \tag{2}$$

This equation yields the optimal price when the solution satisfies (a) $r_B^* \geq v_A^L$ (see Lemma 3 (i)), which requires the LHS to be larger than the RHS when evaluated at the minimum possible price v_A^L and (b) $r_B^* \leq w_A$ (from $z_B = 1$ and Lemma 3 (i)), which requires the LHS to be larger than the RHS when evaluated at w_A. Note also that, from Lemma 3 (iv), $z_B = 1$ may only be optimal when $w_A \geq v_A^L$. Trivially, if $w_A > v_A^H$ then the condition $r_B^* \leq w_A$ will never bind as $r_B^* \leq v_A^H$ holds by Lemma 3 (i). Finally, there can also exist corner solutions with resale offers $\left[v_A^L, 1\right]$ if (a) fails, and $[w_A, 1]$ if (b) fails.

When selling only part of the object, $z_B < 1$, then $r_B z_B = w_A$ (Lemma 3 (iii)) and r_B must solve

$$\max_{r_B} \left\{ \left[w_B - p + w_A + v_B \left(1 - \frac{w_A}{r_B}\right)\right] \left(1 - \hat{F}_A(r_B)\right) + K_B(p)\hat{F}_A(r_B) \right\}$$

In an interior solution, when $z_B < 1$ the optimal resale price $r_B^{**}(p)$ is implicitly defined by:

$$\frac{r_B}{v_B w_A} \left[r_B(w_A - (p + K_B(p) - w_B) + v_B) - v_B w_A \right] = \frac{1}{\hat{h}_A(r_B)} \tag{3}$$

and $z_B^{**} = \frac{w_A}{r_B^{**}}$. The equation for r_B^{**} yields the optimal price when the solution satisfies $r_B^{**} \geq \max\{v_A^L, w_A\}$ (from Lemma 3 (i)) $r_B^{**} \geq v_A^L$, and from Lemma 3 (iii) $r_B^{**} \geq w_A$ is necessary for $r_B^{**} z_B = w_A$ and $z_B < 1$ to hold simultaneously). When $w_A < v_A^L$, the condition $r_B^{**} \geq v_A^L$ requires that the LHS of equation above be larger than its RHS when evaluated at v_A^L, whereas when $w_A \geq v_A^L$ the relevant condition is $r_B^{**} \geq w_A$, which requires that the LHS of equation above be larger than its RHS when evaluated at w_A. We may also have corner solutions with resale offers $\left[v_A^L, \frac{w_A}{v_A^L}\right]$ and $[w_A, 1]$ when the aforementioned conditions fail ($r_B^{**} < v_A^L$ and $r_B^{**} < w_A$, respectively).

The optimal resale offers $[r_B, z_B]$ for player B when $p < \min(w_A, v_A^H) + w_B$ are summarized below:

$$O_B^o(p) = \begin{cases} \left[v_A^L, 1\right] & \text{if } w_A \geq v_A^L,\ p + K_B(p) - w_B \leq \hat{\psi}_A(v_A^L) \\ [r_B^*, 1] & \text{if } v_A^H > w_A \geq v_A^L,\ p + K_B(p) - w_B \in \left(\hat{\psi}_A(v_A^L), \hat{\psi}_A(w_A)\right) \\ [w_A, 1] & \text{if } v_A^H > w_A \geq v_A^L,\ p + K_B(p) - w_B \in \left[\hat{\psi}_A(w_A), w_A - \right. \\ \left. - \frac{v_B}{w_A \hat{h}_A(w_A)}\right] & \\ \left[r_B^{**}, \frac{w_A}{r_B^{**}}\right] & \text{if } v_A^H > w_A \geq v_A^L,\ p + K_B(p) - w_B > w_A - \frac{v_B}{w_A \hat{h}_A(w_A)} \\ [r_B^*, 1] & \text{if } v_A^H \leq w_A,\ p + K_B(p) - w_B > \hat{\psi}_A(v_A^L) \\ \left[v_A^L, \frac{w_A}{v_A^L}\right] & \text{if } w_A < v_A^L,\ p + K_B(p) - w_B \leq w_A + v_B - v_B w_A M_A(v_A^L) \\ \left[r_B^{**}, \frac{w_A}{r_B^{**}}\right] & \text{if } w_A < v_A^L,\ p + K_B(p) - w_B > w_A + v_B - v_B w_A M_A(v_A^L) \end{cases}$$

where $\hat{\psi}_A(x)$ stands for the virtual valuation of player A when her use value is x, i.e., $\hat{\psi}_A(x) = x - \frac{1}{\hat{h}_A(x)}$, and $M_A(v_A^L) = \left(\frac{1+v_A^L \hat{h}_A\left(v_A^L\right)}{\left(v_A^L\right)^2 \hat{h}_A\left(v_A^L\right)}\right)$.

Lemma 4. *At the resale stage, player B will set $O_B = O_B^o(p)$ if $p \leq \min(w_A, v_A^H) + w_B$, and she will default, $O_A = \varnothing$, if $p > \min(w_A, v_A^H) + w_B$.*

Note that B's offers depend on the value of $p + K_B(p) - w_B$, which is

$$p + K_B(p) - w_B = \begin{cases} p - w_B & \text{if } p > w_B \\ \quad v_B & \text{if } p \leq w_B \end{cases}$$

In words, B's offers depend on the amount to be covered at resale to avoid bankruptcy, $p - w_B$, when there is risk of default, and they depend on B's use value for the good, v_B, when there is no risk of default.

Regarding z_B, when all the A types are financially constrained ($w_A \leq v_A^L$) B will always set $z_B = \frac{w_A}{r_B} < 1$, whereas when A is wealthy he may set $z_B = 1$ but he may set $z_B = \frac{w_A}{r_B} < 1$ when $p - w_B$ or v_B are high. In the first case, if $w_A < v_A^L$, the highest price B can set if he were to sell the entire object is w_A so that his payoff will be $w_A + w_B - p$ which is lower than what he can get by setting a higher resale price such as v_A^L and getting w_A from player A while additionally using part of the good.

Regarding r_B, if $w_A < v_A^H$ then resale prices satisfy $v_A^L < r_B^* < w_A < r_B^{**}$. Furthermore, note that $p + K(p)$ is constant with p when $p \leq w_B$ and increasing in p when $p > w_B$, so that the optimal resale prices set by B are non-decreasing in p in (w_B, ∞) and are constant in p in $[0, w_B)$. However, resale prices may decrease with p at $p = w_B$ since $K(p) = 0$ at $p > w_B$ and $K(p) = v_B$ at $p = w_B$.

Detailed calculations and proofs for the claims above may be found in the appendix.

Losers' expected utilities

When the strong buyer wins, her optimal resale offers will depend on $p - w_A$. From Lemma 2, straightforward calculations allow to derive B's expected utility when he loses the auction, $U_B^{WA}(p - w_A)$. Trivially, $U_B^{WA}(p - w_A) = w_B$ when B either does not receive any resale offer or he rejects it. No resale offer will be made whenever A has wealth enough to cover p or whenever A defaults because $p - w_A > \min(w_B, v_B^H)$). For the remaining cases $U_B^{WA}(p - w_A)$ is given by

$$\begin{array}{ll} w_B + w_A - p + v_B\left(\frac{p-w_A}{v_B^L}\right) & \text{if } 0 < p - w_A \leq \frac{1}{M_B\left(v_B^L\right)} \\ \max\left\{w_B + w_A - p + v_B\left(\frac{p-w_A}{r_A^*(p)}\right), w_B\right\} & \text{if } \frac{1}{M_B\left(v_B^L\right)} < p - w_A \leq \min(w_B, v_B^H). \end{array}$$

When the weak buyer wins, his optimal resale offers take into account his rival wealth, particularly whether $w_A \geq v_A^L$ or $w_A < v_A^L$. Because of this, in what follows, when computing A's expected utility when losing the auction we treat the two cases separately.

If $w_A \geq v_A^L$ then $U_A^{WB}(p)$ is given by

$$\begin{array}{ll} w_A + v_A - v_A^L & \text{if } w_A < v_A^H,\ p + K_B(p) - w_B \leq \hat{\psi}_A(v_A^L) \\ \max\{w_A + v_A - r_B^*, w_A\} & \text{if } w_A < v_A^H,\ p + K_B(p) - w_B \in \left(\hat{\psi}_A(v_A^L), \hat{\psi}_A(w_A)\right) \\ \max\{v_A, w_A\} & \text{if } w_A < v_A^H,\ p + K_B(p) - w_B \in \left[\hat{\psi}_A(w_A), w_A - \frac{v_B}{w_A \hat{h}_A(w_A)}\right] \\ \max\left\{v_A\left(\frac{w_A}{r_B^{**}}\right), w_A\right\} & \text{if } w_A < v_A^H,\ p + K_B(p) - w_B > w_A - \frac{v_B}{w_A \hat{h}_A(w_A)} \\ \max\{w_A + v_A - r_B^*, w_A\} & \text{if } w_A \geq v_A^H,\ p + K_B(p) - w_B > \hat{\psi}_A(v_A^L) \end{array}$$

If A faces a financially unconstrained seller because $p < w_B$ then $p + K_B(p) - w_B = v_B$ and B's resale prices will only depend on his type. If this is the case, A is better off the larger is the difference between her use values and that of her rival, being maximal if $\bar{v}_B \leq \hat{\psi}_A(v_A^L)$ as she will then be offered $r_B = v_A^L$. If she faces a financially constrained bidder then $p + K_B(p) - w_B = p - w_B$ and A can affect the resale price by affecting p. Since $v_A^L < r_B^* < w_A < r_B^{**}$, she is always better off, conditional on losing, if B is financially constrained unless $\bar{v}_B \leq \hat{\psi}_A(v_A^L)$ holds.

If $w_A < v_A^L$ then $U_A^{WB}(p)$ is given by

$$\begin{array}{ll} v_A\left(\frac{w_A}{v_A^L}\right) & \text{if } p + K_B(p) - w_B \leq w_A + v_B - w_A v_B M_A(v_A^L), \text{ and} \\ \max\left\{v_A\left(\frac{w_A}{r_B^{**}}\right), w_A\right\} & \text{if } p + K_B(p) - w_B > w_A + v_B - w_A v_B M_A(v_A^L). \end{array}$$

Buyer A's utility is maximal if her rival is financially unconstrained but has very low use values, i.e., if $\bar{v}_B \leq \frac{1}{M_A(v_A^L)} = \frac{\left(v_A^L\right)^2 \hat{h}_A\left(v_A^L\right)}{1 + v_A^L \hat{h}_A\left(v_A^L\right)}$, as then $r_B^{**} = v_A^L$. When $p > w_B$ then $r_B^{**} = v_A^L$ iff $\bar{v}_B \leq \frac{w_A + w_B - p}{w_A M_A(v_A^L) - 1}$, which always holds for a sufficiently low w_A or if p is sufficiently large as compared to the total resources, $w_A + w_B$.

From the above expressions, next result follows.

Lemma 5. (i) *In an equilibrium in which B loses: if $b_A > w_A + \frac{1}{M_B\left(v_B^L\right)}$ then $b_B \geq w_A + \frac{1}{M_B\left(v_B^L\right)}$, with equality if F concave. Similarly, if $b_A > w_A$ then $b_B > w_A$.*

(ii) *If $w_A \geq v_A^L$ and $w_B < \bar{v}_A$, in an equilibrium in which A loses: if $b_B > w_B$ and $\bar{v}_B > \hat{\psi}_A(v_A^L)$ hold, then $b_A > w_B$.*

(iii) *If $w_A < v_A^L$ and $w_B < \bar{v}_A$, in an equilibrium in which A loses: if $b_B > w_B$ and $\bar{v}_B > \frac{1}{M_A(v_A^L)}$ hold, then $b_A \in (w_B, w_B + w_A\left(1 - \bar{v}_B M_A(v_A^L)\right) + \bar{v}_B]$.*

Proof. See Appendix.

Results in Lemma 5 stem from the loser's preference for facing a financially constrained winner. Buyer B will not receive a resale offer unless A is financially constrained, which explains part (i). Similarly, if the loser is buyer A, to induce the best possible resale offer she will make her rival financially constrained (parts (ii) and (iii)). Because of this, whenever A is always ex-ante financially constrained a strategy profile in which A bids no more than her wealth can only be part of a PBE if it results in A losing.

The importance of bidders' use-values asymmetries is also highlighted in Lemma 5. If A loses but her rival has very low use-values as compared to hers, then she will always be offered $r_B = v_A^L$. When this is the case, a loser buyer A will make no attempt to manipulate the auction price (other than avoiding default). The conditions for $r_B = v_A^L$ are

$$\bar{v}_B \leq \hat{\psi}_A(v_A^L) \text{ if } w_A \geq v_A^L, \text{ and } \bar{v}_B \leq \frac{1}{M_A(v_A^L)} \text{ if } w_A < v_A^L.$$

If asymmetries are less severe (as would be the case for example with $\bar{v}_B = \underline{v}_A$) then those inequalities will not hold for, at least, $v_A^L = \underline{v}_A$.

3. Results

The possibility of default due to the presence of financial constraints creates a link between the resale price and the auction price. This link can affect equilibrium behavior at the auction stage, an issue we explore next. We have seen in the previous section that the presence of financial constraints may induce a potential loser to increase the auction price in order to fine-tune the winner's resale offer. Similarly, the presence of a resale market affects the possibility of default as the auction winner can get extra resources from reselling.

To disentangle the impact of resale from that of financial constrains, in what follows we first explore if the equilibria when there is resale among unconstrained buyers remain equilibria under financial constraints. We then analyze whether the equilibria in no-resale auctions with financial constraints remain equilibria when there is resale. Finally, we explore the equilibria emerging from the interplay between resale and financial constraints.

3.1. The resale equilibria

Garrat and Tröger (2006) have shown that in second-price and English auctions resale creates a role for a speculator -a bidder with zero use value for the good on sale. Because of this, in a second price auction with resale the truth-telling equilibrium coexists with a continuum of inefficient equilibria in which the speculator wins the auction and makes positive profits. In our set-up no buyer is a pure-speculator as their use values are strictly positive. Nevertheless if $\bar{v}_B$ is sufficiently low, buyer B could be considered as a speculator.

We first show that financial constraints preclude the existence of a truth-telling equilibria.

Proposition 1. *Truth-telling at the bidding stage is part of a PBE of the auction with resale if and only if $w_A \geq \bar{v}_A$.*

Proof. See Appendix.

Proposition above shows that when there are potential financial constraints, truth-telling fails to be an equilibrium. Moreover, as soon as there exist "potential" financial constraints, a separating perfect revealing equilibrium fails to be an equilibrium. The reason is that whenever the strong buyer can be made financially constrained the weak one takes advantage of it by raising the price at least up to $w_A + \epsilon$. Thus, a strategy by bidder A such as $b_A = \bar{v}_A > w_A$ can only be part of an equilibrium if $b_B > w_A$. Recall from Lemma 5 that If $b = (b_A, b_B)$ is such that $b_A(v_A) > w_A$ and $b_A(v_A) \geq b_B(v_B)$ for some pair (v_A, v_B), then $p > w_A$ for b to be part of a PBE equilibrium.

We study next if a profitable "speculative-like" equilibrium à la Garrat and Tröger can exist when the strong buyer is financially constrained. Garrat and Tröger (2006) show that the following strategies constitute an equilibrium of a second-price auction with resale among two buyers: a non-speculator buyer i with use values in the range $[0,1]$ bids 0 if $v_i < \theta^*$ and bids her use value $(b_i^{\theta^*}(v_i) = v_i)$ if $v_i \geq \theta^*$, and a speculator with zero use value bids θ^*. They show that there is an equilibrium in this family for each $\theta^* \in [0,1]$. Thus, there is a continuum of inefficient speculative θ^*-equilibria.

Considering the weak buyer as a speculator, and considering the strategy profile in which buyer A bids $\underline{v}_A$ if $v_i < \theta^*$ and bids her use value $(b_i^{\theta^*}(v_i) = v_i)$ if $v_i \geq \theta^*$, and buyer B bids θ^* no matter his type (this candidate to equilibrium will be referred to as the speculative θ^*-equilibrium, as displayed below)

$$b_A : \underline{v}_A \underbrace{-\,-\,-\,-\,-\,-\,-\,-\,-}_{\underline{v}_A} \theta^* \underbrace{-\,-\,-\,-\,-\,-\,-\,-\,-}_{v_A} \bar{v}_A$$

$$b_B : \underline{v}_B \underbrace{-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-}_{\theta^*} \bar{v}_B$$

We next show that a "speculative-like" θ^*-equilibrium can only exist if $\theta^* = \bar{v}_A$ and $w_B > \bar{v}_A$. For any other $\theta^* \in (\underline{v}_A, \bar{v}_A)$, a speculative θ^*-equilibrium (in which buyer A bids $\underline{v}_A$ if $v_i < \theta^*$ and bids v_i if $v_i \geq \theta^*$, and buyer B bids θ^*) fails to exist due to the combination of budget constraints and the fact that $\underline{v}_A \geq \bar{v}_B > 0$.

Proposition 2. (i) *A speculative θ^*-equilibrium, $\theta^* < \bar{v}_A$, does not exist for any $w_A < \bar{v}_A$.*

(ii) *The speculative $\bar{v}_A$-equilibrium with bids $(\underline{v}_A, \bar{v}_A)$ is part of a PBE in which the low valuation bidder wins (and makes positive profits if $\bar{v}_B > \psi_A(\underline{v}_A)$)*[8] *if $w_B \geq \bar{v}_A > w_A \geq \underline{v}_A$, while it is not otherwise.*

Proof. See Appendix.

Note that financial constraints destroy any speculative θ^*-equilibrium other than $\theta^* = \bar{v}_A$ (Proposition 2 part (i)). To see this, note that if $\theta^* < \bar{v}_A$ then in a speculative θ^*-equilibrium some types of buyer A are bidding her value while being left financially unconstrained as $\underline{v}_A < p = \theta^* < w_A$ so that buyer B is better off deviating to force his rival to be financially constrained. When $w_A < \underline{v}_A$ a speculative θ^*-equilibrium ($\theta^* < \bar{v}_A$) fails to exist as A types above θ^* win the

[8] Trivially, it is an equilibrium for buyer B to bid sufficiently high, say $\bar{v}_A$, and for buyer A to bid $\underline{v}_A$, provided that at the resale stage B sets $r_B = \underline{v}_A$ which will be the case if $\bar{v}_B \leq \psi_A(\underline{v}_A)$. At such an equilibrium B breaks even.

auction but face the risk of default and are hence better off by deviating to lose and then buying at the resale market.

For $\theta^* = \bar{v}_A$ at the speculative $\bar{v}_A$-equilibrium buyer B always wins and if $w_B \geq \bar{v}_A$ holds, then he cannot be made financially constrained. His only possible deviation is to lose which is unprofitable if $w_A \geq \underline{v}_A$. In contrast, if $w_B < \bar{v}_A$ and $\bar{v}_B > \psi_A(\underline{v}_A)$ the strategy profile $(\underline{v}_A, \bar{v}_A)$ cannot be part of an equilibrium, given that, conditional on losing, buyer A is always better off forcing his rival to be constrained.

Note that if $\bar{v}_B$ is sufficiently low so that B could be considered a speculator, when he wins he prefers to resell the entire good. At $p = \underline{v}_A$, he makes a loss from any fraction he gets to consume as $p > v_B$. Because of this, whenever he offers a resale price larger than the auction price, his offer is unaccepted with positive probability, and he will be forced to consume the good. For some types of buyer B to find it profitable to set $r_B > \underline{v}_A$ it must be the case that $\bar{v}_B > \psi_A(\underline{v}_A)$ holds (which will be the case if $\bar{v}_B = \underline{v}_A$). In Garrat and Tröger's θ^*-equilibrium $p = v_B = 0$ so that the disutility from a price higher than the use value does not arise. Furthermore, $\bar{v}_B > \psi_A(\underline{v}_A)$ does always hold in their set-up as $\underline{v}_A = \bar{v}_B = 0$ and $\psi_A(0) = -\frac{1}{h_A(0)} < 0$.

In sum, financial constraints destroy all the resale equilibria unless one of the buyers be wealthy enough so that $w_i \geq \bar{v}_A$ holds for some i. When it is the strong buyer, then truth-telling is the unique resale equilibria that survives the introduction of financial constraints. When it is the weak one then the speculative θ^*-equilibrium with $\theta^* = \bar{v}_A$ is the only surviving equilibrium.

3.2. The financial constraints equilibria

In a static one-round SPA with budget constraints it is a dominant strategy to bid $\min(w_i, v_i)$ (see Che and Gale, 1998). The reason is that if the second highest bid is above the winner's budget, he will renege, will not get the object and will pay the fine, resulting in a negative surplus. With the possibility of resale this argument breaks down if the winner can resell the good and, by doing so, can get more than the auction price. This is the case here as long as the potential buyer at the resale market does not follow dominated strategies.

Arguably, truth-telling is not an equilibrium in the auction with resale as it is not an equilibrium either in an auction without resale among financially constrained bidders. Recall that the equilibrium bids in undominated strategies of the auction without resale requires each buyer to bid $b_i = \min\{v_i, w_i\}$ (in what follows we refer to this as the no-resale equilibrium). Acknowledging this fact, we explore next the conditions needed for the equilibrium in the auction without resale to be an equilibrium under resale.

The possible realizations of bidders' wealths with their corresponding bidding strategies in the no-resale equilibrium are displayed in table below.

wealths	$\underline{v}_B \leq w_B < \underline{v}_A$	$w_B \geq \underline{v}_A$
$\underline{v}_B \leq w_A < \underline{v}_A$	$(w_A, \min\{w_B, v_B\})$	(w_A, v_B)
$\underline{v}_A \leq w_A$	$(\min\{w_A, v_A\}, \min\{w_B, v_B\})$	$(\min\{w_A, v_A\}, v_B)$

The no-resale equilibrium is not an equilibrium under resale in the first row, i.e., if $w_A < \underline{v}_A$, because any loser bidder B with $v_B < w_A$ is better off deviating so as to win and then reselling part of the good. Note that whenever buyer A wins, since

$w_A > \min\{v_B, w_B\}$ for A to win, she will make no resale offer so that B's utility will be w_B. By deviating to $w_A + \epsilon$, B wins and by setting $[r_B, z_B] = [\underline{v}_A, \frac{w_A}{\underline{v}_A}]$ he gets a larger expected utility since he recovers the price w_A and is able to consume a fraction of the good.

If $w_A \geq \underline{v}_A$ (second row) then at the no-resale equilibrium buyer B follows a weakly dominated strategy as he is bidding below his lowest possible *endogenous valuation* $\underline{v}_A$. His *endogenous* valuations takes into account the overall surplus from winning and reselling the object. Since $w_A \geq \underline{v}_A$ buyer B can always get $\underline{v}_A$ from the resale market so that $b_B = \underline{v}_A$ weakly dominates $b_B = \min\{w_B, v_B\}$. If either winning or losing with the two of them they yield the same profits. But if winning with $b_B = \underline{v}_A$ while losing with $\min\{w_B, v_B\}$ then the former yields larger expected profits as $p < \underline{v}_A \leq r_B$ so that

$$U_B^{WB}(b_A, \underline{v}_A) > w_B = U_B^{WA}(b_A, \min\{w_B, v_B\}).$$

Next proposition summarizes this discussion.

Proposition 3. *(i) If $w_A < \underline{v}_A$ then bidding $b_i = \min\{w_i, v_i\}$ $i = A, B$ at the first stage is never part of a PBE.*

(ii) If $w_A \geq \underline{v}_A$ then bidding $b_i = \min\{w_i, v_i\}$ $i = A, B$ at the first stage can never be part of a PBE in which bidders do not employ weakly dominated strategies.

3.3. Resale and Financial constraints

We have seen that buyers' attempts to make their rivals financially constrained give rise to profitable deviations which may destroy the equilibria under resale. Similarly, we have seen that the existence of a resale market can generate buyers' valuations above the use values making some strategies weakly dominated.

Since buyers will anticipate their rivals' incentives to make them financially constrained, they can incorporate these incentives in their bidding. Similarly, bids must account for the prospects at the post-auction resale, which, in turn, depend on wealth. We next characterize equilibrium bids depending of the strong buyer wealth.

High budget buyer A: $w_A \geq \underline{v}_A$

We consider that player A has a relatively high budget, $w_A \geq \underline{v}_A$, while maintaining the assumption that she is at least potentially financially constrained so that $w_A \leq \bar{v}_A$.

By weak dominance arguments, any $b_B < \underline{v}_A$ is now weakly dominated.

Consider the family of cut-off strategies $b_A^{\lambda^*}(v_A)$, parameterized by λ^* with $\lambda^* \in [\underline{v}_A, \bar{v}_A)$, consisting of strategies for which any bidder A with a use value below λ^* bids her value, while any bidder with use value above λ^* bids w_A, i.e.,

$$b_A^{\lambda^*}(v_A) = \begin{cases} v_A & \text{if } v_A \leq \lambda^* \\ w_A & \text{if } v_A > \lambda^* \end{cases}$$

$$b_A : \underline{v}_A \underbrace{---------}_{v_A} \lambda^* \underbrace{---------}_{w_A} \bar{v}_A.$$

Note that $\lambda^* = \bar{v}_A$ generates the truth-telling strategy, $\lambda^* = w_A$ gives the equilibrium strategy of the auction without resale,[9] and $\lambda^* = \underline{v}_A$ generates the pooling strategy $b_A(v_A) = w_A$.

We show that the strategy profile $(b_A^{\lambda^*}, \underline{v}_A)$ allows to generate a continuum of equilibria, in which buyer A wins at the first stage and there is no resale, if one of the two conditions below hold (this requirement will be referred to as **condition (w)**)

(i) $w_B \in \left[\frac{\bar{v}_B}{w_A h_A(w_A)}, w_A\right]$ or

(ii) $w_A \in [\frac{\bar{v}_B}{h_A(w_A)(w_A - \bar{v}_B)}, w_B)$.

Condition (w) ensures that player B will not raise his bid to win and profit from resale: either w_B is not too low so that the risk of bankruptcy is a serious threat, or w_A (and hence p) is high enough leaving no profits to be made at resale. It is worth noting that with symmetric wealths ($w_A = w_B$) or when they are close enough, condition (w) always holds. When $w_A < w_B$, part *(ii)* may be rewritten as $\bar{v}_B \leq \frac{1}{M_A(w_A)}$ and when $w_A > w_B$, part *(i)* may be expressed as $\bar{v}_B \leq \frac{w_A w_B}{h_A(w_A)}$.

Proposition 4. *If condition (w) holds, then for any $\lambda^* \in [\underline{v}_A, w_A]$ there exists a Perfect Bayesian equilibrium of the SPA with resale in which players bid $b = (b_A^{\lambda^*}, \underline{v}_A)$ at the auction stage and they follow the resale strategies described in Lemma 2 and Lemma 4, respectively.*

Proof. See Appendix.

To further clarify the role of (w) take $\lambda^* = w_A$ so that $b_A = \min\{v_A, w_A\}$. At the candidate equilibrium buyer A wins and does not resell since $p \leq w_A$ so that $U_B = w_B$. Deviations that will keep B as a loser while forcing A to be financially constrained are not feasible, in other words, Lemma 5 (i) does not bite as $b_A \leq w_A$. The only payoff relevant deviations for B involve winning by bidding more than w_A (if winning at any $p < w_A$ he learns v_A so that $r_B = p = v_A$ and $U_B = w_B$). Consider hence deviations to $b'_B > w_A$.

If $w_B < w_A$, he will need to resell to avoid defaulting as $p = w_A > w_B$. For any resale price $r_B > w_A = p$ he earns $(r_B - p)$ when his offer is accepted but loses w_B when unaccepted. For the gains to outweigh the loses it must be the case that w_B is sufficiently small. Thus, there is a profitable deviation if and only if $w_B < \frac{\bar{v}_B}{w_A h_A(w_A)}$ holds. The reason for low-budget bidders to find it profitable to deviate is reminiscent of the results in Zheng (2001). A low-budget bidder B has less to lose from bankruptcy than a high-budget bidder. That is why bidder B must have a sufficiently low budget to find it profitable to bid high as to win and then resale while facing the risk of default.[10]

If $w_B > w_A$ when buyer B deviates to win then he can always recover the auction price by setting $r_B = v_A^L = w_A$ whenever $p = w_A$. Thus, he is indifferent between bidding $\underline{v}_A$ and losing or bidding $b'_B > w_A$ winning and reselling at $r_B = v_A^L = w_A$. Furthermore, the deviation is worth it if $\bar{v}_B > \frac{1}{M_A(w_A)}$ holds, (i.e., if

[9] The assumption $w_A \geq \underline{v}_A$ guarantees that these bidding strategies are weakly increasing. If $w_A < \underline{v}_A$ then $b_A^{\lambda^*}$ is weakly decreasing unless $\lambda^* = \underline{v}_A$ or $\lambda^* = \bar{v}_A$.

[10] In a model without resale Zheng (2001) does also find that the symmetric Bayes Nash equilibrium bidding strategy is not monotonic as a function of the bidder's budget .

$w_A < \frac{\bar{v}_B}{h_A(w_A)(w_A - \bar{v}_B)}$). For a large w_A the loses when a resale offer is rejected, given by $(v_B - w_A)$, are too large, making it less attractive to become the winner.

Remark: $\lambda^* \leq w_A$ *is needed for the above strategies to constitute a PBE. If* $\lambda^* > w_A$ *instead, from Lemma 5* (i), there will be a profitable deviation as $p < w_A$.

We have provided a set of equilibria when $w_A \geq \underline{v}_A$ which yield (despite the multiplicity of equilibrium strategies) a unique equilibrium outcome characterized by the strong buyer winning, and consuming the entire good. In equilibrium there is no resale and $p = \underline{v}_A$.

Whenever B is wealthier than A, the above equilibrium relies on B not deviating when he is indifferent between winning or losing. If he bids w_B to take advantage of his (relative) strength it is a best response for buyer A to bid $\underline{v}_A$ provided that $\bar{v}_B \leq \frac{1}{M_A(w_B)}$ holds. Note that $\bar{v}_B > \psi_A(\underline{v}_A)$ is necessary for the above strategies to yield a profitable outcome for at least some types of buyer B. But even if A is wealthier than B, buyer B can prevent his rival from making him financially constrained, by bidding w_B. When he does so, no type of A wants to deviate so as to win provided that $r_B^{**} \leq w_B$.

Proposition 5. *Assume B is wealthier than A so that $w_B > w_A \geq \underline{v}_A$. The strategy profile $(\underline{v}_A, w_B)$ is part of a PBE (in which B wins) if $\bar{v}_B \leq \frac{1}{M_A(w_B)}$. If A is wealthier than B with $w_A > w_B \geq \underline{v}_A$ then the strategy profile $(\underline{v}_A, w_B)$ is part of a PBE (in which B wins) iff $\bar{v}_B \leq \psi_A(w_B)$.*

Proof. See Appendix.

When B is wealthier than A and $\bar{v}_B \leq \frac{1}{M_A(w_A)}$ we have found two equilibria both with $p = \underline{v}_A$ which differ in the identity of the winner. If $\frac{1}{M_A(w_A)} < \bar{v}_B \leq \frac{1}{M_A(w_B)}$ the equilibria in which A wins disappears and if $\bar{v}_B > \frac{1}{M_A(w_B)}$ both equilibria disappear. Given that $\frac{1}{M_A(w)}$ is increasing in w with a minimum value of 0 at $w = \underline{v}_A$, and tends to $\bar{v}_A$ when w tends to $\bar{v}_A$, we have that $\bar{v}_B = \frac{1}{M_A(w)}$ for some $w \in (\underline{v}_A, \bar{v}_A)$ and for higher levels of wealth the condition $\bar{v}_B \leq \frac{1}{M_A(w)}$would hold. Thus, these equilibria disappear when wealths are close to $\underline{v}_A$.

Low budget buyer A: $w_A < \underline{v}_A$

At the resale market buyer B will get at most w_A from buyer A. Because of this, now weak dominance arguments only eliminate bids $b_B < w_A$.

Whenever $w_B \geq \underline{v}_A > w_A$ buyer B can take advantage of his (relative) strength (his larger wealth) and behave more aggressively incorporating in his bid his expected profits from resale (he can always get w_A by setting $r_B = \underline{v}_A$). Furthermore, he can prevent his rival from making him financially constrained, by bidding w_B. Since the best response by the strong buyer to a sufficiently high bid is to hide her type, a candidate to equilibrium is $b = (b_A, b_B) = (w_A, w_B)$.

Proposition 6. *If $w_B \geq \underline{v}_A > w_A$, the strategy profile (w_A, w_B) is part of a PBE (in which B wins and makes positive profits).*

Proof. See Appendix.

4. Conclusion

This paper has analyzed the effect of two important features related to auctions for big facilities or contracts: partial resale and binding budget constraints. In these auctions bidders usually face severe financial constraints that may force them to resell part of the property of the good (or subcontract part of the project) at a resale market. We show that the interaction between resale and financial constraints changes previous results on auctions with financial constraints and those in auctions with resale. The reason is the link between the resale price and the auction price introduced by the presence of financial constraints. Such link induces a potential loser to increase the auction price in order to fine-tune the winner's resale offer. We also characterize equilibria of the auction with resale and financial constraints that do not involve speculative behavior by the low use value player and yield efficient outcomes.

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5. Appendix

Proof of lemma 3 (i) Since buyer A will never pay more than her use value, $v_A^H \geq r_B^o$, and since any type will pay at least v_A^L then $r_B^o \geq v_A^L$. Finally, buyer A total payment cannot exceed her wealth, $r_B^o z_B^o \leq w_A$.

(ii) If $r_B^o z_B^o < w_A$ it is possible for B to increase his utility by increasing z_B^o and keeping r_B^o constant, so that $r_B^o z_B^o < w_A$ still holds. Since his utility increases ($r_B^o \geq v_A^L \geq \bar{v}_B$ and the probability of acceptance does not change) it must be the case that $z_B^o = 1$ if $r_B^o z_B^o < w_A$. The same arguments show (iii).

(iv) Assume not, $z_B^o = 1$. Then $r_B^o = r_B^o z_B^o \leq w_A < v_A^L$, which contradicts (i).■

Proof of Lemma 4 We derive the optimal resale offers by player B.

Case 1. $p \leq w_B$.

If $w_A \geq v_A^L$ bidder B can set $z_B = 1$ and $r_B \in \left[v_A^L, \min(w_A, v_A^H)\right)$ to maximize her expected utility,

$$\max_{r_B} \left\{w_B - p + r_B \Pr\left(r_B \leq v_A/p\right) + v_B \Pr\left(r_B \geq v_A/p\right)\right\}.$$

Solving the maximization problem above, B's optimal resale price r_B^* is the solution to equation below:

$$r_B - v_B = \frac{1}{\hat{h}_A\left(r_B\right)} \tag{4}$$

which has a unique solution as the standard hazard rate condition holds. Since the LHS of (4) is strictly increasing while the RHS is strictly decreasing a necessary condition for r_B^* to be interior is that $v_A^L - v_B < \frac{1}{\hat{h}_A\left(v_A^L\right)}$. Finally, the solution to equation above fails to be lower than w_A if $w_A - v_B > \frac{1}{\hat{h}_A(w_A)}$. The constrained maximum in then $r_B^* = w_A$.

Alternatively, B can set z_B be such that $z_B r_B = w_A$, so that $r_B \in \left[v_A^L, v_A^H\right]$ if $w_A < v_A^L$ and $r_B \in \left[w_A, v_A^H\right]$ if $w_A \geq v_A^L$ will solve

$$\begin{aligned}
&\max_{r_B}\left\{w_B - p + \left(w_A + v_B\left(1 - \frac{w_A}{r_B}\right)\right)\left(1 - \hat{F}_A\left(r_B\right)\right) + v_B \hat{F}_A\left(r_B\right)\right\} \\
&= \max_{r_B}\left\{w_B - p + v_B + w_A\left(1 - \frac{v_B}{r_B}\right)\left(1 - \hat{F}_A\left(r_B\right)\right)\right\}
\end{aligned}$$

The optimal resale price $\hat{r}_B^*$ is the solution to:

$$\frac{r_B}{v_B}\left(r_B - v_B\right) = \frac{1}{\hat{h}_A\left(r_B\right)} \tag{5}$$

For the same arguments as before, the optimal (unconstrained) resale price is v_A^L if $\left(v_A^L - v_B\right)\frac{v_A^L}{v_B} > \frac{1}{\hat{h}_A\left(v_A^L\right)}$. It will be larger than w_A if $\frac{w_A}{v_B}\left(w_A - v_B\right) < \frac{1}{\hat{h}_A(w_A)}$.

By comparing the two offers it follows that $[r_B = \hat{r}_B^*,\, z_B < 1]$ dominates $[r_B = w_A,\, z_B = 1]$ if $w_A < v_A^L$ as $U_B\left(r_B = \hat{r}_B^*,\, z_B < 1\right) \geq U_B\left(r_B = v_A^L,\, z_B < 1\right)$ by the optimality of $\hat{r}_B^*$, and the fact that $U_B\left(r_B = v_A^L,\, z_B < 1\right) > U_B\left(r_B = w_A,\, z_B = 1\right)$. The optimal resale offers by B

when $w_A \leq v_A^L$ are hence

$$\hat{O}_B^*(p) = \begin{cases} \left[r_B = v_A^L,\ z_B = \frac{w_A}{v_A^L}\right] \text{ if } v_B < \frac{\left(v_A^L\right)^2 \hat{h}_A\left(v_A^L\right)}{1+v_A^L \hat{h}_A\left(v_A^L\right)} \\ \left[r_B = \hat{r}_B^*,\ z_B = \frac{w_A}{\hat{r}_B^*}\right] \text{ if } v_B \geq \frac{\left(v_A^L\right)^2 \hat{h}_A\left(v_A^L\right)}{1+v_A^L \hat{h}_A\left(v_A^L\right)} \end{cases}$$

If $w_A \geq v_A^L$ since $r_B^* > \hat{r}_B^*$ (the LHS of (4) is steeper than the corresponding one in (5) while they are both equal to zero at $r_B = v_B$), the resale offer $[r_B = r_B^* < w_A,\ z_B = 1]$ dominates the offer $[r_B = \hat{r}_B^*,\ z_B < 1]$. But if $\hat{r}_B^* > w_A$ (i.e. if $w_A \hat{h}_A(w_A) < \frac{v_B}{(w_A - v_B)}$) then $[r_B = \hat{r}_B^* > w_A,\ z_B < 1]$ dominates $[r_B = r_B^* = w_A,\ z_B = 1]$ as with both B gets the same resources, namely w_A, whereas with the former he gets to consume part of the good in return for the risk of getting more often his offer rejected. Because of this w_A cannot be too large. In sum, the optimal resale offers if B is unconstrained are

$$\begin{array}{lll} \left[r_B = v_A^L,\ z_B = 1\right] & \text{if} & v_B \leq \hat{\psi}_A(v_A^L) \\ \left[r_B = r_B^{**},\ z_B = 1\right] & \text{if} & v_B \in \left[\hat{\psi}_A(v_A^L), \hat{\psi}_A(w_A)\right] \\ \left[r_B = w_A,\ z_B = 1\right] & \text{if } v_B \in & \left[\hat{\psi}_A(w_A), w_A - \frac{v_B}{w_A \hat{h}_A(w_A)}\right] \\ \left[r_B = \hat{r}_B^{**} > w_A,\ z_B < 1\right] & \text{if} & v_B \geq w_A - \frac{v_B}{w_A \hat{h}_A(w_A)}. \end{array}$$

Case 2. $p \in \left(w_B,\ \min(w_A, v_A^H) + w_B\right]$.

Bidder B will maximize:

$$\max_{r_B, z_B} \left\{(w_B - p + r_B z_B + v_B(1 - z_B)) \left(1 - \hat{F}_A(r_B)\right)\right\}$$

As in case 1, whenever $w_A \geq v_A^L$ bidder B may sell the entire object,[11] $z_B = 1$, for a resale price $r_B \in \left[v_A^L, v_A^H\right)$ which maximizes his expected utility,

$$\max_{r_B} \left\{(w_B - p + r_B) \left(1 - \hat{F}_A(r_B)\right)\right\}$$

The optimal resale price r_B^{**} is the solution to equation below

$$r_B - (p - w_B) = \frac{1}{\hat{h}_A(r_B)} \tag{6}$$

The optimal resale offers by B when $w_A > v_A^L$ and $z_B = 1$ are as follows

$$\begin{cases} \left[r_B = v_A^L,\ z_B = 1\right] \text{ if } p - w_B \leq \hat{\psi}_A(v_A^L) \\ \left[r_B = r_B^{**}(p),\ z_B = 1\right] \text{ if } p - w_B \in \left[\hat{\psi}_A(v_A^L), \hat{\psi}_A(w_A)\right] \\ \left[r_B = w_A,\ z_B = 1\right] \text{ if } p - w_B \geq \hat{\psi}_A(w_A). \end{cases}$$

[11] Assume that the optimal (r_B^*, z_B^*) is such that $z_B^* < 1$.Then, if $r_B^* z_B^* < w_A$ it is possible to increase utility by increasing z_B^* and keeping r_B^* constant, since $r_B^* > v_B$ and the probability of acceptance does not change. If $r_B^* z_B^* = w_A$, increasing z_B requires decreasing r_B and this changes the objective function with the sign of $[-(1-F) + \frac{z_B}{r_B}(1 - z_B) f'(r_B)]$.

Alternatively, buyer B can set $z_B r_B = w_A$ and $r_B \in \left[v_A^L, v_A^H\right)$ to maximize

$$\left(w_B - p + w_A + v_B\left(1 - \frac{w_A}{r_B}\right)\right)\left(1 - \hat{F}_A\left(r_B\right)\right) + 0$$

The optimal resale price $\hat{r}_B^{**}(p)$ is now the solution to equation below:

$$\frac{r_B}{v_B}\left(r_B\left(w_A + w_B - p + v_B\right) - w_A v_B\right) = \frac{w_A}{\hat{h}_A\left(r_B\right)}, \tag{7}$$

where $\hat{r}_B^{**}(p) > w_A$ for $w_A > v_A^L$ iff $p - w_B > w_A - \frac{v_B}{w_A \hat{h}_A(w_A)}$, whereas for $w_A < v_A^L$ it always holds that $\hat{r}_B^{**}(p) > w_A$.

If $w_A \leq v_A^L$ then the optimal resale offers are

$$\begin{cases} \left[r_B = v_A^L,\ z_B = \frac{w_A}{v_A^L}\right] & \text{if } \frac{v_A^L}{v_B}\left(v_A^L\left(w_A + w_B - p + v_B\right) - w_A v_B\right) > \frac{w_A}{\hat{h}_A\left(v_A^L\right)} \\ \left[r_B = r_B^{**}(p),\ z_B = \frac{w_A}{r_B^{**}}\right] & \text{if } \frac{v_A^L}{w_A}\left(v_A^L\left(w_A + w_B - p + v_B\right) - w_A v_B\right) \leq \frac{v_B}{\hat{h}_A\left(v_A^L\right)} \end{cases}$$

whereas when $w_A > v_A^L$ they are as follows

$$\begin{array}{ll} \left[r_B = v_A^L,\ z_B = 1\right] & \text{if } \quad p - w_B \leq \hat{\psi}_A(v_A^L) \\ \left[r_B = r_B^{**},\ z_B = 1\right] & \text{if } \quad p - w_B \in \left[\hat{\psi}_A(v_A^L), \hat{\psi}_A(w_A)\right] \\ \left[r_B = w_A,\ z_B = 1\right] & \text{if } p - w_B \in \left[\hat{\psi}_A(w_A), w_A - \frac{v_B}{w_A \hat{h}_A(w_A)}\right] \\ \left[r_B = \hat{r}_B^{**} > w_A,\ z_B < 1\right] & \text{if } \quad p - w_B \geq w_A - \frac{v_B}{w_A \hat{h}_A(w_A)}. \end{array}$$

Combining the different cases, O_B^o follows.■

Proof of Lemma 5 (i) Let us write for short $p - w_A$ as y. Since for any $y \leq \frac{\left(v_B^L\right)^2 \hat{f}_B\left(v_B^L\right)}{1 + v_B^L \hat{f}_B\left(v_B^L\right)} = \frac{1}{\hat{M}_B\left(\underline{v}_B\right)}$ the expected utility by loser B is strictly increasing in y as

$$U_B^{WA}(y) = w_B + (v_B - v_B^L)\frac{y}{v_B^L},$$

it trivially follows that $y = \frac{1}{\hat{M}_B\left(\underline{v}_B\right)}$ yields larger expected utility to buyer B than any lower y. Thus, if $b = (b_A, b_B)$ is such that $b_A(v_A) > w_A$ and $b_A(v_A) \geq b_B(v_B)$ for some pair (v_A, v_B), then $p > w_A$ for b to be part of a PBE equilibrium, as B is always better off if A is financially constrained. Note that if A wins and has wealth enough to cover the auction price she will not resell, so that $U_B^{WA}(v_A, v_B) = w_B$. In contrast, if B deviates to bidding $b'_B = w_A + \epsilon$ he will get an expected utility larger than w_B, as player A still wins but he must now resell $z_A = \frac{\epsilon}{r_A^*}$. As buyer A's posterior will coincide with her prior, her optimal resale offer employs $\hat{f}_B = f_B$ and $v_B^L = \underline{v}_B$. By making ϵ arbitrarily small so that $\epsilon \leq \frac{1}{M_B\left(\underline{v}_B\right)}$ holds, it follows that $r_A^* = \underline{v}_B$, and $U_B^{WA}(v_A, w_A + \epsilon) > w_B$.

We next show that for larger values of y bounded above by $\min(w_B v_B^H)$ there exists $\hat{v}_B > r_A$ such that $\frac{\partial U_B^{WA}(y)}{\partial y} > 0$ if $v_B > \hat{v}_B$ whereas $\frac{\partial U_B^{WA}(y)}{\partial y} < 0$ otherwise, where

$$U_B^{WA}(y) = \max\left\{w_B + (v_B - r_A^*(y))\frac{y}{r_A^*(y)}, w_B\right\} \text{ so that for any } v_B \geq r_A$$

it follows that

$$\frac{\partial U_B^{WA}(y)}{\partial y} = \frac{v_B\left(r_A^* - y r_A'^*\right) - \left(r_A^*\right)^2}{\left(r_A^*\right)^2}. \tag{8}$$

Total differentiation of $r_A\left(\frac{r_A}{y} - 1\right) = \frac{1}{\hat{h}_B(r_A)}$ gives

$$\frac{dr_A}{dy} = \frac{r_A^2\left(\hat{h}_B(r_A)\right)^2}{y\left[(2r_A - y)\left(\hat{h}_B(r_A)\right)^2 + y\hat{h}_B'(r_A)\right]} > 0$$

where the sign of the denominator equals the sign of $-\frac{\partial^2 U_A(r_A)}{\partial r_A^2}$ and is hence positive by the concavity of U_A. Plugging its value into $(r_A^* - y r_A'^*)$ and simplifying we get $v_B r_A^*\left(1 - \frac{y r_A'^*}{r_A^*}\right) - (r_A^*)^2 = r_A^*\left(v_B\left(1 - \frac{y r_A'^*}{r_A^*}\right) - r_A^*\right)$ so that the sign of (8) is given by the sign of $v_B\left(1 - \frac{y r_A'^*}{r_A^*}\right) - r_A^*$, with

$$v_B\left(1 - \frac{y r_A'^*}{r_A^*}\right) - r_A^* = \frac{\left(\left(\hat{h}_B(r_A^*)\right)^2 (r_A^* - y) + y\hat{h}_B'(r_A^*)\right)[v_B - r_A^*] - r_A^*\left(\hat{h}_B(r_A^*)\right)^2}{\left(\hat{h}_B(r_A^*)\right)^2 (2r_A^* - y) + y\hat{h}_B'(r_A^*)}$$

Since $v_B \geq r_A$ must hold for buyer B to accept r_A, it follows that $\frac{\partial U_B^{WA}(y)}{\partial y} < 0$ for $v_B = r_A$, whereas $\frac{\partial U_B^{WA}(y)}{\partial y}$ could be positive for v_B sufficiently above r_A^*. Since $v_B - r_A^* < \bar{v}_B - \underline{v}_B = 1$ we have that

$$v_B\left(1 - \frac{y r_A'^*}{r_A^*}\right) - r_A^* < \frac{y\left[\hat{h}_B'(r_A^*) - \left(\hat{h}_B(r_A^*)\right)^2\right]}{\left(\hat{h}_B(r_A^*)\right)^2 (2r_A^* - y) + y\hat{h}_B'(r_A^*)} =$$

$$\frac{y\left[\frac{\hat{f}_B'(r_A^*)}{1 - \hat{F}_B(r_A^*)}\right]}{\left(\hat{h}_B(r_A^*)\right)^2 (2r_A^* - y) + y\hat{h}_B'(r_A^*)} < 0$$

where the equality follows from the fact that $\hat{h}_B'(r_A^*) = \left(\hat{h}_B(r_A^*)\right)^2 + \frac{\hat{f}_B'(r_A^*)}{1 - \hat{F}_B(r_A^*)}$, and the last inequality from the concavity of F.

(ii). Assume by contradiction that there exists an equilibrium in which A loses by bidding $b_A < w_B$ while $b_B > w_B$ and $\bar{v}_B > \hat{\psi}_A(v_A^L)$ hold. Since $v_A^L < r_B^* < w_A < r_B^{**}$ then A expected utility is bounded above by $w_A + v_A - r_B^*$. Since by deviating to $b_A = w_B + \epsilon < b_B$ her expected utility is $w_A + v_A - v_A^L > w_A + v_A - r_B^*$, we have found a profitable deviation contradicting equilibrium behavior.

(iii). Assume by contradiction that there exists an equilibrium in which A loses by bidding $b_A < w_B$ while $b_B > w_B$ and $\bar{v}_B > \frac{1}{M_A(v_A^L)}$ hold. At such an equilibrium her expected utility equals $w_A + (v_A - r_B^{**})\frac{w_A}{r_B^{**}}$ which is smaller than $w_A + (v_A - v_A^L)\frac{w_A}{v_A^L}$ the utility he would get if $b_A = w_B + \epsilon < b_B$. The same arguments apply for $b_A > w_B + w_A\left(1 - \bar{v}_B M_A(v_A^L)\right) + \bar{v}_B$ which shows our claim.■

Proof of Proposition 1 The *if* part was first shown in Haile (1999). For the *only if* part assume first that $w_A \in [\underline{v}_A, \bar{v}_A)$. By following the truth-telling equilibrium strategies player B loses and sets the auction price at $p = v_B \leq \underline{v}_A$. Player A wins and as she has wealth enough to cover the auction price she does not resell, so that $U_B^{WA}(v_A, v_B) = w_B$. In contrast, if B deviates to bidding $b'_B = w_A + \epsilon$ he will get an expected utility larger than w_B. If $v_A \geq w_A + \epsilon$ then the result follows from Lemma 5 (i). If $v_A < w_A + \epsilon$ buyer B wins and can infer v_A from the auction price. He will optimally resell the entire object at a price v_A so that $U_B^{WB}(v_A, w_A + \epsilon) = w_B$. Summing up over all the possible types of bidder A it follows that $b'_B = w_A + \epsilon$ constitutes a profitable deviation so that (v_A, v_B) do not constitute equilibrium bids. Assume next that $w_A < \underline{v}_A$. By deviating to $w_A + \epsilon$ player B remains a loser (as with v_B) and gets larger expected profit as he will get to use part of the good that he will acquire at the resale market. Since for any $w_A < \bar{v}_A$ there is a profitable deviation, the statement follows. Finally, note that at $w_A = \bar{v}_A$ we have $U_B^{WB}(v_A, w_A + \epsilon) = U_B^{WA}(v_A, v_B) = w_B$ so that the deviation does not yield larger profits.■

Proof of Proposition 2 (i) At any θ^*-equilibrium $(\theta^* < \bar{v}_A)$ buyer B wins if facing types $v_A < \theta^*$ and loses otherwise.

Assume first $w_A \in [\theta^*, \bar{v}_A)$ so that buyer B gets utility w_B if he loses as types $v_A \geq \theta^*$ will not resell. In contrast, if deviating to bid $b'_B = w_A + \epsilon$ he will get a larger expected utility. Profits from types $v_A < \theta^*$ will coincide with the two strategies. If he remains a loser with b'_B because $v_A \geq w_A + \epsilon$ then he now makes larger profits (the result follows from Lemma 5 (i)). If $v_A < w_A + \epsilon$ buyer B wins with the deviation, he infers v_A from the auction price. He will optimally resell the entire object at a price v_A so that $U_B^{WB}(v_A, w_A + \epsilon) = w_B$ if $p = v_A > \underline{v}_A$. Summing up over all the possible types of bidder A it follows that $b'_B = w_A + \epsilon$ constitutes a profitable deviation.

Assume next $w_A \in [\underline{v}_A, \theta^*)$ so that types $v_A \geq \theta^*$ are financially constrained when winning. We first note that $v_A > w_A + \frac{1}{M_B(\underline{v}_B)}$ for some $v_A \in (\theta^*, \bar{v}_A)$ suffices for B to find profitable to deviate to bid $w_A + \frac{1}{M_B(\underline{v}_B)}$ (recall Lemma 5 (i)). Even if $\bar{v}_A < w_A + \frac{1}{M_B(\underline{v}_B)}$ holds then high use-values types prefer to get the entire object at the resale market rather than winning and being forced to resell as $p > w_A$.

Let us finally consider the case $w_A < \underline{v}_A$. A buyer A with type $v_A > \theta^*$ wins and will have to resell so that her utility is bounded above by $\left(1 - \frac{p - w_A}{r_A}\right) v_A$ where $p = \theta^*$ so that $U_A^{WA} \leq \left(1 - \frac{\theta^* - w_A}{\bar{v}_B}\right) v_A$ as $r_A < \bar{v}_B$.

If deviating to lose, by bidding $\underline{v}_A$ so as to pool with types $v_A \leq \theta^*$ she gets $U_A^{WB} = \frac{w_A}{r_B} v_A$, where we are using the fact that $r_B < \theta^*$ so that $\frac{w_A}{r_B} v_A > \frac{w_A}{\theta^*} v_A > w_A$. The deviation is profitable as

$$\begin{aligned}\frac{w_A}{\theta^*} - \left(\frac{\bar{v}_B - \theta^* + w_A}{\bar{v}_B}\right) &= \frac{(\bar{v}_B - \theta^*)\, w_A + \theta^*\,(\theta^* - \bar{v}_B)}{\theta^* \bar{v}_B} \\ &= \frac{(\theta^* - w_A)\,(\theta^* - \bar{v}_B)}{\theta^* \bar{v}_B} > 0.\end{aligned}$$

(ii) For $\theta^* = \bar{v}_A$ at the speculative $\bar{v}_A$-equilibrium buyer B always wins and as $w_B > \bar{v}_A$ holds, he cannot be made financially constrained. Because of this, buyer A

is trivially best-responding. As for buyer B, if $\bar{v}_B > \psi_A(\underline{v}_A)$ he will profitably resell, i.e., $p = \underline{v}_A = v_A^L < r_B$. His only possible deviation is to lose which is unprofitable if $w_A \geq \underline{v}_A$ as then buyer A will not resell.

We next show that if $w_B < \bar{v}_A$ and $\bar{v}_B > \psi_A(\underline{v}_A)$ holds then the strategy profile $(\underline{v}_A, \bar{v}_A)$ cannot be part of an equilibrium. Conditional on losing, buyer A is always better-off forcing his rival to be constrained and to set the best offer from her viewpoint, i.e., $z_B = 1$ and $r_B = v_A^L$. Buyer A can do so if by bidding $b_A = w_B + \underline{v}_A - \frac{1}{h_A(\underline{v}_A)}$. But then $p > r_B$ and B is better off deviating to lose.

Finally If $w_A < \underline{v}_A$, buyer B will partially resell getting w_A at the resale market and consuming part of the good. Let $\hat{v}_B$ be such that $\hat{v}_B = \frac{1}{M_A(\underline{v}_A)}$ ($\hat{v}_B = \bar{v}_B$ if $\bar{v}_B \leq \frac{1}{M_A(\underline{v}_A)}$). For any $v_B \leq \hat{v}_B$ we have $r_B^{**}(v_B) = \underline{v}_A$ so that $U_B^{WB}(p = \underline{v}_A) = w_B - \underline{v}_A + v_B + w_A \left(1 - \frac{v_B}{\underline{v}_A}\right) < w_B$ as $\frac{v_B}{\underline{v}_A}(\underline{v}_A - w_A) < \underline{v}_A - w_A$. Buyer B with use value below $\hat{v}_B$ is hence better-off deviating to lose. Since $r_B^{**}(v_B)$ is increasing in v_B for types sufficiently close to $\hat{v}_B$ such that $r_B^{**}(v_B) > \underline{v}_A$ it remains true that buyer B prefers to deviate so that $(\underline{v}_A, \bar{v}_A)$ is not an equilibrium as claimed. ■

Proof of Proposition 4 At the purported strategies bidder A wins the auction, keeps the entire good and pays $p = \underline{v}_A$ so that $U_B^{WA}(b_A^{\lambda^*}, b_B) = w_B$ and $U_A^{WA}(b_A^{\lambda^*}, b_B) = v_A + w_A - \underline{v}_A$. Consider first deviations by player A. The only payoff-relevant deviations require losing the auction and buying the entire good at the resale market at a cheaper price which is unfeasible as $r_B^* \geq \underline{v}_A = p$. It hence follows that A will not deviate. Consider now player B. If he deviates to any $b'_B < w_A$ he will win the auction if $v_A \leq \lambda^*$, will pay the use value of its rival and will resell at a price equal to that use value so that his expected utility will be equal to that under b_B, namely w_B. If he deviates to any $b'_B > w_A$, he will win to any buyer A. Now if $p < w_A$ then his expected payoff will be w_B as $r_B^* = p = v_A$. If $p = w_A$ he will infer that $v_A^L = \lambda^*$. The deviation will be profitable if and only if B finds optimal reselling part of the good at $r_B > w_A$ (i.e., if $r_B = r_B^{**}$). For $w_B > p = w_A$ since $\hat{F}_A(x) = \frac{F_A(x) - F_A(\lambda^*)}{1 - F_A(\lambda^*)}$, the deviation is unprofitable iff $r_B \neq r_B^{**}$, or, iff $(w_A - v_B)\frac{w_A}{v_B} \geq \frac{1}{h_A(w_A)}$ (see the computations of B resale offers).[12] Since the LHS is decreasing in v_B then $(w_A - \underline{v}_B)\frac{w_A}{\underline{v}_B} \geq \frac{1}{h_A(w_A)}$ must hold for $(b_A^{\lambda^*}, b_B)$ to be part of a PBE, i.e., $\overline{v}_B \leq \frac{1}{M_A(w_A)}$

For $v_B < w_B < p = w_A$ deviation is unprofitable iff $r_B^{**} \in [\lambda^*, w_A]$ and $z_B^{**} = 1$ which is the case if $\frac{w_A}{v_B} w_B \geq \frac{1}{h_A(w_A)}$ as shown in Lemma 4. Consequently, since $w_B > v_B$ and $w_A h_A(w_A) \geq 1$, the deviation is not profitable in this case either.■

Proof of Proposition 5 At the purported equilibrium profile $(\underline{v}_A, w_B)$, buyer B wins and pays $p = \underline{v}_A$. At the resale stage he will follow the resale offers in Lemma 4 for $\hat{F}_B(v_A) = F_B(v_A)$. The only types that can (potentially) profitably deviate are types of buyer A with $v_A > w_B$.

[12] If $\hat{F}_A$ is a truncation of F then $\hat{F}_A(x) = \frac{F_A(x) - F_A(v_A^L)}{1 - F_A(v_A^L)}$ so that

$$\frac{1}{\hat{h}_A(w_A)} = \frac{1 - \hat{F}_A(w_A)}{\hat{f}_A(w_A)} = \frac{1 - F_A(w_A)}{f_A(w_A)} = \frac{1}{h_A(w_A)}$$

If they deviate to win they will while $w_B > w_A$ they will have to resell part of the good so that $U_A^{WA} < \left(1 - \frac{w_B - w_A}{r_A^*}\right) v_A$. Since $\left(1 - \frac{w_B - w_A}{r_A^*}\right) v_A < v_A - r_B^* + w_A$ provided that $r_B^* < w_A + \left(\frac{w_B - w_A}{r_A^*}\right) v_A$ then $\bar{v}_B < \psi_A(w_A)$ guarantees that the deviation is unprofitable as $r_B^* \leq w_A$. Similarly, if she only consumes part of the good but $r_B^{**} \leq w_B$ player A will not deviate as

$$\begin{aligned}
U_A^{WB}(\underline{v}_A, w_B) - U_A^{WA}(b_A, w_B) &= v_A \frac{w_A}{r_B^{**}} - v_A \left(\frac{r_A^* - (w_B - w_A)}{r_A^*}\right) \\
&= v_A \left[\frac{r_A^* (w_A - r_B^{**}) + (w_B - w_A) r_B^{**}}{r_B^{**} r_A^*}\right] \\
&> v_A \left[\frac{r_A^* (w_A - r_B^{**}) + (w_B - w_A) r_A^*}{r_B^{**} r_A^*}\right] = v_A \left[\frac{(w_B - r_B^{**})}{r_B^{**}}\right] \geq 0,
\end{aligned}$$

where the first inequality follows from $r_A^* < \bar{v}_B < \underline{v}_A < r_B^{**}$ and the second one from $r_B^{**} \leq w_B$. Note that $r_B^{**} \leq w_B$ requires $\frac{w_B}{v_B} > \frac{w_B h_A(w_B)+1}{h_A(w_B) w_B}$ (see equation (5) in the appendix). It hence follows that $\bar{v}_B < \frac{w_B^2 h_A(w_B)}{w_B h_A(w_B)+1} = \frac{1}{M_A(w_B)}$ suffices for the deviation by the strong buyer to be unprofitable.

If $w_A > w_B$, the expected utilities are given by

$$U_A^{WB}(\underline{v}_A, w_B) = \begin{cases} w_A & \text{if } v_A < r_B^* \\ v_A - r_B^* + w_A & \text{if } v_A \geq r_B^* \end{cases} \quad \text{and}$$

$$U_B^{WB}(\underline{v}_A, w_B) > w_B \text{ as } r_B^* > v_A^L = \underline{v}_A.$$

If B deviates the only payoff relevant deviation is to bid as to lose but then $U_B^{WA} = w_B$ as A will not resale. Since $U_B^{WB}(\underline{v}_A, w_B) > w_B$ the deviation is unprofitable.

Focus next on buyer A. If her use value is below w_B she is better-off losing than winning so she will not deviate. If her use value is above w_B she will deviate if and only if $r_B^* > w_B$ so that $\bar{v}_B \leq \psi_A(w_B)$ is a necessary and sufficient condition for the above strategies to be an equilibrium whenever $w_B < w_A$. ■

Proof of Proposition 6 At the purported equilibrium profile buyer B wins and must pay $p = w_A$. At the resale stage B will follow the resale offers in Lemma 4 for $\hat{F}_B(v_A) = F_B(v_A)$. B only payoff relevant deviation is to lose but then his expected utility will be w_B as A will not resell. The only other deviation to check is that of types $v_A > w_B$. Since if deviating to win, she will have to resell, the deviation is unprofitable. More precisely, at the candidate equilibrium $U_A^{WB} = \max\{\frac{w_A}{r_B^{**}} v_A, w_A\}$. If deviating to win, she will have to resell so that her utility is bounded above by

$$U_A^{WA} \leq \left(1 - \frac{p - w_A}{r_A^*}\right) v_A.$$

Since $\left(1 - \frac{w_B - w_A}{r_A^*}\right) v_A \leq \left(1 - \frac{w_B - w_A}{\bar{v}_B}\right) v_A \leq \frac{w_A}{r_B^{**}} v_A$ the result follows as $r_B^{**} > w_A$ implies

$$\begin{aligned}
\frac{w_A}{r_B^{**}} - \left(1 - \frac{w_B - w_A}{\bar{v}_B}\right) &= \frac{(\bar{v}_B - r_B^{**}) w_A + r_B^{**} (w_B - \bar{v}_B)}{r_B^{**} \bar{v}_B} \\
&> \frac{(\bar{v}_B - \underline{v}_A) w_A + w_A (w_B - \bar{v}_B)}{r_B^{**} \bar{v}_B} = \frac{(w_B - \underline{v}_A) w_A}{r_B^{**} \bar{v}_B} > 0. ■
\end{aligned}$$

Numerical Study of a Linear Differential Game with Two Pursuers and One Evader*

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Abstract. A linear pursuit-evasion differential game with two pursuers and one evader is considered. The pursuers try to minimize the final miss (an ideal situation is to get exact capture), the evader counteracts them. Two case are investigated. In the first case, each one pursuer is dynamically stronger than the evader, in the second one, they are weaker. Results of numerical study of value function level sets (Lebesgue sets) for these cases are given. A method for constructing optimal feedback controls is suggested on the basis of switching lines. Results of numerical simulation are shown.

Keywords: pursuit-evasion differential game, linear dynamics, value function, optimal feedback control.

1. Introduction and Problem Formulation

1. In the paper, a model differential game with two pursuers and one evader is studied. Three inertial objects moves in the straight line. The dynamics descriptions for pursuers P_1 and P_2 are

$$\begin{aligned} &\ddot{z}_{P_1} = a_{P_1}, && \ddot{z}_{P_2} = a_{P_2}, \\ &\dot{a}_{P_1} = (u_1 - a_{P_1})/l_{P_1}, && \dot{a}_{P_2} = (u_2 - a_{P_2})/l_{P_2}, \\ &|u_1| \le \mu_1, && |u_2| \le \mu_2, \\ &a_{P_1}(t_0) = 0, && a_{P_2}(t_0) = 0. \end{aligned} \tag{1}$$

Here, z_{P_1} and z_{P_2} are the geometric coordinates of the pursuers, a_{P_1} and a_{P_2} are their accelerations generated by the controls u_1 and u_2. The time constants l_{P_1} and l_{P_2} define how fast the controls affect the systems.

The dynamics of the evader E is similar:

$$\begin{aligned} &\ddot{z}_E = a_E, \;\; \dot{a}_E = (v - a_E)/l_E, \\ &|v| \le \nu, \quad\;\; a_E(t_0) = 0. \end{aligned} \tag{2}$$

Let us fix some instants T_1 and T_2. At the instant T_1, the miss of the first pursuer with the respect to the evader is computed, and at the instant T_2, the miss of the

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second one is computed:

$$r_{P_1,E}(T_1) = |z_E(T_1) - z_{P_1,E}(T_1)|, \quad r_{P_2,E}(T_2) = |z_E(T_2) - z_{P_2,E}(T_2)|. \tag{3}$$

Assume that the pursuers act in coordination. This means that we can join them into one player P (which will be called the *first player*). This player governs the vector control $u = (u_1, u_2)$. The evader is counted as the *second player*. The result miss is the following value:

$$\varphi = \min\{r_{P_1,E}(T_1),\ r_{P_2,E}(T_2)\}. \tag{4}$$

At any instant t, all players know exact values of all state coordinates z_{P_1}, $\dot{z}_{P_1}$, a_{P_1}, z_{P_2}, $\dot{z}_{P_2}$, a_{P_2}, z_E, $\dot{z}_E$, a_E. The first player choosing its feedback control minimizes the miss φ, the second one maximizes it.

Relations (1)–(4) define a standard antagonistic differential game. One needs to construct the value function of this game and optimal strategies of the players.

2. Nowadays, there are a lot of publications dealing with differential games where one group of objects pursues another group; see, for example, the following works (in some order): (Stipanovic et al., 2009), (Blagodatskih and Petrov, 2009), (Chikrii, 1997), (Levchenkov and Pashkov, 1990), (Abramyantz and Maslov, 2004), (Pshenichnii, 1976), (Grigorenko, 1991), (Breakwell, 1976). The problem under consideration has two pursuers and one evader. So, from the point of view of number of objects, it is the simplest one. On the other hand, strict mathematical studies of problems "group-on-group" usually include quite strong assumptions for the dynamics of objects, dimension of the state vector and conditions of termination. Conversely, this paper considers the problem without any assumptions of this type. Solution of the problem can be interesting for the group differential games.

3. Now, let us describe a practical problem, whose reasonable simplification gives model game (1)–(4). Suppose that two pursuing objects attacks the evading one on collision courses. They can be rockets or aircrafts in the horizontal plane. A nominal motion of the first pursuer is chosen such that at the instant T_1 the exact capture occurs. In the same way, a nominal motion of the second pursuer is chosen (the capture is at the instant T_2). But indeed, the real positions of the objects differ from the nominal ones. Moreover, the evader using its control can change its trajectory in comparison with the nominal one (but not principally, without sharp turns). Correcting coordinated efforts of the pursuers are computed during the process by the feedback method to minimize the result miss, which is the minimum of absolute values of deviations at the instants T_1 and T_2 from the first and second pursuers, respectively, to the evader.

The passage from the original non-linear dynamics to a dynamics, which is linearized with the respect to the nominal motions, gives (Shima and Shinar, 2002), (Shinar and Shima, 2002) the problem under considerations.

4. The paper includes results of numerical study of game (1)–(4) for two marginal cases: 1) both pursuers P_1 and P_2 are dynamically stronger than the evader E; 2) both pursuers are dynamically weaker. Results for intermediate situations will be published in another work.

Difficulty of the solution is stipulated by the fact that the payoff function φ is not convex (even for the case $T_1 = T_2$). In the paper (Le Ménec, 2011), a case of "stronger" pursuers is considered and analytically methods are applied to the problem of solvability set construction in the game with zero result miss. For $T_1 = T_2$, an exact solution is obtained; if $T_1 \neq T_2$, then some upper approximation for the set is given. In general case, the exact analytical solution cannot be got, in the authors opinion.

The numerical study is based on algorithms and programs for solving linear differential games worked out in the Institute of Mathematics and Mechanics (Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russia). The central procedure is the backward constructing level sets (Lebesgue sets) of the value function. Optimal strategies of the players are constructed by some processing of the level sets.

2. Passage to Two-Dimensional Differential Game

At first, let us pass to relative geometric coordinates $y_1 = z_E - z_{P_1}$, $y_2 = z_E - z_{P_2}$ in dynamics (1), (2) and payoff function (4). After this, we have the following notations:

$$\begin{array}{ll} \ddot{y}_1 = a_E - a_{P_1} & \ddot{y}_2 = a_E - a_{P_2} \\ \dot{a}_{P_1} = (u_1 - a_{P_1})/l_{P_1} & \dot{a}_{P_2} = (u_2 - a_{P_2})/l_{P_2} \\ \dot{a}_E = (v - a_E)/l_{P_1} & |u_2| \leq \mu_2 \\ |u_1| \leq \mu_1,\ |v| \leq \nu & \varphi = \min\{|y_1(T_1)|,\ |y_2(T_2)|\}. \end{array} \tag{5}$$

State variables of system (5) are $y_1, \dot{y}_1, a_{P_1}, y_2, \dot{y}_2, a_{P_2}, a_E$; u_1 and u_2 are controls of the first player; v is the control of the second one. The payoff function φ depends on the coordinate y_1 at the instant T_1 and on the coordinate y_2 at the instant T_2. From general point of view (existence of the value function, positional type of the optimal strategies), differential game (5) is a particular case of a differential game with a positional functional (Krasovskii and Krasovskii).

A standard approach, which is set forth in (Krasovskii and Subbotin, 1974) and (Krasovskii and Subbotin, 1988) for study linear differential games with fixed terminal instant and payoff function depending on some state coordinates at the terminal instant is to pass to new state coordinates. They can be treated as values of the target coordinates forecasted to the terminal instant under zero controls. In our situation, we have two instants T_1 and T_2, but coordinates computed at these instants are independent; namely, at the instant T_1, we should take into account $y_1(T_1)$ only, and at the instant T_2, we use the value $y_2(T_2)$. This fact allows us to use the mentioned approach when solving differential game (5). With that, we pass to new state coordinates x_1 and x_2, where $x_1(t)$ is the value of y_1 forecasted to the instant T_1, and $x_2(t)$ is the value of y_2 forecasted to the instant T_2.

The forecasted values are computed by formula

$$x_i = y_i + \dot{y}_i\tau_i + a_{P_i} l_{P_i}^2 h(\tau_i/l_{P_i}) + a_E l_E^2 h(\tau_i/l_E), \quad i = 1, 2. \tag{6}$$

Here, x_i, y_i, and $\dot{y}_i$ depends on t; $\tau_i = T_i - t$; $h(\alpha) = e^{-\alpha} + \alpha + 1$. Emphasize that the values τ_1 and τ_2 are connected to each other by the relation $\tau_1 - \tau_2 = \text{const} = T_1 - T_2$. One has $x_i(T_i) = y_i(T_i)$.

The dynamics in the new coordinates x_1, x_2 is the following (Le Ménec, 2011):

$$\begin{aligned}
&\dot{x}_1 = -l_{P_1} h(\tau_1/l_{P_1})u_1 + l_E h(\tau_1/l_E)v,\\
&\dot{x}_2 = -l_{P_2} h(\tau_2/l_{P_2})u_2 + l_E h(\tau_2/l_E)v,\\
&|u_1| \le \mu_1,\ |u_2| \le \mu_2,\ |v| \le \nu,\\
&\varphi\big(x_1(T_1), x_2(T_2)\big) = \min\{|x_1(T_1)|,\ |x_2(T_2)|\}.
\end{aligned} \tag{7}$$

The first player governs the controls u_1, u_2 and minimizes the payoff φ; the second one has the control v and maximizes φ.

Note that the control u_1 (u_2) affects only the horizontal (vertical) component $\dot{x}_1$ ($\dot{x}_2$) of the velocity vector $\dot{x} = (\dot{x}_1, \dot{x}_2)$. When $T_1 = T_2$, the second summand in dynamics (7) is the same for $\dot{x}_1$ and $\dot{x}_2$. Thus, the component of the velocity vector $\dot{x}$ depending on the second player control is directed at any instant t along the bisectrix of the first and third quadrants of the plane x_1, x_2. When $v = +\nu$, the angle between the axis x_1 and the velocity vector of the second player is 45°; when $v = -\nu$, the angle is 225°. This property simplifies the dynamics in comparison with the case $T_1 \neq T_2$.

Let $x = (x_1, x_2)$ and $V(t, x)$ be the value function at the position (t, x). For any $c \ge 0$, the value function level set

$$W_c = \big\{(t, x) : V(t, x) \le c\big\}$$

coincides with the maximal stable bridge (see (Krasovskii and Subbotin, 1974) and (Krasovskii and Subbotin, 1988)) built from the terminal set

$$M_c = \big\{(t, x) : t = T_1,\ |x_1| \le c;\ t = T_2,\ |x_2| \le c\big\}.$$

The set W_c can be treated as the solvability set for the considered game with the result not greater than c. When $c = 0$, one has the situation of the exact capture. The exact capture means equality to zero, at least, one of $x_1(T_1)$ and $x_2(T_2)$.

Comparing dynamics capabilities of each of pursuers P_1 and P_2 and the evader E, one can introduce parameters (Le Ménec, 2011) $\eta_i = \mu_i/\nu_i$ and $\varepsilon_i = l_E/l_{P_i}$, $i = 1, 2$. They define the shape of the maximal stable bridges in the individual games P_1 against E and P_2 against E.

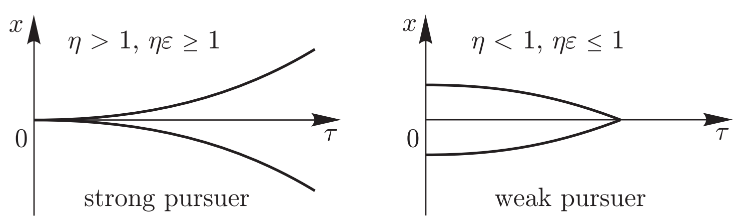

Fig. 1. Different variants of the stable bridges evolution in an individual game

Consider two cases: 1) $\eta_i > 1$, $\eta_i\varepsilon_i \ge 1$, $i = 1, 2$; 2) $\eta_i < 1$, $\eta_i\varepsilon_i \le 1$, $i = 1, 2$. In the first case, each of pursuers P_1 and P_2 is stronger than the evader E; in the second one, both pursuers are weaker. The maximal stable bridges in the individual games in the first case look as it is shown in Fig. 1 (at the left); the right subfigure in Fig. 1 gives the outline for the second case. The horizontal axis is the backward time τ, the vertical axis is the one-dimensional state variable x.

3. Level Sets of the Value Function

As it was mentioned above, a level set W_c of the value function is the maximal stable bridge for dynamics (7) built in the space t, x from the target set M_c. A time section (t-section) $W_c(t)$ of the bridge W_c at the instant t is a set in the plane of two-dimensional variable x.

To be definite, let $T_1 \geq T_2$. Then, for any $t \in (T_2, T_1]$, the set $W_c(t)$ is a vertical stripe around the axis x_2. Its width along the axis x_1 equals the width of the bridge in the individual game P_1–E at the instant $\tau = T_1 - t$ of the backward time. At the instant $t = T_1$, half-width of $W_c(T_1)$ is equal to c.

Denote by $W_c(T_2 + 0)$ the right limit of the set $W_c(t)$ as $t \to T_2 + 0$. Then, the set $W_c(T_2)$ is cross-like, obtained by union of the vertical stripe $W_c(T_2 + 0)$ and a horizontal one around the axis x_1 with the width equal $2c$ along the axis x_2.

When $t \leq T_2$, the backward construction of the sets $W_c(t)$ is made starting from the set $W_c(T_2)$.

The algorithm, which is suggested by the authors for constructing the approximating sets $\widetilde{W}_c(t)$, uses a time grid in the interval $[0, T_1]$: $t_N = T_1$, t_{N-1}, $\ldots$, $t_S = T_1$, t_{S-1}, t_{S-2}, $\ldots$. For any instant t_k from the taken grid, the set $\widetilde{W}_c(t_k)$ is built on the basis of the previous set $\widetilde{W}_c(t_{k+1})$ and a dynamics obtained from (7) by fixing its value at the instant t_{k+1}. So, dynamics (7), which varies in the interval $(t_i, t_{i+1}]$, is changed by a dynamics with simple motions (Isaacs, 1965). The set $\widetilde{W}_c(t_k)$ is treated as a collection of all positions at the instant t_k, where from the first player guarantees guiding the system to the set $\widetilde{W}_c(t_{k+1})$ under "frozen" dynamics (7) and discrimination of the second player, that is, when the second player announces its constant control v, $|v| \leq \nu$, in the interval $[t_i, t_{i+1}]$.

Due to symmetry of dynamics (7) and the sets $W_c(T_1)$, $W_c(T_2)$ with the respect to the origin, one gets that for any $t \leq T_1$ the t-section $W_c(t)$ is symmetric also.

3.1. Maximal Stable Bridges for the Case of Strong Pursuers

Simultaneous dynamic advantage of P_1 and P_2 with the respect to E implies that for any c, $W_c(\overline{t}) \subset W_c(\underline{t})$ if $\underline{t} < \overline{t}$. This means that the bridge W_c expands in the backward time. The latter allows to make independent constructions in all four quadrants. And due to the central symmetry, it is sufficient to make the constructions in the I and II quadrants only.

Let us give results of constructing t-sections $W_c(t)$ for the following values of game parameters:

$$\begin{aligned} &\mu_1 = 2, \quad \mu_2 = 3, \qquad \nu = 1, \\ &l_{P_1} = 1/2, \; l_{P_2} = 1/0.857, \; l_E = 1. \end{aligned}$$

Equal terminal instants. Let $T_1 = T_2 = 6$. Fig. 2 shows results of constructing the set W_0 (that is, with $c = 0$). In the figure, one can see several time sections $W_0(t)$ of this set. The bridge has a quite simple structure. At the initial instant $\tau = 0$ of the backward time (when $t = 6$), its section coincides with the target set M_0, which is the union of two coordinate axes. Further, at the instants $t = 4$, 2, 0, the cross thickens, and two triangles are added to it. The widths of the vertical and horizontal parts of the cross correspond to sizes of the maximal stable bridges in the individual games with the first and second pursuers. These triangles are located in the II and IV quadrants (where the signs of x_1 and x_2 are different, in other words, when the

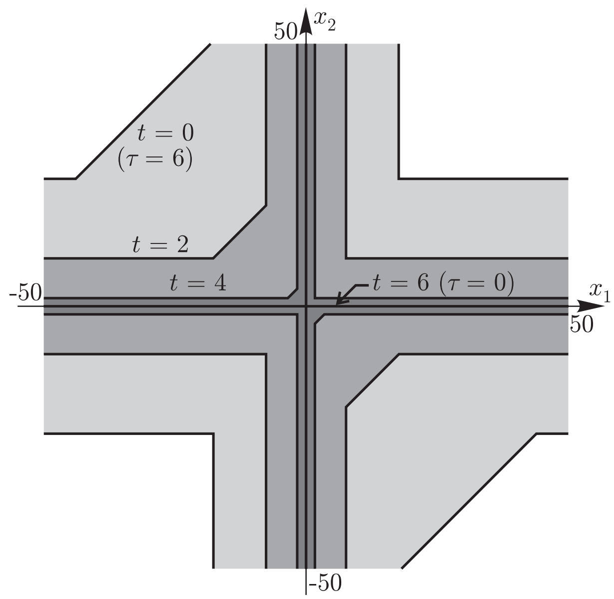

Fig. 2. Two strong pursuers, equal terminal instants: time sections of the bridge W_0

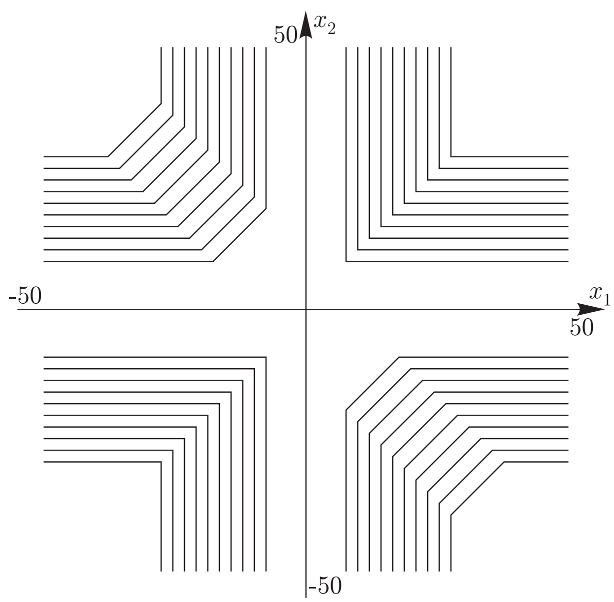

Fig. 3. Two strong pursuers, equal terminal instants: level sets of the value function, $t = 2$

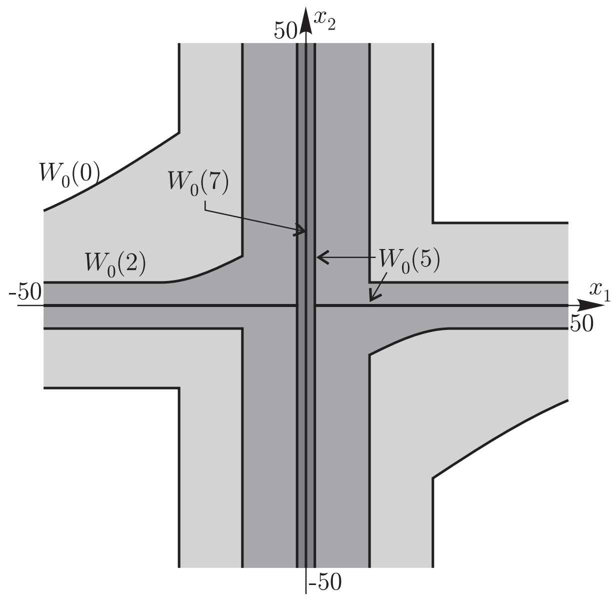

Fig. 4. Two strong pursuers, different terminal instants: time sections of the bridge W_0

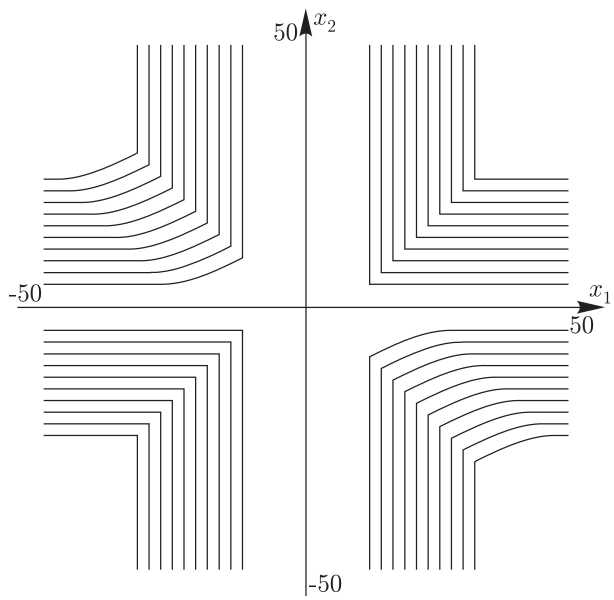

Fig. 5. Two strong pursuers, different terminal instants: level sets of the value function, $t = 2$

evader is between the pursuers) give the zone where the capture is possible only under collective actions of both pursuers.

Time sections $W_c(t)$ of other bridges W_c, $c > 0$, have a shape similar to $W_0(t)$. In Fig. 3, one can see the sections $W_c(t)$ at $t = 2$ ($\tau = 4$) for a collection $\{W_c\}$ corresponding to some series of values of the parameter c. For other instants t, the structure of the sections $W_c(t)$ is similar.

Different terminal instants. Let $T_1 = 7$, $T_2 = 5$. Results of construction of the set W_0 are given in Fig. 4. When $t < 5$, time sections $W_0(t)$ grow both horizontally and vertically; two additional triangles appear, but now they are curvilinear.

Total structure of the sections $W_c(t)$ at $t = 2$ is shown in Fig. 5.

3.2. Maximal Stable Bridges for the Case of Weak Pursuers

Now, we consider a variant of the game when both pursuers are weaker than the evader. Let us take the parameters

$$\mu_1 = 0.9, \quad \mu_2 = 0.8, \quad \nu = 1, \quad l_{P_1} = l_{P_2} = 1/0.7, \quad l_E = 1.$$

Let us show results for the case of different terminal instants only: $T_1 = 7$, $T_2 = 5$.

Since in this variant the evader is more maneuverable than the pursuers, they cannot guarantee the exact capture.

Fix some level of the miss, namely, $\bigl|x_1(T_1)\bigr| \le 2.0$, $\bigl|x_2(T_2)\bigr| \le 2.0$. Time sections $W_{2.0}(t)$ of the corresponding maximal stable bridge are shown in Fig. 6. The upper-left subfigure corresponds to the instant when the first player stops to pursue. The upper-right subfigure shows the picture for the instant, when the second pursuer finishes its pursuit. At this instant, the horizontal strip is added, which is a bit wider than the vertical one contracted during the passed period of the backward time. Then, the bridges contracts both in horizontal and vertical directions, and two additional curvilinear triangles appear (see middle-left subfigure). The middle-right subfigure gives the view of the section when the vertical strip collapses, and the lower-left subfigure shows the configuration just after the collapse of the horizontal strip. At this instant, the section loses connectivity and disjoins into two parts symmetrical with respect to the origin. Further, these parts continue to contract (as it can be seen in the lower-right subfigure) and finally disappear.

Time sections $\bigl\{W_c(t)\bigr\}$ are given in Fig. 7 at the instant $t = 0$ ($\tau_1 = 7$, $\tau_2 = 5$).

4. Optimal Feedback Control

Using knowledge of the value function provided by its level sets W_c, we can construct optimal strategies of the first and second players. Let us do it dividing the plane x_1, x_2 for every instant t to some cells. Inside each cell, the optimal control takes some extremal values.

Rewrite system (7) as

$$\begin{gathered} \dot{x} = \mathscr{D}_1(t)u_1 + \mathscr{D}_2(t)u_2 + \mathscr{E}(t)v, \\ |u_1| \le \mu_1, \ |u_2| \le \mu_2, \ |v| \le \nu. \end{gathered}$$

Here, $x = (x_1, x_2)$; vectors $\mathscr{D}_1(t)$, $\mathscr{D}_2(t)$, and $\mathscr{E}(t)$ look like

$$\begin{aligned} &\mathscr{D}_1(t) = \bigl(-l_{P_1} h((T_1 - t)/l_{P_1}),\, 0\bigr), \qquad \mathscr{D}_2(t) = \bigl(0,\, -l_{P_2} h((T_2 - t)/l_{P_2})\bigr), \\ &\mathscr{E}(t) = \bigl(l_E h((T_1 - t)/l_E),\, l_E h((T_2 - t)/l_E)\bigr). \end{aligned}$$

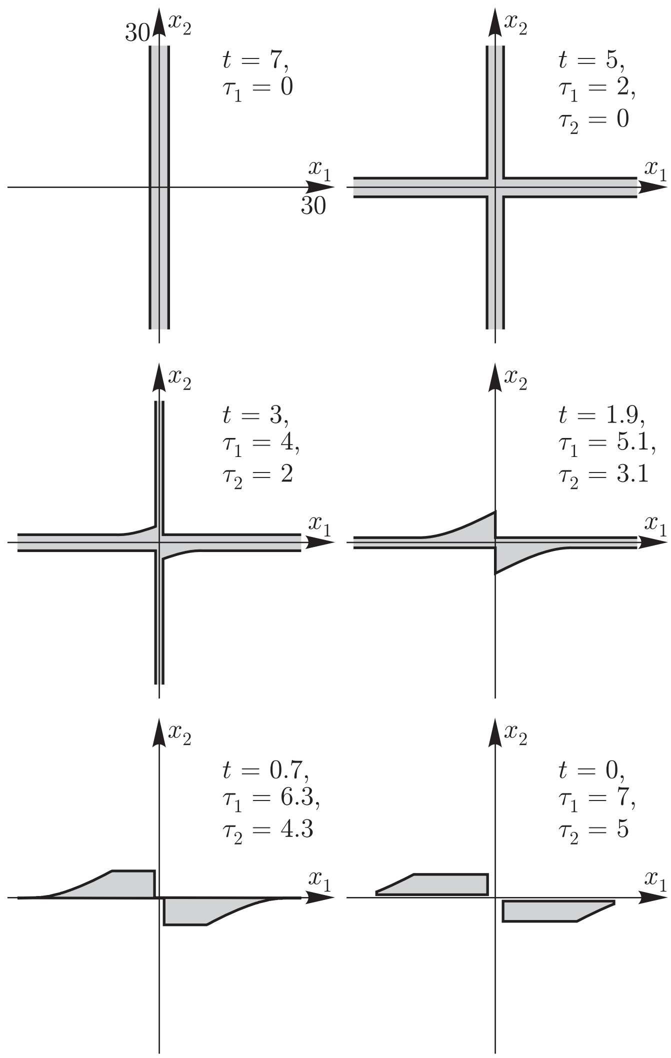

Fig. 6. Two weak pursuers, different termination instants: time sections of the maximal stable bridge $W_{2,0}$

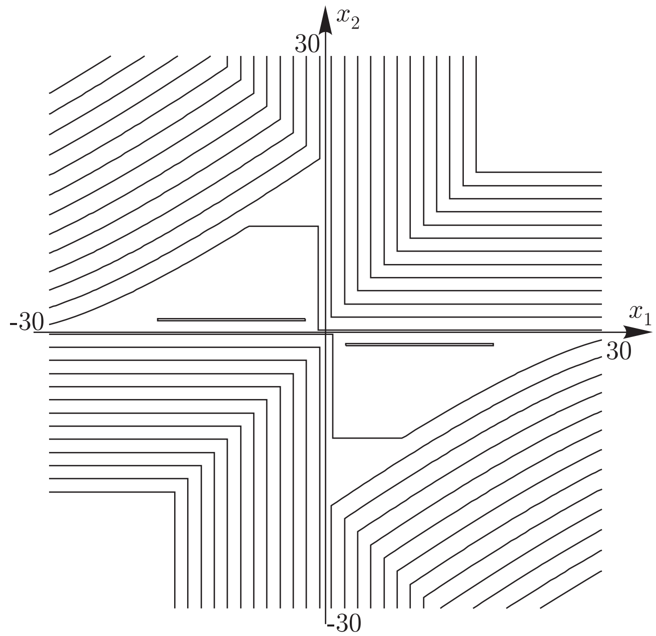

Fig. 7. Two weak pursuers, different terminal instants: level sets of the value function, $t = 0$

We see that the vector $\mathscr{D}_1(t)$ ($\mathscr{D}_2(t)$) is directed along the horizontal (vertical) axis; when $T_1 = T_2$, the angle between the axis x_1 and the vector $\mathscr{E}(t)$ equals 45°; when $T_1 \neq T_2$, the angle changes in time.

4.1. Switching Lines in the Case of Strong Pursuers

Feedback control of the first player. Analyzing the change of the value function along a horizontal line in the plane x_1, x_2 for a fixed instant t, one can conclude that the minimum of the function is reached in the segment of intersection of this line and the set $W_0(t)$. With that, the function is monotonic at both sides of the segment. For points at the right (at the left) from the segment, the control $u_1 = \mu_1$ ($u_1 = -\mu_1$) directs the vector $\mathscr{D}_1(t)u_1$ to the minimum.

Splitting the plane into horizontal lines and extracting for each line the segment of minimum of the value function, one can gather these segments into a set in the plane and draw a switching line through this set, which separates the plane into two parts at the instant t. At the right from this switching line, we choose the control $u_1 = \mu_1$, and at the left the control is $u_1 = -\mu_1$. On the switching line, the control u_1 can be arbitrary obeying the constraint $|u_1| \leq \mu_1$. The easiest way is to take the vertical axis x_2 as the switching line.

In the same way, using the vector $\mathscr{D}_2(t)$, we can conclude that the horizontal axis x_1 can be taken as the switching line for the control u_2.

Thus,

$$u_i^*(t, x) = \begin{cases} \mu, & \text{if } x_i > 0, \\ -\mu, & \text{if } x_i < 0, \\ \text{any } u_i \in [-\mu, \mu] & \text{if } x_i = 0. \end{cases} \tag{8}$$

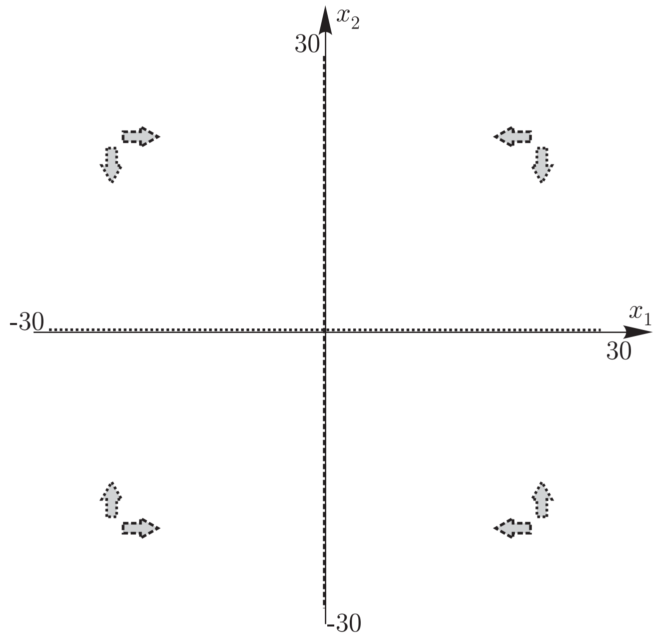

Fig. 8. Two strong pursuers, equal terminal instants: switching lines for the first player

The switching lines (the coordinate axes) at any t divide the plane x_1, x_2 into 4 cells. In each of these cells, the optimal control of the first player is constant. The synthesis of the first player optimal control is the same for all time instants and is shown in Fig. 8. Arrows denote the direction of the vectors $\mathscr{D}_i(t)u_i^*$, $i = 1, 2$.

Feedback control of the second player. For a fixed instant t, consider a split of the plane x_1, x_2 into lines parallel to the vector $\mathscr{E}(t)$. Take segments of local minimum and local maximum of the value function on all lines. One can easily see that for any line (except lines passing near the origin), there are two segments of local minimum and one of local maximum located between them. The segments of minimum appear by intersection of the line with the set $W_0(t)$. The segment of maximum for the case $T_1 = T_2$ coincides with the rectilinear part of the boundary of some set $W_c(t)$ and has slope angle equal to 45°. If $T_1 \neq T_2$, then the segment of maximum degenerates to a point coinciding with the corner point of a curvilinear triangle. For any point in the line outside all the segments, the control v is chosen in such a way that the vector $\mathscr{E}(t)v$ is oriented to the direction of growth of the value function. So, there are two parts of the line, where $v = \nu$, and two parts, where $v = -\nu$.

For a fixed instant t, the switching lines for the second player comprise of the coordinate axes and some line $\Pi(t)$, which passes through the middles of the segments of local minimum, if $T_1 = T_2$, and through the corner points of curvilinear triangles, if $T_1 \neq T_2$. An unpleasant peculiarity is that if $T_1 \neq T_2$, then one should take $v = \pm\nu$ in the switching line $\Pi(t)$; choices $|v| < \nu$ are not optimal.

Inside each of 6 cells, to which the plane is separated by the switching lines of the second player, the control is taken either $v = \nu$ or $v = -\nu$.

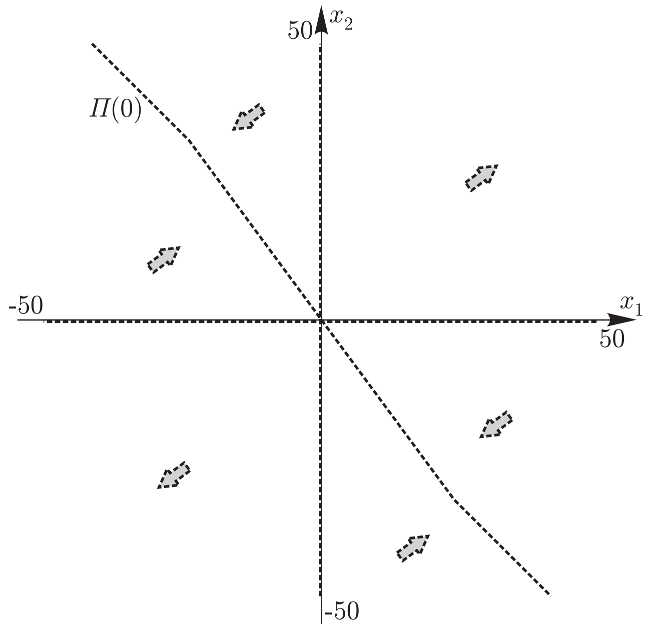

Fig. 9. Two strong pursuers, equal terminal instants: switching lines for the second player, $t = 0$

The second player optimal synthesis for the case $T_1 = 7$, $T_2 = 5$ is shown in Fig. 9 for $t = 0$. Arrows denote direction of the vectors $\mathscr{E}(t)v^*$.

4.2. Switching Lines in the Case of Weak Pursuers

In the case of pursuers weaker than the evader, the structure of the sets W_c is more complex in some neighborhood of the origin. This leads to more complicated shape of the switching lines both for the first and second players.

Switching lines of the first player are given in Fig. 10 at the instant $t = 0$ ($\tau_1 = 7$, $\tau_2 = 5$). The dashed line is the switching line for the component u_1; the dotted one is for the component u_2. The switching lines are obtained as a result of the analysis of the function $x \to V(t, x)$ in horizontal (for u_1 in accordance with the direction of the vector $\mathscr{D}_1(t)$) and vertical (for u_2 in accordance with the direction of the vector $\mathscr{D}_2(t)$) lines. If in the considered horizontal (vertical) line the minimum of the value function is attained in a segment, then the middle of such a segment is taken as a point for the switching line. Arrows show the directions of the vectors $\mathscr{D}_1(t)u_1^*$ and $\mathscr{D}_2(t)u_2^*$ in 4 cells.

In Fig. 11 switching lines and the directions of the vectors $\mathscr{E}(t)v^*$ are shown for $t = 0$. In this picture, we have 4 cells with constant values of the second player control.

4.3. Generating Feedback Controls. Discrete Scheme of Control

Switching lines are built as a result of processing the boundary of the sets $W_c(t)$. With that, some grid of instants t_k, where the t-sections $W_{c_j}(t_k)$ of the maximal stable bridges W_{c_j} are constructed by the backward procedure. The values c_j are also taken in some grid. For any instant t_k, approximating switching lines are stored as polygonal lines in the memory of a computer.

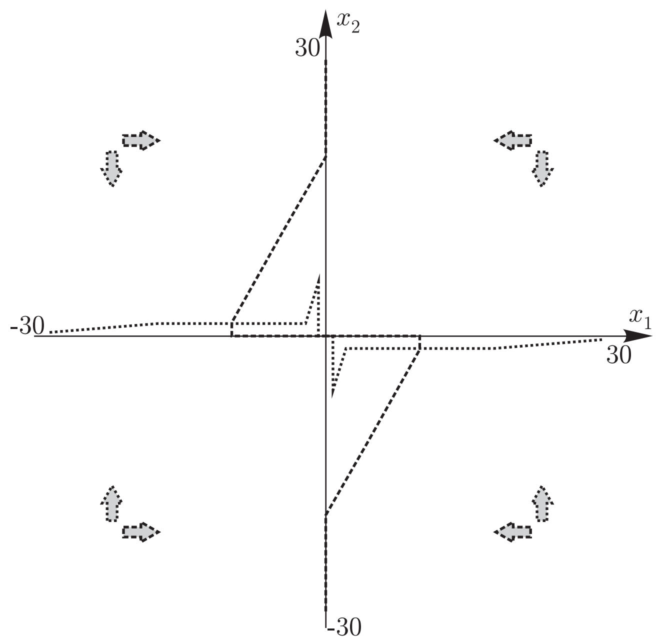

Fig. 10. Two weak pursuers, equal terminal instants: switching lines for the first player, $t = 0$

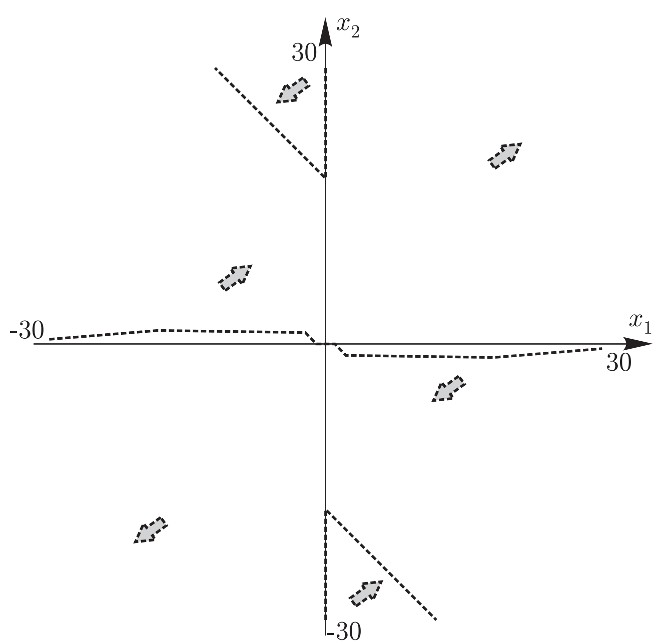

Fig. 11. Two weak pursuers, equal terminal instants: switching lines for the second player, $t = 0$

Having a position $x(t_k)$ at the instant t_k, it is possible to compute the controls $u_1^*\big(t_k, x(t_k)\big)$ and $u_2^*\big(t_k, x(t_k)\big)$ analyzing location of the point $x(t_k)$ with the respect to the switching lines for u_1 and u_2. The vectors $\mathscr{D}_1(t_k)$ and $\mathscr{D}_2(t_k)$ are used for this. In the case of strong pursuers, the axis x_2 is the switching line for the control u_1, and the axis x_1 is the switching line for the control u_2. The values of u_1^* and u_2^* are defined by formula (8). In the case of weak pursuers, the switching line is unique for each component u_i of the control too. Drawing a ray from the point $x(t_k)$ with the directing vector $\mathscr{D}_i(t_k)$, one can decide whether it crosses a switching line corresponding to the index i. If it does not, then $u_i^*\big(t_k, x(t_k)\big) = -\mu_i$; if it crosses, then $u_i^*\big(t_k, x(t_k)\big) = \mu_i$.

The first player control chosen at the instant t_k is kept until the instant t_{k+1}. At the position $\big(t_{k+1}, x(t_{k+1})\big)$, a new control value is chosen, etc. So, the feedback control generated by the switching lines is applied in a discrete control scheme (Krasovskii and Subbotin, 1974, Krasovskii and Subbotin, 1988).

To construct $v^*\big(t_k, x(t_k)\big)$ we use the vector $\mathscr{E}(t_k)$. Compute how many times (even or odd) a ray with the beginning at the point $x(t_k)$ and the directing vector $\mathscr{E}(t_k)$ crosses the second player switching lines. If the number of crosses is even (absence of crosses means that the number equals zero and is even), then we take $v^*\big(t_k, x(t_k)\big) = +\nu$; otherwise, $v^*\big(t_k, x(t_k)\big) = -\nu$. The chosen control is kept until the next instant t_{k+1}. In the position $\big(t_{k+1}, x(t_{k+1})\big)$, a new control is built, etc.

This synthesis for the first (second) player is suboptimal. Analysis of its closeness to an optimal one needs an additional study. Namely, it is necessary to show that under a coordinated choice of diameters Δt and Δc of grids in t and c, the feedback control of the first (second) player built on the basis of switching lines guarantees the limit of result as $\Delta t \to 0$ and $\Delta c \to 0$, which is not greater (not less) than $V(t_0, x_0)$ for any initial position (t_0, x_0). Such a study for linear differential games with convex t-sections $W_c(t)$ of maximal stable bridges is made in the works (Botkin and Patsko, 1982, Zarkh, 1990, Patsko, 2006). In the problem under consideration the sections $W_c(t)$ are not convex, and this fact preconditions the difficulty of this problem.

5. Simulation Results

Let the pursuers P_1, P_2, and the evader E move in the plane. At the initial instant $t_0 = 0$, velocities of all objects are parallel (Fig. 12) and sufficiently greater than the possible changes of the lateral velocity components. The instant of longitudinal coincidence of objects P_1 and E is T_1; the instant of coincidence of the objects P_2 and E is T_2. The dynamics of lateral motion is described by relations (1), (2); the resulting miss is given by formula (4).

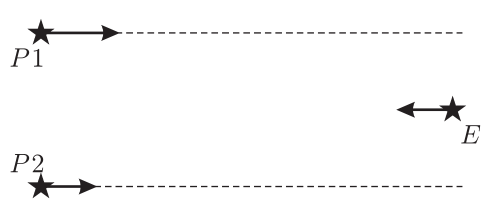

Fig. 12. Schematic initial positions of the pursuers and evader

In all following results, the initial lateral velocities and accelerations are assumed to be zero:

$$\dot{z}^0_{P_1} = z^0_{P_2} = \dot{z}^0_E = 0; \quad a^0_{P_1} = a^0_{P_2} = a^0_E = 0.$$

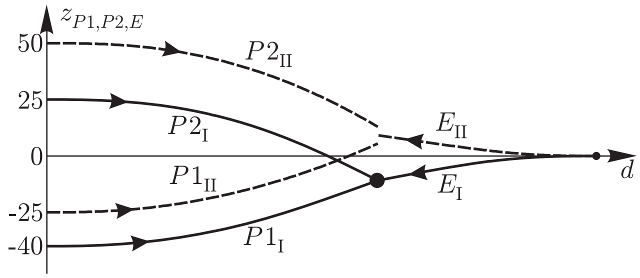

Fig. 13. Two strong pursuers, equal termination instants: trajectories in the original space

In Fig. 13, one can see the trajectories of the objects in the original space for the case of strong pursuers and equal terminal instants for the following game parameters:

$$\mu_1 = 2,\ \mu_2 = 3,\ \nu = 1,\ l_{P_1} = 1/2,\ l_{P_2} = 1/0.857,\ l_E = 1,\ T_1 = T_2 = 6.$$

The pursuers P_1, P_2, and the evader E act optimally. The trajectories drawn by solid lines correspond to the following initial data:

$$z^0_{P_1} = -40,\ z^0_{P_2} = 25,\ z^0_E = 0.$$

The dashed lines denote the trajectories for the following initial lateral parameters:

$$z^0_{P_1} = -25,\ z^0_{P_2} = 50,\ z^0_E = 0.$$

In the first case, the evader is successfully captured (at the terminal instant, the positions of both pursuers are the same as the position of the evader). In the second variant of initial positions, the evader escapes: at the terminal instant no one of the pursuers superposes with the evader. In this case, one can see as the evader aims itself to the middle between the terminal positions of the pursuers (this guarantees to him the maximum of the payoff function φ).

Figs. 14, 15, and 16 correspond to the case of weak pursuers and different terminal instants:

$$\mu_1 = 0.9,\ \mu_2 = 0.8,\ \nu = 1,\ l_{P_1} = l_{P_2} = 1/0.7,\ l_E = 1,\ T_1 = 7,\ T_2 = 5.$$

The initial positions are taken as follows:

$$z^0_{P_1} = -12,\ z^0_{P_2} = 12,\ z^0_E = 0.$$

Trajectories in Fig. 14 are built for the optimal controls of all objects. At the beginning of the pursuit, the evader closes to the first (lower) pursuer. It is done to increase the miss from the second (upper) pursuer at the instant T_2. Further closing is not reasonable, and the evader switches its control to increase the miss from the first pursuer at the instant T_1.

Fig. 15 gives the trajectories, when the pursuers use their optimal feedback controls generated by switching lines, but the evader applies a constant control $v \equiv \nu$ escaping from P_1 and ignoring P_2. In Fig. 16, the situation is given, when the evader, vice versa, keeps control $v \equiv -\nu$ escaping from P_2 and ignoring P_1. In both these situations, the payoff is less than in the case when the second player uses optimal control. When a constant control $v = +\nu$ is applied, the miss to the second pursuer at the instant T_2 is less; when the second player keeps $v = -\nu$, the miss to the first pursuer at the instant T_1 decreases.

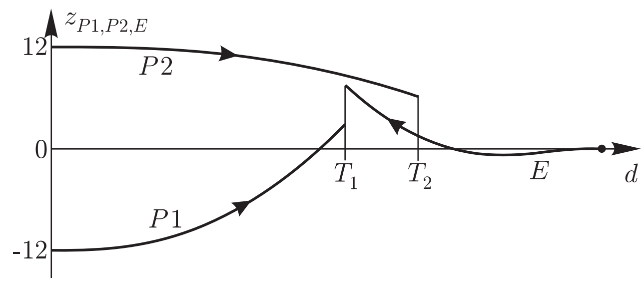

Fig. 14. Two weak pursuers, different termination instants: trajectories of the objects in the original space, optimal control of the second player

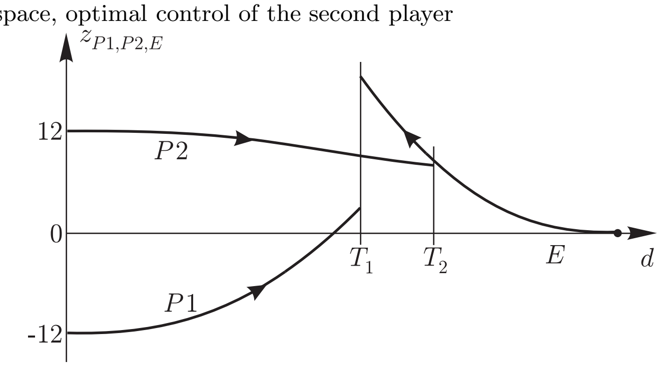

Fig. 15. Two weak pursuers, different termination instants: trajectories of the objects in the original space, constant control of the second player $v = +\nu$

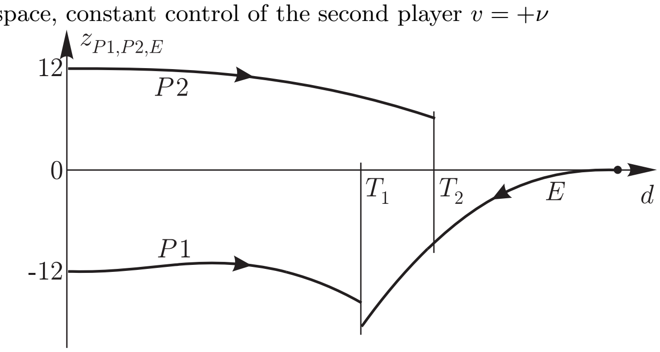

Fig. 16. Two weak pursuers, different termination instants: trajectories of the objects in the original space, constant control of the second player $v = -\nu$

6. Conclusion

A problem of pursuit-evasion with two pursuing objects and one evading object is considered as a two-dimensional antagonistic differential game. Difficulty of numerical solution of this problem is conditioned by the fact that the t-sections of the value function level sets are not convex. For two qualitatively different types of parameters ("strong" pursuers, "weak" pursuers), an analysis of the value function level sets is worked out in the paper. On the basis of this analysis, optimal strategies of players are built.

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The Dynamic Procedure of Information Flow Network*

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Abstract. The characterization of the architecture of information flow equilibrium networks and the dynamics of network formation are studied under the premise of local information flow. The main result of this article is that it gives the dynamic formation random procedure in the local information flow network. The research shows that core-periphery structure is the most representative equilibrium network in the case of the local information flow without information decay whatever the cost of information is homogeneous or heterogeneous and it obtains the theory of equilibrium network architecture on two kinds of typical core-periphery architecture networks. If profits and link costs are homogeneous under the premise of local information flow and with the presence of decay beside empty network complete network is typical equilibrium network when the cost of linking is small enough and they are tried to prove theoretically.

Keywords: local information flow, information decay, random dynamic procedure, core-periphery architecture, equilibrium network.

1. Introduction

The main actors of information flow network are players. The difference between information flow network and regular network is that the equilibrium network of information flow network not only depends on its topological structure and players have the need and competence of acquiring information personally and getting information from others in the given network structure. In 2010 Goyal S suggested that the dynamic formation procedure of information flow network is one of the most worthy of concern question in his latest article (Galeotti and Goyal, 2010).

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([original text] In actual practice individuals decide on information acquisition and links with others over time, and it is important to understand these dynamics). Similar to "Two-way" flow network model in common sense (Bala and Goyal, 2000), an important assumption is that "unilateral formation connection and bilateral information exchange", that is to say, a link is formed once some player pays for it and it allows both players to access the information personally acquired by the other player.

2. The local information flow network without decay

Let $N = \{1, 2, ..., n\}$ with $n \geq 3$ be the set of players and let i and j be typical members of this set. Each player chooses a level of personal information acquisition $x_i \in X = [0, +\infty]$ and a set of links with others to access their information, which is represented as a (row) vector:

$$g_i = (g_{i1}, ..., g_{ii-1}, g_{ii}, g_{ii+1}, ..., g_{in})$$

where $g_{ii} = 0, \forall i \in N$ and $g_{ij} \in \{0, 1\}$, for each $j \in N \backslash \{i\}$. We say that player i has a link with player j if $g_{ij} = 1$, and the cost of linking with one other person is $k > 0$. Otherwise we have $g_{ij} = 0$. Our paper assumes that the cost of linking with one other person is homogeneous, and the link between player i and j allow both players share information. The set of strategies of player i is denoted by $S_i = X \times G_i$. Define $S = S_1 \times ... \times S_n$ as the set of strategies of all players. A strategy profile $s = (x, g) \in S$ specifies the personal information acquired by each player $x = (x_1, x_2, ..., x_n)$, and the network of relations (connection Matrix) $g = (g_1, g_2, ..., g_n)^T$, where the T specifies a transposition.

The network g is a directed graph, where the arrow from i to j specifies $g_{ij} = 1$. Let G be the set of all possible directed graphs on n vertices. Define $N_i(g) = \{j \in N : g_{ij} = 1\}$ as the set of players with whom i has formed a link. Let $\eta_i(g) = |N_i(g)|$. The closure of g is an undirected network denoted by $\bar{g} = cl(g)$, where $\bar{g}_{ij} = max\{g_{ij}, g_{ji}\}$. In words, the closure of a directed network involves replacing every directed edge of by an undirected one. Define $N_i(\bar{g}) = \{j \in N : \bar{g}_{ij} = 1\}$ as the set of players directly connected to i. The undirected link between two players reflects bilateral information exchange between them.

The payoffs to player i under strategy profile $s = (x, g)$ are:

$$\Pi_i(s) = f(x_i + \sum_{j \in N_i(\bar{g})} x_j) - c_i x_i - \eta_i(g)k \tag{1}$$

Where $c_i > 0$ the cost of information. We assume the cost of information that personally acquired is heterogeneous. We will assume that $f(y)$ is twice continuously differentiable, increasing, and strictly concave in y. To focus on interesting cases we will assume that

$$f(0) = 0, f'(0) > c^0 = \max_{1 \leq i \leq n} \{c_i\}$$

and

$$\lim_{y \to +\infty} f'(y) = m < c_0 = \min_{1 \leq i \leq n} \{c_i\}$$

Under these assumptions there exists a number $\hat{y}_i > 0$ such that

$$\hat{y}_i = \arg\max_{y \in X} [f(y) - c_i y]$$

i.e $\hat{y}_i$ solves $f^{'}(\hat{y}_i) = c_i$.

A Nash equilibrium is a strategy profile $s^* = (x^*, g^*)$ such that

$$\Pi_i(s_i^*, s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*), \forall s_i \in S_i, \forall i \in N \tag{2}$$

2.1. Equilibrium networks in static model

An example of two-core-periphery architecture.

Example 1. Let $N = \{1, 2, 3, 4, 5, 6\}$ be the set of players, the cost of linking with one other person is homogeneous $k = \frac{1}{5}$, the cost of information that personally acquired is heterogeneous and denoted by $c_1 = c_2 = c_3 = c_4 = \frac{1}{2}, c_5 = c_6 = \frac{10}{21}$. Suppose payoffs are given by (1), where $f(y) = ln(1 + y)$.

We can check that $s^* = (x^*, g^*)$ is a Nash equilibrium, where $x^* = (x_1^*, ..., x_6^*) = (\frac{12}{70}, \frac{12}{70}, \frac{12}{70}, \frac{12}{70}, \frac{29}{70}, \frac{29}{70})$, the corresponding connection matrix and the network structure are shown below:

$$g^* = \begin{pmatrix} g_1^* \\ g_2^* \\ g_3^* \\ g_4^* \\ g_5^* \\ g_6^* \end{pmatrix} = \begin{pmatrix} 0\,0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0 \end{pmatrix}$$

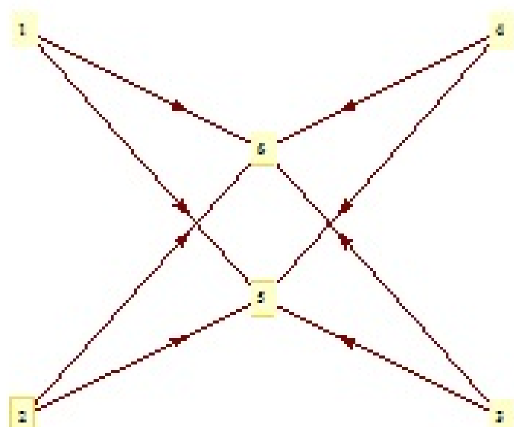

Fig. 1. Connection matrix and corresponding network

Notice that equilibrium network in this example has typical "two-core-periphery" architecture. Players 5,6 become two hubs because players 5,6 have slightly lower costs of acquiring information, the information that they acquire personally is $\frac{29}{70}$. Players 1,2,3,4 become spokes. Because the cost of linking with one other person is low, so the information that they acquire personally is $\frac{12}{70}(x_1^* = x_2^* = x_3^* = x_4^* = \frac{12}{70})$ and aggregate information is 1 through linking with two hubs, in fact $\hat{y}_1 = \hat{y}_2 = \hat{y}_3 = \hat{y}_4 = 1$ at this moment, the aggregate information which players 5,6 own is

$\frac{11}{10} = \hat{y}_5 = \hat{y}_6$. Under the equilibrium network, the payoff of spokes is

$$\Pi_i(s^*) = ln(1+\hat{y}_i) - c_i x_i^* - 2k = ln(1+1) - \frac{1}{2} \times \frac{6}{35} - 2 \times \frac{1}{5} = 0.2074, i = 1,2,3,4$$

However, when spokes acquire the optimal aggregate information 1 personally and don't link with hubs, the payoff is

$$\Pi_i(s) = ln(1+\hat{y}_i) - c_i = ln(1+1) - \frac{1}{2} = 0.193, i = 1,2,3,4$$

Similarly, we can verify $\Pi_i(s) < \Pi_i(s^*), \forall s \in S_i, \forall i \in N$. So the spokes form links with two hubs while don't form links between them and two hubs don't form links between them is the equilibrium strategy for the players.

In this example, there is no link between two hubs because of $k > c_5\hat{y}_6 = c_6\hat{y}_5 (0.2 > \frac{10}{21} \times \frac{29}{70} = 0.1973)$. Spokes choose to form links with hubs because of $k < c_i\hat{y}_5 = c_i\hat{y}_6 (i = 1,2,3,4), i.e. 0.2 < \frac{1}{2} \times \frac{29}{70} = 0.2071$.

Network characteristics with "multi-core-periphery" architecture. As we see in Example 1, the cost heterogeneity of information acquisition is based on two levels and slight differences. The primary cause lies in the high complexity of the algorithm of general equilibrium network. In fact, slight cost difference can help us distinguish which players attain information actively and which players become spokes.

To simplify symbolic system, we mark $\hat{y}_1$ and $\hat{y}$ as the optimal value of information, which are obtained by players who have information cost advantage and haven't advantage. We use $\tilde{c} = c - \varepsilon < c$ to express the slight advantage of attaining information cost, where $\varepsilon > 0$ is a small number.

In the network g with core-periphery architecture, we will assume that $N_c(g)$ be the set of hubs, where $|N_c(g)| = m$, then $N \backslash N_c(g)$ be the set of spokes, and we have $|N \backslash N_c(g)| = mq$, where $q \in N_+$, namely $n = (q+1)m$. The homogeneous cost of linking with one other person is k which satisfy $f(\hat{y}) - c\hat{y} < f(m\hat{y}_1) - mk$.

Lemma 1. *In the equilibrium networks with core-periphery architecture, if for each $l \in N$, we have $x_l > 0$, then $x_i + y_i = \hat{y}_1$, for each $i \in N_c(g)$. Moreover, $x_p + y_p = \hat{y}$, for each $p \in N \backslash N_c(g)$.*

The proof of this lemma is similar to the lemma 1 in [1].

Definition 1. The local information flow network is called *core-empty-periphery*, if for any pair of player $i, i' \in N_c(g)$, we have $\bar{g}_{ii'} = 0$, and for each player $p \in N \backslash N_c(g)$, we have $g_{p1} = g_{p2} = ... = g_{pm} = 1$.

Definition 2. The local information flow network is called *core-completely-periphery*, if for any pair of player $i, i' \in N_c(g)$, we have $\bar{g}_{ii'} = 1$, and for each player $p \in N \backslash N_c(g)$, we have $g_{p1} = g_{p2} = ... = g_{pm} = 1$.

Theorem 1. *The local information flow network is a equilibrium network with "core-empty-periphery" architecture, if*
1) The personal information acquired by each player i is:

$$x_i = \hat{y}\left(\frac{mq}{m^2q-1}\right) - \left(\frac{1}{m^2q-1}\right)\hat{y}_1, \forall i \in N_c(g)$$

2) The personal information acquired by each player p is:

$$x_p = \frac{m\hat{y}_1 - \hat{y}}{m^2q - 1}, \forall p \in N \backslash N_c(g)$$

3) $cx_p < k < cx_i, k > \widetilde{c}x_i, \frac{c}{k} > \frac{m}{\hat{y} - x_p}$, *where* $i \in N_c(g), p \in N \backslash N_c(g)$.

In fact, the first inequality in term 3) assures that any spokes are not being connected and spokes are favorable to link with hubs; the second inequality assures that any hubs are not being connected; the third inequality assures that spokes link with hubs if they acquire partial information actively. Meeting with information level of terms 1) and 2) can guarantee hubs own the aggregate information $\hat{y}_1$ in the network, while the aggregate information owned by spokes is $\hat{y}$. As shown in Example 1.

Theorem 2. *The local information flow network is a equilibrium network with "core-completely-periphery" architecture, if*
1)The personal information acquired by each player i is:

$$\frac{q}{mq - 1}\hat{y} - \frac{1}{m^2q - m}\hat{y}_1, \forall i \in N_c(g)$$

2)The personal information acquired by each player p is:

$$x_p = \frac{\hat{y}_1 - \hat{y}}{mq - 1}, \forall p \in N \backslash N_c(g)$$

3)$cx_p < k < \widetilde{c}x_i < cx_i, \frac{c}{k} > \frac{m}{\hat{y} - x_p}$, *where* $i \in N_c(g), p \in N \backslash N_c(g)$.

Similar to Theorem 1, the first inequality in term 3) assures that spokes don't form links between them while hubs are favorable to form links with each other and spokes are favorable to form links with hubs; the second inequality assures that spokes link with hubs if they acquire partial information actively. Meeting with information level of terms 1) and 2) can guarantee hubs own the aggregate information $\hat{y}_1$ in the network, while the aggregate information owned by spokes is $\hat{y}$.

We can see from the above situation, after fixing the amount of hubs, the information acquisition of spokes is degressive. So if $q \to +\infty$then $x_p \to 0$. Relatively, the information acquisition of hubs is increasing.

The result also reflects contents of "The law of the few", that is a lot of information will be grasped in a few hubs, while most of other players, that is, the spokes will choose to link with hubs to get information, but themselves will choose little "personal information acquisition", or even entirely depends on the connection to get information, making their "personal information acquisition" to zero.

An example of multi-core-periphery architecture.

Example 2. Let $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the set of players, the cost of linking with one other person is homogeneous, we denote by $k = 0.15$, the cost of information that personally acquired is heterogeneous and denote by $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = \frac{1}{2}, c_7 = c_8 = c_9 = \frac{10}{21}$. Suppose payoffs are given by (1), where $f(y) = ln(1 + y)$. The initial matrix is a zero Matrix, and the initial information vector is $(0, 0, 0, 0, 0, 0, \frac{11}{30}, \frac{11}{30}, \frac{11}{30})$.

We can check that $s^* = (x^*, g^*)$ is a Nash equilibrium, where $x^* = (0, 0, 0, 0, 0, 0, \frac{11}{30}, \frac{11}{30}, \frac{11}{30})$.

$$g^* = \begin{pmatrix} g_1^* \\ g_2^* \\ g_3^* \\ g_4^* \\ g_5^* \\ g_6^* \\ g_7^* \\ g_8^* \\ g_9^* \end{pmatrix} = \begin{pmatrix} 0\,0\,0\,0\,0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \end{pmatrix}$$

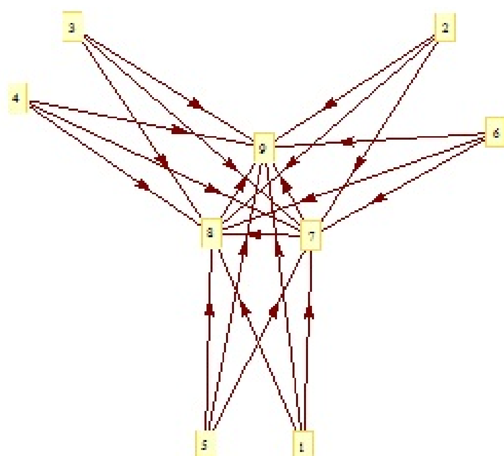

Fig. 2. Connection matrix and three-core-completely-periphery structure of equilibrium network

2.2. Dynamic procedure and algorithm

The algorithm that we study in this paper is built on the dynamic procedure (Gao, et al, 2010, Gao and Petrosyan, 2009, Hojman and Szeidl, 2008). Considered that the algorithmic which even for some player build his best strategy will have n varies, it is no doubt that it is very complicated.

Dynamic procedure: Given the set of players, at each stage, agents who are chosen at random play their best response, they adjust their links in response to the network structure and the personal information acquired by oneself in previous period and then compose the network structure and the personal information.(it is called the player's non-coordinated behavior)

Algorithm: At stage t, first select a subset R of the set of players N at random. For every selected player $i \in R$, we calculate his optimal personal information and the best link set by comparing all of his possible links, namely the player's best response strategy. Each player who is selected at random chooses a pure strategy best response to the strategy of all other agents in the previous period and for the player who isn't selected he maintains the strategy chosen in the previous period. Compose all strategies, and then dynamic procedure comes to stage $t+1$.

The algorithm of dynamic procedure. Based on above dynamic procedure the advantage of the algorithm is that the optimization question has only one variable. The network reaches a stable state when no one chooses to change his links and personal information during dynamic procedure, and then we can get equilibrium strategy profile $s^* = (x^*, g^*)$ and relative information equilibrium network.

Given an initial stage t_0, an initial information vector $x^{t_0} = (x_1^{t_0}, ..., x_n^{t_0})$, initial link matrix $[\alpha_{ij}^{t_0}] \in \{0,1\}^{N\times N}$, where $\alpha_{ii}^{t_0} = 0, i \in N$ and $\alpha_{ij}^{t_0} \in \{0,1\}, \forall j \in N\backslash\{i\}$. Homogeneous link costs $k \geq 0$ and heterogeneity profits $c_i \geq 0$ are given.

At stage t, let information vector be $x^t = (x_1^t, ..., x_n^t)$, link matrix $[\alpha_{ij}^t] = (\alpha_1^t, ..., \alpha_n^t)^T \in \{0,1\}^{N\times N}$, where α_i^t is the row i of the link matrix $[\alpha_{ij}^t]$.

Select randomly the set of players $R = (i_1, ..., i_r) \subseteq N$. For every $i_k \in R$, the row i_k of link matrix has 2^{n-1} possibilities if we fix the rest $n-1$ rows. For every possibility, first calculate $\bar{\alpha}_{i_k j}^t = \max\{\alpha_{i_k j}^t, \alpha_{j i_k}^t\} \in \{0,1\}$, it is easy to see $\bar{\alpha}_{i_k i_k}^t = 0$.

Consider that

$$\max \theta_{i_k}^t = f(x_{i_k}^t + \sum_{j\in N} \bar{\alpha}_{i_k j}^t x_j^t) - c_{i_k} x_{i_k}^t - \sum_{j\in N} \alpha_{i_k j}^t k \qquad (3)$$

s.t.

$$x_{i_k}^t \geq 0$$

Notice that $\sum_{j\in N} \bar{\alpha}_{i_k j}^t x_j^t$ and $\sum_{j\in N} \alpha_{i_k j}^t k$ in objective function are constant for given stage t and all probably value of line i_k in link matrix $[\alpha_{ij}^t]$. So the variable in (3) only is $x_{i_k}^t$.

Suppose that the optimal value is reached at $x_{i_k}^t$. For all the possibilities in row i_k we calculate relative optimal value $x_{i_k}^t$ and $\theta_{i_k}^t$. At last by comparing above 2^{n-1} optimal value $\theta_{i_k}^t$, we can find $\max \theta_{i_k}^t$, and then output relative optimal value $\widetilde{x}_{i_k}^t$ and $\widetilde{\alpha}_{i_k}^t$.

Based on the algorithm we compose above results, let

$$[\alpha_{ij}^{t+1}] = (\alpha_1^t, ..., \widetilde{\alpha}_{i_1}^t, ..., \widetilde{\alpha}_{i_r}^t, ..., \alpha_n^t)^T$$

be link matrix at stage $t+1$, and information vector

$$x^{t+1} = (x_1^{t+1}, ..., x_n^{t+1}) = (x_1^t, ..., \widetilde{x}_{i_1}^t, ..., \widetilde{x}_{i_r}^t, ..., x_n^t)^T$$

The dynamic procedure comes into stage $t+1$. Select the set of players at random again and the network formation procedure is repeated.

The dynamic procedure is end until no one chooses to change his links and personal information. Notice that some examples show that the dynamic procedure may be a circulation, when we need special terminator program. At last we get the information flow equilibrium network and equilibrium strategy profile will comprise the link matrix of the equilibrium network and the optimal personal acquired information vector.

Program: Compute programming language is maple.

Examples of dynamic procedure. Corresponding to Theorem 2 we give two examples and model the dynamic procedure using the program.

Example 3. Given the set of players $N = \{1, 2, 3, 4, 5, 6\}$, homogeneous link costs $k = 0.04$ and personal information heterogeneity profits $c_1 = c_2 = c_3 = c_4 = \frac{1}{10}, c_5 = c_6 = \frac{1}{11}$. Payoff function is given by (1), where $f(y) = ln(1+y)$. Initial link matrix and initial information vector are 0.

It is different to reach final stage for each implementation of the dynamic process program and convergence to the equilibrium state because the non-coordinated behavior of the players caused by the randomness of process of forming dynamic network. Therefore there is not practical significance for number of stages to achieve a equilibrium state.

We only had one interception of several typical stages in some run results in Figure 3, and let K be the number of stages needed to achieve a equilibrium state.

The network structure and dynamics of information vectors in example 3:

1) At initial stage select the set of players $R = \{2, 3, 4, 5, 6\}$ at random, because of the initial matrix is zero, they all choose to acquire their optimal aggregate information and don't form any link. And then it will come into an empty network by the first round of iteration. But the information level vector is $x^1 = (x_1^1, ..., x_6^1) = (0, 9, 9, 9, 10, 10)$.

2) Owing to the selected set of players is $R = \{3, 5, 6\}$ at stage 1, we can infer that it is the best response for player 3 to form links with player 2,4,5,6 and he don't acquire information personally, the payoff $\Pi_3(.) = ln[1+(9+9+10+10)]-4\times 0.04 \approx 3.5036$ is larger than the payoff in any other situations. Because player 5,6 have the same acquired advantage, it is the best response for player 5,6 to form links with 2,3,4,6 and he don't acquire information. The dynamic procedure comes into stage 2.

3) When the dynamic procedure comes into stage t_1 it is two-core-periphery architecture. The information level vector of players is $x^{t_1} = (0, 9, 0, 0, 0, 0)$. Select set of the players $R = \{1, 4, 5, 6\}$ randomly. Although here the two-core-periphery architecture is the same as the eventual equilibrium structure formally. (player 2,4 are temporal core), obviously, for player 5,6 their aggregate information don't reach their optimal value $\hat{y}_5 = \hat{y}_6 = 10$, so here the network isn't equilibrium network and not stable. For player 5,6 they will delete their link with player 4 and maintain their link with player 2 and acquire information 1 by themselves, the payoff $\Pi_i(.) = ln[1 + (9 + 1)] - \frac{1}{11} \times 1 - 0.04 \approx 2.2669, i = 5, 6$ is lager than any other situations, so it is the best response for them on this stage. Because player 4 don't acquire any information personally, player 1,5,6 will delete their link with 4 and player 1,4 can acquire their optimal aggregate information 9 by maintaining their link with 2.

4) When the dynamic procedure comes into stage t_2, it is three-core-completely-periphery architecture. The information level of players is $x^{t_2} = (0, 3, 1, 0, 5, 1)$. Selected set of the players $R = N$. So here for player 1,4 it is their best response to maintain their link with player 2,3,5,6, player 2,3 will maintain their link and reduce their information 1 by themselves and for player 5,6 they maintain their strategy in pre-period. Notice that here the network is four-core-completely-periphery architecture, it is not stable.

5) When the dynamic procedure comes into stage $K - 1$, the information level of players is $x^{K-1} = (0, 0, 0, 0, 8, 2)$. Selected set of the players $R = \{1\}$ randomly. For player 1 it is his best response to delete his link with player 3 and maintain

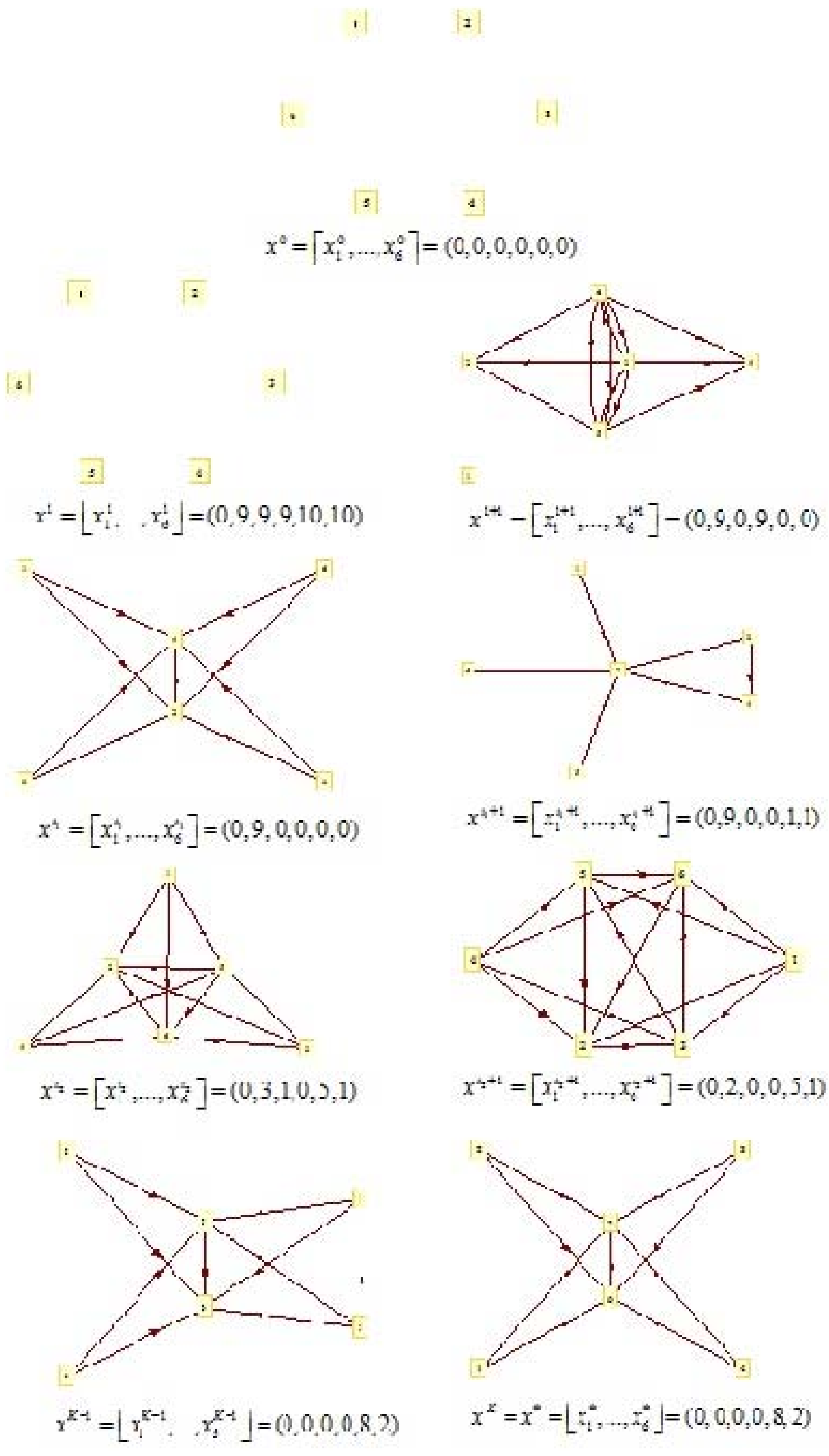

Fig. 3. Example of the stage of the dynamic process

his link with 5,6 while he acquires no information. So "two-core-periphery" equilibrium network is formed, the hub is composed of two players with information cost advantage acquired actively. The dynamic procedure is over.

2.3. Conclusion

Core-periphery structure is the most representative equilibrium network in the case of the local information flow without information loss. In fact, under the premise of players have two levels of information each person who has a comparative advantage of the cost of acquiring information would become the hub possibly. The statistical results show that it is the easiest to form "single-core-periphery" structure; but the "multi-core-periphery" structure which contains all players who has the same advantage on the cost as the hubs (maximum) has the smallest probability; those which between these two kinds become harder and harder with the increasing of the players' number in the core.

We believe that there are several reasons why the dynamics are important. One reason is that a dynamic model allows us to study the process by which individual agents learn about the network and adjust their links in response to their learning. Relatedly, dynamic may help select among different equilibria of the static game.

There is very important prospect to study the characterization of the architecture of information flow equilibrium networks and the dynamics of network formation under the premise of local information flow and without the presence of decay. In addition, consider that some of the players may coordinate and prepare to maximize the common interests of members by some local cooperation; the model and algorithm in this essay can be transformed to an incomplete information cooperative game model. This article assumes that the players have the need and competence of acquiring information personally and getting information from others in the given network structure, in fact, this can also be diversified.

3. The local information flow equilibrium network with homogeneous costs and information decay

3.1. The characteristic of equilibrium network architecture

Lemma 2. *In the local information flow equilibrium network with decay g, we have $x_i + y_i \geq \hat{y}$, for all $i \in N$, and if $x_i > 0$, then $x_i + y_i = \hat{y}$. Here $y_i = \sum_{j \in N_i(\bar{g})} \delta x_j$, that is to say y_i represents the information that player i acquires from his neighbors.*

Proof. Suppose not, then there must exist some player i such that $x_i + y_i < \hat{y}$ in equilibrium network g. Under the assumptions that $f(y)$ is twice continuously differentiable, increasing, and strictly concave in y, we know $f'(x_i+y_i) > c$, so player i can strictly increase his payoffs by increasing personal information acquisition, a contradiction with equilibrium. So we know $x_i + y_i \leq \hat{y}, \forall i \in N$. Next suppose that $x_i > 0$ and $x_i + y_i > \hat{y}$. Under our assumptions on $f(.)$ and c, if $x_i + y_i > \hat{y}$ then $f'(x_i+y_i) < c$, and then player i can strictly increase his payoffs by lowering personal information acquisition, which contradicts equilibrium. Therefore, if $x_i > 0$, then $x_i + y_i = \hat{y}$.

Theorem 3. *In the local information flow equilibrium network with decay, if $k > c\delta\hat{y}$, the empty network is unique equilibrium. every player acquires information $\hat{y}$ personally and no one forms links.*

Proof. 1) Suppose that the strategy profile $s = (x, g)$ corresponds to an empty network. If player i forms m_1 links with other m_1 players and personally acquires information x_i', here the strategy profile is $s|s_i'$. Lemma 2 implies that the aggregate information of player i should be $\hat{y}$, and then his payoff is

$$\Pi_i(s|s_i') = f(\hat{y}) - c(\hat{y} - \delta m_1 \hat{y}) - m_1 k$$

however, in the empty networks, his payoff is

$$\Pi_i(s) = f(\hat{y}) - c\hat{y}$$

Since $k > c\delta\hat{y}$, we have

$$\Pi_i(s) > \Pi_i(s|s_i')$$

2) Suppose player i forms m_2 links with other m_2 players initiatively based on the empty network, and $x_i'' = 0$, the strategy profile is $s|s_i''$, payoff is

$$\Pi_i(s|s_i'') = f(\delta m_2 \hat{y}) - m_2 k$$

. Here $\delta m_2 \hat{y} \geq \hat{y}$, then $\delta m_2 \ \geq 1$. Therefore

$$\Pi_i(s|s_i'') = f(\delta m_2 \hat{y}) - m_2 k < f(\delta m_2 \hat{y}) - c m_2 \delta \hat{y} \leq f(\hat{y}) - c\hat{y} = \Pi_i(s)$$

Due to $k > c\delta\hat{y}$, the first inequality is strict obviously, since $f(y)$ is twice continuously differentiable, increasing, and strictly concave in y, the second is hold. Then the empty network is equilibrium network.

Finally, we will show that the empty network is unique equilibrium. As we all know, every player's personal information acquisition are no more than $\hat{y}$, if some player i want to form a link with other player (say j), player i can obtain δx_j from j, and satisfied $c\delta x_j > k$, this would contradict with $k > c\delta\hat{y}$(because $\hat{y} > x_j$).

Lemma 3. *If complete network is the local information flow equilibrium network with decay, the personal information acquisition of each player is equal and* $x = \frac{\hat{y}}{1+(n-1)\delta}$.

Proof. For every player he acquires information personal $x_i \geq 0$. If there are $m(1 \leq m < n)$ players who personally acquires information 0, we let $x_1 = x_2 = ... = x_m = 0$, according to Lemma 2, we conclude following inequalities:

$$\begin{cases} \sum_{i \in N \setminus \{1,\dots,m\}} \delta x_i \geq \hat{y} \\ x_j + \sum_{i \in N \setminus \{j\}} \delta x_i = \hat{y}, j = m+1, ..., n \end{cases}$$

We can obtain $x_{m+1} = ... = x_n = \frac{\hat{y}}{1+(n-m-1)\delta}$ from the second inequality, hence, $\delta \geq 1$ a contradiction. Hence, in complete network, we have $x_i > 0, \forall i \in N$. As Lemma 2, we have

$$x_i + \sum_{j \in N \setminus \{i\}} \delta x_j = \hat{y}, i = 1, ..., n$$

therefore

$$x_1 = x_2 = ... = x_n = x = \frac{\hat{y}}{1+(n-1)\delta}$$

Theorem 4. *In the local information flow equilibrium network with decay, if* $k < c\delta x = \frac{c\delta\hat{y}}{1+(n-1)\delta}$*, complete network is the unique equilibrium.*

Proof. Firstly prove that if the personal information acquisition of each player is equal and $x = \frac{\hat{y}}{1+(n-1)\delta}$ in complete network, complete network is equilibrium structure. Let $s^* = (s_1^*, ..., s_n^*)$ be the strategy profile in complete network, where $s_i^* = (g_i^*, x)$. If player i have $m(0 \le m \le n-1)$ links which he formed with others, since every player can obtain information from other $n-1$ players, they would not form links with others.

Secondly, we show that it is not the best response for player i of deleting links or changing his personal information level.

Now consider the case $m = 0$, player i can change their personal information acquisition to be x', we can write strategy profile $s^*|s_i'$, hence

$$\Pi_i(s^*|s_i') = f(\hat{y} - x + x') - cx' < f(\hat{y}) - cx = \Pi_i(s^*)$$

so changing strategies are not the best response for player i.

Consider the case $m = 1$, if player i deletes links and acquires information $(1+\delta)x$ personally, hence

$$\Pi_i(s^*|s_i') = f(\hat{y}) - c(1+\delta)x$$

Since $k < c\delta x$

$$\Pi_i(s^*|s_i') = f(\hat{y}) - c(1+\delta)x < f(\hat{y}) - cx - k = \Pi_i(s^*)$$

so changing strategies are not the best response for player i.

Similarly, we can show that the best response for player i is to maintain current strategy and completeness of network when $m = 2, ..., n-1$ and $k < c\delta x$.

Following we show that if $k < c\delta x = \frac{c\delta\hat{y}}{1+(n-1)\delta}$, each player acquires information $x = \frac{\hat{y}}{1+(n-1)\delta}$ personally, complete network is the unique equilibrium.

Lemma 3 implies that if complete network is equilibrium structure, the personal information acquisition of each player is equal and $x = \frac{\hat{y}}{1+(n-1)\delta}$.

Two steps: the first step is to prove the connected but not completely network is not equilibrium, the second step is to prove not connected network is not equilibrium.

Step 1.

Suppose not. Then suppose the connected but not completely network is equilibrium. (These connected but not completely network is seen as the sub-network which is formed by deleting several links at random based on a complete network. Actually, the network must not be equilibrium where the number of links between two players are more than 1.)(the link number is $n-1$ (Bala and Goyal, 2000) in the minimal connected network).

We delete any link in complete network, $g_{ij} = 0$ or $g_{ji} = 0$, let g_1 be the network. Suppose g_1 is equilibrium. Since Lemma 2, every player's aggregate information should be more than or equal to $\hat{y}$. Since $\bar{g}_{ij} = 0$, we have $k > c\delta x_i$ and $k > c\delta x_j$. Players who are in $N\backslash\{i,j\}$ have links with each other, let $N_1 = \{i,j\}, N_2 = N\backslash\{i,j\}, I(s) = \{p|x_p > 0, p \in N\}$. For $q \in N_2 \cap I(s)$, we have

$$x_q + \sum_{p\in(N_2\cap I(s))\backslash\{q\}} \delta x_p + \sum_{t\in N_1} \delta x_t = \hat{y}$$

If $x_i = x_j = 0$, and $i, j \in N_1$, so that

$$x_i + \sum_{p \in N_2 \cap I(s)} \delta x_p = 0 + \sum_{p \in N_2 \cap I(s)} \delta x_p < \hat{y}$$

$$x_j + \sum_{p \in N_2 \cap I(s)} \delta x_p = 0 + \sum_{p \in N_2 \cap I(s)} \delta x_p < \hat{y}$$

a contradiction with equilibrium.

If $x_i = 0, x_j > 0$, and $i \in N_1$, hence

$$x_i + \sum_{p \in N_2 \cap I(s)} \delta x_p = 0 + \sum_{p \in N_2 \cap I(s)} \delta x_p < \hat{y}$$

Then player i can strictly increase his payoffs by increasing personal information acquisition. It contradicts with equilibrium.

The case of $x_j = 0, x_i > 0$ is similarly to the above.

Hence, we can conclude that personal information acquisition of i and j more than 0. Otherwise, for $l \in N_2 \backslash I(s)$, we have

$$\begin{aligned} & x_l + \sum_{p \in N_2 \cap I(s)} \delta x_p + \sum_{t \in N_1} \delta x_t \\ & = 0 + \sum_{p \in N_2 \cap I(s)} \delta x_p + \sum_{t \in N_1} \delta x_t \\ & = \sum_{p \in (N_2 \cap I(s)) \backslash \{q\}} \delta x_p + \delta x_q + \sum_{t \in N_1} \delta x_t \\ & < x_q + \sum_{p \in (N_2 \cap I(s)) \backslash \{q\}} \delta x_p + \sum_{t \in N_1} \delta x_t = \hat{y} \end{aligned}$$

That is to say, the aggregate information of l is less than $\hat{y}$, contradiction. Hence, for any $i \in N$, every player's personal information acquisition is $x_i > 0$. According to Lemma 2, the following inequalities are hold

$$\begin{cases} x_t + \sum_{p \in N_2} \delta x_p = \hat{y}, \forall t \in N_1 \\ x_q + \sum_{p \in N_2 \backslash \{q\}} \delta x_p + \sum_{t \in N_1} \delta x_t = \hat{y}, \forall q \in N_2 \end{cases}$$

So we have every player personal information acquisition in N_1 equal to $x_i = x_j = x'$, and personal information acquisition in N_2 equal to $x_q = x'', \forall q \in N_2$, so that

$$\begin{cases} x' = \frac{(1-\delta)\hat{y}}{1+\delta-2n\delta^2+4\delta^2} \\ x'' = \frac{(1-2\delta)\hat{y}}{1+\delta-2n\delta^2+4\delta^2} \end{cases}$$

For any $n \geq 3$, any network is not equilibrium network.

Similarly, we can show that except complete network, connected network is not the equilibrium network.

Step 2.

Firstly, suppose there is an isolated player in equilibrium network at least, the set of these players is N^0. The personal information acquisition is $\hat{y}$ in N^0. Due to

$k < c\delta x < c\delta\hat{y}$, there must exist some player in $N \setminus N^0$ who will strictly increase his payoffs by lowering his personal information acquisition and switching to link with players in N^0, contradiction with equilibrium.

Secondly, suppose $C_1(\bar{g})$ and $C_2(\bar{g})$ are two part of the network $\bar{g}$, and every part has more than one player.

From Step 1, we know that the two parts are complete.

Suppose $|C_1(\bar{g})| = n_1 > 1, |C_2(\bar{g})| = n_2 > 1$, and assume $n_1 \leq n_2$. Lemma 2 and Lemma 3 claimed that the information is equal in $C_1(\bar{g})$ and $C_2(\bar{g})$, respectively:

$$x_{n_1} = \frac{\hat{y}}{1 + (n_1 - 1)\delta}, x_{n_2} = \frac{\hat{y}}{1 + (n_2 - 1)\delta}$$

Then $x_{n_1} \geq x_{n_2} > x$. We can concluded that $k < c\delta x < c\delta x_{n_2} \leq c\delta x_{n_1}$, the players in $C_2(\bar{g})$ will link with the players in $C_1(\bar{g})$ initiatively. A contradiction completes the proof.

Similarly, we can show that non-connected network which contains multiple parts is not the equilibrium.

Theorem 5. *In the local information flow network equilibrium structure with decay, if $c\delta x < k < c\delta\hat{y}, x = \frac{\hat{y}}{1+(n-1)\delta}$, no equilibrium structure exist.*

All of the examples show the results of Theorem 5 are correct, but the strict proof is not yet complete.

3.2. Examples of Equilibrium Network

Example 4. Let $N = \{1, 2, 3, 4, 5, 6\}$ be the set of players, the homogeneous cost of linking with one other person is denoted by k, the homogeneous cost of information that players acquire is denoted by c, the index of local information decay is denoted by δ and $0 < \delta < 1$, where payoffs are given by (1), $f(y) = ln(1 + y)$.

As shown in Fig.4, δ is x-coordinate, k is y-coordinate, where $n = 6, c = \frac{1}{3}, \hat{y} = 2$, the curve equation is $k = \frac{2\delta}{3+15\delta}$, and the linear equation is $k = \frac{2}{3}\delta$. If $k < \frac{2\delta}{3+15\delta}$, complete network (Fig.5) is the unique equilibrium, write C; if $k > \frac{2}{3}\delta$, empty network is the unique equilibrium, write A; if $\frac{2\delta}{3+15\delta} < k < \frac{2}{3}\delta$, there exists no equilibrium, write B.

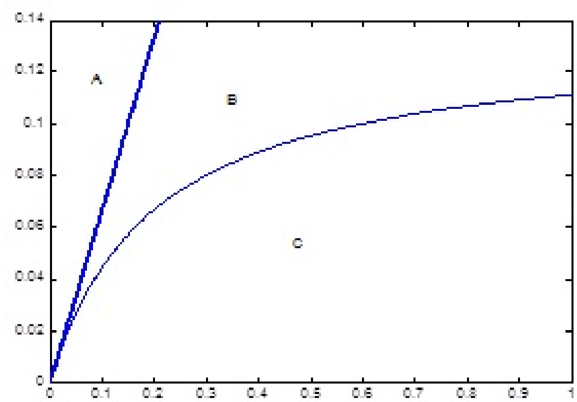

Fig. 4. Distribution of equilibrium network

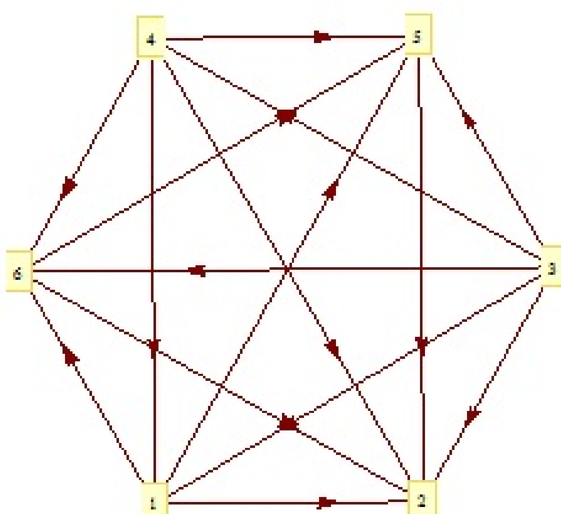

Fig. 5. Completely network

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Game Theory Approach for Supply Chains Optimization

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Abstract. In this paper game theoretic mathematical models of inventory systems are treated. We consider a market, where several distributors are acting. Each distributor has warehouse for storage goods before supply to customers. Assume that demand for their goods has deterministic nature and depends on total supply or on prices of distributors. So we will consider quantitative and price competition among distributors. Distributors are considered as players in non-cooperative game. First we treat quantitative competition in context of model of Cournot. Then to consider price competition we use modified model of Bertrand. For modeling of control of inventory system we use the relaxation method of inventory regulation with admission of deficiency.

Keywords: Nash equilibrium, optimal, internal strategy, external strategy, demand, distributor.

1. Basic Model for Inventory Policy with Backordering

Let's consider a single-product inventory control system of stocks with a deficiency assumption. We will use relaxation method of regulation of stocks to minimize long-run inventory costs (Grigoriev M.N., Dolgov A.P., Uvarov S.A., 2006), (Hedly J., Uaitin T., 1969). Assume the demand for items of product is known and uniform during a period of planning. The dynamic of inventory system is illustrated in Figure 1.

The following parameters are used to establish a mathematical model for this problem

K – fixed cost per order.
c – unit cost of procurement an item of product.
h – cost per holding item in inventory during the period of planning.
g – cost of being short one item during the entire period of planning.
a – demand per inventory circle.
y – order quantity.
S – maximal inventory level.
D – demand for the period of planning.

Variables y and S are controlled variables in the problem and may be assumed as strategies.

The usual objective of an inventory policy is to minimize cost of inventory system. To meet a given constant demand for the period of planning distributor has to

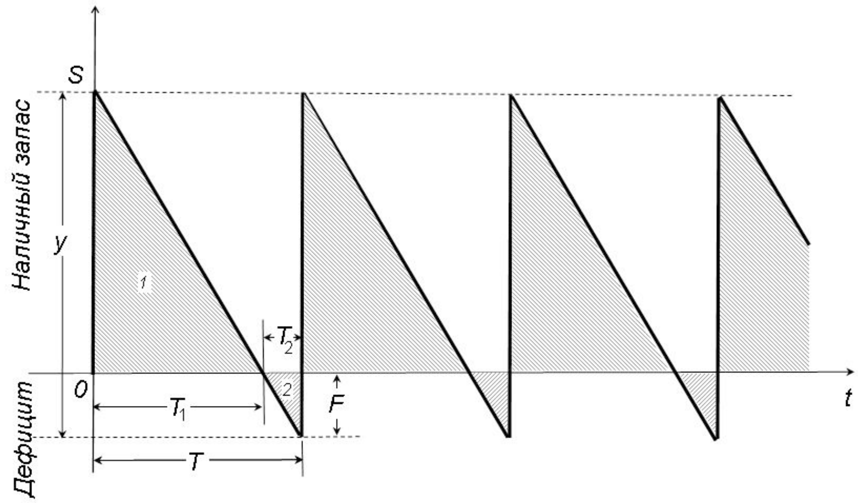

Fig. 1.

make $m = D/y$ orders. The total cost of inventory system for the period of planning T_{plan} is the following

$$L(y,S) = mL_{cycle}(y,S) = \frac{D}{y}\left(K + cy + h\frac{S^2}{2a} + g\frac{(y-S)^2}{2a}\right) \tag{1}$$

2. Description of Non-cooperetive Game

Following (Vorobiev N.N., 1985) a system

$$\Gamma = \langle N, \{X_i\}_{i\in N}, \{\Pi_i\}_{i\in N}\rangle\,, \tag{2}$$

is called a non-cooperative game, where
$N = \{1, 2, ..., n\}$ – set of players,
X_i – set of strategies of player i,
Π_i – payoff function of player $i$which provides a mapping from the set of strategies of players $X = \prod_{i=1}^{n} X_i$ to R^1.

Players make an interactive decisions simultaneously choosing their strategies x_i from strategy sets X_i. Vector $x = (x_1, x_2, ..., x_n)$is called situation in the game. As a result players are paid payoff $\Pi_i = \Pi_i(x)$. We call situation $x^* = (x_1^*, x_2^*, ..., x_n^*)$ a Nash equilibrium if for all admissible strategies $x_i \in X_i$, $i = 1, ..., n$ the following inequalities hold

$$\Pi_i(x^*) \geq \Pi_i(x_1^*, x_2^*, ..., x_{i-1}^*, x_i, x_{i+1}^*, ..., x_n^*). \tag{3}$$

Theorem 1 (Kukushkin N.S., Morozov V.V., 1984). *In game (4) there exists Nash equilibrium in pure strategies if for each $i \in N$ strategy set X_i is compact and convex, and payoff function $\Pi_i(x)$ is concave with respect to x_i and continuous on X.*

Assume for any $i \in N$ the function $\Pi_i(x)$ is twice continuously differentiable with respect to x_i. From (Jean Tirole, 2000) we can see that first-order necessary condition for Nash equilibrium is the following

$$\frac{\partial \Pi_i(x^*)}{\partial x_i} = 0, i \in N. \tag{4}$$

Suppose the payoff function $\Pi_i(x)$, $i = 1, ..., n$ is concave for all $x_i \in X_i$, that is

$$\frac{\partial^2 \Pi_i(x^*)}{\partial x_i^2} \leq 0, , i \in N. \tag{5}$$

In this case solution of system (4) appears to be a Nash equilibrium in pure strategies in non-cooperative game $\Gamma = \langle N, \{X_i\}_{i \in N}, \{\Pi_i\}_{i \in N} \rangle$.

Let us consider two type of oligopoly: quantitave competition (model of Kournot) and price competition (model of Bertrand) (Jean Tirole, 2000).

In the model of Kournot n players (distributors) make simultaneously interactive decisions about $Q_i \in \Omega_i$, quantities of product to be supplied (produced for) to the market. Their cost functions $L_i(Q_i)$, $i = 1, ..., n$ are the same like in (3). Suppose $Q_{-i} = (Q_1, Q_2, ..., Q_{i-1}, Q_{i+1}, ..., Q_n)$ is a vector of expected value of quantities of product of other players. Demand is defined by decreasing inverse demand function $p(Q_i, Q_{-i}) = p\left(\sum_{k=1}^{n} Q_k\right)$. Than payoff function of player i can be expressed as follows

$$\Pi_i(Q_i, Q_{-i}) = p\left(Q_i, Q_{-i}\right) - L_i(Q_i). \tag{6}$$

Thus we have non-cooperative game

$$\Gamma_K = \left\langle N, \{\Omega_i\}_{i=1}^n, \{\Pi_i\}_{i=1}^n \right\rangle. \tag{7}$$

In modified model of Bertrand we use demand function $Q_i = D_i(p_i, p_{-i})$of price $p_i \in \Omega_i$ assigned by player $i$and prices of competitors $p_{-i} = (p_1, p_2, ..., p_{i-1}, p_{i+1}, ..., p_n)$. In this case payoff function of player i will be of the form

$$\Pi_i(p_i, p_{-i}) = p_i D_i\left(p_i, p_{-i}\right) - L_i(D_i(p_i, p_{-i})). \tag{8}$$

We denote the game (4) with payoff functions (8):

$$\Gamma_B = \left\langle N, \{\Omega_i\}_{i=1}^n, \{\Pi_i\}_{i=1}^n \right\rangle. \tag{9}$$

3. Game Theory Model for Inventory Decision. Quantity Competition

3.1. Statement of the Problem

Let us consider Kournot model (7). Suppose that n distributors supply uniformly product to market making decisions about quantitatis Q_i and variables (y_i, S_i) to maximize profit (payoff functions)

$$\Pi_i(Q_i, Q_{-i}) = p\left(Q_i, Q_{-i}\right) Q_i - L_i(Q_i), i = 1, ..., n, \tag{10}$$

where total cost $L_i(Q_i)$ is the following

$$L_i(Q_i) = L_i(Q_i, y_i, S_i) = \frac{Q_i}{y_i}\left(K_i + c_i y_i + h_i \frac{S_i^2}{2a_i} + g_i \frac{(y_i - S_i)^2}{2a_i}\right). \tag{11}$$

Notice that demand per inventory circle $a_i(\cdot)$may be constant or considered as a function of price per item of product. Let $a_i(p) = a_i(p(Q_i, Q_{-i})) = b_i(Q_i, Q_{-i})$, $i = 1, ..., n$. In this case cost function (11) will be in the form

$$L_i(Q_i, Q_{-i}, y_i, S_i) = \frac{Q_i}{y_i}\left(K_i + c_i y_i + h_i \frac{S_i^2}{2b_i(Q_i, Q_{-i})} + g_i \frac{(y_i - S_i)^2}{2b_i(Q_i, Q_{-i})}\right). \quad (12)$$

Substituting (12) to expression (11) we get

$$\Pi_i(Q_i, Q_{-i}) = \Pi_i(Q_i, Q_{-i}, y_i, S_i) =$$
$$= p(Q_i, Q_{-i})Q_i - \frac{Q_i}{y_i}(K_i + c_i y_i + h_i \frac{S_i^2}{2b_i(Q_i, Q_{-i})} + g_i \frac{(y_i - S_i)^2}{2b_i(Q_i, Q_{-i})}). \quad (13)$$

Function (13) for each i, $i = 1, ..., n$ is continuously differentiable on y_i and S_i and concave with respect to vector of variables (y_i, S_i) on the set $[0, \infty) \times [0, \infty)$ when admissible vector $(Q_i, Q_{-i})$is fixed. Notice also that according to (13) profit of a distributor i is influenced by supply strategy of all competitors and variables (y_i, S_i) as well.

Definition 1. We will call pair (y_i, S_i) internal strategy and quantity Q_i external strategy of player i.

Denote strategy sets of player i, $i = 1, ..., n$ by : $\Omega_i^{(1)} = \{Q_i \mid Q_i \in [a_i^{(1)}, b_i^{(1)}] \subset [0, \infty)\}$; $\Omega_i^{(2)} = \{y_i \mid y_i \in [a_i^{(2)}, b_i^{(2)}] \subset (0, \infty)\}$; $\Omega_i^{(3)} = \{S_i \mid S_i \in [a_i^{(3)}, b_i^{(3)}] \subset [0, \infty)\}$. Let $a_i^{(j)} \ll b_i^{(j)}$ for all $i = 1, ..., N \;\; j = 1, 2, 3$.

3.2. Internal optimization problem

Considering vector of external strategies $(Q_i, Q_{-i}) \in \Omega^{(1)} = \Omega_1^{(1)} \times \Omega_2^{(1)} \times ... \times \Omega_N^{(1)} \subset R_+^n$ as given player i chooses internal strategy $(y_i, S_i) \in \Omega_i^{(2)} \times \Omega_i^{(3)}$ to miximize payoff function

$$\Pi_i(Q_i, Q_{-i}, y_i, S_i) \to_{(y_i, S_i)} max, \;\; y_i \in \Omega_i^{(2)}, \;\; S_i \in \Omega_i^{(3)}. \quad (14)$$

Solving problem (14) for each $i = 1, ..., n$ we get

$$y_i^* = y_i^*(Q_i, Q_{-i}) = \sqrt{\frac{2K_i(g_i + h_i)}{h_i g_i} b_i(Q_i, Q_{-i})}, \quad (15)$$

$$S_i^* = S_i^*(Q_i, Q_{-i}) = \sqrt{\frac{2K_i g_i}{h_i(g_i + h_i)} b_i(Q_i, Q_{-i})}. \quad (16)$$

3.3. Existence of Nash equilibrium in pure strategies

Substituting optimal internal strategy of player i (y_i^*, S_i^*), $i = 1, ..., n$ defined by (15) and (16) we get the following expression for payoff functions

$$\Pi_i(Q_i, Q_{-i}, y_i^*(Q_i, Q_{-i}), S_i^*(Q_i, Q_{-i})) = \Phi_i(Q_i, Q_{-i}) =$$
$$= Q_i p(Q_i, Q_{-i}) - \sqrt{\frac{2K_i g_i h_i}{h_i + g_i}} \cdot \frac{Q_i}{\sqrt{b_i(Q_i, Q_{-i})}} - Q_i c_i. \quad (17)$$

These functions are only influenced by strategies (Q_i, Q_{-i}). Thus we have now non-cooperative n-person game of type (7) (model of Kournot)

$$\Gamma_K = \langle N, \{\Omega_i\}_{i=1}^n, \{\Phi_i\}_{i=1}^n \rangle . \quad (18)$$

The following theorem takes place

Theorem 2. *Suppose the following conditions hold for $i = 1, ..., n$:*

1) reverse demand function $p(Q_i, Q_{-i})$ is twice differentiable, decreasing and concave with respect to $Q_i \in \Omega_i^{(1)}$ for any fixed $Q_{-i} \in \Omega^{(1)}/\Omega_i^{(1)}$;

2) function $\frac{Q_i}{\sqrt{b_i(Q_i,Q_{-i})}}$ is continuous on $\Omega^{(1)}$, convex with respect to $Q_i \in \Omega_i^{(1)}$ for any fixed $Q_{-i} \in \Omega^{(1)}/\Omega_i^{(1)}$ or the following inequality holds (if $b_i(Q_i, Q_{-i})$ – twice differentiable)

$$3Q_i \left(\frac{\partial b_i(Q_i, \boldsymbol{Q}_{-i})}{\partial Q_i} \right)^2 \geq 4b_i(Q_i, \boldsymbol{Q}_{-i}) \frac{\partial b_i(Q_i, \boldsymbol{Q}_{-i})}{\partial Q_i} + 2Q_i b_i(Q_i, \boldsymbol{Q}_{-i}) \frac{\partial^2 b_i(Q_i, \boldsymbol{Q}_{-i})}{\partial Q_i^2}$$

3) there exists $\underset{i}{\widetilde{Q}} \in (a_i^{(1)}, b_i^{(1)})$such that

$$p(Q_i, Q_{-i}) < \sqrt{\frac{2K_i g_i h_i}{g_i + h_i}} \cdot \frac{b_i(Q_i, Q_{-i}) - 1/2 Q_i \frac{\partial b_i(Q_i, Q_{-i})}{\partial S_i}}{b_i^{3/2}(Q_i, Q_{-i})} + c_i \quad (19)$$

for $Q_i \geq \underset{i}{\widetilde{Q}}$.

Then in game (18) there exists Nash equilibrium in pure strategies $(Q_1^*, Q_2^*, \ldots, Q_N^*)$, and $Q_i^* \in [a_i^{(1)}, \underset{i}{\widetilde{Q}})$, $i = 1, ..., n$.

The proof of this theorem is based on the first-order (4) and second-order (5) conditions. According to the proof of the theorem Nash equilibrium is a solution of the following system

$$\frac{\partial \Pi_i(Q_i, Q_{-i})}{\partial Q_i} = 0, \;\; i = 1, ..., n$$

or the same

$$\frac{\partial p(Q_i, Q_{-i})}{\partial Q_i} Q_i + p(Q_i, Q_{-i}) - \sqrt{\frac{2K_i g_i h_i}{g_i + h_i}} \frac{b(Q_i, Q_{-i}) - 1/2 Q_i \frac{\partial b_i(Q_i, Q_{-i})}{\partial Q_i}}{b_i^{3/2}(Q_i, Q_{-i})} - c_i = 0, \quad (20)$$

$$i = 1, ..., n.$$

Substituting solution of this system to (15) and (16) to calculate optimal internal strategies (y_i^*, S_i^*) where $y_i^* = y_i^*(Q_1^*, Q_2^*, \ldots, Q_n^*)$, $S_i^* = S_i^*(Q_1^*, Q_2^*, \ldots, Q_n)$, we will finally get optimal distributor's strategies $U_i^* = (Q_i^*, y_i^*, S_i^*)$, $i = 1, ..., n$, which maximize profit and support Nash equilibrium in the game.

4. Game Theory Model for Inventory Decision. Price Competition

4.1. Statement of the problem

To formalize game theory model of price competition we consider model of Bertrand described in chapter 2. According to (8) profit of distributor $i = 1, ..., n$is expressed as follows

$$\Pi_i(p_i, p_{-i}) = D_i(p_i, p_{-i})p_i - L_i(D_i(p_i, p_{-i})), \tag{21}$$

where $L_i(D_i(p_i, p_{-i}))$ is the cost function of inventory system. From (3) we get the following expression for the cost function

$$L_i(D_i(p_i, p_{-i})) = \overline{L}_i(p_i, p_{-i}, y_i, S_i) =$$
$$= \frac{D_i(p_i, p_{-i})}{y_i}\left(K_i + c_i y_i + h_i \frac{S_i^2}{2a_i} + g_i \frac{(y_i - S_i)^2}{2a_i}\right). \tag{22}$$

In a similar manner as in chapter 3 we consider demand function per inventory circle $a_i(\cdot)$ of the form $a_i = b_i(p_i, p_{-i})$, $i = 1, ..., n$. Thus function (22) can be rewritten in the form

$$L_i(D_i(p_i, p_{-i})) = \frac{D_i(p_i, p_{-i})}{y_i}\left(K_i + c_i y_i + h_i \frac{S_i^2}{2b_i(p_i, p_{-i})} + g_i \frac{(y_i - S_i)^2}{2b_i(p_i, p_{-i})}\right). \tag{23}$$

Substituting (23) in (21) we get profit function as follows

$$\Pi_i(p_i, p_{-i}) = \Pi_i(p_i, p_{-i}, y_i, S_i) =$$
$$= D_i(p_i, p_{-i})p_i - \frac{D_i(p_i, p_{-i})}{y_i}\left(K_i + c_i y_i + h_i \frac{S_i^2}{2b_i(p_i, p_{-i})} + g_i \frac{(y_i - S_i)^2}{2b_i(p_i, p_{-i})}\right). \tag{24}$$

Analogously to chapter 3 we call (y_i, S_i) internal strategy, p_i – external strategy of distributor i, $i = 1, ..., n$.
Strategy sets of player i, $i = 1, ..., n$ are as follows
$\Omega_i^{(1)} = \{p_i \mid p_i \in [a_i^{(1)}, b_i^{(1)}] \subset [0, \infty)\}$; $\Omega_i^{(2)} = \{y_i \mid y_i \in [a_i^{(2)}, b_i^{(2)}] \subset (0, \infty)\}$; $\Omega_i^{(3)} = \{S_i \mid S_i \in [a_i^{(3)}, b_i^{(3)}] \subset [0, \infty)\}$. Thus we have two level optimization problem for each distributor. It is also can be presented as combination of internal and external problems.

4.2. Internal optimization problem

In internal optimization problem player i chooses internal strategy $(y_i, S_i) \in \Omega_i^{(2)} \times \Omega_i^{(3)}$ with fixed given prices of all players $(p_i, p_{-i}) \in \Omega^{(1)} = \Omega_1^{(1)} \times \Omega_2^{(1)} \times ... \times \Omega_n^{(1)}$ to maximize profit function

$$\Pi_i(p_i, p_{-i}, y_i, S_i) \rightarrow_{(y_i, S_i)} max, \ \ y_i \in \Omega_i^{(2)}, \ \ S_i \in \Omega_i^{(3)}. \tag{25}$$

Note that profit function (24) for each i is continuously differentiable with respect to y_i and S_i taken separately and also is convex with respect to (y_i, S_i) on the set $[0, \infty) \times [0, \infty)$ (in particular on $\Omega_i^{(2)} \times \Omega_i^{(3)}$) when vector of prices (p_i, p_{-i}) is fixed.

Solving problem (25) we get the following internal strategies of players

$$y_i^* = y_i^*(p_i, p_{-i}) = \sqrt{\frac{2K_i(g_i + h_i)}{h_i g_i} b_i(p_i, p_{-i})}, \tag{26}$$

$$S_i^* = S_i^*(p_i, p_{-i}) = \sqrt{\frac{2K_i g_i}{h_i(g_i + h_i)} b_i(p_i, p_{-i})}, \tag{27}$$

$$i = 1, ..., n.$$

4.3. Nahs equilibrium in external problem

Substituting internal strategies (26), (27) in (24) we can find payoff functions of players in external game

$$\Pi_i(p_i, p_{-i}, y_i^*(p_i, p_{-i}), S_i^*(p_i, p_{-i})) = \Phi_i(p_i, p_{-i}) =$$

$$= p_i D_i(p_i, p_{-i}) - \xi_i \frac{D_i(p_i, p_{-i})}{\sqrt{b_i(p_i, p_{-i})}} - D_i(p_i, p_{-i}) c_i, \tag{28}$$

where

$$\xi_i = \sqrt{\frac{2K_i g_i h_i}{h_i + g_i}},\ i = 1, ..., n.$$

Notice that functions (28) depend only on external strategies of players $(p_i, p_{-i}) \in \Omega^{(1)} = \Omega_1^{(1)} \times \Omega_2^{(1)} \times ... \times \Omega_N^{(1)} \subset R_+^n$. Thus we are getting modified Bertrand model as non-cooperative game

$$\Gamma_B = \left\langle N, \{\Phi_i\}_{i=1}^n, \left\{\Omega_i^{(1)}\right\}_{i=1}^n \right\rangle. \tag{29}$$

To calculate Nash equilibrium in this game we can be guided by the following theorem

Theorem 3. *Suppose the following conditions hold for $i = 1, ..., n$:*

1) demand function $D_i(p_i, p_{-i})$ is differentiable, decreasing, convex with respect to p_i on the set $\Omega^{(1)}$ for any fixed admissible p_{-i} and continuous on the set $\Omega^{(1)}$;

2) function $D_i(p_i, p_{-i})/\sqrt{b_i(p_i, p_{-i})}$ is convex with respect to p_i on the set $\Omega_i^{(1)}$ for any fixed admissible p_{-i} and continuous on the set $\Omega^{(1)}$;

3) function $p_i D_i(p_i, p_{-i})$ is cocave with respect to p_i on the set $\Omega_i^{(1)}$ for any fixed admissible p_{-i}.

4) if for each $i = 1, ..., n$ there exists $\bar{p}_i \in \left(a_i^{(1)}, b_i^{(1)}\right)$ such that the following inequality holds

$$D_i(p_i, p_{-i}) < \xi_i \frac{2\frac{\partial D_i(p_i, p_{-i})}{\partial p_i} b_i(p_i, p_{-i}) - D_i(p_i, p_{-i}) \frac{\partial b_i(p_i, p_{-i})}{\partial p_i}}{2 b_i^{3/2}(p_i, p_{-i})} + \frac{\partial D_i(p_i, p_{-i})}{\partial p_i} c_i \tag{30}$$

Then in game (29) there exists Nash equilibrium in pure strategies $(p_1^*, p_2^*, \ldots, p_N^*)$ and $p_i^* \in [a_i^{(1)}, \underset{i}{\widetilde{p}})$, $i = 1, ..., n.$.

As in Theorem 2 the proof of Theorem 3 is based on the first-order (4) and second-order (5) conditions.

Nash equilibrium $(p_1^*, p_2^*, \ldots, p_N^*)$ can be found as a solution of the following system

$$\frac{\partial D_i(p_i, p_{-i})}{\partial p_i} b_i(p_i, p_{-i}) + D_i(p_i, p_{-i}) =$$

$$= \xi_i \frac{2\frac{\partial D_i(p_i,p_{-i})}{\partial p_i} b_i(p_i, p_{-i}) - D_i(p_i, p_{-i}) \frac{\partial b_i(p_i,p_{-i})}{\partial p_i}}{2b_i^{3/2}(p_i, p_{-i})} + \frac{\partial D_i(p_i, p_{-i})}{\partial p_i} c_i, \ \ i = 1, ..., n. \tag{31}$$

In the case when period of planning is constant, that is $T_i = \frac{D_i(p_i,p_{-i})}{b_i(p_i,p_{-i})}$ =const , we have

$$\frac{D_i(p_i, p_{-i})}{\sqrt{b_i(p_i, p_{-i})}} = \frac{D_i(p_i, p_{-i})}{\sqrt{D_i(p_i, p_{-i})/T_i}} = \sqrt{T_i D_i(p_i, p_{-i})},$$

and profit function (28) will be in the form

$$\Phi_i(p_i, p_{-i}) = p_i D_i(p_i, p_{-i}) - \rho_i \sqrt{D_i(p_i, p_{-i})} - D_i(p_i, p_{-i}) c_i, \tag{32}$$

where

$$\rho_i = \sqrt{\frac{2T_i K_i g_i h_i}{h_i + g_i}}, \ \ i = 1, ..., n.$$

Then Nash equilibrium $(p_1^*, p_2^*, \ldots, p_n^*)$ could be defined as a solution of the system

$$\frac{\partial D_i(p_i, p_{-i})}{\partial p_i} b_i(p_i, p_{-i}) + D_i(p_i, p_{-i}) = \rho_i \frac{\partial D_i(p_i, p_{-i})}{2\partial p_i} \frac{1}{\sqrt{D_i(p_i, p_{-i})}} + \frac{\partial D_i(p_i, p_{-i})}{\partial p_i} c_i, \tag{33}$$

$$i = 1, ..., n$$

if there exists $\bar{p}_i \in \left(a_i^{(1)}, b_i^{(1)}\right)$, $i = 1, ..., n$ such that

$$D_i(p_i, p_{-i}) < \rho_i \frac{\partial D_i(p_i, p_{-i})}{2\partial p_i} \frac{1}{\sqrt{D_i(p_i, p_{-i})}} + \frac{\partial D_i(p_i, p_{-i})}{\partial p_i} c_i$$

for $p_i > \bar{p}_i$. Moreover $p_i^* \in [a_i^{(1)}, \bar{p}_i)$, $\forall \ i$.

Internal strategies of players as optimal reaction to equilibrium external strategies $(p_1^*, p_2^*, \ldots, p_n^*)$ by the formulas

$$y_i^* = y_i^*(p_i^*, p_{-i}^*) = \sqrt{\frac{2K_i(g_i + h_i) b_i(p_i^*, p_{-i}^*)}{h_i g_i}}, \tag{34}$$

$$S_i^* = S_i^*(p_i^*, p_{-i}^*) = \sqrt{\frac{2K_i g_i b_i(p_i^*, p_{-i}^*)}{h_i(g_i + h_i)}}, \tag{35}$$

$$i = 1, ..., n.$$

5. Example of inventory system in case of price competition

Suppose that demand function $D_i(p_i, p_{-i})$ and demand per inventory circle function $b_i(p_i, p_{-i})$ for each distributor i has the same properties of depending on external strategies. Analogously to (Jean Tirole, 2000) we introduce these functions as follows

$$D_i(p_i, p_{-i}) = d_i \frac{p_1^{\beta_{i1}} p_2^{\beta_{i2}} \dots p_{i-1}^{\beta_{i,i-1}} p_{i+1}^{\beta_{i,i+1}} \dots p_n^{\beta_{in}}}{p_i^{1+\alpha_i}}, \tag{36}$$

$$b_i(p_i, p_{-i}) = e_i \frac{p_1^{\beta_{i1}} p_2^{\beta_{i2}} \dots p_{i-1}^{\beta_{i,i-1}} p_{i+1}^{\beta_{i,i+1}} \dots p_n^{\beta n}}{p_i^{1+\alpha_i}}, \tag{37}$$

where d_i and e_i – a positive constants , $\alpha_i > \beta_{ij} > 0 \ \forall \ j \neq i, \, i = 1, ..., n$.

Elasticity of distributor's demand function $D_i(p_i, p_{-i})$with respect to his price is negative $\varepsilon_{ii} = -1 - \alpha_i < 0$, $i = 1, ..., n$, and with respect to competitors prices are positive and the following inequalities hold $\varepsilon_{ij} = \beta_{ij} > 0 . \forall \ j \neq i, \, i = 1, ..., n$

So demand is uniform during the period of planning and a circle as well we notice that period of planning is equal to $D_i(p_i, p_{-i})/b_i(p_i, p_{-i}) = d_i/e_i$.

Let us consider internal and external problems in this case.

Internal problem. We can rewrite formulas for y_i and S_i in the form

$$y_i^*(p_i, p_{-i}) = \sqrt{\frac{2K_i(g_i + h_i) e_i p_1^{\beta_{i1}} p_2^{\beta_{i2}} \dots p_{i-1}^{\beta_{i,i-1}} p_{i+1}^{\beta_{i,i+1}} \dots p_n^{\beta n}}{h_i g_i p_i^{1+\alpha_i}}},$$

$$S_i^*(p_i, p_{-i}) = \sqrt{\frac{2K_i g_i e_i p_1^{\beta_{i1}} p_2^{\beta_{i2}} \dots p_{i-1}^{\beta_{i,i-1}} p_{i+1}^{\beta_{i,i+1}} \dots p_n^{\beta n}}{h_i(g_i + h_i) p_i^{1+\alpha_i}}}.$$

Denote $\gamma_i(p_{-i}) = p_1^{\beta_{i1}} p_2^{\beta_{i2}} \dots p_{i-1}^{\beta_{i,i-1}} p_{i+1}^{\beta_{i,i+1}} \dots p_n^{\beta n}$. Than we get for $i = 1, ..., n$

$$y_i^*(p_i, p_{-i}) = \sqrt{\frac{2K_i(g_i + h_i) e_i \gamma_i(p_{-i})}{h_i g_i p_i^{1+\alpha_i}}}, \tag{38}$$

$$S_i^*(p_i, p_{-i}) = \sqrt{\frac{2K_i g_i e_i \gamma_i(p_{-i})}{h_i(g_i + h_i) p_i^{1+\alpha_i}}}. \tag{39}$$

External problem. We substitute internal strategies (38), (39) in (32). Due to (36) and (37) we get

$$\Phi_i(p_i, p_{-i}) = \frac{d_i \gamma_i(p_{-i})}{p_i^{\alpha_i}} - \frac{d_i}{p_i^{(1+\alpha_i)/2}} \sqrt{\frac{2K_i g_i h_i \gamma_i(p_{-i})}{e_i(g_i + h_i)}} - \frac{d_i c_i \gamma_i(p_{-i})}{p_i^{1+\alpha_i}}. \tag{40}$$

Thus we state model of Bertrand for our case as non-cooperative game

$$\Gamma_K^1 = \left\langle N, \{\Omega_i^{(1)}\}_{i=1}^n, \{\Phi_i\}_{i=1}^n \right\rangle, \tag{41}$$

where $\Omega_i^{(1)}$ – the set of external strategies of player i, $\Omega_i^{(1)} = \{p_i | a_i^{(1)} \leq p_i \leq b_i^{(1)}\}$, $i = 1, ..., n$.

One can notice that for our case the following corollary takes place

Corollary 1. *For any $\alpha_i > 0$, $i = 1, ..., n$ in game Γ_K^1 (41) there exists unique Nash equilibrium in pure strategies. If $0 < \alpha_i < 1$, $i = 1, ..., n$, then payoff functions on equilibrium strategies has positive values.*

Solving of the problem . It is easy to see that second-order condition for function (40) is fulfilled $\frac{\partial^2 \Phi_i(p_i, p_{-i})}{\partial p_i^2} \leq 0$ and from the first order condition (4) we get the following system

$$\frac{\partial \Phi_i(p_i, p_{-i})}{\partial p_i} = -\frac{\alpha_i d_i \gamma_i(p_{-i})}{p_i^{1+\alpha_i}} + \frac{d_i(1+\alpha_i)}{2}\sqrt{\frac{2K_i g_i h_i \gamma_i(p_{-i})}{e_i(g_i + h_i)}}\frac{1}{p_i^{(3+\alpha_i)/2}} +$$

$$+\frac{(1+\alpha_i) d_i \gamma_i(p_{-i}) c_i}{p_i^{2+\alpha_i}} = 0, \ \ i = 1, ..., n. \tag{42}$$

It can also be written in the form

$$-\alpha_i p_i + \frac{1+\alpha_i}{2}\sqrt{\frac{2K_i g_i h_i}{e_i(g_i + h_i)\gamma_i(p_{-i})}} p_i^{(1+\alpha_i)/2} + (1+\alpha_i) c_i = 0, \ \ i = 1, ..., n. \tag{43}$$

Denoting

$$\xi_i = \frac{1+\alpha_i}{2}\sqrt{\frac{2K_i g_i h_i}{e_i(g_i + h_i)}}, \tag{44}$$

we get system

$$\xi_i p_i^{\frac{1+\alpha_i}{2}} = \sqrt{p_1^{\beta_{i1}} p_2^{\beta_{i2}} \dots p_{i-1}^{\beta_{i-1,1}} p_{i+1}^{\beta_{i+1,1}} \dots p_n^{\beta_{in}}} \left(\alpha_i p_i - (1+\alpha_i) c_i\right), \ \ i = 1, ..., n.$$

This system can be only solved by numerical methods.

Let us consider game of two players i.e. $n = 2$. For this case we will have the following system to find Nash equilibrium

$$\begin{aligned} \xi_1 p_1^{\frac{\alpha_1+1}{2}} - \sqrt{p_2^{\beta_{12}}} \left(\alpha_1 p_1 - (\alpha_1 + 1) c_1\right) = 0, \\ \xi_2 p_2^{\frac{\alpha_2+1}{2}} - \sqrt{p_1^{\beta_{21}}} \left(\alpha_2 p_2 - (\alpha_2 + 1) c_2\right) = 0. \end{aligned} \tag{45}$$

Let parameters of the model are as follows
$K_1 = 400$ USD, $c_1 = 10$ USD, $h_1 = 10$ USD/h, $d_1 = 100000$, $e_1 = 10000$, $g_1 = 5$ USD/h, $\alpha_1 = 1/2$, $\beta_{12} = 1/4$, $K_2 = 400$ USD, $c_2 = 8$ USD, $h_2 = 8$ USD/h, $d_2 = 100000$, $e_2 = 10000$, $g_2 = 6$ USD/h, $\alpha_2 = 1/2$, $\beta_{21} = 1/4$.

System (45) will be of the form

$$\begin{aligned} 0,3872983344 p_1^{\frac{3}{4}} - p_2^{\frac{1}{8}} \left(\tfrac{1}{2} p_1 - 15\right) = 0, \\ 0,3927922024 p_2^{\frac{3}{4}} - p_1^{\frac{1}{8}} \left(\tfrac{1}{2} p_2 - 12\right) = 0. \end{aligned}$$

With the solution $p_1^* = 37,68643585$ USD and $p_2^* = 30,47300925$ USD. These prices are values of external strategies in Nash equilibrium. Profit of distributors for this situation are $\Phi_1(p_1^*, p_2^*) = 26744,43112$ and $\Phi_2(p_1^*, p_2^*) = 22422,48488$.

Using (38) and (39) we also calculate internal strategies y_1^*, S_1^* ? y_2^*, S_2^*: $y_1^* \approx 156$, $S_1^* \approx 52$ and $y_2^* \approx 185$, $S_2^* \approx 79$.

6. Conclusion

In this paper we have discussed two models of control of inventory systems in case of quantitative and price competition.

For each model necessary and sufficient conditions for existence of Nash equilibrium in pure strategies are proposed. Methods for finding Nash equilibrium in pure strategies are discussed.

Using these methods it is possible to get analytical expressions for internal and external strategies in deterministic models. When it is not the case we can find solutions numerically.

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Stochastic Coalitional Games with Constant Matrix of Transition Probabilties

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Abstract. The stochastic game Γ under consideration is repetition of the same stage game G which is played on each stage with different coalitional partitions. The transition probabilities over the coalitional structures of stage game depends on the initial stage game G in game Γ. The payoffs in stage games (which is a simultaneous game with a given coalitional structure) are computed as components of the generalized PMS-vector (see (Grigorieva and Mamkina, 2009), (Petrosjan and Mamkina, 2006)). The total payoff of each player in game Γ is equal to the mathematical expectation of payoffs in different stage games G (mathematical expectation of the components of PMS-vector). The concept of solution for such class of stochastic game is proposed and the existence of this solution is proved. The theory is illustrated by 3-person 3-stage stochastic game with changing coalitional structure.

Keywords: stochastic games, coalitional partition, Nash equilibrium, Shapley value, PMS-vector.

1. Introduction.

In the papers (Grigorieva, 2010) a class of multistage stochastic games with different coalitional partitions where the transition probability from some coalitional game to another depends from coalitional partition in the initial game and from the n-tuple of strategies which realizes in initial game is examined. A new mathematical method for solving stochastic coalitional games, based on constructing Nash equilibrium (NE) in a stochastic game similarly scheme of constructing of absolute NE in a multistage game with perfect information ((Zenkevich et al., 2009), (Petrosjan et al., 1998)), and based on calculation of the generalized PMS-value introduced in (Grigorieva and Mamkina, 2009), (Petrosjan and Mamkina, 2006), for the first time, is proposed along with the proof of the solution existence. In this paper transition probability from some coalitional game to another depends only from coalitional partition in the initial game. So the matrix transition probabilities which is constant during of whole multistage game is form.

The theory is illustrated by 3-person 3-stage stochastic game with changing coalitional structure.

Remind that *coalitional game* is a game where players are united in fixed coalitions in order to obtain the maximum possible payoff, and *stochastic game* is a multistage game with random transitions from state to state, which is played by one or more players.

2. State of problem.

Suppose finite graph tree $\Gamma = (Z,\ L)$, where Z is the set of vertices in the graph and L is point-to-set mapping, defined on the set Z: $L(z) \subset Z,\ z \in Z$. Finite graph tree with the initial vertex z_0 will be denoted by $\Gamma(z_0)$.

In each vertex $z \in Z$ of the graph $\Gamma\ (z_0)$ simultaneous N-person game is defined in a normal form

$$G(z) = \langle N, X_1, ..., X_n,\ K_1, ..., K_n \rangle ,$$

where

- $N = \{1, \ldots, n\}$ is the set of players identical for all vertices $z \in Z$;
- X_j is the set of pure strategies x_j^z of player $j \in N$, identical for all vertices $z \in Z$;
  $-x^z = (x_1^z, ..., x_n^z)$, $x_j^z \in X_j$, $j = \overline{1, n}$**, is the n-tuple of pure strategies in the game $G(z)$ at vertex $z \in Z$;**
  $-\ \mu^z = (\mu_1^z, ..., \mu_n^z)$, $j = \overline{1, n}$**, is the n-tuple of mixed strategies in the game $G(z)$ in mixed strategies at vertex $z \in Z$;**
- $K_j(x^z)$, **is the payoff function of the player j identical for all vertices $z \in Z$; it is supposed that $K_j(x^z) \geq 0\ \forall\, x^z \in X$ and $\forall\ j \in N$.**

Furthermore, let in each vertex $z \in Z$ of the graph $\Gamma\ (z_0)$ the coalitional partition of the set N be defined

$$\Sigma_z = \{S_1, \ldots, S_l\}\ ,\ \ l \leq n\ ,\ \ S_i \cap S_j = \emptyset\ \ \forall\ \ i \neq j\ ,\ \ \bigcup_{i=1}^{l} S_i = N\ ,$$

i. e. the set of players N is divided into l coalitions each acting as one player. Coalitional partitions can be different for different vertices z.

Then in each vertex $z \in Z$ we have the *simultaneous l-person coalitional game in a normal form associating with the game $G(z)$*

$$G(z\ ,\ \Sigma_z) = \left\langle N, \tilde{X}_{S_1}^z, ..., \tilde{X}_{S_l}^z,\ H_{S_1}^z, ..., H_{S_l}^z \right\rangle ,$$

where

- $\tilde{X}_{S_i}^z = \prod_{j \in S_i} X_j$ is the set of strategies $\tilde{x}_{S_i}^z$ of coalition $S_i, i = \overline{1, l}$, where the strategy $\tilde{x}_{S_i}^z \in \tilde{X}_{S_i}^z$ of coalition S_i is n-tuple of strategies of players from coalition S_i, i. e. $\tilde{x}_{S_i}^z = \left\{\ x^j \in X_j \middle|\ j \in S_i \right\}$;
  $-\tilde{x}^z = \left(\tilde{x}_{S_1}^z, ..., \tilde{x}_{S_l}^z\right)$, $\tilde{x}_{S_i}^z \in \tilde{X}_{S_i}^z$, $i = \overline{1, l}$, **is n-tuple of strategies in the game $G(z\ ,\ \Sigma_z)$;**
  $-\tilde{\mu}^z = (\tilde{\mu}_1^z, ..., \tilde{\mu}_l^z)$, $i = \overline{1, l}$, **is n-tuple of mixed strategies in the game $G(z)$ in mixed strategies at the vertex $z \in Z$; however notice that $\mu^z \neq \tilde{\mu}^z$;**
- **the payoff of coalition S_i is defined as a sum of payoffs of players from coalition S_i, i. e.**

$$H_{S_i}^z(\tilde{x}^z) = \sum_{j \in S_i} K_j(x)\ ,\ \ i = \overline{1, l}\,. \tag{1}$$

For each vertex $z \in Z$ of the graph $\Gamma\,(z_0)$ by matrix of transition probabilities the probabilities $p\,(z,\ y)$ of transition to the next vertices $y \in L\,(z)$ of the graph $\Gamma\,(z_0)$ are defined:

$$p\,(z,\ y) \geq 0\,,\ \sum_{y \in L(z)} p\,(z,\ y) = 1\,.$$

Definition 1. The game defined on the finite graph tree $\Gamma\,(z_0)$ with initial vertex z_0 is called the finite step coalitional stochastic game $\tilde{\Gamma}\,(z_0)$ with constant matrix of transition probabilities:

$$\tilde{\Gamma}\,(z_0) = \left\langle N,\ \Gamma\,(z_0)\ ,\ \{G\,(z,\ \Sigma_z)\}_{z\in Z}\,,\ \{p\,(z,\ y)\}_{z\in Z\,,\ y\in L(z)}\,,\ k_{\tilde{\Gamma}} \right\rangle\,,$$

where

- $N = \{1,\ \ldots,\ n\}$ is the set of players identical for all vertices $z \in Z$;
- $\Gamma\,(z_0)$ is the graph tree with initial vertex z_0;
- $\{G\,(z,\ \Sigma_z)\}_{z\in Z}$ is the simultaneous coalitional l-person game defined in a normal form in each vertex $z \in Z$ of the graph $\Gamma\,(z_0)$;
- $\{p\,(z,\ y)\}_{z\in Z\,, y\in L(z)\,,}$ is the realization probability of the coalitional game $G\,(y\,,\ \Sigma_y)$ at the vertex $y \in L\,(z)$ under condition that the simultaneous game $G\,(z\,,\ \Sigma_z)$ was realized at the previous step at vertex z;
- $k_{\tilde{\Gamma}}$ is the finite and fixed number of steps in the stochastic game $\tilde{\Gamma}\,(z_0)$; the step $k\,,\ k = \overline{0,\ k_{\tilde{\Gamma}}}$ at the vertex $z_k \in Z$ is defined according to the condition of $z_k \in (L\,(z_0))^k$, i. e. the vertex z_k is reached from the vertex z_0 in k stages.

States in the multistage stochastic game $\tilde{\Gamma}$ are vertices of graph tree $z \in Z$ with the defined coalitional partitions in each vertex Σ_z, i. e. pair $(z\,,\ \Sigma_z)$. Game $\tilde{\Gamma}$ is stochastic, because transition from state $(z\,,\ \Sigma_z)$ to state $(y\,,\ \Sigma_y)$, $y \in L\,(z)$, is defined by the given probability $p\,(z,\ y)$.

Multistage stochastic coalitional game $\tilde{\Gamma}\,(z_0)$ is realized as follows. At moment t_0 the game $\tilde{\Gamma}\,(z_0)$ starts at the vertex z_0, where the game $G\,(z_0\,,\ \Sigma_{z_0})$ with a certain coalitional partition Σ_{z_0} is realized. Players choose their strategies, thus n-tuple of strategies x^{z_0} is formed. Then on the next stage with given probabilities $p\,(z_0,\ z_1)$ the transition from vertex z_0 on the graph tree $\Gamma\,(z_0)$ to the game $G\,(z_1\,,\ \Sigma_{z_1})\ ,\ z_1 \in L\,(z_0)$ is realized. In the game $G\,(z_1\,,\ \Sigma_{z_1})$ players choose their strategies again, n-tuple of strategies x^{z_1} is formed. Then from vertex $z_1 \in L\,(z_0)$ the transition to the vertex $z_2 \in (L\,(z_0))^2$ is made, again n-tuple of strategies x^{z_2} is formed. This process continues until at the end of the game the vertex $z_{k_{\tilde{\Gamma}}} \in (L\,(z_0))^{k_{\tilde{\Gamma}}}\ ,\ \ L\left(z_{k_{\tilde{\Gamma}}}\right) = \emptyset$ is reached.

Denote by $\tilde{\Gamma}\,(z)$ the subgame of game $\tilde{\Gamma}\,(z_0)$, starting at the vertex $z \in Z$ of the graph $\Gamma\,(z_0)$, i. e. at coalitional game $G\,(z\,,\ \Sigma_z)$. Obviously the subgame $\tilde{\Gamma}\,(z)$ is a stochastic game as well.

Denote by:

- $u_j^z\,(\cdot)$ the strategy of player $j,\ j = \overline{1,\ n}$, in the subgame $\tilde{\Gamma}\,(z)$, which to each vertex $y \in Z$ assigns the strategy x_j^y of player j in each simultaneous game $G\,(y,\ \Sigma_y)$ at all vertices $y \in \Gamma\,(z)$, i. e.

$$u_j^z\,(y) = \left\{\ x_j^y\ \middle|\ \ y \in \Gamma\,(z)\ \right\};$$

– $u^z_{S_i}(\cdot)$ the strategy of coalition S_i in the subgame $\tilde{\Gamma}(z)$, which is a set of strategies $u^z_j(\cdot)\,,\ j \in S_i$;
– $u^z(\cdot) = (u^z_1(\cdot)\,, \ldots, u^z_n(\cdot)) = \left(u^z_{S_1}(\cdot)\,, \ldots, u^z_{S_n}(\cdot)\right)$ the n-tuple in the game $\tilde{\Gamma}(z)$.

It's easy to show that the payoff $E^z_j(u^z(\cdot))$ of player $j,\ j = \overline{1,\, n}$, in any game $\tilde{\Gamma}(z)$ is defined as the mathematical expectation of payoffs of player j in all its subgames, i. e. by the following formula (Zenkevich et al., 2009, p. 158):

$$E^z_j(u^z(\cdot)) = K_j(x^z) + \sum_{y \in L(z)} \left[p(z,\, y)\ E^y_j(u^y(\cdot))\right]\,. \tag{2}$$

Thus, a coalitional stochastic game $\tilde{\Gamma}(z_0)$ with constant matrix of transition probabilities can be written as a game in normal form

$$\tilde{\Gamma}(z_0) =$$
$$= \left\langle N,\, \Gamma(z_0)\,,\, \{G(z,\, \Sigma_z)\}_{z \in Z}\,,\, \{p(z,\, y)\}_{\substack{z \in Z,\\ y \in L(z)}}\,,\, \left\{U^z_j\right\}_{j=\overline{1,n}}\,,\, \left\{E^z_j\right\}_{j=\overline{1,n}}\,, k_{\tilde{\Gamma}} \right\rangle,$$

where U^z_j is the set of the strategies $u^z_j(\cdot)$ of the player $j,\ j = \overline{1,\, n}$.

The payoff $H^z_{S_i}(x^z)$ of coalition $S_i \in \Sigma_z,\ i = \overline{1,\, l}$, in each coalitional game $G(z\,,\ \Sigma_z)$ at the vertex $z \in Z$ is defined as the sum of payoffs of players from the coalition S_i, see formula (1):

$$H^z_{S_i}(x^z) = \sum_{j \in S_i} K_j(x^z)\,.$$

The payoff $H^z_{S_i}(u^z(\cdot))\,,\ S_i \in \Sigma_z,\ i = \overline{1,\, l}$, in the subgame $\tilde{\Gamma}(z)$ of the game $\tilde{\Gamma}(z_0)$ at the vertex $z \in Z$ is defined as the sum of payoffs of players from the coalition S_i in the subgame $\tilde{\Gamma}(z)$ at the vertex $z \in Z$:

$$H^z_{S_i}(u^z(\cdot)) = \sum_{j \in S_i} E^z_j(u^z(\cdot)) =$$

$$= \sum_{j \in S_i} \left\{ K_j(x^z) + \sum_{y \in L(z)} \left[p(z,\, y)\ E^y_j(u^y(\cdot))\right] \right\} =$$

$$= \sum_{j \in S_i} K_j(x^z) + \sum_{j \in S_i} \left\{ \sum_{y \in L(z)} \left[p(z,\, y)\ E^y_j(u^y(\cdot))\right] \right\} =$$

$$= \sum_{j \in S_i} K_j(x^z) + \sum_{y \in L(z)} \left[p(z,\, y) \sum_{j \in S_i} E^y_j(u^y(\cdot)) \right] =$$

$$= H^z_{S_i}(x^z) + \sum_{y \in L(z)} \left[p(z,\, y)\, H^y_{S_i}(u^y(\cdot))\right]. \tag{3}$$

It's clear, that in any vertex $z \in Z$ under the coalitional partition Σ_z the game $\tilde{\Gamma}(z)$ with payoffs E_j^z of players $j = \overline{1,\, n}$ defined by (2), is a non-coalitional game between coalitions with payoffs $H_{S_i}^z\left(u^z(\cdot)\right)$ defined by (3). For finite non-coalitional games the existence of the NE (Petrosjan et al., 1998, p. 137) in mixed strategies is proved.

However, as the payoffs of players $j,\ j = \overline{1,\, n}$, are not partitioned from the payoff of coalition in the subgame $\tilde{\Gamma}(z)$, it may occur at the next step in the subgame $\tilde{\Gamma}(y)$, $y \in L(z)$, with another coalitional partition at the vertex y, the choice of player j will be not trivial and will be different from the corresponding choice of equilibrium strategy $\bar{u}_j^z(\cdot)$ in the subgame $\tilde{\Gamma}(z)$.

3. Nash Equilibrium in a multistage stochastic game with constant matrix of transition probabilities

Remind the algorithm of constructing the generalized PMS-value in a coalitional game. Calculate the values of payoff $H_{S_i}^z(x^z)$ for all coalitions $S_i \in \Sigma_z,\ i = \overline{1,\, l}$, for each coalitional game $G(z,\ \Sigma_z)$ by formula (1):

$$H_{S_i}^z(x^z) = \sum_{j \in S_i} K_j(x^z)\ .$$

In the game $G(z,\ \Sigma_z)$ find n-tuple NE $\bar{x}^z = \left(\bar{x}_{S_1}^z, ..., \bar{x}_{S_l}^z\right)$ or $\bar{\mu}^z = \left(\bar{\mu}_{S_1}^z, ..., \bar{\mu}_{S_l}^z\right)$.

In case of $l = 1$ the problem is the problem of finding the maximal total payoff of players from the coalition S_1, in case of $l = 2$ it is the problem of finding of NE in bimatrix game, in other cases it is the problem of finding NE n-tuple in a non-coalitional game. In the case of multiple NE (Nash, 1951) the solution of the corresponding coalitional game will be not unique.

The payoff of each coalition in NE n-tuple $H_{S_i}^z(\bar{\mu}^z)$ is divided according to Shapley's value (Shapley, 1953) $Sh(S_i) = (Sh(S_i : 1), \ldots, Sh(S_i : s))$:

$$Sh(S_i : j) = \sum_{\substack{S' \subset S_i \\ S' \ni j}} \frac{(s'-1)\,!\,(s-s')\,!}{s\,!}\left[v(S') - v(S' \setminus \{j\})\right]\ \ \forall\, j = \overline{1,\, s}\,, \tag{4}$$

where $s = |S_i|\ \ (s' = |S'|)$ is the number of elements of set $S_i\ \ (S')$ and $v(S')$ is the total maximal guaranteed payoff of subcoalition $S' \subset S_i$. We have

$$v(S_i) = \sum_{j=1}^{s} Sh(S_i : j)\,.$$

Then PMS-vector in the NE in mixed strategies in the game $G(z,\ \Sigma_z)$ is defined as

$$\text{PMS}(\bar{\mu}^z) = \left(\text{PMS}_1(\bar{\mu}^z)\ , ..., \text{PMS}_n(\bar{\mu}^z)\right)\ ,$$

where

$$\text{PMS}_j(\bar{\mu}^z) = Sh(S_i : j)\,, j \in S_i,\ i = \overline{1, l}.$$

Remark. If the calculation of PMS-vector is difficult, then any other "optimal" solution can be proposed to be used as a PMS-solution, for example, Pareto-optimality or Nash arbitration scheme (Grigorieva, 2009).

We apply the known algorithm of constructing NE n-tuple in a stochastic coalitional game to the stochastic coalitional game $\tilde{\Gamma}(z_0)$ with constant matrix of transition probabilities (Grigorieva, 2010).

4. Examples.

Example 1. Let there be 3 players in the game each having 2 strategies, and let payoffs of each player be defined, see table 1. Consider all possible combinations of coalitional partitions, cooperative and non-coalitional cases.

Table 1.

The strategies			The payoffs			The payoffs of coalition			
I	II	III	I	II	III	(I, II)	(II, III)	(I, III)	(I, II, III)
1	1	1	4	2	1	6	3	5	7
1	1	2	1	2	2	3	4	3	5
1	2	1	3	1	5	4	6	8	9
1	2	2	5	1	3	6	4	8	9
2	1	1	5	3	1	8	4	6	9
2	1	2	1	2	2	3	4	3	5
2	2	1	0	4	3	4	7	3	7
2	2	2	0	4	2	4	6	2	6

1. Solve coalitional game $G(\Sigma_1)$, $\Sigma_1 = \{S = \{\mathrm{I,II}\},\ N\backslash S = \{\mathrm{III}\}\}$, by calculating PMS-value (Grigorieva and Mamkina, 2009) as follows.

1.1. Find NE in mixed strategies in the bimatrix game:

$$\begin{array}{ccccc} & & \eta = 3/7 & 1-\eta = 4/7 \\ & & 1 & 2 \\ 0 & (1,\ 1) & [6,\ 1] & [3,\ 2] \\ 0 & (2,\ 2) & [4,\ 3] & [4,\ 2] \\ \xi = 1/3 & (1,\ 2) & [4,\ 5] & [6,\ 3] \\ 1-\xi = 2/3 & (2,\ 1) & [8,\ 1] & [3,\ 2]. \end{array}$$

First and second rows are dominated by the last and third ones correspondingly. Find n-tuple of NE in the mixed strategies in the bimatrix game

$$\bar{\mu}^2 = \begin{pmatrix} 3/7 \ 4/7 \end{pmatrix},\ \bar{\mu}^1 = \begin{pmatrix} 0\ 0\ 1/3\ 2/3 \end{pmatrix},$$

by using the theorem about complete mixed equilibrium [(Petrosjan et al., 1998), p. 135].

Realization of payoffs of coalitions S and $N\backslash S$ in mixed strategies take place with follows probabilities:

$$\begin{array}{ccc} & \eta_1 & \eta_2 \\ \xi_1 & 0 & 0 \\ \xi_2 & 0 & 0 \\ \xi_3 & 1/7 & 4/21 \\ \xi_4 & 2/7 & 8/21 \end{array}.$$

Calculate mathematical expectation of payoffs in NE in mixed strategies:

$$E\left(\bar{\mu}^1,\ \bar{\mu}^2\right) = \frac{1}{7}[4,\ 5] + \frac{2}{7}[8,\ 1] + \frac{4}{21}[6,\ 3] + \frac{8}{21}[3,\ 2] = \left[\frac{36}{7},\ \frac{7}{3}\right] = \left[5\frac{1}{7},\ 2\frac{1}{3}\right].$$

1.2. Find guaranteed payoffs $v\{\mathrm{I}\}$ and $v\{\mathrm{II}\}$ of players I and II, see table 2, as follows. Fix strategy of player III

$$\bar{\mu}^2 = \begin{pmatrix} 3/7 \ 4/7 \end{pmatrix}.$$

Then mathematical expectation of payoffs of players of coalition S under fix strategy of coalition $N\backslash S$ looks as:

$$
\begin{aligned}
E_{S(1,\,1)}\left(\bar{\mu}^2\right) &= \left(\tfrac{3}{7}\cdot 4+\tfrac{4}{7}\cdot 1;\ \tfrac{3}{7}\cdot 2+\tfrac{4}{7}\cdot 2\right) = \left(2\tfrac{2}{7};\, 2\right);\\
E_{S(1,\,2)}\left(\bar{\mu}^2\right) &= \left(\tfrac{3}{7}\cdot 3+\tfrac{4}{7}\cdot 5;\ \tfrac{3}{7}\cdot 1+\tfrac{4}{7}\cdot 1\right) = \left(4\tfrac{1}{7};\, 1\right);\\
E_{S(2,\,1)}\left(\bar{\mu}^2\right) &= \left(\tfrac{3}{7}\cdot 5+\tfrac{4}{7}\cdot 1;\ \tfrac{3}{7}\cdot 3+\tfrac{4}{7}\cdot 2\right) = \left(2\tfrac{5}{7};\, 2\tfrac{3}{7}\right);\\
E_{S(2,\,2)}\left(\bar{\mu}^2\right) &= \left(\tfrac{3}{7}\cdot 0+\tfrac{4}{7}\cdot 0;\ \tfrac{3}{7}\cdot 4+\tfrac{4}{7}\cdot 4\right) = (0;\, 4)\,.
\end{aligned}
$$

Hence, guaranteed payoffs are calculated as follows:

$$
\left.\begin{aligned}
\min H_1\left(x_1=1,\, x_2,\, \bar{\mu}^2\right) &= \min\left\{2\tfrac{2}{7};\, 4\tfrac{1}{7}\right\} = 2\tfrac{2}{7};\\
\min H_1\left(x_1=2,\, x_2,\, \bar{\mu}^2\right) &= \min\left\{2\tfrac{5}{7};\, 0\right\} = 0;
\end{aligned}\right|\ v\{\mathrm{I}\} = \max\left\{2\tfrac{2}{7};\, 0\right\} = 2\tfrac{2}{7};
$$

$$
\left.\begin{aligned}
\min H_2\left(x_1,\, x_2=1,\, \bar{\mu}^2\right) &= \min\left\{2;\, 2\tfrac{3}{7}\right\} = 2;\\
\min H_2\left(x_1,\, x_2=2,\, \bar{\mu}^2\right) &= \min\{1;\, 4\} = 1;
\end{aligned}\right|\ v\{\mathrm{II}\} = \max\{2;\, 1\} = 2.
$$

Thus, guaranted payoffs equals: $v\{\mathrm{I}\} = 2\frac{2}{7}$, $v\{\mathrm{II}\} = 2$.

Table 2.

					The strategies of coalitions N\S, the payoffs of coalitions S					
				μ^2	0.43			0.57		
	Math.Expect.		μ^1		1		S	2		S
The strategies of coalition S	2.286	2.000	0.00	1, 1	4	2	6	1	2	3
	4.143	1.000	0.33	1, 2	3	1	4	5	1	6
	2.714	2.429	0.67	2, 1	5	3	8	1	2	3
	0.000	4.000	0.00	2, 2	0	4	4	0	4	4
	v1	v2			v1	v2		v1	v2	
	2.286	2.000		min 1	3	2		1	2	
	0.000	1.000		min 2	0	1		0	1	
	2.286	2		max	3	2		1	2	

1.3. Divide the payoff $E_1\left(\bar{\mu}^1,\, \bar{\mu}^2\right) = 5\frac{1}{7}$ according to the Shapley's value (Shapley, 1953):

$$
\begin{gathered}
Sh_1 = v\{\mathrm{I}\} + \tfrac{1}{2}\left(v\{\mathrm{I,II}\} - v\{\mathrm{II}\} - v\{\mathrm{I}\}\right) = 2\tfrac{2}{7} + \tfrac{1}{2}\left(5\tfrac{1}{7} - 2\tfrac{2}{7} - 2\right) = 2\tfrac{5}{7};\\
Sh_2 = v\{\mathrm{II}\} + \tfrac{1}{2}\left(v\{\mathrm{I,II}\} - v\{\mathrm{II}\} - v\{\mathrm{I}\}\right) = 2\tfrac{3}{7}.
\end{gathered}
$$

Then PMS-vector (Grigorieva and Mamkina, 2009) will be:

$$
\mathrm{PMS}_1 = 2\frac{5}{7};\ \ \mathrm{PMS}_2 = 2\frac{3}{7};\ \ \mathrm{PMS}_3 = 2\frac{1}{3}\,.
$$

2. Solve coalitional game $G\left(\Sigma_2\right)$, $\Sigma_2 = \left\{S = \{\mathrm{II,III}\},\ \ N\backslash S = \{\mathrm{I}\}\right\}$, by calculating PMS-vector (Grigorieva and Mamkina, 2009) as follows.

2.1. Find NE in mixed strategies in the bimatrix game:

$$\begin{array}{ccc} & & \eta = 1 & 1-\eta = 0 \\ & & 1 & 2 \\ 0 & (1,\,1) & [3,\,4] & [4,\,5] \\ 0 & (2,\,2) & [4,\,5] & [6,\,0] \\ \xi = 0 & (1,\,2) & [4,\,1] & [4,\,1] \\ 1-\xi = 1 & (2,\,1) & [6,\,3] & [7,\,0]\,. \end{array}$$

First three rows are dominated by the last, second column is dominated by first. Hence,

$$\bar{\mu}^2 = \begin{pmatrix} 1 & 0 \end{pmatrix},\ \bar{\mu}^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}\ ,$$

and vector of coalitional payoffs is $E\left(\bar{\mu}^1,\, \bar{\mu}^2\right) = [6,\, 3]$

2.2. Find guaranteed payoffs $v\,\{\mathrm{II}\} = 2$ and $v\,\{\mathrm{III}\} = 2$ of players II and III. Fix strategy of player I

$$\bar{\mu}^2 = \begin{pmatrix} 1 & 0 \end{pmatrix}\ .$$

Then:

$$\left.\begin{array}{l} \min H_2\left(\bar{\mu}^2,\, x_2 = 1,\, x_3\right) = \min\{2;\, 2\} = 2\,; \\ \min H_2\left(\bar{\mu}^2,\, x_2 = 2,\, x_3\right) = \min\{1;\, 1\} = 1; \\ \min H_3\left(\bar{\mu}^2,\, x_2,\, x_3 = 1\right) = \min\{1;\, 5\} = 1\,; \\ \min H_3\left(\bar{\mu}^2,\, x_2,\, x_3 = 2\right) = \min\{2;\, 3\} = 2; \end{array}\right| \begin{array}{l} v\,\{\mathrm{II}\} = \max\{2;\, 1\} = 2; \\ \\ v\,\{\mathrm{III}\} = \max\{1;\, 2\} = 2. \end{array}$$

Thus, guaranted payoffs are: $v\,\{\mathrm{II}\} = 2$, $v\,\{\mathrm{III}\} = 2$.

2.3. Divide the payoff $E_1\left(\bar{\mu}^1,\, \bar{\mu}^2\right) = 6$ according to the Shapley's value (Shapley, 1953):

$$\begin{array}{r} Sh_2 = v\,\{\mathrm{II}\} + \frac{1}{2}\left(v\,\{\mathrm{II,III}\} - v\,\{\mathrm{II}\} - v\,\{\mathrm{III}\}\right); \\ Sh_3 = v\,\{\mathrm{III}\} + \frac{1}{2}\left(v\,\{\mathrm{II,III}\} - v\,\{\mathrm{II}\} - v\,\{\mathrm{III}\}\right). \end{array}$$

Then PMS-vector ((Grigorieva and Mamkina, 2009), (Petrosjan and Mamkina, 2006)):

$$\mathrm{PMS}_1 = \mathrm{PMS}_2 = \mathrm{PMS}_3 = 3\,.$$

3. Solve coalitional game $G\left(\Sigma_3\right)$, $\Sigma_3 = \{S = \{\mathrm{I,III}\},\ N\backslash S = \{\mathrm{II}\}\}$, by calculating PMS-value (Grigorieva and Mamkina, 2009) as follows.

3.1. Find NE in mixed strategies in the bimatrix game:

$$\begin{array}{cccc} & & \eta = 5/6 & 1-\eta = 1/6 \\ & & 1 & 2 \\ \xi = 1/2 & (1,\,1) & [5,\,2] & [8,\,1] \\ 0 & (2,\,2) & [3,\,2] & [2,\,4] \\ 0 & (1,\,2) & [3,\,2] & [8,\,1] \\ 1-\xi = 1/2 & (2,\,1) & [6,\,3] & [3,\,4]\,. \end{array}$$

Second and third rows are dominated by first. Find NE in the mixed strategies in the bimatrix game

$$\bar{\mu}^2 = \begin{pmatrix} 5/6 & 1/6 \end{pmatrix},\ \bar{\mu}^1 = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \end{pmatrix}\ ,$$

vector of coalitional payoffs is

$$E\left(\bar{\mu}^1,\, \bar{\mu}^2\right) = \frac{5}{12}\,[5,\, 2] + \frac{1}{12}\,[8,\, 1] + \frac{5}{12}\,[6,\, 3] + \frac{1}{12}\,[3,\, 4] = \left[\frac{66}{12},\, \frac{30}{12}\right] = \left[5\frac{1}{2},\, 2\frac{1}{2}\right].$$

3.2. Fix strategy of player II $\bar{\mu}^2 = \begin{pmatrix} 5/6 \ 1/6 \end{pmatrix}$ and find guaranteed payoffs $v\{\mathrm{I}\} =$

$= 1.68$ and $v\{\mathrm{III}\} = 2$ of players I and III, see table 3.

Table 3.

The strategies of coalitions N\S, the payoffs of coalitions S

The strategies of coalition S	Math.Expect.		μ^1	μ^2	0.5			0.5		
					1		S	2		S
	3.83	1.68	0.5	1, 1	4	1	5	3	5	8
	1.68	2.17	0	1, 2	1	2	3	5	3	8
	4.15	1.34	0.5	2, 1	5	1	6	0	3	3
	0.83	2.00	0	2, 2	1	2	3	0	2	2
	v1	v3			v1	v3		v1	v3	
	1.68	1.34		min 1	1	1		3	3	
	0.83	2.00		min 2	1	2		0	2	
	1.68	2.00		max	1	2		3	3	

3.3. Divide the payoff $E_1\left(\bar{\mu}^1,\, \bar{\mu}^2\right) = 5.5$ according to the Shapley's value (Shapley, 1953):

$$Sh_1 = v\{\mathrm{I}\} + \tfrac{1}{2}\left(v\{\mathrm{I},\mathrm{III}\} - v\{\mathrm{I}\} - v\{\mathrm{III}\}\right);$$
$$Sh_3 = v\{\mathrm{III}\} + \tfrac{1}{2}\left(v\{\mathrm{I},\mathrm{III}\} - v\{\mathrm{I}\} - v\{\mathrm{III}\}\right).$$

Then PMS-vector in mixed strategies (Grigorieva and Mamkina, 2009):

$$\mathrm{PMS}_1 = 2.59;\ \ \mathrm{PMS}_2 = 2.5;\ \ \mathrm{PMS}_3 = 2.91\,.$$

4. Solve cooperative game $G\left(\Sigma_4\right)$, $\Sigma_4 = \{N = \{\mathrm{I}, \mathrm{II},\, \mathrm{III}\}\}$, see table 2. Find the maximum payoff of coalition N and divide it according to Shapley's value (Shapley, 1953):

$$Sh_1 = \frac{1}{6}\left[v\{\mathrm{I},\mathrm{II}\} + v\{\mathrm{I},\mathrm{III}\} - v\{\mathrm{II}\} - v\{\mathrm{III}\}\right] + \frac{1}{3}\left[v\{\mathrm{I},\mathrm{II},\mathrm{III}\} - v\{\mathrm{II},\mathrm{III}\} + v\{\mathrm{I}\}\right];$$

$$Sh_2 = \frac{1}{6}\left[v\{\mathrm{II},\mathrm{I}\} + v\{\mathrm{II},\mathrm{III}\} - v\{\mathrm{I}\} - v\{\mathrm{III}\}\right] + \frac{1}{3}\left[v\{\mathrm{I},\mathrm{II},\mathrm{III}\} - v\{\mathrm{I},\mathrm{III}\} + v\{\mathrm{II}\}\right];$$

$$Sh_3 = \frac{1}{6}\left[v\{\mathrm{III},\mathrm{I}\} + v\{\mathrm{III},\mathrm{II}\} - v\{\mathrm{I}\} - v\{\mathrm{II}\}\right] + \frac{1}{3}\left[v\{\mathrm{I},\mathrm{II},\mathrm{III}\} - v\{\mathrm{I},\mathrm{II}\} + v\{\mathrm{III}\}\right].$$

Find guaranteed payoffs:

$$v\{\mathrm{I},\,\mathrm{II}\} = \max\{4,\,3\} = 4\,,\ v\{\mathrm{I},\,\mathrm{III}\} = \max\{3,\,2\} = 3\,,\ v\{\mathrm{II},\,\mathrm{III}\} = \max\{3,\,4\} = 4\,,$$

$$v\{\mathrm{I}\} = \max\{1,\,0\} = 1\,,\ v\{\mathrm{II}\} = \max\{2,\,1\} = 2\,,\ v\{\mathrm{III}\} = \max\{1,\,2\} = 2\,.$$

Then

$$Sh_1^{(2,\,1,\,1)} = Sh_1^{(1,\,2,\,2)} = Sh_1^{(1,\,2,\,1)} = \frac{1}{3} + \frac{1}{6} + \frac{1}{3}\left[9 - 4\right] + \frac{1}{3} = \frac{1}{3} + \frac{1}{6} + \frac{5}{3} + \frac{1}{3} = 2\frac{1}{2},$$

$$Sh_2^{(2,\,1,\,1)} = Sh_2^{(1,\,2,\,2)} = Sh_2^{(1,\,2,\,1)} = \frac{1}{2} + \frac{1}{3} + \frac{1}{3}\left[9 - 3\right] + \frac{2}{3} = \frac{1}{2} + \frac{1}{3} + \frac{6}{3} + \frac{2}{3} = 3\frac{1}{2},$$

Table 4.

The strategies of players			The payoffs of players			The payoff of coalition	Shapley's value		
I	II	III	I	II	III	H_N(I, II, III)	$\lambda_1 H_N$	$\lambda_2 H_N$	$\lambda_3 H_N$
1	1	1	4	2	1	7			
1	1	2	1	2	2	5			
1	2	1	3	1	5	9	2.5	3.5	3
1	2	2	5	1	3	9	2.5	3.5	3
2	1	1	5	3	1	9	2.5	3.5	3
2	1	2	1	2	2	5			
2	2	1	0	4	3	7			
2	2	2	0	4	2	6			

$$Sh_3^{(2,\,1,\,1)} = Sh_3^{(1,\,2,\,2)} = Sh_3^{(1,\,2,\,1)} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\,[9-4] + \frac{2}{3} = \frac{1}{3} + \frac{1}{3} + \frac{5}{3} + \frac{2}{3} = 3.$$

5. Solve the non-coalitional game $G\,(\Sigma_3)$, $\Sigma_3 = \{S_1 = \{\mathrm{I}\}\,,\, S_2 = \{\mathrm{II}\}\,,\, S_3 = \{\,\mathrm{III}\}\}$. NE does not exist in pure strategies.

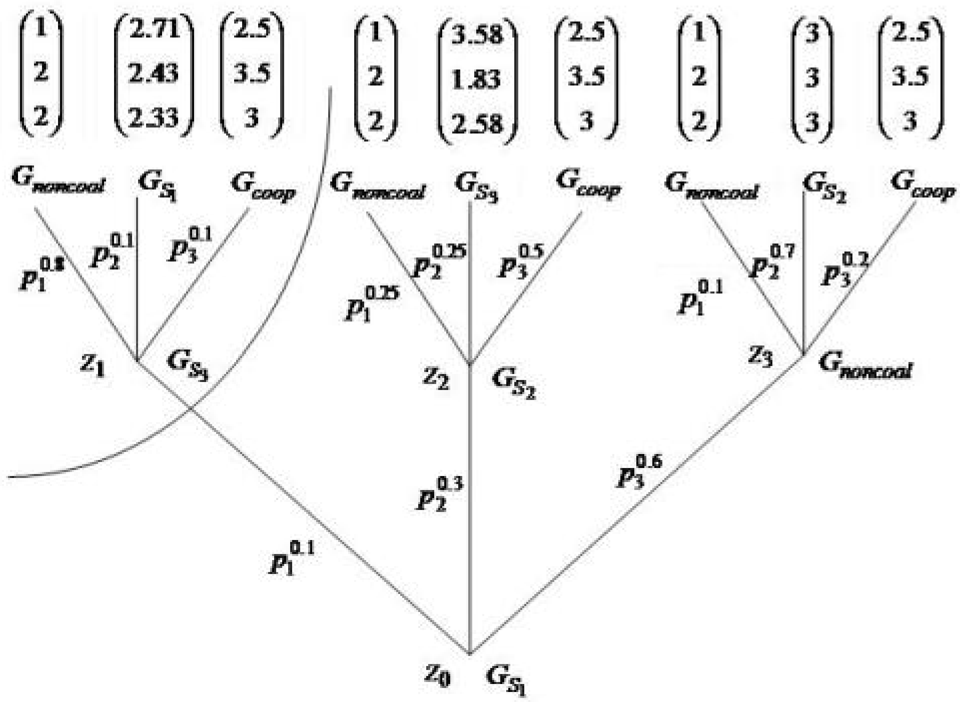
Use maximal guaranteed payoffs calculated in item 4 $v\,\{\mathrm{I}\} = 1$; $v\,\{\mathrm{II}\} = 2$; $v\,\{\mathrm{III}\} = 2$. Find optimal strategies according to the Nash arbitration scheme (Grigorieva, 2009), see table 5, where "−" means that strategies are not Pareto optimal, but "+" − are Pareto optimal. Then we have optimal n-tuples (1, 1, 2) and (2, 1, 2) which provide identical payoff (1, 2, 2) in both n-tuples.

Table 5.

The strategies of players			The payoffs of players			Optimality by Pareto (P) and Nash arbitration scheme	
I	II	III	I	II	III	Nash arbitration scheme	P
1	1	1	4	2	1	$(4-1)\,(2-2)\,(1-2) < 0$	-
1	1	2	1	2	2	$(1-1)\,(2-2)\,(2-2) = 0$	+
1	2	1	3	1	5	$(3-1)\,(1-2)\,(5-2) < 0$	-
1	2	2	5	1	3	$(5-1)\,(1-2)\,(3-2) < 0$	-
2	1	1	5	3	1	$(5-1)\,(3-2)\,(1-2) < 0$	-
2	1	2	1	2	2	$(1-1)\,(2-2)\,(2-2) = 0$	+
2	2	1	0	4	3	$(0-1)\,(4-2)\,(3-2) < 0$	-
2	2	2	0	4	2	$(0-1)\,(4-2)\,(2-2) < 0$	-

Conclusion.

- For $\Sigma_1 = \{S_1 = \{\mathrm{I,II}\}\,,\ N\backslash S_1 = \{\mathrm{III}\}\}$ we have payoff ((2.71, 2.43), 2.33).
- For $\Sigma_2 = \{S_2 = \{\mathrm{II,III}\}\,,\ N\backslash S_2 = \{\mathrm{I}\}\}$ we have payoff (3, (3, 3)).
- For $\Sigma_3 = \{S_3 = \{\mathrm{I,III}\}\,,\ N\backslash S_3 = \{\mathrm{II}\}\}$ we have payoff (2.59, (2.5), 2.91).
- For $\Sigma_4 = \{N = \{\mathrm{I,II,\ III}\}\}$ we have (2.5, 3.5, 3).
- For $\Sigma_5 = \{S_1 = \{\mathrm{I}\}\,,\, S_2 = \{\mathrm{II}\}\,,\, S_3 = \{\,\mathrm{III}\}\}$ we have optimal payoff (1, 2, 2) in n-tuples (1, 1, 2) and (2, 1, 2).



(1, 2, 2)
(2.71, 2.43, 2.33)
(2.5, 3.5, 3)
(1, 2, 2)
(3.58, 1.83, 2.58)
(2.5, 3.5, 3)
(1, 2, 2)
(3, 3, 3)
(2.5, 3.5, 3)
$G_{noncoal}$
G_{S_1}
G_{coop}
$G_{noncoal}$
G_{S_3}
G_{coop}
$G_{noncoal}$
G_{S_2}
G_{coop}
$p_1^{0.8}$
$p_2^{0.1}$
$p_3^{0.1}$
$p_1^{0.25}$
$p_2^{0.25}$
$p_3^{0.5}$
$p_1^{0.1}$
$p_2^{0.7}$
$p_3^{0.2}$
z_1 G_{S_3}
z_2 G_{S_2}
z_3 $G_{noncoal}$
$p_1^{0.1}$
$p_2^{0.3}$
$p_3^{0.6}$
z_0 G_{S_1}

Fig. 1.

Example 2. Consider the following stochastic game, see picture 1. Transition probabilities (p_1, p_2, p_3) from one game to another game are shown on the graph shown in the picture 1.

Transition probabilities are determined by table 6. Table 6 defines the matrix of transition probabilities as follows. In the left column the games from which is done transition to the games locating by the first row of the table with probabilities in corresponding rows are located.

Table 6.

The probabilities	G_{noncoal}	G_{S_1}	G_{S_2}	G_{S_3}	G_{coop}
G_{noncoal}	0.1	0	0.7	0	0.2
G_{S_1}	0.6	0	0.3	0.1	0
G_{S_2}	0.25	0	0	0.25	0.5
G_{S_3}	0.8	0.1	0	0	0.1
G_{coop}	0.1	0.1	0.5	0.1	0.2

Payoffs of players in n-tuple of NE of the following simultaneous games G are equal to (see example 1)

$$\text{PMS}_{\text{noncoal}} = (1,\ 2,\ 2)\,,\ \text{PMS}_{G_{S_1}} = (2.71,\ 2.43,\ 2.33)\,,\ \text{PMS}_{G_{S_2}} = (3,\ 3,\ 3)\,,$$

$$\text{PMS}_{G_{S_3}} = (2.59,\ 2.5,\ 2.91)\,,\ \text{PMS}_{\text{coop}} = \left(2\frac{1}{2},\ 3\frac{1}{2},\ 3\right),$$

Algorithm for solving the problem.

1. Consider two step game $\tilde{\Gamma}(G_{S_3})$, shown at the very left and upper angle of the graph in the picture 1:

$$\text{PMS}_{\tilde{\Gamma}(G_{S_3})}(\bar{\mu}) = \text{PMS}_{G_{S_3}}(\bar{\mu}) + p_1(G_{S_3}||\,G_{\text{noncoal}})\,\text{PMS}_{G_{\text{noncoal}}} +$$
$$+ p_2(G_{S_3}||\,G_{S_1})\text{PMS}_{G_{S_1}} + p_3(G_{S_3}||\,G_{\text{coop}})\,\text{PMS}_{G_{\text{coop}}} =$$
$$= \text{PMS}_{G_{S_3}}(\bar{\mu}) + \begin{pmatrix} 1.32 \\ 2.19 \\ 2.13 \end{pmatrix} = \begin{pmatrix} 2.59 \\ 2.5 \\ 2.91 \end{pmatrix} + \begin{pmatrix} 1.32 \\ 2.19 \\ 2.13 \end{pmatrix} = \begin{pmatrix} 3.91 \\ 4.69 \\ 5.04 \end{pmatrix},$$

$$\bar{\mu} = \{\sigma(1,\,1,\,1) = 0.42,\, \sigma(1,\,2,\,1) = 0.08,\, \sigma(2,\,1,\,1) = 0.42,\, \sigma(2,\,2,\,1) = 0.08\},$$

where $\sigma(\bar{x})$ is realization probability of NE n-tuple $\bar{x}$ in pure strategies.

2. Similarly solve games $\tilde{\Gamma}(G_{S_2})$ and $\tilde{\Gamma}(G_{\text{noncoal}})$, see picture 1.

$$\text{PMS}_{\tilde{\Gamma}(G_{S_2})}(\bar{x}) = \text{PMS}_{G_{S_2}}(\bar{x}) + p_1(G_{S_2}||\,G_{\text{noncoal}})\,\text{PMS}_{G_{\text{noncoal}}} +$$
$$+ p_2(G_{S_2}||\,G_{S_3})\text{PMS}_{G_{S_3}} + p_3(G_{S_2}||\,G_{\text{coop}})\,\text{PMS}_{G_{\text{coop}}} =$$
$$= \text{PMS}_{G_{S_2}}(\bar{x}) + 0.25 \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + 0.25 \cdot \begin{pmatrix} 2.59 \\ 2.5 \\ 2.91 \end{pmatrix} + 0.5 \cdot \begin{pmatrix} 2.5 \\ 3.5 \\ 3 \end{pmatrix} =$$

$$= \mathrm{PMS}_{G_{S_2}}(\bar{x}) + \begin{pmatrix} 2.15 \\ 2.88 \\ 2.73 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 2.15 \\ 2.88 \\ 2.73 \end{pmatrix} = \begin{pmatrix} 5.15 \\ 5.88 \\ 5.73 \end{pmatrix},$$

$$\bar{x} = (1,\ 2,\ 1);$$

$$\mathrm{PMS}_{\tilde{\Gamma}(G_{\mathrm{noncoal}})}(\bar{x}) = \mathrm{PMS}_{G_{\mathrm{noncoal}}}(\bar{x}) + p_1(G_{\mathrm{noncoal}} || \, G_{\mathrm{noncoal}})\, \mathrm{PMS}_{G_{\mathrm{noncoal}}} +$$

$$+ p_2(G_{\mathrm{noncoal}} || \, G_{S_2}) \mathrm{PMS}_{G_{S_2}} + p_3(G_{\mathrm{noncoal}} || \, G_{\mathrm{coop}})\, \mathrm{PMS}_{G_{\mathrm{coop}}} =$$

$$= \mathrm{PMS}_{G_{\mathrm{noncoal}}}(\bar{x}) + 0.1 \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + 0.7 \cdot \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + 0.2 \cdot \begin{pmatrix} 2.5 \\ 3.5 \\ 3 \end{pmatrix} =$$

$$= \mathrm{PMS}_{G_{\mathrm{noncoal}}}(\bar{x}) + \begin{pmatrix} 2.7 \\ 3 \\ 2.9 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2.7 \\ 3 \\ 2.9 \end{pmatrix} = \begin{pmatrix} 3.7 \\ 5 \\ 4.9 \end{pmatrix},$$

$$\bar{x} = (1,\ 1,\ 2) \ \text{ or } \ \bar{x} = (2,\ 1,\ 2).$$

3. Now solve the game $\tilde{\Gamma}(\tilde{\Gamma}(G_{S_3}), \tilde{\Gamma}(G_{S_2}), \tilde{\Gamma}(G_{\mathrm{noncoal}}))$, see picture 1.

$$\mathrm{PMS}_{\tilde{\Gamma}(\tilde{\Gamma}(G_{S_3}), \tilde{\Gamma}(G_{S_2}), \tilde{\Gamma}(G_{\mathrm{noncoal}}))}(\bar{\mu}) = \mathrm{PMS}_{G_{S_1}}(\bar{\mu}) +$$

$$+ p_1(G_{S_1} || \, \tilde{\Gamma}(G_{\mathrm{noncoal}}))\, \mathrm{PMS}_{\tilde{\Gamma}(G_{\mathrm{noncoal}})} +$$

$$+ p_2(G_{S_1} || \, \tilde{\Gamma}(G_{S_2})) \mathrm{PMS}_{\tilde{\Gamma}(G_{S_2})} + p_3(G_{S_1} || \, \tilde{\Gamma}(G_{S_3}))\, \mathrm{PMS}_{\tilde{\Gamma}(G_{S_3})} =$$

$$= \mathrm{PMS}_{G_{S_1}}(\bar{\mu}) + 0.6 \cdot \begin{pmatrix} 3.7 \\ 5 \\ 4.9 \end{pmatrix} + 0.3 \cdot \begin{pmatrix} 5.15 \\ 5.88 \\ 5.73 \end{pmatrix} + 0.1 \cdot \begin{pmatrix} 3.91 \\ 4.69 \\ 5.04 \end{pmatrix} =$$

$$= \mathrm{PMS}_{G_{S_1}}(\bar{\mu}) + \begin{pmatrix} 4.16 \\ 5.23 \\ 5.16 \end{pmatrix} = \begin{pmatrix} 2.71 \\ 2.43 \\ 2.33 \end{pmatrix} + \begin{pmatrix} 4.16 \\ 5.23 \\ 5.16 \end{pmatrix} = \begin{pmatrix} 6.87 \\ 7.66 \\ 7.49 \end{pmatrix},$$

$$\bar{\mu} = \{\sigma(1,\ 2,\ 1) = 0.14,\ \sigma(1,\ 2,\ 2) = 0.19,\ \sigma(2,\ 1,\ 1) = 0.29,\ \sigma(2,\ 1,\ 2) = 0.38\}.$$

Since the game $\tilde{\Gamma}(\tilde{\Gamma}(G_{S_3}), \tilde{\Gamma}(G_{S_2}), \tilde{\Gamma}(G_{\mathrm{noncoal}}))$ is three stage game, then mean payoff of each player at one step can be calculated by formula:

$$(6.87,\ 7.66,\ 7.49)/3 = (2.29,\ 2.55,\ 2.5).$$

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Cash Flow Optimization in ATM Network Model

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Abstract. The main purpose of this work is to optimize cash flow in case of the encashment process in the ATM network with prediction of ATM refusal. The solution of these problems is based on some modified algorithms for the Vehicle Routing Problem and use statistical methods to compile the requests from the ATM network. A numerical example is considered.

Keywords: ATM network, route optimization, Vehicle Routing Problem, cash flow, statistical methods.

1. Introduction

Nowadays ATM network and credit cards are the essential parts of modern lifestyle, consequently one of the most actual problem in the bank's ATM network is optimization of cash flow and organization of uninterrupted work. For the bank it is important to prevent the rush demand for cash withdrawals by customers, that can be provoked by the delayed loading of ATMs, and reduce the service expenses. Serving the ATMs network is a costly task: it takes employees' time to supervise the network and make decisions about cash management and it involves high operating costs (financial, transport, etc.). At the present time more and more banks are turning their attention to have greater efficiency in how they manage their cash in ATMs.

Through cash management optimization banks can avoid falling into the trap of maintaining too much cash and begin to profit by mobilizing idle cash. Effective cash management and control starts with accurate prediction of ATMs refusal, allowing banks to forecast cash demand for the network, and find an optimal routes, which manage to reduce servicing costs. The increase of transportation and servicing cost can be substantial for banks. Route optimization for the collector teams is allow to reduce bank expenses and to control the encashment process (Simutis et al., 2007).

In this work we consider a problem in which a set of geographically dispersed ATMs with known requirements must be served with a fleet of money collector teams stationed in the depot in such a way as to minimize some distribution objective. This problem is combined with the problem of composition of service requests from the ATM network. We assume that the money collector teams are identical with the equal capacity and must start and finish their routes at the depot. To estimate an average cash amount in each ATM and form the requests for the money collector teams we use following factors:

– ATM location and open hours;
– Irregular number of operations per week;
– Increasing of operation number in holidays and salary days.

Moreover we define the necessity of servicing each ATM and predict the future requests for the collector teams based on the statistical data and restrictions, which are proposed above. The optimal routes to load ATMs depend on the current requests and predictable requests.

The main purpose of this work is to optimize cash flow in case of the encashment process in the ATM network with prediction of ATM refusal. To solve the problem we base on some modified algorithms for the Vehicle Routing Problem and use statistical methods to compile the requests from the ATM network. In the paper a numerical simulation for ATM network of one bank of St. Petersburg is considered.

2. Route optimization for collector teams

In this section we consider a problem in which a set of geographically dispersed ATMs must be served with a fleet of money collector teams stationed in the bank. This problem will be solved based on the Capacitated Vehicle Routing Problem (CVRP) see (Hall, 2003), which is the problem of designing feasible routes for a set of homogeneous vehicles that make up a vehicle fleet. The routes are formed minimizing total travel costs of all of the routes. Each route begins and ends at the depot and contains a subset of the stops requiring service. A solution to this problem is feasible if the vehicle capacity on each route is not exceeded and all stops are assigned to a route. In the simplest statement of the CVRP there are no lower and upper bounds on the duration of each route. As such, there are no route balance considerations.

To describe the basic CVRP we suppose, that a complete undirected network (or graph) $G = (V, E)$ is given where $V = (0, 1, \dots, n)$ is the set of vertices and E is the set of undirected arcs (edges). A non-negative cost c_{ij} is associated with every edge. $V' = V \setminus \{0\}$ is a set of n vertices, each vertex corresponds to a stop, vertex 0 corresponds to the depot. Henceforth, i will be used interchangeably to refer both to a stop and to its vertex location. Each stop i requires a supply of q_i units from depot 0. A set of M identical vehicles of capacity Q is located at the depot and is used to service the stops; these M vehicles comprise the homogeneous vehicle fleet. It is required that every vehicle route starts and ends at the depot and that the load carried by each vehicle is no greater than Q.

The route cost corresponds to the distance travelled on the route and is computed as the sum of the costs of the edges forming the route.

An optimal solution for the CVRP is a set of M feasible routes, one for each vehicle, in which all stops are visited, the capacity of each vehicle is not exceeded and the sum of the route costs is minimized.

In the case of street routing the stops are located on a street network and the travel time between stops is computed as the shortest travel time path between stops. If the travel time matrix is symmetric, then we have the symmetric or basic CVRP. In the symmetric CVRP it can be assumed that there are no one-way streets and turn and street crossing difficulties are not considered in setting up the travel time matrix. In this paper we consider one procedure for solving the symmetrical CVRP, using statistical methods to reduce model dimension.

2.1. Integer programming formulation of the CVRP

Consider the presentation of the CVRP, where $V = (0, 1, \ldots, n)$ is the complete set of vertices, subset $V' = V \setminus \{0\}$ is a set of n vertices without depot, each vertex corresponds to a stop, vertex 0 corresponds to the depot. Let x_{ij} be an integer variable representing the number of vehicles traversing the undirected arc (edge) $\{i, j\}$, and let $r(S)$ be the number of vehicles needed to satisfy the demand of ATMs in S, c_{ij} are costs.

Here we present the formalization of the basic CVRP problem:

$$\min \sum_{i<j} c_{ij} x_{ij}, \tag{1}$$

$$\sum_{i<j} x_{ij} + \sum_{i>j} x_{ji} = 2, \quad i \in V', \tag{2}$$

$$\sum_{i \in S} \sum_{j \in V \setminus S} x_{ij} \geq 2r(s), \quad S \subseteq V', \quad |S| > 1, \tag{3}$$

$$\sum_{j \in V'} x_{0j} = 2M, \tag{4}$$

$$x_{ij} \in \{0, 1\}, \quad i \in V', \quad j \in V', \quad i < j, \tag{5}$$

$$x_{0j} \in \{0, 1, 2\}, \quad j \in V'. \tag{6}$$

Constraints (2) are the degree constraints for each ATM. Constraints (3) are the capacity constraints which, for any subset S of ATMs, that does not include the depot, impose that $r(S)$ vehicles enter and leave S, where $r(S)$ is the minimum number of vehicles of capacity Q required for servicing the ATMs in S. Constraints (3) are also called generalized subtour elimination constraints. It is NP-hard to compute $r(S)$, since it corresponds to solve a bin-packing problem where $r(S)$ is the minimum number of bins of capacity Q that are needed for packing the quantities. However, inequalities (3) remain valid if $r(S)$ is replaced by a lower bound to its value, such as $[\sum_{i \in S} q_i / Q]$ where $[y]$ denotes the smallest integer not less than y and q_i is required amount of cash for each ATM. Constraint (4) states that M vehicles must leave and return to the depot while constraints (5) and (6) are the integrality constraints. Finally, $x_{0j} = 2$ corresponds to a route containing only stop j.

This formulation cannot be solved directly by a general purpose integer programming algorithm because constraints (3) are too numerous to be enumerated a priori. However if the dimension of model will be reduced, this formulation will be solved.

At the present time various approaches to the solution of the CVRP have been proposed in different papers. In our work we focused on several papers and handbooks. The main sources is (Hall, 2003), which presents several formulations of Vehicle Routing Problem and it's modification, as well as some ideas and methods of solution are described. Nowadays Vehicle Routing Problem (VRP) is very actual and many authors concentrate on application of different methods and their extensions to solve VRP. This paper is based on some previous works, for instance, (Ralphs et al., 2003), in their work, suggest a decomposition-based separation methodology for the capacity constraints. Meanwhile the paper (Lysgaard et al., 2003) describes a branch-and-cut algorithm based on application of variety of cutting planes.

In current work we offer to reduce dimension of the model and simplify constraints (3) using statistical analysis on the ATM network.

3. Forecasting of Cash Flows in ATM Network

In current section we consider a problem of cash flow forecasting in the ATM network, especially focused on forecasting of cash balance in ATMs and the moment of ATM upload. Moreover we find the moment of each ATM refusal and compile the requests for the bank's processing center (Vasin, 2005). We analyzed cash flows of each ATM, using statistical data, and found that cash withdrawal is heterogeneous process which depends on following factors:

- Paydays, weekends, holidays, etc.;
- ATM location;
- ATM open hours.

Consider the set of geographically dispersed ATMs and a bank, defined in Section 2. Here $V' = V \setminus \{0\}$ is a subset of n ATMs, without bank, as in previous section vertex 0 corresponds to the bank. According to the factors mentioned above the whole set of ATMs V' is divided into m subsets $S_k \subseteq V'$, $S_k \cap S_l = \emptyset$, $k \neq l$, $k, l \in \{1, \dots, m\}$. Lets define y_{it} as the amount of cash withdrawal from ATM i at day t.

Consider each subset S_k consisting of n_k ATMs separately. Using statistical cash withdrawal data for each ATM over N days, we describe an average amount of cash withdrawal per day for subset S_k by following expression:

$$\bar{y}_{kt} = \frac{1}{n_k} \sum_{i \in S_k} y_{it}, \quad t = 1, \dots, N, \quad k = 1, \dots, m. \tag{7}$$

Thus we obtain cash withdrawal time series for each subset S_k: $(\bar{y}_{k1}, \bar{y}_{k2}, \dots, \bar{y}_{kN})$. Estimate trend and periodic component of the time series, for that purpose choose linear trend and find equation of linear regression (Bure, Evseev, 2004):

$$tr_{kt} = a_k + b_k t, \quad t = 1, \dots, N, \quad k = 1, \dots, m, \tag{8}$$

where the evaluations of unknown parameters a_k, b_k are calculated with the least-squares procedure.

The periodic component is estimated by the the following formula, removing trend from the time series:

$$P_{kt} = \frac{1}{z_k} \sum_{j=0}^{z_k - 1} (\bar{y}_{k,t+j\cdot T_k} - tr_{k,t+j\cdot T_k}), \quad t = 1, \dots, T_k, \quad k = 1, \dots, m, \tag{9}$$

where N – number of observations in subset S_k, T_k – period length, $z_k = \frac{N}{T_k}$ – integer number of periods.

Thereby we define cash withdrawal time series for each subset S_k:

$$\bar{y}_{kt} = tr_{kt} + P_{kt}, \quad t = 1, \dots, N, \quad k = 1, \dots, m. \tag{10}$$

The cash withdrawal forecast for two weeks is evaluated with the formula (10):

$$Y_{kt} = tr_{kt} + P_{kt}, \quad t = N+1, \ldots, N+14, \quad k = 1, \ldots, m. \tag{11}$$

Proceed the forecasting of cash balance in ATMs network and extend this procedure for all subgroups S_k. Assume B_{it} is the cash balance in ATM i at day t. If we know the cash balance for exact day, we can make the cash balance forecast using cash withdrawal forecast calculated by formula (11), i.e. find B_{it} when $t = N+1, \ldots, N+14$. Now we can detect the moment of each ATM refusal, i.e. when the cash balance approaches zero, and generate requests for the money collector teams.

Our purpose is to form the requests for service senter for each day of the forecasting period. Every request should contain the information about ATMs needed to be serviced and the quantity of cartridges (Diebold). We do not take into account the nominal of banknotes.

The algorithm for cash balance forecast was simulated in the MATLAB system and for each date of the forecasting period we received the requests for ATM service with numbers of ATMs, their addresses and number of cartridges. Requests are transmitted to the encashment department and can be used to reduce CVRP dimension and respectively the number on constraints (3), while constructing optimal encashment routes.

3.1. Numerical simulation

Consider one bank of St. Petersburg with ATM network consisting of 49 items and apply the procedure offered above. Divide subset V' into several subsets S_k which are presented in the Table 1:

Table 1. Subsets of ATM network

Subset S_k	Name (location)	Quantity of ATMs	Open hours
S_1	Subway	21	Open daily
S_2	Shop	8	Open daily
S_3	Shopping Mall	9	Open daily
S_4	Educational Organization	2	Mo–Fr
S_5	Enterprise	2	Mo–Fr
S_6	Bank	7	Mo–Fr

Consider each group separately and analyze cash withdrawal time series using formulas (7)-(9). For example, examine the subset S_1, which is the group of ATMs situated at the subway. We construct time series for the period from 10 may 2010 to 20 june 2010. The diagram of the average cash withdrawal time series for the subset S_1 is presented in Fig.1. It shows that on Friday and Saturday the amount of cash withdrawal increases rapidly.

We construct equation of linear regression with estimated trend and periodic component: $tr_t = 369,5t + 167022$. We have 42 observations, then the length of time period $T_1 = 7$ and hence the integer number of periods is $z_1 = \frac{n}{T_1} = 6$. Make the cash withdrawal forecast for two weeks. The diagram of the time series estimation and the cash withdrawal forecast for the period from 21 june 2010 to 4 july 2010 is presented in Fig.2.

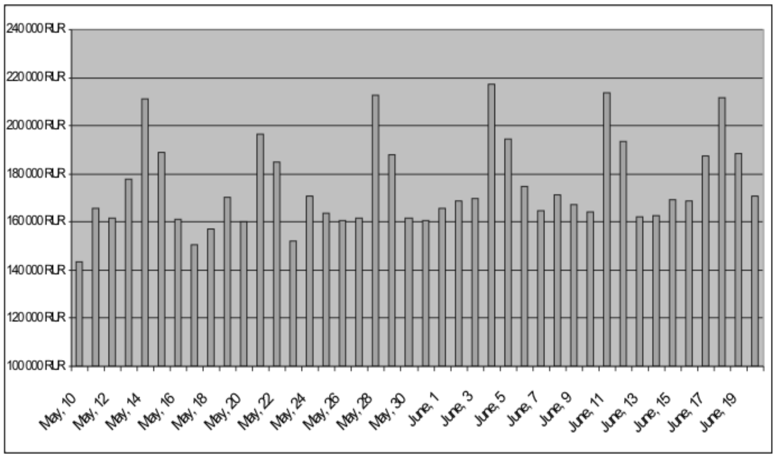

Fig. 1. Cash withdrawal time series for the subset S_1

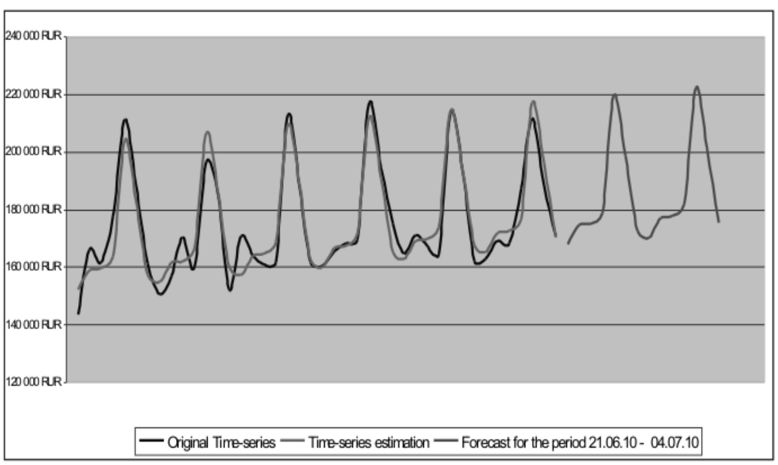

Fig. 2. Cash withdrawal time series, time series estimation and forecast for the subset S_1

Analogous, the cash withdrawal time series were analyzed and the forecast of cash withdrawal were done for each subset of ATMs. Cash withdrawal forecast for all groups is presented in Table 2.

Table 2. Forecast of withdrawals cash for the period 21.06.10 – 04.07.10, RUR

Date	Subway	Shop	Shopping Mall	Educational Organization	Enterprize	Bank
21.06	167900	90800	94400	27000	48200	180000
22.06	174500	88200	92600	25900	88200	169100
23.06	175200	89200	96800	27300	89200	171200
24.06	179200	91000	96000	27000	49500	171500
25.06	219700	118400	122100	46400	90100	208500
26.06	198700	161300	174400	94300	0	0
27.06	172900	133900	151600	0	0	0
28.06	170500	92400	98100	108900	49400	180800
29.06	177100	89900	96300	97400	174100	169900
30.06	177800	90800	100400	135000	255400	171900
01.07	181800	92700	99700	102200	135000	172200
02.07	222300	120100	125700	47600	91300	209200
03.07	201300	163000	178100	53100	0	0
04.07	175500	135500	155300	0	0	0

With the data from the Table 2 and knowing the cash balance in each ATM on 20 june 2010 we compile the cash balance forecast for the period from 21 june 2010 to 4 july 2010. The service requests for the money collector teams were generated in MATLAB system. All requests contain the information about the number of ATMs to service, their location, cash amount, etc. Now we can reduce the CVRP dimension and generate the routes for the money collector teams using the concrete request.

For instance, let we receive request for 3 July 2010 to service 9 ATMs located at the different subway stations of St.Petersburg: 2 – Tekhnologicheskiy Institut, 3 – Moskovskie Vorota, 4 – Lomonosovskaya, 5 – Vasileostrovskaya, 6 – Prospekt Bol'shevikov, 7 – Ploschad' Lenina, 8 – Narvskaja, 9 – Chkalovskaja and 10 – Sennaja Ploschad'. Construct optimal routes for the current request. Firstly we assume that the bank has three collector teams with equal vehicle capacity $Q = 12$ cartridges and each ATM requires $q_i = 3$ cartridges. Distances between ATMs and the Bank are given in the Table 3.

We received the solution for this example in Maple system. Routes, which were constructed are represented in the Figure 3.

The optimal solution in the current model consists of three routes, one for each collector team. The first team drives through ATMs 4-6 (subway stations: Lomonosovskaya, Vasileostrovskaya, Prospekt Bol'shevikov), the second team goes through ATMs 2-3-8-10 (subway stations: Tekhnologicheskiy Institut, Moskovskie Vorota, Narvskaja, Sennaja Ploschad') and the third team goes through ATMs 5-9-7 (subway stations: Vasileostrovskaya, Ploschad' Lenina, Chkalovskaja). Every route begins and ends at the bank, vehicle capacity on each route is not exceeded, all ATMs are assigned to a route and total travel costs are minimized. Thus, we got optimal routes for the current request. The distance travelled on this optimal route

Table 3. Distances between ATMs and bank, m

	Bank	?2	?3	?4	?5	?6	?7	?8	?9	?10
Bank	0	3250	6530	9000	5005	10007	6680	7810	7650	3940
?2	3250	0	2930	10000	4870	13500	5480	3860	6770	1280
?3	6530	2930	0	10120	7940	13070	10610	5410	9180	4050
?4	9000	10000	10120	0	13690	6000	11900	14500	15100	10540
?5	5005	4870	7940	13690	0	15300	5990	5970	2750	4030
?6	10007	13500	13070	6000	15300	0	11100	14560	14600	10480
?7	6680	5480	10610	11900	5990	11100	0	9070	4690	6500
?8	7810	3860	5410	14500	5970	14560	9070	0	8300	4670
?9	7650	6770	9180	15100	2750	14600	4690	8300	0	5010
?10	3940	1280	4050	10540	4030	10480	6500	4670	5010	0

corresponds to 64 332 meters, this is a minimal length of all possible routes for the money collector teams.

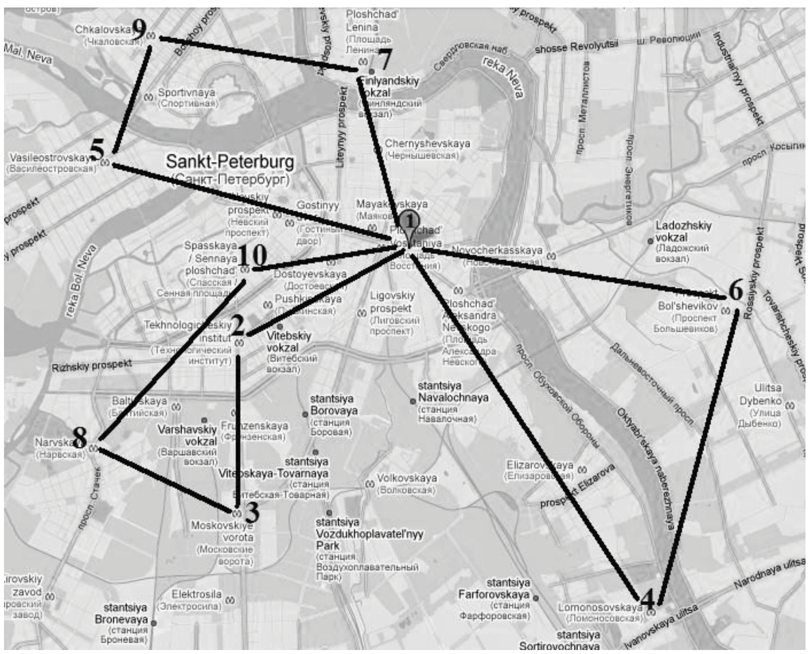

Fig. 3. Routes of the money collector teams.

4. Discussions

This paper's main contribution is in using general results from the theory of CVRP and statistical analysis to simulate the optimal bank strategy for the encashment process and predict the ATM failure in the network.

In discussion section we would like to consider one application of the optimization procedure for the encashment process, in particular case, in two different situations and compare the average costs of servicing the ATM network in these situations. The **situation 1** is: the upload amount of cash is small, this involves the bank to increase the currency of ATM network service so the transportation costs arise and can be substantial. In **situation 2** the uploaded cash is rather big, it means that the bank can incur losses related to the opportunity cost of cash uploading.

Consider average service costs as the average sum of the following costs:

– route costs in cash equivalent;
– uploaded cash;
– opportunity cost of cash uploading.

It should be noticed that route costs are minimized during the CVRP optimization. Also, route costs RC, particularly the distances c_{ij} between ATMs in optimal route $\hat{R}$, can be converted to cash equivalent with the following formula:

$$RC = f \cdot p \cdot \sum_{i,j \in \hat{R}} c_{ij}, \tag{12}$$

where f is the average fuel rate, measured in liter per meter, p is the fuel price. For example, in numerical simulation we got the optimal route corresponds to 64 332 m. In cash equivalent it is 139 RUR under the assumption that the average price of gasoline is 27 RUR for liter and the vehicle's fuel rate comes up to 8 liters per 100 km (in average).

Consider our two situations and assume that the uploaded cash can possess two values: 2 mln or 4,25 mln rubles. In the **situation 1** bank upload 2 mln rubles to the ATM network and in the **situation 2** – 4,25 mln rubles, respectively. The calculation of average total costs of servicing the whole ATM network were simulated in MATLAB subject to different upload values and presented in the Table 4.

Table 4. Average service costs, RUR

Upload value (one ATM)	Route costs	Cash uploading (ATM network)	Opportunity cost of cash uploading	Average total costs
2 mln	7 243 050	174 mln	24 467 090	205 710 140
4,25 mln	4 991 400	225,25 mln	19 757 714	249 999 114

Thus we can conclude that in this particular case, if uploaded amount of cash in the ATM network is small then servicing expenses (Route costs) arise for the bank and equal to 7 243 050 RUR, but negative profit from the idle (nonused) cash amount increases because Average Total Costs are equal to 205 710 140 RUR. In the same time if uploaded amount of cash in the ATM network is large, in our case 4,25 mln RUR, then servicing expenses are going down and Route Costs are 4 991 400 RUR. Hence bank store large amount of cash in his ATM network then it can fail into the trap of maintaining too much idle (unused) cash and so Average Total Costs are equal to 249 999 114 RUR.

So we can see that optimization of cash distribution through ATM network could be a very actual problem. Therefore in the future work we plan to define the optimal cash amount to store in ATM network, which will guarantee optimal servicing, including the reducing expenses of the bank. Also we plan to improve our simulation procedures.

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Stable Families of Coalitions for Network Resource Allocation Problems

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Abstract. A very common question appearing in resource management is: what is the optimal way of behaviour of the agents and distribution of limited resources. Is any form of cooperation more preferable strategy than pure competition? How cooperation can be treated in the game theoretic framework: just as one of a set of Pareto optimal solutions or cooperative game theory is a more promising approach? This research is based on results proving the existence of a non-empty K -core, that is, the set of allocations acceptable for the family K of all feasible coalitions, for the case when this family is a set of subtrees of a tree.

A wide range of real situations in resource management, which include optimal water, gas and electricity allocation problems can be modeled using this class of games. Thus, the present research is pursuing two goals: 1. optimality and 2. stability.

Firstly, we suggest to players to unify their resources and then we optimize the total payoff using some standard LP technique. The same unification and optimization can be done for any coalition of players, not only for the total one. However, players may object unification of resources. It may happen when a feasible coalition can guarantee a better result for every coalitionist. Here we obtain some stability conditions which ensure that this cannot happen for some family K. Such families were characterized in Boros et al. (1997) as Berge's normal hypergraphs. Thus, we obtain a solution which is optimal and stable. From practical point of view, we suggest a distribution of profit that would cause no conflict between players.

'What to suggest?. . . They write and write [. . .] . It makes my head reel. Just take everything and share equally. . .'
Michail Bulgakov, ***The heart of a dog***

1. Introduction

1.1. Rationale

Optimal performance of multi-agent economic systems is a pivotal problem of the modern economics. Performance quality of such systems can be described by a set of numerical characteristics reflecting the way these systems operate. These characteristics could include profit, cost, gross margin, quantity or number of produced commodities, and will be referred to henceforth as Objective Functions (OF). The primal managerial task is to optimize (maximize or minimize) these OF. Without loosing the generality, the maximization formulation will be considered.

There are two possible approaches to selecting the optimal strategy. The first one (henceforth 'collective' approach) is used when an overall system performance is considered and individual agents' performances are not the main focus of the research. In this case, the most common tool employed for the system optimization is the non-linear or linear programming, usually integer or mixed. The second approach implies optimization considered from each of the agents' point of view (henceforth referred to as 'individual'). With this approach, the system optimization tool is the game theoretic methods. The fist approach (overall optimization) is equivalent to the second approach (game) when all agents form one large coalition. The value of overall maximized OF is not less than the sums of the individual OFs obtained by individual players or coalitions as solutions of a game. It is tempting to suggest that agents form one coalition and then divide the total payoff between them. However, some players (or coalitions of players) might be better off pursuing strategies different from the strategy formulated for the overall optimization for 'one large coalition' game. Moreover, players or coalitions of players, which are better off playing separately, exist in the general case.

This paper describes quite a wide range of systems for which the non-conflicting distribution of optimal 'collective' OFs between a system's agents is possible. These systems belong to the class of the so-called network systems and are characterized by the existence of carriers connecting individual players who are located in nodes. The performance of such systems can be optimized using the network LP methods, containing a set of very specific constraints, related to the capacity of the carriers and the existence of paths between different nodes. A wide range of real economic systems (water supply networks, electricity and gas supply networks, telecommunications, etc.) can be described as such network systems, which makes the results of the reported research highly demanded by managerial practices in these areas. The fundamental result employed in this paper states that for such systems there exists a non-conflicting redistribution of total income between the players.

1.2. Mathematical formulation

We consider the situation when a group of agents I that has an access to the set of limited resources R has a goal to optimize some objective function. P is the set of products (industries). We ask the following question: what optimal cooperation strategy, or coalition structure, should be chosen by the agents in order to optimize their economic outputs.

The linear case appears when for each player $i \in I$ his strategies x_i are the shares of his activities in industry (product) $p \in P$. Let

$$A = \|a_{rp}\|$$

be a matrix representing the amount of resource $r \in R$ needed for producing a unit of product $p \in P$, and

$$b^i = (b^i_1, b^i_2, \dots, b^i_{|R|})$$

be a vector of resources available for player i.

Then we can formulate the production strategy choice for each player as a LP problem:

Maximize the revenue function $f(x) = \sum_{p \in P} c_p x_p$

Subject to $Ax \leq b, \quad x \geq 0$, where b is a vector of available resources.

This formulation can be settled for each player $i \in I$ or coalition $K \subseteq I$.

Respectively, we replace b by b^i or b^K. For each coalition K vector b^K is additively defined:

$$b^K = \sum_{i \in K} b^i$$

The corresponding LP solution will be denoted x^*.

Let us define function v on the set of coalitions as:

$$v(K) = Cx^*(b^K)$$

This function is superadditive, that is, for each two disjoint coalitions K_1 and K_2

$$v(K_1 + K_2) \geq v(K_1) + v(K_2)$$

This result was proven by Johnson (1973), Gomory and Johnson (1973) and further developed by Blair and Jeroslow (1977), Jeroslow (1977, 1978) and Schrijver (1980). It is referred in the literature as superadditive (or subadditive) duality. We can combine it with BGV theorem (see Boros, Gurvich, Vasin (1997) and the originating papers of Gurvich and Vasin (1977, 1978)) stating that family of coalitions $\mathcal{K} \subseteq 2^I$ is stable (that is the $\mathcal{K}$ -core $C(v, \mathcal{K})$ is not empty for any superadditive characteristic function v) if and only if $\mathcal{K}$ is a normal hypergraph, according to Berge (1970).

As an application, we plan to consider an example of particular game, in which the corresponding graph is a water allocation system represented by a network of gravitationally driven water carriers (rivers, canals, pipelines). We show that if a coalition is created by the neighbour players (farmers) whose properties are located along the same water supply carrier, or along a tree formed by such carriers, then the core of the obtained cooperative game is not empty. Let us now define accurately the concepts considered above.

2. Stable families of coalitions, normal hypergraph and TU-games

For the beginning, let us consider cooperative games with transferable utility (TU-games).

The notation is formalized as follows: Let I be a set of players. Its subsets $K \subseteq I$ are called coalitions. A TU game is defined by a characteristic function $v: \quad 2^I \to R$.

Function v (and the corresponding game) is called superadditive if

$$v(K^{'} \bigcup K^{''}) \geq v(K^{'}) + v(K^{''}) \quad \forall K^{'}, K^{''} \subseteq I \text{ such that } K^{'} \bigcap K^{''} = \emptyset$$

Vector $(x_i : i \in I)$ is called an allocation if $\sum_{i \in I} x_i \leq v(I)$. Here x_i is interpreted as a payoff of player $i \in I$.

Furthermore,

$x_K = \sum_{i \in I} x_i$ is a payoff of coalition $K \subseteq I$.

The core $C(v)$ of the obtained superadditive TU game is defined by the system of linear inequalities

$C(v) = \{x \in R^{|I|} | x_I \leq v(I) \text{ and } x_K \geq v(K)\ \forall K \subseteq I\}$

It can be interpreted as a set of allocations $(x_i\colon i \in I)$ acceptable for all coalitions $K \subseteq I$.

In other words, the core of a TU-game is empty whenever each allocation is rejected by a coalition. The core is a natural and probably the simplest concept of solution in cooperative game theory. Yet it has an important disadvantage: $C(v)$ is frequently empty, because it must be acceptable for all $2^{|I|}$ coalitions.

Yet in real life not every coalition has a chance to appear, because some agents may not know or not like each other. The following relaxation of the concept is common in the literature:

Given a family of coalitions $\mathcal{K} \subseteq 2^I$ the $\mathcal{K}$–core is defined as a family of allocations acceptable for all coalitions $K \in \mathcal{K}$, that is

$C(v, \mathcal{K}) = \{x \in R^{|I|} |\ x_I \leq v(I) \text{ and } x_K \geq v(K)\ \forall K \in \mathcal{K}\}$

Family $\mathcal{K} \subseteq 2^I$ is called *stable* if the $\mathcal{K}$–core is not empty for any superadditive TU-game $v : 2^I \rightarrow R$.

An important result obtained by Boros et al. (1997), which is referred to as the BGV theorem, claims that family of coalitions $\mathcal{K}$ is stable if and only if $\mathcal{K}$ is a Berge normal hypergraph (Berge (1970)). The definitions of terms 'perfect graph', 'normal hypergraph' and explanation of links between games, coalitions and hypergraphs are given in Appendix to Gurvich and Schreider (2010), accessible online.

For the present paper the following example is most important: Let T be a tree, assign a player to every vertex of T. Then an arbitrary family of subtrees of T forms a stable family of coalitions.

This special case of the BGV theorem, considered already in Gurvich and Vasin (1977), is important in applications. As it has been already mentioned above, we consider the family of LP problems whose resource constraints (right-hand side) b_K depends on a coalition. Hence, the optimal solution $v(K) = Cx^*(b^K) = Cx^*(K)$ is a superadditive function of K. Thus the BGV theorem is applicable. Hence, it exist such a distribution of the total optimal payoff $v(I)$ among the players that will cause no objection from any coalition. Which coalitions are feasible will be discussed in the Section 4.

3. Network supply systems and games

Water, electricity and gas supply systems fall into the class of the so-called network systems. The common characteristics of all these allocation systems is that they can be represented as a set of nodes (for instance, reservoirs, junctions and users) interconnected by a system of carriers with different capacities (wires, channels, pipelines). The last two decades a number of publications appeared when these types of systems were described by linear and non-linear optimization methods. For water supply system this approach was employed for instance in Perera et al. (2005). Electricity supply systems and associated electricity market, as well as traffic and telecommunication systems, were considered in Hu and Ralph (2007) and Ralph (2008). Gas supply system for Europe and placeNorth America were also modelled by similar methods of network LP optimization (Egging et al. (2008) and Gabriel et al. (2005)).

In order to be more specific, the REALM (Resource Allocation Model) illustrates how this network LP works for the irrigation water supply systems. More detailed description of this LP network optimization can be found in Dixon et al. (2005).

Following Dixon et al. (2005) in stylized, form REALM models can be represented as:

Choose non-negative values for

$$\begin{aligned} &F(i,r,t), \quad \text{for } i \in D,\ r \in D \bigcup E, \quad t = 1,2,\dots,T, \\ &S(e,t), \quad \text{for } e \in E, \quad t = 1,2,\dots,T \quad \text{and} \\ &W(i,t) \quad \text{for } i \in D,\ t = 1,2,\dots,T \end{aligned}$$

to minimize

$$\sum_{i\in D}\sum_{r\in D\bigcup E}\sum_{t} c_{i,r}(t)^{*}F(i,r,t) + \sum_{e\in E}\sum_{t}\beta_e(t)^{*}\left|d(e,t) - S(e,t)\right| +$$
$$+\sum_{i\in D}\sum_{t} g_{i,t}[W(i,t) - W_{\min}(i,t)]$$

subject to

$$W(i,t+1) \le W(i,t) - \sum_{r\in D\bigcup E} F(i,r,t) + \sum_{k\in D} F(k,i,t) * [1 - l(k,i,t)] +$$
$$+X(i,t) - \theta_{i,t}\,[W(i,t)] \quad \text{for } i \in D,\ t = 1,2,\dots,T$$

$W(i,t) \le C(i)$ for $i \in D$, t = 1, 2, ..., T and

$S(e,t) = \sum\limits_{i\in D} F(i,e,t)^{*}\,[1 - l(i,e,t)]$ for $e \in E$, t = 1, 2, ..., T

where:

D is the set of dams. Dams include not only water storage facilities but also junctions in the water network. A junction has either more than one inlet or more than one outlet. It can be treated as a dam with zero capacity;

E is the set of end users;

$F(i,r,t)$ is the flow in period t from dam i to dam or end-use r;

T is the last period of interest. If the model were solved for one year with periods of one month, then $T = 12$;

$W(i,t)$ is the amount of water in dam i at the beginning of period t. $W(i,0)$ is exogenous;

$S(e,t)$ is the amount of water supplied to end-user e in period t;

$d(e,t)$ is the exogenously determined ideal water requirements of end-user e in period t;

$c_{i,r}(t)$ is the cost of sending a unit of water from dam i to dam or end-user r in period t. If it is physically impossible to send water from i to r, then $c_{i,r}(t)$ can be set at an arbitrarily large number;

$\beta_e(t)$ is the penalty or cost per unit of shortfall in meeting the water demands of end-user e in period t;

$W_{min}(i,t)$ is the minimum level of water for dam i that is desirable from an environmental or aesthetic point of view;

$g_{i,t}$ is a penalty function. It takes positive values if $W(i,t) - W_{min}(i,t)$ is negative;

$l(k,i,t)$ is losses per unit of flow from dam k to dam or end-user i in period t (exchange losses);

$X(i,t)$, specified exogenously, is the natural inflow to dam i in period t;

$\theta_{i,t}$ is a function giving evaporation from dam i in period t;

$C(i)$ is the capacity of dam i. If dam i is a junction then $C(i) = 0$.

Models such as REALM can be used to plan flows in a hydrological area and to decide how these flows should be varied in response to changes in rainfall [reflected in $X(i,t)$], changes in demands [$d(e,t)$], and changes in a myriad of technical and cost coefficients.

Similar optimization formulations can be used for electricity and gas supply systems.

4. Superadditive characteristic functions related to linear programming

Here we refer to an important result obtained by Jonson (1973), Gomory and Jonson (1973), Blair and Jeroslow (1977), Jeroslow (1977, 1978) and Schrijver (1980).

Let a LP formulation be given in the following form:

Maximize the revenue function $f(x) = \sum_{p \in P} c_p x_p = Cx$

Subject to $Ax \leq b,\ x \geq 0$

where b is a vector of resources and c is a vector of prices.

The optimal solution is a vector $x^* = (x_i^*, i \in I)$

For each coalition K, the function b^K is defined additively:

$$b^K = \sum_{i \in K} b_i$$

Then function v is defined on the set of coalitions as $v(K) = Cx^*(b^K)$, which is the optimal value of objective function subject to resources b^K available for the coalition K.

Then, it appears that $v(K)$ is a superadditive function, that is,

$$v(K_1 + K_2) \geq v(K_1) + v(K_2) \text{ for any pair of disjoint coalitions, } K_1 \bigcap K_2 = \emptyset$$

5. Stability of optimal solutions

Let us consider a network LP optimization problem for an acyclic graph. Referring to the results mentioned in Section 2 , we can state that a family of coalitions is stable if and only if it is a normal hypergraph. In particular any family subtrees of a tree form a stable family of coalitions. Thus, there exists an allocation corresponding to the optimal LP solution, which is stable. It means that there is a distribution of the total income among players which is acceptable for all feasible coalitions.

6. Concluding remarks

In nutshell, this paper suggests a new approach to optimization and fair distribution of total payoff among players. This method is based on integration of two fundamental results, which are

1. Superadditive duality for LP optimization, and
2. The BGV theorem claiming that a family of coalitions $\mathcal{K}$ is stable (that is the $\mathcal{K}$–core $C(v, \mathcal{K})$ is not empty for any superadditive characteristic function v) if and only if $\mathcal{K}$ is a normal hypergraph.

Then BGV theorem can be applied for acyclic network systems. The following two assumptions are crucial: 1. **acyclicity** of corresponding graph and 2. **superadditivity** of the optimal value. It allows us to conclude that some 'natural' families of coalitions are stable, that is, admit a non-empty core. Therefore, it is possible to conclude that a fair distribution of payoff among the payers is possible. Thus, the existence of an optimal and stable solution for such class of games is proven.

A few words should be written about possible further continuation of this research. Firstly, we plan to illustrate these results by considering the real network supply system (say, the water allocation system modelled by REALM model) and demonstrate that for the full set of possible coalitions the game is unstable, that is, the core is empty. Then we will select a set of 'natural' coalitions, say, coalitions of farmers whose properties are located on the same water carrier, and demonstrate that such families are stable. The next step of this research is consideration of more sophisticated network supply systems and demonstration what are the optimal and stable solution in this case. The key point is formulation of appropriate managerial advises about redistribution of optimal income between real players constituting these systems. We also plan to consider not only TU but also more general classes of cooperative games such as games with non-transferable utilities, so-called NTU-games, also games in normal form, in effectivity function form, etc.

It should be also mentioned that the "superadditive duality" holds not only for LP but for integer programming as well. Hence, the same analyses of stability is applicable for these, more general, models.

Finally, we plan to consider the non-linear case related to the objective function in form of Cobb-Douglass' production function:

$CB(u_1, u_2, \ldots, u_{|R|}) = \prod_{i=1,..,n} u_i^{\gamma_i}$ where u_i are production factors and $\gamma_1 + \gamma_2 + ... + \gamma_{|P|} = 1$

Schreider, Zeephongsekul and Abbasi (2010) proved that this function is also superadditive. This result gives us a potential to employ similar approach for the game on non-cyclic graphs with Cobb-Douglass objective function.

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Signaling Managerial Objectives to Elicit Volunteer Effort*

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Abstract. We examine a nonprofit organization (npo) with a manager and a motivated volunteer, assume that the manager has private information about the volunteer's utility function, that the volunteer has no private information on the manager's utility function, and propose a signaling solution for the volunteer's effort decision. We focus on the general properties of a signal in a npo, specifically taking into account the effects of the volunteer's motivations on the signal. We find that, due to the diversity of volunteer motivations, the set of signals can be large. We relate this to signals that can be used by nonprofit managers in practice and we reinterpret existing practices as signals.

Keywords: signaling; volunteers; nonprofit organizations; motivation, incentives

1. Introduction

Volunteers are an important resource for many nonprofit organizations (npos). Therefore, appropriate management of volunteers can help these npos to achieve their mission. A volunteer (he) is by definition unpaid for his work, but can receive non-monetary rewards from the output that is produced by his effort (Besley and Ghatak, 2003), thus reaching a number of his goals, e.g. values (helping others), understanding (acquiring new skills), career (increasing future job opportunities), enhancement (increasing self-esteem), social (establishing social relationships) and protective (addressing personal problems) functions (Clary et al., 1996). The volunteer's goals are the functions that he wants to satisfy and he can do this by producing output. The volunteer's valuation of output depends on the manager (she). For example, managers can be of different types (competent, incompetent, honest, dishonest, etc.) which affects output. Managers have private information about their own type, so there exists a problem of asymmetric information. This problem is aggravated because the volunteer's reward is non-monetary. Promises about some kinds of non-monetary rewards cannot be legally binding: in these cases the volunteer cannot sue the npo to pay compensation when the npo reneges on its promises (Suazo et al., 2009).

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In the presence of asymmetric information, the manager can steer the volunteer's behavior with incentives. Incentives (such as controlling, monitoring, monetary rewards, threats, deadlines, etc.) may be used to influence the volunteer's effort. Examining a wide range of compensation and incentive schemes such as promotions, deferred compensation, and efficiency wages, Prendergast (1999) finds that agents (in a profit context) do respond positively to these schemes in certain cases.

In other cases, when performance measurement is noisy or biased, incentives may be less useful, and can even be contra-productive, resulting in sabotage (Gibbons, 1998). A possible explanation for the negative effects of incentives especially useful in a nonprofit setting is given by Frey (1997) and Frey and Jegen (2001) who state that certain incentives can crowd out intrinsic motivation. A literature overview on the interaction effects between monetary and non-monetary motives is given by Fehr and Falk (2002). There is a need to solve the problem of asymmetric information in npos, but due to the specific role of motivations in the nonprofit sector, incentives may not be very useful. Instead we discuss an alternative solution: signaling. A signal is an action that the manager can undertake to give the volunteer information about his expected utility. The difference between signaling and incentives is that in the case of signaling the volunteer feels that his actions are a personal choice while in the case of control he feels like he is obeying a regulation. Indeed, the volunteer's intrinsic motivation can be influenced by feelings of autonomy or control. If he feels he is being controlled, then his intrinsic motivation will decrease (Ryan and Deci, 2000) and he might leave the organization, which is easier for a volunteer than for a (paid) employee. The literature shows that non-monetary motivation (such as intrinsic motivation) is important for volunteers. When this is the case it is often better to use non-controlling motivation tools to steer the volunteer's behavior. Therefore, we feel that it is important to recognize the important role that signaling may play in volunteer behavior and motivation. Signaling in the unique context of a npo is the focus of this paper.

We follow Bénabou and Tirole (2003) who state that any action can have three effects on the manager's utility: 1) the direct effect of the action on the manager's utility, for example the cost of the action, 2) the direct effect of the action on the utility of the employee or volunteer, he can like or dislike the action, 3) the indirect signaling effect, the employee or volunteer learns new information which influences the expected utility of his effort.

Several authors have considered different actions that can serve as a signal. We discuss briefly the signaling effects of rewards, contracts and trust. A reward can be a signal of task attractiveness when it gives information about the manager's estimation of the agent's intrinsic motivation (Bénabou and Tirole, 2003). For example, a high reward may lead the agent to believe that the task is very risky or unpleasant. Other possible signals are the properties of the contract between agent and principal. Contracts that are based on measured output can be a signal of the manager's ability to distort output (Allen and Gale, 1993). Similarly to Inderst (2001), Spier (1993) states that a contract that insures the principal against possible negative outcomes may be a signal that negative outcomes are likely. Kreps (1997) considers the length of an employment contract as a signal, for example when the manager chooses a contract that makes it very costly for her to fire the employee, the employee may interpret this as a signal of job security. Besides rewards or contracts, trust can also be a signal. When the principal decides to trust the agent, she shows that she believes that most agents are trustworthy. If some agents are conformists and act like the majority of agents, then these agents may react positively to the manager's decision to trust (Slikwa, 2007). If the agent considers trust as the preferred action of the type of manager that gives him the highest utility of effort, a principal who trusts can give a signal about her type (Ellingsen and Johannesson, 2008).

A common element in these studies is that the principal has private information about the state of the world that determines the agent's utility. The principal's actions have different costs in different states. We show that an action can be used as a signal if the action has the lowest cost for the principal in the state that is optimal from the viewpoint of the agent, given that action. We discussed authors that use this property in a model with a specific signal, and show that this property also holds for a general signal. Our model shows that the cost of the signal in the optimal state must not be strictly lower than in any other state. Even if the cost of the action is only weakly lower in the optimal state, then this action may still be a signal. We therefore take a broader view than the studies above in the sense that we do not focus on any particular signal. Instead we focus on the set of actions that possess the characteristics necessary to be a signal and we apply this to the specific environment of a npo with volunteers. The requirements of a signal are: 1) the manager must know which state the volunteer considers to be optimal; 2) the action must be observed by the volunteer; and 3) the signal must be reliable, the signal must have weakly lower costs for the manager in the optimal state.

A mission driven organization, like a npo, does not only rely on the competence of the manager but also on her goals, so the manager has to signal her goals or mission. Volunteers, who commit specific resources to the npo, need to have information about the manager's goals. The signals that are sent by the manager to give this information have a specific nature, as they are signals about intangible and subjective properties of the manager.

The set of actions that can be a possible signal is large. Many management practices may have signaling effects, though some of these effects may not be known to the manager. Therefore, we stress the importance for managers to asses their practices so that they can eliminate unwanted signals. Viewing management practices as a signal allows us to reinterpret their effect on the organization. To illustrate this we discuss some potential signals. We first discuss an example of current management practices (formal control) and offer an alternative explanation for its existence, compared to what is commonly found in the literature. Then we make two suggestions of signals that managers of a npo could use in practice (non-mission related activities and volunteering in similar organizations).

According to Frey (1997) using a formal control system in an organization to reduce agency problems between managers and employees may be problematic in itself because formal control may undermine the employees' intrinsic motivation to perform well. Intrinsic motivation may be lower because formal control systems can have a negative effect on the employee's feelings of competence or autonomy (Ryan and Deci, 2000). Our paper may offer an alternative explanation. If without formal control volunteers exhibit some form of non-productive behavior, it may be profitable for the npo to increase formal control so that this non-productive behavior is removed. The absence of formal control can be viewed as a signal. Compassionate and understanding managers may have fewer problems with non-productive behavior than others. These managers may be preferred by a value-driven volunteer who works for a npo for altruistic or humanitarian reasons. These managers may use this insight to send a signal to the volunteer. Other managers, having a cost disadvantage, may not be able or willing to bear the extra (utility) cost of non-productive behavior. By not increasing formal control, and so bearing the extra cost of non-productive behavior, the manager may convince a value-driven volunteer that she is compassionate and understanding. This, in turn, may increase volunteer effort if he believes that a compassionate and understanding manager is also value-driven.

Caers et al. (2009) suggest that the goals of managers and volunteers in a npo can be divided in three categories: individual goals (income, reputation and effort), organizational goals (only help those clients whose profile matches the organization's mission statement) and client goals (help the client no matter what his profile is). If the volunteer is client-driven (he wants to do good for other people), then he may prefer to work for a manager who finds client goals important. The amount of resources that the organization uses for client goals may be a signal of how well the volunteer will be able to help others. There may be an offsetting increase in volunteer effort when the manager diverts resources from organizational goals to client goals.

Managers who share the same goals as the volunteer might also like to do similar volunteer work in other organizations. A manager who is less caring or altruistic receives fewer (utility) benefits from volunteering, so she will be less willing to volunteer. When the volunteer observes that the manager volunteers in other (comparable) organizations, he may interpret this as a signal that the manager is caring or altruistic. This may induce the volunteer to increase his effort. If the volunteer wants to increase his skills then he may prefer to work for a manager who is also interested in acquiring skills through volunteering. Such a manager may indeed possess the skills that the volunteer wants to acquire. If the volunteer wants to increase or strengthen his social relationships, the manager can show by participating in other organizations that she can be a good link towards other social groups.

This result is not only important for volunteers but is also important for other stakeholders of the organization who want to interpret the organization's behavior. Board members need to take into account the signaling behavior of managers when they evaluate the manager. Which signals does the manager send? Are these signals optimal for the organization? Are there other signals the manager could send? What do these signals tell about the true preferences of the manager? Donors can interpret signals to know whether their donations will be put to good use. Governments and other funding authorities can use signals to verify whether the organization will keep its promises or obligations.

In the next section we formally present our model and Section 3 is the conclusion. It is not our ambition to develop a complete model of a npo. Therefore, we deliberately keep our model simple by only considering signaling effects, not more than two types of managers, one-dimensional motivations, and we ignore dynamics. Whereas a case can be made that a more realistic model can lead to interesting conclusions, we believe that it is better to focus on the basic principles of signaling within a npo.

2. The model

2.1. Introduction

In this section, we develop a model of a nonprofit manager who tries to convince the volunteer to perform high effort. The volunteer has no knowledge about the manager's type, there is asymmetric information. We propose that managers can send signals that give information to the volunteer about their type.

We assume there are two types of managers, the 'identical' type who shares the same goals as the volunteer and the 'non-identical' type who does not share the same goals. Whether the volunteer is able to satisfy his goals when he is working for the npo depends on the type of manager. We assume that the manager's type is the only determinant of the state of the world that is relevant for the volunteer. The volunteer only prefers high effort when he is working for an identical manager. The manager can try to signal that this is her type. The order of the game between the manager and the volunteer is as follows: nature chooses the type of the manager which is not revealed to the volunteer. The manager can send a signal to the volunteer to convince him that her type is the identical type. Given the signal, the volunteer chooses his optimal amount of effort.

If a managerial action has the lowest cost in the state of the world that is optimal for the volunteer then it can lead to a separating equilibrium. Whether or not it does depends on the manager's incentive compatibility constraints. If these constraints are satisfied then the identical manager can distinguish herself from the non-identical manager. This increases the volunteer's effort she receives. A manager in the optimal state can send a signal so that the volunteer knows with certainty that the manager is in this state.

In the signaling models we discussed in the introduction, the actions always had the lowest cost in the state optimal for the agent. In reality, this criterion includes a large amount of possible actions. When judging management practices, one therefore must be careful to examine whether these management practices do not have some kind of signaling function.

First, we describe the manager's utility function and the differences between different types of managers. Second, we propose the volunteer's utility function, and how he obtains information about the manager's type. Finally, we summarize our model in three propositions and discuss the results.

2.2. The manager

The manager of a npo wants to convince a volunteer to exert high effort, and therefore produces a signal. If the volunteer exerts effort (e) then he produces output y(e). There are two types of managers, the identical manager (i) and the non-identical manager (ni). The set of types t is therefore $\{i, ni\}$. Volunteers observe a signal if $S > 0$. When S = 0 there is no signal. The cost of sending a signal is $C_i(S)$ for the identical manager and $C_{ni}(S)$ for the non-identical manager. We make the following assumptions about the cost functions of signaling:

Assumption 1: $C_i(S), C_{ni}(S)$ are non-negative, continuous and non-decreasing functions in S, $S \in [0, +\infty]$.

Assumption 2: $C_i(S) < C_{ni}(S), \forall S > 0$.

Assumption 3: $C_i(S) \geq 0$ and $C_{ni}(S) > 0, \forall S > 0, C_i(0) = C_{ni}(0) = 0$.

Assumption 4: S_0 is the set of signals: $C_i(S) = 0$. S^* is the maximum of this set.

Assumption 2 states that the identical manager has a cost advantage. For each strictly positive signal, the identical manager has lower signaling costs. Assumption 3 says that the signaling costs must be larger than or equal to 0. Assumption 4 says that $C_i(S)$ 'leaves' the horizontal axis at some positive S called S^*, this is the strongest signal that identical manager can send without cost. S^* is either zero or greater than zero. Both cases are compatible with the other assumptions.

The utility function of the manager, given in Equation (1), combines two additively separable elements: the output produced by the effort of the volunteer and the signaling costs.

$$U_{M,t} = y(e) - C_t(S) \tag{1}$$

The manager wants to send a signal because this message may influence the volunteer's choice of effort. We will elaborate on this in the next section when we discuss the volunteer's utility function. The manager chooses the level of S so that her expected utility is maximized. The term y(e) can be interpreted as the manager's valuation of the volunteer's effort.

2.3. The volunteer

The volunteer wants to produce output to satisfy his goals (e.g. values, understanding, career, enhancement, social and protective goals (Clary et al., 1996)). Whether or not he will get the chance to produce such output depends on the type of the manager. If the manager is identical, then the volunteer's goals will be achieved if he chooses high effort. The volunteer's utility function (U_V) is increasing in the volunteer's valuation of the amount of output produced (($v_t(e)$) and decreasing in the amount of effort exerted. We call the cost of effort c(e). We make the following assumptions (and use ' to indicate first derivates) about the volunteer's utility function:

Assumption 5: U_V has a unique maximum for each t $\in \{i, ni\}$,

Assumption 6: $\forall e_i : v_i(e) > v_{ni}(e)$,

Assumption 7: $\forall e_i : v'_i(e) > v'_{ni}(e)$,

Assumption 8: $v_i(e)$ and $v_{ni}(e)$ are concave and increasing functions in e.

Assumption 9: c(e) is increasing in e and is non-concave.

The volunteer's valuation depends on the type of the manager:

$$U_V = v_t(e) - c(e) \tag{2}$$

The volunteer chooses the amount of e that maximizes the expected value of equation (2). It is easily shown that under our assumptions the optimal effort when volunteering for an identical manager is higher than when working for a non-identical manager, an implication compatible with intuition: if e_l is the volunteer's optimal choice of effort when he works for the non-identical manager then $v'_{ni}(e_l) = c'(e_l)$. According to assumption 7: $v'_i(e_l) > c'(e_l)$, from which, with Assumption 5, U_V is maximized for a value $e_h > e_l$. This is also implied by the definition of the identical manager, which is the manager for which the volunteer wants to exert high effort.

However, the volunteer is uncertain about the manager's type. The manager's type is assumed to be her private information. Both types of managers have an incentive to try to convince the volunteer that they are of the identical type. Therefore, the volunteer's maximization problem is given in (3):

$$\max_e E[U_V] = E[v_t(e)|S] - c(e) \tag{3}$$

If the manager's type is expected to be i then the volunteer chooses high effort e_h. If the manager's type is expected to be non-identical, then the volunteer chooses low effort e_l. For any probability $p \in [0, 1]$ that the manager is (perceived to be) identical the volunteer chooses his effort level $e \in [e_l, e_h]$. We call e^* the effort of the volunteer in a pooling equilibrium:

$$c'(e^*) = pv_i'(e^*) + (1-p)v_{ni}'(e^*) \tag{4}$$

2.4. Results

We summarize the results of our model in two propositions.

Proposition 1. *There is a separating equilibrium at $S = \underline{S}$ where $y(e^*) - y(e_l) = C_{ni}(\underline{S})$ when $y(e_h) - y(e^*) > C_i(\underline{S})$ holds.*

See Appendix A for the proof.

Proposition 1 states that in a separating equilibrium the identical manager sends a signal that is just costly enough to deter the non-identical manager from sending that signal.

Proposition 2. *There is a pooling equilibrium at S^* when $y(e_h) - y(e^*) < C_i(\underline{S})$ and $y(e^*) - y(e_l) > C_{ni}(S^*)$ hold.*

The signal in a pooling equilibrium does not have an influence on p and is equal to S_0 (see Appendix B for the proof). Proposition 2 states that in a pooling equilibrium the identical manager sends a signal that minimizes her signaling cost. It is just high enough so that the non-identical manager cannot be identified with certainty as being non-identical.

These results show that if certain incentive compatibility constraints are satisfied, the identical manager can send a signal that convinces the volunteer that she is indeed identical. In this case, even if there is asymmetric information and managers have an incentive to lie, the volunteer still has reason to believe the manager's signal. Signaling allows the volunteer to have greater confidence in an otherwise uncertain, hard to measure and subjective outcome.

The signal does not have to be a single action, it can also be a bundle of several actions. We found a separating equilibrium because there are differences in costs between identical and non-identical managers. However, we do not require that the signal is a unique action. The signal can also be a bundle of actions and not every action in this bundle needs to have strictly lower costs for the identical manager. Some actions may have weakly lower costs. This enlarges the set of possible signals that the manager can choose from and allows her to be creative to find a unique combination of actions that can serve as the best signal for her.

3. Conclusion

We started from the assumption that different types of managers have different cost functions for certain actions ('signals'). If these actions satisfy assumptions 1, 2 and 3 then for some incentive compatibility constraints, these actions are signals that lead to a separating equilibrium. The cost is high enough to deter the non-identical manager from sending it, but not high enough to deter the identical manager. In our model there can be separating equilibrium and a pooling equilibrium. This allows us to shed some light on practices by managers. Are current management practices a signal? In the introduction we have discussed several examples of how management practices can be reinterpreted as a signal. Managers can try to convince volunteers to exert high effort in many ways. If incentives are not an option, then maybe signals are. The manager can signal her values, skills or social capital by doing volunteer work in another organization, she can signal that she is kind and understanding by allowing some slack behavior by her subordinates by not using formal control methods, she can signal that volunteering for the organization may lead to a good career within the organization by spending money on emoluments or she can signal her values by diverting money to client goals.

These examples show that managerial signaling can be diverse in the nonprofit sector. There are many signals conceivable because volunteers have different motives. The volunteer's interpretation of the signal depends on his underlying motivation to volunteer. Therefore, we stress the importance for the manager to know the voluntee's motivation. Once the manager knows the volunteer's motivation, she must find the appropriate signal. Thus our model predicts that there is a relationship between the signal sent by the manager and the (perceived) motivation of the volunteer. This prediction can be verified empirically; therefore a possible direction for future research is to empirically investigate whether volunteers try to overcome uncertainty by looking for signals and whether managers provide these signals. The answer to this question is not only useful for volunteer but for all stakeholders of the organization.

Our model started from the assumption that the volunteer was already working for the npo. We did not model the volunteer's decision to join the npo. This question is also important but needs to be addressed in a different way, the results of which can be used to verify the conclusions of the present model. Of course, the model that we considered made a number of simplifying assumptions (only signaling effects are important, only two types of managers, only one dimension of the volunteer's and manager's motivation). Relaxing these assumptions may lead to furhter interesting insights.

Appendix

1. Appendix A

If a separating equilibrium is to occur at , then the identical manager must always send a signal and the non-identical never sends . We compare eh with e^* for the identical manager and e^* with e_l for the non-identical manager. The non-identical manager never receives e_h, because if that were the case the volunteer would believe that p = 1, which is false. For the same reason, the identical manager never receives e_l.

The identical manager always sends when (ICC1):$y(e_h) - y(e^*) < C_i(\underline{S})$.

The non-identical manager never sends when (ICC2):$y(e^*) - y(e_l) < C_{ni}(\underline{S})$.

The identical manager chooses such that is the smallest value that satisfies $y(e^*) - y(e_l) < C_{ni}(\underline{S})$. This satisfies ICC2 and minimizes the signaling costs for the identical manager. If this value for $\underline{S}$ also satisfies ICC1, then there is a separating equilibrium.

2. Appendix B

If, in a pooling equilibrium, $S > S^*$ both types can increase their utility by choosing a smaller S. If $S \leq S^*$, the identical manager is indifferent between all $S \leq S^*$, the non-identical manager can increase her utility by choosing the signal $S : C_{ni}(S) = 0$ meaning S = 0. In this case e cannot be independent of S. No non-identical manager would ever choose S^*, this would increase the effort at S^* to the same level as the separating equilibrium. All identical managers would choose S^* and the non-identical manager would have to mimic this behavior. This means that the pooling equilibrium exists and is unique when the following ICCs are satisfied:

The identical manager sends S^* when (ICC3): $y(e_h) - y(e^*) < C_i(\underline{S})$.

The non-identical manager sends S^* when (ICC4): $y(e^*) - y(e_l) \geq C_{ni}(S^*)$.

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Two Solution Concepts for TU Games with Cycle-Free Directed Cooperation Structures*

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Abstract. For arbitrary cycle-free directed graph games tree-type values are introduced axiomatically and their explicit formula representation is provided. These values may be considered as natural extensions of the tree and sink values as has been defined correspondingly for rooted and sink forest graph games. The main property for the tree value is that every player in the game receives the worth of this player together with his successors minus what these successors receive. It implies that every coalition of players consisting of one of the players with all his successors receives precisely its worth. Additionally their efficiency and stability are studied. Simple recursive algorithms to calculate the values are also provided. The application to the water distribution problem of a river with multiple sources, a delta and possibly islands is considered.

Keywords: TU game, cooperation structure, Myerson value, efficiency, deletion link property, stability

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1. Introduction

In standard cooperative game theory it is assumed that any coalition of players may form. However, in many practical situations the collection of coalitions that can be formed is restricted by some social, economical, hierarchical, communication, or technical structure. The study of games with transferable utility and limited cooperation introduced by means of communication graphs was initiated by Myerson (Myerson, 1977). In this paper we restrict our consideration to the class of cycle-free digraph games in which the players are partially ordered and the communication via bilateral agreements between players is represented by a directed graph without directed cycles. A cycle-free digraph cooperation structure allows modeling of various flow situations when several links may merge at a node, while other links split at a node into several separate ones.

It is assumed that a directed link represents a one-way communication situation. This restricts the set of coalitions that can be formed. In the paper we consider two

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different scenarios possible for controlling cooperation in case of directed communication. First it is assumed that players can only control their successors and if in the underlying graph structure a player is a successor of another player and both players are members of some coalition, then also within this coalition the former player must be a successor of the last player. Another scenario assumes that players can only control their predecessors and nobody accepts that one of his predecessors becomes his equal partner if a coalition forms.

We introduce tree-types values for cycle-free digraph games axiomatically and provide their explicit formula representation. On the class of cycle-free digraph games the (root-)tree value is completely characterized by maximal-tree efficiency (MTE) and successor equivalence (SE), where a value is maximal-tree efficient if for every root of the graph, being a player without predecessors, it holds that the payoff for him and his successors is equal to the worth they can get by their own, and a value is successor equivalent if when a link towards a player is deleted this player and all his successors will get the same payoff. It implies that every player receives what he contributes when he joins his successors in the graph and that the total payoff for any player together with all his successors is equal to the worth they can get by their own. Similarly, we introduce the sink-tree value which on the class of cycle-free digraph games is completely characterized by maximal-sink efficiency (MSE) and predecessor equivalence (PE). At the sink value every player receives what he contributes when he joins his predecessors in the graph and the total payoff for this player and all his predecessors is equal their worth. It is worth to emphasize that both values should not be considered as personal payment by one player to another one (the boss to his subordinate) but as distribution of the total worth according to the proposed scheme. We also provide simple recursive computational methods for computing these values and study their efficiency and when possible their stability. The introduced tree and sink values for arbitrary cycle-free digraph games may be considered as natural extensions of the tree and sink values defined correspondingly for rooted and sink forest digraph games (cf. (Demange, 2004), (Khmelnitskaya, 2010)). Furthermore, we extend the Ambec and Sprumont line-graph river game model of sharing a river (Ambec and Sprumont, 2002) to the case of a river with multiple sources, a delta and possibly islands by applying the results obtained to this more general problem of sharing a river among different agents located at different levels along the river bed restated in terms of a cycle-free digraph game.

The structure of the paper is as follows. Basic definitions and notation are introduced in Sect. 2.. Sect. 3. provides an axiomatic characterization of the tree value for a rooted-tree digraph game via component efficiency and subordinate equivalence. In Sect. 4. we discuss application to the water distribution problem of a river with multiple sources, a delta and possibly islands.

2. Preliminaries

A *cooperative game with transferable utility (TU game)* is a pair $\langle N, v\rangle$, where $N = \{1,\dots,n\}$ is a finite set of n, $n \geq 2$, players and $v\colon 2^N \to \mathrm{I\!R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. A subset $S \subseteq N$ is called a *coalition* and the associated real number $v(S)$ represents the *worth* of coalition S. The set of TU games with fixed player set N we denote $\mathcal{G}_N$. For simplicity of notation and if no ambiguity appears, we write v instead of $\langle N, v\rangle$ when we refer to a

TU game. A game $v \in \mathcal{G}$ is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$, such that $S \cap T = \emptyset$, and $v \in \mathcal{G}$ is *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$, for all $S, T \subseteq N$. A *value* on a subset $\mathcal{G}$ of $\mathcal{G}_N$ is a function $\xi \colon \mathcal{G} \to \mathbb{R}^N$ that assigns to every game $v \in \mathcal{G}$ a vector $\xi(v) \in \mathbb{R}^N$; the number $\xi_i(v)$ represents the *payoff* to player i, $i \in N$, in the game v. In the sequel we use standard notation $x(S) = \sum_{i \in S} x_i$, $x_S = (x_i)_{i \in S}$ for any $x \in \mathbb{R}^N$ and $S \subseteq N$, $|A|$ for the cardinality of a given set A, and omit brackets when writing one-player coalitions such as i instead of $\{i\}$, $i \in N$.

A payoff vector $x \in \mathbb{R}^N$ in a game $v \in \mathcal{G}$ is *efficient* if it holds that $x(N) = v(N)$. We also say that a coalition $S \subseteq N$ is *efficient* in a game $v \in \mathcal{G}$ with respect to a payoff vector $x \in \mathbb{R}^N$ if $x(S) = v(S)$.

The *core* (Gillies, 1953) of a game $v \in \mathcal{G}_N$ is defined as

$$C(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N),\ x(S) \geq v(S), \text{ for all } S \subseteq N\}.$$

For a game $v \in \mathcal{G}_N$, together with the core, we may consider the *weak core* defined as

$$\tilde{C}(v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N),\ x(S) \geq v(S), \text{ for all } S \subsetneq N\}.$$

A value ξ on a subset $\mathcal{G}$ of $\mathcal{G}_N$ is *stable* if for any game $v \in \mathcal{G}$ it holds that $\xi(v) \in C(v)$, and a value ξ on $\mathcal{G}$ is *weakly stable* if for any game $v \in \mathcal{G}$ it holds that $\xi(v) \in \tilde{C}(v)$.

The *cooperation structure* on the player set N is specified by a graph, directed or undirected, on N. An *undirected graph* on N consists of a set of nodes, being the elements of N, and a collection of unordered pairs of nodes $\Gamma \subseteq \Gamma_N^c = \{\,\{i,j\} \mid i,j \in N,\ i \neq j\}$, where Γ_N^c is the complete undirected graph without loops on N and an unordered pair $\{i,j\} \in \Gamma$ is a *link* between $i, j \in N$. A *directed graph*, or *digraph*, on N is given by a collection of directed links $\Gamma \subseteq \bar{\Gamma}_N^c = \{(i,j) \mid i,j \in N, i \neq j\}$. A subset Γ' of a graph Γ on N is a *subgraph* of Γ. For a subgraph Γ' of Γ, $S(\Gamma') \subseteq N$ is the set of nodes in Γ', i.e., $S(\Gamma') = \{i \in N \mid \exists j \in N \colon \{i,j\} \in \Gamma'\}$, if Γ is undirected, and $S(\Gamma') = \{i \in N \mid \exists j \in N \colon \{(i,j),(j,i)\} \cap \Gamma' \neq \emptyset\}$, if Γ is a digraph. For a graph Γ on N and a coalition $S \subseteq N$, the *subgraph of Γ on S* is the graph $\Gamma|_S = \{\{i,j\} \in \Gamma \mid i,j \in S\}$, if Γ is undirected, and $\Gamma|_S = \{(i,j) \in \Gamma \mid i,j \in S\}$, if Γ is directed.

In a graph Γ on N a sequence of different nodes $p = (i_1, \ldots, i_r)$ is a *path* in Γ from node i_1 to node i_r if $r \geq 2$ and for $h = 1, \ldots, r-1$ it holds that $\{i_h, i_{h+1}\} \in \Gamma$ when Γ is undirected and $\{(i_h, i_{h+1}), (i_{h+1}, i_h)\} \cap \Gamma \neq \emptyset$ when Γ is directed. In a digraph Γ on N a path $\boldsymbol{p} = (i_1, \ldots, i_r)$ is a *directed* path from node i_1 to node i_r if for all $h = 1, \ldots, r-1$ it holds that $(i_h, i_{h+1}) \in \Gamma$. For a digraph Γ on N and any $i, j \in N$ we denote by $\boldsymbol{P}_\Gamma(i,j)$ the set of all directed paths from i to j in Γ. Any node i of a (directed) path p we denote as an element of p, i.e., $i \in p$. Moreover, when for a directed path $\boldsymbol{p}$ in a digraph Γ we write $(i,j) \in \boldsymbol{p}$, we assume that i and j are consecutive nodes in $\boldsymbol{p}$. For any set P of (directed) paths, by $S(P) = \{i \in p \mid p \in P\}$ we denote the set of nodes determining the paths in P. A directed link $(i,j) \in \Gamma$ for which there exists a directed path $\boldsymbol{p}$ in Γ from i to j such that $\boldsymbol{p} \neq (i,j)$ is *inessential*, otherwise (i,j) is an *essential* link. A directed path $\boldsymbol{p}$ is a *proper path* if it contains only essential links.

In a graph Γ on N, undirected or directed, a sequence of nodes $(i_1, \ldots, i_{r+1})$ is a *cycle* if $r \geq 3$, $(i_1, \ldots, i_r)$ and $(i_2, \ldots, i_{r+1})$ are paths and $i_1 = i_{r+1}$. In a digraph Γ a sequence of nodes $(i_1, \ldots, i_{r+1})$ is a *directed cycle* if $r \geq 2$, $(i_1, \ldots, i_r)$ and

$(i_2, \ldots, i_{r+1})$ are directed paths, and $i_1 = i_{r+1}$. An undirected graph Γ is *cycle-free* if it contains no cycles. A digraph Γ on N is *cycle-free* if it contains no directed cycles, i.e., no node is a successor of itself. A digraph Γ on N is *strongly cycle-free* if it is cycle-free and contains no cycles. Remark that in a strongly cycle-free digraph all links are essential.

For a directed link $(i,j) \in \Gamma$, i is the *origin* and j is the *terminus*, i is a *superior* of j and j is a *subordinate* or *follower* of i. If a directed link (i,j) is essential, then j is a *proper subordinate* of i and i is a *proper superior* of j. All nodes having the same superior in Γ are called *brothers*. Besides, for $i, j \in N$, j is a *(proper) successor* of i and i is a *(proper) predecessor* of j if there is a directed (proper) path from i to j. For $i \in N$, we denote by $P_\Gamma(i)$ the set of all predecessors of i in Γ, by $O_\Gamma(i)$ the set of all superiors of i in Γ, by $O^*_\Gamma(i)$ the set of all proper superiors of i, by $F_\Gamma(i)$ the set of all subordinates of i in Γ, by $F^*_\Gamma(i)$ the set of all proper subordinates of i, by $S_\Gamma(i)$ the set of all successors of i in Γ, and by $B_\Gamma(i)$ the set of all brothers of i in Γ. Moreover, for $i \in N$, we define $\bar{P}_\Gamma(i) = P_\Gamma(i) \cup i$, $\bar{S}_\Gamma(i) = S_\Gamma(i) \cup i$, and $\bar{B}_\Gamma(i) = B_\Gamma(i) \cup i$. A coalition $S \subseteq N$ is a *full successors set in* Γ, if $S = \bar{S}_\Gamma(i)$ for some $i \in N$, and is a *full predecessors set in* Γ, if $S = \bar{P}_\Gamma(i)$ for some $i \in N$. A node $i \in N$ having no superior in Γ, i.e., $O_\Gamma(i) = \emptyset$, is a *root* in Γ. A node $i \in N$ having no subordinate in Γ, i.e., $F_\Gamma(i) = \emptyset$, is a *leaf* in Γ. For any $S \subseteq N$ denote by $R_\Gamma(S)$ the set of all roots in $\Gamma|_S$ and by $L_\Gamma(S)$ the set of all leaves in $\Gamma|_S$. For simplicity of notation, for a digraph Γ on N and $i \in N$, by Γ^i we denote the subgraph $\Gamma|_{\bar{S}_\Gamma(i)}$ and by Γ_i the subgraph $\Gamma|_{\bar{P}_\Gamma(i)}$. Given a digraph Γ on N and $i \in N$, the *in-degree* of i is defined as $d_\Gamma(i) = |O^*_\Gamma(i)|$ and the *out-degree* of i as $\tilde{d}_\Gamma(i) = |F^*_\Gamma(i)|$, while for any $i \in N$ and $j \in S_\Gamma(i)$ the *in-degree of j with respect to* i is equal to $d^i(j) = |O^*_{\Gamma^i}(j)|$ and for any $j \in P_\Gamma(i)$ the *out-degree of j with respect to* i is equal to $d_i(j) = |F^*_{\Gamma_i}(j)|$. Given a digraph Γ on N, $i \in N$ and $j \in P_\Gamma(i)$, a node $h \in S(\boldsymbol{P}_\Gamma(i,j))$ such that $d^i(h) \cdot d_j(h) > 1$ is called a *proper intersection point* in $S(\boldsymbol{P}_\Gamma(i,j))$.

Given a graph Γ on N, two nodes i and j in N are *connected* if there exists a path from node i to node j. Graph Γ on N is *connected* if any two nodes in N are connected. Given a graph Γ on N, a coalition $S \subseteq N$ is *connected* if the subgraph $\Gamma|_S$ is connected. For a graph Γ on N and coalition $S \subseteq N$, $C^\Gamma(S)$ is the set of all connected subcoalitions of S, S/Γ is the set of maximally connected subcoalitions of S, called the *components of* S, and $(S/\Gamma)_i$ is the component of S containing player $i \in S$.

A directed graph Γ on N is a *(rooted) tree* if it has precisely one root, denoted $r(\Gamma)$, and there is a unique directed path in Γ from this node to any other node in N. In a tree the root plays the role of the source of the stream presented via this graph. A directed graph Γ on N is a *sink tree* if the directed graph composed by the same set of links as Γ but with the opposite orientation is a rooted tree; in this case a root of a tree changes its meaning to an absorbing sink. A directed graph Γ is a *(rooted/sink) forest* if it is composed by a number of disjoint (rooted/sink) trees. A *line-graph* is a forest that contains links only between subsequent nodes. Both a rooted tree and a sink tree, and in particular a line-graph, are strongly cycle-free. For a directed graph Γ, a subgraph T is a *subtree* of Γ if T is a tree on $S(T)$. A subtree T of a digraph Γ is a *full subtree* if it contains together with its root $r(T)$ all successors of $r(T)$, in other words, $S(T) = \bar{S}_\Gamma(r(T))$. A full subtree T of Γ is a *maximal subtree* if its root is a root of Γ.

In what follows it is assumed that the cooperation structure on the player set N is specified by a cycle-free directed graph, not necessarily being strongly cycle-free. A pair $\langle v, \Gamma\rangle$ of a game $v \in \mathcal{G}_N$ and a cycle-free directed communication graph Γ on N constitutes a *game with cycle-free limited cooperation* or *cycle-free digraph structure* and is also called a *directed cycle-free graph game* or just a *cycle-free digraph game*. The set of all cycle-free digraph games on a fixed player set N is denoted $\mathcal{G}_N^\Gamma$. A *value* on a subset $\mathcal{G}$ of $\mathcal{G}_N^\Gamma$ is a function $\xi\colon \mathcal{G} \to \mathbb{R}^N$ that assigns to every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}$ a vector of payoffs $\xi(v, \Gamma) \in \mathbb{R}^N$. For any graph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$, a payoff vector $x \in \mathbb{R}^N$ is *component efficient* if for every component $C \in N/\Gamma$ it holds that $x(C) = v(C)$.

3. Main results

In this section we introduce two values for the class of cycle-free digraph games, not being necessarily strongly cycle-free.

For a directed link in an arbitrary digraph there are two different interpretations possible. One interpretation is that a link is directed to indicate which player has initiated the communication, but at the same time it represents a fully developed communication link. In such a case, following Myerson (Myerson, 1977), it is assumed that cooperation is possible among any set of connected players, i.e., the coalitions in which players are able to cooperate, the *productive coalitions*, are all the connected coalitions. In this case the focus is on component efficient values. Another interpretation of a directed link assumes that a directed link represents the only one-way communication situation. In that case not every connected coalition might be productive. In this paper we abide by the second interpretation of a directed link and consider two different options for creation of the productive coalitions.

3.1. Tree connectedness

In a cycle-free digraph Γ there is at least one node having no superior. A node without superior, i.e., any root in the graph, can be seen as a *topman* of the communication structure given by Γ. There are different scenarios possible for controlling cooperation in case of directed communication. First we assume that in any coalition each player can be controlled only by his predecessors and that nobody accepts that one of his subordinates becomes his equal partner if a coalition forms. This entails the assumption that the only productive coalitions are the so-called *tree connected*, or simply *t-connected*, coalitions, being the connected coalitions $S \in C^\Gamma(N)$ that also meet the condition that for every root $i \in R_\Gamma(S)$ it holds that $i \notin S_\Gamma(j)$ for any other root $j \in R_\Gamma(S)$. It is not difficult to see that the latter condition guarantees that every t-connected coalition inherits the subordination of players prescribed by Γ in N. Obviously, every component $C \in N/\Gamma$ is t-connected. Moreover, any full successors set in Γ is t-connected. A t-connected coalition is *full t-connected*, if it together with its roots contains all successors of these roots. Observe that a full t-connected coalition is the union of several full successors sets.

In what follows for a cycle-free digraph Γ on N and a coalition $S \subseteq N$, let $C_t^\Gamma(S)$ denote the set of all t-connected subsets of S, $[S/\Gamma]^t$ the set of maximally t-connected subsets of S, called the *t-connected components of* S, and $[S/\Gamma]_i^t$ the t-connected component of S containing player $i \in S$.

Since the communication is assumed to be one-way, we require for efficiency of a value that the t-connected coalition consisting of one of the roots of the graph together with all his successors realizes its worth. This gives the first property a value must satisfy, what we call maximal-tree efficiency.

A value ξ on $\mathcal{G}_N^\Gamma$ is *maximal-tree efficient* (MTE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ it holds that

$$\sum_{j\in \bar{S}_\Gamma(i)} \xi_j(v,\Gamma) = v(\bar{S}_\Gamma(i)), \qquad \text{for all } i \in R_\Gamma(N).$$

MTE generalizes the usual definition of efficiency for a tree. In a digraph with only one topman, the maximal-tree efficiency just says that the total payoff should be equal to the worth of the grand coalition. Still, MTE is not the productive component efficiency condition. Different from the Myerson case with undirected communication graph (Myerson, 1977) we assume that not every productive component is able to realize its exact capacity but only those with a tree structure. For example if one worker works in two different divisions, the two managers of these firms and the worker create a productive coalition. Yet, it is impossible to guarantee the efficiency of this coalition because there is no communication link between the managers of the two divisions.

The next property, what we call successor equivalence, says that if a link is deleted, each successor of the terminus of this link still receives the same payoff.

A value ξ on $\mathcal{G}_N^\Gamma$ is *successor equivalent* (SE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ it holds that for all $(i,j) \in \Gamma$

$$\xi_k(v, \Gamma\backslash(i,j)) = \xi_k(v,\Gamma), \qquad \text{for all } k \in \bar{S}_\Gamma(j).$$

SE means that the payoff to any member in the full successors set of a player does not change if any of the superiors of that player breaks his link to that player. It implies that for each successors set the payoff distribution is completely determined by the players of this set.

Along with MTE we consider a stronger efficiency property, what we call full-tree efficiency, that requires that every full successors set realizes its worth.

A value ξ on $\mathcal{G}_N^\Gamma$ is *full-tree efficient* (FTE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ it holds that

$$\sum_{j\in \bar{S}_\Gamma(i)} \xi_j(v,\Gamma) = v(\bar{S}_\Gamma(i)), \qquad \text{for all } i \in N. \tag{1}$$

Proposition 1. *On the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ MTE and SE together imply FTE.*

Proof. Let ξ be a value on $\mathcal{G}_N^\Gamma$ that meets MTE and SE, and let a cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ be arbitrarily chosen. For every $i \in N$ the subgraph Γ^i is a maximal tree in the subgraph $\Gamma\backslash \bigcup_{j\in O_\Gamma(i)}\{(j,i)\}$. Hence, due to MTE,

$$\sum_{j\in \bar{S}_\Gamma(i)} \xi_j(v, \Gamma\backslash \bigcup_{k\in O_\Gamma(i)} \{(k,i)\}) \overset{\text{MTE}}{=} v(\bar{S}_\Gamma(i)).$$

By successive application of SE,

$$\xi_j(v, \Gamma \setminus \bigcup_{k \in O_\Gamma(i)} \{(k,i)\}) \overset{\text{SE}}{=} \xi_j(v, \Gamma), \qquad \text{for all } j \in \bar{S}_\Gamma(i).$$

Whence,

$$\sum_{j \in \bar{S}_\Gamma(i)} \xi_j(v, \Gamma) = v(\bar{S}_\Gamma(i)), \qquad \text{for all } i \in N,$$

i.e., the value ξ meets FTE. ∎

It turns out that MTE and SE uniquely define a value on the class of cycle-free digraph games.

Theorem 1. *On the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ there is a unique value t that satisfies MTE and SE. For every cycle-free digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$, the value $t(v, \Gamma)$ satisfies the following conditions:*

(i) it obeys the recursive equality

$$t_i(v, \Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} t_j(v, \Gamma), \qquad \textit{for all } i \in N; \tag{2}$$

(ii) it admits the explicit representation in the form

$$t_i(v, \Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} \kappa_i(j) v(\bar{S}_\Gamma(j)), \qquad \textit{for all } i \in N, \tag{3}$$

where for all $i \in N$, $j \in S_\Gamma(i)$

$$\kappa_i(j) = \sum_{r=0}^{n-2} (-1)^r \kappa_i^r(j), \tag{4}$$

and $\kappa_i^r(j)$ is the number of tuples $(i_0, \ldots, i_{r+1})$ such that $i_0 = i$, $i_{r+1} = j$, $i_h \in S_\Gamma(i_{h-1})$, $h = 1, \ldots, r+1$.

Proof. Due to Proposition 1 the value t on $\mathcal{G}_N^\Gamma$ that satisfies MTE and SE meets FTE as well, wherefrom the recursive equality (2) follows straightforwardly. Next, we show that the representation in the form (2) is equivalent to the representation in the form (3). According to (2) it holds for the value t that every player receives what this player together with his successors can get on their own, their worth, minus what all his successors will receive by themselves. Since the same property holds for these successors as well, it is not difficult to see that (3) follows directly from (2) by successive substitution. Indeed,

$$t_i(v, \Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} t_j(v, \Gamma) \overset{(2)}{=}$$

$$v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} v(\bar{S}_\Gamma(j)) + \sum_{j \in S_\Gamma(i)} \sum_{k \in S_\Gamma(j)} t_k(v, \Gamma) \overset{(2)}{=}$$

$$v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} v(\bar{S}_\Gamma(j)) + \sum_{j \in S_\Gamma(i)} \sum_{k \in S_\Gamma(j)} v(\bar{S}_\Gamma(k)) - \sum_{j \in S_\Gamma(i)} \sum_{k \in S_\Gamma(j)} \sum_{h \in S_\Gamma(k)} t_h(v, \Gamma) \overset{(2)}{=}$$

$$\ldots = v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} \sum_{r=0}^{n-2} (-1)^r \kappa_i^r(j) v(\bar{S}_\Gamma(j)) = v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} \kappa_i(j) v(\bar{S}_\Gamma(j)).$$

From (3), we obtain immediately that the value t meets SE, because in any digraph Γ for all $(i,j) \in \Gamma$ and for every $k \in \bar{S}_\Gamma(j)$ the full subtrees Γ^k and $(\Gamma\backslash(i,j))^k$ coincide. This completes the proof, since MTE follows from FTE automatically. ∎

Corollary 1. According to (2) the value t assigns to every player the worth of his full successors set minus the total payoff to his successors. Wherefrom we obtain a simple recursive algorithm for computing the value t going upstream from the leaves of the given digraph.

Observe that the computation of the coefficients $\kappa_i(j)$, $i \in N$, $j \in S_\Gamma(i)$, in the explicit formula representation (3) requires, in general, the enumeration of quite a lot of possibilities. We show below that in many cases the coefficients $\kappa_i(j)$ can be easily computed and the value t can be presented in a computationally more transparent and simpler form. Before formulating the next theorem we introduce some additional notation.

For any digraph Γ on N and $i \in N$ the set $S_\Gamma(i)$ of all successors of i can be partitioned into three disjoint subsets $F^*_\Gamma(i)$, $S^1_\Gamma(i)$, and $S^2_\Gamma(i)$, i.e.,

$$S_\Gamma(i) = F^*_\Gamma(i) \cup S^1_\Gamma(i) \cup S^2_\Gamma(i),$$

where both sets $S^1_\Gamma(i)$ and $S^2_\Gamma(i)$ are composed by successors of i that are not proper subordinates of i. $S^1_\Gamma(i)$ consists of any of them for which all paths from i to that node j can be partitioned into a number of separate groups, might be only one group, such that all paths in the same group have at least one common node different from i and j and paths from different groups do not intersect between i and j, namely,

$$\begin{aligned} S^1_\Gamma(i) = \Big\{ j \in S_\Gamma(i)\backslash F^*_\Gamma(i) \mid \boldsymbol{P}_\Gamma(i,j) = \bigcup_{h=1}^{q} \boldsymbol{P}_h, \boldsymbol{P}_h \cap \boldsymbol{P}_l = \emptyset, h \neq l\colon \\ \forall h = 1,...,q, \exists k_h \in S(\boldsymbol{P}_h)\backslash\{i,j\}\colon \\ k_h \in \boldsymbol{p}, \forall \boldsymbol{p} \in \boldsymbol{P}_h \text{ and } \boldsymbol{p}_h \cap \boldsymbol{p}_l = \{i,j\}, \forall \boldsymbol{p}_h \in \boldsymbol{P}_h, \forall \boldsymbol{p}_l \in \boldsymbol{P}_l, h \neq l \Big\} \end{aligned}$$

and

$$S^2_\Gamma(i) = S_\Gamma(i) \setminus \big(F^*_\Gamma(i) \cup S^1_\Gamma(i)\big).$$

Remark that all $j \in S_\Gamma(i) \setminus F^*_\Gamma(i)$ with $d^i(j) = 1$ belong to $S^1_\Gamma(i)$ since the unique proper superior of j belongs to all paths $\boldsymbol{p} \in \boldsymbol{P}_\Gamma(i,j)$; in particular, it holds that $j \in S^1_\Gamma(i)$, when there is only one path from i to j, i.e., when $|\boldsymbol{P}_\Gamma(i,j)| = 1$. From here besides it follows that for all $j \in S^2_\Gamma(i)$, $d^i(j) > 1$. For every $j \in S^1_\Gamma(i)$ we define the *proper in-degree* $\tilde{d}^i(j)$ of j with respect to i as the number of groups $\boldsymbol{P}_h$, $h = 1,...,q$, in the partition of $\boldsymbol{P}_\Gamma(i,j)$.

Next, observe that for a given digraph Γ on N, for any $i \in N$ and $j \in S_\Gamma(i)$, all nodes forming a tuple $(i_0,...,i_{r+1})$ in which $i_0 = i$, $i_{r+1} = j$, $i_h \in S_\Gamma(i_{h-1})$, $h = 1,...,r+1$, belong to the same directed path $\boldsymbol{p} \in \boldsymbol{P}_\Gamma(i,j)$. Wherefrom it easily follows that for all $i \in N$ and $j \in S_\Gamma(i)$, $\kappa_i(j)$ given by (4) in fact is defined via tuples of nodes from the set of nodes $S(\boldsymbol{P}_\Gamma(i,j))$ that determine the set of directed paths $\boldsymbol{P}_\Gamma(i,j)$. Similar to the definition of $\kappa_i(j)$ given by (4), for any subset of nodes $M \subseteq S(\boldsymbol{P}_\Gamma(i,j))$ containing nodes i and j, we may define

$$\kappa_i(M;j) = \sum_{r=0}^{n-2} (-1)^r \kappa^r_i(M;j), \tag{5}$$

where $\kappa_i^r(M;j)$ counts only the tuples $(i_0,...,i_{r+1})$ for which $i_0 = i$, $i_{r+1} = j$, and $i_h \in S_\Gamma(i_{h-1}) \cap M$, $h = 1,\dots,r+1$. Remark that $\kappa_i(j) = \kappa_i(S(\boldsymbol{P}_\Gamma(i,j));j)$. The subset of $S(\boldsymbol{P}_\Gamma(i,j))$ composed by i, j, all proper subordinates $h \in F_\Gamma^*(i) \cap S(\boldsymbol{P}_\Gamma(i,j))$ and all proper intersection points in $S(\boldsymbol{P}_\Gamma(i,j))$ is called the *upper covering set* for $\boldsymbol{P}_\Gamma(i,j)$ and denoted $M_\Gamma(i,j)$. It turns out that on $F_\Gamma^*(i)$ and $S_\Gamma^1(i)$ the exact value of $\kappa_i(j)$ can be simply computed, while on $S_\Gamma^2(i)$ the computation of $\kappa_i(j)$ can be reduced to the enumeration only over the nodes from the upper covering set for $\boldsymbol{P}_\Gamma(i,j)$. For simplicity of notation we denote $\kappa_i(M_\Gamma(i,j);j)$ by $\kappa_i^M(j)$.

Theorem 2. *For every cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$ the value t given by (3) admits the equivalent representation in the form*

$$t_i(v,\Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j\in F_\Gamma^*(i)} v(\bar{S}_\Gamma(j)) +$$

$$+ \sum_{j\in S_\Gamma^1(i)} (\tilde{d}^i(j)-1)v(\bar{S}_\Gamma(j)) - \sum_{j\in S_\Gamma^2(i)} \kappa_i^M(j)v(\bar{S}_\Gamma(j)), \quad \textit{for all } i \in N. \tag{6}$$

If the consideration is restricted to only strongly cycle-free digraph games, then the above representation reduces to

$$t_i(v,\Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j\in F_\Gamma(i)} v(\bar{S}_\Gamma(j)), \quad \textit{for all } i \in N. \tag{7}$$

For rooted-forest digraph games defined by rooted forest digraph structures that are strongly cycle-free, the value given by (7) coincides with the tree value introduced first under the name of hierarchical outcome in (Demange, 2004), where it is also shown that under the mild condition of superadditivity it belongs to the core of the restricted game defined in (Myerson, 1977). More recently, the tree value for rooted-forest games was used as a basic element in the construction of the average tree solution for cycle-free undirected graph games in (Herings et al., 2008). In (Khmelnitskaya, 2010) it is shown that on the class of rooted-forest digraph games the tree value can be characterized via component efficiency and successor equivalence; moreover, it is shown that the class of rooted-forest digraph games is the maximal subclass in the class of strongly cycle-free digraph games where this axiomatization holds true. It is worth to recall that by definition for a rooted-tree digraph game every connected component is a tree. Hence, on the class of rooted-forest digraph games every connected component is productive and maximal-tree efficiency coincides with component efficiency.

From now on we refer to the value t given by (3), or equivalently by (6), as to the *root-tree value*, or simply the *tree value*, for cycle-free digraph games. The tree value assigns to every player the payoff equal to the worth of his full successors set minus the worths of all full successors sets of his proper subordinates plus or minus the worths of all full successors sets of any other of his successors that are subtracted or added more than once. Moreover, for any player $i \in N$ and his successor $j \in N$ that is not his proper subordinate, the coefficient $\kappa_i(j)$ indicates the number of overlappings of full successors sets of all proper subordinates of i at node j. In fact a player receives what he contributes when he joins his successors when only the full successors sets, that are the only efficient productive coalitions, are counted. Since a leaf has no successors, a leaf just gets his own worth. Besides, it is worth to note and not difficult to check that the right sides of both formulas (6) and (7) being considered with respect not to coalitional worths but to players in these coalitions contain only player i when taking into account all pluses and minuses.

The validity of the first statement of Theorem 2 follows directly from Theorem 1 and Lemma 1 below. The second statement follows easily from the first one. Indeed, in any strongly cycle-free digraph Γ all links are essential and $d^i(j) = 1$ for all $i \in N$, $j \in S_\Gamma(i)$. Whence it easily follows that $F^*_\Gamma(i) = F_\Gamma(i)$, $S^2_\Gamma(i) = \emptyset$, and $\tilde{d}^i(j) = d^i(j) = 1$ for all $j \in S^1_\Gamma(i)$.

Lemma 1. *For a given digraph Γ on N, the coefficients $\kappa_i(j)$, $i \in N$, $j \in S_\Gamma(i)$, defined by (4) satisfy the following properties:*

- *(i) if a link $(k,l) \in \Gamma$ is inessential, then for all $i \in N$ and $j \in S_\Gamma(i)$, $\kappa_i(j)$ defined on Γ is equal to $\kappa_i(j)$ defined on $\Gamma \backslash (k,l)$;*
- *(ii) $\kappa_i(j) = 1$ for all $i \in N$, $j \in F^*_\Gamma(i)$;*
- *(iii) $\kappa_i(j) = -\tilde{d}^i(j) + 1$ for all $i \in N$, $j \in S^1_\Gamma(i)$;*
- *(iv) $\kappa_i(j) = \kappa^M_i(j)$ for all $i \in N$ and $j \in S_\Gamma(i)$.*

Proof. (i). It is sufficient to prove the statement only in case when $k \in S_\Gamma(i)$ and $j \in S_\Gamma(l)$. Let $\boldsymbol{p} \in \boldsymbol{P}_\Gamma(i,j)$ be such that $\boldsymbol{p} \ni (k,l)$. By definition of an inessential link there exists $\boldsymbol{p}_0 \in \boldsymbol{P}_\Gamma(k,l)$ such that $\boldsymbol{p}_0 \neq (k,l)$. It is not difficult to see that the path $\boldsymbol{p}_1 = \boldsymbol{p} \backslash (k,l) \cup \boldsymbol{p}_0$ obtained from the path $\boldsymbol{p}$ by replacing the link (k,l) by the path $\boldsymbol{p}_0$ belongs to $\boldsymbol{P}_\Gamma(i,j)$, and moreover, all tuples $(i_0, ..., i_{r+1})$ in the definition of $\kappa_i(j)$ that belong to $\boldsymbol{p}$ also belong to $\boldsymbol{p}_1$. Whence it follows straightforwardly that deleting an inessential link does not change the value of $\kappa_i(j)$.

(ii). If $j \in F^*_\Gamma(i)$, then $\boldsymbol{P}_\Gamma(i,j)$ contains only the path $\boldsymbol{p} = (i,j)$ and the only tuple $(i_0, ..., i_{r+1})$ is (i,j) with $r = 0$. Wherefrom it follows that $\kappa_i(j) = 1$.

(iii). Let $j \in S^1_\Gamma(i)$. First consider the case when $\tilde{d}^i(j) = 1$. Then there exists $k \in S(\boldsymbol{P}_\Gamma(i,j))$, $k \neq i, j$, such that $k \in \boldsymbol{p}$ for all $\boldsymbol{p} \in \boldsymbol{P}_\Gamma(i,j)$. By definition, $\kappa^r_i(j)$ is equal to the number of tuples $(i_0, \ldots, i_{r+1})$ such that $i_0 = i$, $i_{r+1} = j$, $i_h \in S_\Gamma(i_{h-1})$, $h = 1, \ldots, r+1$, or equivalently, $\kappa^r_i(j)$ is equal to the number of these tuples $(i_0, \ldots, i_{r+1})$ that do not contain k plus the number of these tuples $(i_0, \ldots, i_{r+1})$ that contain k. Notice that since $k \in \boldsymbol{p}$ for all $\boldsymbol{p} \in \boldsymbol{P}_\Gamma(i,j)$, for every r-tuple $(i_0, \ldots, i_{r+1})$ that does not contain k there exists a uniquely defined $(r+2)$-tuple composed by the same nodes plus the node k. From which together with equality (4) it follows that $\kappa_i(j) = 0$.

Let now $\tilde{d}^i(j) > 1$. Then $\boldsymbol{P}_\Gamma(i,j) = \bigcup\limits_{h=1}^{q} \boldsymbol{P}_h$ with $q = \tilde{d}^i(j)$ and there exist nodes $k_h \neq i, j$, $h = 1, ..., q$ such that $k_h \in \boldsymbol{p}$ for all paths $\boldsymbol{p} \in \boldsymbol{P}_h$ and all paths $\boldsymbol{p}_h \in \boldsymbol{P}_h$ and $\boldsymbol{p}_l \in \boldsymbol{P}_l$, $h \neq l$, intersect only at i and j. We may split the computation of $\kappa_i(j)$ by the computation via groups of paths $\boldsymbol{P}_h$ excluding on each step the tuples that were counted at the previous steps, i.e.,

$$\kappa_i(j) = \kappa_i(S(\boldsymbol{P}_1); j) + [\kappa_i(S(\boldsymbol{P}_2); j) - \kappa_i(S(\boldsymbol{P}_1 \cap \boldsymbol{P}_2; j)] + \ldots$$

$$\ldots + [\kappa_i(S(\boldsymbol{P}_q); j) - \kappa_i(S(\bigcap_{h=1}^{q} \boldsymbol{P}_h); j)].$$

Applying the same argument as in the proof of the case when $\tilde{d}^i(j) = 1$, we obtain that for all $h = 1, \ldots, q$,

$$\kappa_i(S(\boldsymbol{P}_h); j) = 0.$$

Since the paths from different groups $\boldsymbol{P}_h$ do not intersect between i and j, only tuple $(i_0, i_1) = (i, j)$ with $r = 0$ belongs to all $\boldsymbol{p} \in \boldsymbol{P}_\Gamma(i, j)$. Therefore, for all $h = 2, \ldots, q$,

$$\kappa_i(S(\bigcap_{h=1}^{h} \boldsymbol{P}_h); j) = -1.$$

Then the validity of (iii) follows immediately from the last three equalities.

(iv). If $M_\Gamma(i, j) \neq S(\boldsymbol{P}_\Gamma(i, j))$, consider arbitrary $k \in S(\boldsymbol{P}_\Gamma(i, j)) \setminus M_\Gamma(i, j)$. We may split the computation of $\kappa_i(j)$ into two parts:

$$\kappa_i(j) = \kappa_i(j; k) + \kappa_i(S(\boldsymbol{P}_\Gamma(i, j)) \setminus \{k\}; j),$$

where $\kappa_i(j; k)$ is computed via tuples $(i_0, \ldots, i_{r+1}) \ni k$ and $\kappa_i(S(\boldsymbol{P}_\Gamma(i, j)) \setminus \{k\}; j)$ is computed via tuples $(i_0, \ldots, i_{r+1}) \subset S(\boldsymbol{P}_\Gamma(i, j)) \setminus \{k\}$, i.e., tuples $(i_0, \ldots, i_{r+1}) \not\ni k$. By definition of a covering set, $M_\Gamma(i, j)$ contains predecessors of k, i.e., $M_\Gamma(i, j) \cap P_\Gamma(k) \neq \emptyset$. Moreover, since $k \notin M_\Gamma(i, j)$, i.e., k is neither a proper subordinate of i nor a proper intersection point in the subgraph $\Gamma|_{S(\boldsymbol{P}_\Gamma(i,j))}$, there exists $h \in M_\Gamma(i, j) \cap P_\Gamma(k)$ that belongs to all paths $\boldsymbol{p} \in \boldsymbol{P}_\Gamma(i, j)$, $\boldsymbol{p} \ni k$. Applying the same argument as above in the proof of the statement (iii), now with respect to h, we obtain that $\kappa_i(j; k) = 0$. Thus $\kappa_i(j) = \kappa_i(S(\boldsymbol{P}_\Gamma(i, j)) \setminus \{k\}; j)$. Repeating the same reasoning successively with respect to all $k' \in S(\boldsymbol{P}_\Gamma(i, j)) \setminus \big(M_\Gamma(i, j) \cup \{k\}\big)$ we obtain $\kappa_i(j) = \kappa_i(M_\Gamma(i, j); j) \stackrel{def}{=} \kappa_i^M(j)$. ∎

Example 1. The examples of digraphs given in Figure 1 demonstrate the more complicated situation with the computation of coefficients $\kappa_i(j)$ when $j \in S^2_\Gamma(i)$ for some $i \in N$.

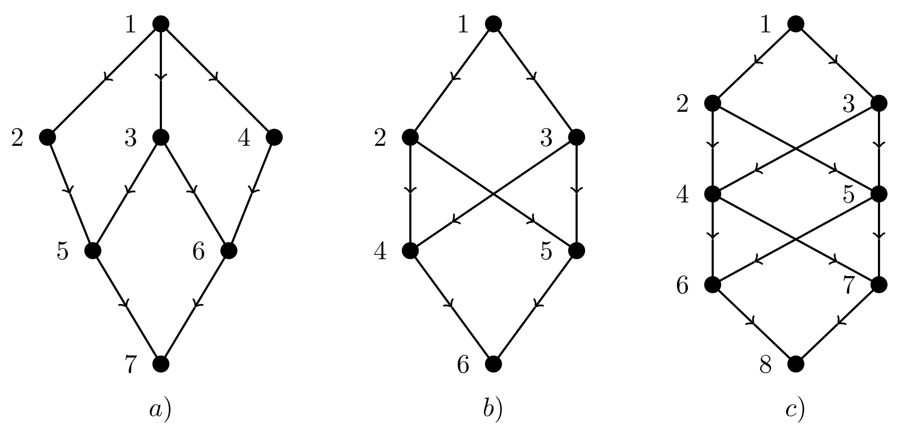

Fig. 1.

Figure 1,a: $7 \in S^2_\Gamma(1)$, $d^1(7) = 2$, $\kappa_1(7) = 0$;
Figure 1,b: $6 \in S^2_\Gamma(1)$, $d^1(6) = 2$, $\kappa_1(6) = 1$.
Figure 1,c: $8 \in S^2_\Gamma(1)$, $d^1(8) = 2$, $\kappa_1(8) = -1$.

Example 2. Figure 2 provides an example of the tree value for a 10-person game with cycle-free but not strongly cycle-free digraph structure.

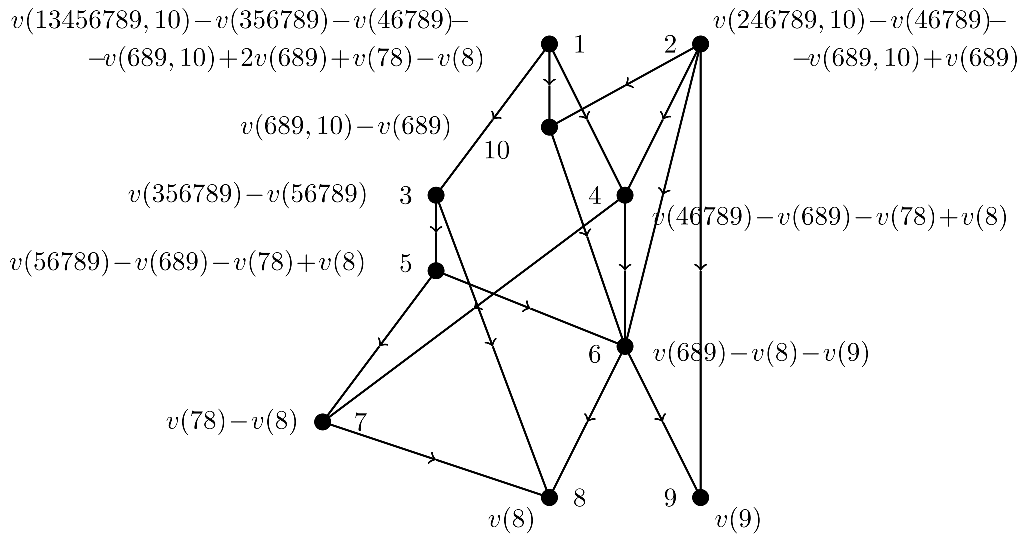

Fig. 2.

The tree value may be computed in two different ways, either by the recursive algorithm based on the recursive equality (2) or using the explicit formula representation (6).

We explain in detail the computation of $t_1(v,\Gamma)$ based on the explicit representation given by (6):

$\bar{S}_\Gamma(1) = \{1,3,4,5,6,7,8,9,10\}$.

$3,4,10 \in F^*_\Gamma(1) \Longrightarrow\ \kappa_1(3) = \kappa_1(4) = \kappa_1(10) = 1$.

$5,6,7,9 \in S^1_\Gamma(1), d^1(5) = d^1(9) = 1,\ d^1(6) = 3,\ d^1(7) = 2 \Longrightarrow\ \ \kappa_1(5) = \kappa_1(9) = 0,\ \kappa_1(6) = -2,\ \kappa_1(7) = -1$.

$8 \in S^2_\Gamma(1)$:

$\boldsymbol{P}_\Gamma(1,8) = \big\{\boldsymbol{p}_1 = (1,3,5,7,8),\ \boldsymbol{p}_2 = (1,3,5,6,8),\ \boldsymbol{p}_3 = (1,10,6,8),$
$\boldsymbol{p}_4 = (1,4,7,8),\ \boldsymbol{p}_5 = (1,4,6,8),\ \boldsymbol{p}_6 = (1,3,8)\big\}$;

we eliminate the path $\boldsymbol{p}_6$ since it contains inessential link $(3,8)$;

$M = \{1,4,5,6,7,8,10\}$ is a minimal covering set for $\boldsymbol{P}_\Gamma(1,8)$;

$\kappa_1(\boldsymbol{p}_1;8) = 0$;

$\boldsymbol{p}_2\backslash\boldsymbol{p}_1$ contains tuples $(1,6,8)$ and $(1,5,6,8)$ $\Longrightarrow\ \kappa_1(\boldsymbol{p}_2\backslash\boldsymbol{p}_1;8) = 0$;

$\boldsymbol{p}_3\backslash(\boldsymbol{p}_1\cup\boldsymbol{p}_2)$ contains tuples $(1,10,8)$, $(1,10,6,8))$ $\Longrightarrow\ \kappa_1(\boldsymbol{p}_3\backslash(\boldsymbol{p}_1\cup\boldsymbol{p}_2);8) = 0$;

$\boldsymbol{p}_4\backslash(\boldsymbol{p}_1\cup\boldsymbol{p}_2\cup\boldsymbol{p}_3)$ contains $(1,4,8)$, $(1,4,7,8))$ $\Longrightarrow\ \kappa_1(\boldsymbol{p}_4\backslash(\boldsymbol{p}_1\cup\boldsymbol{p}_2\cup\boldsymbol{p}_3);8) = 0$;

$\boldsymbol{p}_5\backslash(\boldsymbol{p}_1\cup\boldsymbol{p}_2\cup\boldsymbol{p}_3\cup\boldsymbol{p}_4)$ contains $(1,4,6,8))$ $\Longrightarrow\ \kappa_1(\boldsymbol{p}_5\backslash(\boldsymbol{p}_1\cup\boldsymbol{p}_2\cup\boldsymbol{p}_3\cup\boldsymbol{p}_4);8) = 1$;

$\Longrightarrow\ \kappa_1(8) = 1$.

$t_1(v,\Gamma) = v(13456789,10) - v(356789) - v(46789) - v(689,10) + 2v(689) + v(78) - v(8)$.

Example 3. Figure 3 gives an example of the tree value for a 10-person game with strongly cycle-free digraph structure.

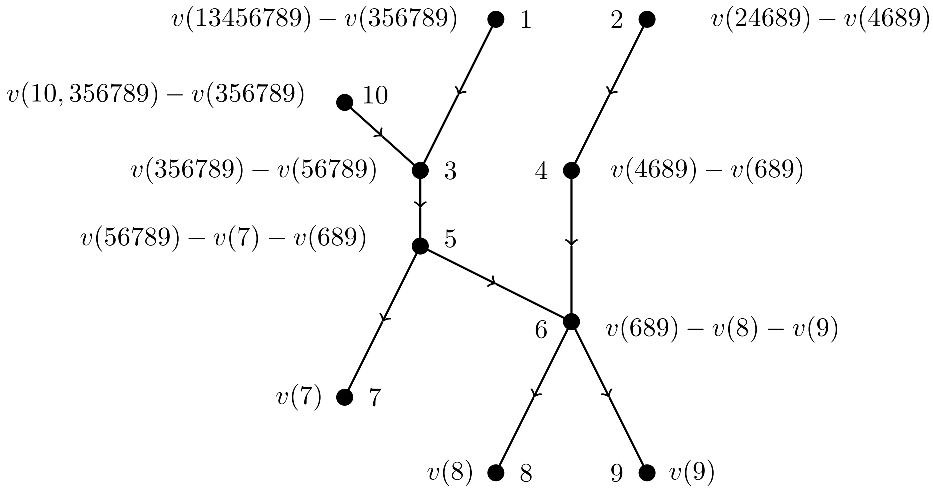

Fig. 3.

It turns out that the tree value not only meets FTE but FTE alone uniquely defines the tree value on the class of cycle-free digraph games.

Theorem 3. *On the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ the tree value is the unique value that satisfies FTE.*

Proof. Since the tree value satisfies FTE, to prove the theorem it is enough to show that the tree value is the unique value that meets FTE on $\mathcal{G}_N^\Gamma$. Let a value ξ on $\mathcal{G}_N^\Gamma$ satisfy axiom FTE. Then, because of FTE, (1) holds for every $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$. Every digraph Γ under consideration is cycle-free, i.e., no player in N appears to be a successor of itself. Hence, due to the arbitrariness of game $\langle v, \Gamma\rangle$, the n equalities in (1) are independent. Therefore, we have a system of n independent linear equalities with respect to n variables $\xi_j(v, \Gamma)$ which uniquely determines the value $\xi(v, \Gamma)$ that in this case coincides with $t(v, \Gamma)$. ∎

Corollary 2. FTE on the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ implies not only MTE but SE as well.

3.2. Overall efficiency and stability

In this subsection we consider efficiency and stability of the tree value. First we derive for the tree value the total payoff for any t-connected coalition.

Theorem 4. *In a cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$, for any t-connected coalition $S \in C_t^\Gamma(N)$ it holds that*

$$\sum_{i\in S} t_i(v,\Gamma) = \sum_{i\in R_\Gamma(S)} v(\bar{S}_\Gamma(i)) - \\ - \sum_{i\in S\setminus R_\Gamma(S)} \big(\kappa_S(i)-1\big) v(\bar{S}_\Gamma(i)) - \sum_{i\in \bar{S}_\Gamma(S)\setminus S} \kappa_S(i) v(\bar{S}_\Gamma(i)), \qquad (8)$$

where

$$\bar{S}_\Gamma(S) = \bigcup_{i \in R_\Gamma(S)} \bar{S}_\Gamma(i),$$

$$\kappa_S(i) = \sum_{j \in \bar{P}_\Gamma(i) \cap \bar{S}_\Gamma(S)} \kappa_j(i), \quad \textit{for all } i \in \bar{S}_\Gamma(S),$$

while $\kappa_S(i) = 1$ *when* $d_N(i) = 1$, *where for any* t*-connected coalition* $S \in C_t^\Gamma(N)$, *for all* $i \in \bar{S}_\Gamma(S)$, $d_S(i)$ *is the in-degree of* i *in the subgraph* $\Gamma|_{\bar{S}_\Gamma(S)}$, *i.e.,*

$$d_S(i) = |O_\Gamma(i) \cap \bar{S}_\Gamma(S)|,$$

in particular, $d_N(i) = d_\Gamma(i)$ *for all* $i \in N$.

If the consideration is restricted to only strongly cycle-free digraph games, then for any t*-connected coalition* $S \in C_t^\Gamma(N)$ *it holds that*

$$\sum_{i \in S} t_i(v, \Gamma) = \sum_{i \in R_\Gamma(S)} v(\bar{S}_\Gamma(i)) - \\ - \sum_{i \in S \setminus R_\Gamma(S)} \big(d_S(i) - 1\big) v(\bar{S}_\Gamma(i)) - \sum_{i \in R_\Gamma(\bar{S}_\Gamma(S) \setminus S)} d_S(i) v(\bar{S}_\Gamma(i)). \qquad (9)$$

Proof. Let $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$ be a cycle-free digraph game and let S be any t-connected coalition $S \in C_t^\Gamma(N)$. Then it holds that

$$\sum_{i \in S} t_i(v, \Gamma) \overset{(3)}{=} \sum_{i \in S} \big(v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} \kappa_i(j) v(\bar{S}_\Gamma(j)) \big) =$$

$$= \sum_{i \in R_\Gamma(S)} v(\bar{S}_\Gamma(i)) - \sum_{i \in S \setminus R_\Gamma(S)} \big(\sum_{j \in S_\Gamma(i)} (\kappa_i(j) - 1) v(\bar{S}_\Gamma(i)) \big) - \sum_{i \in \bar{S}_\Gamma(S) \setminus S} \big(\sum_{j \in S_\Gamma(i)} \kappa_i(j) v(\bar{S}_\Gamma(i)) \big).$$

Since for all $i, j \in S$ with $j \in S_\Gamma(i)$ every path from i to j belongs to S, (8) follows straightforwardly from the last equality.

Next, if $d_N(i) = 1$, then due to Lemma 1 for all $j \in \big(\bar{P}_\Gamma(i) \cap \bar{S}_\Gamma(S)\big) \setminus F_\Gamma(i)$, $d^j(i) = 0$ and therefore $\kappa_j(i) = 0$, and for $j \in F_\Gamma(i) \cap \bar{S}_\Gamma(S)$, $\kappa_j(i) = 1$.

In case Γ is a strongly cycle-free digraph, it holds that

$$\sum_{i \in S} t_i(v, \Gamma) \overset{(7)}{=} \sum_{i \in S} \big(v(\bar{S}_\Gamma(i)) - \sum_{j \in F_\Gamma(i)} v(\bar{S}_\Gamma(j)) \big) =$$

$$= \sum_{i \in R_\Gamma(S)} v(\bar{S}_\Gamma(i)) - \sum_{i \in S \setminus R_\Gamma(S)} \big(d_S(i) - 1\big) v(\bar{S}_\Gamma(i)) - \sum_{\substack{j \in F_\Gamma(i) \\ i \in S,\, j \notin S}} d_S(j) v(\bar{S}_\Gamma(j)).$$

To complete the proof of (9) it suffices to notice that, since Γ a strongly cycle-free digraph, every subordinate $j \in F_\Gamma(i)$ of $i \in S$ that does not belong to S is a root in $\bar{S}_\Gamma(S) \setminus S$. ∎

From Theorem 4 it follows that for any cycle-free digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$ the overall efficiency is given by

$$\sum_{i \in N} t_i(v, \Gamma) = \sum_{i \in R_\Gamma(N)} v(\bar{S}_\Gamma(i)) - \sum_{i \in N \setminus R_\Gamma(N)} \big(\kappa_N(i) - 1\big) v(\bar{S}_\Gamma(i)), \qquad (10)$$

while if the consideration is restricted to only strongly cycle-free digraph games, (10) reduces to

$$\sum_{i\in N} t_i(v,\Gamma) = \sum_{i\in R_\Gamma(N)} v(\bar{S}_\Gamma(i)) - \sum_{i\in N\setminus R_\Gamma(N)} \big(d_\Gamma(i)-1\big) v(\bar{S}_\Gamma(i)). \tag{11}$$

To support these expressions we recall the Myerson model in of a game with undirected cooperation structure (Myerson, 1977), in which the component efficiency entails the equality

$$\sum_{i\in N} \xi_i(v,\Gamma) = \sum_{C\in N/\Gamma} v(C). \tag{12}$$

While the right-side expression in (12) is composed by connected components that are the only efficient productive elements in the Myerson's model, the building bricks in (10) and (11) are the full successors sets which are the only efficient productive coalitions under the assumption of tree connectedness. Observe also that for strongly cycle-free rooted-forest digraph games (11) reduces to (12),

$$\sum_{i\in N} t_i(v,\Gamma) = \sum_{i\in R_\Gamma(N)} v(\bar{S}_\Gamma(i)) = \sum_{C\in N/\Gamma} v(C).$$

For a cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$, we define the *t-core* $C^t(v,\Gamma)$ as the set of component efficient payoff vectors that are not dominated by any t-connected coalition,

$$C^t(v,\Gamma) = \{x\in \mathbb{R}^N \mid x(C)=v(C), \forall C\in N/\Gamma;\ x(S)\geq v(S), \forall S\in C_t^\Gamma(N)\}, \tag{13}$$

while the *weak t-core* $\tilde{C}^t(v,\Gamma)$ is the set of weakly component efficient payoff vectors that are not dominated by any t-connected coalition,

$$\tilde{C}^t(v,\Gamma) = \{x\in \mathbb{R}^N \mid x(C)\leq v(C), \forall C\in N/\Gamma;\ x(S)\geq v(S), \forall S\in C_t^\Gamma(N)\}. \tag{14}$$

Theorem 5. *The tree value on the subclass of superadditive rooted-forest digraph games is t-stable.*

Proof. Let $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$ be a superadditive rooted-forest digraph game arbitrarily chosen. We show that the tree value $t(v,\Gamma)$ belongs to the core $C^t(v,\Gamma)$. Consider arbitrary $C\in N/\Gamma$, then C is a tree. Let $i\in C$ be a root in Γ, then $C=\bar{S}_\Gamma(i)$ because of the rooted-forest structure of Γ. Due to the full-tree efficiency of the tree value, it holds that

$$\sum_{j\in \bar{S}_\Gamma(i)} t_j(v,\Gamma) \overset{FTE}{=} v(\bar{S}_\Gamma(i)),$$

wherefrom it follows that

$$\sum_{j\in C} t_j(v,\Gamma) = v(C).$$

Let now $S\in C_t^\Gamma(N)$. Because of the rooted-forest structure of Γ, it holds that $d_N(i)=1$ for all $i\in N\setminus R_\Gamma(N)$. Wherefrom it follows that $\Gamma|_S$ contains exactly one root, say, node i, $\Gamma|_S$ is a subtree, and $S\subseteq \bar{S}_\Gamma(i)$. Moreover, since Γ is strongly cycle-free, $\Gamma|_{\bar{S}_\Gamma(i)}$ is a full subtree, and because of the tree structure of $\Gamma|_S$, $\Gamma|_{\bar{S}_\Gamma(i)\backslash S}$ consists of a collection (might be empty) of disconnected full subtrees, i.e., $\Gamma|_{\bar{S}_\Gamma(i)\backslash S} = \bigcup_{k=1}^q T_k$ where $T_k \bigcap T_l = \emptyset$, $k \neq l$, and $q = |[\bar{S}_\Gamma(i)\backslash S]/\Gamma|$ is the number of components in $\bar{S}_\Gamma(i)\backslash S$. Hence,

$$\bar{S}_\Gamma(i) = S \cup \bigcup_{k=1}^{q} T_k.$$

Applying again the full-tree efficiency of the tree value, we obtain that

$$\sum_{j \in \bar{S}_\Gamma(i)} t_j(v,\Gamma) \overset{FTE}{=} v(\bar{S}_\Gamma(i)),$$

and

$$\sum_{j \in T_k} t_j(v,\Gamma) \overset{FTE}{=} v(T_k), \qquad \text{for all } \; k = 1, \dots, q.$$

From the superadditivity of v and the last three equalities, it follows that

$$\sum_{j \in S} t_j(v,\Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{k=1}^{q} v(T_k) \geq v(S). \qquad \blacksquare$$

Remark 1. The statement of Theorem 5 can also be obtained as a corollary of the stability result proved in (Demange, 2004). Indeed, in a rooted forest every connected component has a tree structure and, therefore, is t-connected. Whence, for any rooted-forest digraph game the t-core coincides with the core of the Myerson restricted game.

However, the following examples show that for t-stability of a superadditive digraph game the requirement on the digraph to be a rooted forest is non-reducible. In Example 4 the tree value of a superadditive cycle-free but not strongly cycle-free digraph game violates individual rationality and, therefore, does not meet the second constraint of the weak t-core, while in Example 5 the tree value of a superadditive strongly cycle-free game in which the graph contains two roots violates weak efficiency.

Example 4. Consider a 4-person cycle-free superadditive digraph game $\langle v,\Gamma\rangle$ with $v(24) = v(34) = v(234) = v(N) = 1, v(S) = 0$ otherwise, and Γ given in Figure 4.

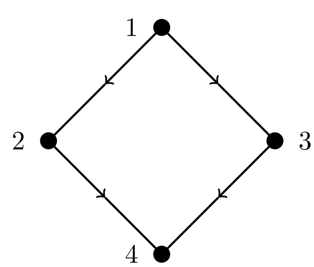

Fig. 4.

Then $t(v,\Gamma) = (-1, 1, 1, 0)$, whence $t_1(v,\Gamma) = -1 < 0 = v(1)$. Remark that every singleton coalition, in particular $S = \{1\}$, is t-connected.

Example 5. Consider a 3-person cycle-free superadditive digraph game $\langle v, \Gamma\rangle$ with $v(12) = v(13) = v(N) = 1, v(S) = 0$ otherwise, and Γ given in Figure 5.

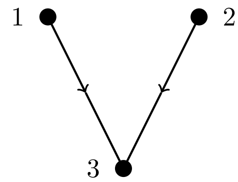

Fig. 5.

Then $t(v, \Gamma) = (1, 1, 0)$, whence $t_1(v, \Gamma) + t_2(v, \Gamma) + t_3(v, \Gamma) = 2 > 1 = v(N)$.

A cycle-free digraph game $\langle v, \Gamma\rangle$ is *t-convex*, if for all t-connected coalitions $T, Q \subset C_t^{\Gamma}(N)$ such that T is a full t-connected set, Q is a full successors set, and $T \cup Q \in C_t^{\Gamma}(N)$, it holds that

$$v(T) + v(Q) \leq v(T \cup Q) + v(T \cap Q). \tag{15}$$

Theorem 6. *The tree value on the subclass of t-convex strongly cycle-free digraph games is weakly efficient.*

Proof. Let $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$ be any t-convex strongly cycle-free digraph game. Assume that Γ is connected, otherwise we apply the same argument to any component $C \in N/\Gamma$. If there is only one root in Γ, it holds that $\sum_{i=1}^{n} t_i(v, \Gamma) = v(N)$ and the tree value is even efficient. So, suppose that there are q different roots $r_1, \ldots, r_q$ in Γ for some $q \geq 2$. Since Γ is connected, the roots in Γ can be ordered in such a way that

$$\bigcup_{h=1}^{j-1} \bar{S}_{\Gamma}(r_h) \cap \bar{S}_{\Gamma}(r_j) \neq \emptyset, \qquad \text{for } j = 2, ..., q.$$

For $j = 1, ..., q$ let $T_j = \bigcup_{h=1}^{j} \bar{S}_{\Gamma}(r_h)$. Then from the strongly cycle-freeness of Γ it follows that for $j = 2, ..., q$ there exists a unique $i_j \in N$ such that

$$T_{j-1} \cap \bar{S}_{\Gamma}(r_j) = \bar{S}_{\Gamma}(i_j).$$

By t-convexity of the digraph game $\langle v, \Gamma\rangle$ it holds that

$$v(T_{j-1}) + v(\bar{S}_{\Gamma}(r_j)) \leq v(T_j) + v(\bar{S}_{\Gamma}(i_j)), \qquad \text{for } j = 2, ..., q.$$

Since $T_1 = \bar{S}_{\Gamma}(r_1)$ and $T_q = N$, then applying the last inequality successively $q - 1$ times we obtain

$$\sum_{j=1}^{q} v(\bar{S}_{\Gamma}(r_j)) \leq v(N) + \sum_{j=2}^{q} v(\bar{S}_{\Gamma}(i_j)).$$

Hence,

$$v(N) \geq \sum_{j=1}^{q} v(\bar{S}_{\Gamma}(r_j)) - \sum_{j=2}^{q} v(\bar{S}_{\Gamma}(i_j)).$$

Since Γ is strongly cycle-free, for any $i \in N \backslash R_\Gamma(N)$, node i has $d_\Gamma(i)$ different roots as predecessor, which implies that the term $v(\bar{S}_\Gamma(i))$ appears precisely $d_\Gamma(i) - 1$ times. Therefore,

$$v(N) \geq \sum_{i \in R_\Gamma(N)} v(\bar{S}_\Gamma(i)) - \sum_{i \in N \backslash R_\Gamma(N)} \big(d_\Gamma(i) - 1\big) v(\bar{S}_\Gamma(i)). \qquad \blacksquare$$

The following example of a convex strongly cycle-free digraph game shows that even under the assumption of convexity of a given digraph game, which is stronger than t-convexity, one or more constraints for not being dominated in the definition of the week t-core might be violated by the tree value, and therefore, the tree value is not weakly t-stable.

Example 6. Consider a 5-person cycle-free convex digraph game $\langle v, \Gamma \rangle$ with $v(N) = 10$, $v(123) = v(1234) = v(1235) = 3$, $v(1345) = v(2345) = 2$, $v(S) = 0$ otherwise, and the strongly cycle-free digraph Γ given in Figure 6.

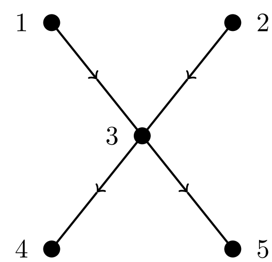

Fig. 6.

Then $t(v, \Gamma) = (1, 1, 0, 0, 0)$, whence, the total payoff of t-connected coalition $S = \{1, 2, 3\}$ $t_1(v, \Gamma) + t_2(v, \Gamma) + t_3(v, \Gamma) = 2 < 3 = v(123)$.

From (10) it follows that for a cycle-free (for simplicity connected) digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$ a necessary and sufficient condition for the weak efficiency of the tree value is that

$$\sum_{i \in R_\Gamma(N)} v(\bar{S}_\Gamma(i)) \leq v(N) + \sum_{i \in N \backslash R_\Gamma(N)} \big(\kappa_N(i) - 1\big) v(\bar{S}_\Gamma(i)). \tag{16}$$

Since $N = \bigcup_{i \in R_\Gamma(N)} \bar{S}_\Gamma(i)$, the grand coalition equals the union of the successors sets of all roots in the graph Γ. In case there is only one root in Γ, condition (16) is redundant, because the left side is then equal to $v(N)$. In case there is more than one root in Γ, the different successors sets of the roots of Γ will intersect each other and for any $i \in N \backslash R_\Gamma(N)$ the number $\kappa_N(i) - 1$ is the number of times that the successors set $\bar{S}_\Gamma(i)$ of node i equals the intersection of successors sets of the roots of Γ. Therefore, condition (16) is a kind of convexity condition for the grand coalition saying that the sum of the worths of the successors sets of all the roots of the graph should be less than or equal to the worth of the grand coalition (their union) plus the total worths of their intersections. In a firm where any full successors set of a root is a division within the firm and subdivisions that are intersections of several divisions are shared by these divisions, in (16) the left-side minus the sum in the right-side can be economically interpreted as the total worths of the divisions when they do not cooperate, while $v(N)$ is the worth of the firm when the divisions do cooperate. To have weak efficiency the latter value should be at least equal to the former value. Remark that $v(N)$ minus the total payoff at the tree value can be interpreted as the net profit of the firm (or the synergy effect from cooperation) that can be given to its shareholders.

3.3. Sink connectedness

We consider now another scenario of controlling cooperation in case of directed communication and assume that in any coalition each player may be controlled only by his successors and that nobody accepts that his former superior becomes his equal partner if a coalition forms. This entails the assumption that the only productive coalitions are the so-called *sink connected*, or simply *s-connected*, being the connected coalitions $S \in C^{\Gamma(N)}$ that meet also the condition that for every leaf $i \in L_\Gamma(S)$ it holds that $i \notin P_\Gamma(j)$ for another leaf $j \in L_\Gamma(S)$. Similar to the case of tree connectedness, every s-connected coalition inherits the subordination of players prescribed by Γ in N, every component $C \in N/\Gamma$ is s-connected, and any full predecessors set in Γ is s-connected. We say that an s-connected coalition is *full s-connected*, if it together with its leaves contains all predecessors of these leaves. Observe that a full s-connected coalition is the union of several full predecessors sets. For a cycle-free digraph Γ on N and a coalition $S \subseteq N$, let $C_s^\Gamma(S)$ denote the set of all s-connected subcoalitions of S, $[S/\Gamma]^s$ the set of maximally s-connected subcoalitions of S, called the *s-connected components of* S, and $[S/\Gamma]_i^s$ the s-connected component of S containing player $i \in S$.

For efficiency of a value we require that each leaf of the given communication digraph together with all his predecessors realizes the total worth they possess. This generates the first property a value must satisfy, what we call maximal-sink efficiency.

A value ξ on $\mathcal{G}_N^\Gamma$ is *maximal-sink efficient* (MSE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ it holds that

$$\sum_{j \in \bar{P}_\Gamma(i)} \xi_j(v,\Gamma) = v(\bar{P}_\Gamma(i)), \qquad \text{for all } i \in L_\Gamma(N).$$

The next property, called the predecessor equivalence, says that if a directed link is broken, each member of the full predecessors set of the origin of this link still receives the same payoff.

A value ξ on $\mathcal{G}_N^\Gamma$ is *predecessor equivalent* (PE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ it holds that for all $(i,j) \in \Gamma$

$$\xi_k(v, \Gamma\backslash(i,j)) = \xi_k(v,\Gamma), \qquad \text{for all } k \in \bar{P}_\Gamma(i).$$

Along with MSE we consider a stronger efficiency property, what we call full-sink efficiency, that requires that every full predecessors set realizes its worth.

A value ξ on $\mathcal{G}_N^\Gamma$ is *full-sink efficient* (FSE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ it holds that

$$\sum_{j \in \bar{P}_\Gamma(i)} \xi_j(v,\Gamma) = v(\bar{P}_\Gamma(i)), \qquad \text{for all } i \in N.$$

It is easy to see that the assumption of sink connectedness in digraph Γ is equivalent to the assumption of tree connectedness in the digraph $\tilde{\Gamma}$ composed by the same set of links as Γ but with the opposite orientation. Moreover, each of axioms MSE, FSE and PE with respect to any cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ is equivalent to the corresponding MTE, FTE or SE axiom with respect to the digraph game $\langle v, \widetilde{\Gamma}\rangle$. In case of sink connectedness the last two observations allow to obtain the following results straightforwardly from the results proved above in Subsections 3.1. and 3.2. under the assumption of tree connectedness.

Proposition 2. *On the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ MSE and PE together imply FSE.*

MSE and PE uniquely define a value on the class of cycle-free digraph games.

Theorem 7. *On the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ there is a unique value s that satisfies MSE and PE. For every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$, the value $s(v, \Gamma)$ satisfies the following conditions:*

(i) *it obeys the recursive equality*

$$s_i(v,\Gamma) = v(\bar{P}_\Gamma(i)) - \sum_{j\in P_\Gamma(i)} s_j(v,\Gamma), \qquad \text{for all } i \in N; \tag{17}$$

(ii) *it admits the explicit representation in the form*

$$s_i(v,\Gamma) = v(\bar{P}_\Gamma(i)) - \sum_{j\in P_\Gamma(i)} \tilde{\kappa}_i(j) v(\bar{P}_\Gamma(j)), \qquad \text{for all } i \in N, \tag{18}$$

where for all $i \in N$, $j \in P_\Gamma(i)$,

$$\tilde{\kappa}_i(j) = \sum_{r=0}^{n-2} (-1)^r \tilde{\kappa}_i^r(j), \tag{19}$$

and $\tilde{\kappa}_i^r(j)$ is the number of tuples $(i_0, \ldots, i_{r+1})$ such that $i_0 = j$, $i_{r+1} = i$, $i_h \in P_\Gamma(i_{h-1})$, $h = 1, \ldots, r+1$.

Before stating the next theorem providing simpler explicit representation of the value s we introduce some additional notions and notation. Let

$$P_\Gamma^1(i) = \big\{ j \in P_\Gamma(i) \backslash O_\Gamma^*(i) \mid \boldsymbol{P}_\Gamma(j,i) = \bigcup_{h=1}^{q} \boldsymbol{P}_h, \boldsymbol{P}_h \cap \boldsymbol{P}_l = \emptyset, h \neq l:$$
$$\forall h = 1,...,q, \exists k_h \in S(\boldsymbol{P}_h) \backslash \{j,i\}:$$
$$k_h \in \boldsymbol{p}, \forall \boldsymbol{p} \in \boldsymbol{P}_h \text{ and } \boldsymbol{p}_h \cap \boldsymbol{p}_l = \{j,i\}, \forall \boldsymbol{p}_h \in \boldsymbol{P}_h, \forall \boldsymbol{p}_l \in \boldsymbol{P}_l, h \neq l \big\};$$

and

$$P_\Gamma^2(i) = P_\Gamma(i) \setminus \big(O_\Gamma^*(i) \cup P_\Gamma^1(i)\big).$$

Both sets $P_\Gamma^1(i)$ and $P_\Gamma^2(i)$ are composed by predecessors of i that are not proper superiors of i. $P_\Gamma^1(i)$ consists of any such j for which all paths from j to i can be partitioned into a number of separate groups, might be only one group, such that all paths in the same group have at least one common node different from

j and i and paths from different groups do not intersect between j and i. Notice that all $j \in P_\Gamma(i) \setminus O^*_\Gamma(i)$ with $d_i(j) = 1$ belong to $P^1_\Gamma(i)$ since the unique proper subordinate of j belongs to all paths $\boldsymbol{p} \in \boldsymbol{P}_\Gamma(j,i)$; in particular, it holds that $j \in P^1_\Gamma(i)$, when there is only one path from j to i, i.e., when $|\boldsymbol{P}_\Gamma(j,i)| = 1$. From here besides it follows that for all $j \in P^2_\Gamma(i)$, $d_i(j) > 1$. For every $j \in P^1_\Gamma(i)$ we define the *proper out-degree* $\tilde{d}_i(j)$ of j with respect to i as the number of groups $\boldsymbol{P}_h$, $h = 1, ..., q$, in the partition of $\boldsymbol{P}_\Gamma(j,i)$. The subset of $\tilde{M}_\Gamma(j,i) \subseteq S(\boldsymbol{P}_\Gamma(j,i))$, $j \in P_\Gamma(i)$, composed by j, i, all proper intersection points in $S(\boldsymbol{P}_\Gamma(j,i))$ and all proper superiors $h \in O^*_\Gamma(i) \cap S(\boldsymbol{P}_\Gamma(j,i))$ we call the *lower covering set* for $\boldsymbol{P}_\Gamma(j,i)$. Similarly to the definition of $\tilde{\kappa}_i(j)$ given by (19) we define

$$\tilde{\kappa}^M_i(j) = \sum_{r=0}^{n-2} (-1)^r \tilde{\kappa}^{r,M}_i(j),$$

where $\tilde{\kappa}^{r,M}_i(j)$ counts only the tuples $(i_0, ..., i_{r+1})$ for which $i_0 = j$, $i_{r+1} = i$, and $i_h \in P_\Gamma(i_{h-1}) \cap \tilde{M}_\Gamma(j,i)$, $h = 1, \ldots, r+1$.

Theorem 8. *For every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}^\Gamma_N$ the value s given by (18) admits the equivalent representation in the form*

$$s_i(v,\Gamma) = v(\bar{P}_\Gamma(i)) - \sum_{j \in O^*_\Gamma(i)} v(\bar{P}_\Gamma(j)) +$$

$$+ \sum_{j \in P^1_\Gamma(i)} (\tilde{d}_i(j) - 1) v(\bar{P}_\Gamma(j)) - \sum_{j \in P^2_\Gamma(i)} \tilde{\kappa}^M_i(j) v(\bar{P}_\Gamma(j)), \quad \text{for all } i \in N. \qquad (20)$$

If the consideration is restricted to only strongly cycle-free digraph games, then the above representation reduces to

$$s_i(v,\Gamma) = v(\bar{P}_\Gamma(i)) - \sum_{j \in O_\Gamma(i)} v(\bar{P}_\Gamma(j)), \quad \text{for all } i \in N. \qquad (21)$$

For sink-forest digraph games defined by sink forest digraph structures that are strongly cycle-free, the value given by (21) coincides with the sink value introduced in (Khmelnitskaya, 2010). By that reason from now on we refer to the value s given by (18), or equivalently by (20), as to the *sink-tree value*, or simply the *sink value*, for cycle-free digraph games.

The sink value assigns to every player the payoff equal to the worth of his full predecessors set minus the worths of all full predecessors sets of his proper superiors plus or minus the worths of all full predecessors sets of any other of his predecessors that are subtracted or added more than once. Moreover, for any player $i \in N$ and his predecessor $j \in N$ that is not his proper superior, the coefficient $\tilde{\kappa}_i(j)$ indicates the number of overlappings of full predecessors sets of all proper superiors of i at node j. In fact a player receives what he contributes when he joins his predecessors when only the full predecessors sets, that are the only efficient productive coalitions under given assumptions, are counted. Since a root has no predecessors, a root just gets his own worth. Furthermore, it is not difficult to check that the right-sides of both formulas (20) and (21) being considered with respect not to coalitional worths but to players in these coalitions contain only player i when taking into account all pluses and minuses. Besides, according to (17) the sink value assigns to every player the worth of his full predecessors set minus the total payoff to his predecessors. Wherefrom we obtain a simple recursive algorithm for computing the sink value going downstream from the roots of the given digraph.

Example 7. Figure 7 provides an example of the sink value for a 10-person game with cycle-free but not strongly cycle-free digraph structure.

The sink value may be computed in two different ways, either by the recursive algorithm based on the recursive equality (17), or using the explicit formula representation (20).

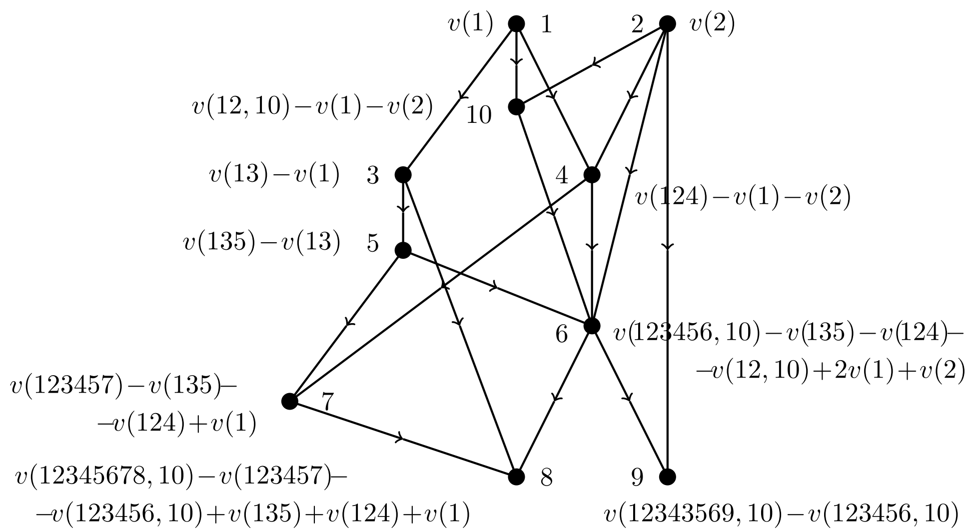

Fig. 7.

The sink value not only meets FSE but FSE alone uniquely defines the sink value on the class of cycle-free digraph games.

Theorem 9. *On the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ the sink value is the unique value that satisfies FSE.*

Corollary 3. FSE on the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ implies not only MSE but PE as well.

The next theorem derives the total sink value payoff for any s-connected coalition.

Theorem 10. *In a cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$, for any s-connected coalition $S \in C_s^\Gamma(N)$ it holds that*

$$\sum_{i\in S} s_i(v,\Gamma) = \sum_{i\in L_\Gamma(S)} v(\bar{P}_\Gamma(i)) - \sum_{i\in S\setminus L_\Gamma(S)} \big(\tilde{\kappa}_S(i)-1\big) v(\bar{P}_\Gamma(i)) - \sum_{i\in \bar{P}_\Gamma(S)\setminus S} \tilde{\kappa}_S(i) v(\bar{P}_\Gamma(i)),$$

where

$$\bar{P}_\Gamma(S) = \bigcup_{i\in L_\Gamma(S)} \bar{P}_\Gamma(i),$$

$$\tilde{\kappa}_S(i) = \sum_{j \in \bar{S}_\Gamma(i) \cap \bar{P}_\Gamma(S)} \tilde{\kappa}_j(i), \quad \textit{for all } i \in \bar{P}_\Gamma(S),$$

while $\tilde{\kappa}_S(i) = 1$ *when* $\tilde{d}_N(i) = 1$, *where for any s-connected coalition* $S \in C_s^\Gamma(N)$, *for all* $i \in \bar{P}_\Gamma(S)$, $\tilde{d}_S(i)$ *is the out-degree of* i *in the subgraph* $\Gamma|_{\bar{P}_\Gamma(S)}$, *i.e.,*

$$\tilde{d}_S(i) = |F_\Gamma(i) \cap \bar{P}_\Gamma(S)|,$$

in particular, $\tilde{d}_N(i) = \tilde{d}_\Gamma(i)$ *for all* $i \in N$.

If the consideration is restricted to only strongly cycle-free digraph games, then for any s-connected coalition $S \in C_s^\Gamma(N)$ *it holds that*

$$\sum_{i \in S} s_i(v, \Gamma) = \sum_{i \in L_\Gamma(S)} v(\bar{P}_\Gamma(i)) - \sum_{i \in S \setminus L_\Gamma(S)} \big(\tilde{d}_S(i) - 1\big) v(\bar{P}_\Gamma(i)) - \sum_{i \in L_\Gamma(\bar{P}_\Gamma(S) \setminus S)} \tilde{d}_S(i) v(\bar{P}_\Gamma(i)).$$

For any cycle-free digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$ the overall efficiency is given by

$$\sum_{i \in N} s_i(v, \Gamma) = \sum_{i \in L_\Gamma(N)} v(\bar{P}_\Gamma(i)) - \sum_{i \in N \setminus L_\Gamma(N)} \big(\tilde{\kappa}_N(i) - 1\big) v(\bar{P}_\Gamma(i)),$$

while if the consideration is restricted to only strongly cycle-free digraph games, the last equality reduces to

$$\sum_{i \in N} s_i(v, \Gamma) = \sum_{i \in L_\Gamma(N)} v(\bar{P}_\Gamma(i)) - \sum_{i \in N \setminus L_\Gamma(N)} \big(\tilde{d}_\Gamma(i) - 1\big) v(\bar{P}_\Gamma(i)).$$

For a cycle-free digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$, the *s-core* $C^s(v, \Gamma)$ is defined as the set of component efficient payoff vectors that are not dominated by any s-connected coalition,

$$C^s(v, \Gamma) = \{x \in \mathbb{R}^N \mid x(C) = v(C), \forall C \in N/\Gamma;\ x(S) \geq v(S), \forall S \in C_s^\Gamma(N)\},$$

while the *weak s-core* $\tilde{C}^s(v, \Gamma)$ as the set of weakly component efficient payoff vectors that are not dominated by any s-connected coalition,

$$\tilde{C}^s(v, \Gamma) = \{x \in \mathbb{R}^N \mid x(C) \leq v(C), \forall C \in N/\Gamma;\ x(S) \geq v(S), \forall S \in C_s^\Gamma(N)\}.$$

Theorem 11. *The sink value on the subclass of superadditive sink-forest digraph games is s-stable.*

A cycle-free digraph game $\langle v, \Gamma \rangle$ is *s-convex*, if for all s-connected coalitions $T, Q \subset C_s^\Gamma(N)$ such that T is a full s-connected set, Q is a full predecessors set, and $T \cup Q \in C_s^\Gamma(N)$, it holds that

$$v(T) + v(Q) \leq v(T \cup Q) + v(T \cap Q).$$

Theorem 12. *The sink value on the subclass of s-convex strongly cycle-free digraph games is weakly efficient.*

4. Sharing a river with multiple sources, a delta and possible islands

In (Ambec and Sprumont, 2002) the problem of optimal water allocation for a given river with certain capacity over the agents (cities, countries) located along the river is approached from the game theoretic point of view. Their model assumes that between each pair of neighboring agents there is an additional inflow of water. Each agent, in principal, can use all the inflow between itself and its upstream neighbor, however, this allocation in general is not optimal in respect to total welfare. To obtain a more profitable allocation it is allowed to allocate more water to downstream agents which in turn can compensate the extra water obtained by side-payments to upstream ones. The problem of optimal water allocation is approached as the problem of optimal welfare distribution. In (van den Brink et al., 2007) it is shown that the Ambec-Sprumont river game model can be naturally embedded into the framework of a graph game with line-graph cooperation structure. In (Khmelnitskaya, 2010) the line-graph river model is extended to the rooted-tree and sink-tree digraph model of a river with a delta or with multiple sources, respectively. We extend the line-graph, rooted-tree or sink-tree model of a river to the cycle-free digraph model of a river with both multiple sources and a delta, and also possible islands along the river bed as well.

Let N be a set players (users of water) located along the river from upstream to downstream. Let $e_{ki} \geq 0$, $i \in N$, $k \in O(i)$, be the inflow of water in front of the most upstream player(s) (in this case $k = 0$) or the inflow of water entering the river between neighboring players in front of player i. Figure 8 provides a schematic representation of the model.

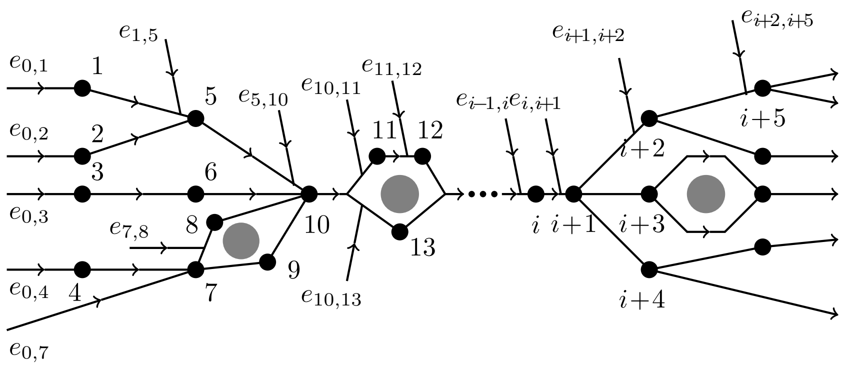

Fig. 8.

A river with multiple sources, a delta, and several islands along the river bed

Following (Ambec and Sprumont, 2002) it is assumed that each player $i \in N$ has a quasi-linear utility function given by $u^i(x_i, t_i) = b^i(x_i) + t_i$ where t_i is a monetary compensation to player i, x_i is the amount of water allocated to player i, and $b^i \colon \mathbb{R}_+ \to \mathbb{R}$ is a continuous nondecreasing function providing benefit $b^i(x_i)$ to player i when he consumes the amount x_i of water. Moreover, in case of a river with a delta it is also assumed that if a splitting of the river into branches happens to occur after a certain player, then this player takes, besides his own quota, also the responsibility to split the rest of the water flow to the branches such to guarantee the realization of the water distribution plan x^* to his successors.

The superadditive river game $v \in \mathcal{G}_N$ introduced under the same assumptions in (Khmelnitskaya, 2010) for a river with multiple sources or a delta defined as:

for any connected coalition $S \subseteq N$, $v(S) = \sum_{i \in S} b^i(x_i^S)$, where $x^S \in \mathbb{R}^s$ solves

$$\max_{x \in \mathbb{R}^s_+} \sum_{i \in S} b^i(x_i) \;\; s.t. \; \begin{cases} \sum\limits_{j \in \bar{P}_\Gamma(i)} x_j \leq \sum\limits_{j \in \bar{P}_\Gamma(i)} \sum\limits_{k \in O(j)} e_{kj}, \\ \sum\limits_{j \in P_\Gamma(i) \cup \bar{B}_\Gamma(i)} x_j \leq \sum\limits_{j \in P_\Gamma(i) \cup \bar{B}_\Gamma(i)} \sum\limits_{k \in O(j)} e_{kj}, \end{cases} \quad \forall i \in S,$$

and for any disconnected coalition $S \subset N$, $v(S) = \sum\limits_{T \in C^\Gamma(S)} v(T)$,

suits to the case of a river with both multiple sources and a delta, and also possible islands along the river bed as well. The tree and sink values proposed above can be applied for the solution of the river game in the general case.

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Tax Auditing Models with the Application of Theory of Search

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Abstract. One of the most important aspects of modeling of taxation — tax control — is studied in the network of game theoretical attitude. Realization of systematic tax audits is considered as the interaction of two players – hiding (T – taxpayer) and seeking (A – authority) — in different statements of the search game.
In the problem statement it is assumed, that tax authority knows due to indirect signs, that some taxpayer T evaded from taxation in the given period (it put down lower than real income in the tax declaration). Also it is assumed, that this taxpayer is quite a large company, having many branches. Company can distribute the hidden income among all or some of its branches or can hide it in the income of one of its branches. Different game cases, which conditions are determined by various possibilities of the evading taxpayer and the tax authority, are considered. In the conditions of the games considered, tax authority A solves the problem of maximization of its income, using mixed strategies, which allow making optimal (in sense of evasion search) tax audits. Taxpayers T aspire to hide their income with the help of mixed strategies, allowing to reduce a probability of revealing of their evasion.
Along with search games, in which tax authority is defined as a search unit or, at best, as several search units, the task of tax audits using the theory of search of immobile chain with the help of big search system is considered. This model takes into account, that practically tax authority is hierarchic structure and is solving tasks on different levels (federal, regional, distirct etc.).
In this model tax authority solves the task of maximizing the probability of revealing of evasion with the help of optimal (in given sense) distribution of search efforts on the given period of time. Company solves the opposite task: to distribute evasion in the way, that minimizes probability of revealing of evasion.
Thus, different possibilities of solving the task of tax evasion search are considered.

Keywords: tax control, taxpayers, tax authority, tax evasions, search games, theory of search.

1. Introduction

One of the most important aspects of tax control is revelation of tax evasions. Gathering and analysis of indirect information about taxpayers' income (see, for example, Vasin and Agapova, 1993, Macho-Stadler and Perez-Castrillo, 2002 and others) allows to realize systematic tax audits to reveal tax evasions.

Another important task of tax authority is to distribute its resources to improve the net income, which is realized to the state budget (see, for example, Sanchez and Sobel, 1993).

Tools for solving both of these problems is given by the theory of search, in which the probability of revealing of the subject in finite time interval is defined as the function of the search strategy or search efforts (spent search resources).

At the end of the given tax period each taxpayer declares income and pays tax in compliance with them.

Let's assume, that tax authority knows due to indirect signs, that some firm evaded from taxation in the given period (it put down lower than real income in the tax declaration). Let's also assume, that this firm is quite a large company, having many branches. The company can distribute the concealed income among all or some or one of its branches. The tax authority doesn't know in which.

It should be noted that the solution of mentioned problem can be considered in another statement:

- the evasion of big company can be distributed among its different business fields, but not among different branches (it is spoken about some total tax payments, including, for example, transport tax, property tax, value-added tax, profit tax, excises etc.);
- a set of taxpayers, acting together in the network of a coalition agreement, can be considered as evader.

A problem of optimal distribution of resources on tax audit was studied before by such authors as Sanchez and Sobel, 1993. Tax authority has hierarchical structure and solves tasks on different levels (federal, regional, distirct etc.). Taking this feature into account, use of the described by Hellman, 1985, theory of search of immobile chain with the help of a big search system becomes acceptable to the task of tax audits.

2. The application of theory of search for a big search system to the task of revelation of tax evasions

Let's suppose, that the firm conceals income only in one of n branches. Let k be the number of this branch; the tax authority doesn't know it. As opposed to the theory, described by Hellman, 1985, the search of the object (concealed income) is realized in the discrete space of n branches. Speaking, that the tax authority is a big search system, following Hellman, 1985, let's assume, that the activity of the system's separate unit can be defined with the help of function $\lambda_i(t)$ with the next characteristics:

$$\lambda_i(t) \geq 0,\ i = \overline{1,n}, t > 0; \tag{1}$$

$$\sum_{i=1}^{n} \lambda_i(t) = L(t),\ t > 0; \tag{2}$$

$$\lambda_i(t)\Delta t + o(\Delta t) \tag{3}$$

– is the probability of revelation of the evasion in the i-th branch in time interval $(t, t + \Delta t)$ in the condition that in it there is a real evasion, which was not revealed until the moment t. $\lambda_i(t)$ is called as density or strategy of search.

Consider the vector $u = (u_1, \dots, u_n)$, which characterizes a distribution of a tax evasion among branches. Each its component u_i is the probability of evasion of the i-th branch.

It should be noted that the distribution u can be improper (see Feller, 1967). Then the inequality $\sum_{i=1}^{n} u_i < 1$ is fulfilled. It is due to the information about the tax evasion (which stimulates the tax authority to audit the taxpayer's branches) has an estimation character and can turn out false with some nonnegative probability. (In the paper by Macho-Stadler and Perez-Castrillo, 2002, such information called as signal). If this information is true, then the condition $\sum_{i=1}^{n} u_i = 1$ is hold. So, in the common case (when the tax authority a priori doesn't know how the signal corresponds to the reality) let's suppose that

$$\sum_{i=1}^{n} u_i \leq 1.$$

Let k be the number of the branch, where the income concealment was. Let's define $\{(t_1, t_2)\}$ as the event consists in that there was no revelation of the evasion in the time interval (t_1, t_2). Let for fixed k

$$p_i(t) = P\{(i = k)|(0, t)\}$$

be the probability of the revealing of the tax evasion in the i-th branch in the condition that there was the evasion and it was unrevealed until the time t. From the definition of search density obtain:

$$P\{(t, t + \varDelta t)|(i = k)(0, t)\} = 1 - \lambda_i(t)\varDelta t + o(\varDelta t)$$

– is the probability, that in time interval $(t, t + \varDelta t)$ there is no revelation of the evasion in the branch in which this evasion was in the condition that it was not revealed until the time t. Then the probability of unrevealing of the evasion of the whole company in time interval $(t, t + \varDelta t)$:

$$P\{(t, t + \varDelta t)|(0, t)\} = 1 - \varDelta t \sum_{i=1}^{n} u_i \lambda_i(t) + o(\varDelta t).$$

Let's denote by $P(t)$ the probability of that the evasion was revealed in time interval $(0; t)$: $P(t) = 1 - P\{(0, t)\}$. Then, similar to obtained in the book by Hellman, 1985, the equality for the probability of revealing of the object in time T, let's define the probability of the existing tax evasion of the company in $(0; T)$, which is a period of limitation for tax crimes:

$$P(T) = 1 - \sum_{i=1}^{n} u_i \exp(-\int_0^T \lambda_i(\tau) d\tau). \tag{4}$$

The strategy of search λ_i must be defined such way as it fulfills the conditions (1) – (3) maximizes the probability (4) of the revealing of the evasion in the given time period $(0; T)$.

Let's take into consideration a function $\varphi_i(t)$, which fulfills the conditions

$$\varphi_i(T) = \int_0^T \lambda_i(\tau)d\tau; \tag{5}$$

$$\varphi_i(T) \geq 0,\ i = \overline{1,n}; \tag{6}$$

$$\sum_{i=1}^{n} \varphi_i(T) = \int_0^T L(\tau)d\tau \tag{7}$$

and call it as the number of search resources, spent on the revelation of the tax evasion in the i-th branch during the time interval $(0;T)$. Then

$$P(T) = 1 - \sum_{i=1}^{n} u_i \exp(-\varphi_i(T)). \tag{8}$$

The tax authority's aim is to distribute the number of search resources $\varphi_i(T)$ on the given tax period $(0;T)$ in order to fulfill the conditions (5) – (7) and maximize the probability (8).

The company solves the opposite task: to minimize probability of revealing of evasion $P(T)$, i.e. to distribute evasion in the way, that the function probability of revealing of evasion get its maximum.

Each proper task requires an individual solution, which depends on a lot of reasons: a number of branches, involved in the evasion, a geography of branches, resources (time, finances, search units) and the ways helping to realize the auditing etc.

Consider several examples of the search tasks depending on the ways of distribution of the concealed income and the ways of searching it by the tax authority. These models are based on the search games, studied by Petrosyan and Garnaev, 1992, Petrosyan and Zenkevich, 1987. Let's consider the tax authority (player A – *authority*) as a searcher and taxpayer (taxpayers) (player T – *taxpayers*) as a concealer.

The model of search of evasion, which is concentrated in the one branch. The taxpayer T conceals its income in one of its n branches. The tax authority A also searches in one of them. Both of the players — A and T — have n pure strategies, identifiable with numbers of the branches, chosen for evasion or auditing correspondingly. If there is an evasion in the k-th branch and audit is in the i-th branch, the A's benefit is equal to

$$K(i,k) = \begin{cases} \sigma_i,\ if\ p_k < p_k^* \\ 0,\ if\ p_k \geq p_k^*, \end{cases}$$

where k is the number of the branch, where the evasion was, i is the number of the branch, where the audit was. The quantity σ_i $(0 < \sigma_i \leq 1,\ k = \overline{1,n})$ is interpreted as the probability of revelation of the evasion in the k-th branch.

In the terms of this game the event consists in the revelation of the total evasion of the company, is equal to the revelation of existing evasion in the one (the k-th) branch; $u_k = 1$, that's why with (8):

$$\sigma_k = 1 - e^{-\varphi_k(T)}, \tag{9}$$

where T can be interpreted as the time interval, given for realization of the audit. As in the terms of this model the audit can be realized only in one branch, it is possible to suppose that the tax authority will use all of its resources, i. e. total search effort $\int_0^T L(\tau)d\tau$. Then (9) will turn out

$$\sigma_k = 1 - e^{-\int_0^T L(\tau)d\tau}. \tag{10}$$

The game considered is the matrix game, and the matrix of the A's benefits is

$$\begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}.$$

This game's value v and optimal mixed strategies are $\mu^* = (\mu_1^*, \dots, \mu_n^*)$ $\nu^* = (\nu_1^*, \dots, \nu_n^*)$:

$$v = \frac{1}{\sum_{k=1}^{n} \frac{1}{\sigma_k}},$$

$$\mu_i^* = \nu_i^* = \frac{\frac{1}{\sigma_i}}{\sum_{k=1}^{n} \frac{1}{\sigma_k}}, \ i = \overline{1, n}.$$

If $\sigma_i = \sigma_j$ for $i \neq j$, then the choice of the branches, in which there were evasion and audit, are equiprobables for each player. Then the optimal strategies and the game value get the form

$$\mu^* = \nu^* = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}), \quad v = \frac{1}{n}.$$

Then let's consider the situations in which the company can evade in several branches.

The model of search of evasion, which is concentrated in one or two branches. Let's suppose, that the company either concealed its income in one branch or divide it into two parts and distribute on two branches.

The set of T's pure strategies contains n^2 elements and consists of various pares (i, j), where i, j are the numbers of the branches, in which there exist an evasion.

Player A can make an audit in one of n branches. Therefore, he has n pure strategies. Player A's aim is to maximize the probability of revealing of even one of the parts of evasion. In this case the probability of revelation of existing evasion in the k-th branch during time T is defined similar to the game of search of the one evasion from (9).

The tax authority's profit $K(k, (i, j))$ in the situation $(k, (i, j))$ is:

$$K(k, (i, j)) = \begin{cases} 0, \ if \ i = j, i \neq k; \\ \sigma_i, \ if \ i = k, i \neq j; \\ \sigma_j, \ if \ j = k, i \neq j; \\ \sigma_i(2 - \sigma_i), \ if \ i = j = k; \\ 0, \ if \ i \neq k, j \neq k. \end{cases}$$

Using the solution of corresponding game (see Petrosyan and Garnaev, 1992), obtain the optimal mixed strategies $\mu^* = (\mu_1^*, \dots, \mu_n^*)$ and $\nu^* = \{\nu_{ij}^*\}$ of the players A and T correspondingly and the game value v:

$$\mu_i^* = \nu_{i,i}^* = \frac{\frac{1}{\sigma_i(2-\sigma_i)}}{\sum\limits_{l=1}^{n} \frac{1}{\sigma_l(2-\sigma_l)}}, \; if \; i = \overline{1,n};$$

$$\nu_{i,j}^* = 0, \; if \; i \neq j;$$

$$v = \frac{1}{\sum\limits_{l=1}^{n} \frac{1}{\sigma_l(2-\sigma_l)}}.$$

It should be noted that from the statement of T's strategy it is reasonable for this player to concentrate total concealed income in one of its branches.

The model with the distribution of the concealed income among m branches. Assume, that the company T evades in m branches ($m \leq n$). It means that concealed income is divided into m equal parts and distributed among corresponding number of T's branches. The tax authority A can make audits in l branches at the same time.

Let p_i be the probability of revelation of the evasion in the i-th branch when it is audited. First let's consider an ideal situation, when

$$p_i = \begin{cases} 1, \; if \; there \; is \; an \; evasion \; in \; this \; branch; \\ 0, \; if \; there \; is \; no \; evasion \; in \; this \; branch. \end{cases} \; i = \overline{1,n}$$

The number of the taxpayer's pure strategies is C_n^m (various samples without replacement m of n branches to conceal income there), the number of the tax authority's pure strategies is C_n^l (various samples without replacement l of n branches to audit them simultaneously).

The player A's profit is defined as a tax from total income revealed in one or several branches (or a number of revealed parts of the income). This player's aim is to maximize his income. The game matrix has dimension $C_n^m \times C_n^l$.

The solution of this game is found using the results, obtained by Petrosyan and Garnaev, 1992: the player A's (T's) optimal mixed strategies $\mu^*(\nu^*)$ consist in the choice of one of the $C_n^l(C_n^m)$ possible pure strategies with probability $\frac{1}{C_n^l}(\frac{1}{C_n^m})$. The game value is $v = \frac{ml}{n}$.

The model with the distribution of equal or different parts of the concealed income among m branches. Consider another situation: the taxpayer T divides an income, he wants to conceal, into m different parts. But, distributing this income among its branches, T can conceal either one or several parts of it in each branch.

First, let's suppose that player A can make an audit only in one of T's branches (this fact can be reasoned by time, financial and official restrictions for the unit of tax authority), but this audit reveals total tax evasion in the branch. In this game players T and A have n^m and n pure strategies correspondingly. So, the A's profit matrix has dimension $n \times n^m$.

Using the results, obtained for the base game by Petrosyan and Garnaev, 1992, define the tax authority's optimal mixed strategy μ^* as auditing one of n branches of the company with the probability $\frac{1}{n}$. The taxpayer's optimal mixed strategy ν^* is to choose one of n^m possible placements of m parts of the evasion into n branches with the probability $\frac{1}{n^m}$. The game value is equal to $\frac{m}{n}$.

The generalization of considered situation is the case when the tax authority can make simultaneous audit in l branches, revealing every part of existing evasion in them. The A's profit is the tax from total concealed income of the company, revealed as the result of auditing. In this game player T (A) has $n^m(C_n^l)$ pure strategies, therefore, the A's profit matrix has a dimension $C_n^l \times n^m$.

The tax authority's optimal mixed strategy μ^* is to choose one of C_n^l samples without replacement l of n company's branches for simultaneous auditing with the probability $\frac{1}{C_n^l}$. The taxpayer's optimal mixed strategy ν^* is to choose one of n^m possible placements of m parts of the evasion into n branches with the probability $\frac{1}{n^m}$. The game value is equal to $\frac{lm}{n}$.

It should be noted that in the case of distribution of the evasion among m branches the game value is the same as in the last case (when there can be revealed one or several parts of the evasion in audited branch). That is, the possibility to conceal in one branch grater or smaller part of the evasion does not influence on player A's profit.

To approach the models with distribution of concealed income among several branches to the practical tax auditing it is necessary to decline an assumption that the audit is always effective. That is, the probability of revelation of the tax evasion in the i-th branch during time T, given for the tax audit, can be calculated with use of (10):

$$p_i(T) = \begin{cases} 1 - e^{-\varphi_i(T)}, \; if\ there\ is\ an\ evasion; \\ 0, \; if\ there\ is\ no\ evasion, \end{cases} \quad i = \overline{1,n}.$$

This probability depends on the search efforts, spent in the i-th branch on revelation of the tax evasion, if it exists there.

There are two principally different ways of the searching of tax evasion in one of l branches — in consecutive order and simultaneous realizing of l audits.

The tax authority doesn't know in which branches there is concealed income. It makes the estimation of spent resources more difficult and, therefore, it becomes more difficult to choose the most preferable way of auditing in this sense.

It is also obviously that the consecutive audits have no preferences in comparison with parallel auditing in sense of spending time: simultaneous search in l branches takes as long as an audit of one branch when the consecutive search is realized.

Everything told above lets draw a conclusion that parallel auditing in l branches is more rational in comparison with consecutive audits.

Total search resources, spent by the tax authority on l simultaneous tax audits, is defined as the amount of search resources in each branch. Then, with (8), the probability of revelation of the tax evasion when there realized a parallel auditing of l branches in time T is defined as:

$$P(T) = 1 - \sum_{i=j}^{j+l-1} u_i \exp(-\varphi_i(T)). \tag{11}$$

If the company's evasion is really distributed among l branches, then the vector $u = (u_1, \ldots, u_n)$ is such as

$$\sum_{i=j}^{j+l-1} u_i = 1.$$

In this case the statement (11) becomes:

$$P(T) = 1 - \exp(-\int_0^T L(\tau)d\tau),$$

where $L(T)$ is the total cost of simultaneous search of evasion in l branches.

Thus, the application of theory of search to the task of tax auditing is considered. The results, obtained above, allows to solve tasks of tax evasion search and optimal distribution of tax authority's resources to improve its net income.

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Bargaining Powers, a Surface of Weights, and Implementation of the Nash Bargaining Solution ⋆

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Abstract. In the present paper a new approach to the Nash bargaining solution (N.b.s.) is proposed. (Shapley, 1969) introduced weights of individual utilities and linked the N.b.s. with utilitarian and egalitarian solutions. This equivalence leaves open a positive question of a possible mechanism of weights formation. Can the weights be constructed in result of a recurrent procedure of reconciliation of utilitarian and egalitarian interests? Can a set of feasible bundles of weights be a result of a procedure or a game independent on a concrete bargaining situation? We answer these questions in the paper. A two-stage n-person game is considered, where on the first stage the players on base of their bargaining powers elaborate a set of all possible bundles of weights $\Lambda = \{(\lambda_1, \ldots, \lambda_n)\}$. This surface of weights can be used by an arbitrator for evaluation outcomes in different concrete bargains. On the second stage, for a concrete bargain, the arbitrator chooses a vector of weights and an outcome by use of a maximin criterion. We prove that this two-stage game leads to the well-known asymmetric N.b.s.

Keywords: Bargaining powers, Weights of individual utilities, Nash bargaining solution, Imlementation, Egalitarian solution, Utilitarian solution
JEL classification C78, D74

1. Introduction

An n-person *bargaining problem* is defined by a pair (S, d) where S is a convex set in $\mathbb{R}^n$ (the feasible set) and $d \in S$ (the status quo, disagreement point, threat point). Each point in S is interpreted as a bundle of utilities of the players; d is a point of utilities which can be achieved without cooperation, other points in S are available in case of cooperation. It is required to choose a "good", in one sense or another, outcome $x \in S$.

Instead of $x \in S$ it is convenient to consider differences $u = x - d$. However, it is often assumed that $d = 0$, in such case $u = x$; this assumption is accepted in the present paper.

(Nash, 1950) considered the bargaining problem from a normative point of view: he formulated a list of properties (axioms) of a good solution and found a unique solution satisfying these axioms. The set of axioms includes Pareto optimality, anonymity (or its weaker version, symmetry), scale invariance and contraction independence (also known as independence of irrelative alternatives). The solution, referred as *Nash bargaining solution* (hereafter *N.b.s.*), is the point of maximum

$$\max_{x \in S} u_1 u_2 \ldots u_n.$$

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In an asymmetric case, when the players have different *bargaining powers*, the axiom of symmetry is not true, and the outcome satisfying the remaining three axioms can be found as a point of maximum

$$\max_{x \in S} u_1^{b_1} u_2^{b_2} \ldots u_n^{b_n},$$

where $b_i \geq 0$ are bargaining powers of the players, (Harsanyi and Selten, 1972, Kalai, 1977a).

For reviews of results developing the axiomatic approach to the bargaining problem see (Roth, 1979a), (Thomson and Lendsberg, 1989), (Thomson, 1994), (Serrano, 2008).

Nash himself saw a shortage of the axiomatic approach in a lack of an explicit description of a process of negotiations or of dynamics of formation of the outcome. A problem of constructing a non-cooperative strategic game resulting in an outcome satisfying definite axioms is referred now as *Nash program.* Contributions to the program were provided by (Nash, 1953), (Rubinstein, 1982), (Binmore et al., 1986), (Carlsson, 1991) and many others. Later many authors tried to connect the Nash program with the theory of mechanisms. A term *implementation* appeared, we understand it here as a rather broad approach collecting games and mechanisms supporting the N.b.s.

The asymmetric N.b.s. finds applications in very different branches of Economics such as wage formation (Grout, 1984 and many others), pricing mechanisms (Bester, 1993), macroeconomic policy (Alesina, 1987), circulation of ideas (Hellman and Perotti, 2006). A recent paper (Eismont, 2010) is devoted to analysis of the European gas market.

The N.b.s. is closely related with the problem of interpersonal comparison of utilities and with the use of weights of individual utilities. Historically, the usage of cardinal utilities of players and sums of individual utilities in Game Theory (in particular in games with transferable utilities) induced a dissatisfaction. (Harsanyi, 1963) and (Shapley, 1969) introduced intrinsically-defined utility weights. Shapley (Shapley, 1969) interpreted them as endogenously determined transfer rates between players' utilities, i.e. as a kind of prices (analogues of exchange rates, i.e. prices of currencies). Shapley allowed a possibility of a change in weights and considered as solutions both a point of maximum $\max_{x \in S} (\mu_1 x_1 + \ldots + \mu_n x_n,)$ and a point of equality $\lambda_1 x_1 = \ldots = \lambda_n x_n$ (equivalent to finding $\max\limits_{x \in S} \min\limits_i \lambda_i x_i$). Correspondingly, he spoke about 'efficiency' weights and 'equity' weights. These two criteria correspond two basic concepts of optimality, "competing" in economic policies. Later (Yaari, 1981) interpreted these criteria as 'utilitarian' and 'egalitarian', correspondingly. A review of utilitarian and egalitarian approaches to moral theory and, in particular, a discussion of papers (Shapley, 1969) and (Yaari, 1981) is provided by Binmore (Binmore, 1998, Binmore, 2005).

In relation with the weights of utilities (Shapley, 1969) advanced the following principle: "(III) An outcome is acceptable as a "value of the game" only if there exist scaling factors for the individual (cardinal) utilities under which the outcome is both equitable and efficient". So, of principle for Shapley was the existence of a vector of weights which could for some outcome serve efficiency and equity simultaneously. For the bargaining problem such solution really exists and it is nothing else but the symmetric N.b.s. The fact can be formulated in the following way.

Theorem 1. *Let the set of feasible utilities, S, be restricted by coordinate planes and by a surface $g(x_1, \ldots, x_n) = C$, where g is a strictly convex function. Then the following two statements are equivalent.*

1.For a point $\overline{x} \in S$ there exists such a vector of weights $\lambda_1, \ldots, \lambda_n$ that $\overline{x}$ is simultaneously (i) a point of maximum of 'utilitarian' function $\lambda_1 x_1 + \ldots + \lambda_n x_n$ and (ii) a point where $\max_x \min_i \lambda_i x_i$ is achieved or, what is the same, where 'egalitarian' condition $\lambda_1 \overline{x}_1 = \ldots = \lambda_n \overline{x}_n$ takes place.

2.Point $\overline{x} \in S$ is a symmetric N.b.s., i.e. the point of maximum of the product $x_1 \ldots x_n$.

A proof will be given for a more general Theorem 3 below.

In the present paper a new approach to the problem of implementation of the Nash bargaining solution (N.b.s.) is proposed. The equivalence described in Theorem 1 (or in Theorem 3 in an asymmetric case) leaves open a positive question of a possible mechanism of weights formation. Can the weights be formed in result of a recurrent procedure of reconciliation of utilitarian and egalitarian interests? Can a set of feasible bundles of weights be a result of a procedure or a game independent on a concrete bargaining situation?

We consider a two-stage game, where on the first stage the players, on base of their bargaining powers, elaborate a set of all admissible bundles of weights $\Lambda = \{(\lambda_1, \ldots, \lambda_n)\}$ which can be used by an arbitrator to assess outcomes in different concrete bargains. On the second stage, for a concrete bargain, the arbitrator chooses a vector of weights and an outcome by use of a Rawlsian-type maximin criterion. We prove that this game leads to the asymmetric N.b.s.

We interpret the arbitrator as a society (or a community) in a framework of which the bargains are fulfilled. In fact, in reality very often bargains take place not in a "vacuum" but depend on an encirclement and on a choice of moral-ethical assessments.

We suppose that the society confesses a maximin Rawlsian-type principle of justice but attaches weights to individual utilities and is manipulated. The participants, by use of all available means draw up a set Λ of admissible bundles of weights (reputational moral-ethical assessments); bargaining powers of participants become apparent in this process. (The bargaining powers depend on an access to political power and to media, on image-makers' abilities, on previous reputations, etc.) In each concrete situation of bargaining the society chooses weights from the set Λ. The assessments (weights) entering the set Λ are not unambiguous, and the society chooses those weights which it finds to be fair in a concrete case. Thus, generally, the society is conformist and possesses a whole spectrum of assessments which can be used in case of need. In each concrete bargaining problem there is a correspondence $\Lambda \leftrightarrow ParS$ between admissible bundles of weights $\lambda \in \Lambda$ and outcomes $x \in ParS$, where $ParS$ is the Pareto border of S.

In Section 2 we consider a relation between 'efficiency' weights, 'equity' weights and bargaining powers as well as different characterizations of the N.b.s. The concepts of weights as moral-ethical assessments and of a two-stage game are introduced in Section 3. In Section 4 the same point of view is applied to two other well-known bargaining solutions: Kalai-Smorodinsky and egalitarian (Kalai). In Section 5 the moral-ethical assessments are considered as a mechanism working in a concrete bargaining situation. In Section 6 a relation between the N.b.s. and the Cobb-Douglas production functions is studied. Section 7 concludes.

2. The weights of individual utilities and characterizations of the N.b.s.

The question of existence of a "good" system of weights is a normative one. However, not less interesting is a positive question of what is the system of weights acting in a concrete bargaining situation. In this relation a question arises about a presence of an iterative process of weights formation similar to a cobweb rule and leading to the N.b.s.

The set S plays a role similar to the production-possibilities set in the theory of trade (see, e.g. (Caves et al., 2006)). For the case of 2 players let the set S be bounded by the coordinate axes and a curve $g(x_1, x_2) = C$, where g is differentiable. Following (Shapley, 1969) the weights can be considered as coefficients of conversion, i.e. prices. 'Efficient' prices are being formed similarly to commodity prices in the trade model: if the economy is interested in achieving a point x then prices λ_1, λ_2 are established that the marginal rate of substitution is equal to the relative price:

$$-\frac{dx_2}{dx_1}(x) = \frac{\frac{\partial g(x)}{\partial x_1}}{\frac{\partial g(x)}{\partial x_2}} = \frac{\lambda_1}{\lambda_2}. \tag{1}$$

Thus, the relative price is a function of the relative utility:

$$\frac{\lambda_1}{\lambda_2} = \psi\left(\frac{x_1}{x_2}\right). \tag{2}$$

On the other hand, given current 'efficient' prices λ_1, λ_2, the society tries to redistribute the utility fairly. If the utility is transferable, then an outcome $\hat{x}$ would be chosen to satisfy the system of equations:

$$\begin{cases} \lambda_1\hat{x}_1 = \lambda_2\hat{x}_2 \\ \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 = \lambda_1 x_1 + \lambda_2 x_2 \end{cases}. \tag{3}$$

However, as soon as in reality the utility is not transferable, an outcome is chosen not by use of (3) but using the system:

$$\begin{cases} \lambda_1\hat{x}_1 = \lambda_2\hat{x}_2 \\ g(\hat{x}_1, \hat{x}_2) = C \end{cases}.$$

Consequently, the new relative utility is connected with the relative price by an inverse relation:

$$\frac{\hat{x}_1}{\hat{x}_2} = \frac{\lambda_2}{\lambda_1}$$

The point $(\hat{x}_1, \hat{x}_2)$ is used on the next iteration to find new 'efficient' prices $\hat{\lambda}_1, \hat{\lambda}_2$ such that

$$\frac{\hat{\lambda}_1}{\hat{\lambda}_2} = \psi\left(\frac{\lambda_2}{\lambda_1}\right) = \varphi\left(\frac{\lambda_1}{\lambda_2}\right),$$

where $\varphi(k) = \psi\left(\frac{1}{k}\right)$.

So, an iterative process takes place:

$$\left(\frac{\lambda_1}{\lambda_2}\right)_{t+1} = \varphi\left(\frac{\lambda_1}{\lambda_2}\right)_t. \tag{4}$$

A convergence of this process is of interest.

Theorem 2. *A condition of a local stability of the process (4) is inequality* $|E| > 1$, *where* E *is the elasticity of substitution of function* $g(x_1, x_2)$ *at point* $(\bar{x}_1, \bar{x}_2)$ *of the symmetric N.b.s.*

A proof of a more general Theorem 6 is provided below.

EXAMPLE. Let set S be constrained by coordinate axes and by curve

$$\delta x_1^p + (1-\delta)\, x_2^p = C, \tag{5}$$

where $0 < \delta < 1$, $p > 1$, $C > 0$. Then the condition of stability of the cobweb rule (4) is $p < 2$.

Indeed, (5) can be written as

$$\left(\delta x_1^p + (1-p)\, x_2^p\right)^{\frac{1}{p}} = \overline{C}.$$

The LHS, the well-known CES function, has elasticity of substitution $\sigma = \frac{1}{1-p}$. The stability condition $|\sigma| > 1$ reduces to $p < 2$. So, the stability takes place under $1 < p < 2$.

The natural generalization of Theorem 3 is the following statement.

Theorem 3. *Let* S *and* g *satisfy conditions of Theorem 1, and let* $b_1, \ldots, b_n$ *be positive bargaining powers. Then the following two statements are equivalent.*

1. For a point $\overline{x}$ *there exists such a vector of weights* $\lambda_1, \ldots, \lambda_n$ *, that* $\overline{x}$ *is simultaneously*

(i) a point of maximum of 'utilitarian' function $\mu_1 x_1 + \ldots + \mu_n x_n$ *with weights*

$$\mu_i = b_i \lambda_i \quad (i = 1, \ldots, n) \tag{6}$$

and

(ii) a point where $\max\limits_{x \in S} \min\limits_i \lambda_i x_i$ *is achieved or, what is the same, where 'egalitarian' condition*

$$\lambda_1 \overline{x}_1 = \ldots = \lambda_n \overline{x}_n \tag{7}$$

takes place.

2. Point $\overline{x}$ *is the asymmetric N.b.s. with bargaining powers* $b_1, \ldots, b_n$, *i.e. the point of maximum of function*

$$x_1^{b_1} \ldots x_n^{b_n}.$$

Proof. For the problem of maximization of the 'utilitarian' function a necessary and sufficient condition of optimality is the equation

$$\frac{\mu_i}{\mu_j} = \frac{\frac{\partial g}{\partial x_i}(\overline{x})}{\frac{\partial g}{\partial x_j}(\overline{x})} \quad (i, j = 1, \ldots, n). \tag{8}$$

For the problem of maximization of function $x_1^{b_1} \ldots x_n^{b_n}$ a necessary and sufficient condition of optimality is the equation

$$\frac{b_i x_j}{b_j x_i} = \frac{\frac{\partial g}{\partial x_i}(\overline{x})}{\frac{\partial g}{\partial x_j}(\overline{x})} \quad (i, j = 1, \ldots, n). \tag{9}$$

Equations (6), (7) and (8) imply (9), i.e. Statement 1 implies Statement 2.

Let Statement 2 be true. Then (9) takes place. For point $\overline{x}$ weights $\lambda_1, \ldots, \lambda_n$ can be found for which equations (7) are true. Equations (6), (7) and (9) imply (8). Thus, Statement 1 follows from Statement 2.

Binmore (Binmore, 2009) writes that "A small school of psychologists who work on "modern equity theory" come closest to my own findings. They find experimental support for Aristotle's ancient contention that "what is fair is what is proportional". More precisely, they argue that an outcome is regarded as fair when each person's gain over the status quo is proportional to that person's "social index". This conclusion is consistent, for example, with the widespread concern about preserving differentials between the wages paid to different trades (such as electricians or carpenters) when there is a general wage increase."

According to (7), in role of a 'social index' of player i the value $\frac{1}{\lambda_i}$ reverse to her 'egalitarian' weight appears.

The relation (6) between 'utilitarian' weights μ_i and 'egalitarian' weights λ_i can be interpreted in the following way. The weights $\lambda_i \;\; (i = 1, \ldots, n)$ are constructed in a framework of the 'egalitarian' criterion and are not suitable for the 'utilitarian' function because they too strongly reflect personalities of the players: players with higher bargaining powers b_i have understated weights λ_i while for players with lower bargaining powers the weights are overstated. The products $\mu_i = b_i\lambda_i$ contain "correcting" coefficients and provide objective, in some sense, estimates of utilities of players for inclusion into the 'utilitarian' function.

Theorem 4. *The asymmetric N.b.s. can be characterized as a point in the Pareto border, where the pair-wise elasticity of the border is equal to the corresponding ratio of the bargaining powers.*

Proof. From (9)

$$\frac{\frac{\partial g}{\partial x_i}(\overline{x})}{\frac{\partial g}{\partial x_j}(\overline{x})}\frac{x_i}{x_j} = \frac{b_i}{b_j} \quad (i, j = 1, \ldots, n).$$

Geometrically the point of the N.b.s., $\overline{x}$, is characterized as follows. Consider a (hyper)plane tangent to surface $g(x_1, \ldots, x_n) = C$ at point $\overline{x}$. Let $X_i \;\; (i = 1, ..., n)$ be points of intersection of the plane with coordinate axes.

Theorem 5.

$$\overline{x} = \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_n X_n,$$

where $\beta_i = \frac{b_i}{b_1 + \ldots + b_n}$ *is the relative bargaining power of player* i $(i = 1, \ldots, n)$, $\sum\limits_{i=1}^{n} \beta_i = 1$.

Proof. Let $\mu_1 x_1 + \ldots + \mu_n x_n = \mu_1 \overline{x}_1 + \ldots + \mu_n \overline{x}_n \equiv \overline{C}$ be an equation of the tangent plane at point $\overline{x}$. The point X_i of intersection of the plane with the i-th coordinate axis has the i-th coordinate

$$\frac{\overline{C}}{\mu_i} = \frac{\mu_1}{\mu_i}\overline{x}_1 + \ldots + \frac{\mu_n}{\mu_i}\overline{x}_n,$$

and other coordinates are zero. By Theorem 3, the N.b.s. satisfies equations

$$\overline{x}_j = \frac{b_j}{b_i}\frac{\mu_i}{\mu_j}\overline{x}_i \quad (j = 1, \ldots, n).$$

Hence,

$$\frac{\overline{C}}{\mu_i} = \frac{b_1 + \ldots + b_n}{b_i}\overline{x}_i.$$

From here

$$b_1 X_1 + \ldots + b_n X_n = \left(b_1 \frac{\overline{C}}{\mu_1}, \ldots, b_n \frac{\overline{C}}{\mu_n} \right) = (b_1 + \ldots + b_n)\,\overline{x}.$$

Vice versa, an arbitrary point $\widetilde{x} \in S$ with positive coordinates satisfies equation $\widetilde{x} = \beta_1 \widetilde{X}_1 + \ldots + \beta_n \widetilde{X}_n$ with some coefficients $\beta_i > 0$, $\sum_{i=1}^{n} \beta_i = 1$, where $\widetilde{X}_t$ is the point of intersection of the tangent plane at point $\widetilde{x}$ with i-th coordinate axis. These coordinates define point $\widetilde{x}$ uniquely. According to Theorem 4, point $\widetilde{x}$ is the asymmetric N.b.s. with bargaining powers $\beta_i > 0$ $(i = 1, \ldots, n)$ (or any proportional bargaining powers).

In particular, in case of two players with equal bargaining powers, the N.b.s. $\overline{x}$ is characterized by the property that the segment of the straight line tangent to curve $g(x_1, x_2) = C$ at point $\overline{x}$ and contained between coordinate axes is divided in halves by point $\overline{x}$. In case of three players with equal bargaining powers, $\overline{x}$ is the point of intersection of medians of the triangle formed by intersections of the coordinate planes with the plane tangent to surface $\varphi(x_1, x_2, x_3) = 0$ at point $\overline{x}$.

Similarly to the previous one, consider an iterative "cobweb" process of weights formation. Let μ_1, μ_2 be 'efficient' prices under which the society tries to redistribute utilities x_1, x_2. In case of transferable utility, an outcome $(\hat{x}_1, \hat{x}_2)$ would be chosen to satisfy the system of equations

$$\begin{cases} \lambda_1 \hat{x}_1 = \lambda_2 \hat{x}_2 \\ \mu_1 \hat{x}_1 + \mu_2 \hat{x}_2 = \mu_1 x_1 + \mu_2 x_2 \end{cases}, \tag{10}$$

where $\lambda_i = \mu_i / b_i$, and b_i is the bargaining power of player i $(i = 1, 2)$. As soon as utility is not in fact transferable, the outcome $(\hat{x}_1, \hat{x}_2)$ has to satisfy the system of equations

$$\begin{cases} \lambda_1 \hat{x}_1 = \lambda_2 \hat{x}_2 \\ g\left(\hat{x}_1, \hat{x}_2\right) = C \end{cases}. \tag{11}$$

Hence, the relative utility is linked with the relative price by equations:

$$\frac{\hat{x}_1}{\hat{x}_2} = \frac{\lambda_2}{\lambda_1} = \frac{b_1}{b_2}\frac{\mu_2}{\mu_1}$$

On the next iteration the point $(\hat{x}_1, \hat{x}_2)$ is used to define new 'efficient' prices $(\hat{\mu}_1, \hat{\mu}_2)$ such that

$$\frac{\hat{\mu}_1}{\hat{\mu}_2} = \frac{\frac{\partial g(\hat{x})}{\partial x_1}}{\frac{\partial g(\hat{x})}{\partial x_2}} = \psi\left(\frac{\hat{x}_1}{\hat{x}_2}\right) = \psi\left(\frac{b_1}{b_2}\frac{\mu_2}{\mu_1}\right).$$

So, the following iterative process takes place:

$$\left(\frac{\mu_1}{\mu_2}\right)_{t+1} = \varphi\left(\frac{\mu_1}{\mu_2}\right)_t, \tag{12}$$

where $\varphi(k) = \psi\left(\frac{b_1}{b_2}\frac{1}{k}\right)$.

Theorem 6. *A condition of a local stability of the process (12) is the inequality $|E| > 1$, where E is the elasticity of substitution of function $g(x_1, x_2)$ at point $\bar{x}$ of the asymmetric N.b.s.*

Proof. A condition of stability is inequality $\left|\varphi'\left(\frac{\bar{\mu}_1}{\bar{\mu}_2}\right)\right| < 1$ at the steady state. The weights $\bar{\mu}_1, \bar{\mu}_2$ are responded by utilities $\bar{x}_1, \bar{x}_2$ such that

$$\frac{\bar{\mu}_1}{b_1}\bar{x}_1 = \frac{\bar{\mu}_2}{b_2}\bar{x}_2.$$

The following is true:

$$\left|\varphi'\left(\frac{\bar{\mu}_1}{\bar{\mu}_2}\right)\right| = \left|\psi'\left(\frac{\bar{x}_1}{\bar{x}_2}\right)\frac{b_1}{b_2}\left(\frac{\bar{\mu}_2}{\bar{\mu}_1}\right)^2\right| = \left|\frac{d\left(\frac{\frac{\partial g(\bar{x})}{\partial x_1}}{\frac{\partial g(\bar{x})}{\partial x_2}}\right)}{d|_{\frac{\bar{x}_1}{\bar{x}_2}}\left(\frac{x_1}{x_2}\right)} \cdot \frac{\bar{x}_1}{\bar{x}_2} \cdot \frac{\frac{\partial g(\bar{x})}{\partial (x_2)}}{\frac{\partial g(\bar{x})}{\partial x_1}}\right| = \left|\frac{1}{E}\right|,$$

where E is the elasticity of substitution of function $g(x_1, x_2)$ at point $\bar{x}$. Thus, the stability condition is the inequality $|E| > 1$.

3. Weights as reputational moral-ethical assessments. Formation of weights

Many real bargains and negotiations are characterized by two common features. Firstly, it is a presence of a community in role of an arbitrator in a framework of which the bargains are fulfilled. For example, in case of international trade and international relations it is a so called "international community" including governments of countries, and different international organizations. In case of labor relations, it is a "collective" of a firm (or, in the West, a union). In other cases it can be a local community, a "scientific community", an "artistic community", and so on.

Secondly, bargains between a fixed set of participants are often not 'one-shot' but represent a routine repeated process, and, along with bargains, a "public opinion" of the community is being formed. The community acts as an arbitrator realizing a control for bargains in such way that unfair, from the point of view of the community, bargains are not possible, at least as routine ones. An outcome of an unfair bargain can be revised, if not formally, than in result of a conflict. Such conflicts often arise, both in local communities and organizations (conflicts between separate members of community or between a worker and administration), and on a national level (conflicts between social groups) and in international relations (conflicts between countries).

The process of formation of the public opinion goes uninterruptedly, but often one does not know what bargains will take place in the future, so the public opinion is elaborated by parties not in conformity to a concrete bargain but in relation to all admissible situations of bargaining. Moral-ethical assessments formed by use of propaganda emphasize an insufficiency of utilities received by some participants and an excessiveness of utilities of others. In other words, the essence of the moral-ethical assessments is that they act as decreasing or increasing coefficients (weights) applied to utilities of the participants.

As an example we can compare judgments concerning the annual income of President D. Medvedev on site *Compromat.ru* of media *Gazeta.ru* (December 2008) and in *Izvestia* newspaper (April 2009). The site emphasizes that D. Medvedev has a largest, in comparison with other members of the government, apartment. *Izvestia*, on the contrary, stresses that D. Medvedev, by the World measure, has one of the lowest wages among state managers of such level.

It is important to emphasize that moral-ethical assessments are usually not univalent but allow a variance: the public opinion practically always stresses both positive and negative features of participants.

Formation of public opinion is performed by means of propaganda and by use of any available tools (e.g. media, Internet, political meetings, creation of gossips). Possibilities of formation of public opinion are limited by an access to media and by professional abilities of "image-makers" as well as by a presence of adaptive or rational expectations of the audience: it is rather difficult to instill an opinion which does not correspond to expectations.

Concrete weights can differ in different bargains depending on circumstances. Thus, the community deals not with a unique vector of assessments but with a curve (in case of two bargainers) or with a surface of possible assessments. It can be supposed that the community in its approval or disapproval of an outcome of a bargain acts in accordance with a Rawlsian-type maximin principle, paying attention to the most infringed participant and taking into account the whole spectrum of admissible weights of utilities formed in result of propaganda. In result a parity is reached in the model: weighted utilities of the participants coincide.

Curve of weights in case of two participants. Let us consider a two-stage game. On the first stage the participants form a curve of weights, i.e. a spectrum of all possible assessments of utilities. When the curve of bundles of weights, $\Lambda = \{(\lambda_1, \lambda_2)\}$, is formed it is passed to the arbitrator. On the second stage, for a concrete bargain, by use of the system of assessments, Λ, the arbitrator finds the maximin

$$\max_{x\in S, \lambda\in\Lambda} \min\{\lambda_1 x_1, \lambda_2 x_2\}.$$

What is the same, the problem

$$\max_{x\in S, \lambda\in\Lambda} \{v : v = \lambda_1 x_1 = \lambda_2 x_2\}$$

is solved.

Under such mechanism of partition, utility x_i gained by player i is negatively connected with her own weight λ_i and is positively connected with $\lambda_j,\ \ j \neq i$. That is why on the first stage each participant is interested in diminishing the weight of her own utility and in increasing the weight of the others' utility. However, in negotiations on the system of weights the player i would agree to a decrease in the other's weight λ_j in some part of the curve Λ at the expense of an increase of her own weight λ_i as soon as her partner similarly temporizes in another part of the curve Λ.

So far as the system of weights is essential only to within a multiplier, the participants may start bargaining from an arbitrary vector of weights.

To what increase in her own weight (under a decrease in the other's weight) will a player agree? Bargaining powers become apparent here. We suppose that a constancy of bargaining powers of participants means a constancy of elasticities of λ_i with respect to λ_j. In other words, player j for 1% decrease in the other's utility will agree to an increase in her own utility only to b_j/b_i per cent, where b_i, b_j are bargaining powers of the players. The more the relative bargaining power of player i is, the fewer the increase in her utility is, when the opponent attacks. Thus, the following differential equation is fulfilled:

$$\frac{d\lambda_i}{d\lambda_j}\frac{\lambda_j}{\lambda_i} = -\frac{b_j}{b_i} = const. \tag{13}$$

By solving this differential equation we receive:

$$\frac{d\lambda_1}{\lambda_1}b_1 = -\frac{d\lambda_2}{\lambda_2}b_2,$$

$$\ln \lambda_1^{b_1} = -\ln \lambda_2^{b_2} + const,$$

$$\lambda_1^{b_1}\lambda_2^{b_2} = C.$$

The curve Λ is defined.

Differential game of the weights curve formation. The process of the formation of Λ can be described more explicitly as a differential game. We assume that the absolute values of the growth rates of weights, $|g_1|$ and $|g_2|$, are constants chosen by the players. Evidently, $g_i < 0$ when player i attacks, and $g_i > 0$ when she defends. The growth rates g_i will be considered as *control variables* of the players. It is assumed that when player i defends, she solves the following problem:

$$\min g_i$$

$$s.t.\ \ g_i \geq |g_j|\,\frac{b_j}{b_i}.$$

The meaning of the coefficients in the RHS is following: the opponent's bargaining power, b_j, sharpens the constraint and leads to increasing the weight λ_i, while the own bargaining power, b_i relaxes the constraint and thus prevents increasing λ_i.

Thus the differential game is described by a pair of problems:

$$\min g_1$$

$$s.t.\ \ g_1 \geq |g_2|\,\frac{b_2}{b_1}.$$

and

$$\min g_2$$

$$s.t.\ \ g_2 \geq |g_1|\,\frac{b_1}{b_2}.$$

A continuum of Nash equilibria exists, each of them satisfying equation

$$\frac{g_i}{g_j} = -\frac{b_j}{b_i}$$

which is equivalent to (13).

Properties of the curve of weights. Let us see, how does the curve of weights depend on the relative bargaining power. Let the players start formation of the curve of weights from a point $(\widehat{\lambda}_1, \widehat{\lambda}_2)$. The equation of the curve of weights is:

$$\lambda_1^{b_1} \lambda_2^{b_2} = C = \widehat{\lambda}_1^{b_1} \hat{\lambda}_1^{b_2}.$$

Hence,

$$\lambda_1 = \left(\frac{\hat{\lambda}_2}{\lambda_2} \right)^{\frac{b_2}{b_1}} \hat{\lambda}_1. \tag{14}$$

When player 2 diminishes her weight (attacks) and player 1 prevents increasing her weights (defends), under $\lambda_2 < \hat{\lambda}_2$, an increase in the relative bargaining power of player 1 would lead, according to (14), to a decrease in her weight. In other words, player 1, defending, reaches the higher success (i.e. the lower weight) the higher her relative bargaining power is.

Under $\lambda_1 < \hat{\lambda}_1$, if player 1 attacks, to see the role of her relative bargaining power we have to look at the weight of player 2:

$$\lambda_2 = \left(\frac{\hat{\lambda}_1}{\lambda_1} \right)^{\frac{b_1}{b_2}} \hat{\lambda}_2. \tag{15}$$

With an increase in the relative bargaining power of player 1, the weight of player 2 increases, i.e., while attacking, player 1 achieves also a higher (i. e. worse) weight of her opponent.

If $b_2 \to 0$, then, according to (14), $\lambda_1 \to \hat{\lambda}_1$. It means that if player 1 possesses a very high bargaining power then, when player 2 attacks, the weight of player 1 increases negligibly. When player 1 attacks, as (15) shows, with an increase in $\hat{\lambda}_1/\lambda_1$, the weight of player 2 increases significantly.

Utility of outcome and transformation of scale. Let us consider the case of two participants. The curve Λ is described as above. Then to any outcome $x \in ParS$ a unique vector of weights $\lambda \in \Lambda$ corresponds such that $\lambda_1 x_1 = \lambda_2 x_2$. Namely, if $\lambda_1^{b_1} \lambda_2^{b_2} =$ then

$$\lambda_1 = c^{\frac{1}{b_1+b_2}} \left(\frac{x_2}{x_1} \right)^{\frac{b_2}{b_1+b_2}}, \quad \lambda_2 = c^{\frac{1}{b_1+b_2}} \left(\frac{x_1}{x_2} \right)^{\frac{b_1}{b_1+b_2}}.$$

The value $v(x) = \lambda_1 x_1 = \lambda_2 x_2$ will be called a *value of outcome* x. Thus,

$$v(x) = c^{\frac{1}{b_1+b_2}} x_1^{\frac{b_1}{b_1+b_2}} x_2^{\frac{b_2}{b_1+b_2}}.$$

Evidently, $\max_{x \in S} v(x)$ is reached at the point $\bar{x}$ of the asymmetric N.b.s.

Let us see, how the value of outcome changes under a change in scales of utilities, when each possible outcome $x = (x_1, x_2)$ is transformed into $\tilde{x} = (a_1 x_1, a_2 x_2)$, where

a_1, a_2 are constant. Let $\lambda, \tilde{\lambda} \in \Lambda$ be vectors of weights before and after the change in scales:

$$v(x) = \lambda_1 x_1 = \lambda_2 x_2,$$

$$v(\tilde{x}) = \tilde{\lambda}_1 a_1 x_1 = \tilde{\lambda}_2 a_2 x_2.$$

Consequently,

$$\frac{\lambda_1}{\tilde{\lambda}_1} = \frac{a_1 \lambda_2}{a_2 \tilde{\lambda}_2}. \tag{16}$$

On the other hand, $\lambda_1^{b_1} \lambda_2^{b_2} = \tilde{\lambda}_1^{b_1} \tilde{\lambda}_2^{b_2} = c$ and, therefore,

$$\left(\frac{\lambda_1}{\tilde{\lambda}_1}\right)^{b_1} = \left(\frac{\lambda_2}{\tilde{\lambda}_2}\right)^{b_2}. \tag{17}$$

It follows from (16) and (17) that

$$\tilde{\lambda}_2 = \left(\frac{a_1}{a_2}\right)^{\frac{b_1}{b_1+b_2}} \lambda_2.$$

Hence,

$$v(\tilde{x}) = a_1^{\frac{b_1}{b_1+b_2}} a_2^{\frac{b_2}{b_1+b_2}} v(x).$$

Thus, under a change in scales of utilities with coefficients a_1, a_2, the value of outcome is multiplied by $a_1^{\frac{b_1}{b_1+b_2}} a_2^{\frac{b_2}{b_1+b_2}}$. It is a fundamental property of the N.b.s. ($\bar{x} = \arg\max_{x \in s} v(x)$) to stay invariant under changes in scales of utilities.

Proposition 1. $v(x) = \max\limits_{\lambda \in \Lambda} \min\limits_{i=1,2} \lambda_i x_i$ *for any* $x \in S$.

Proof. Let λ be a vector of weights satisfying $v(x) = \lambda_1 x_1 = \lambda_2 x_2$ and $p \in \Lambda$ an arbitrary vector of weights, $p \neq \lambda$. Let $p_i < \lambda_i, p_j > \lambda_j$. Then $p_i x_i < \lambda_i x_i = \lambda_j x_j$ and, hence, $\min\limits_{i=1,2} p_i x_i < v(x) = \min\limits_{i=1,2} \lambda_i x_i$.

The problem of the arbitrator is to find

$$\max_{x \in S} \max_{\lambda \in \Lambda} \min \lambda_i x_i.$$

Theorem 7. *For each outcome* $x = (x_1, x_2) \in S$

$$v(x) = \max_{\lambda \in \Lambda} \min \lambda_i x_i = A x_1^{b_1} x_2^{b_2},$$

where $A = const$, *and* β_1, β_2 *are relative bargaining powers.*

See below a proof for the case of n players (Theorem 8).

In such way the arbitrator's problem reduces to finding

$$\max_{x \in S} A x_1^{b_1} x_2^{b_2}.$$

The solution is the asymmetric N.b.s.

Surface of weights in case of n players. Similarly to the case of two players, negotiations of n players apropos the structure of the surface of weights (assessments) of utilities start from an arbitrary bundle $\hat{\lambda}$ and process in such way that player j for a 1 per cent decrease in utility x_i agrees to an increase in her own utility x_j to b_j/b_i per cent only, where b_i, b_j are bargaining powers. This leads to the following system of differential equations:

$$\frac{d\lambda_j}{d\lambda_i}\frac{\lambda_i}{\lambda_j} = -\frac{b_j}{b_i} = const \quad (i, j = 1, ..., n, \quad i \neq j)$$

Solving this system, as in the case of two players, we receive

$$\lambda_i^{b_i}\lambda_i^{b_j} = const$$

Multiplying out these equations, we come to equation

$$\lambda_1^{b_1}...\lambda_n^{b_n} = C,$$

which defines the surface of weights Λ .

The problem of the arbitrator is to find

$$\max_{x\in S}\max_{\lambda\in\Lambda}\min\{\lambda_1 x_1, ..., \lambda_n x_n\},$$

where $x = (x_1, \dots, x_n)$ are admissible outcomes.

Theorem 8. *For each vector of utilities $x = (x_1, ..., x_n)$ the following equality takes place:*

$$\max_{\lambda\in\Lambda}\min\{\lambda_1 x_1, ..., \lambda_n x_n\} = A x_1^{\beta_1}...x_n^{\beta_n},$$

where A is a constant independent on x and $\beta_1, ..., \beta_n$ are relative bargaining powers.

Proof. Let us fix an arbitrary utility vector $x > 0$. There is a unique vector $\tilde{\lambda} \in \Lambda$ proportional to the vector of inverse elements of vector x:

$$\tilde{\lambda} = (\tilde{\lambda}_1, ..., \tilde{\lambda}_n) = c(x_1^{-1}, ..., x_n^{-1}).$$

Hence,

$$c^{b_1+...+b_n} x_1^{-b_1}...x_n^{-b_n} = C.$$

From here

$$c = C^{-\frac{1}{b_1+\dots+b_n}} x_1^{\beta_1}...x_n^{\beta_n},$$

where $\beta_i = \frac{b_1}{b_1+...+b_n}$ are relative bargaining powers of players $i = 1, ..., n$.

Evidently, $\min\{\tilde{\lambda}_1 x_1, ..., \tilde{\lambda}_n x_n\} = c$. Let us show that if $\lambda \in \Lambda, \lambda \neq \tilde{\lambda}$ then $\min\{\lambda_1 x_1, ..., \lambda_n x_n\} < c$. Assume the opposite: $\min\{\lambda_1 x_1, ..., \lambda_n x_n\} \geq c$. This implies $\lambda_i \geq c x_i^{-1} = \tilde{\lambda}_i \quad (i = 1, ..., n)$. But, on the other hand, for any vector $\lambda \in \Lambda$ such that $\lambda \neq \tilde{\lambda}$ there exists an element λ_k , such that $\lambda_k < \tilde{\lambda}_k$. We come to a contradiction. Hence,

$$\max_{\lambda\in\Lambda}\min\{\lambda_1 x_1, ..., \lambda_n x_n\} = c = A x_1^{\beta_1}...x_n^{\beta_n}$$

where $A = C^{-\frac{1}{b_1+\dots+b_n}} = const$.

In such way the arbitrator's problem reduces to finding

$$\max_{x\in S} Ax_1^{b_1}...x_n^{b_n},$$

and we come to the asymmetric N.b.s.

The maximin criterion, used here, reminds the Rawlsian criterion of fairness. However, in our case the society (the arbitrator) looking in each point x for maximum in λ plays in favor of a wealthier player by making worse (i.e. increasing) the weight of a more restrained player.

Similarly to the proof of Theorem 8, it can be proved that

$$\min_{\lambda\in\Lambda}\max\{\lambda_1 x_1, ..., \lambda_n x_n\} = c = Ax_1^{\beta_1}...x_n^{\beta_n}.$$

It means that a "Pharisee just" society for which the criterion is

$$\max_{x\in S}\max_{\lambda\in\Lambda}\min\{\lambda_1 x_1, ..., \lambda_n x_n\}$$

does not differ in its accepted outcome from a society searching for

$$\max_{x\in S}\min_{\lambda\in\Lambda}\max\{\lambda_1 x_1, ..., \lambda_n x_n\}$$

and, thus, openly acting in favor of the most wealthy player by improving (i.e. decreasing) her weight.

In such way the model demonstrates that societies with different political mechanisms may negligibly differ in economic and other objective characteristics of their activities.

Now let us show that the same surface of weights Λ provides the N.b.s. as a utilitarian solution.

Theorem 9. *Solution to the problem*

$$\max_{x\in S}\min_{\lambda\in\Lambda}\sum_{i=1}^{n}\mu_i x_i, \tag{18}$$

where $\mu_i = b_i\lambda_i \quad (i = 1, ..., n)$, *is*

$$\bar{x}, \bar{\lambda} = (\bar{x}_1^{-1}, ..., \bar{x}_n^{-1}),$$

where $\bar{x}$ *is the asymmetric N.b.s.*

Proof. For problem $\min_{\lambda\in\Lambda}\sum_{i=1}^{n}\mu_i x_i$ (with a fixed x) we construct a Lagrangian

$$L = \sum_{i=1}^{n}\mu_i x_i - \nu(\lambda_1^{b_1}...\lambda_n^{b_n} - C)$$

and derive the first order optimality conditions

$$b_i x_i - \nu b_i \lambda_i^{b_i-1}\prod_{k\neq i}\lambda_k^{b_k} = 0 \quad (i = 1, ..., n).$$

It follows that

$$x_i\lambda_i = const \quad (i = 1, ..., n),$$

and, by use of Theorem 3, the solution of problem (18) is the asymmetric N.b.s.

4. Other bargaining solutions

It is interesting to compare, from the point of view of our two-stage game, the N.b.s. with two other well-known solutions of the bargaining problem: the Kalai-Smorodinsky solution (Kalai and Smorodinsky, 1975) and the Kalai egalitarian solution (Kalai, 1977b). In both of them, instead of a surface of weights, a fixed bundle of weights $\hat{\lambda}_1, ..., \hat{\lambda}_n$ is used. The arbitrator chooses an outcome by solving the problem

$$\max_{x \in S} \min \{\hat{\lambda}_1 x_1, ..., \hat{\lambda}_n x_n\}. \tag{19}$$

Proposition 2. *Problem (19) has a unique solution. This is vector $\tilde{x} \in ParS$ proportional to $\left(\frac{1}{\hat{\lambda}_1}, ..., \frac{1}{\hat{\lambda}_n}\right)$.*

Proof. The vector $\tilde{x}$ satisfies

$$\hat{\lambda}_1 \tilde{x}_1 = ... = \hat{\lambda} \tilde{x}_n = c.$$

Let us show that $\min\{\hat{\lambda}_1 x_1, ..., \hat{\lambda}_n x_n\} < c$ for any $x \in S, x \neq \tilde{x}$. If the opposite is true then $\hat{\lambda}_i x_i \geq c$ for all $i = 1, ..., n$. On the other hand, there exists i such that $\tilde{x}_i > x_i$ and, hence, $c/\hat{\lambda}_i > x_i$. The contradiction implies that

$$\max_{x \in S} \min\{\hat{\lambda}_1 x_1, ..., \hat{\lambda}_n x_n\} = c,$$

and solution is reached at the unique point $\tilde{x}$.

The logic of the Kalai-Smorodinsky solution (Kalai and Smorodinsky, 1975) is that 'social indexes' of players are defined by their maximal possible gains. Then in problem (18) such weights $\hat{\lambda}_1, ..., \hat{\lambda}_n$ are used that the elements of vector $(\frac{1}{\hat{\lambda}_1}, ..., \frac{1}{\hat{\lambda}_n})$ are proportional (with a common coefficient) to maximal values $\hat{x}_1, ..., \hat{x}_n$ of coordinates in set S. The solution is a vector $x \in ParS$ proportional to $\hat{x} = (\hat{x}_1, ..., \hat{x}_n)$

The egalitarian solution (Kalai, 1977b) starts from assumption that all players have equal 'social indexes'. So, the arbitrator chooses a point solving the problem

$$\max_{x \in S} \min\{x_1, ..., x_n\}.$$

It follows from Proposition 2 that the solution is a point x^* with equal elements:

$$x_1^* = ... = x_n^*.$$

In case of weighted egalitarian solutions (Roth, 1979b) the surface of weights also consists of a single point.

5. Moral-ethical assessments as a mechanism of bargaining

Earlier we supposed that on the first stage of the game the surface (or curve) of weights Λ is constructed, and on the second stage the arbitrator (society) in a concrete bargaining situation takes a maximin solution. Now let us consider another version of the model: in a concrete bargain participants change weights gradually. Each player tries to diminish the weight of her own utility (or not to allow it to increase) and to increase the weights of the others' utilities. The arbitrator controls the bargaining process and does not allow the value of outcome, $v(x)$, to diminish.

The following scheme can serve as a model. In a two-player game, if for a current relative weight, λ_i/λ_j, the value of outcome decreases in λ_i then player i attacks by decreasing her weight, λ_i , and increasing the other's weight, λ_j. The arbitrator allows this as long as the value of outcome, $v(x)$, increases. In these circumstances a counter-attack of player j is impossible because it would lead to a decrease in $v(x)$ what is not allowed by the arbitrator. Player j in this situation can only defend herself using her bargaining power to restrict the decrease in weight λ_i and the increase in weight λ_j. The attack of player i can continue only as long as the value of outcome, $v(x)$, increases. With continuous time, this process stops as soon as the value of outcome $v(x)$ reaches its maximum at point $\bar{x}$ of the asymmetric N.b.s.

The process can be modeled, for example, by a differential game reducing to the following differential equation which describes a path in the curve Λ as well as a corresponding path in the Pareto frontier of the set S:

$$\frac{\dot{\lambda}_1}{\lambda_1} = \begin{cases} -\beta,\ if\ \dot{\lambda}_1 x_1 + \lambda_1 \dot{x}_1 > 0 \Big(and\ \dot{\lambda}_2 x_2 + \lambda_2 \dot{x}_2 < 0 \Big) \\ \ \ \beta,\ if\ \dot{\lambda}_1 x_1 + \lambda_1 \dot{x}_1 < 0 \Big(and\ \dot{\lambda}_2 x_2 + \lambda_2 \dot{x}_2 > 0 \Big). \\ \ \ 0,\ if\ \dot{\lambda}_1 x_1 + \lambda_1 \dot{x}_1 = 0 \Big(and\ \dot{\lambda}_2 x_2 + \lambda_2 \dot{x}_2 = 0 \Big) \end{cases}$$

This game leads to a point $\bar{x}$ described by equations

$$\dot{\lambda}_1 x_1 + \lambda_1 \dot{x}_1 = 0, \qquad (20)$$

$$\dot{\lambda}_2 x_2 + \lambda_2 \dot{x}_2 = 0.$$

Hence,

$$\frac{d\lambda_1}{d\lambda_2}\frac{\lambda_2}{\lambda_1} = \frac{dx_1(\bar{x})}{dx_2}\frac{\bar{x}_2}{\bar{x}_1}.$$

The LHS is equal to $-b_2/b_1$. According to Theorem 4, $\bar{x}$ is the asymmetric N.b.s.

A more straightforward proof follows. Let the Pareto frontier of set S be defined by curve $x_2 = \bar{g}(x_1)$. The curve of weights, Λ, is $\lambda_1^{b_1}\lambda_2^{b_2} = 1$. To each vector $\lambda \in \Lambda$ an outcome $x \in ParS$ corresponds satisfying equation $\lambda_1 x_1 = \lambda_2 x_2$. It follows that

$$\lambda_1 = \left(\frac{\bar{g}(x_1)}{x_1}\right)^{\frac{b_2}{b_1+b_2}}.$$

Transforming, we come to the following form of equation (20):

$$b_2\bar{g}'(x_1)x_1 + b_1\bar{g}(x_1) = 0. \qquad (21)$$

Equation (21) defines the symmetric N.b.s., as soon as $\max_{x\in S} x_1^{b_1} x_2^{b_2}$ is reached under (21).

6. N.b.s. and Cobb-Douglas production function

One more possible area of application of the N.b.s. is the theory of economic growth. Remind that, according to the neoclassical theory, the Cobb-Douglas production function $Y = AK^\alpha L^{1-\alpha}$ (where Y is output, K is capital, and L is labor) reflects not only production but also distribution of product: under conditions of perfect competition, parameters α and $1-\alpha$ are not only elasticities of output with respect to production factors but are also shares of the owners of production factors. In recent time a series of research appeared demonstrating a difference in labor and capital shares in countries of the World (in average, in developing countries the capital share is higher than in industrial countries) and changes in the factor shares in time (the capital share increases in all groups of countries). It is possible to use the N.b.s. concept to explain the differences between countries and the tendency of changes in the shares. A constancy of bargaining powers can explain a constancy of factor shares (a validity of the corresponding Kaldor's stylized fact) in some countries on a definite stage of their development.

Let us consider a model with two social groups: workers and owners of capital (entrepreneurs) who possess bargaining powers b_L and b_K, correspondingly, and bargain apropos partition of the national product. The N.b.s. is realized by use of the mechanism of propaganda and moral-ethical assessments described above and leads to a distribution of the product in shares b_L and b_K. To ensure a possibility of such distribution the social groups choose those production technologies (and fields of specialization of the economy) for which factor elasticities $1-\alpha=\frac{b_L}{b_K+b_L}$ and $\alpha=\frac{b_K}{b_K+b_L}$ are used in the Cobb-Douglas production function. The choice of technologies can take place both on stage of R&D and on stage of production (e.g., workers can violate technologies to receive in result a demanded share of output. This can explain, for example, a tolerance of society to small and big plundering in manufacturing in the USSR).

The process can be modeled by the two-stage game described above. On the first stage two players (entrepreneurs and workers) form a curve $\Lambda=(\lambda_K,\lambda_L)$ of admissible moral-ethical assessments (weights). On the second stage an arbitrator (society) chooses an admissible pair of weights and divides the product Y among the players ($Y=Y_K+Y_L$) to achieve the maximin

$$\max_{Y,\lambda}\min\{\lambda_K Y_K,\lambda_L Y_L\}.$$

The curve of weights formed on the first stage of the game is

$$\lambda_K^{b_K}\lambda_L^{b_L}=C.$$

For the second stage of the game, Theorem 7 implies the following statement.

Theorem 10. *For each outcome* (Y_K,Y_L) *(where* $Y_K+Y_L=const$*):*

$$\max_{\lambda\in\Lambda}\min\{\lambda_K Y_K,\lambda_L Y_L\}=AY_K^{\frac{b_K}{b_K+b_L}}Y_L^{\frac{b_L}{b_K+b_L}},$$

where $A=const$.

Thus, the arbitrator's problem reduces to the asymmetric N.b.s. The solution, as can be easily seen, implies that the players receive shares proportional to their bargaining powers:

$$\frac{Y_K}{Y_L}=\frac{b_K}{b_L}.$$

For such distribution to be possible, the players will choose those technologies (and those fields of specialization of the economy) under which factor elasticities used in a Cobb-Douglas production function are proportional to the bargaining powers.

In relation with globalization, there is a question of stability of a product partition between participants of the global production process. The basic argument of anti-globalists is a reduction in the labor share in advanced countries which is explained by extension of production in developing countries.

Let us consider a model with three participants: an entrepreneur (a transnational company) acting in two countries and workers (unions) of these two countries. Capital is mobile, in contrast to labor. Will an equilibrium reached inside the countries on base of 'inner' bargaining powers of the participants coincide with an equilibrium which would arise if the three participants meet in 'global' negotiations? (In case of a non-coincidence, one of the social groups would be interested in a revision of the "World order", and the equilibrium would not be stable). We will find bargaining powers under which such coincidence takes place.

Let inner bargaining powers of the entrepreneur and the workers be $\alpha_1, 1-\alpha_1$ in country 1 and $\alpha_2, 1-\alpha_2$ in country 2. As a result of a choice of technologies corresponding to the desired partition of the product, the production functions are $A_1 K_1^{\alpha_1} L_1^{1-\alpha_1}$ and $A_2 K_2^{\alpha_2} L_2^{1-\alpha_2}$. These functions can also be written as $K_1^{\alpha_1}\tilde{L}_1^{1-\alpha_1}$ and $K_2^{\alpha_2}\tilde{L}_2^{1-\alpha_2}$ where $\tilde{L}_i = A_i^{\frac{1}{1-\alpha}} L_i$ is the effective labor in country i $(i = 1, 2)$. The entrepreneur will place the total capital K in a way to equalize the marginal products of capital in the countries:

$$\alpha_1 K_1^{\alpha_1 - 1}\tilde{L}_1^{1-\alpha_1} = \alpha_2 K_2^{\alpha_2 - 1}\tilde{L}_2^{1-\alpha_2}.$$

Denote α, β, γ bargaining powers of participants in a global bargain, $\alpha + \beta + \gamma = 1$. A coincidence of the equilibria achieved in the inner and in the global bargains means holding equations:

$$\alpha = \alpha_1 \frac{Y_1}{Y_1 + Y_2} + \alpha_2 \frac{Y_2}{Y_1 + Y_2}, \beta = (1-\alpha_1)\frac{Y_1}{Y_1 + Y_2}, \gamma = (1-\alpha_2)\frac{Y_2}{Y_1 + Y_2}.$$

Thus, in equilibrium, the workers of country i possess in the global bargain a bargaining power equal to their inner bargaining power, $1 - \alpha_i$ with a discounting weight $\chi_i = \frac{Y_i}{Y_1 + Y_2}$ equal to the country's share in the global output. The global bargaining power of the entrepreneur is equal to the convex combination of her inner bargaining powers α_i with the weights χ_i $(i = 1, 2)$.

Let us consider a case when inner bargaining powers in both countries are the same: $\bar{\alpha}$ and $1 - \bar{\alpha}$. Then

$$K_1^{\bar{\alpha}-1}\tilde{L}_1^{1-\bar{\alpha}} = K_2^{\bar{\alpha}-1}\tilde{L}_2^{1-\bar{\alpha}}.$$

From here,

$$K_1 = \frac{K\tilde{L}_1}{\tilde{L}_1 + \tilde{L}_2}, K_2 = \frac{K\tilde{L}_2}{\tilde{L}_1 + \tilde{L}_2}.$$

Thus, the entrepreneur places the capital proportionally to the effective labor. In the global bargaining,

$$\alpha = \bar{\alpha},$$

$$\beta = (1-\bar{\alpha})\frac{Y_1}{Y_1 + Y_2} = (1-\bar{\alpha})\frac{\tilde{L}_1}{\tilde{L}_1 + \tilde{L}_2},$$

$$\gamma = (1-\bar{\alpha})\frac{\tilde{L}_2}{\tilde{L}_1 + \tilde{L}_2}.$$

Therefore, for holding the equilibrium, the bargaining power of the entrepreneur in the global bargaining has to be the same as in the inner bargaining, and the bargaining powers of the workers have to be proportional to their effective labor. Each of the two groups of workers has in the global bargaining a bargaining power less than in the inner bargaining. If the bargaining powers of the two groups of workers are summed up, the "united labor" would have the same bargaining power as the workers have in inner bargains in each of the countries. In general, this does not contradict an intuitive idea of a global bargaining and a partition of the world gross product.

7. Conclusion

The Nash bargaining solution (N.b.s.) takes a central place in the theory of bargaining. Research in this paper differs essentially from known approaches to the N.b.s. which were based, as a rule, either on a choice of axioms of a fair distribution when participants play a purely passive role, or on a choice of a stopping rule in negotiations with reference to a concrete bargain. In the present model negotiations relate to a system of weights, Λ. In the negotiations participants exhibit their bargaining powers. The set Λ can be used not in a single bargain but in a set of bargains of the same participants. A special role is plaid by an arbitrator interpreted here as a community (society).

A relation between bargaining powers, 'utilitarian' weights and 'egalitarian weights is studied. In particular, a cobweb procedure leading to the N.b.s. is considered.

We studied a two-stage game, where on the first stage the players form a set Λ of all possible bundles of weights, and on the second stage, for a concrete bargain, the arbitrator finds a maximin solution. We showed that this game leads to the asymmetric N.b.s. Also it is shown that two other well-known bargaining solutions, Kalai-Smorodinsky and egalitarian, can be received as special cases with Λ consisting of a single bundle of weights.

As an example of application of the ideas elaborated in the paper we propose a model where the asymmetric N.b.s. leads to the Cobb-Douglas production function.

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On Games with Constant Nash Sum

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Abstract. A class of games in strategic form with the following property is identified: for every $\mathbf{n} \in E$, i.e. Nash equilibrium, the (Nash) sum $\sum_l n^l$ is constant. For such a game sufficient conditions for E to be polyhedral and semi-uniqueness (i.e. $\#E \leq 1$) are given. The abstract results are illustrated by applying them to a class of games that covers various types of Cournot oligopoly and transboundary pollution games. The way of obtaining the results is by analysing so-called left and right marginal reductions.

Keywords: Oligopoly, transboundary pollution, Hahn conditions, aggregative game, co-strategy mapping, marginal reduction, non-differentiable payoff function, structure of set of Nash equilibria, game in strategic form, convex analysis.

1. Introduction

In Szidarovszky and Yakowitz (1982) the following result was proven for the set of Cournot equilibria E of a homogeneous Cournot oligopoly game with N oligopolists:[1]

Theorem 1. *Suppose that each oligopolist has a capacity constraint, the inverse demand function p is concave, decreasing and continuous, and each cost function c^i is convex, increasing and continuous. Then:*

I. $\#E \geq 1$.

II. If each c^i is strictly convex or p is strictly decreasing, then:

1. *the Nash sum is constant, i.e. there exists $\Psi \geq 0$ such that $\sum_{l=1}^{N} n^l = \Psi$ for all $\mathbf{n} \in E$;*
2. *E is polyhedral.*

III. In case p is differentiable, each of the following conditions separately is sufficient for $\#E \leq 1$:

1. *each c^i is strictly convex;*
2. *$p' < 0$.* $\diamond$

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[1] In this article I understand by a *Cournot oligopoly game* a game in strategic form with $N (\geq 1)$ players where a player i has strategy set $X^i = [0, m^i]$ (*capacity constraint* $m^i > 0$) or $X^i = \mathbb{R}_+$ and payoff function $\pi^i(\mathbf{x}) = p^i(\sum_{l=1}^{N} t_l x^l) x^i - c^i(x^i)$. Here $c^i : X^i \to \mathbb{R}$ is his cost function and, with $Y := \sum_{l=1}^{N} t_l X^l$ and the $t_l > 0$, $p^i : Y \to \mathbb{R}$ is his *inverse demand function*. With the *differentiable case* I refer to a Cournot oligopoly game where all cost functions and inverse demand functions are differentiable. If $p^1 = \cdots = p^N =: p$ and $t_1 = \cdots = t_N = 1$, then the Cournot oligopoly game is *homogeneous*.

Because in this theorem for each oligopolist each conditional payoff function[2] is concave, statement I about existence directly follows from the Nikaido-Isoda theorem.[3] Therefore, the new contribution in Theorem 1 were statement II about the geometric structure of the set of Cournot equilibria and III about their semi-uniqueness (and in fact, because of I, even about their uniqueness).[4] The proof of II and III given in Szidarovszky and Yakowitz (1982) can be characterized as 'technical and by hand'. The aim of my article is to obtain similar results as in II and III with conceptual relatively simple (but abstract) proofs, for a broad class of games in strategic form, with player dependent one-dimensional action sets. Note that in Theorem 1 only for semi-uniqueness a differentiability assumption (for p) is made.

A predecessor of I is Theorem 1.1.1.1 in Okuguchi (1976). There are many complementary results to and improvements of I. A recent one is Theorem 3.2 in Ewerhart (2009) which implies important results of Novshek (1985) and Amir (1996).

Both statements in II seem to be isolated results in the literature. I am only aware of the convexity of the set of Nash equilibria for strict competitive games between two players with quasi-concave conditional payoff functions (see Theorem 3.3 in Friedman (1991)).[5]

Predecessors of III are Theorem 2.1 in Okuguchi (1976) and Theorem 1 in Szidarovszky and Yakowitz (1977). Also here various complementary results are known, but I am not aware of substantial improvements. However, for the differentiable case various improvements exist (for instance, Murphy et al.(1982), Gaudet and Salant (1991), Watts (1996), Corchón (2001) and Cornes and Hartley (2005)). For this case there exists a simple proof of (a even stronger statement than) III. To see how and what, I define for each player i the function $t^i : X^i \times Y \to \mathbb{R}$ by

$$t^i(x^i, y) \; := \; p'(y)x^i + p(y) - c^{i\prime}(x^i)$$

and note that the following condition is sufficient for semi-uniqueness:[6]

for every i and $\mathbf{a}, \mathbf{b} \in E$ with $\sum_l b^l \geq \sum_l a^l$ it holds that

$$b^i > a^i \;\Rightarrow\; t^i(b^i, \sum_l b^l) < t^i(a^i, \sum_l a^l). \tag{1}$$

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[2] I.e. the payoff as a function of his own action, given the actions of the others.

[3] This is the following result: if each strategy set is a compact convex subset of a finite dimensional normed real linear space, each payoff function is continuous and each conditional payoff function is quasi-concave, then there exists a Nash equilibrium.

[4] Theorem 1 (like various others in oligopoly theory) does not refer to the market satiation point $v \; := \; \inf\{y \in Y \mid p(y) \leq 0\} \in Y \cup \{+\infty\}$ of p. However, by referring to it, an improved version of the theorem can be obtained where the overall assumption that p is concave can be replaced by the concavity of $p \restriction Y \cap [0, v\,[$. However, such improvements are generally considered as not so substantial.

[5] Also see Lemma 5 in Murphy et al.(1982) where sufficient conditions for a homogeneous Cournot oligopoly game in the differentiable case are given for the set of Nash sums to be convex.

[6] Indeed: I have $D_i f^i(\mathbf{x}) = t^i(x^i, \sum_l x^l)$. And in a Cournot equilibrium $\mathbf{n}$ I have $D_i f^i(\mathbf{n}) = 0$ if n^i is an interior point of X^i, $D_i f^i(\mathbf{n}) \leq 0$ if n^i is a left-boundary point of X^i and $D_i f^i(\mathbf{n}) \geq 0$ if n^i is a right-boundary point of X^i. Having this, the existence of two Cournot equilibria $\mathbf{a}, \mathbf{b}$ easily leads with (1) to a contradiction.

Next note that sufficient (but not necessary) for (1) to hold is that t^i is strictly decreasing in its first variable and decreasing in its second.[7] Sufficient for this in turn is that p and the c^i are twice differentiable and $D_1 t^i = p'(y) - c^{i''}(x^i) < 0$ (*first Hahn condition*) and $D_2 t^i = p'(y) + p''(y)x^i \leq 0$ (*second Hahn condition*) hold.

I will refer to the above t^i as *marginal reductions*. They first were systematically used in Corchón (2001) in the context of uniqueness results for aggregative games. The given proof in footnote 6 is obtained by a slight adaptation of the proof (of the semi-uniqueness part) of Proposition 1.3 in Corchón (2001). Note that in (1) in fact only local conditions at Nash equilibria matter: it is not necessary that the definition of t^i globally makes sense. By studying for the differentiable case[8] the global properties of the marginal reductions t^i by means of the correspondences $b^i(y) \ := \ \{x^i \in X^i \mid t^i(x^i, y) = 0\}$ a very strong uniqueness result was obtained. The conditions in this result involve a strong variant of the first Hahn-condition and, a local condition at Nash equilibria (involving a sum of quotients $D_2 t^i / D_1 t^i$).

The games I will deal with encompass various aggregative games with non-differentiable conditional payoff functions. I give with Corollary 1 sufficient (local) conditions for the Nash sum to be constant, and more generally for a function $\Phi : \mathbf{X} \to \mathbb{R}^N$, to be referred as *co-strategy mapping*, to be constant on the set of Nash equilibria E. Corollary 2 gives sufficient (local) conditions for $\#E \leq 1$. Theorem 2 provides a class of games with concave conditional payoff functions for which E is polyhedral. The power of these results is illustrated by applying them to what I call *transboundary Cournot games*. In particular such a game allows for a Cournot oligopoly game and for a transboundary pollution game with uniform pollution (see, for instance, Finus (2001) and Folmer and von Mouche (2002)). Transboundary pollution games are in some sense dual to Cournot oligopoly games. The main result for transboundary Cournot games is Theorem 3; it implies and intrinsically improves Theorem 1.

In order to obtain my results, I use some standard results of convex analysis and refine the ideas in Folmer and von Mouche (2004) and von Mouche (2009). In general I do not suppose that conditional payoff functions are differentiable. But I generally make the (weak) assumption that their left and right derivatives exist (in $\overline{\mathbb{R}} \ := \ \mathbb{R} \cup \{-\infty, +\infty\}$) in Nash equilibria.[9] This leads to the concept of right marginal reductions $\mathcal{T}^i_+$ and left marginal reductions $\mathcal{T}^i_-$. My method consists in a mainly local analysis of them; special care is needed to deal in a correct way with boundaries and infinite derivatives.

2. Notations and Co-strategy Mappings

I use the following notations for a subset C of $\mathbb{R}$:

$$\partial_-(C) := \begin{cases} \{\min(C)\} \text{ if } \min(C) \text{ exists} \\ \emptyset \text{ if } \min(C) \text{ does not exist,} \end{cases} \quad \partial_+(C) := \begin{cases} \{\max(C)\} \text{ if } \max(C) \text{ exists} \\ \emptyset \text{ if } \max(C) \text{ does not exist;} \end{cases}$$

$$C_- \ := \ (C \setminus \partial_+(C)) \cup \partial_-(C), \quad C_+ \ := \ (C \setminus \partial_-(C)) \cup \partial_+(C);$$

[7] Note that then all conditional payoff functions are strictly concave (and therefore existence is guaranteed in case of capacity constraints).

[8] Strictly speaking for a situation with a market satiation point $v > 0$ and p differentiable on $[0, v[$. In fact there $b^i : [0, v[\multimap \mathbb{R}$.

[9] Note that this assumption implicitly even globally holds in Theorem 1 because each conditional payoff function is concave.

$$\mathrm{Int}(C) \;:=\; \text{the topological interior of } C.$$

Now[10]

$$\partial_+(C) \subseteq C_+,\;\; \partial_-(C) \subseteq C_-,\; \mathrm{Int}(C) \subseteq C_- \subseteq C,\;\; \mathrm{Int}(C) \subseteq C_+ \subseteq C;$$
$$C = C_- \cup \partial_+(C) = C_+ \cup \partial_-(C). \tag{2}$$

For the whole article I fix a positive integer N, and writing,

$$\mathcal{N} \;:=\; \{1, \ldots, N\},$$

I also fix subsets

$$X^i \;(i \in \mathcal{N})$$

of $\mathbb{R}$ and assume

$$\text{every } X^i \text{ is a non-degenerate interval.}^{11}$$

I write

$$\mathbf{X} \;:=\; X^1 \times \cdots \times X^N,$$
$$\mathbf{X}^{\hat{\imath}} \;:=\; X^1 \times \cdots \times X^{i-1} \times X^{i+1} \times \cdots \times X^N \;(i \in \mathcal{N})$$

and sometimes identify $\mathbf{X}$ with $X^i \times \mathbf{X}^{\hat{\imath}}$ and then write $\mathbf{x} \in \mathbf{X}$ as $\mathbf{x} = (x^i; \mathbf{x}^{\hat{\imath}})$.

A *co-strategy function* for player i is a function

$$\varphi^i : \mathbf{X} \to \mathbb{R}$$

and a *co-strategy mapping* is a mapping

$$\varPhi = (\varphi^1, \ldots, \varphi^N) : \mathbf{X} \to \mathbb{R}^N.$$

Given a co-strategy function φ^i, I write

$$Y^i \;:=\; \varphi^i(\mathbf{X}).$$

For a subset Z of $\mathbf{X}$, I write

$$\mathcal{Z}(Z) \;:=\; \{i \in \mathcal{N} \mid \text{ there exist } \mathbf{a}, \mathbf{b} \in Z \text{ with } a^i \neq b^i\}.$$

For a subset Z of $\mathbf{X}$, a co-strategy mapping $\varPhi$ and $i \in \mathcal{N}$, the following (weak) Property $F^i_w(Z, \varPhi)$ and (strong) Property $F^i_s(Z, \varPhi)$ will be dealt with:

Property $F^i_w(Z, \varPhi)$. For all $\mathbf{a}, \mathbf{b} \in Z$: $\varPhi(\mathbf{b}) \gg \varPhi(\mathbf{a}) \;\Rightarrow\; b^i \leq a^i$.

Property $F^i_s(Z, \varPhi)$. For all $\mathbf{a}, \mathbf{b} \in Z$: $\varPhi(\mathbf{b}) \geq \varPhi(\mathbf{a}) \;\Rightarrow\; b^i \leq a^i$.

Lemma 1. *Let Z be a subset of $\mathbf{X}$ and $\varPhi$ a co-strategy mapping. Each of the following conditions separately is sufficient for $\varPhi \restriction Z$ to be constant:*

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[10] Example: if $C = \,]-3, 1]$, then $\mathrm{Int}(C) = \,]-3, 1[$, $\partial_-(C) = \emptyset$, $\partial_+(C) = \{1\}$, $C_- = \,]-3, 1[$ and $C_+ = \,]-3, 1]$. And if $C = \{3\}$, then $\mathrm{Int}(C) = \emptyset$ and $\partial_-(C) = \partial_+(C) = C_- = C_+ = \{3\}$.

[11] I.e. a convex subset of $\mathbb{R}$ containing more than one and therefore infinitely many elements.

1. *Properties $F^i_w(Z,\Phi)$ ($i \in \mathcal{Z}(Z)$) hold and $\Phi \restriction Z$ is increasing*[12] *and strictly ordered.*[13]
2. *Properties $F^i_s(Z,\Phi)$ ($i \in \mathcal{Z}(Z)$) hold and $\Phi \restriction Z$ is increasing and ordered. In this case even $\#Z \leq 1$.* $\diamond$

Proof. 1. By contradiction. So suppose that $\mathbf{a}, \mathbf{b} \in Z$ such that $\Phi(\mathbf{a}) \neq \Phi(\mathbf{b})$. Let $j \in \mathcal{N}$ such that $\varphi^j(\mathbf{a}) \neq \varphi^j(\mathbf{b})$. It may be supposed that the strict inequality $\varphi^j(\mathbf{b}) > \varphi^j(\mathbf{a})$ holds. Because $\Phi \restriction Z$ is strictly ordered, this strict inequality implies $\Phi(\mathbf{b}) \gg \Phi(\mathbf{a})$. Because Property $F^i_w(Z,\Phi)$ ($i \in \mathcal{Z}(Z)$) holds, I have $b^i \leq a^i$ ($i \in \mathcal{Z}(Z)$). Also $b^i = a^i$ ($i \in \mathcal{N} \setminus \mathcal{Z}(Z)$). So $\mathbf{b} \leq \mathbf{a}$. Because $\varphi^j \restriction Z$ is increasing, $\varphi^j(\mathbf{b}) \leq \varphi^j(\mathbf{a})$ follows, which is a contradiction.

2. Suppose that $\mathbf{a}, \mathbf{b} \in Z$. Because $\Phi \restriction Z$ is ordered, one has $\Phi(\mathbf{b}) \geq \Phi(\mathbf{a})$ or $\Phi(\mathbf{b}) \leq \Phi(\mathbf{a})$. It may be assumed that $\Phi(\mathbf{b}) \geq \Phi(\mathbf{a})$. By Property $F^i_s(Z,\Phi)$ ($i \in \mathcal{Z}(Z)$), I have $b^i \leq a^i$ ($i \in \mathcal{Z}(Z)$). Also $b^i = a^i$ ($i \in \mathcal{N} \setminus \mathcal{Z}(Z)$). So $\mathbf{b} \leq \mathbf{a}$. Because $\Phi \restriction Z$ is increasing, $\Phi(\mathbf{a}) \geq \Phi(\mathbf{b})$ follows. By Property $F^i_s(Z)$ ($i \in \mathcal{Z}(Z)$), I have $a^i \leq b^i$ ($i \in \mathcal{Z}(Z)$). Thus $\mathbf{a} \leq \mathbf{b}$ and therefore $\mathbf{a} = \mathbf{b}$ holds. $\square$

Given, $i \in \mathcal{N}$, a co-strategy function $\varphi^i : \mathbf{X} \to \mathbb{R}$ and a subset Z of $\mathbf{X}$, I define

$$\mathcal{W}^i_-(Z) := \{(x^i, \varphi^i(x^i; \mathbf{z})) \mid (x^i; \mathbf{z}) \in Z \text{ and } x^i \in X^i_-\},$$

$$\mathcal{W}^i_+(Z) := \{(x^i, \varphi^i(x^i; \mathbf{z})) \mid (x^i; \mathbf{z}) \in Z \text{ and } x^i \in X^i_+\}.$$

Given a subset Z of $\mathbb{R}^N$ with $Z \subseteq \mathbf{X}$, $i \in \mathcal{N}$, $y^i \in Y^i$ and functions $\mathcal{T}^i_+ : \mathcal{W}^i_-(Z) \to \overline{\mathbb{R}}$ and $\mathcal{T}^i_- : \mathcal{W}^i_+(Z) \to \overline{\mathbb{R}}$, I write

$$\begin{aligned} N^i_{y^i}(Z) := \; & \{x^i \in \mathrm{Int}(X^i) \mid (x^i, y^i) \in \mathcal{W}^i_-(Z) \cap \mathcal{W}^i_+(Z),\ \mathcal{T}^i_+(x^i, y^i) \leq 0 \leq \mathcal{T}^i_-(x^i, y^i)\} \\ & \cup \{x^i \in \partial_+(X^i) \mid (x^i, y^i) \in \mathcal{W}^i_+(Z),\ \mathcal{T}^i_-(x^i, y^i) \geq 0\} \\ & \cup \{x^i \in \partial_-(X^i) \mid (x^i, y^i) \in \mathcal{W}^i_-(Z),\ \mathcal{T}^i_+(x^i, y^i) \leq 0\}. \end{aligned} \tag{3}$$

For this situation:

Lemma 2. *If Z is a cartesian product of intervals, $\mathcal{T}^i_+$ and $\mathcal{T}^i_-$ are decreasing in their first variable and φ^i is increasing in its i-th variable and continuous, then $N^i_{y^i}(Z)$ is an interval.* $\diamond$

Proof. Suppose that $a^i, b^i \in N^i_{y^i}(Z)$ with $a^i < b^i$. Let $\lambda \in]0,1[$ and $c^i = \lambda a^i + (1-\lambda) b^i$. I have $a^i < c^i < b^i$ and thus $c^i \in \mathrm{Int}(X^i)$. As $a^i, b^i \in N^i_{y^i}(Z)$, $a^i \in X^i_-$ and $b^i \in X^i_+$, I have

$$(a^i, y^i) \in \mathcal{W}^i_-(Z) \text{ and } \mathcal{T}^i_+(a^i, y^i) \leq 0, \quad (b^i, y^i) \in \mathcal{W}^i_+(Z) \text{ and } \mathcal{T}^i_-(b^i, y^i) \geq 0.$$

[12] Given a positive integer n, the relations $\geq, >, \gg$ on $\mathbb{R}^n$ are defined by: $\mathbf{x} \geq \mathbf{y} :\ x_k \geq y_k$ ($1 \leq k \leq n$); $\mathbf{x} > \mathbf{y} :\ \mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$; $\mathbf{x} \gg \mathbf{y} : x_k > y_k$ ($1 \leq k \leq n$). And $\leq, <, \ll$ denote the dual relations of respectively $\geq, >, \gg$.

Consider a mapping $F = (F^1, \ldots, F^N) : Z \to \mathbb{R}^N$, where $Z \subseteq \mathbb{R}^m$. F is called *increasing* if for all $\mathbf{a}, \mathbf{b} \in Z$ one has $\mathbf{a} \leq \mathbf{b} \;\Rightarrow\; F(\mathbf{a}) \leq F(\mathbf{b})$. Note that this is equivalent with: every $F^i \restriction Z$ is increasing. Also: if Z is a cartesian product, then $F : Z \to \mathbb{R}$ is increasing if and only if F is increasing in each variable.

[13] Consider a mapping $F : Z \to \mathbb{R}^n$, where $Z \subseteq \mathbb{R}^m$. In this article, F is called – *ordered* if for all $\mathbf{a}, \mathbf{b} \in Z$ it holds that $F(\mathbf{a}) \geq F(\mathbf{b})$ or that $F(\mathbf{a}) \leq F(\mathbf{b})$; – *strictly ordered* if for all $\mathbf{a}, \mathbf{b} \in Z$ it holds that $F(\mathbf{a}) \gg F(\mathbf{b})$ or that $F(\mathbf{a}) = F(\mathbf{b})$ or that $F(\mathbf{a}) \ll F(\mathbf{b})$.

Also

$$(c^i, y^i) \in \mathcal{W}^i_-(Z) \cap \mathcal{W}^i_+(Z).$$

Indeed: as $(a^i, y^i) \in \mathcal{W}^i_-(Z)$, there exists $\mathbf{z} \in \mathbf{X}^{\hat{\imath}}$ such that $(a^i; \mathbf{z}) \in Z$ and $y^i = \varphi^i(a^i; \mathbf{z})$. As $(b^i, y^i) \in \mathcal{W}^i_+(Z)$, there exists $\mathbf{z}' \in \mathbf{X}^{\hat{\imath}}$ such that $(b^i; \mathbf{z}') \in Z$ and $y^i = \varphi^i(b^i; \mathbf{z}')$. As φ^i is increasing in its i-th variable, $\varphi^i(c^i; \mathbf{z}) \geq \varphi^i(a^i; \mathbf{z}) = y^i = \varphi^i(b^i; \mathbf{z}') \geq \varphi^i(c^i; \mathbf{z}')$. So $\varphi^i(c^i; \mathbf{z}) \geq y^i \geq \varphi^i(c^i; \mathbf{z}')$. As φ^i is continuous, it follows that there exists $\mu \in [0, 1]$ such that for $\mathbf{z}'' := \mu\mathbf{z} + (1 - \mu)\mathbf{z}'$ I have

$$y^i = \varphi^i(c^i; \mathbf{z}'').$$

As Z is a cartesian product of intervals, I have $(c^i; \mathbf{z}'') \in Z$. It follows that $(c^i, y^i) \in \mathcal{W}^i_-(Z) \cap \mathcal{W}^i_+(Z)$. Because $\mathcal{T}^i_+ : \mathcal{W}^i_-(Z) \to \overline{\mathrm{I\!R}}$ and $\mathcal{T}^i_- : \mathcal{W}^i_+(Z) \to \overline{\mathrm{I\!R}}$ are increasing in their first variable, I have

$$\mathcal{T}^i_+(c^i, y^i) \leq \mathcal{T}^i_+(a^i, y^i) \leq 0 \leq \mathcal{T}^i_-(b^i, y^i) \leq \mathcal{T}^i_-(c^i, y^i).$$

Thus $c^i \in N^i_{y^i}(Z)$. □

3. Marginal Reductions

In this section and in the next three I always consider a game in strategic form, with $\mathcal{N}$ as the set of players, with X^i as strategy set for player i and with $f^i : \mathbf{X} \to \mathrm{I\!R}$ his payoff function. The set of of Nash equilibria is denoted by

$$E.$$

Notation: with $(D^+_i f^i(\mathbf{x})$ $(D^-_i f^i(\mathbf{x}))$ I denote the right (left) partial derivative of f^i w.r.t. its i-th variable in $\mathbf{x}$.

Definition 1. Let $i \in \mathcal{N}$ and Z a subset of $\mathbf{X}$.

A two-tuple $(\mathcal{T}^i_+, \varphi^i)$ with

$$\varphi^i : \mathbf{X} \to \mathrm{I\!R} \text{ a (co-strategy) function and } \mathcal{T}^i_+ : W^i_- \to \overline{\mathrm{I\!R}} \text{ a function}$$

where $W^i_- \subseteq X^i \times Y^i$ such that $\mathcal{W}^i_-(Z) \subseteq W^i_-$, is called a *right marginal reduction* of f^i on Z (with domain W^i_-) if for every $\mathbf{x} \in Z$ with $x^i \in X^i_-$:

- $x^i \in \mathrm{Int}(X^i) \Rightarrow f^i$ is right partially differentiable w.r.t. its i-th variable in $\mathbf{x}$;
- $x^i \in \partial_-(X^i) \Rightarrow$ the right partial derivative of f^i w.r.t. its i-th variable exists in $\mathbf{x}$ as element of $\overline{\mathrm{I\!R}}$;
- $D^+_i f^i(\mathbf{x}) = \mathcal{T}^i_+(x^i, \varphi^i(\mathbf{x}))$.

A two-tuple $(\mathcal{T}^i_-, \varphi^i, Z)$, with

$$\varphi^i : \mathbf{X} \to \mathrm{I\!R} \text{ a (co-strategy) function and } \mathcal{T}^i_- : W^i_+ \to \overline{\mathrm{I\!R}} \text{ a function}$$

where $W^i_+ \subseteq X^i \times Y^i$ such that $\mathcal{W}^i_+(Z) \subseteq W^i_+$, is called a *left marginal reduction* of f^i on Z (with domain W^i_+) if for every $\mathbf{x} \in Z$ with $x^i \in X^i_+$:

- $x^i \in \mathrm{Int}(X^i) \Rightarrow f^i$ is left partially differentiable w.r.t. its i-th variable in $\mathbf{x}$;
- $x^i \in \partial_+(X^i) \Rightarrow$ the left partial derivative of f^i w.r.t. its i-th variable exists in $\mathbf{x}$ as element of $\overline{\mathrm{I\!R}}$;

– $D_i^- f^i(\mathbf{x}) = \mathcal{T}_-^i(x^i, \varphi^i(\mathbf{x}))$.

A three-tuple $(\mathcal{T}_+^i, \mathcal{T}_-^i, \varphi^i)$ is called a *marginal reduction* of f^i on Z (with domain (W_-^i, W_+^i)) if $(\mathcal{T}_+^i, \varphi^i)$ is a right marginal reduction of f^i on Z (with domain W_-^i) and $(\mathcal{T}_-^i, \varphi^i)$ is a left marginal reduction of f^i on Z (with domain W_+^i). $\diamond$

Note: it may be well possible to allow in the above definition infinity derivatives also in interior points and to develop the theory for this more general situation. However, I will not allow for this in order to facilitate the presentation.

Also note that if $(\mathcal{T}_+^i, \varphi^i)$ is a right marginal reduction of f^i on Z with domain W_-^i, and I change the definition of $\mathcal{T}_+^i$ on $W_-^i \setminus \mathcal{W}_-^i(Z)$, then I still keep a right marginal reduction of f^i on Z. A same remark applies to $(\mathcal{T}_-^i, \varphi^i)$.

In practice for a given game one often quickly can see, by studying (the form of) $D_i^+ f^i$ and $D_i^- f^i$, that there exists a marginal reduction $(\mathcal{T}_+^i, \mathcal{T}_-^i, \varphi^i)$ of f^i, may be even on $\mathbf{X}$. In particular, this is the case for various aggregative games, i.e. for games in strategic form where each payoff function f^i is of the form

$$f^i(x^1, \ldots, x^N) = \pi^i(x^i, \sum_{l=1}^N T_l x^l).$$

For $i \in \mathcal{N}$ and $\mathbf{z} \in \mathbf{X}^{\hat{\imath}}$, I define the *conditional payoff function* $f_{\mathbf{z}}^i : X^i \to \mathbb{R}$ by

$$f_{\mathbf{z}}^i(x^i) \ := \ f^i(x^i; \mathbf{z}).$$

Suppose that $\mathbf{n} \in E$. If $(\mathcal{T}_+^i, \varphi^i)$ is a right marginal reduction of f^i on $\{\mathbf{n}\}$, then, because $\mathbf{n}$ is a Nash equilibrium,

$$n^i \in X_-^i \ \Rightarrow \ D_i^+ f^i(\mathbf{n}) = D^+ f_{\mathbf{n}^{\hat{\imath}}}^i(n^i) = \mathcal{T}_+^i(n^i, \varphi^i(\mathbf{n})) \leq 0. \tag{4}$$

And if $(\mathcal{T}_-^i, \varphi^i)$ is a left marginal reduction of f^i on $\{\mathbf{n}\}$, then

$$n^i \in X_+^i \ \Rightarrow \ D_i^- f^i(\mathbf{n}) = D^- f_{\mathbf{n}^{\hat{\imath}}}^i(n^i) = \mathcal{T}_-^i(n^i, \varphi^i(\mathbf{n})) \geq 0. \tag{5}$$

Of course, sufficient for $\mathbf{n} \in \mathbf{X}$ to be a Nash equilibrium is that for all $i \in \mathcal{N}$:

- the function $f_{\mathbf{n}^{\hat{\imath}}}^i$ is concave;
- there exists a marginal reduction $(\mathcal{T}_+^i, \mathcal{T}_-^i, \varphi^i)$ of f^i on $\{\mathbf{n}\}$;
- $n^i \in X_-^i \ \Rightarrow \ \mathcal{T}_+^i(n^i, \varphi^i(\mathbf{n})) \leq 0$ and $n^i \in X_+^i \ \Rightarrow \ \mathcal{T}_-^i(n^i, \varphi^i(\mathbf{n})) \geq 0$.

For the proof of the next proposition I use the following result from convex analysis:

Lemma 3. *Let I be a non-degenerate open interval of $\mathbb{R}$ and $g : I \to \mathbb{R}$ a continuous function.*

1. *If g is right differentiable and $D^+ g : I \to \mathbb{R}$ is decreasing, then g is concave.*
2. *If g is left differentiable and $D^- g : I \to \mathbb{R}$ is decreasing, then g is concave.* $\diamond$

Proof. There are almost no appropriate references for this result. The best that I know is Theorem 5.3.1 in Hiriart-Urruty and Lemaréchal (1993). However, the result there uses a lemma that is left as a (not so straightforward) exercise. $\square$

Proposition 1. *Let $i \in \mathcal{N}$ and $\mathbf{z} \in \mathbf{X}^{\hat{\imath}}$. Suppose that the conditional payoff function $f^i_{\mathbf{z}}$ is lower semi-continuous and continuous in each interior point of X^i. For $f^i_{\mathbf{z}}$ to be concave it is sufficient that φ^i is increasing in its i-th variable and that one of the following two properties holds:*

I. There exists a right marginal reduction $(\mathcal{T}^i_+, \varphi^i)$ of f^i on $\mathrm{Int}(X^i) \times \{\mathbf{z}\}$ with domain W^i_- such that
  a. If $(a^i, y^i) \in W^i_-$ and $b^i \in \mathrm{Int}(X^i)$ with $b^i > a^i$, then also $(b^i, y^i) \in W^i_-$;
  b. $\mathcal{T}^i_+ : W^i_- \to \overline{\mathrm{I\!R}}$ is decreasing in both variables.

II. There exists a left marginal reduction $(\mathcal{T}^i_-, \varphi^i)$ of f^i on $\mathrm{Int}(X^i) \times \{\mathbf{z}\}$ with domain W^i_+ such that
  a. If $(a^i, y^i) \in W^i_+$ and $b^i \in \mathrm{Int}(X^i)$ with $b^i > a^i$, then also $(b^i, y^i) \in W^i_+$;
  b. $\mathcal{T}^i_- : W^i_+ \to \overline{\mathrm{I\!R}}$ is decreasing in both variables. $\diamond$

Proof. I assume that condition I holds; if II holds, the proof is analogous. I will prove that the function $g := f^i_{\mathbf{z}} \restriction \mathrm{Int}(X^i)$ is concave; the lower semi-continuity of $f^i_{\mathbf{z}}$ then implies its concavity.

Because $(\mathcal{T}^i_+, \varphi^i)$ is a right marginal reduction of f^i on $\mathrm{Int}(X^i) \times \{\mathbf{z}\}$, it follows that for $x^i \in \mathrm{Int}(X^i)$, g is right differentiable in x^i and $D^+ g(x^i) = \mathcal{T}^i_+(x^i, \varphi^i(x^i; \mathbf{z}))$. By Lemma 3, g is concave if $D^+ g$ is decreasing. So take $a^i, b^i \in \mathrm{Int}(X^i)$ with $a^i < b^i$. As φ^i is increasing in its i-th variable, the inequality $\varphi^i(b^i; \mathbf{z}) \geq \varphi^i(a^i; \mathbf{z})$ holds. I have $(a^i, \varphi^i(a^i; \mathbf{z})) \in \mathcal{W}^i_-(\mathrm{Int}(X^i) \times \{\mathbf{z}\}) \subseteq W^i_-$. By a also $(b^i, \varphi^i(a^i; \mathbf{z})) \in W^i_-$. Now with b,

$$D^+ g(a^i) = \mathcal{T}^i_+(a^i, \varphi^i(a^i; \mathbf{z})) \geq \mathcal{T}^i_+(b^i, \varphi^i(a^i; \mathbf{z})) \geq \mathcal{T}^i_+(b^i, \varphi^i(b^i; \mathbf{z})) = D^+ g(b^i). \quad \square$$

4. Constant Nash Sums

For a co-strategy mapping Φ, a subset Z of $\mathbf{X}$ and $i \in \mathcal{N}$ the following fundamental Properties $\mathcal{F}^i_w(Z, \Phi)$ and $\mathcal{F}^i_s(Z, \Phi)$ will be dealt with:

Property $\mathcal{F}^i_w(Z, \Phi)$. There exists a marginal reduction $(\mathcal{T}^i_+, \mathcal{T}^i_-, \varphi^i)$ of f^i on Z such that for all $\mathbf{a}, \mathbf{b} \in Z$ with $\Phi(\mathbf{b}) \gg \Phi(\mathbf{a})$: $\quad b^i > a^i \quad \Rightarrow \quad \mathcal{T}^i_-(b^i, \varphi^i(\mathbf{b})) < \mathcal{T}^i_+(a^i, \varphi^i(\mathbf{a}))$.

Property $\mathcal{F}^i_s(Z, \Phi)$. There exists a marginal reduction $(\mathcal{T}^i_+, \mathcal{T}^i_-, \varphi^i)$ of f^i on Z such that for all $\mathbf{a}, \mathbf{b} \in Z$ with $\Phi(\mathbf{b}) \geq \Phi(\mathbf{a})$: $\quad b^i > a^i \quad \Rightarrow \quad \mathcal{T}^i_-(b^i, \varphi^i(\mathbf{b})) < \mathcal{T}^i_+(a^i, \varphi^i(\mathbf{a}))$.

Property $\mathcal{F}^i_w(Z, \Phi)$ ($\mathcal{F}^i_s(Z, \Phi)$) is in terms of properties of left and right marginal reductions and, in case $Z \subseteq E$ sufficient for Property $F^i_w(Z, \Phi)$ ($F^i_s(Z, \Phi)$) to hold:

Proposition 2. *Suppose that Z is a subset of E and $i \in \mathcal{N}$. Then:*

1. $\mathcal{F}^i_w(Z, \Phi) \;\Rightarrow\; F^i_w(Z, \Phi)$.
2. $\mathcal{F}^i_s(Z, \Phi) \;\Rightarrow\; F^i_s(Z, \Phi)$. $\diamond$

Proof. 1. By contradiction. So suppose that I have $\mathbf{a}, \mathbf{b} \in Z$ with $\Phi(\mathbf{b}) \gg \Phi(\mathbf{a})$ and $b^i > a^i$. By (4) and (5), I have $\mathcal{T}^i_-(b^i, \varphi^i(\mathbf{b})) \geq 0 \geq \mathcal{T}^i_+(a^i, \varphi^i(\mathbf{a}))$. Because of Property $\mathcal{F}^i_w(Z, \Phi)$, also $\mathcal{T}^i_-(b^i, \varphi^i(\mathbf{b})) < \mathcal{T}^i_+(a^i, \varphi^i(\mathbf{a}))$, which is a contradiction.

2. If $\#Z \leq 1$, then the statement is (logically) trivial true. Now suppose that $\#Z \geq 2$. I prove by contradiction that Property $F_s^i(Z, \Phi)$ holds. So suppose that $i \in \mathcal{N}$, $\mathbf{a}, \mathbf{b} \in Z$ with $\Phi(\mathbf{b}) \geq \Phi(\mathbf{a})$ and $b^i > a^i$. Now $a^i \in X_-^i$ and $b^i \in X_+^i$. By (4) and (5), $\mathcal{T}_-^i(b^i, \varphi^i(\mathbf{b})) \geq 0 \geq \mathcal{T}_+^i(a^i, \varphi^i(\mathbf{a}))$. Because of Property $\mathcal{F}_s^i(Z, \Phi)$, also $\mathcal{T}_-^i(b^i, \varphi^i(\mathbf{b})) < \mathcal{T}_+^i(a^i, \varphi^i(\mathbf{a}))$, which is a contradiction. □

Sufficient for Property $\mathcal{F}_s^i(Z, \Phi)$ to hold is that there exists a marginal reduction reduction $(\mathcal{T}_+^i, \mathcal{T}_-^i, \varphi^i)$ of f^i on Z with domain $(X^i \times Y^i, X^i \times Y^i)$ such that $\mathcal{T}_+^i = \mathcal{T}_-^i =: \mathcal{T}^i$ and $\mathcal{T}^i$ is strictly decreasing in its first variable and decreasing in its second.[14]

Lemma 1(1) and Proposition 2(1) imply:

Corollary 1. *Let Φ be a co-strategy mapping and suppose that $Z \subseteq E$. Sufficient for $\Phi \restriction Z$ to be constant is that $\Phi \restriction Z$ is increasing and strictly ordered and that for every $i \in \mathcal{Z}(Z)$ there exists a marginal reduction $(\mathcal{T}_+^i, \mathcal{T}_-^i, \varphi^i)$ of f^i on Z such that for all $\mathbf{a}, \mathbf{b} \in Z$ with $\Phi(\mathbf{b}) \gg \Phi(\mathbf{a})$: $\quad b^i > a^i \;\Rightarrow\; \mathcal{T}_-^i(b^i, \varphi^i(\mathbf{b})) < \mathcal{T}_+^i(a^i, \varphi^i(\mathbf{a}))$.* ⋄

Note that if $\Phi \restriction Z$ is strictly ordered and non-constant it holds that $\#Y^i \geq 2$ $(i \in \mathcal{N})$.[15]

Given a non-empty subset Z of $\mathbf{X}$, I denote in case $\varphi^i \restriction Z$ is constant, by

$$\Psi^i$$

its constant value. In case $\Phi \restriction Z$ is constant, I denote it by $\mathbf{\Psi}$. And if in this case $\varphi^1 = \cdots = \varphi^N$, then each coefficient of $\mathbf{\Psi}$ is the same and $\mathbf{\Psi}$ is identified with this coefficient and denoted by $\Psi \in \mathbb{R}$.

Lemma 1(2) and Proposition 2(2) imply:

Corollary 2. *Let Φ be a co-strategy mapping and suppose that $Z \subseteq E$. Sufficient for $\#Z \leq 1$ is that $\Phi \restriction Z$ is increasing and ordered and that for every $i \in \mathcal{Z}(Z)$ there exists a marginal reduction $(\mathcal{T}_+^i, \mathcal{T}_-^i, \varphi^i)$ of f^i on Z such that for all $\mathbf{a}, \mathbf{b} \in Z$ with $\Phi(\mathbf{b}) \geq \Phi(\mathbf{a})$: $\quad b^i > a^i \;\Rightarrow\; \mathcal{T}_-^i(b^i, \varphi^i(\mathbf{b})) < \mathcal{T}_+^i(a^i, \varphi^i(\mathbf{a}))$.* ⋄

5. Polyhedral Structure

Let $\Phi = (\varphi^1, \ldots, \varphi^N)$ be a co-strategy mapping and Z a subset of $\mathbb{R}^N$. Suppose that for $i \in \mathcal{N}$, $(\mathcal{T}_+^i, \mathcal{T}_-^i, \varphi^i)$ is a marginal reduction of f^i on Z. Then, with $N_{y^i}^i(Z)$ as in (3), I define for $\mathbf{y} \in \mathbf{Y} := Y^1 \times \cdots \times Y^N$

$$E(\mathbf{y}; Z) := (N_{y^1}^1(Z) \times \cdots \times N_{y^N}^N(Z)) \cap \Phi^{<-1>}(\mathbf{y}).$$

For this situation:

Proposition 3. *1. If $\mathbf{n} \in Z \cap E$, then $\mathbf{n} \in E(\Phi(\mathbf{n}); Z)$.*

[14] A similar result hold for Property $\mathcal{F}_s^i(Z, \Phi)$.

[15] Indeed: because $\Phi \restriction Z$ is non-constant, there exist $\mathbf{a}, \mathbf{b} \in Z$ with $\Phi(\mathbf{a}) \neq \Phi(\mathbf{b})$. Let $j \in \mathcal{N}$ be such that $\varphi^j(\mathbf{a}) \neq \varphi^j(\mathbf{b})$. Without loss of generality assume $\varphi^j(\mathbf{a}) < \varphi^j(\mathbf{b})$. Because $\Phi \restriction Z$ is strictly ordered, $\Phi(\mathbf{a}) \ll \Phi(\mathbf{b})$ follows and therefore $\varphi^i(\mathbf{a}) < \varphi^i(\mathbf{b})$ $(i \in \mathcal{N})$. Thus $\#Y^i \geq 2$ $(i \in \mathcal{N})$.

2. *Suppose* $\mathbf{n} \in \mathbf{X}$ *is such that* $\mathbf{n} \in E(\Phi(\mathbf{n}); Z)$. *Then sufficient for* $\mathbf{n}$ *to be a Nash equilibrium is that every* $f^i_{\mathbf{n}^{\hat{\imath}}}$ *is concave.* $\diamond$

Proof. 1. Of course, $\mathbf{n} \in \Phi^{<-1>}(\Phi(\mathbf{n}))$. Fix $i \in \mathcal{N}$. Because $(\mathcal{T}^i_+, \mathcal{T}^i_-, \varphi^i)$ is a marginal reduction of f^i on $\{\mathbf{n}\}$ and $\mathbf{n} \in E$, (4) and (5) give with $y^i = \varphi^i(\mathbf{n})$

$$n^i \in \mathrm{Int}(X^i) \;\Rightarrow\; [(n^i, y^i) \in \mathcal{W}^i_-(Z) \cap \mathcal{W}^i_+(Z) \wedge \mathcal{T}^i_+(n^i, y^i) \leq 0 \leq \mathcal{T}^i_-(n^i, y^i)], \quad (6)$$

$$n^i \in \partial_+(X^i) \;\Rightarrow\; [(n^i, y^i) \in \mathcal{W}^i_+(Z) \wedge \mathcal{T}^i_-(n^i, y^i) \geq 0], \quad (7)$$

$$n^i \in \partial_-(X^i) \;\Rightarrow\; [(n^i, y^i) \in \mathcal{W}^i_-(Z) \wedge \mathcal{T}^i_+(n^i, y^i) \leq 0]. \quad (8)$$

This implies $n^i \in N^i_{y^i}(Z) = N^i_{\varphi^i(\mathbf{n})}(Z)$.

2. Now $n^i \in N^i_{\varphi^i(\mathbf{n})}(Z)$ $(i \in \mathcal{N})$. This implies for every i, that (6) - (8) hold. So, because $f^i_{\mathbf{n}^{\hat{\imath}}}$ is concave and $(\mathcal{T}^i_+, \mathcal{T}^i_-, \varphi^i)$ is a marginal reduction of f^i on $\{\mathbf{n}\}$, it follows that $\mathbf{n} \in E$. $\square$

Proposition 3 (with $Z = \mathbf{X}$) implies:

Corollary 3. *Suppose for every* $i \in \mathcal{N}$ *that*

a. *there exists a marginal reduction* $(\mathcal{T}^i_+, \mathcal{T}^i_-, \varphi^i)$ *of* f^i *on* $\mathbf{X}$*;*
b. *for every* $\mathbf{n} \in E$ *the conditional payoff* $f^i_{\mathbf{n}^{\hat{\imath}}}$ *is concave.*

And further suppose that

c. $E \neq \emptyset$*;*
d. $\Phi \upharpoonright E$ *is a constant* $\mathbf{\Psi}$.

Then $E = E(\mathbf{\Psi}; \mathbf{X})$. $\diamond$

The following theorem deals with a class of games where (according to Proposition 1) each conditional payoff function is concave.

Theorem 2. *Let* $\Phi = (\varphi^1, \ldots, \varphi^N)$ *be an affine,*[16] *increasing and strictly ordered co-strategy mapping. Suppose that for each player* i *the following Properties a–d hold:*

a. *each conditional payoff function* $f^i_{\mathbf{z}}$ *is lower semi-continuous and is continuous in each interior point of* X^i*;*
b. *there exists a marginal reduction* $(\mathcal{T}^i_+, \mathcal{T}^i_-, \varphi^i)$ *of* f^i *on* $\mathbf{X}$ *with domain* $(X^i_- \times Y^i_-, X^i_+ \times Y^i_+)$*;*
c. $\mathcal{T}^i_+ : X^i_- \times Y^i_- \to \overline{\mathbb{R}}$ *or* $\mathcal{T}^i_- : X^i_+ \times Y^i_+ \to \overline{\mathbb{R}}$ *is decreasing in both variables;*
d. *for all* $\mathbf{a}, \mathbf{b} \in E$ *with* $\Phi(\mathbf{b}) \gg \Phi(\mathbf{a})$*:*

$$[i \in \mathcal{Z}(E),\ b^i > a^i] \;\;\Rightarrow\;\; \mathcal{T}^i_-(b^i, \varphi^i(\mathbf{b})) < \mathcal{T}^i_+(a^i, \varphi^i(\mathbf{a})).$$

Then:

1. $\Phi \upharpoonright E$ *is constant.*

[16] I.e. there exist a linear mapping $A : \mathbb{R}^N \to \mathbb{R}^N$ and $\mathbf{a} \in \mathbb{R}^N$ such that $\Phi(\mathbf{x}) = A(\mathbf{x}) + \mathbf{a}$ $(\mathbf{x} \in \mathbf{X})$.

2. *E is a convex subset of $\mathbb{R}^N$ and closed in $\mathbf{X}$. Even: E is the intersection of an interval*[17] *of $\mathbb{R}^N$ and an affine subset of $\mathbb{R}^N$.*
3. *If each strategy set is closed and each payoff function is continuous, then E is polyhedral.* $\diamond$

Proof. If $\#E \leq 1$, then 1–3 hold. Now suppose that $\#E \geq 2$.

1. Having Properties b and d, this follows from Corollary 1 with $Z = E$.

2. The set $\{\mathbf{x} \in \mathbb{R}^N \mid A(\mathbf{x}) = \mathbf{\Psi} - \mathbf{a}\}$ is affine and therefore convex and even polyhedral. Every $f^i_{\mathbf{n}^{\hat{\imath}}}$ $(i \in \mathcal{N}, \mathbf{n} \in E)$ is concave. By Corollary 3,

$$E = (N^1_{\Psi^1}(\mathbf{X}) \times \cdots \times N^N_{\Psi^N}(\mathbf{X})) \cap \Phi^{<-1>}(\mathbf{\Psi}).$$

By Lemma 2, $N^1_{\Psi^1}(\mathbf{X}) \times \cdots \times N^N_{\Psi^N}(\mathbf{X})$ is a cartesian product of intervals. Note that

$$\Phi^{<-1>}(\mathbf{\Psi}) = \{\mathbf{x} \in \mathbf{X} \mid \Phi(\mathbf{x}) = \mathbf{\Psi}\} = \{\mathbf{x} \in \mathbb{R}^N \mid A(\mathbf{x}) = \mathbf{\Psi} - \mathbf{a}\} \cap \mathbf{X}.$$

Thus

$$E = (N^1_{\Psi^1}(\mathbf{X}) \times \cdots \times N^N_{\Psi^N}(\mathbf{X})) \cap \{\mathbf{x} \in \mathbb{R}^N \mid A(\mathbf{x}) = \mathbf{\Psi} - \mathbf{a}\},$$

i.e. E is the intersection of an interval of $\mathbb{R}^N$ and an affine subset of $\mathbb{R}^N$ and therefore convex.

3. Now $\mathbf{X}$ is closed. As each payoff function f^i is continuous, E is a closed subset of $\mathbf{X}$. Therefore E is a closed subset of $\mathbb{R}^N$ that is an intersection of an interval of $\mathbb{R}^N$ and an affine subset of $\mathbb{R}^N$. This implies that E is polyhedral. $\square$

Property b in Theorem 2 presupposes that $\mathcal{W}^i_+(\mathbf{X}) \subseteq X^i_+ \times Y^i_+$ and $\mathcal{W}^i_-(\mathbf{X}) \subseteq X^i_- \times Y^i_-$. Sufficient for this to hold is that φ^i is *i-regular*. This notion is defined as follows: for all $\mathbf{x} \in \mathbf{X}$

$$[x^i \in X^i_- \;\Rightarrow\; \varphi^i(\mathbf{x}) \in Y^i_-] \;\;\wedge\;\; [x^i \in X^i_+ \;\Rightarrow\; \varphi^i(\mathbf{x}) \in Y^i_+].$$

Note that each constant co-strategy function is i-regular for all $i \in \mathcal{N}$. Another sufficient condition for a co-strategy function φ^i to be i-regular is that

$$\varphi^i \text{ is increasing in its } i\text{-th variable, } \#\mathrm{argmax}\, \varphi^i \leq 1 \text{ and } \#\mathrm{argmin}\, \varphi^i \leq 1.^{18} \quad (9)$$

6. Transboundary Cournot Games

The next definition deals with a special class of aggregative games for which the above abstract results will become more concrete.

Definition 2. A *transboundary Cournot game* is a game in strategic form where each player i has strategy set $X^i = [0, m^i]$ (*capacity constraint*) or $X^i = \mathbb{R}_+$ and payoff function

$$f^i(\mathbf{x}) = A^i(x^i) - (x^i)^{\beta^i} B^i(q(\mathbf{x})),$$

[17] I.e. a cartesian product of N intervals of $\mathbb{R}$.

[18] To see this, suppose that $x^i \in X^i_-$ and $\varphi^i(\mathbf{x}) \notin Y_-$. By (2), $\varphi^i(\mathbf{x}) \in \partial_+(Y)$. Because X^i is a non-degenerate interval, there exists $b^i \in X^i$ with $b^i > x^i$. Because φ^i is increasing in its i-th variable, it holds that $\varphi^i(x^i; \mathbf{x}^{\hat{\imath}}) \leq \varphi^i(b^i; \mathbf{x}^{\hat{\imath}})$. It follows that also $\varphi^i(b^i; \mathbf{x}^{\hat{\imath}}) \in \partial_+(Y)$. Thus $(b^i; \mathbf{x}^{\hat{\imath}})$ and $(x^i; \mathbf{x}^{\hat{\imath}})$ are two maximisers of φ^i, which is a contradiction with (9). The proof of the other statement is similar.

where $\beta^i \in \{0,1\}$, $A^i : X^i \to \mathbb{R}$ and $B^i : Y \to \mathbb{R}$. Here $Y = q(\mathbf{X})$ with $q : \mathbf{X} \to \mathbb{R}$ given by

$$q(\mathbf{x}) \;:=\; \sum_{l=1}^{N} T_l x^l$$

with $T_1, \ldots, T_N > 0$. $\diamond$

Note that $\partial_-(Y) = \{0\}$ and $\partial_+(Y) \neq \emptyset \;\Rightarrow\; \partial_+(Y) = \{\sum_l T_l m^l\}$.

An example of a transboundary Cournot game is the Cournot oligopoly game (as defined in footnote 1): take

$$A^i = -c^i,\; \beta^i = 1,\; B^i = -p^i,\; T_l = t_l.$$

Of course, often one assumes that $p^i, c^i \geq 0$, that the c^i are increasing and that the p^i are decreasing. (But this I only will do when I need it.)

Another example is the transboundary pollution game with uniform pollution as in for instance Finus (2001) and Folmer and von Mouche (2002): take

$$A^i = \mathcal{P}^i,\; \beta^i = 0,\; B^i = \mathcal{D}^i,\; T_l = t_l.$$

So here

$$f^i(\mathbf{x}) = \mathcal{P}^i(x^i) - \mathcal{D}^i(\sum_{l=1}^{N} t_l x^l).$$ [19]

Note that in a transboundary Cournot game some players may have a capacity constraint while others may not have. Also: some players may have a payoff function of 'Cournot type' while others have one of 'transboundary pollution type'.

Below q will serve as co-strategy function. (9) implies that q is i-regular for every $i \in \mathcal{N}$.

Definition 3. A transboundary Cournot game is *quasi-smooth* for player i on $\mathbf{X}$ if the following properties hold.

a. For every $x^i \in X^i$:
   i. if $x^i \in \mathrm{Int}(X^i)$, then A^i is left and right differentiable in x^i;
   ii. if $x^i = 0$, then the right derivative of A^i in 0 exists as element of $\overline{\mathbb{R}}$;
   iii. if $x^i \in \partial_+(X^i)$, then the left derivative of A^i in m^i exists as element of $\mathbb{R} \cup \{-\infty\}$.
b. For every $y \in Y$:
   i. if $y \in \mathrm{Int}(Y)$, then B^i is left and right differentiable in y;
   ii. if $y = 0$, then B^i is right differentiable in 0;
   iii. if $y \in \partial_+(Y)$, then the left derivative of B^i in $\sum_{l=1}^{N} T_l m^l$ exists as element of $\mathbb{R} \cup \{+\infty\}$. $\diamond$

[19] An action of a player in a transboundary pollution game has the real-world interpretation of the emission level of a country, the t_l are transport coefficients and the weighted sum $\sum_l t_l x^l$ of emission levels across the countries is interpreted as the deposition level of emissions.

The special conditions in a(ii,iii) and b(ii,iii) are chosen such that a lot of important cases below can be dealt with.

Given a transboundary Cournot game which is quasi-smooth for player i on $\mathbf{X}$, the function $t^i_+ : X^i_- \times Y_- \to \overline{\mathrm{I\!R}}$ is well-defined by[20]

$$t^i_+(x^i, y) \;:=\; D^+A^i(x^i) - T_i(x^i)^{\beta^i} D^+B^i(y) - \beta^i B^i(y). \tag{10}$$

and the function $t^i_- : X^i_+ \times Y_+ \to \mathrm{I\!R} \cup \{-\infty\}$ is well-defined by

$$t^i_-(x^i, y) \;:=\; D^-A^i(x^i) - T_i(x^i)^{\beta^i} D^-B^i(y) - \beta^i B^i(y). \tag{11}$$

Because q is i-regular it follows that $\mathcal{W}^i_-(\mathbf{X}) \subseteq X^i_- \times Y_-$ and $\mathcal{W}^i_+(\mathbf{X}) \subseteq X^i_+ \times Y_+$. Using this, I obtain:

Proposition 4. *If a transboundary Cournot game is quasi-smooth for player i on $\mathbf{X}$, then (t^i_+, t^i_-, q) is a marginal reduction of f^i on $\mathbf{X}$ with domain $(X^i_- \times Y_-, X^i_+ \times Y_+)$.* $\diamond$

I denote the co-strategy mapping $(q, \ldots, q)$ by

$$Q.$$

For a transboundary Cournot game and $i \in \mathcal{N}$, I define the function $R^i : Y \to \mathrm{I\!R}$ by

$$R^i(y) \;:=\; -y^{\beta^i} B^i(y).$$

For a Cournot oligopoly game I refer to $R^i (= yp^i(y))$ as *revenue function* of oligopolist i.

Given a transboundary Cournot game and $i \in \mathcal{N}$, I consider the following properties:

- CONT(i): A^i and B^i are continuous.
- BINC(i): B^i is increasing.
- ACONC(i): A^i is concave.
- BCONV(i): B^i is convex.
- STRIC(i): A^i is strictly concave, or $\beta^i = 1$ and B^i is strictly increasing.
- RCONC(i): R^i is concave.

I note that for player i to have concave conditional payoff functions it is sufficient that Properties ACONC(i), BINC(i) and RCONC(i) hold.[21] Also I note that [BCONV(i) $\wedge$ BINC(i)] $\Rightarrow$ RCONC(i) holds.

Lemma 4. *Let $i \in \mathcal{N}$. Suppose that Properties* BINC(i), ACONC(i), BCONV(i) *and* STRIC(i) *hold. Then:*

1. *The game is quasi-smooth for player i on $\mathbf{X}$.*
2. *Property $\mathcal{F}^i_w(\mathbf{X}, Q)$ holds.*

[20] Indeed: note that the right hand side of (10) is well-defined: undefined operations like $\infty - \infty$ and $0 \cdot \infty$ do not occur; fortunately I do not need to define $t^i_+(0, \sum_l T_l m^l)$. A same remark applies to the definition of t^i_-.

[21] For a proof (in case of a Cournot oligopoly game) see Lemma 1 in Murphy et al.(1982).

3. *Let $U \subseteq \mathbf{X}$. Sufficient for Property $\mathcal{F}^i_s(U,Q)$ to hold is that B^i is differentiable in each point of $q(U)$.* $\diamond$

Proof. 1. A^i is concave. Therefore this function is left and right differentiable in each point of $\mathrm{Int}(X^i)$ and its right derivative in 0 exists as element of $\mathbb{R} \cup \{+\infty\}$. If $x^i \in \partial_+(X^i)$, then $x^i = m^i$ and $A^i : [0, m^i] \to \mathbb{R}$ is concave. This implies that the left derivative of A^i in m^i exists as element of $\mathbb{R} \cup \{-\infty\}$. Thus Property a in Definition 3 follows.

B^i is convex. Therefore this function is left and right differentiable in each point of $\mathrm{Int}(Y)$. As B^i is convex and increasing, B^i is right differentiable in 0. As B^i is convex, I have if $y \in \partial_+(Y)$ that its left derivative in $\sum_l T_l m^l$ exists as element of $\mathbb{R} \cup \{+\infty\}$. Thus Property b in Definition 3 follows. Thus the game is quasi-smooth for player i on $\mathbf{X}$.

2. By 1 and Proposition 4, (t^i_+, t^i_-, q) is a marginal reduction of f^i on $\mathbf{X}$ with domain $(X^i_- \times Y_-, X^i_+ \times Y_+)$. Suppose that Property $\mathcal{F}^i_w(\mathbf{X}, Q)$ would not hold, so suppose that $\mathbf{a}, \mathbf{b} \in \mathbf{X}$, $q(\mathbf{b}) > q(\mathbf{a})$, $b^i > a^i$ and $t^i_-(b^i, q(\mathbf{b})) \geq t^i_+(a^i, q(\mathbf{a}))$. By (10) and (11)

$$D^- A^i(b^i) - T_i (b^i)^{\beta^i} D^- B^i(q(\mathbf{b})) - \beta^i B^i(q(\mathbf{b}))$$

$$\geq D^+ A^i(a^i) - T_i (a^i)^{\beta^i} D^+ B^i(q(\mathbf{a})) - \beta^i B^i(q(\mathbf{a})).$$

I.e.

$$D^- A^i(b^i) - D^+ A^i(a^i)$$

$$\geq t^i \Big((b^i)^{\beta^i} D^- B^i(q(\mathbf{b})) - (a^i)^{\beta^i} D^+ B^i(q(\mathbf{a})) \Big) + \beta^i (B^i(q(\mathbf{b})) - B^i(q(\mathbf{a}))).$$

Now I derive a contradiction by determining the signs of the left and right hand side in the above inequality. Because A^i is concave, $a^i < b^i$, the term on the left hand side is non-positive. If this function is strictly concave, then this term is negative. Because B^i is convex and increasing and $q(\mathbf{b}) > q(\mathbf{a})$, I have

$$+\infty \geq D^- B^i(q(\mathbf{b})) \geq D^+ B^i(q(\mathbf{a})) \geq 0. \qquad (\star)$$

From this, noting that $b^i > 0$,

$$(b^i)^{\beta^i} D^- B^i(q(\mathbf{b})) \geq (b^i)^{\beta^i} D^+ B^i(q(\mathbf{a})) \geq (a^i)^{\beta^i} D^+ B^i(q(\mathbf{a})).$$

Therefore the first term on the right hand side is non-negative. Because B^i is increasing, I have $B^i(q(\mathbf{b})) \geq B^i(q(\mathbf{a}))$. Therefore also the second term there is non-negative. This term is positive if $\beta^i = 1$ and B^i is strictly increasing. So the right hand side is non-negative and positive if $\beta^i = 1$ and B^i is strictly increasing. Thus a contradiction follows.

3. By contradiction. So suppose that B^i is differentiable in each point of $q(U)$, $\mathbf{a}, \mathbf{b} \in U$, $q(\mathbf{b}) \geq q(\mathbf{a})$, $b^i > a^i$ and $t^i_-(b^i, \varphi^i(\mathbf{b})) \geq t^i_+(a^i, \varphi^i(\mathbf{a}))$. Almost the same proof as in 2 applies: everywhere replace there $D^- B^i$ and $D^+ B^i$ by DB^i. I just note that $(\star)$ becomes

$$(DB^i)(q(\mathbf{b})) \geq (DB^i)(q(\mathbf{a})) \geq 0.$$

Indeed: $q(\mathbf{b}) \geq q(\mathbf{a})$, B^i is convex and increasing and B^i is differentiable. $\square$

Theorem 3. *Consider a transboundary Cournot game.*

I. Suppose that each player i has a capacity constraint and that the properties

$$\mathrm{CONT}(i), \mathrm{BINC}(i), \mathrm{ACONC}(i) \text{ and } \mathrm{RCONC}(i)$$

hold. Then $\#E \geq 1$.

II. 1. Suppose that for every $i \in \mathcal{Z}(E)$ the properties

$$\mathrm{BINC}(i), \mathrm{ACONC}(i), \mathrm{BCONV}(i) \text{ and } \mathrm{STRIC}(i)$$

hold. Then there exists $\Psi \geq 0$ such that $\sum_{l=1}^{N} T_l n^l = \Psi$ for all $\mathbf{n} \in E$.

2. Suppose that for every $i \in \mathcal{N}$ the properties

$$\mathrm{CONT}(i), \mathrm{BINC}(i), \mathrm{ACONC}(i), \mathrm{BCONV}(i)] \text{ and } \mathrm{STRIC}(i)$$

hold. Then E is polyhedral.

III. Suppose that for every $i \in \mathcal{Z}(E)$ the properties

$$\mathrm{BINC}(i, V), \mathrm{ACONC}(i), \mathrm{BCONV}(i) \text{ and } \mathrm{STRIC}(i)$$

hold.[22] *Then:*

1. *B^i $(i \in \mathcal{Z}(E))$ are differentiable $\Rightarrow$ $\#E \leq 1$.*
2. *If $E \neq \emptyset$, then: B^i $(i \in \mathcal{Z}(E))$ are differentiable in Ψ $\Rightarrow$ $\#E \leq 1$.*
3. *$\#E \geq 2$ $\Rightarrow$ there exists $i \in \mathcal{Z}(E)$ for which B^i is not differentiable in Ψ.*
4. *$\#E \geq 2$ $\Rightarrow$ for every $i \in \mathcal{Z}(E)$ for which A^i is strictly concave, B^i is not differentiable in Ψ.* $\diamond$

Proof. I. This follows from the Nikaido-Isoda theorem (see footnote 1).

II. Applying Lemma 4(1) and Proposition 4 for $i \in \mathcal{N}$ gives that the game is quasi-smooth for i on $\mathbf{X}$ and that (t^i_+, t^i_-, q) is a marginal reduction of f^i on $\mathbf{X}$ with domain $(X^i_- \times Y_-, X^i_+ \times Y_+)$.

(1). Applying Lemma 4(2) gives that Properties $\mathcal{F}^i_w(\mathbf{X}, Q)$ $(i \in \mathcal{Z}(E))$ hold. Therefore also Properties $\mathcal{F}^i_w(E, Q)$ $(i \in \mathcal{Z}(E))$ hold. This implies that Corollary 1 applies with $\Phi = Q$ and $Z = E$. So $Q \restriction E$ is constant. Thus $q \restriction E$ is constant. Of course, if $E \neq \emptyset$ then this constant belongs to $[0, +\infty\,[$.

(2). Because the game is quasi-smooth for i on $\mathbf{X}$, it holds that $D^+B^i(y) \in \mathbb{R}$ $(y \in Y_-)$ and $D^-B^i(y) \in \mathbb{R} \cup \{+\infty\}$ $(y \in Y_+)$. Also noting that A^i is concave, I see from (10) and (11) that the functions $t^i_+ : X^i_- \times Y_- \to \overline{\mathbb{R}}$ and $t^i_- : X^i_+ \times Y_+ \to \mathbb{R} \cup \{-\infty\}$ are decreasing in their first variable. Because B^i is increasing and convex, it follows that these functions also are decreasing in its second variable. Lemma 4(2) guarantees that Properties $\mathcal{F}^i_w(E, Q)$ $(i \in \mathcal{N})$ hold. Theorem 2(3) applies. Thus E is polyhedral.

III. Applying Lemma 4(1) and Proposition 4 for $i \in \mathcal{Z}(E)$ gives that (t^i_+, t^i_-, q) is a marginal reduction of f^i on $\mathbf{X}$.

(2). I have $q(E) = \{\Psi\}$. By Lemma 4(3), Properties $\mathcal{F}^i_s(E, Q)$ $(i \in \mathcal{Z}(E))$ hold. Corollary 2 now implies that $\#E = 1$.

(1,3). By III(2).

(4). By contradiction. So suppose that $i \in \mathcal{Z}(E)$, A^i is strictly concave and B^i is differentiable in Ψ. There exist $\mathbf{a}, \mathbf{b} \in E$ with $a^i \neq b^i$. It may be supposed that

[22] Thus, by II(1), $q \restriction E$ is a constant $\Psi \in [0, +\infty\,[$.

$a^i < b^i$. Note that $q(\mathbf{a}) = q(\mathbf{b}) = \Psi$. Because $a^i \in X^i_-$, $b^i \in X^i_+$, (10) and (11) give $t^i_+(a^i, \Psi) \leq 0$ and $t^i_-(b^i, \Psi) \geq 0$. Therefore,

$$D^- A^i(b^i) \geq T_i(b^i)^{\beta^i} DB^i(\Psi) - \beta^i B^i(\Psi), \; D^+ A^i(a^i) \leq T_i(a^i)^{\beta^i} DB^i(\Psi) - \beta^i B^i(\Psi).$$

Because $a^i < b^i$ and A^i is strictly concave, $D^+A^i(a^i) > D^-A^i(b^i)$ holds. It follows that

$$T_i(a^i)^{\beta^i} DB^i(\Psi) > T_i(b^i)^{\beta^i} DB^i(\Psi),$$

which is impossible. □

7. Theorem 1 Revisited

Theorem 3 implies:

Corollary 4. *Consider a Cournot oligopoly game where each inverse demand function p^i is decreasing.*

I. *If each oligopolist i has a capacity constraint, p^i and c^i are continuous, c^i is convex and the revenue function R^i is concave, then $\#E \geq 1$.*

II. 1. *Suppose that for each oligopolist $i \in \mathcal{Z}(E)$: c^i is convex, p^i is concave, and c^i is strictly convex or p^i is strictly decreasing. Then there exists $\Psi \in [0, +\infty[$ such that $\sum_{l=1}^{N} t_l n^l = \Psi$ for all $\mathbf{n} \in E$.*
    2. *Suppose that for each oligopolist i: p^i and c^i are continuous, c^i is convex, p^i is concave, and c^i is strictly convex or p^i is strictly decreasing. Then E is polyhedral.*

III. *Suppose that for each oligopolist $i \in \mathcal{Z}(E)$: c^i is convex, p^i is concave, and c^i is strictly convex or p^i is strictly decreasing.*[23] *Then:*
    1. *p^i $(i \in \mathcal{Z}(E))$ are differentiable $\Rightarrow$ $\#E \leq 1$.*
    2. *If $E \neq \emptyset$, then: p^i $(i \in \mathcal{Z}(E))$ are differentiable in Ψ $\Rightarrow$ $\#E \leq 1$.*
    3. *$\#E \geq 2$ $\Rightarrow$ there exists $i \in \mathcal{Z}(E)$ for which p^i is not differentiable in Ψ.*
    4. *$\#E \geq 2$ $\Rightarrow$ for every $i \in \mathcal{Z}(E)$ for which c^i is strictly convex, p^i is not differentiable in Ψ.* ◇

Remarks:

1. Of course, II(1), III(1-3) also hold if one replaces there everywhere $\mathcal{Z}(E)$ by $\mathcal{N}$.
2. Corollary 4 implies Theorem 1 and shows among other things that this theorem even holds without the overall assumptions that each oligopolist has a capacity constraint and all cost functions are increasing.
3. As each oligopolist has his own inverse demand function p^i, Corollary 4 is also valid for a (small) class of oligopolies with product differentiation.

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[23] Thus, by II(1), the function $\sum_l t_l x^l$ is on E a constant $\Psi \in [0, +\infty[$.

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Claim Problems with Coalition Demands

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Abstract. We consider a generalized claim problem, where each member of a fixed collection of coalitions of agents $\mathcal{A}$ has its claim. Several generalizations of the Proportional method and of the Uniform Losses method for claim problems are examined.
For claim problems, the proportional solution, the proportional nucleolus, and the weighted entropy solution give the same results. For generalized claim problem conditions on $\mathcal{A}$ that provide the existence of the proportional solution and the existence of the weakly proportional solution are obtained. The condition on $\mathcal{A}$ for coincidence the weighted entropy solution and the weakly proportional solution is obtained. For such $\mathcal{A}$, we give an axiomatic justification for a selector of these solutions.
For claim problems, the uniform losses solution, the nucleolus, and the least square solution give the same results, but for generalized claim problems conditions on $\mathcal{A}$ concerning existence results and inclusion results are similar to the case of proportional solution.

Keywords:claim problem, cooperative games, proportional solution, weighted entropy, nucleolus, constrained equal losses solution.

1. Introduction

A claim problem is a problem of division a positive amount of resourses among n agents taking into account nonnegative claims of agents. A share of each agent must be nonnegative. Claim methods and their axiomatic justifications are described in the surveys (Moulin, 2002) and (Thomson, 2003).

Here we consider a generalized claim problem. For a fixed collection of coalitions of agents $\mathcal{A}$, the claims of agents are replaced by the claims of coalitions of agents from this collection. We assume that $\mathcal{A}$ covers the set of agents. A solution of this problem is a set of vectors of agent's shares. A method is a map that associates to any problem its solution. We may interpret a generalized claim problem as a TU–cooperative game with limited cooperation possibilities and a method as a value on the set of such games.

We consider generalizations of the Proportional method for claim problem and the Uniform Losses method (also called the Constrained Equal Losses method).

For claim problems, the proportional solution, the proportional nucleolus, and the weighted entropy solution give the same result. Weighted entropy solution for bargaining problems with goal points was introduced first in (Bregman and Romanovskij, 1975) and was examined in (Bregman and Naumova, 1984, 2002). Naumova (2008a) proposed this solution for cooperative games with unlimited cooperation.

In the case of generalized claim problems, for each $\mathcal{A}$, the proportional nucleolus and the weighted entropy solution are always nonempty and define uniquely total shares of coalitions in $\mathcal{A}$. The proportional solution is nonempty for all possible claims of coalitions in $\mathcal{A}$ iff $\mathcal{A}$ is a minimal covering of the set of agents.

We define a weakly proportional solution, where the ratios of total shares of coalitions to their claims are equal only for disjoint coalitions in $\mathcal{A}$. Necessary and sufficient condition on $\mathcal{A}$ that ensures nonemptyness of the weakly proportional solution for all possible claims of coalitions in $\mathcal{A}$ is obtained.

Since the proportional nucleolus and the weighted entropy solution are generalizations of the proportional solution, we are interested to obtain conditions on $\mathcal{A}$ that ensure the inclusion of these solutions either in the proportional solution or in the weakly proportional solution.

The proportional nucleolus is contained in the proportional solution for all possible claims of coalitions in $\mathcal{A}$ iff $\mathcal{A}$ is a partition of the set of agents. The weighted entropy solution is contained in the proportional solution for all possible claims of coalitions in $\mathcal{A}$ iff $\mathcal{A}$ is a partition of the set of agents.

Necessary condition on $\mathcal{A}$ for inclusion the proportional nucleolus in the weakly proportional solution for all possible claims of coalitions in $\mathcal{A}$ is obtained. This condition is also necessary for inclusion the weighted entropy solution in the weakly proportional solution.

Moreover, we obtain necessary and sufficient condition on $\mathcal{A}$ for coincidence the weighted entropy solution with the weakly proportional solution. For such $\mathcal{A}$, we give an axiomatic justification of a selector of these solutions.

In the uniform losses solution of the claim problem, if an agent's share is positive, then his excess is not less that the excess of any other agent. This property defines this solution uniquely. The nucleolus and the least square solution on the set of nonnegative sharings of the total amount give the same result.

The extensions of these methods to generalized claim problems give different results. Conditions on $\mathcal{A}$ concerning existence results and inclusion results are similar to the case of proportional solution.

2. Generalizations of the Proportional Solution

A *claim problem* is a triple (N,t,x), where N is a finite set of agents, $t>0$ represents the amount of resources to be divided by agents, $x=\{x_i\}_{i\in N}$, $x_i>0$ for all i, x_i is a claim of agent i.

A *solution to the claim problem* is a vector $y=\{y_i\}_{i\in N}$ such that $y_i\geq 0$ for all $i\in N$, $\sum_{i\in N} y_i=t$. A *claim method* is a map that associates to any claim problem its solution.

For a fixed collection of nonempty coalitions of agents $\mathcal{A}\subset 2^N$, we replace the vector x (the set of claims of agents) by $\{v(T)\}_{T\in\mathcal{A}}$, where $v(T)$ is a claim of T. We may consider a generalized claim problem as a TU–cooperative game with limited cooperation possibilities.

A *generalized claim problem* is a family $(N,\mathcal{A},t,\{v(S)\}_{S\in\mathcal{A}})=(N,\mathcal{A},t,v)$. We assume that $\mathcal{A}$ covers N.

A *generalized claim method* is a map that associates to any problem $(N,\mathcal{A},t,v)$ a subset of the set $\{\{y_i\}_{i\in N}: y_i\geq 0,\ \sum_{i\in N} y_i=t\}$.

Let $X\subset R^n$, $f_1,\ldots,f_k$ be functions defined on X. For $z\in X$, let π be a permutation of $\{1,\ldots,n\}$ such that $f_{\pi(i)}(z)\leq f_{\pi(i+1)}(z)$, $\theta(z)=\{f_{\pi(i)}(z)\}_{i=1}^n$.

Then $y \in X$ belongs to the *nucleolus with respect to* $f_1, \ldots, f_k$ *on* X iff

$$\theta(y) \geq_{lex} \theta(z) \quad \text{for all} \quad z \in X.$$

We suppose that $v(T) > 0$ for all $T \in \mathcal{A}$.

1. A vector $y = \{y_i\}_{i \in N}$ belongs to the *proportional solution* of $(N, \mathcal{A}, t, v)$ iff there exists $\alpha > 0$ such that $y(T) = \alpha v(T)$ for all $T \in \mathcal{A}$, $y_i \geq 0$ for all $i \in N$, $\sum_{i \in N} y_i = t$. We denote by $\mathcal{P}(N, \mathcal{A}, t, v)$ the proportional solution of $(N, \mathcal{A}, t, v)$.

2. A vector $y = \{y_i\}_{i \in N}$ belongs to the *proportional nucleolus* of $(N, \mathcal{A}, t, v)$ iff y belongs to the nucleolus w.r.t. $\{f_T\}_{T \in \mathcal{A}}$ with $f_T(z) = z(T)/v(T)$ on the set $\{z = \{z_i\}_{i \in N} : z_i \geq 0,\ \sum_{i \in N} z_i = t\}$. We denote by $\mathcal{N}(N, \mathcal{A}, t, v)$ the proportional nucleolus of $(N, \mathcal{A}, t, v)$.

3. A vector $y = \{y_i\}_{i \in N}$ belongs to the *weighted entropy solution* of $(N, \mathcal{A}, t, v))$ iff y minimizes

$$\sum_{S \in \mathcal{A}} z(S)(ln(z(S)/v(S)) - 1)$$

on the set $\{z = \{z_i\}_{i \in N} : z_i \geq 0,\ \sum_{i \in N} z_i = t\}$. We denote by $\mathcal{H}(N, \mathcal{A}, t, v)$ the weighted entropy solution of $(N, \mathcal{A}, t, v))$.

For each $\mathcal{A}$, $t > 0$, v with $v(T) > 0$, the proportional nucleolus and the weighted entropy solution of $(N, \mathcal{A}, t, v))$ are nonempty and define uniquely $y(T)$ for each $T \in \mathcal{A}$.

Theorem 1. *The proportional solution of* $(N, \mathcal{A}, t, v)$ *is nonempty for all* $t > 0$*,* v *with* $v(T) > 0$ *iff* $\mathcal{A}$ *is a minimal covering of* N*.*

Proof. Let $\mathcal{A}$ be a minimal covering of N. Then for each $S \in \mathcal{A}$ there exists $j(S) \in S \setminus \cup_{Q \in \mathcal{A} \setminus \{S\}} Q$. Denote $J = \{j(S) : S \in \mathcal{A}\}$. Take $y = \{y_i\}_{i \in N}$ such that $y_i = 0$ for all $i \in N \setminus J$, $\sum_{i \in N} y_i = t$, and $y_{j(S)}$ are proportional to $v(S)$. Then $y \in \mathcal{P}(N, \mathcal{A}, t, v)$.

Suppose that there exists $S \in \mathcal{A}$ such that $\mathcal{A} \setminus \{S\}$ covers N. Take $v(S) = t$, $v(Q) = \epsilon$, where $0 < \epsilon < t/|\mathcal{A}|$ for all $Q \in \mathcal{A} \setminus \{S\}$. Let $y \in \mathcal{P}(N, \mathcal{A}, t, v)$. Then for each $Q \in \mathcal{A} \setminus \{S\}$, $y(Q) \leq \epsilon$, hence $\sum_{i \in N} y_i \leq |\mathcal{A}|\epsilon < t$, but this contradicts to $\sum_{i \in N} y_i = t$. □

A vector $y = \{y_i\}_{i \in N}$ belongs to the *weakly proportional solution* of $(N, \mathcal{A}, t, v)$ iff $y_i \geq 0$ for all $i \in N$, $\sum_{i \in N} y_i = t$, $y(S)/v(S) = y(Q)/v(Q)$ for $S, Q \in \mathcal{A}$ with $S \cap Q = \emptyset$. We denote by $\mathcal{WP}(N, \mathcal{A}, t, v)$ the weakly proportional solution of $(N, \mathcal{A}, t, v)$.

We describe conditions on $\mathcal{A}$ that ensure existence of weakly proportional solutions. The following result of the author will be used.

Theorem 2 (Naumova, 1978, Theorem 2) or (Naumova, 2008b)). *Let* $c > 0$*,* $I(c) = \{x \in R^{|N|} : x_i \geq 0,\ x(N) = c\}$*, Gr be an undirected graph with the set of nodes* $\mathcal{A}$*,* $\{\succ_x\}_{x \in I(c)}$ *be a family of relations on* $\mathcal{A}$*, and for each* $K \in \mathcal{A}$

$$F^K = \{x \in I(c) : L \not\succ_x K \text{ for all } L \in \mathcal{A}\}.$$

Let $\{\succ_x\}_{x\in I(c)}$ satisfy the following 5 conditions.
1. $\succ_x$ is acyclic on $\mathcal{A}$.
2. If $K \in \mathcal{A}$ and $x_i = 0$ for all $i \in K$, then $x \in F^K$.
3. The set F^K is closed for each $K \in \mathcal{A}$.
4. If $K \succ_x L$, then K and L are adjacent in the graph Gr.
5. If a single node is taken out from each component of Gr, then the remaining elements of $\mathcal{A}$ do not cover N.

Then there exists $x^0 \in I(c)$ such that $K \not\succ_{x^0} L$ for all $K, L \in \mathcal{A}$.

This theorem is a generalization of Peleg's theorem (Peleg, 1967), where $\mathcal{A}$ is the set of all singletons.

A set of coalitions $\mathcal{A}$ generates the undirected graph $G = G(\mathcal{A})$, where $\mathcal{A}$ is the set of nodes and $K, L \in \mathcal{A}$ are adjacent iff $K \cap L = \emptyset$.

Theorem 3. *The weakly proportional solution of $(N, \mathcal{A}, t, v)$ is nonempty set for all $t > 0$, v with $v(T) > 0$ if and only if $\mathcal{A}$ satisfies the following condition: if a single node is taken out from each component of $G(\mathcal{A})$, then the remaining elements of $\mathcal{A}$ do not cover N.*

Proof. For each imputation x, consider the following relation on $\mathcal{A}$: $P \succ_x Q$ iff $P \cap Q = \emptyset$ and $x(P)/v(P) < x(Q)/v(Q)$. Then $x^0 \in \mathcal{WP}(N, \mathcal{A}, t, v)$ iff $K \not\succ_{x^0} L$ for all $K, L \in \mathcal{A}$. This family of relations and the graph $G(\mathcal{A})$ satisfy all conditions of Theorem 2, hence $\mathcal{WP}(N, \mathcal{A}, t, v) \neq \emptyset$ for all $t > 0$, v.

Now suppose that $\mathcal{A}$ does not satisfy the condition of the Theorem. Let m be the number of components of $G(\mathcal{A})$, $S_1, \dots, S_m$ be the nodes taken out from each component of $G(\mathcal{A})$ such that $\mathcal{A} \setminus \{S_1, \dots, S_m\}$ cover N.

Let us take $c > 0$, $v(S_i) = c$ for all $i = 1, \dots, m$, $v(Q) = \epsilon$ for remaining $Q \in \mathcal{A}$, where $\epsilon|\mathcal{A}| < c$. Suppose that $y \in \mathcal{WP}(N, \mathcal{A}, c, v)$, then $y(Q) \leq \epsilon$ for $Q \neq S_i$, and as such Q cover N, we get $y(N) \leq |\mathcal{A}|\epsilon < c = y(N)$. This contradiction completes the proof. $\square$

Example 1. $|N| = 4$, $\mathcal{A}$ consists of no more than 5 two-person coalitions. Then $\mathcal{A}$ satisfies the existence condition of Theorem 3.

Example 2. $|N| = 4$, $\mathcal{A}$ consists of all two-person coalitions. Then $\mathcal{A}$ does not satisfy the existence condition of Theorem 3. Indeed, if we take out from $\mathcal{A}$ the coalitions $\{1, 2\}, \{1, 3\}, \{2, 3\}$ then the remaining coalitions cover N.

Proposition 1. *The proportional nucleolus of $(N, \mathcal{A}, t, v)$ is contained in the proportional solution of $(N, \mathcal{A}, t, v)$ for all $t > 0$, all v with $v(T) > 0$ iff $\mathcal{A}$ is a partition of N.*

Proof. Let $\mathcal{A}$ be a partition of N, then the proportional nucleolus always coincides with the proportional solution.

Let the proportional nucleolus be always contained in the proportional solution. Suppose that there exist $P, Q \in \mathcal{A}$ such that $P \cap Q \neq \emptyset$. We take the following v: $v(P) = 1$, $v(T) = \epsilon$ otherwise, where $\epsilon < 1/(4|N|)$.

Let $x \in \mathcal{N}(N, \mathcal{A}, 1, v)$ then $x \in \mathcal{P}(N, \mathcal{A}, 1, v)$ and $x(T) = \epsilon x(P)$ for all $T \in \mathcal{A} \setminus \{P\}$, hence $x_i \leq \epsilon$ for all $i \in N \setminus P$. As long as $\mathcal{A}$ covers N, $x(P) > 3/4$. Since x belongs to the proportional nucleolus and $\mathcal{A}_P = \{T \in \mathcal{A} : T \cap P \neq \emptyset\} \neq \emptyset$, $x_i = 0$ for all $i \in P \setminus \cup_{T \in \mathcal{A}_P} T$. Then $x(S) \geq x(P)/|P|$ for some $S \in \mathcal{A}_P$. Therefore,

$$x(S) \geq 3/(4|N|) > \epsilon > \epsilon x(P),$$

but this contradicts to $x(S) = \epsilon x(P)$. □

Proposition 2. *The weighted entropy solution of* $(N, \mathcal{A}, t, v)$ *is contained in the proportional solution of* $(N, \mathcal{A}, t, v)$ *for all* $t > 0$*, all* v *with* $v(T) > 0$ *iff* $\mathcal{A}$ *is a partition of* N*.*

Proof. Let $\mathcal{A}$ be a partition of N, then, as for claim problems, the weighted entropy solution of $(N, \mathcal{A}, t, v)$ coincides with the proportional solution.

Let the weighted entropy solution be always contained in the proportional solution. Suppose that $\mathcal{A}$ is not a partition of N, then there exist $P, Q \in \mathcal{A}$ such that $P \cap Q \neq \emptyset$. We take the following v: $v(P) = 2$, $v(T) = \epsilon$ otherwise, where $\epsilon < 1/|N|$.

Let $x \in \mathcal{H}(N, \mathcal{A}, 1, v)$. Since $x \in \mathcal{P}(N, \mathcal{A}, t, v)$, $x(T) = \epsilon x(P)/2 \leq \epsilon/2$ for all $T \in \mathcal{A} \setminus \{P\}$, hence $x_i \leq \epsilon/2$ for all $i \in N \setminus P$. If $x_i \leq \epsilon$ for all $i \in P$, then $x(N) \leq \epsilon|N| < 1$, hence there exists $j_0 \in P \setminus \cup_{T \in \mathcal{A} \setminus \{P\}} T$ such that $x_{j_0} > \epsilon$.

Let $i_0 \in P \cap Q$, $y \in R^{|N|}$, $y_{j_0} = x_{j_0} - \delta$, $y_{i_0} = x_{i_0} + \delta$, otherwise $y_i = x_i$. If $\delta > 0$, then $y(P) = x(P)$, $y(Q) > x(Q)$, $y(T) \geq x(T)$ for all $T \in \mathcal{A} \setminus \{P, Q\}$, $y(N) = x(N)$. For sufficiently small δ, $y_i \geq 0$, $y(T) \leq \epsilon$ at each $T \in \mathcal{A} \setminus \{P\}$, hence

$$\sum_{T \in \mathcal{A}} y(T)(ln(y(T)/v(T)) - 1) < \sum_{T \in \mathcal{A}} x(T)(ln(x(T)/v(T)) - 1)$$

and $x \notin \mathcal{H}(N, \mathcal{A}, 1, v)$. □

Proposition 3. *Let the proportional nucleolus of* $(N, \mathcal{A}, t, v)$ *be contained in the weakly proportional solution of* $(N, \mathcal{A}, t, v)$ *for all* $t > 0$*, all* v *with* $v(T) > 0$*. Then the case* $P, Q, S \in \mathcal{A}$*,* $P \neq Q$*,* $P \cap S = Q \cap S = \emptyset$*,* $P \cap Q \neq \emptyset$ *is impossible.*

Proof. Suppose that there exist $P, Q, S \in \mathcal{A}$ such that $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$. Let us take the following v: $v(S) = v(P) = 1$, $v(T) = \epsilon$ for all $T \in \mathcal{A} \setminus \{S, P\}$, where $0 < \epsilon < 1/|N|$. Let $x \in \mathcal{N}(N, \mathcal{A}, 1, v)$.

We prove that $x(Q) \geq x(P)/|P|$. Assume the contrary, then $x(P \cap Q) < x(P)/|P|$, hence there exists $i_0 \in P \setminus Q$ with $x_{i_0} > x(P)/|P|$. Let $j \in P \cap Q$.

Let $i_0 \notin T$ for all $T \in \mathcal{A} \setminus \{P\}$ then we take $y \in R^{|N|}$: $y_{i_0} = 0$, $y_j = x_j + x_{i_0}$, $y_i = x_i$ otherwise. Then $y(P) = x(P)$, $y(Q) > x(Q)$, $y(T) \geq x(T)$ for all $T \in \mathcal{A}$, hence $x \notin \mathcal{N}(N, \mathcal{A}, 1, v)$.

Let $i_0 \in T$ for some $T \in \mathcal{A} \setminus \{P\}$, then $T \neq S$ and $x(T) > x(P)/|P| > x(Q)$. This implies $x(T)/v(T) = x(T)/\epsilon > x(Q)/v(Q)$. Let $z = z(\delta) \in R^{|N|}$, $z_{i_0} = x_{i_0} - \delta$, $z_j = x_j + \delta$, $z_i = x_i$ otherwise. If $\delta > 0$ and δ is sufficiently small then $z(T)/v(T) > z(Q)/v(Q) > x(Q)/v(Q)$ for $i_0 \in T \in \mathcal{A} \setminus \{P\}$, otherwise $z(T) \geq x(T)$, hence $\theta(z(\delta))\ _{lex}\theta(x)$. Thus $x(Q) \geq x(P)/|P|$.

Since x is weakly proportional, $x(Q)/\epsilon = x(S) = x(P)$, then $x(Q) \geq x(P)/|P|$ implies $x(P) \geq x(P)/(\epsilon|P|)$. Since x belongs to the proportional nucleolus, $x(T) > 0$ for all $T \in \mathcal{A}$, so we have $\epsilon|P| \geq 1$, but this contradicts to $\epsilon < 1/|N|$. □

Proposition 4. *Let the weighted entropy solution of $(N, \mathcal{A}, t, v)$ be contained in the weakly proportional solution of $(N, \mathcal{A}, t, v)$ for all $t > 0$, all v with $v(T) > 0$. Then the case $P, Q, S \in \mathcal{A}$, $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$ is impossible.*

Proof. Suppose that there exist $P, Q, S \in \mathcal{A}$ such that $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$. Let $i_0 \in P \cap Q$, $\mathcal{A}_0 = \{T \in \mathcal{A} : i_0 \in T, T \cap S \neq \emptyset\}$.

We take the following v: $v(T) = 1$ for $T \in \mathcal{A}_0 \cup \{P, S\}$, otherwise $v(T) = \epsilon$, where $0 < \epsilon < 1/(4|N|)$. Let $x \in \mathcal{H}(N, \mathcal{A}, 1, v)$. Since x is weakly proportional and $S \cap P = \emptyset$, we have $x(S) = x(P) \leq 1/2$. Then $x(Q)/v(Q) = x(S)/v(S) \leq 1/2$. As $v(Q) = \epsilon$, $x(Q) \leq \epsilon/2$. Consider 2 cases.

1. $x_i \leq \epsilon$ for all $i \in P$. Then $x(S \cup P) = x(S) + x(P) \leq 2\epsilon|P| < 1/2$. If $x_i \leq \epsilon$ for all $i \in N \setminus (P \cup S)$ then $x(N) < 3/4$, hence $x_{j_0} > \epsilon$ for some $j_0 \in N \setminus (P \cup S)$.

2. There exists $j_0 \in P$ with $x_{j_0} > \epsilon$. Then $j_0 \neq i_0$ because $x_{i_0} \leq x(Q) \leq \epsilon/2$.

Thus, there always exists $j_0 \in N$ such that $j_0 \notin S$, $j_0 \neq i_0$, $x_{j_0} > \epsilon$. Let $0 < \delta < \min\{\epsilon/2, x_{j_0} - \epsilon\}$. Let $y \in R^{|N|}$, $y_{j_0} = x_{j_0} - \delta$, $y_{i_0} = x_{i_0} + \delta$, $y_i = x_i$ otherwise. For $T \in \mathcal{A}$, if either $i_0 \in T$, $j_0 \in T$ or $i_0 \notin T$, $j_0 \notin T$ then $y(T) = x(T)$.

Let $j_0 \in T$, $i_0 \notin T$. Then $T \notin \mathcal{A}_0 \cup \{P, S\}$, hence $v(T) = \epsilon$. As $y(T) \geq y_{j_0} > \epsilon$, we have $x(T) > y(T) > v(T)$, then

$$y(T)(ln(y(T)/v(T)) - 1) < x(T)(ln(x(T)/v(T)) - 1).$$

Let $j_0 \notin T$, $i_0 \in T$. If $v(T) = \epsilon$ then $T \cap S = \emptyset$ and $x(T)/\epsilon = x(S)/v(S) \leq 1/2$. Thus,

$$v(T) = \epsilon > \epsilon/2 + \delta \geq x(T) + \delta = y(T) > x(T).$$

If $v(T) = 1$, then $1 \geq y(T) > x(T)$. Therefore in these cases $v(T) \geq y(T) > x(T)$, hence

$$y(T)(ln(y(T)/v(T)) - 1) < x(T)(ln(x(T)/v(T)) - 1).$$

Note that $j_0 \notin Q$ since $x(Q) \leq \epsilon/2 < x_{j_0}$, hence

$$\sum_{T \in \mathcal{A}} y(T)(ln(y(T)/v(T)) - 1) < \sum_{T \in \mathcal{A}} x(T)(ln(x(T)/v(T)) - 1)$$

and $x \notin \mathcal{H}(N, \mathcal{A}, 1, v)$. □

A collection of coalitions $\mathcal{A}$ is called *totally mixed at* N if $\mathcal{A} = \cup_{i=1}^{k} \mathcal{P}^i$, where $\mathcal{P}^i$ are partitions of N and for each collection $\{S_i\}_{i=1}^{k}$ where $S_i \in \mathcal{P}^i$, we have $\cap_{i=1}^{k} S_i \neq \emptyset$.

Theorem 4. *The weighted entropy solution of* $(N, \mathcal{A}, t, v)$ *coincides with the weakly proportional solution of* $(N, \mathcal{A}, t, v)$ *for all* $t > 0$*, all* v *with* $v(T) > 0$ *iff* $\mathcal{A}$ *is totally mixed at* N*.*

Proof. Let $\mathcal{A}$ be totally mixed at N. Then in view of Theorem 3, the weakly proportional solution of $(N, \mathcal{A}, t, v)$ is always nonempty. As long as all $\mathcal{P}^i$ are partitions of N, $x(S)$ are uniquely defined for all $S \in \mathcal{A}$, $x \in \mathcal{WP}(N, \mathcal{A}, t, v)$.

Let $y \in \mathcal{H}(N, \mathcal{A}, t, v)$. Suppose that $y \notin \mathcal{WP}(N, \mathcal{A}, t, v)$ then there exists $i \in \{1, \ldots, k\}$ such that $y(S)/v(S) > t(Q)/v(Q)$ for some $S, Q \in \mathcal{P}^i$. Since $\mathcal{A}$ is totally mixed, there exist sufficiently small $\delta > 0$ and a vector $z = z(\delta)$ with $z_j \geq 0$ such that $z(Q) = y(Q) + \delta$, $z(S) = y(S) - \delta$, $z(T) = y(T)$ for remaining $T \in \mathcal{A}$. Denote

$$f(\delta) = \sum_{T \in \mathcal{A}} z(\delta)(T)(ln(z(\delta)(T)/v(T)) - 1).$$

Then

$$f'(0) = \ln(y(Q)/v(Q)) - \ln(y(S)/v(S)) < 0,$$

hence for sufficiently small $\delta > 0$,

$$\sum_{T \in \mathcal{A}} z(\delta)(T)(ln(z(\delta)(T)/v(T)) - 1) < \sum_{T \in \mathcal{A}} y(T)(ln(y(T)/v(T)) - 1),$$

but this contradicts $y \in \mathcal{H}(N, \mathcal{A}, t, v)$. Thus $\mathcal{H}(N, \mathcal{A}, t, v) \subset \mathcal{WP}(N, \mathcal{A}, t, v)$. Since $x(S)$ are uniquely defined for all $x \in \mathcal{WP}(N, \mathcal{A}, t, v)$, this implies $\mathcal{H}(N, \mathcal{A}, t, v) = \mathcal{WP}(N, \mathcal{A}, t, v)$.

Now suppose that $\mathcal{WP}(N, \mathcal{A}, t, v) = \mathcal{H}(N, \mathcal{A}, t, v)$ for all $t > 0$, all v with $v(T) > 0$. By Proposition 4, $\mathcal{A} = \bigcup_{i=1}^{k} \mathcal{B}^i$, where $\mathcal{B}^i$ are subsets of partitions of N. If each $\mathcal{B}^i$ is a partition $\mathcal{P}^i$ of N then by Theorem 3, for each collection $\{S_i\}_{i=1}^k$ with $S_i \in \mathcal{P}^i$, we have $\cap_{i=1}^k S_i \neq \emptyset$, so $\mathcal{A}$ is totally mixed at N.

Let some $\mathcal{B}^i$ be not a partition of N. Then without loss of generality, there exists $q < k$ such that $\cup_{i=1}^q \mathcal{B}^i$ does not cover N and $\cup_{i=1}^q \mathcal{B}^i \cup \mathcal{B}^j$ covers N for each $j > q$. Denote $N^0 = \bigcup_{S \in \cup_{i=1}^q \mathcal{B}^i} S$. We consider 2 cases.

Case 1. For each $j = q+1, \ldots, k$, there exists $S_j \in \mathcal{B}^j$, such that $\cup_{j=q+1}^k (\mathcal{B}^j \setminus S_j)$, covers $(N \setminus N^0)$.

Take the following $v = v^\epsilon$:
$v^\epsilon(S) = \epsilon$ for all $S \in \cup_{j=1}^q \mathcal{B}^j$,
$v^\epsilon(S_j) = 1$, $j = q+1, \ldots, k$,
$v^\epsilon(T) = 1/(2|N|)$ otherwise.
Let $x^\epsilon \in \mathcal{H}(N, \mathcal{A}, 1, v^\epsilon)$. Since x^ϵ is weakly proportional, $x^\epsilon(T) \leq 1/(2|N|)$ for all $T \in \mathcal{B}^j$, $T \neq S_j$, $j = q+1, \ldots, k$, hence $x^\epsilon(N \setminus N^0) \leq 1/2$ and $x^\epsilon(N^0) \geq 1/2$. Consider

$$\sum_{T \in \cup_{j=1}^q \mathcal{B}^j} x^\epsilon(T)(ln(x^\epsilon(T)/v^\epsilon(T)) - 1) =$$

$$\sum_{T \in \cup_{j=1}^q \mathcal{B}^j} x^\epsilon(T)(\ln x^\epsilon(T) - 1) - \sum_{T \in \cup_{j=1}^q \mathcal{B}^j} x^\epsilon(T) ln(\epsilon).$$

Note that $\sum_{T \in \cup_{j=1}^q \mathcal{B}^j} x^\epsilon(T) \geq x(N^0) \geq 1/2$ implies

$$\sum_{T \in \cup_{j=1}^q \mathcal{B}^j} x^\epsilon(T) ln(\epsilon) \to -\infty \text{ as } \epsilon \to 0.$$

Moreover, $z(T)(lnz(T)-1)$ for $T\in\cup_{j=1}^{q}\mathcal{B}^j$ and $z(T)(lnz(T)/v^{\epsilon}(T)-1)$ for $T\in\cup_{j=q+1}^{k}\mathcal{B}^j$ are bounded below, therefore

$$\sum_{T\in\mathcal{A}} x^{\epsilon}(T)(ln(x^{\epsilon}(T)/v(T))-1)\to+\infty \text{ as } \epsilon\to 0.$$

Let $y\in R^{|N|}$, $\sum_{i\in N} y_i=1$, $y_i>0$ for all i. Define $z^{\epsilon}\in R^{|N|}$:
$z_i^{\epsilon}=\epsilon y_i$ for $i\in N^0$,
$z_i^{\epsilon}=(1-\epsilon y(N^0))/y(N\setminus N^0)y_i$ for $i\in N\setminus N^0$.
Let $\epsilon\to 0$, then

$$\sum_{T\in\cup_{j=1}^{q}\mathcal{B}^j} z^{\epsilon}(T)(ln(z^{\epsilon}(T)/v^{\epsilon}(T))-1)=\epsilon\sum_{T\in\cup_{j=1}^{q}\mathcal{B}^j} y(T)(ln(y(T)-1)\to 0,$$

$$\sum_{T\in\cup_{j=q+1}^{k}\mathcal{B}^j} z^{\epsilon}(T)(ln(z^{\epsilon}(T)/v^{\epsilon}(T))-1)\to\sum_{T\in\cup_{j=q+1}^{k}\mathcal{B}^j} u(T)(ln(u(T)/v^{\epsilon}(T))-1),$$

where $u=\{u_j\}_{j\in N}$, $u_j=0$ for $j\in N^0$, $u_j=y_j/y(N\setminus N^0)$ for $j\notin N^0$. Thus,

$$\sum_{T\in\mathcal{A}} z^{\epsilon}(T)(ln(z^{\epsilon}(T)/v^{\epsilon}(T))-1)\to\sum_{T\in\cup_{j=q+1}^{k}\mathcal{B}^j} u(T)(ln(u(T)/v^{\epsilon}(T))-1)<+\infty.$$

Therefore for sufficiently small ϵ,

$$\sum_{T\in\mathcal{A}} x^{\epsilon}(T)(ln(x^{\epsilon}(T)/v(T))-1)>\sum_{T\in\mathcal{A}} z^{\epsilon}(T)(ln(z^{\epsilon}(T)/v^{\epsilon}(T))-1),$$

and this contradicts $x^{\epsilon}\in\mathcal{H}(N,\mathcal{A},1,v^{\epsilon})$.

Case 2. If $S_j\in\mathcal{B}^j$ is taken out from $\mathcal{B}^j$, $j=q+1,\ldots,k$, then the remaining elements of $\cup_{j=q+1}^{k}\mathcal{B}^j$ do not cover $N\setminus N^0$.

For each $j=q+1,\ldots,k$, $S_j\in\mathcal{B}^j$, we have $S_j\cap(N\setminus N^0)\neq\emptyset$. Indeed, suppose that $S_{j_0}\subset N^0$ for some $j_0>q$. Then if we take S_{j_0} and arbitrary $S_j\in\mathcal{B}^j$ for $j>q$, $j\neq j_0$ out from $\cup_{j=q+1}^{k}\mathcal{B}^j$, the remaining elements of $\cup_{j=q+1}^{k}\mathcal{B}^j$ cover $N\setminus N^0$ as if $\{N^0\}\cup\mathcal{B}^{j_0}$ covers N.

Let

$$\mathcal{C}=\{(N\setminus N^0)\cap S:\ S\in\cup_{j=q+1}^{k}\mathcal{B}^j,\ S\cap P=\emptyset \text{ for some } P\in\mathcal{A}\}.$$

Then $P,S\in\cup_{j=q+1}^{k}\mathcal{B}^j$, $P\neq S$, $P\cap(N\setminus N^0)\in\mathcal{C}$ imply $P\cap(N\setminus N^0)\neq S\cap(N\setminus N^0)$. Indeed, suppose that $P\cap(N\setminus N^0)=S\cap(N\setminus N^0)$. There exists $P^1\in\mathcal{A}$ such that $P\cap P^1=\emptyset$. If we take S,P^1 and arbitrary $S_j\in\mathcal{B}^j$ for $j>q$ with $P\notin\mathcal{B}^j$ out from $\cup_{j=q+1}^{k}\mathcal{B}^j$, the remaining elements of $\cup_{j=q+1}^{k}\mathcal{B}^j$ cover $N\setminus N^0$ because $\{N^0\}\cup\mathcal{B}^{j_0}$ covers N, where $\mathcal{B}^{j_0}\ni S$.

For arbitrary problem $(N,\mathcal{A},t,v)$, where $\mathcal{A}$ is under the case 2, consider the problem $(N\setminus N^0,\mathcal{C},t,w)$, where $w(T)=v(S)$ for $T=S\cap(N\setminus N^0)\in\mathcal{C}$. As was proved above, w is well defined. Under the case 2, due to Theorem 3, there exists $y\in\mathcal{WP}(N\setminus N^0,\mathcal{C},t,w)$. Let $x\in R^{|N|}$, $x_i=0$ for $i\in N^0$, $x_i=y_i$ for $i\in N\setminus N^0$, then $x\in\mathcal{WP}(N,\mathcal{A},t,v)$, $x(N^0)=0$.

Let $\tilde{v}(S)=|S|/|N|$ for all $S\in\mathcal{A}$, $\tilde{x}_i=1/|N|$ for all $i\in N$, then $\tilde{x}\in\mathcal{H}(N,\mathcal{A},1,\tilde{v})$ as if $\tilde{x}(S)=\tilde{v}(S)$ for all $S\in\mathcal{A}$, but $\tilde{x}(N^0)>0$. Thus in case 2 the weighted entropy solution does not coincide with the weakly proportional solution for some problem. $\square$

Corollary 1. *The proportional nucleolus of $(N, \mathcal{A}, t, v)$, the weighted entropy solution of $(N, \mathcal{A}, t, v)$, and the weakly proportional solution of $(N, \mathcal{A}, t, v)$ coincide for all $t > 0$, all v with $v(T) > 0$ iff $\mathcal{A}$ is totally mixed at N.*

3. Axiomatic Justification of a Single–valued Solution

Note that all solutions described above are set valued. In the case of totally mixed set $\mathcal{A}$ we propose a selector for these solutions and give an axiomatic justification to it.

Let $\mathcal{A} = \cup_{i=1}^{k} \mathcal{P}^i$, where $\mathcal{P}^i$ are partitions of N, be a totally mixed at N collection of coalitions, $t > 0$. Then $y = \{y_i\}_{i \in N}$ is the *totally proportional solution* of $(N, \mathcal{A}, t, v))$ if for each collection $\{S_i\}_{i=1}^{k}$ with $S_i \in \mathcal{P}^i$,

$$y(\cap_{i=1}^{k} S_i) = t \prod_{i=1}^{k} v(S_i) / \prod_{i=1}^{k} \alpha_i,$$

where $\alpha_i = \sum_{S \in \mathcal{P}^i} v(S)$ and $y_p = y_q$ for all $p, q \in \cap_{i=1}^{k} S_i$.

Example 3. Let $N = \{1, 2, 3, 4, 5\}$, $\mathcal{A} = \{\{1, 2\}, \{3, 4, 5\}, \{1, 3\}, \{2, 4, 5\}\}$, then the following $x \in R^5$ is the totally proportional solution of $(N, \mathcal{A}, 1, v)$.
$x_1^0 = [v(1, 2)v(1, 3)]/[(v(1, 2) + v(3, 4, 5))(v(1, 3) + v(2, 4, 5))]$,
$x_2^0 = [v(1, 2)v(2, 4, 5)]/[(v(1, 2) + v(3, 4, 5))(v(1, 3) + v(2, 4, 5))]$,
$x_3^0 = [v(3, 4, 5)v(1, 3)]/[(v(1, 2) + v(3, 4, 5))(v(1, 3) + v(2, 4, 4))]$,
$x_4^0 = x_5^0 = [v(3, 4, 5)v(2, 4, 5)]/[2(v(1, 2) + v(3, 4, 5))(v(1, 3) + v(2, 4, 5))]$.

Consider the following properties of methods that generate single-valued solutions.

A single-valued method F is *anonymous* if for each permutation π of N, $F(\pi N, \pi\mathcal{A}, t, \pi v) = \pi F(N, \mathcal{A}, t, v)$.

Players i and j are *symmetric with respect to* $\mathcal{A}$ if for all $S \in \mathcal{A}$, either $i \in S$ and $j \in S$ or $i \notin S$, $j \notin S$.

If i and j are symmetric w.r.to $\mathcal{A}$, then *the result of glueing* i *and* j *at* $(N, \mathcal{A}, t, v)$ is a game $(N_{[i,j]}, \mathcal{A}_{[i,j]}, t, v_{[i,j]})$, where
$N_{[i,j]} = N \setminus \{i, j\} \cup \{i^0\}$,
$Q \in \mathcal{A}_{[i,j]}$ iff either $Q \in \mathcal{A}$ and $i \notin Q$ or $i^0 \in Q$ and $(Q \setminus \{i^0\} \cup \{i, j\}) \in \mathcal{A}$,
$v_{[i,j]}(Q) = v(Q)$ for $Q \in \mathcal{A}$, or else $v_{[i,j]}(Q) = v(Q \setminus \{i^0\} \cup \{i, j\})$.

F is *independent of glueing* if for all symmetric w.r.to $\mathcal{A}$ i and j, $F(N, \mathcal{A}, t, v)_k = F(N_{[i,j]}, \mathcal{A}_{[i,j]}, t, v_{[i,j]})_k$ for all $k \in N \setminus \{i, j\}$.

A single-valued method F is *independent of merging* if for all $S, T \in \mathcal{A}$ with $S \cap T = \emptyset$ and $(S \cup T) \notin \mathcal{A}$,

$$F(N, \mathcal{A}, t, v)_i = F(N,\ \mathcal{A}_{S,T},\ t,\ v_{S,T})_i \ \forall\ i \notin (S \cup T),$$

where $\mathcal{A}_{S,T} = \mathcal{A} \setminus \{S, T\} \cup \{S \cup T\}$,
$v_{S,T}(S \cup T) = v(S) + v(T)$, otherwise $v_{S,T}(Q) = v(Q)$.

A single-valued method F defined on $\mathcal{G}$ is *consistent* if for each $Q \in \mathcal{A}$, $Q \neq N$, $(N \setminus Q,\ \mathcal{A}^Q,\ t - F(N, \mathcal{A}, t, v) \in \mathcal{G}$ implies

$$F(N \setminus Q,\ \mathcal{A}^Q,\ t - F(N, \mathcal{A}, t, v)(Q),\ v^Q)_i = F(N, \mathcal{A}, t, v)_i \quad \text{for all } i \in N \setminus Q,$$

where $\mathcal{A}^Q = \{S \setminus Q : S \in \mathcal{A},\ S \setminus Q \neq \emptyset\}$,

$$v^Q(T) = \max_{S \in \mathcal{A}: S \setminus Q = T} (v(S) - F(N, \mathcal{A}, t, v)(S \cap Q)).$$

Let $\mathcal{G}_+$ be the class of problems $(N, \mathcal{A}, t, v)$ such that N is finite, $\mathcal{A}$ is totally mixed at N, $t > 0$, $v(S) > 0$ for all $S \in \mathcal{A}$, $\sum_{S \in \mathcal{P}} v(S) > t$ for all partitions $\mathcal{P}$ of N with $\mathcal{P} \subset \mathcal{A}$.

Lemma 1. *Let F be a single-valued method defined on $\mathcal{G}_+$.*

Then F is consistent, anonymous, $F(N, \mathcal{A}, t, v)(S) < v(S)$ for all $S \in \mathcal{A}$, $F(N, \mathcal{A}, t, v)(S) < t$ for all $S \in \mathcal{A}$, $S \neq N$, and $F(N, \mathcal{A}, t, v) \in \mathcal{WP}(N, \mathcal{A}, t, v)$ for all partitions $\mathcal{A}$ of N iff F is totally proportional.

Proof. Let F be totally proportional. Then for each problem $(N, \mathcal{A}, t, v)$, for each $S \in \mathcal{P}^i \subset \mathcal{A}$, $F(N, \mathcal{A}, t, v)(S) = tv(S)/\alpha_i$, hence $F(N, \mathcal{A}, t, v)(S) < v(S)$ for all $S \in \mathcal{A}$ and $F(N, \mathcal{A}, t, v) \in \mathcal{WP}(N, \mathcal{A}, t, v)$ for all partitions $\mathcal{A}$ of N. Moreover, F is anonymous.

Let us verify the consistency condition. Let $Q \in \mathcal{A}$, $Q \neq N$, then $N \setminus Q \neq \emptyset$, $\mathcal{A}^Q = \{S \setminus Q : S \in \mathcal{A},\ S \setminus Q \neq \emptyset\}$ is totally balanced at $N \setminus Q$,

$$v^Q(S \setminus Q) = v(S) - F(N, \mathcal{A}, t, v)(S \cap Q) \text{ under } S \setminus Q \neq \emptyset.$$

Let $x = F(N, \mathcal{A}, t, v)$. Since F is totally proportional, for each collection $\{S_i\}_{i=1}^k$, where $S_i \in \mathcal{P}^i$, we have

$$x(\cap_{i=1}^k S_i) = M \prod_{i=1}^k v(S_i). \tag{1}$$

We need to prove that for each collection $\{S_i\}_{i=1}^k$, where $S_i \in \mathcal{P}^i$ and $S_i \neq Q$, we have

$$x(\cap_{i=1}^k (S_i \setminus Q) = M^Q \prod_{i=1}^k v^Q(S_i \setminus Q). \tag{2}$$

Here

$$\cap_{i=1}^k (S_i \setminus Q) = \cap_{i=1}^k S_i. \tag{3}$$

If $S \cap Q = \emptyset$ then $v^Q(S) = v(S)$. Let $S \cap Q \neq \emptyset$, $S \in \mathcal{P}^{i_0}$. Then since F is totally proportional, $x(S \cap Q) = v(S)v(Q)\mu_{i_0}$ and $v^Q(S \setminus Q) = v(S) - x(S \cap Q) = v(S)(1 - \mu_{i_0} v(Q))$, hence

$$\prod_{i=1}^k v^Q(S_i \setminus Q) = \prod_{i=1}^k (1 - \mu_i v(Q)) \prod_{i=1}^k v(S_i).$$

In view of (1) and (3) this implies (2). Thus the totally proportional solution satisfies all conditions of Lemma.

Now let F satisfy all conditions of Lemma. Let $x = F(N, \mathcal{A}, t, v)$, $Q \in \mathcal{A}$, $Q \neq N$. First we verify that the problem $(N \setminus Q,\ \mathcal{A}^Q,\ t - x(Q), v^Q) \in \mathcal{G}_+$.

Obviously, the collection of coalitions $\mathcal{A}^Q$ is totally mixed, $t - x(Q) > 0$, For each $T \in \mathcal{A}^Q$, $T = S \setminus Q$ for some $S \in \mathcal{A}$ and such S is unique as if $\mathcal{A}$ is totally balanced, hence $v^Q(T) = v(S) - x(S \cap Q) \geq v(S) - x(S) > 0$.

For each partition $\mathcal{P}^Q$ of $N \setminus Q$ with $\mathcal{P}^Q \subset \mathcal{A}^Q$,

$$\sum_{T \in \mathcal{P}^Q} v^Q(T) > t - x(Q). \tag{4}$$

Indeed,

$$\sum_{T \in \mathcal{P}^Q} v^Q(T) = \sum_{S \in \mathcal{P}} v^Q(S \setminus Q) \tag{5}$$

for some partition $\mathcal{P}$ of N, $\mathcal{P} \subset \mathcal{A}$.

$$\sum_{S \in \mathcal{P}} v^Q(S \setminus Q) = \sum_{S \in \mathcal{P}} (v(S) - x(S \cap Q)) = \sum_{S \in \mathcal{P}} v(S) - x(N \cap Q) > t - x(Q).$$

In view of (5) this implies (4). Thus the problem $(N \setminus Q,\ \mathcal{A}^Q,\ t - x(Q), v^Q)$ belongs to $\mathcal{G}_+$.

Let $\mathcal{A} = \cup_{i=1}^k \mathcal{P}^i$, where $\mathcal{P}^i$ are partitions of N. We prove by induction on k that for each collection $\{S_i\}_{i=1}^k$ with $S_i \in \mathcal{P}^i$,

$$F(N, \mathcal{A}, t, v)(\cap_{i=1}^k S_i) = t \prod_{i=1}^k v(S_i) / \prod_{i=1}^k \alpha_i, \text{ where } \alpha_i = \sum_{S \in \mathcal{P}^i} v(S). \tag{6}$$

If $k = 1$ then (6) is a condition of Lemma. Suppose that (6) is proved for $k \leq m$. Let $\mathcal{A} = \cup_{i=0}^m \mathcal{P}^i$, where $\mathcal{P}^i$ are partitions of N. Let $\mathcal{P}^0 = \{S_1, \ldots, S_r\}$. Let we take $S_2, \ldots, S_r$ out of $\mathcal{A}$. Then due to consistency, we get the game $(S_1, \mathcal{A}^{S_2, \ldots S_r}, t - \sum_{j=2}^r x(S_j), v_{S_1})$, where $x = F(N, \mathcal{A}, t, v)$, $v_{S_1} = v^{S_2, \ldots S_r}$, and for each $P_j^i \in \mathcal{P}^i$, $i \geq 1$,

$$v_{S_1}(S_1 \cap P_j^i) = v(P_j^i) - \sum_{q=2}^r x(S_q \cap P_j^i) = v(P_j^i) - x(P_j^i) + x(S_1 \cap P_j^i). \tag{7}$$

By induction assumption, for all $P_{j_i}^i, P_{q_i}^i \in \mathcal{P}^i$, $i = 1, \ldots, m$, we have

$$\frac{x(\cap_{i=1}^m P_{j_i}^i \cap S_1)}{\prod_{i=1}^m v_{S_1}(S_1 \cap P_{j_i}^i)} = \frac{x(\cap_{i=1}^m P_{q_i}^i \cap S_1)}{\prod_{i=1}^m v_{S_1}(S_1 \cap P_{q_i}^i)}. \tag{8}$$

Let $q_i = j_i$ for $i \neq 1$, then (8) and (7) imply

$$\begin{aligned} &x(\cap_{i=1}^m P_{j_i}^i \cap S_1)[v(P_{q_1}^1) - x(P_{q_1}^1) + x(S_1 \cap P_{q_1}^1)] = \\ &x(\cap_{i=2}^m P_{j_i}^i \cap S_1 \cap P_{q_1}^1)[v(P_{j_1}^1) - x(P_{j_1}^1) + x(S_1 \cap P_{j_1}^1)] \end{aligned} \tag{9}$$

If $m = 1$ we obtain

$$x(S_1 \cap P_{j_1}^1)[v(P_{q_1}^1) - x(P_{q_1}^1) + x(S_1 \cap P_{q_1}^1)] = x(S_1 \cap P_{q_1}^1)[v(P_{j_1}^1) - x(P_{j_1}^1) + x(S_1 \cap P_{j_1}^1)].$$

If $m > 1$ then summing equations (9) over all collections $j_2, \ldots, j_m$, we get the same formula. Hence

$$x(S_1 \cap P^1_{j_1})[v(P^1_{q_1}) - x(P^1_{q_1})] = x(S_1 \cap P^1_{q_1})[v(P^1_{j_1}) - x(P^1_{j_1})]. \tag{10}$$

Substituting S_i for S_1 we get

$$x(S_i \cap P^1_{j_1})[v(P^1_{q_1}) - x(P^1_{q_1})] = x(S_i \cap P^1_{q_1})[v(P^1_{j_1}) - x(P^1_{j_1})] \text{ for all } i = 1, \ldots, r.$$

Summing these equations over $i = 1, \ldots, r$, we obtain

$$x(P^1_{j_1})[v(P^1_{q_1}) - x(P^1_{q_1})] = x(P^1_{q_1})[v(P^1_{j_1}) - x(P^1_{j_1})],$$

then

$$x(P^1_{j_1})/v(P^1_{j_1}) = x(P^1_{q_1})/v(P^1_{q_1}) = t/\alpha_1. \tag{11}$$

Since $t/\alpha_1 < 1$, (10) and (11) imply

$$x(S_1 \cap P^1_{j_1}/v(P^1_{j_1}) = x(S_1 \cap P^1_{q_1}/v(P^1_{q_1}) \tag{12}$$

Due to $t/\alpha_1 < 1$, combining (11) and (12) with (9), we obtain

$$x(\cap^m_{i=1} P^i_{j_i} \cap S_1)v(P^1_{q_1}) = x(\cap^m_{i=2} P^i_{j_i} \cap S_1 \cap P^1_{q_1})v(P^1_{j_1}),$$

then

$$\frac{x(\cap^m_{i=1} P^i_{j_i} \cap S_1)}{v(S_1)\prod^m_{i=1} v(P^i_{j_i})} = \frac{x(\cap^m_{i=2} P^i_{j_i} \cap S_1 \cap P^1_{q_1})}{v(S_1)v(P^1_{q_1})\prod^m_{i=2} v(P^i_{j_i})}.$$

Thus we can obtain by m steps that for all $P^i_{j_i}, P^i_{q_i} \in \mathcal{P}^i$, $i = 0, \ldots, m$,

$$\frac{x(\cap^m_{i=0} P^i_{j_i})}{\prod^m_{i=0} v(\cap P^i_{j_i})} = \frac{x(\cap^m_{i=0} P^i_{q_i})}{\prod^m_{i=0} v(\cap P^i_{q_i})},$$

and this implies (6). Since F is anonymous, F is totally proportional. $\square$

Theorem 5. *Let F be a single-valued method defined on the class of problems $\mathcal{G}_+$.*

Then F is continuous with respect to v, anonymous, consistent, independent of glueing, independent of merging, $F(N, \mathcal{A}, t, v)(S) < v(S)$ for all $S \in \mathcal{A}$, and $F(N, \mathcal{A}, t, v)(S) < t$ for all $S \in \mathcal{A}$, $S \neq N$ iff F is totally proportional.

Proof. In view of Lemma, the totally proportional solution defined on $\mathcal{G}_+$ satisfies all conditions of the Theorem.

Let F satisfy all conditions of the Theorem then continuity, independence of glueing, and independence of merging imply that $F(N, \mathcal{A}, t, v) \in \mathcal{WP}(N, \mathcal{A}, t, v)$ for all partitions $\mathcal{A}$ of N. Therefore, by Lemma, F is totally proportional. $\square$

4. Generalizations of the Uniform Losses Solution

A vector $y = \{y_i\}_{i \in N}$ is the *uniform losses solution* of the claim problem

(N, t, x), where $x_i \geq 0$, $t > 0$, if there exists $\mu \in R^1$ such that $y_i = (x_i - \mu)_+$, $\sum_{i \in N} y_i = t$, $y_i \geq 0$, where $a_+ = \max\{a, 0\}$.

Then y can also be defined by each of the following 3 conditions.

C1. $y_i \geq 0$, $\sum_{i \in N} y_i = t$, and if $x_i - y_i > x_j - y_j$, then $y_j = 0$.

C2. $\{y\}$ is the nucleolus w.r.t. $\{f_i\}_{i \in N}$, where $f_i(z) = z_i - x_i$, on the set $\{z = \{z_i\}_{i \in N} : z_i \geq 0,\ \sum_{i \in N} z_i = t\}$.

C3. y minimizes

$$\sum_{i \in N} (z_i - x_i)^2$$

on the set $\{z = \{z_i\}_{i \in N} : z_i \geq 0,\ \sum_{i \in N} z_i = t\}$.

We suppose that $v(T) \geq 0$ for all $T \in \mathcal{A}$.

1. A vector $y = \{y_i\}_{i \in N}$ belongs to the *uniform losses solution* of $(N, \mathcal{A}, t, v)$ iff $y_i \geq 0$ for all $i \in N$, $\sum_{i \in N} y_i = t$, and for all $S, T \in \mathcal{A}$, $v(S) - y(S) > v(T) - y(T)$ implies $y(T) = 0$.

2. A vector $y = \{y_i\}_{i \in N}$ belongs to the *nucleolus* of $(N, \mathcal{A}, t, v)$ iff y belongs to the nucleolus w.r.t. $\{f_T\}_{T \in \mathcal{A}}$ with $f_T(z) = z(T) - v(T)$ on the set $\{z = \{z_i\}_{i \in N} : z_i \geq 0,\ \sum_{i \in N} z_i = t\}$.

3. A vector $y = \{y_i\}_{i \in N}$ belongs to the *least square solution* of $(N, \mathcal{A}, t, v))$ iff y minimizes

$$\sum_{T \in \mathcal{A}} (z(T) - v(T))^2$$

on the set $\{z = \{z_i\}_{i \in N} :\ z_i \geq 0,\ \sum_{i \in N} z_i = t\}$.

A vector $y = \{y_i\}_{i \in N}$ belongs to the *weakly uniform losses solution* of $(N, \mathcal{A}, t, v)$ iff $y_i \geq 0$ for all $i \in N$, $\sum_{i \in N} y_i = t$, and for all $S, T \in \mathcal{A}$ with $S \cap Q = \emptyset$,

$$v(S) - y(S) > v(T) - y(T) \text{ implies } y(T) = 0.$$

Theorem 6. *The uniform losses solution of $(N, \mathcal{A}, t, v)$ is nonempty for all $t > 0$, v with $v(T) \geq 0$ iff $\mathcal{A}$ is a minimal covering of N.*

Proof. Let $\mathcal{A}$ be a minimal covering of N. Then for each $S \in \mathcal{A}$ there exists $j(S) \in S \setminus \cup_{Q \in \mathcal{A} \setminus \{S\}} Q$. Denote $J = \{j(S) : S \in \mathcal{A}\}$. Let z_J be the uniform losses solution of the claim problem $(J, t, \{v(S)\}_{S \in \mathcal{A}})$, then $y = \{y_i\}_{i \in N}$ such that $y_i = 0$ for all $i \in N \setminus J$, and $y_{j(S)} = z_{j(S)}$ belongs to the uniform losses solution of $(N, \mathcal{A}, t, v)$.

Suppose that there exists $S \in \mathcal{A}$ such that $\mathcal{A} \setminus \{S\}$ covers N. We take $v(S) = 1$, $v(Q) = 0$ for all $Q \in \mathcal{A} \setminus \{S\}$. Let y belong to the uniform losses solution of $(N, \mathcal{A}, 1, v)$. Then for each $Q \in \mathcal{A} \setminus \{S\}$, $y(Q) = 0$, hence $\sum_{i \in N} y_i = 0$, but this contradicts $\sum_{i \in N} y_i = 1$. □

Theorem 7. *The weakly uniform losses solution of $(N, \mathcal{A}, t, v)$ is nonempty set for all $t > 0$, v with $v(T) \geq 0$ if and only if $\mathcal{A}$ satisfies the following condition:*
if a single node is taken out from each component of $G(\mathcal{A})$, then the remaining elements of $\mathcal{A}$ do not cover N.

The Theorem was proved in (Naumova, 1978). The proof is based on Theorem 2.

Proposition 5. *The nucleolus of $(N, \mathcal{A}, t, v)$ is contained in the uniform losses solution of $(N, \mathcal{A}, t, v)$ for all $t > 0$, all v with $v(T) \geq 0$ iff $\mathcal{A}$ is a partition of N.*

Proof. Let $\mathcal{A}$ be a partition of N, then, as for claim problems, the nucleolus always coincides with the uniform losses solution.

Let the nucleolus be always contained in the uniform losses solution. Suppose that $\mathcal{A}$ is not a partition of N, then there exist $P, Q \in \mathcal{A}$ such that $P \cap Q \neq \emptyset$. We take the following v: $v(P) = 1$, $v(T) = 0$ otherwise. Let x belong to the nucleolus of $(N, \mathcal{A}, 1, v)$. Since x belongs to the uniform losses solution, $x(T) = 0$ for all $T \in \mathcal{A} \setminus \{P\}$, hence $x(P) = 1$. Since x belongs to the nucleolus of $(N, \mathcal{A}, 1, v)$ and $\mathcal{A}_P = \{T \in \mathcal{A} : T \cap P \neq \emptyset\} \neq \emptyset$, $x_i = 0$ for all $i \in P \setminus \cup_{T \in \mathcal{A}_P} T$. Then there exists $j_0 \in P \cap S$ for some $S \in \mathcal{A}_P$ such that $x_{j_0} > 0$. Then $x(S) > 0$, but this contradicts to $x(S) = 0$. □

Proposition 6. *The least square solution of $(N, \mathcal{A}, t, v)$ is contained in the uniform losses solution of $(N, \mathcal{A}, t, v)$ for all $t > 0$, all v with $v(T) \geq 0$ iff $\mathcal{A}$ is a partition of N.*

Proof. Let $\mathcal{A}$ be a partition of N, then, as for claim problems, the least square solution always coincides with the uniform losses solution.

Let the least square solution be always contained in the uniform losses solution. Suppose that $\mathcal{A}$ is not a partition of N, then there exist $P, Q \in \mathcal{A}$ such that $P \cap Q \neq \emptyset$. We take the following v: $v(P) = 2$, $v(T) = \epsilon$ otherwise, where $0 < \epsilon < 1$. Let x belong to the least square solution of $(N, \mathcal{A}, 1, v)$.

Then $v(P) - x(P) \geq 1$, $v(T) - x(T) \leq \epsilon$ for all $T \in \mathcal{A} \setminus \{P\}$. Since x belongs to the uniform losses solution, $x(T) = 0$ for all $T \in \mathcal{A} \setminus \{P\}$, hence there exists $j_0 \in P \setminus \cup_{T \in \mathcal{A} \setminus \{P\}} T$ such that $x_{j_0} > 0$.

Let $i_0 \in P \cap Q$, $y \in R^{|N|}$, $y_{j_0} = x_{j_0} - \delta$, $y_{i_0} = x_{i_0} + \delta$, $y_i = x_i$ otherwise. If $\delta > 0$, then $y(P) = x(P)$, $y(Q) > x(Q)$, $y(T) \geq x(T)$ for all $T \in \mathcal{A} \setminus \{P, Q\}$, $y(N) = x(N)$. If δ is sufficiently small then $y_i \geq 0$, $y(T) \leq \epsilon$ for all $T \in \mathcal{A} \setminus \{P\}$, hence

$$\sum_{T \in \mathcal{A}} (y(T) - v(T))^2 < \sum_{T \in \mathcal{A}} (x(T) - v(T))^2$$

and x does not belong to the least square solution of $(N, \mathcal{A}, 1, v)$. □

Proposition 7. *Let the nucleolus of $(N, \mathcal{A}, t, v)$ be contained in the weakly uniform losses solution of $(N, \mathcal{A}, t, v)$ for all $t > 0$, all v with $v(T) \geq 0$. Then the case $P, Q, S \in \mathcal{A}$, $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$ is impossible.*

Proof. Suppose that there exist $P, Q, S \in \mathcal{A}$ such that $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$. Let us take the following v: $v(S) = v(P) = 1$, $v(T) = 0$ for all $T \in \mathcal{A} \setminus \{S, P\}$. Let x belong to the nucleolus of $(N, \mathcal{A}, 1, v)$. Then $x(Q) = 0$ because otherwise $v(S) - x(S) \geq 0 > v(Q) - x(Q)$. If $x(P) = 0$ then $x(P) - v(P) = -1$ and x does not belong to nucleolus because if $z(P) = z(S) = 1/2$ then $\min_{T \in \mathcal{A}}(z(T) -$

$v(T)) \geq -1/2$. Hence $x(P) > 0$ and there exists $i_0 \in P \setminus Q$ such that $x_{i_0} > 0$. Let $j \in P \cap Q$.

If $i_0 \notin T$ for all $T \in \mathcal{A} \setminus \{P\}$ then we take $y \in R^{|N|}$: $y_{i_0} = 0$, $y_j = x_j + x_{i_0}$, $y_i = x_i$ otherwise. Then $y(P) = x(P)$, $y(Q) > x(Q)$, $y(T) \geq x(T)$ for all $T \in \mathcal{A}$, hence x does not belong to the nucleolus of $(N, \mathcal{A}, 1, v)$.

Let $i_0 \in T$ for some $T \in \mathcal{A} \setminus \{P\}$, then $T \neq S$, $v(T) = 0$, and $x(T) - v(T) > 0 = x(Q) - v(Q)$. Let $z = z(\delta) \in R^{|N|}$, $z_{i_0} = x_{i_0} - \delta$, $z_j = x_j + \delta$, $z_i = x_i$ otherwise. If $\delta > 0$ and δ is sufficiently small then $z(T) - v(T) > z(Q) - v(Q) > x(Q) - v(Q)$ for $i_0 \in T \in \mathcal{A} \setminus \{P\}$, otherwise $z(T) \geq x(T)$, hence $\theta(z(\delta)) >_{lex} \theta(x)$ and x does not belong to the nucleolus of $(N, \mathcal{A}, 1, v)$. □

Proposition 8. *Let the least square solution of $(N, \mathcal{A}, t, v)$ be contained in the weakly uniform losses solution of $(N, \mathcal{A}, t, v)$ for all $t > 0$, all v with $v(T) \geq 0$. Then the case $P, Q, S \in \mathcal{A}$, $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$ is impossible.*

Proof. Suppose that there exist $P, Q, S \in \mathcal{A}$ such that $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$. Let $i_0 \in P \cap Q$, $\mathcal{A}_0 = \{T \in \mathcal{A} : i_0 \in T, T \cap S \neq \emptyset\}$.

We take the following v:
$v(T) = 1$ for $T \in \mathcal{A}_0 \cup \{P, S\}$,
otherwise $v(T) = \epsilon$, where $0 < \epsilon < 1/(4|N|)$.

Let x belong to the least square solution of $(N, \mathcal{A}, 1, v)$. Since x is weakly uniform losses, for $S \cap T = \emptyset$ and $T \neq P$, either $x(T) = 0$ or $x(T) > 0$, $x(S) < 1$, and $v(T) - x(T) \geq v(S) - x(S) > 0$. As $v(T) = \epsilon$, $x(T) < \epsilon$. In particular, $x(Q) < \epsilon$. Consider 2 cases.

1. $x_i \leq \epsilon$ for all $i \in P$. Then $x(S \cup P) = x(S) + x(P) \leq 2\epsilon|P| < 1/2$. If $x_i \leq \epsilon$ for all $i \in N \setminus (P \cup S)$ then $x(N) < 3/4$, hence $x_{j_0} > \epsilon$ for some $j_0 \in N \setminus (P \cup S)$.

2. There exists $j_0 \in P$ with $x_{j_0} > \epsilon$. Then $j_0 \neq i_0$ because $x_{i_0} \leq x(Q) < \epsilon$.

Thus, there always exists $j_0 \in N$ such that $j_0 \notin S$, $j_0 \neq i_0$, $x_{j_0} > \epsilon$. Let $0 < \delta$. Let $y = y(\delta) \in R^{|N|}$, $y_{j_0} = x_{j_0} - \delta$, $y_{i_0} = x_{i_0} + \delta$, $y_i = x_i$ otherwise. For $T \in \mathcal{A}$, if either $i_0 \in T$, $j_0 \in T$ or $i_0 \notin T$, $j_0 \notin T$ then $y(T) = x(T)$.

Let $j_0 \in T$, $i_0 \notin T$. Then $T \notin \mathcal{A}_0 \cup \{P, S\}$, hence $v(T) = \epsilon$. Let $\delta < x_{j_0} - \epsilon$. Since $y(T) \geq y_{j_0} > \epsilon$, we have $x(T) > y(T) > v(T)$, then

$$(y(T) - v(T))^2 < (x(T) - v(T))^2.$$

Let $j_0 \notin T$, $i_0 \in T$. If $v(T) = \epsilon$ then $T \cap S = \emptyset$ and $x(T) < \epsilon$. Thus, for sufficiently small δ,

$$v(T) = \epsilon \geq x(T) + \delta = y(T) > x(T).$$

If $v(T) = 1$, then $1 \geq y(T) > x(T)$. Therefore in these cases $v(T) \geq y(T) > x(T)$, hence

$$(y(T) - v(T))^2 < (x(T) - v(T))^2.$$

Note that $j_0 \notin Q$ since $x(Q) < \epsilon < x_{j_0}$, $i_0 \in Q$, hence for sufficiently small δ,

$$\sum_{T \in \mathcal{A}} (y(T) - v(T))^2 < \sum_{T \in \mathcal{A}} (x(T) - v(T))^2$$

and x does not belong to the least square solution of $(N, \mathcal{A}, 1, v)$. □

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Games with Differently Directed Interests and Their Application to the Environmental Management

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Abstract. The resolution of numerous ecological problems on different levels should be realized on the base of sustainable development concept that determines the conditions to the state of environmental-economic systems and impacting control actions. Those conditions can't be realized by themselves and require special collaborative efforts of different agents using both cooperation and hierarchical control. To formalize the inevitable trade-offs it is natural to use game theoretic models including the games with differently directed interests.

Keywords: differential games, sustainable development, cooperation, dynamic stability, hierarchical control

JEL classification codes: C70, Q56

1. Introduction

A rapid growth of production and human population, urbanization and activity of transnational corporations exert an essential influence on the environment and result in many ecological problems on local, regional, and global level. Those problems are environmental pollution, water and wind erosion, acid rains, global warming, extinction, non-renewable resource exploitation, and so on. A successful solution of the problems is possible only within a sustainable development due to which an economic growth doesn't violate the ecological equilibrium. Sustainable development requires strong collaborative efforts of states, corporations, social organizations and individuals. Because of many concerned agents it is evident that sustainable development is a conflict controlled process which must be described by game theoretic models. A special attention has to be given to the classes of differential games describing hierarchical relations, cooperation and dynamic stability (time consistency).

A general survey of non-cooperative differential games is given by Basar and Olsder (1995), hierarchical games are considered by Fudenberg and Tirole (1991). Group optimal cooperative solutions in differential games are proposed in Dockner and Jorgensen (1984), Dockner and Long (1993), Tahvonen (1994), Maler and de Zeeuw (1998), Rubio and Casino (2002). Haurie and Zaccour (1986, 1991), Kaitala and Pohjola (1988, 1990, 1995), Kaitala et al (1995), Jorgensen and Zaccour (2001) presented classes of transferable-payoff cooperative games with solutions which satisfy group optimality and individual rationality. In Leitmann (1974, 1975), Tolwinsky et al (1986), Hamalainen et al (1986), Haurie and Pohjola (1987), Gao et al (1989), Haurie (1991), Haurie et al (1994) are presented solutions satisfying group optimality and individual rationality at the initial time in cooperative games with non-transferable payoffs. In some papers (see Hamalainen et al (1986), Tolwinsky et al

(1986)) threats are used to ensure that no players will deviate from the initial cooperative strategies throughout the game horizon. Environmental applications are considered in Dockner and Long (1993), Hamalainen et al (1986), Jorgensen and Zaccour (2001), Kaitala and Pohjola (1988, 1995), Kaitala et al (1995), Mazalov and Rettieva (2005, 2005, 2006, 2007), Yeung (2009) and others.
The problem of dynamic stability in differential games has been intensively explored in the past three decades (Yeung 2009). Haurie (1976) raised the problem of instability when the Nash bargaining solution is extended to differential games. Petrosyan (1977) formalized the notion of dynamic stability in differential games. Kidland and Prescott (1977) introduced the notion of time consistency related to economic problems. Petrosyan and Danilov (1979, 1982) proposed an "imputation distribution procedure" for cooperative solution. Petrosyan (1991, 1993b) studied the dynamic stability of optimality principles in non-zero sum cooperative differential games. Petrosyan (1993a) and Petrosyan and Zenkevich (1996, 2007) have presented a detailed analysis of dynamic stability in cooperative differential games, in which the method of regularization was used to construct time consistent solutions. Yeung and Petrosyan (2001) designed time consistent solutions in differential games and characterized the conditions that the allocation-distribution procedure must satisfy. Petrosyan and Zenkevich (2009) proposed the conditions of sustainable cooperation. Those and other results permit to propose a mathematical formalization of the concept of sustainable development on the base of a game theoretic model (Ougolnitsky 2002).

2. Games with differently directed interests

A general model of game with differently directed interests may be represented as follows (case $n = 2$ is considered for simplicity):

$$\begin{aligned} u_1(x_1,x_2) &= f(x_1,x_2) + g(x_1,x_2) + f_1(x_1,x_2) \to \max_{x_1 \in X_1}; \\ u_2(x_1,x_2) &= f(x_1,x_2) - g(x_1,x_2) + f_2(x_1,x_2) \to \max_{x_2 \in X_2}. \end{aligned} \tag{1}$$

where terms f describe coincident interests, terms g - antagonistic interests, and terms f_1 , f_2 - independent interests of the players. Specific cases of the model (1) are games with partly coincident interests

$$\begin{aligned} u_1(x_1,x_2) &= f(x_1,x_2) + f_1(x_1,x_2) \to \max_{x_1 \in X_1}; \\ u_2(x_1,x_2) &= f(x_1,x_2) + f_2(x_1,x_2) \to \max_{x_2 \in X_2}. \end{aligned} \tag{2}$$

and games with partly antagonistic interests

$$\begin{aligned} u_1(x_1,x_2) &= g(x_1,x_2) + f_1(x_1,x_2) \to \max_{x_1 \in X_1}; \\ u_2(x_1,x_2) &= -g(x_1,x_2) + f_2(x_1,x_2) \to \max_{x_2 \in X_2}. \end{aligned} \tag{3}$$

For example, a well-known Germeyer-Vatel model (1974)

$$\begin{aligned} u_1(x_1,x_2) &= f(a_1 - x_1, a_2 - x_2) + f_1(x_1) \to \max_{x_1 \in X_1}; \\ u_2(x_1,x_2) &= f(a_1 - x_1, a_2 - x_2) + f_2(x_2) \to \max_{x_2 \in X_2}. \end{aligned} \tag{4}$$

is a specific case of a game with partly coincident interests (2) for which an existence of strong equilibrium is proved.
There are many natural applications of models (1)-(3) to the management problems. Thus, a model of motivation (impulsion) management

$$\begin{aligned} u_1(x_1,x_2) &= x_1 f(x_2) + f_1(x_1,x_2) \to \max_{0 \le x_1 \le 1}; \\ u_2(x_1,x_2) &= (1-x_1) f(x_2) + f_2(x_1,x_2) \to \max_{0 \le x_2 \le 1}. \end{aligned} \tag{5}$$

belongs to the type (3). In our previous paper (Ougolnitsky 2010) a specific case of the model (2) was considered, namely a pricing model of hierarchical control of the sustainable development of the regional construction works complex

$$\begin{aligned} &u_L(\overline{p}, p_0, p) = \delta p \alpha(p) - M \rho(p, p_0) \to \max \\ &0 < p_0 \le \overline{p} \le p_{max} \\ &u_F(\overline{p}, p_0, p) = p\alpha(p) + p\xi(p)(1-\alpha(p)) \to \max \\ &0 \le p \le p_{max}. \end{aligned}$$

Here p is sales price; p_0 - normative price of social class residential real estate development; $\overline{p}$ - limit price of social class residential real estate development; $M >> 1$ - penalty constant; $\rho(p,p_0) = \{ \begin{matrix} 0, p \le p_0, \\ 1, p > p_0, \end{matrix}$; δ - Administration bonus parameter for social class residential real estate development sales; p_{max} - "overlimit" price of social class residential real estate development (there are no sales if $p > p_{max}$); $\alpha(p)$ is share of residential real estate development bought by Administration with warranty; $\xi(p)$ - share of another residential real estate development successfully sold by Developer without help.
There are many other examples of game theoretic models of the described type in management, such as models of resource allocation considering private interests, models of organizational monitoring optimization, models of reward systems, corruption models and so on. The games are considered as hierarchical ones.

3. Cooperative differential game of resource allocation with partly coincident interests on the base of compulsion

A dynamical model of hierarchical control by resource allocation in a tree-like system may be represented as follows:

$$f_0(x(t), u_1(\bullet), \dots, u_n(\bullet)) = \sum_{i \in M} \int_{t_0}^{T} g_i(x(t), u_i(t))dt \to \max;$$
$$0 \le q_i(t) \le 1; r_i(t) \ge 0, i \in M; \sum_{i \in M} r_i(t) = 1, t \in [t_0, T];$$
$$f_i(x(t), u_i(t)) = g_i(x(t), u_i(t)) + h_i(x(t), u_i(t)), i \in M, t \in [t_0, T];$$
$$q_i(t) \le u_i(t) \le r_i(t), i \in M, t \in [t_0, T];$$
$$\frac{dx}{dt} = F(x(t), u_1(t), \dots, u_n(t)), x(0) = x_0.$$

The tree-like hierarchical structure consists of $n+1$ elements: one element of the higher level (Leader) designated by index 0 and n elements of the lower level (Followers). Let's denote the whole set of elements by $N = \{0, 1, \dots, n\}$, and the set of elements on lower level by $M = \{1, \dots, n\}$. The Leader controls the Followers separately by control variables of compulsion q_i (administrative impacts) and control variables r_i of impulsion (resources) (Ougolnitsky 2002). Without lack of generality we may consider the total resource equal to one. After receiving the values of q_i and r_i each Follower $i \in M$ chooses the control value u_i (environmental protection efforts). The objective of Leader is to maximize the payoff function f_0, and the objective of each Follower is to maximize f_i . We suppose that g_i represents the ecological interests for the whole system, and h_i represents private economic interests of the i-th Follower; functions g_i are non-negative, continuous, differentiable, monotonically increase on u_i , $g_i(x(t), 0) = 0$; functions h_i are non-negative, continuous, differentiable, monotonically decrease on u_i , $h_i(x(t), r_i(t)) = 0$ for each $t \in [t_0, T]$.

To define a cooperative differential game with initial state x_0 and time horizon $[t_0, T]$ on the base of compulsion it is sufficient to build a characteristic function $v : 2^N \to R$ using the definition of compulsion (Ougolnitsky 2002). In this case the values of r_i are fixed, and the Leader chooses q_i , $i \in M$, as open-loop strategies $q_i(\cdot) = \{q_i(t)\}$, $t \in [t_0, T]$. We have

$$\nu(\{0\}; x_0, T - t_0) =$$
$$= \max_{0 \le q_i(\bullet) \le r_i(\bullet), i \in M} \; \min_{u(t) \in R_i(q_i(\bullet), r_i(\bullet)), i \in M} \; \sum_{i \in M} \int_{t_0}^{T} g_i(x(t), u_i(t))dt =$$
$$= \sum_{i \in M} \int_{t_0}^{T} g_i(x(t), r_i(t))dt;$$
$$R_i(q_i(t), r_i(t)) = Arg \max_{q_i(\bullet) \le u_i(t) \le r_i(\bullet)} f_i(x(t), u_i(t)), \;\; i \in M, t \in [t_0, T];$$

the compulsion mechanism has a form $q_i(t) = r_i(t) \Rightarrow u_i^*(t) = r_i(t)$, $i \in M$, where $u_i^*(t)$ is an optimal reaction of i-th Follower on $q_i(t)$, $t \in [t_0, T]$. Respectively,

$$\nu(\{i\}; x_0, T-t_0) = \int_{t_0}^{T} g_i(x(t), r_i(t))dt + \int_{t_0}^{T} h_i(x(t), r_i(t))dt =$$
$$= \int_{t_0}^{T} g_i(x(t), r_i(t))dt, i \in M;$$
$$\nu(K; x_0, T-t_0) = \sum_{i\in K} \int_{t_0}^{T} f_i(x(t), r_i(t))dt = \sum_{i\in K} \int_{t_0}^{T} g_i(x(t), r_i(t))dt, K \subseteq M.$$

So, if Leader has full possibilities of compulsion, he can compel all Followers to follow the ecological interests only. Farther we get

$$\nu(\{0\} \cup K; x_0, T-t_0) =$$
$$= \max_{0\le q_i(\bullet)\le r_i(\bullet), i\in M} \max_{q_i(\bullet)\le u_i(t)\le r_i(\bullet), i\in K} \min_{q_i(\bullet)\le u_i(t)\le r_i(\bullet), i\in M\setminus K} [\sum_{i\in M} \int_{t_0}^{T} g_i(x(t), u_i(t))dt$$
$$+ \sum_{i\in K} \int_{t_0}^{T} g_i(x(t), u_i(t))dt + \sum_{i\in K} \int_{t_0}^{T} h_i(x(t), u_i(t))dt] =$$
$$= \max_{0\le q_i(\bullet)\le r_i(\bullet), i\in M} \max_{q_i(\bullet)\le u_i(t)\le r_i(\bullet), i\in K} \min_{q_i(\bullet)\le u_i(t)\le r_i(\bullet), i\in M\setminus K} [\sum_{i\in K} (2 \int_{t_0}^{T} g_i(x(t), u_i(t))dt$$
$$+ \int_{t_0}^{T} h_i(x(t), u_i(t))dt) + \sum_{i\in M\setminus K} \int_{t_0}^{T} g_i(x(t), u_i(t))dt] =$$
$$= \sum_{i\in K} \int_{t_0}^{T} (2g_i(x(t), u_i^*(t))dt + h_i(x(t), u_i^*(t)))dt +$$
$$+ \sum_{i\in M\setminus K} \int_{t_0}^{T} g_i(x(t), r_i(t))dt,$$

where $\sum_{i\in K} \int_{t_0}^{T} (2g_i(u_i^*(t)) + h_i(u_i^*(t)))dt =$

$$= \max_{0\le q_i(\bullet)\le r_i(\bullet), i\in K} \max_{q_i(\bullet)\le u_i(\bullet)\le r_i(\bullet), i\in K} \sum_{i\in K} \int_{t_0}^{T} (2g_i(x(t), u_i(t)) + h_i(x(t), u_i(t)))dt,$$

the mechanism of compulsion is $q_i^*(\bullet) = \{ \begin{matrix} u_i^*(\bullet), \ i \in K, \\ r_i(\bullet), \ i \in M \setminus K. \end{matrix}$ At last,

$$\nu(N; x_0, T-t_0) = \max_{0\le q_i(\bullet)\le r_i(\bullet), i\in M} \max_{q_i(\bullet)\le u_i(t)\le r_i(\bullet), i\in M} \sum_{i\in M} \int_{t_0}^{T} [2g_i(x(t), u_i(t)) +$$
$$+ h_i(x(t), u_i(t))]dt = \sum_{i\in M} \int_{t_0}^{T} [2g_i(x(t), u_i^*(t))dt + h_i(x(t), u_i^*(t))]dt.$$

So, if the Leader forms a coalition with Followers, he begins to consider their private interests, and the point of maximum shifts.

Lemma 1. *Function v is superadditive.*

Proof (of lemma). It is sufficient to consider three cases:

1. $\forall K, L \subseteq M (K \cap L = \emptyset)$

$$\nu(K; x_0, T - t_0) + \nu(L; x_0, T - t_0) = \sum_{i \in K} \int_{t_0}^{T} g_i(x(t), r_i(t))dt +$$
$$+ \sum_{i \in L} \int_{t_0}^{T} g_i(x(t), r_i(t))dt = \sum_{i \in K \cup L} \int_{t_0}^{T} g_i(x(t), r_i(t))dt = \nu(K \cup L; x_0, T - t_0).$$

2. $\forall K \subseteq M$

$$\nu(\{0\} \cup K; x_0, T - t_0) - \nu(\{0\}; x_0, T - t_0) - \nu(K; x_0, T - t_0) =$$
$$= \sum_{i \in K} \int_{t_0}^{T} [2g_i(x(t), u_i^*(t)) + h_i(x(t), u_i^*(t))]dt + \sum_{i \in M \setminus K} \int_{t_0}^{T} g_i(x(t), r_i(t))dt -$$
$$- \sum_{i \in M} \int_{t_0}^{T} g_i(x(t), r_i(t))dt - \sum_{i \in K} \int_{t_0}^{T} g_i(x(t), r_i(t))dt =$$
$$= \sum_{i \in K} \int_{t_0}^{T} [2g_i(x(t), u_i^*(t)) + h_i(x(t), u_i^*(t)) - 2g_i(x(t), r_i(t))]dt \geq 0;$$

3. for $\forall K, L \subseteq M (K \cap L = \emptyset)$ as far $M \setminus K = M \setminus (K \cup L) \cup L$ we get

$$\nu(\{0\} \cup K \cup L; x_0, T - t_0) - \nu(\{0\} \cup K; x_0, T - t_0) - \nu(L; x_0, T - t_0) =$$
$$= \sum_{i \in K \cup L} \int_{t_0}^{T} [2g_i(x(t), u_i^*(t)) + h_i(x(t), u_i^*(t))]dt +$$
$$+ \sum_{i \in M \setminus (K \cup L)} \int_{t_0}^{T} g_i(x(t), r_i(t))dt -$$
$$- \sum_{i \in K} \int_{t_0}^{T} [2g_i(x(t), u_i^*(t)) + h_i(x(t), u_i^*(t))] - \sum_{i \in M \setminus K} \int_{t_0}^{T} g_i(x(t), r_i(t))dt -$$
$$- \sum_{i \in L} \int_{t_0}^{T} g_i(x(t), r_i(t))dt =$$
$$= \sum_{i \in L} \int_{t_0}^{T} [2g_i(x(t), u_i^*(t)) + h_i(x(t), u_i^*(t)) - 2g_i(x(t), r_i(t))]dt \geq 0;$$

Thus, the function v is superadditive and generates a cooperative differential game on the base of compulsion $\Gamma_\nu = \langle N; \nu; x_0, T - t_0 \rangle = \Gamma_\nu(x_0, T - t_0)$ Let's remind that the set of non-dominated imputations in the game $\Gamma_\nu(x_0, T - t_0)$ is called C-core and denoted as $C_\nu(x_0, T - t_0)$, and the Shapley value $\Phi^\nu(x_0, T - t_0)$ is determined by formulas

$$\Phi_i^\nu(x_0, T - t_0) = \sum_{K \subset N (i \in K)} \gamma(k)[\nu(K; x_0, T - t_0) - \nu(K \setminus \{i\}; x_0, T - t_0)], \quad (6)$$

$i = 1, \ldots, n,$

$$\gamma(k) = \frac{(n - k)!(k - 1)!}{n!}, k = |K|, n = |N|. \quad (7)$$

□

Theorem 1. *In the game $\Gamma_\nu(x_0, T-t_0)$ it is true that $\Phi^\nu(x_0, T-t_0) \in C_\nu(x_0, T-t_0)$.*

Proof (of theorem). Let's calculate the components of Shapley value subject to (6):

$$\begin{aligned}
&\Phi^\nu_0(x_0, T-t_0) = \gamma(1)\nu(\{0\}; x_0, T-t_0) + \\
&+\gamma(2) \sum_{\{0\}\in K, |K|=2} [\nu(K; x_0, T-t_0) - \nu(K \setminus \{0\}; x_0, T-t_0)] + \cdots + \\
&+\gamma(n) \sum_{\{0\}\in K, |K|=n} [\nu(K; x_0, T-t_0) - \nu(K \setminus \{0\}; x_0, T-t_0)] + \\
&+\gamma(n+1)[\nu(N; x_0, T-t_0) - \nu(M; x_0, T-t_0)] = \\
&= \gamma(1) \sum_{i\in M} \int_{t_0}^{T} g_i(x(t), r_i(t))dt + \gamma(2) \sum_{\{0\}\in K, |K|=2} \int_{t_0}^{T} [\sum_{i\in K} (2g_i(x(t), u_i^*(t)) + \\
+\ & h_i(x(t), u_i^*(t))) + \sum_{i\in M\setminus K} g_i(x(t), r_i(t)) - \sum_{i\in K} g_i(x(t), r_i(t))]dt + \cdots + \\
&+\gamma(n) \sum_{\{0\}\in K, |K|=n} \int_{t_0}^{T} [\sum_{i\in K} (2g_i(x(t), u_i^*(t)) + h_i(x(t), u_i^*(t))) + \\
&+ \sum_{i\in M\setminus K} g_i(x(t), r_i(t)) - \sum_{i\in K} g_i(x(t), r_i(t))]dt + \\
&+\gamma(n+1)[\sum_{i\in M} \int_{t_0}^{T} (2g_i(x(t), u_i^*(t)) + h_i(x(t), u_i^*(t)))dt + \sum_{i\in M} \int_{t_0}^{T} g_i(x(t), r_i(t))dt] \\
&= \gamma(2) \sum_{\{0\}\in K, |K|=2} \sum_{i\in K} \int_{t_0}^{T} (2g_i(x(t), u_i^*(t)) + h_i(x(t), u_i^*(t)))dt + \\
&+\gamma(3) \sum_{\{0\}\in K, |K|=3} \sum_{i\in K} \int_{t_0}^{T} (2g_i(x(t), u_i^*(t)) + h_i(x(t), u_i^*(t)))dt + \cdots + \\
&+\gamma(n) \sum_{\{0\}\in K, |K|=n} \sum_{i\in K} \int_{t_0}^{T} (2g_i(x(t), u_i^*(t)) + h_i(x(t), u_i^*(t)))dt + \\
&+\gamma(n+1) \sum_{i\in N} \int_{t_0}^{T} (2g_i(x(t), u_i^*(t)) + h_i(x(t), u_i^*(t)))dt] = \\
&= \sum_{s=2}^{n+1} \gamma(s) \sum_{\{0\}\in K, |K|=s} \sum_{i\in K} \int_{t_0}^{T} A_i(t)dt, \\
&A_i(t) = 2g_i(x(t), u_i^*(t)) + h_i(x(t), u_i^*(t)).
\end{aligned}$$

As each of n players of lower level takes part C_{n-1}^{s-1} times in the set of coalitions of s players than

$\Phi_0^\nu(x_0,T-t_0)=\sum_{s=2}^{n+1}\gamma(s)C_{n-1}^{s-2}\sum_{i\in M}\int_{t_0}^{T}A_i(t)dt=0,5\sum_{i\in M}\int_{t_0}^{T}A_i(t)dt$. Thus,

$$\Phi_0^\nu(x_0,T-t_0)=\sum_{i\in M}\int_{t_0}^{T}(g_i(x(t),u_i^*(t))+0,5h_i(x(t),u_i^*(t)))dt=0,5\nu(N;x_0,T-t_0)$$

As far Shapley value is Pareto-optimal we get

$$\sum_{i\in M}\Phi_i^\nu(x_0,T-t_0)=0,5\nu(N;x_0,T-t_0)=\Phi_0^\nu(x_0,T-t_0)$$

and as far all Followers are completely symmetrical we get

$$\Phi_i^\nu(x_0,T-t_0)=\int_{t_0}^{T}(g_i(x(t),u_i^*(t))+0,5h_i(x(t),u_i^*(t)))dt, i\in M.$$

To complete the proof it is sufficient to verify directly the inequalities of three types:

1. $\sum_{i\in K}\Phi_i^\nu(x_0,T-t_0)+\sum_{j\in L}\Phi_j^\nu(x_0,T-t_0)\geq\nu(K\cup L;x_0,T-t_0)$;
2. $\Phi_0^\nu(x_0,T-t_0)+\sum_{i\in K}\Phi_i^\nu(x_0,T-t_0)\geq\nu(\{0\}\cup K;x_0,T-t_0)$;
3. $\Phi_0^\nu(x_0,T-t_0)+\sum_{i\in K}\Phi_i^\nu(x_0,T-t_0)+\sum_{j\in L}\Phi_j^\nu(x_0,T-t_0)\geq\nu(\{0\}\cup(K\cup L))$,
   $K\cap L=\emptyset, K,L\subseteq M$. □

A cooperative game on the base of impulsion is built similarly. Unfortunately, the optimality principle $\Phi^\nu(x_0,T-t_0)\in C_\nu(x_0,T-t_0)$ is not dynamically stable (time consistent). To provide the dynamic stability a regularization procedure (payoff distribution procedure) has to be used (Petrosyan and Zenkevich 2007). Define

$$\Phi_i^\nu(x_0,T-t_0)=\int_{t_0}^{T}B_i(s)ds, B_i(t)\geq0,\sum_{i\in N}B_i(t)=1, t\in[t_0,T].$$

The quantity $B_i(t)=\dfrac{d\phi_i^\nu(x_0,T-t_0)}{dt}$ is the instantaneous payoff to the player i at the moment t. The vector $B(t)=(B_0(t),B_1(t),\ldots,B_n(t))$ determines a distribution of the total gain among all players. By the proper choice of $B(t)$ it is possible to ensure that at each instant $t\in[t_0,T]$ there will be no objections against realization of the original agreement $\Phi^\nu(x_0,T-t_0)$, i.e. the imputation $\Phi^\nu(x_0,T-t_0)$ is time consistent. It is proved under general conditions that the regularization procedure $B(t)$, $t\in[t_0,T]$, leading to the time consistent cooperative solution, exists and is realizable (Petrosyan and Zenkevich 1996).

4. Conclusion

The resolution of numerous ecological problems on different levels must be implemented on the base of sustainable development concept that determines the conditions to the state of environmental-economic systems and impacting control actions. Those conditions can't be realized by themselves and require special collaborative efforts of different agents using both cooperation and hierarchical control. To formalize the inevitable trade-offs it is natural to use game theoretic models including the games with differently directed interests. Unfortunately, the main optimality principles of hierarchical control (compulsion, impulsion) are not dynamically stable and therefore can't be recommended as the direct base for collective solutions. The most prospective is the conviction method which is formalized as a transition from hierarchy to cooperation and allows a regularization that provides the dynamic stability. However, in current social conditions other methods of hierarchical control also keep their actuality. To provide the dynamic stability of those optimality principles it is necessary to build cooperative differential games on their base. An example of the approach is considered in this paper. The following directions of the game theoretic modeling of the concept of sustainable development are of specific interest: investigation of general game theoretic models of the hierarchical control of sustainable development (Ugolnitsky 2002a, 2005; Ougolnitsky 2009); modeling of the hierarchical control of sustainable development of the ecological-economic systems (Ugolnitsky 1999; Ugolnitsky and Usov 2007a, 2009; Ougolnitsky 2002; Ougolnitsky and Usov 2009); modeling of the hierarchical control of sustainable development of the systems of another types (Ugolnitsky 2002b); modeling of the multilevel hierarchical systems of different structure (Ugolnitsky and Usov 2007b, 2010); modeling of the corruption in hierarchical control systems (Rybasov and Ugolnitsky 2004; Denin and Ugolnitsky 2010); development of the information-analytical systems of decision support in the hierarchically controlled dynamical systems (Ugolnitsky and Usov 2008).

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Memento Ludi: Information Retrieval from a Game-Theoretic Perspective*

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Abstract. We develop a macro-model of information retrieval process using Game Theory as a mathematical theory of conflicts. We represent the participants of the Information Retrieval process as a game of two abstract players. The first player is the 'intellectual crowd' of users of search engines, the second is a community of information retrieval systems. In order to apply Game Theory, we treat search log data as Nash equilibrium strategies and solve the inverse problem of finding appropriate payoff functions. For that, we suggest a particular model, which we call Alpha model. Within this model, we suggest a method, called shifting, which makes it possible to partially control the behavior of massive users.
The paper is asddressed to researchers in both game theory (providing a new class of real life problems) and information retrieval, for whom we present new techniques to control the IR environment.

Introduction

The techniques we present are inspired by the success of macro-approach in both natural and social science. In thermodynamics, starting from a chaotic motion of billions of billions of microparticles, we arrive a simple transparent strongly predictive theory with few macro-variables, such as temperature, pressure, and so on. In models of market behavior the chaotic motion is present as well, but there are two definite parties, each consisting of a big number of individuals with common interests, whose behavior is not concorded.

From a global perspective, information retrieval looks similar: there are many individual seekers of knowledge, on one side, and a number of knowledge providers, on the other: each are both chaotic and non-concorded. There are two definite parties, whose members have similar interests, and every member of each party tends to maximally fulfill his own interests. How could a Mathematician help them? At first sight, each party could be suggested to solve a profit *maximization* problem. But back in 1928 it was J. von Neumann who realized this approach to be inadequate: you can not maximize the value you do not know (Von Neumann, 1928). In fact, the profit gained by each agent depends *not only on its* actions, but also on the activities

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of its counterpart, which are not known. Then the game theory was developed replacing the notion of optimality by that of acceptability. Similarly, the crucial point of information retrieval, in contrast to data retrieval, is to get some satisfaction (feeling of relevance) rather than retrieve something exact. The analogy

$$\boxed{\begin{array}{ll}\textbf{Data Retrieval} & \longrightarrow \text{matching} \\ \textbf{Information Retrieval} & \longrightarrow \text{relevance}\end{array}} \simeq \boxed{\begin{array}{ll}\textbf{Optimization} & \longrightarrow \text{maximum} \\ \textbf{Game Theory} & \longrightarrow \text{equilibrium}\end{array}}$$

was a starting point for us to explore applications of game theory to the problems of information retrieval.

The standard problem of game theory is seeking for reasonable (in various senses) strategies. When the rules of the game are given, there is a vast machinery, which makes it possible to calculate such strategies. In information retrieval we have two parties whose interaction is of exactly game nature, but the rules of this game are not explicitly formulated. However, we may observe the consequence of these rules as users behavior, that is, we deal with the inverse problem of game theory, studied by Dragan (Dragan, 1966) for cooperative games. In this paper we expand it to non-cooperative case.It turns out that the solution of the inverse problem is essentially non-unique: different rules can produce the same behavior. We suggest a particular class of models, called Alpha models describing an idealized search system similar to Wolfram Alpha engine.

What can search engine managers benefit of these techniques? Game theory can work out definite recommendations how to control the interaction between the parties of the information retrieval process. This sounds unrealistic: can one control massive chaotic behavior? Thermodynamics shows us that the answer is yes. We can not control individual molecules, but in order to alter their collective behavior we are able to change macroparameters: the engine of your car reminds it to you. In our case the payoff functions of the Alpha model are just those parameters.

In Section 1. we introduce (only the necessary) basic notion from game theory, in Section 2. we formulate the information retrieval process in terms of game theory and formulate our method as the inverse problem in game theory. In Section 3. we suggest its particular solution, which we call Alpha model as it resembles Wolfram Alpha engine and in Section 4. we suggest a method to control massive users' behavior.

1. Direct problem: classical game theory

Game theory is a mathematical theory studying conflicts and trade-offs. It involves rational participants who follow formal rules. A game is specified by its players, players' strategies and players' payoffs. Begin with a well-known example (a reformulated Prisonners' dilemma (Tucker, 1950)).

There are two players A and B. The player A can choose color: **Red** or **Green** , while B chooses direction: **Left** or **Right** . The rules of the game are specified by the following pair of *payoff matrices* (Table 1)

The Mathematician can predict the outcome of this game provided the players are rational, namely, wishing to gain more: the rational player A will necessarily choose **Red** and B will choose **Left** .

However, both players know the payoff matrices, so, being rational, why can't they *agree* for A to choose **Green** and for B to choose **Right** ? The point is that

	Left	**Right**
Red	10	25
Green	5	20

The gain of A

	Left	**Right**
Red	11	4
Green	23	17

The gain of B

Table 1. A game with domination, defined by its pair of payoff matrices having the following meaning: if A chooses **Green** and B chooses **Right** , A gains 20 and B gains 17, and so on.

they are acting independently, which exclude any agreement. This kind of games are called *non-cooperative* and this is the case for the IR community.

The peculiarity of the above mentioned example is that it has a unique (and therefore straightforward) solution. However, such kind of examples does not describe the generic situation. Now let us consider a more general example (Table 2).

	Left	**Right**
Red	10	20
Green	5	25

The gain of A

	Left	**Right**
Red	11	4
Green	17	23

The gain of B

Table 2. A non-dominating case: two Nash equilibria.

First note that no player has a dominating strategy here, so the outcome of the game is at first glance unpredictable. However the Mathematician predicts us the outcome of this game as well. First, we see that both (**Green** , **Left**) and (**Red** , **Right**) will not[1] be realized by rational players. One of the following two pairs (just according to the maritime Rules of the Road) will necessary occur: (**Red** , **Left**) or (**Green** , **Right**). Why so? The motivation for a rational player to be abide of certain strategy is that leaving it *unilaterally* reduces his gain:

$$\begin{cases} H_A(\mathbf{Red}\,, \mathbf{Left}\,) \geqslant H_A(\alpha, \mathbf{Left}\,) \\ H_B(\mathbf{Red}\,, \mathbf{Left}\,) \geqslant H_B(\mathbf{Left}\,, \beta) \end{cases} \tag{1}$$

where $H_A(\alpha, \beta)$ $(H_B((\alpha, \beta)$, resp.) is the gain of A (B, resp.) when A chooses strategy α and B chooses β. The relations (1) are the famous Nash inequalities. A pair of strategies is said to form the *Nash equilibrium*, if they satisfy these inequalities. In the above example the pair of strategies (**Red** ,**Left**) is Nash equilibrium, but so is the pair (**Green** ,**Right**) as well! So, what will be the Mathematician's prediction for the outcome of this game? He will point out what will not occur and what will take place stably.

Now let us pass to the next example (Table 3), which is generic.

We see that there is no equilibrium pairs of strategies in this game, that is, if the players are represented by individuals, the outcome of an instance of the game

[1] How it works: suppose A chooses **Green** , observes that he gains only 5 and then switches to **Red** , which brings him 10.

	Left	Right
Red	10	20
Green	5	25

The gain of A

	Left	Right
Red	4	11
Green	23	17

The gain of B

Table 3. No Nash equilibria.

can not be predicted. What can the Mathematician tell us now? He will suggest to consider players represented by communities. A choice of the strategy by the collective player A is described by the distribution of the individuals with respect to the strategies they choose:

$$\begin{cases} \boldsymbol{p} = (p_{\mathbf{Red}}\ ,\ p_{\mathbf{Green}}\ ) \\ \boldsymbol{q} = (q_{\mathbf{Left}}\ ,\ q_{\mathbf{Right}}\ ) \end{cases} \tag{2}$$

The gain of the collective players with respect to the chosen pair of strategies is the average:

$$\begin{cases} H_A(\boldsymbol{p},\boldsymbol{q}) = \sum a_{jk} p_j q_k \\ H_B(\boldsymbol{p},\boldsymbol{q}) = \sum b_{jk} p_j q_k \end{cases} \tag{3}$$

where $[a_{jk}]$, $[b_{jk}]$ are the payoff matrices for the players A and B, respectively.

The prediction of the outcome of the game is now a pair of distributions $(\boldsymbol{p_*},\boldsymbol{q_*})$ obtained from the same Nash inequalities (3), but referred now to averages.

$$\begin{cases} H_A(\boldsymbol{p_*},\boldsymbol{q_*}) \geqslant H_A(\boldsymbol{p},\boldsymbol{q_*}) \\ H_B(\boldsymbol{p_*},\boldsymbol{q_*}) \geqslant H_B(\boldsymbol{p_*},\boldsymbol{q}) \end{cases} \tag{4}$$

The fundamental result of game theory is Nash theorem (Owen, 1995), which states that the equilibrium in the sense of (4) always exist. Moreover, when the number of players is two, the answer can be written explicitly:

$$\begin{cases} p_1 = \frac{b_{22}-b_{21}}{b_{11}+b_{22}-b_{12}-b_{21}}\ ;\ p_2 = 1 - p_1 \\ q_1 = \frac{a_{22}-b_{12}}{a_{11}+a_{22}-a_{12}-a_{21}}\ ;\ q_2 = 1 - q_1 \end{cases} \tag{5}$$

Note that the behavior of the player A is completely determined *only* by the payoff matrix of the player B and *vice versa*.

2. Crowd Meets Crowd – Inverse Problem

In this section we describe our IR macromodel as a non-antagonistic conflict of two parties, or, other words, a cooperative game of two players. The first player, call it A, asks questions, the second, call it B, provides answers. The player A stands for the community of users (intellectual crowd) of IR systems, the player B stand for the community of providers of search results (which is symmetrically treated as intellectual crowd).

Each particular strategy α_j of the player A is just typing something in a search-box. Each particular strategy β_k of the player B is to return a page with an answer, which, viewed as, say, HTML code, is a string of symbols as well. An instance of the game is a pair

$$\alpha_j\beta_k = (\texttt{input-string,returned-string})$$

which is somehow evaluated by each participant. For example, the payoff value $H^A(\alpha_j\beta_k)$ for the player A for the pair

$$\alpha_j\beta_k = (\text{`}\texttt{accommodation}\text{'},\text{'}\texttt{No results found}\text{'})$$

is evidently low. In the meantime we do not dare to ascribe any payoff value $H^B(\alpha_j\beta_k)$ of this instance for the player B (we do not know providers' priorities). In more general situations even the evaluations of the player A is not known as well.

However, *numerical* payoff values are needed in order to apply game theory: its basic concept — of Nash equilibrium — is based on comparison of instances (4). As a matter of fact, the participants of IR process do compare instances, but they do it qualitatively. But the Mathematician needs numbers! What data should he proceed in order to get them?

Stability and equilibrium. In a sense this week's World Wide Web is the same as it was a week ago, whatever be the variety of different queries and answers. What is *stable in time* is the statistics of instances $\alpha_j\beta_k$: things frequently asked yesterday repeat today. The Mathematician tells us that from a game-theoretic perspective this stability is not surprising: these are Nash equilibria which are stable, because leaving them is unfavorable.

If we had known the payoff functions, we could find the Nash equilibrium. But in our situation we know the equilibrium (statistics of instances) and we have to find the appropriate payoff functions $H^A(\alpha_j\beta_k)$, $H^B(\alpha_j\beta_k)$ in (3). This is the *inverse problem* in game theory (Dragan, 1966). The inverse problem has multiple solutions: for given frequencies there are many different payoff matrices yielding the same equilibrium[2]. Below, we introduce a specific model, called Alpha model with the smallest number of free parameters.

3. Alpha model

The raw material for us will be a collection of search strings with appropriate frequencies and a collection of returned results with appropriate frequencies as well. According to our model, we interpret it as realized equilibrium. Now we are about to reconstruct the payoff functions. First, according to the remark made above, we assume that the number of different strategies for both players is the same. If not, we may reach it by appropriate prepocessing of data, identifying some data strings.

Note that, given a pair of strategies $(\boldsymbol{p}, \boldsymbol{q})$, there are (infinitely) many different payoff functions, for which this pair of strategies is equilibrium. Among all such models, we consider the simplest one, closest to data retrieval. For this model, the payoff matrices are diagonal:

$$A = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_n \end{pmatrix} ; \qquad B = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_n \end{pmatrix} \tag{6}$$

where $a_1, \dots, a_n; b_1, \dots, b_n$ are positive numbers.

[2] A trivial example of such non-uniqueness is multiplying the payoff matrix by a positive number.

This feature of this model is that the only valuable answer for question α_j is β_j with the same index j, other answers β_k for $k \neq j$ are of zero value. This looks like Wolfram Alpha search engine, which provides the only answer to a query, that is why we call our model Alpha.

The Nash equilibrium for the game is given by:

$$p_j = \frac{b_j^{-1}}{b_1^{-1} + \cdots + b_n^{-1}} \; ; \qquad q_k = \frac{a_k^{-1}}{a_1^{-1} + \cdots + a_n^{-1}} \tag{7}$$

We can check this directly checking Nash inequalities (4). It is sufficient (Owen, 1995) to check it only for pure strategies

$$\leq \tag{8}$$

Recall that we have the inverse problem, that is, we know $(\boldsymbol{p}, \boldsymbol{q})$. Its solution is

$$a_j = \frac{a}{q_j} \; ; \qquad b_k = \frac{b}{p_k} \tag{9}$$

for any fixed positive numbers a, b. The obtained result shows us that:

- The less frequent is a instance, the higher is its value.
- The value of a question is determined by the frequency of the reply, and *vice versa*, the value of a reply is determined by the frequency of the question.

The first statement means that within this model frequently asked questions have low value for the provider B, and, *vice versa*, rarely delivered answers are of high value for the user A.

The magic of Nash theory is captured in the second statement. It means that the behavior of player A is completely determined only by the payoff matrix of player B. In other words, the popularity (=frequency) of users' questions depends on priorities of the answering side rather than on their own priorities.

4. Shifting of users' behavior

So far, we have suggested a quantitative model of IR process. The aim of this model is not just to describe, but also to give some means of control to the overall process. There are two parties involved, each having its own interests. Let us consider what could the provider B do in order to increase its gain.

At first sight, the strategy $\boldsymbol{q}$ should be changed, but the power of Nash theory is that the answer is immediate: it does not make sense, any unilateral deviation from the equilibrium is unfavorable for B. The player B can not directly, by ordering, control the strategy $\boldsymbol{p}$ of player A, nor its payoff matrix. So, the only thing B can do is to *change its own interests*: what remains under control of B, is its own payoff matrix. How it works?

A simple suggestion is to multiply all the elements of B by, say, 1957. This suggestion does not affect, as it follows from (7), the strategy of player A: it is similar to recalculating your wealth from euro to Italian liras: you may feel happy, but your wealth will not grow. So far, we have to accept a normalization condition for the bonuses b_k of B in order to make them scale-invariant. Let us suppose their total amount $\mathfrak{B}$ to be fixed:

$$\sum_k b_k = \mathfrak{B} = \text{const} \tag{10}$$

As it was shown in previous section, the strategy of A depends *only* on the payoffs of B. Hence, changing the matrix B will affect the behavior of its counterpart A. Furthermore, the statistics of instances will change and, therefore, the average gain of B will change. Let us first calculate how the average gain H^B of B depends on the parameters of its payoff matrix (6):

$$H^B(\boldsymbol{p}, \boldsymbol{q}) = \sum_j b_j p_j q_j \tag{11}$$

For any strategies p_j, q_j. Within our model we know, however, that in equilibrium $p_j = \frac{b}{b_j}$ (9), therefore the *optimal* average gain is:

$$H^B(\boldsymbol{p}, \boldsymbol{q}) = \sum_j b q_j = b \tag{12}$$

The value of the multiple b can now be derived from (9) and the condition $\sum p_k = 1$, therefore the optimal gain of the player B reads:

$$H^B = \left(\sum_j b_j^{-1} \right)^{-1} \tag{13}$$

Now let us explore how the optimal gain H^B changes under small variations db_k of the parameters of the Alpha model. It follows from the normalization condition (10) that

$$\sum \delta b_k = 0 \tag{14}$$

and calculate the gradient of the optimal gain H^B:

$$\nabla_k H^B = -\left(\sum_j b_j^{-1} \right)^{-2} \cdot \left(-\frac{1}{b_k^2} \right) = b^2 \cdot \frac{1}{b_k^2} = p_k^2 \tag{15}$$

The variations δb_k are obtained from the gradient $\nabla_k H^B$ by requiring the conditions (14) to be satisfied:

$$\delta b_k = p_k^2 - \frac{1}{n} \sum_j p_j^2 \tag{16}$$

which is unnormalized Yule's characteristic (Yule, 1944), reflecting the diversity of the variety of queries.

The shifting. Now suppose we are in a position to make small changes, of the magnitude ε, of the payoff function of the Alpha Provider. How should we apply them in order to make the gain of B maximally increase? The answer is given by the formula (16), according to which the Alpha Provider has to do the following:

- Find out the relative frequencies p_k of users queries α_k.
- Calculate the average of their squares $w = \frac{1}{n}\sum p_k^2$
- Slightly re-evaluate the instances placing more bonuses on queries, whose frequencies are above the threshold value w, taking them from rarely asked questions, whose frequencies are below w.

As a result, the equilibrium will shift, the frequencies of users' requests will adjust accordingly and the Alpha Provider will increase his gain, as it follows from (11) by

$$\delta b \;=\; \varepsilon \sum_j \delta b_j \, q_j \tag{17}$$

Conclusions

So far, we have described the process of Information retrieval as a non-antagonistic conflict between two parties: Users and providers. The mathematical model of such conflict is a bimatrix cooperative game. Starting from the assumption that *de facto* search log statistics is the Nash equilibrium of certain game, we provide a method of calculating the parameters (9) of this game, thus solving the appropriate inverse problem.

A significant, somewhat counter-intuitive consequence of Nash theory is that in this class of games the equilibrium, *i. e.* stable, behavior of the User is completely determined only by the distribution of priorities of the Provider. From this, we infer suggestions for the provider how to affect the behavior of massive User.

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The Fixed Point Method Versus the KKM Method

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Abstract. In this survey, we compare the fixed point method and the KKM method in nonlinear analysis. Especially, we consider two methods in the proofs of the following important theorems in the chronological order: (1) The von Neumann minimax theorem, (2) The von Neumann intersection lemma, (3) The Nash equilibrium theorem, (4) The social equilibrium existence theorem of Debreu, (5) The Gale-Nikaido-Debreu theorem, (6) The Fan-Browder fixed point theorem, (7) Generalized Fan minimax inequality, and (8) The Himmelberg fixed point theorem.

Keywords: KKM type theorems; Fixed point; Minimax theorem; Nash equilibria.

1. Introduction

In various fields in mathematical sciences, there are many results which can be proved by some fixed point theorems or some KKM type intersection theorems originated by Knaster, Kuratowski, and Mazurkiewicz (KKM, 1929). For example, in our previous work (Park, 2010a) on generalizations of the Nash equilibrium theorem, we found that there are two major methods, that is, the fixed point method (with or without using any continuous selection theorems) and the KKM method with respect to various abstract convexities.

Later we noticed that, for some of the key results in nonlinear analysis or the game theory, the fixed point method and the KKM method are major tools to deduce them. Examples of such key results are given chronologically as follows:

(1) The von Neumann minimax theorem
(2) The von Neumann intersection lemma
(3) The Nash equilibrium theorem
(4) The social equilibrium existence theorem of Debreu
(5) The Gale-Nikaido-Debreu theorem
(6) The Fan-Browder fixed point theorem
(7) Generalized Fan minimax inequality
(8) The Himmelberg fixed point theorem

Recall that many of the above results were originally proved by the Brouwer or Kakutani fixed point theorems. Nowadays, generalized forms of them can be deduced from the KKM theory with respect to various abstract convexities or from the fixed point theorems for acyclic maps.

Our aim in this survey is to compare the fixed point method and the KKM method in the proofs of the above sample results, and to give some of the most general forms of such results.

Recall that such comparison will also work for lots of results other than the above ones in the KKM theory.

2. Abstract Convex Spaces and the KKM Spaces

Multimaps are also called simply maps. Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following in (Park, 2008a,b,c, 2010b):

Definition 1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \to 2^E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \to 2^E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all } A \in \langle D \rangle,$$

then G is called a *KKM map*.

Definition 3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \to 2^E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Space satisfying the partial KKM principle
$\Longrightarrow$ Abstract convex space.

We recall the following in (Park, 1999,2001,2008a,b,c):

Definition 4. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

Definition 5. For an abstract convex space $(E \supset D; \Gamma)$, a function $f : E \to \overline{\mathbf{R}}$ is said to be *quasiconcave* [resp., *quasiconvex*] if $\{x \in E \mid f(x) > r\}$ [resp., $\{x \in E \mid f(x) < r\}$] is Γ-convex for each $r \in \overline{\mathbf{R}}$.

For the basic theory on KKM spaces, see Park (2010b) and the references therein.

3. The KKM type theorems

In 1929, Knaster, Kuratowski, and Mazurkiewicz (simply, KKM) obtained the following celebrated KKM theorem from the Sperner combinatorial lemma in 1928:

Theorem 1. (KKM, 1929) *Let $A_i\,(0 \le i \le n)$ be $n+1$ closed subsets of an n-simplex $p_0p_1\cdots p_n$. If the inclusion relation*

$$p_{i_0}p_{i_1}\cdots p_{i_k} \subset A_{i_0} \cup A_{i_1} \cup \cdots \cup A_{i_k}$$

holds for all faces $p_{i_0}p_{i_1}\cdots p_{i_k}$ $(0 \le k \le n, 0 \le i_0 < i_1 < \cdots < i_k \le n)$, then $\bigcap_{i=0}^n A_i \neq \emptyset$.

The first application of this KKM theorem was to give a simple proof of the Brouwer fixed point theorem; see (KKM, 1929) and (Park, 1999). Later, it is known that the Brouwer theorem, the Sperner lemma, and the KKM theorem are mutually equivalent.

In 1961, Fan extended the KKM theorem as follows:

Lemma 1. (Fan, 1961) *Let X be an arbitrary set in a topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of a finite subset $\{x_1, \cdots, x_n\}$ of X is contained in $\bigcup_{i=1}^n F(x_i)$.*

(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x\in X} F(x) \neq \emptyset$.

From the partial KKM principle for abstract convex spaces, we have a whole intersection property of the Fan type as follows:

Theorem 2. (Park, 2011b) *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle and a map $G : D \to 2^E$ satisfy the following:*

(1) *$\bigcap_{z\in D} \overline{G(z)} = \overline{\bigcap_{z\in D} G(z)}$ (G is intersectionally closed-valued);*

(2) *$\overline{G}$ is a KKM map (that is, $\Gamma_A \subset \overline{G}(A)$ for all $A \in \langle D\rangle$); and*

(3) *there exists a nonempty compact subset K of E such that one of the following holds:*

(i) *$E = K$;*

(ii) *$\bigcap\{\overline{G(z)} \mid z \in M\} \subset K$ for some $M \in \langle D\rangle$; or*

(iii) *for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and*

$$L_N \cap \bigcap_{z\in D'} \overline{G(z)} \subset K.$$

Then $\bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

When G is closed-valued or transfer closed-valued in (1), we obtain the conclusion

$$K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset.$$

Theorem 2 subsumes a very large number of particular KKM type theorems in the literature and has a number of equivalent formulations; see (Park, 2011b).

4. The von Neumann type minimax theorem

In 1928, J. von Neumann obtained the following minimax theorem, which is one of the fundamental results in the theory of games developed by himself. We adopt Kakutani's formulation in (Kakutani, 1941):

Theorem 3. (von Neumann, 1928) *Let $f(x,y)$ be a continuous real-valued function defined for $x \in K$ and $y \in L$, where K and L are arbitrary bounded closed convex sets in two Euclidean spaces $\mathbf{R}^m$ and $\mathbf{R}^n$. If for every $x_0 \in K$ and for every real number α, the set of all $y \in L$ such that $f(x_0, y) \leq \alpha$ is convex, and if for every $y_0 \in L$ and for every real number β, the set of all $x \in K$ such that $f(x, y_0) \geq \beta$ is convex, then we have*

$$\max_{x \in K} \min_{y \in L} f(x,y) = \min_{y \in L} \max_{x \in K} f(x,y).$$

In order to give simple proofs of von Neumann's Lemma in 1937 and the above minimax theorem, Kakutani obtained the following generalization of the Brouwer fixed point theorem to multimaps:

Theorem 4. (Kakutani, 1941) *If $x \mapsto \Phi(x)$ is an upper semicontinuous point-to-set mapping of an r-dimensional closed simplex S into the family of nonempty closed convex subset of S, then there exists an $x_0 \in S$ such that $x_0 \in \Phi(x_0)$.*

Equivalently,

Corollary 1. (Kakutani, 1941) *Theorem 4.2 is also valid even if S is an arbitrary bounded closed convex set in a Euclidean space.*

As Kakutani noted, Corollary 1 readily implies von Neumann's Lemma, and later Nikaido noted that those two results are directly equivalent.

This was the beginning of the fixed point theory of multimaps having a vital connection with the minimax theory in game theory and the equilibrium theory in economics.

In 1958, von Neumann's minimax theorem was extended by Sion to arbitrary topological vector spaces as follows:

Theorem 5. (Sion, 1958) *Let X, Y be a compact convex set in a topological vector space. Let f be a real-valued function defined on $X \times Y$. If*

(1) *for each fixed $x \in X$, $f(x,y)$ is a lower semicontinuous, quasiconvex function on Y, and*

(2) *for each fixed $y \in Y$, $f(x,y)$ is an upper semicontinuous, quasiconcave function on X,*

then we have

$$\min_{y \in Y} \max_{x \in X} f(x,y) = \max_{x \in X} \min_{y \in Y} f(x,y).$$

Sion's proof was based on the KKM theorem and this is the first application of the theorem after (KKM, 1929).

Recently, by the KKM method, we obtained a very general form of the von Neumann minimax theorem as follows; see (Park, 2010a).

Let $(X; \Gamma_1)$ and $(Y; \Gamma_2)$ be abstract convex spaces. For their product, we can define $\Gamma_{X \times Y}(A) := \Gamma_1(\pi_1(A)) \times \Gamma_2(\pi_2(A))$ for $A \in \langle X \times Y \rangle$.

Theorem 6. (Park, 2010a) *Let $(\underline{E};\Gamma) := (X \times Y; \Gamma_{X\times Y})$ be the product abstract convex space, $f, s, t, g : X \times Y \to \overline{\mathbf{R}}$ be four functions,*

$$\mu := \inf_{y\in Y} \sup_{x\in X} f(x,y) \quad \textit{and} \quad \nu := \sup_{x\in X} \inf_{y\in Y} g(x,y).$$

Suppose that

(1) *$f(x,y) \le s(x,y) \le t(x,y) \le g(x,y)$ for each $(x,y) \in X \times Y$;*

(2) *for each $r < \mu$ and $y \in Y$, $\{x \in X \mid s(x,y) > r\}$ is Γ_1-convex; for each $r > \nu$ and $x \in X$, $\{y \in Y \mid t(x,y) < r\}$ is Γ_2-convex;*

(3) *for each $r > \nu$, there exists a finite set $\{x_i\}_{i=1}^m \subset X$ such that*

$$Y = \bigcup_{i=1}^{m} \operatorname{Int}\{y \in Y \mid f(x_i,y) > r\}; \text{ and}$$

(4) *for each $r < \mu$, there exists a finite set $\{y_j\}_{j=1}^n \subset Y$ such that*

$$X = \bigcup_{j=1}^{n} \operatorname{Int}\{x \in X \mid g(x,y_j) < r\}.$$

If $(E;\Gamma)$ satisfies the partial KKM principle, then we have

$$\mu = \inf_{y\in Y} \sup_{x\in X} f(x,y) \le \sup_{x\in X} \inf_{y\in Y} g(x,y) = \nu.$$

5. The von Neumann type intersection theorems

The minimax theorem was later extended by von Neumann in 1937 to the following intersection lemma. We also adopt Kakutani's formulation:

Lemma 2. (von Neumann, 1937) *Let K and L be two bounded closed convex sets in the Euclidean spaces $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively, and let us consider their Cartesian product $K \times L$ in $\mathbf{R}^{m+n}$. Let U and V be two closed subsets of $K \times L$ such that for any $x_0 \in K$ the set U_{x_0}, of $y \in L$ such that $(x_0, y) \in U$, is nonempty, closed and convex and such that for any $y_0 \in L$ the set V_{y_0}, of all $x \in K$ such that $(x, y_0) \in V$, is nonempty, closed and convex. Under these assumptions, U and V have a common point.*

Von Neumann proved this by using a notion of integral in Euclidean spaces and applied this to the problems of mathematical economics. Kakutani gave a simple proof by applying his fixed point theorem. Lemma 2 was generalized by Fan, Ma, and others to various Fan type intersection theorems for sets with convex sections.

Let $\{X_i\}_{i\in I}$ be a family of sets, and let $i \in I$ be fixed. Let

$$X = \prod_{j\in I} X_j \quad \text{and} \quad X^i = \prod_{j\in I\setminus\{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I \setminus \{i\}$, let x^i_j denote the jth coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: Its ith coordinate is x_i and, for $j \ne i$, its jth coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where x^i denotes the projection of x onto X^i.

Some of the most general forms of the von Neumann intersection lemma are the following in the KKM theory:

Theorem 7. (Park, 2010c) *Let $\{(X_i;\Gamma_i)\}_{i=1}^n$ be a finite family of compact abstract convex spaces such that $(X;\Gamma) = (\prod_{i=1}^n X_i;\Gamma)$ satisfies the partial KKM principle and, for each i, let A_i and B_i are subsets of X satisfying*

(1) *for each $x^i \in X^i$, $\emptyset \neq \mathrm{co}_{\Gamma_i} B_i(x^i) \subset A_i(x^i) := \{y_i \in X_i \mid [x^i, y_i] \in A_i\}$; and*

(2) *for each $y_i \in X_i$, $B_i(y_i) := \{x^i \in X^i \mid [x^i, y_i] \in B_i\}$ is open in X^i.*

Then we have $\bigcap_{i=1}^n A_i \neq \emptyset$.

Theorem 8. (Park, 2001) *Let $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of Hausdorff compact G-convex spaces and, for each $i \in I$, let A_i and B_i are subsets of $X = \prod_{i\in I} X_i$ satisfying the following:*

$(1)'$ *for each $x^i \in X^i$, $\emptyset \neq \mathrm{co}_{\Gamma_i} B_i(x^i) \subset A_i(x^i) := \{y_i \in X_i \mid [x^i, y_i] \in A_i\}$; and*

$(2)'$ *for each $y_i \in X_i$, $B_i(y_i) := \{x^i \in X^i \mid [x^i, y_i] \in B_i\}$ is open in X^i.*

Then we have $\bigcap_{i\in I} A_i \neq \emptyset$.

6. The Nash equilibrium theorem

The first remarkable one of generalizations of von Neumann's minimax theorem was Nash's theorem (Nash, 1950, 1951) on equilibrium points of non-cooperative games. The following formulation is given by (Fan, 1966):

Theorem 9. (Nash) *Let $X_1, X_2, \cdots, X_n$ be n (≥ 2) nonempty compact convex sets each in a real Hausdorff topological vector space. Let $f_1, f_2, \cdots, f_n$ be n real-valued continuous functions defined on $\prod_{i=1}^n X_i$. If for each $i = 1, 2, \cdots, n$ and for any given point $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \prod_{j\neq i} X_j$, $f_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ is a quasi-concave function on X_i, then there exists a point $(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) \in \prod_{i=1}^n X_i$ such that*

$$f_i(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) = \max_{y_i \in X_i} f_i(\widehat{x}_1, \cdots, \widehat{x}_{i-1}, y_i, \widehat{x}_{i+1}, \cdots, \widehat{x}_n) \quad (1 \leq i \leq n).$$

The original form of this theorem in (Nash, 1950, 1951) was for Euclidean spaces and its proofs were based on the Brouwer or Kakutani fixed point theorem.

Moreover, based on a generalization of the Kakutani fixed point theorem due to Fan (1952) and Glicksberg (1952), certain generalizations of the Nash theorem were obtained; for example, see (Aliprantis et al., 2006) and (Becker et al., 2006).

The first proof of the Nash theorem by the KKM method was given by Fan (1966). Applying the KKM method or the fixed point method, we obtained some of the most general forms of the Nash theorem as follows:

Theorem 10. (Park, 2010c) *Let $\{(X_i;\Gamma_i)\}_{i=1}^n$ be a finite family of compact abstract convex spaces such that $(X;\Gamma) = (\prod_{i=1}^n X_i;\Gamma)$ satisfies the partial KKM principle and, for each i, let $f_i, g_i : X = X^i \times X_i \to \mathbf{R}$ be real functions such that*

(0) *$f_i(x) \leq g_i(x)$ for each $x \in X$;*

(1) *for each $x^i \in X^i$, $x_i \mapsto g_i[x^i, x_i]$ is quasiconcave on X_i;*

(2) *for each $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ is u.s.c. on X_i; and*

(3) *for each $x_i \in X_i$, $x^i \mapsto f_i[x^i, x_i]$ is l.s.c. on X^i.*

Then there exists a point $\widehat{x} \in X$ such that

$$g_i(\widehat{x}) \geq \max_{y_i \in X_i} f_i[\widehat{x}^i, y_i] \quad \text{for all } i = 1, 2, \ldots, n.$$

Theorem 11. (Park, 2001) *Let $\{(X_i; \Gamma_i)\}_{i\in I}$ be a family of Hausdorff compact G-convex spaces and, for each $i \in I$, let $f_i, g_i : X = X^i \times X_i \to \mathbf{R}$ be real functions satisfying* (0)-(3) *in Theorem 10. Then there exists a point $\widehat{x} \in X$ such that*

$$g_i(\widehat{x}) \geq \max_{y_i \in X_i} f_i[\widehat{x}^i, y_i] \quad \textit{for all} \quad i \in I.$$

7. The social equilibrium existence theorem of Debreu

Acyclic versions of the social equilibrium existence theorem of Debreu (1952) are obtained in (Park, 1998, 2011a) as follows:

A *polyhedron* is a set in $\mathbf{R^n}$ homeomorphic to a union of a finite number of compact convex sets in $\mathbf{R^n}$. The product of two polyhedra is a polyhedron.

A nonempty topological space is said to be *acyclic* whenever its reduced homology groups over a field of coefficients vanish. The product of two acyclic spaces is acyclic by the Künneth theorem.

The following is due to Eilenberg and Montgomery or, more generally, to Begle (1950):

Lemma 3. (Eilenberg and Montgomery, 1946) *Let Z be an acyclic polyhedron and $T : Z \to 2^Z$ an acyclic map* (*that is, u.s.c. with acyclic values*). *Then T has a fixed point $\widehat{x} \in Z$; that is, $\hat{x} \in T(\widehat{x})$.*

Recently, we obtained the following new *collectively fixed point theorem* equivalent to Lemma 3:

Theorem 12. (Park, 2011a) *Let $\{X_i\}_{i\in I}$ be any family of acyclic polyhedra, and $T_i : X \to 2^{X_i}$ an acyclic map for each $i \in I$. Then there exists an $\widehat{x} \in X$ such that $\widehat{x}_i \in T_i(\widehat{x})$ for each $i \in I$.*

From this, we have the following acyclic version of the *social equilibrium existence theorem* of Debreu (1952):

Theorem 13. (Park, 2011a) *Let $\{X_i\}_{i\in I}$ be any family of acyclic polyhedra, $A_i : X^i \to 2^{X_i}$ closed maps, and $f_i, g_i : \mathrm{Gr}(A_i) \to \overline{\mathbf{R}}$ u.s.c. functions for each $i \in I$ such that*

(1) *$g_i(x) \leq f_i(x)$ for all $x \in \mathrm{Gr}(A_i)$;*
(2) *$\varphi_i(x^i) := \max_{y\in A_i(x^i)} g_i[x^i, y]$ is an l.s.c. function of $x^i \in X^i$; and*
(3) *for each $i \in I$ and $x^i \in X^i$, the set*

$$M(x^i) := \{x_i \in A_i(x^i) \mid f_i[x^i, x_i] \geq \varphi_i(x^i)\}$$

is acyclic.

Then there exists an equilibrium point $\widehat{a} \in \mathrm{Gr}(A_i)$ for all $i \in I$; that is,

$$\hat{a}_i \in A_i(\widehat{a}^i) \quad \textit{and} \quad f_i(\widehat{a}) = \max_{a_i \in A(\widehat{a}^i)} g_i[\widehat{a}^i, a_i] \quad \textit{for all} \quad i \in I.$$

Debreu's theorem is the contractible case in (3) for a finite I.

In Park (1998, 2011a), this is applied to deduce acyclic versions of theorems on saddle points, minimax theorems, and the following *Nash equilibrium theorem*:

Theorem 14. (Park, 2011a) *Let $\{X_i\}_{i\in I}$ be a family of acyclic polyhedra, $X = \prod_{i=1}^n X_i$, and for each i, $f_i : X \to \overline{\mathbf{R}}$ a continuous function such that*

(0) *for each $x^i \in X^i$ and each $\alpha \in \overline{\mathbf{R}}$, the set*

$$\{x_i \in X_i \mid f_i[x^i, x_i] \geq \alpha\}$$

is empty or acyclic.

Then there exists a point $\widehat{a} \in X$ such that

$$f_i(\widehat{a}) = \max_{y_i \in X_i} f_i[\widehat{a}^i, y_i] \quad for\ all \quad i \in I.$$

In our previous work (Park, 1998), Theorems 12-14 were obtained for a finite index set I.

8. The Gale-Nikaido-Debreu theorem

A *convex space* (in the sense of Lassonde) is a nonempty convex set (in a vector space) equipped with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls are called *polytopes*. For details, see (Park, 1999) and references therein.

In 1994, we introduced an *admissible* class $\mathcal{A}_c^\kappa(X, Y)$ of maps $T : X \to 2^Y$ between topological spaces X and Y as the one such that, for each T and each compact subset K of X, there exists a map $\Gamma \in \mathcal{A}_c(K, Y)$ satisfying $\Gamma(x) \subset T(x)$ for all $x \in K$; where $\mathcal{A}_c$ is consisting of finite composites of maps in $\mathcal{A}$, and $\mathcal{A}$ is a class of maps satisfying the following properties:

(i) $\mathcal{A}$ contains the class of (single-valued) continuous functions;

(ii) each $F \in \mathcal{A}_c$ is u.s.c. and compact-valued; and

(iii) for any polytope P, each $F \in \mathcal{A}_c(P, P)$ has a fixed point.

Let X be a convex space and Y a Hausdorff space. In 1997, we introduced a more general "better" admissible class $\mathcal{B}$ of multimaps as follows:

$F \in \mathcal{B}(X, Y) \iff F : X \to 2^Y$ such that, for any polytope P in X and any continuous map $f : F(P) \to P$, $f(F|_P)$ has a fixed point.

Here, $\mathcal{A}$ and $\mathcal{B}$ should be denoted by "Fraktur A" and "Fraktur B", resp.

In 1997, we obtained a KKM type theorem related to the better admissible class $\mathcal{B}$ of multimaps. From the theorem, we deduced the following generalization of the so-called Walras excess demand theorem:

Theorem 15. (Park, 1997) *Let X be a convex space, Y a Hausdorff space, $T \in \mathcal{B}(X, Y)$ a compact map, $c \in \mathbf{R}$, and $\phi, \psi : X \times Y \to \overline{\mathbf{R}}$ two extended real-valued functions such that*

(1) *$\phi(x, y) \leq \psi(x, y)$ for each $(x, y) \in X \times Y$;*

(2) *for each $x \in X$, $y \mapsto \psi(x, y)$ is u.s.c. on Y;*

(3) *for each $y \in Y$, $x \mapsto \phi(x, y)$ is quasiconvex on X; and*

(4) *$\phi(x, y) \geq c$ for all $(x, y) \in T$ (Walras law).*

Then there exists a Walras equilibrium; that is, there exists a $y_0 \in Y$ such that

$$c \leq \phi(x, y_0) \quad for\ all \quad x \in X.$$

This generalize a result due to Granas and Liu (1986).

The following is a different version of Theorem 15:

Theorem 16. (Park, 1997) *Let X be a compact convex space, Y a Hausdorff space, and $T \in \mathcal{A}_c^\kappa(X, Y)$. Let $\phi : X \times Y \to \mathbf{R}$ be a continuous function and $c \in \mathbf{R}$ such that*

(1) *for each $y \in Y$, $x \mapsto \phi(x, y)$ is quasiconvex on X; and*

(2) *$\phi(x, y) \geq c$ for all $(x, y) \in T$ (Walras law).*

Then there exists a Warlas equilibrium; that is, there exists an $(x_0, y_0) \in T$ such that

$$c \leq \phi(x_0, y_0) \leq \phi(x, y_0) \quad forall \quad x \in X.$$

This extends results due to Granas-Liu (1986), Gwinner (1981), and Zeidler (1986).

From Theorem 16, we deduced the following generalization of the Gale-Nikaido-Debreu theorem:

Theorem 17. (Park, 1997) *Let $(E, F, \langle\ ,\ \rangle)$ be a dual system of Hausdorff topological vector spaces E and F, where the real bilinear form $\langle\ ,\ \rangle$ is continuous on compact subsets of $E \times F$. Let X be a nonempty compact convex subset of E, P the convex cone $\bigcup\{rX \mid r \geq 0\}$, and $P^+ = \{y \in F \mid \langle p, y\rangle \geq 0,\ p \in P\}$ its positive dual cone. Then for any map $T \in \mathcal{A}_c^\kappa(X, F)$ satisfying $\langle x, y\rangle \geq 0$ for $(x, y) \in T$, there exists an $\overline{x} \in X$ such that $T\overline{x} \cap P^+ \neq \emptyset$.*

This generalizes a result of Gwinner (1981). The Gale-Nikaido-Debreu theorem is the case $P = \{x \in \mathbf{R}^n \mid x_i \geq 0,\ 1 \leq i \leq n\}$, $X = \{x \in P \mid x_1 + \cdots + x_n = 1\}$, the standard $(n-1)$-simplex, and $T \in \mathcal{K}(X, \mathbf{R}^n)$, where $\mathcal{K}$ denotes the class of Kakutani maps (that is, u.s.c. maps with closed convex values). For the references, see (Gwinner, 1981).

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

It is well-known that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are ℓ^p, $L^p(0, 1)$, H^p for $0 < p < 1$, and many others.

In the paper (Park, 2009), from a fixed point theorem for acyclic maps due to Park, we deduced the following generalization of Gwinner's extension of the Walras theorem in Gwinner (1981):

Theorem 18. (Park, 2009) *Let K and L be compact convex subsets of t.v.s. E and F, resp., such that K is admissible. Let $c \in \mathbf{R}$, $f : K \times L \to \mathbf{R}$ a continuous function, and $T : K \to 2^L$ a multimap. Suppose*

(1) *for each $y \in L$, $f(\cdot, y)$ is quasiconvex;*

(2) *T is an acyclic map; and*

(3) *for each $x \in K$ and $y \in T(x)$, we have $f(x, y) \geq c$. (the Walras law)*

Then there exists a Walras equilibrium, that is, there exist $\overline{x} \in K$, $\overline{y} \in L$ such that

$$\overline{y} \in T(\overline{x}) \ \ and \ \ f(x, \overline{y}) \geq c \ \ for\ all \ \ x \in K.$$

Since every convex subset of a locally convex t.v.s. is admissible, from Theorem 18, we immediately have the following:

Corollary 2. *Theorem 18 is also valid even if E is a locally convex t.v.s.*

For a Kakutani map T instead of an acyclic map, Corollary 2 reduces to (Park, 2010d, Theorem 8], where local convexity of F is redundant. To specialize Theorem 18 towards the Gale-Nikaido-Debreu theorem, we boil down the function f to a bilinear form $\langle\cdot,\cdot\rangle$ for a dual system $(E, F, \langle\cdot,\cdot\rangle)$ of t.v.s. E and F.

For a convex cone P of E, the *dual cone* is defined by

$$P^+ := \{y \in F \mid \langle p, y\rangle \geq 0,\ p \in P\}.$$

Theorem 19. (Park, 2010d) *Let $(E, F, \langle\cdot,\cdot\rangle)$ be a dual system of t.v.s. E and F such that the bilinear form $\langle\cdot,\cdot\rangle$ is continuous on compact subsets of $E \times F$. Let K and L be compact convex subsets of t.v.s. E and F, resp., such that K is admissible; and P the convex cone $\bigcup\{rK \mid r \geq 0\}$. Let $T : K \to 2^L$ be an acyclic map such that $\langle x, y\rangle \geq 0$ for all $x \in K$ and $y \in T(x)$. Then there exists $\overline{x} \in K$ such that $T(\overline{x}) \cap P^+ \neq \emptyset$.*

Corollary 3. (Park, 2010d) *Theorem 18 is also valid even if E is a locally convex t.v.s. instead of the admissibility of K.*

For a Kakutani map T instead of an acyclic map, Corollary 3 reduces to (Gwinner, 1981, Corollary to Theorem 8), where local convexity of F is redundant.

With the choice $P := \{x \in \mathbf{R}^n \mid x_i \geq 0;\ i = 1, 2, \ldots, n\}$ and $K = L := \{x \in P \mid x_1 + x_2 + \cdots + x_n = 1\}$ (the standard simplex), the Gale-Nikaido-Debreu theorem can be immediately obtained from Corollary 3.

9. The Fan-Browder type fixed point theorem

In 1968, Browder obtained an equivalent form of Fan's geometric lemma (Fan, 1961). Since then the following is known as the Fan-Browder fixed point theorem:

Theorem 20. (Browder, 1968) *Let K be a nonempty compact convex subset of a Hausdorff topological vector space. Let T be a map of K into 2^K, where for each $x \in K, T(x)$ is a nonempty convex subset of K. Suppose further that for each y in K, $T^-(y) = \{x \in K \mid y \in T(x)\}$ is open in K. Then there exists x_0 in K such that $x_0 \in T(x_0)$.*

Browder proved this by applying the partition of unity argument [this is why Hausdorffness is assumed] and the Brouwer fixed point theorem.

Later the Hausdorffness in the Fan lemma and Browder's theorem was known to be redundant by Lassonde in 1983. Moreover, the Fan lemma and Browder's theorem are known to be equivalent to the KKM theorem. Consequently the Browder theorem can be obtained by a simple KKM proof.

There are several scores of generalizations of Theorem 20. The following are some of recent ones:

Theorem 21. (Park, 2008c, 2010b) *An abstract convex space $(X, D; \Gamma)$ is a KKM space iff for any maps $S : D \to 2^X,\ T : X \to 2^X$ satisfying*

(1) *$S(z)$ is open [resp., closed] for each $z \in D$;*

(2) *for each $y \in X$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$; and*

(3) *$X = \bigcup_{z \in M} S(z)$ for some $M \in \langle D\rangle$,*

T has a fixed point $x_0 \in X$; that is $x_0 \in T(x_0)$.

From Theorem 2, we have the following Fan-Browder type fixed fixed point theorem:

Theorem 22. *Let $(X, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, X is compact, and $S : D \to 2^X$, $T : X \to 2^X$ maps. Suppose that*
(1) *S is unionly open-valued (that is, S^c is intersectionally closed-valued);*
(2) *for each $x \in X$, $M \in \langle S^-(x)\rangle$ implies $\Gamma_M \subset T^-(x)$; and*
(3) *$X = S(D)$.*
Then T has a fixed point.

10. Generalized Fan minimax inequality

One of the most remarkable equivalent formulations of the KKM theorem is the following *minimax inequality* established by Ky Fan from his KKM lemma:

Theorem 23. (Fan, 1972) *Let X be a compact convex set in a t.v.s. Let f be a real-valued function defined on $X \times X$ such that :*
(a) *For each fixed $x \in X$, $f(x, y)$ is a lower semicontinuous function of y on X.*
(b) *For each fixed $y \in X$, $f(x, y)$ is a quasiconcave function of x on X.*
Then the minimax inequality

$$\min_{y\in Y} \sup_{x\in X} f(x, y) \le \sup_{x\in X} f(x, x)$$

holds.

Fan actually assumed the Hausdorffness of the topological vector space in his KKM lemma and his minimax inequality, but it was known to be redundant later. However, the inequality became a crucial tool in proving many existence problems in nonlinear analysis, especially, in various variational inequality problems.

The compactness, convexity, lower semicontinuity, and quasiconcavity in the inequality are extended or modified by a large number of authors. For example, the quasicocavity is extended to γ-DQCV by (Zhou and Chen, 1988). Further, (Lin and Tian, 1993, Theorem 3) defined γ-DQCV in slightly more general form: Let Y be a convex subset of a Hausdorff t.v.s. E and let $\emptyset \ne X \subset Y$. A functional $\varphi(x, y) : X \times Y \to \overline{\mathbf{R}}$ is said to be *γ-diagonally quasi-concave (γ-DQCV)* in x if, for any finite subset $\{x_1, \dots, x_m\} \subset X$ and any $x_\lambda \in \mathrm{co}\,\{x_1, \dots, x_m\}$, we have $\min_{1\le j\le m} \varphi(x_j, x_\lambda) \le \gamma$.

Theorem 24. (Lin and Tian, 1993) *Let Y be a nonempty convex subset of a Hausdorff t.v.s. E, let $\emptyset \ne X \subset Y$, and let $\varphi : X \times Y \to \overline{\mathbf{R}}$ be a functional such that*
(i) *$(x, y) \mapsto \varphi(x, y)$ is l.s.c. in y;*
(ii) *$(x, y) \mapsto \varphi(x, y)$ is γ-DQCV in x;*
(iii) *there exists a nonempty subset C of X such that $\bigcap_{x\in C}\{y \in Y \mid \varphi(x, y) \le \gamma\}$ is compact and C is contained in a compact convex subset B of Y.*
Then there exists a point $y^* \in Y$ such that $\varphi(x, y^*) \le \gamma$ for all $x \in X$.

Lin and Tian (1993) proved Theorem 24 by applying the partition of unity argument [this is why Hausdorffness is assumed] and the Brouwer fixed point theorem. Moreover, they showed that Theorem 24 is equivalent to the Fan KKM lemma [where Hausdorffness is redundant].

Comparing these two proofs, *the KKM method is simpler than the Fixed Point method assuming the redundant Hausdorffness.*

Recently, the $\mathcal{C}$-quasiconcavity due to (Hou, 2009) unifies the diagonal transfer quasiconcavity (weaker than quasiconcavity) and the $\mathcal{C}$-concavity (weaker than concavity). However, S.-Y. Chang (2010) extended the $\mathcal{C}$-quasiconcavity to 0-pair-concavity and obtained a new Fan type inequality.

From the partial KKM principle we can deduce a very general version of the Fan minimax inequality:

Theorem 25. (Park, 2010a,2011b) *Let $(X, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, $f : D \times X \to \overline{\mathbf{R}}$, $g : X \times X \to \overline{\mathbf{R}}$ extended real functions, and $\gamma \in \overline{\mathbf{R}}$ such that*

(1) *for each $z \in D$, $G(z) := \{y \in X \mid f(z, y) \leq \gamma\}$ is intersectionally closed;*

(2) *for each $y \in X$, $\mathrm{co}_\Gamma\{z \in D \mid f(z, y) > \gamma\} \subset \{x \in X \mid g(x, y) > \gamma\}$; and*

(3) *the compactness condition (3) in Theorem 2 holds.*

Then either

(i) *there exists a $\widehat{x} \in X$ such that $f(z, \widehat{x}) \leq \gamma$ for all $z \in D$; or*

(ii) *there exists an $x_0 \in X$ such that $g(x_0, x_0) > \gamma$.*

We found that Chang's Fan type inequality follows from this.

11. The Himmelberg fixed point theorem

In 1972, Himmelberg defined that a subset A of a t.v.s. E is said to be *almost convex* if for any neighborhood $V \in \mathcal{V}$ of the origin 0 in E and for any finite set $\{w_1, \ldots, w_n\}$ of points of A, there exist $z_1, \ldots, z_n \in A$ such that $z_i - w_i \in V$ for all i, and $\mathrm{co}\{z_1, \ldots, z_n\} \subset A$.

Himmelberg derived the following from the Kakutani fixed point theorem:

Theorem 26. (Himmelberg, 1972) *Let K be a nonvoid compact subset of a separated locally convex space L and $G : K \to K$ be a u.s.c. multifunction such that $G(x)$ is closed for all x in K and convex for all x in some dense almost convex subset A of K. Then G has a fixed point.*

Theorem 27. (Himmelberg, 1972) *Let T be a nonvoid convex subset of a separated locally convex space L. Let $F : T \to T$ be a u.s.c. multifunction such that $F(x)$ is closed and convex for all $x \in T$, and $F(T)$ is contained in some compact subset C of T. Then F has a fixed point.*

We stated their original forms. Usually Theorem 27 is called the Himmelberg fixed point theorem which unifies and generalizes historically well-known theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, Hukuhara, and others. There have appeared a very large number of generalizations of the theorem even within the category of topological vector spaces, see (Park, 2008). Especially, there are generalizations of Theorems 26 and 27 which can be deduced from the KKM theorem and its open version; see also (Park, 2008).

Finally, in this section, we show the open version of the KKM principle can be also applied to fixed point theorems for KKM spaces.

Definition 6. An *abstract convex uniform space* $(E, D; \Gamma; \mathcal{U})$ is the one with a basis $\mathcal{U}$ of a uniform structure of E.

A *KKM uniform space* $(E, D; \Gamma; \mathcal{U})$ is a KKM space with a basis $\mathcal{U}$ of a uniform structure of E.

A KKM uniform space $(E \supset D; \Gamma; \mathcal{U})$ is called an *LΓ-space* if D is dense in E and, for each $U \in \mathcal{U}$, the U-neighborhood

$$U[A] := \{x \in E \mid A \cap U[x] \neq \emptyset\}$$

around a given Γ-convex subset $A \subset E$ is Γ-convex.

Theorem 28. (Park, 2009) *Let $(X \supset D; \Gamma; \mathcal{U})$ be a Hausdorff LΓ-space and $T : X \to 2^X$ a compact u.s.c. map with closed Γ-convex values. Then T has a fixed point $x_0 \in X$.*

This is an example of generalizations of the Himmelberg theorem for abstract convex spaces. For more generalizations, see Park (2008d, 2009).

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Proportionality in NTU Games: on a Proportional Excess Invariant Solution

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Abstract. A solution for NTU games which is invariant with respect to proportional excess is defined. It generalizes the corresponding solution for TU games. The existence theorem is proved, and some properties of the solution are studied.

Keywords: NTU games, proportional excess, bargaining game, proportional invariant solution, directional sum of NTU games.

1. Introduction

Our goal in this paper is to generalize to NTU games the proportional excess invariant solution (i.e. the solution which is invariant with respect to proportional excess) for positive TU games defined by E.Yanovskaya in (Pechersky and Yanovskaya, 2004) in a following manner.

A solution Ψ on the set of positive TU games G_{N+} with a set N of players is called *invariant with respect to proportional excess*, if for any two games (N, v), $(N, w) \in G_{N+}$ and any two payoff vectors $x \in X(N, v)$, $y \in X(N, w)$

$$\frac{v(S)}{x(S)} = \frac{w(S)}{y(S)} \text{ for all } S \subset N$$

imply

$$x = \Psi(N, v) \iff y = \Psi(N, w). \tag{1}$$

For example, Ψ defined by

$$\Psi(N, v) = \arg \max_{x \in X(N,v)} \prod_{S \subset N, S \neq N} x(S)^{w(s)v(S)} \tag{2}$$

for some nonnegative numbers $w(s), s \leq n-1, s = |S|, n = |N|$, where $X(N, v) = \{x \in \mathbb{R}^N_{++} : x(N) = v(N)\}$, is a desired solution.

We shall call such a solution *proportional excess invariant solution*, or shortly *proportional invariant solution*. Moreover, in the case of TU games this solution is single-valued, so we call it *proportional invariant value*.

Clearly, (1) is an analogues of shift covariance for the standard TU excess: let Φ be an arbitrary shift covariant value on G_N, and $(N, v), (N, w) \in G_N$. Then

$$v(S) - x(S) = w(S) - y(S) \text{ for all } S \subset N \tag{3}$$

imply

$$x = \Phi(N, v) \iff y = \Phi(N, w). \tag{4}$$

The paper is organized as follows. In Section 2 we recall some properties of the proportional invariant value for TU games. Section 3 introduces some definitions and notations concerning the NTU games. In Section 4 we define a proportional invariant solution for NTU games and prove the existence theorem. Finally, Section 5 studies some properties of the solution defined.

2. Some properties of the proportional invariant value

Firstly we recall some properties of the proportional invariant value for TU games studied by E.Yanovskaya (details and proofs see (Pechersky and Yanovskaya, 2004)). Let G_+ be the set of all positive TU games, and $G_{N+} \subset G_+$ be the set of all positive TU games with a player set N.

Clearly the proportional invariant value is not linear, but we can replace linearity by a weaker property – restricted linearity.

Restricted linearity. A value Ψ for some family of TU games G_N is called *restricted linear*, if for all games $(N, v_k) \in G_N, k = 1, \dots, m$ with the same value $\Psi(N, v_k) = x$ it follows that if the game $(N, \sum_{k=1}^m \alpha_k v_k)$ with $\sum_{k=1}^m \alpha_k = 1$, also belongs to G_N, then it has the same value:

$$\Psi(N, \sum_{k=1}^{m} \alpha_k v_k) = x.$$

The restricted linearity of Ψ means that for every payoff vector $x \in \mathbb{R}^N$ this value is linear on the subclass of games V^x with the value x :

$$V^x = \{(N, v) \in G_N \ : \Psi(N, v) = x\}.$$

Let us call a subclass $G'_N \subset G_N$ *closed under covariant transformations*, if for every game $(N, v) \in G'$ the games $(N, av + b)$ also belong to this class for all $a > 0, b \in \mathbb{R}$.

The following Lemma makes clear the connection between linearity and restricted linearity (the proof see (Pechersky and Yanovskaya, 2004)).

Lemma 1. *If a value Ψ on a class G'_N closed under covariant transformations is restricted linear and covariant then it is linear.*

The following characterization of a proportional invariant value holds (we do not cite here the well-known efficiency and anonimity properties).

Theorem 1. *(Yanovskaya). A proportional invariant value Ψ on G_{N+} satisfies efficiency, anonimity and restricted linearity axioms if and only if there exist such nonnegative numbers $w(s), s \le n-1, s = |S|, n = |N|$, not equal identically to zero, that for every game $(N, v) \in G_{N+}$*

$$\Psi(N, v) = \arg \max_{x \in X(N,v)} \prod_{S \subset N, S \neq N} x(S)^{w(s)v(S)}, \tag{5}$$

where $X(N, v) = \{x \in \mathbb{R}^N_{++} : x(N) = v(N)\}$.

Let us note that the function in (5) is continuous and concave when $w(s)v(S) \geq 0$ and the inequality is strict for at least one $s = 1, \dots, n-1$. Hence the maximum in (5) is attained in a unique *positive* point, and the value Ψ is defined correctly.

It is clear that for every game $(N, v) \in G_{N+}$ this value Ψ can be represented equivalently in a following manner:

$$\Psi(N, v) = \arg \max_{x \in X(N,v)} \sum_{S \subset N} w(s)v(S) \ln x(S). \qquad (6)$$

Therefore $x = \Psi(N, v) > \mathbf{0}$ is a unique solution of the following system:

$$\sum_{S:S \ni i} w(s) \frac{v(S)}{x(S)} = \sum_{S:S \ni j} w(s) \frac{v(S)}{x(S)} \text{ for all } i, j \in N, \qquad (7)$$

and $x(N) = v(N)$.

If $w(s) = 1$ for all s we shall call such proportional invariant value shortly p.i.-value, and in general case of an arbitrary weights system w – p.i.(w)-value.

3. Some definitions and notations

Now recall some definitions concerning the NTU games.

Let $N = \{1, 2, \dots, n\}$ be a non-empty finite set of players. For a subset $S \subset N$ let $\mathbb{R}^S$ denote $|S|$-dimensional Euclidean space with axes indexed by elements of S. A *payoff vector* for S is a vector $x \in \mathbb{R}^S$. For $z \in \mathbb{R}^N$ and $S \subset N$, z^S will denote the projection of z on the subspace

$$\mathbb{R}^{[S]} = \{x \in \mathbb{R}^N \, : \, x_i = 0 \text{ for } i \notin S\},$$

and z_S – the restriction of z on $\mathbb{R}^S$.

Let $x, y \in \mathbb{R}^N$. We will write $x \geq y$, if $x_i \geq y_i$ for all $i \in N$; $x > y$, if $x_i > y_i$ for all $i \in N$. Denote

$$\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \, : \, x \geq \mathbf{0}\},$$

$$\mathbb{R}^N_{++} = \{x \in \mathbb{R}^N \, : \, x > \mathbf{0}\},$$

where $\mathbf{0}$=(0,0,...,0).

Let $A \subset \mathbb{R}^N$. If $x \in \mathbb{R}^N$, then $x + A = \{x + a \, : \, a \in A\}$ and $\lambda A = \{\lambda a \, : \, a \in A\}$. A is *comprehensive*, if $x \in A$ and $x \geq y$ imply $y \in A$. A is *bounded above*, if $A \cap (x + \mathbb{R}^N_+)$ is bounded for every $x \in \mathbb{R}^N$. The boundary of A is denoted by ∂A.

A *nontransferable utility game* (or shortly *NTU game*) is a pair (N, V), where N is the set of players, and V is the set-valued map that assigns to each coalition $S \subset N$ a set $V(S)$, that satisfies:

(1) $V(S) \subset \mathbb{R}^{[S]} = \{x \subset \mathbb{R}^N \, : \, x_i = 0 \text{ for } i \notin S\}$;
(2) $V(S)$ is closed, non-empty, comprehensive and bounded above.

(Usually $V(\emptyset) = \emptyset$.)

Remark 1. Sometimes it is useful to suppose $V(S) \subset \mathbb{R}^S$. Obviously, if $V(S) \subset \mathbb{R}^S$, then $\bar{V}(S) = V(S) \times \mathbf{0}_{N \setminus S} \subset \mathbb{R}^{[S]}$. Conversely, if $V(S) \subset \mathbb{R}^{[S]}$, then $V_S(S) \subset \mathbb{R}^S$, where $V_S(S)$ denotes the restriction of $V(S)$ on $\mathbb{R}^S$.

The following particular cases should be mentioned.

TU game. A TU game v can be considered as a NTU game of the following form:

$$V(S) = \{x \in \mathbb{R}^{[S]} : x(S) \le v(S)\},$$

where $x(S) = \sum_{i \in S} x_i$. The boundary of $V(S)$ is a hyperplane in $\mathbb{R}^{[S]}$ with normal e^S, where $e = (1, 1, \ldots, 1)$.

Bargaining game. A n-person bargaining game is a pair (q, Q), where $q \in \mathbb{R}^N$ is the *status quo* point, $Q \subset \mathbb{R}^N$ and $N = \{1, 2, \ldots, n\}$. When interpreting this pair one can think as follows: if the players act separately the only possible outcome for the players is q giving utility q_i to player $i = 1, 2, \ldots, n$. If *all* players cooperate they can potentially agree on an arbitrary outcome $x \in Q$. The corresponding NTU game can be defined as follows:

$$V(N) = \{x \in \mathbb{R}^N : \text{ there is } y \in Q \text{ such that } x \le y\},$$

$$V(S) = \{x \in \mathbb{R}^{[S]} : x_i \le q_i \text{ for every } i \in S\} \text{ for } S \ne N.$$

Let us define the space $\mathcal{G}_{N+}$. A game $V \in \mathcal{G}_{N+}$ iff for every S

(a) $V(S)$ is positively generated (i.e. $V(S) = (V(S) \cap \mathbb{R}^{[S]}_+) - \mathbb{R}^{[S]}_+$, and $V_+(S) = V(S) \cap \mathbb{R}^{[S]}_+$ is compact), and every ray $L_x = \{\lambda x : \lambda \ge 0\}$, $x \ne \mathbf{0}$ does not intersect the boundary of $V(S)$ more than once;

(b) $\mathbf{0}$ is an interior point of the set $V^\wedge(S) = V(S) + \mathbb{R}^{[N \setminus S]}$.

Recall finally the definition of the proportional excess h for NTU games. It generalizes the TU proportional excess $v(S)/x(S)$ to NTU games (for axiomatic characterization of the excess and its properties see (Pechersky, 2007)).

Let $V \in \mathcal{G}_{N+}$ be an arbitrary game. For $S \subset N$ a set $V(S) \subset \mathbb{R}^{[S]}$ will be called a *game subset*, if it satisfies (a) and (b). The space consisting of all game subsets satisfying (a) and (b) will be denoted by $\mathcal{G}^S_{N+}$. Define a function $h_S : \mathcal{G}^S_{N+} \times \mathbb{R}^N_+ \to \mathbb{R}$ as follows:

$$h_S(V, x) = 1/\gamma(V(S), x^S), \tag{8}$$

where $\gamma(W, y) = \inf\{\lambda > 0 : y \in \lambda W\}$ is the gauge (or Minkowski gauge) function (Rockafellar, 1997).

4. Proportional invariant solution for NTU games: definition and existence

Now turn to the definition of the proportional invariant solution in the case of NTU games.

Definition 1. A solution ψ (set-valued in general) on $\mathcal{G}_{N+}$ is called proportional invariant (i.e. invariant with respect to proportional excess), if for every two games $(N, V), (N, W) \in \mathcal{G}_{N+}$ and every $x \in \partial V_+(N), y \in \partial W_+(N)$ it follows from

$$h_S(V, x) = h_S(W, y) \;\; \forall S \subset N$$

that

$$x \in \psi(N, V) \iff y \in \psi(N, W).$$

Let us show firstly that the *status quo*-proportional bargaining solution (*sq*-proportional solution in what follows) is proportional invariant. SQ-proportional solution is defined as follows (see for details (Pechersky, 2009)).

We suppose that every bargaining game (q, Q) satisfies the following properties:

(a) $Q \subset \mathbb{R}^N_+$ is compact and comprehensive, i.e. $x \in Q$, $y \in \mathbb{R}^N_+$, and $x \geq y$ imply $y \in Q$;
(b) Q is non-leveled, i.e.

$$x, y \in \partial Q, \quad x \geq y \;\Rightarrow\; x = y;$$

(c) $q > \mathbf{0}$, and there is $x \in Q$ such that $x > q$.

Proposition 1. SQ*-proportional solution for bargaining games is invariant with respect to proportional excess.*

Proof. Let (q, Q) and (q^1, Q^1) be two bargaining games, V and V^1 be corresponding NTU games (note that $V_S = P_{q^S}$, where $P_z = \{y \in \mathbb{R}^S : y \leq z\}$ for $z \in \mathbb{R}^S_{++}$). Let $x \in \partial Q$, $y \in \partial Q^1$, $x = \mu(q, Q)q$ and $h_S(V, x) = h_S(V^1, y)$ for every $S \subset N$. Clearly $h_S(V, x) = 1/\mu(q, Q)$. Then $h_S(V^1, y) = 1/\mu(q, Q)$ for every $S \subset N$. In particular $y_1 = \mu(q, Q)q^1_1, \ldots, y_n = \mu(q, Q)q^1_n$. Since $y \in \partial Q^1$, then $\mu(q, Q) = \mu(q^1, Q^1)$, and y is the *sq*-proportional solution of (q^1, Q^1).

We base our construction of proportional invariant solution on the corresponding modification of equations (7), and suppose for simplicity of notations $w(s) \equiv 1$.

Theorem 2. *For every game $(N, V) \in \mathcal{G}_{N+}$ there is $x \in \partial V_+(N)$, which is a solution of the system*

$$\sum_{S: i \in S} h_S(V, x) \;=\; \sum_{S: j \in S} h_S(V, x) \quad \forall i, j \in N. \tag{9}$$

Therefore there exists a proportional invariant solution on $\mathcal{G}_{N+}$.

Proof. Let us consider an arbitrary game $(N, V) \in \mathcal{G}_{N+}$. Note firstly, that if x is a solution of (9) (even if x does not belong to $\partial V_+(N)$), then λx also is a solution of the system for every $\lambda > 0$.

Hence, without loss of generality we can suppose that the set $\partial V_+(N)$ coincides with the standard simplex, i.e.

$$\partial V_+(N) = T^{n-1} = \{y \in \mathbb{R}^n_+ : \sum_i y_i = 1\}.$$

For every $y \in T^{n-1}$ and every coalition $S \neq \emptyset$ there is a unique positive number $\lambda^{(S)}_y$ such that $\lambda^{(S)}_y y \in \partial V(S)$. Note that $\lambda^{(S)}_y$ depend continuously on y.

Let $y \in T^{n-1}$. Define a positive TU game $V_y \in \mathcal{G}_{N+}$ as follows: for every coalition $S \neq \emptyset$

$$\partial V_y(S) = \{z \in \mathbb{R}^S_+ : e_S(z - \lambda^{(S)}_y y) = 0\}.$$

(Recall the definition of the NTU games corresponding to TU game in the case of $\mathcal{G}_{N+}$.) In other words, the part of the boundary of the set $V_y(S)$ belonging to $\mathbb{R}^S_+$ is defined as intersection of a hyperplane in $\mathbb{R}^S$ with unit normal passing thorough the point $\lambda_y^{(S)} y$ and the positive orthant $\mathbb{R}_+^S$. Clearly V_y as an element of $\mathbb{R}_+^{2^n}$ continuously depends on y.

In this case for the game V_y the p.i.-value $\Psi(V_y)$ is uniquely defined.

In accordance with (6) $\Psi(V_y)$ is the unique solution of the maximization problem for the function

$$Q(N, V_y) = \sum_{S \subset N, S \neq N} V_y(S) \ln z(S)$$

on the set $X(N, V_y) = \{z : z \in T^{n-1}\}$. Note that if $z_i \to 0$ for at least one $i \in N$, then $Q(N, V_y) \to -\infty$.

Since for any $z \in T^{n-1}$ and every coalition S we have $z(S) \leq 1$, then all items in the sum are nonpositive (since V_y is a positive game). Hence for every $y \in T^{n-1}$ we have

$$\max_{z \in T^{n-1}} \sum_{S \subset N, S \neq N} V_y(S) \ln z(S) \geq \max_{z \in T^{n-1}} \sum_{S \subset N, S \neq N} (\max_{y \in T^{n-1}} V_y(S)) \ln z(S) \geq$$

$$\geq \max_{z \in T^{n-1}} \sum_{S \subset N, S \neq N} v(S) \ln z(S) \geq \sum_{S \subset N, S \neq N} v(S) \ln(s/n) = a,$$

where $v(S) = \max_{y \in T^{n-1}} V_y(S) > 0, s = \mid S \mid$.

Therefore for every $y \in T^{n-1}$ a solution of the maximization problem of $Q(N, V_y)$ belongs to a (convex) compact set

$$T_a = \{z \in T^{n-1} : \sum_{S \subset N, S \neq N} v(S) \ln z(S) \geq a\} \subset T_o^{n-1},$$

where T_o^{n-1} denotes the relative interior of the set T^{n-1}.

It is not difficult to note that this solution as a solution of (9) depends continuously on y in T_a.

Thus we have constructed a continuous map of the simplex T^{n-1} into itself, and more precisely into T_a: $y \mapsto \Psi(V_y)$. Hence, by Brouwer's fixed point theorem there exists such y, that $y = \Psi(V_y)$.

Show now that y defines a solution of V which is invariant with respect to proportional excess. Indeed, firstly $\gamma_S(V(S), y) = 1/\lambda_y^{(S)}$, but since the proportional excess coincides in TU case with the TU proportional excess $h_S(V_y(S), y) = \frac{e_S \lambda_y^{(S)} y}{e_S y}$, and hence y solves (9).

Thus we have defined a solution ψ on $\mathcal{G}_{N+}$, which associates with every game V the set $\psi(V)$ of the solutions of a system (9). We call this solution also p.i.-solution. Note that in contrast to TU case the solution need not be single-valued. Analogously to TU case the solution corresponding to an arbitrary weights system w, which clearly exist, we call it p.i.(w)-solution.

5. Some properties of p.i.-solution for NTU games

The following propositions follow immediately from the definition and previous theorem.

Proposition 2. *If $V \in \mathcal{G}_{N+}$ corresponds to a TU game, then $\psi(V) = \Psi(v)$.*

Proposition 3. *P.I.-solution ψ possesses efficiency, anonimity and positive homogeneity properties.*

To formulate the property similar to restricted linearity for TU games let us define an operation $\oplus_d$ on $\mathcal{G}_{N+}$. Let $A, B \in \mathcal{G}_{N+}^S$. Then for every $x \in \mathbb{R}_+^S$ there are exactly two points $y \in \partial A$ and $z \in \partial B$ such that $y = \lambda_x x$ and $z = \mu_x x$ for some positive numbers λ_x and μ_x. Now define a *directional sum* of the sets A and B as follows

$$A \oplus_d B = comp\{\bigcup_x (\lambda_x + \mu_x)x\},$$

where $compF$ denotes the comprehensive hull of a set F. Note that since $\lambda_{tx} = \lambda_x/t$, the union can be taken not over all $x \in \mathbb{R}_+$ but over $x \in T^{n-1}$ only.

Let now $V, W \in \mathcal{G}_{N+}$. Define the game $V \oplus_d W$, taking for every coalition S

$$(V \oplus_d W)(S) = V(S) \oplus_d W(S).$$

Obviously $V \oplus_d W \in \mathcal{G}_{N+}$. It is clear also that in TU case the directional sum corresponds to addition of the characteristic functions. Finally, it follows from the definition that

$$h_S(V \oplus_d W, x) = h_S(V, x) + h_S(W, x).$$

Remark 2. It should be noted that the directional sum differs from the *inverse sum* of star-shaped sets, defined by means of addition of gauge functions of the corresponding sets (cf., for example, (Pechersky and Yanovskaya, 2004)).

We can now formulate the analogue of the restricted linearity property.

Proposition 4. *Let $V, W \in \mathcal{G}_{N+}$ and $x \in \psi(V) \bigcap \psi(W)$. Then $x \in \psi(aV \oplus_d (1-a)W)$ for every $0 < a < 1$.*

The proof follows immediately from the definitions and properties of the directional sum.

It is clear that these properties hold also for p.i.(w)-solution for every weights system w.

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On a Multistage Link Formation Game

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Abstract. In the paper we consider a multistage network game with perfect information. In each stage of the game a network connecting players is given. In our setting we suppose that each network edge connecting two players has utility (utility of the first player from the connection with the second player), and players have the right to change the network structure in each stage. We propose a way of finding an optimal players behavior in this type of multistage game.

Keywords: network, network games, characteristic function, Shapley value, Nash equilibrium.

1. Introduction

Originally, classical theory of deterministic multistage games was a theory of non-cooperative games in which each participant involved in it was aimed to maximize his own payoff. As players optimal behavior was proposed a subgame perfect equilibrium. Then, a cooperative way of participants behavior was considered. In that setting it was supposed that all participants jointly choose an n-tuple of strategies which maximizes total payoff of all players. In that cooperative setting the main problem was the allocation of obtained total payoff among the players. The notion of "imputation" was introduced. Among others, core and Shapley value were considered as optimality principles. A few years ago in (Petrosjan and Mamkina, 2005) and later in (Petrosjan et al., 2006) was proposed a way which used the methodology of two previous settings. It was supposed that each participant can choose a coalition in which he wants to belong, and then acts in the interest of the chosen coalition.

In this paper we consider another way of players behavior by adding to the existing theory of multistage games a network component. It is reasonable to add it to the model because in a system of interactions players are connected with each other. We suppose that during the game process players can reconsider current system of interactions by adding or breaking new connections, since as it is supposed each connection contains some utility for the player (positive or negative). In the paper we propose a way of finding players optimal behavior.

Give a short paper overview. In Sec. 2. we define the multistage network game. Particulary, construction of the graph tree of the multistage game and definition of network structure in each vertex of it are given in Sec. 2.1.. In Sec. 2.2. we define stage payments to each participant in each vertex of the graph tree. The definition of the multistage network game with perfect information and other necessary definitions are stated in Sec. 2.3.. Then, in Sec. 3. we propose the algorithm of constructing players optimal behavior in the multistage network game. And, finally, a numerical example illustrating proposed algorithm is considered in Sec. 4..

2. The construction of a multistage network game with perfect information

Let $N = \{1, \ldots, n\}$ be a set of players. Construct a game tree (a finite graph tree (Kuhn, 1953)) $K = (X, F)$ with an initial vertex x_0. The set X is a set of vertices of the graph K, and $F : X \mapsto X$ is a point-to-set mapping which to each vertex $x \in X$ corresponds a set F_x of vertices directly followed by the vertex x. A vertex x of the graph tree K for which $F_x = \emptyset$ is called the terminal vertex.

The set X of vertices of the graph tree K we represent in a common way as a conjunction of $n+1$ disjunctive sets: $X = P_1 \cup \ldots \cup P_n \cup P_{n+1}$, where P_i is a set of personal moves of a player i, $i \in N$, and P_{n+1} is a set of terminal vertices of the graph tree K.

Hereinafter, by $i(x)$ we denote a player who makes a decision in a vertex x of the graph tree K.

Describe the stepwise evolution of the game.

2.1. The construction of a graph tree of the network game

Initial step. In the initial vertex x_0 of the graph tree K a network $G_{x_0} = (N, \theta(x_0))$ is defined. Define by g^{x_0} a set of its edges. Let N be the set of network nodes which coincides with the set of players (node is identified as a player), and $\theta(x_0) : g^{x_0} \mapsto R$ is a real-valued function which can be interpreted as a *utility function.*

Step 1. Player $i(x_0)$ has exactly n alternatives in the vertex x_0:

- not to take any action, and the game process evolves to a vertex $y_{11} \in F_{x_0}$;
- break an edge with a player $j \in N$, $j \neq i(x_0)$, if the edge $(i(x_0), j) \in g^{x_0}$; and the game process evolves to a vertex $y_{1j} \in F_{x_0}$;
- propose to the player k, $k \neq i(x_0)$ a new edge $(i(x_0), k)$, if such edge $(i(x_0), k) \notin g^{x_0}$; and the game process evolves to a vertex $y_{1k} \in F_{x_0}$.

Each of n vertices y_{11}, $\{y_{1j}\}_j$, $\{y_{1k}\}_k$ belongs to F_{x_0}. Subject to player $i(x_0)$ choice, the initial network is changed in vertices of the set F_{x_0}. Thus a set of edges of the new network has the following form:

$$
\begin{array}{ll}
g^{y_{11}} = g^{x_0}, & \text{if the player } i(x_0) \text{ does not take} \\
 & \text{any actions;} \\
g^{y_{1j}} = g^{x_0} \setminus (i(x_0), j), & \text{if the player } i(x_0) \text{ breaks a connection} \\
 & \text{with a player } j; \\
g^{y_{1k}} = g^{x_0} \cup (i(x_0), k), & \text{if the player } i(x_0) \text{ propose a new connection} \\
 & \text{to a player } k.
\end{array}
$$

Then for a vertex $x_1 \in F_{x_0} = \{y_{11}, \{y_{1j}\}_j, \{y_{1k}\}_k\}$ a set of edges g^{x_1} is uniquely defined. If $x_1 \notin P_{n+1}$, we consider the second step for each vertex $x_1 \in F_{x_0}$. This step is fully similar to the step 1, so skipping the description of the second step we consider the step t.

Step t $(1 < t \leq l)$. Suppose we have constructed the graph tree consisting of vertices, which can be reached from the initial vertex x_0 no more than in $t-1$ stages. Let $\{x_0, x_1, \ldots, x_{t-1}\}$ be a path in the constructed graph tree starting from the vertex x_0 and leading to the vertex x_{t-1} in $t-1$ stages. In all vertices $x_0, x_1, \ldots, x_{t-1}$ corresponding sets of edges g^{x_0}, $g^{x_1}, \ldots$, $g^{x_{t-1}}$ are uniquely defined. Define the set g^{x_t}.

A player $i(x_{t-1})$ has exactly n alternatives in the vertex x_{t-1}:

– not to take any action, and the game process evolves to a vertex $y_{t1} \in F_{x_{t-1}}$;
– break an edge with a player $j \in N$, $j \neq i(x_{t-1})$, if the edge $(i(x_{t-1}), j) \in g^{x_{t-1}}$; and the game process evolves to a vertex $y_{tj} \in F_{x_{t-1}}$;
– propose to the player k, $k \neq i(x_{t-1})$ a new edge $(i(x_{t-1}), k)$, if such edge $(i(x_{t-1}), k) \notin g^{x_{t-1}}$; and the game process evolves to a vertex $y_{tk} \in F_{x_{t-1}}$.

Each of n vertices y_{t1}, $\{y_{tj}\}_j$, $\{y_{tk}\}_k$ belongs to $F_{x_{t-1}}$. Subject to player $i(x_{t-1})$ choices, the current network is changed in vertices of the set $F_{x_{t-1}}$. Thus a set of edges of the new network has the following form:

$$\begin{array}{ll} g^{y_{t1}} = g^{x_{t-1}}, & \text{if the player } i(x_{t-1}) \text{ does not take} \\ & \text{any actions;} \\ g^{y_{tj}} = g^{x_{t-1}} \setminus (i(x_{t-1}), j), & \text{if the player } i(x_{t-1}) \text{ breaks a connection} \\ & \text{with a player } j; \\ g^{y_{tk}} = g^{x_{t-1}} \cup (i(x_{t-1}), k), & \text{if the player } i(x_{t-1}) \text{ propose a new connection} \\ & \text{to a player } k. \end{array}$$

For a vertex $x_t \in F_{x_{t-1}} = \{y_{t1}, \{y_{tj}\}_j, \{y_{tk}\}_k\}$ a set of edges g^{x_t} is uniquely defined. If $x_t \notin P_{n+1}$, we consider the next step for each vertex $x_t \in F_{x_{t-1}}$, and the construction of the graph tree is fully similar to the previous stages. When $t = l$ the graph tree K is constructed.

2.2. The definition of an individual payment to the player

Definition 1. Let $S \subseteq N$. A real-valued function $v : X \times 2^N \mapsto R$, defined on a cartesian product of the set X and the set of all subsets of the set N, and specified as

$$v(y, S) = \sum_{(i,j) \in g^y:\ i,j \in S} \theta_{ij}(y), \tag{1}$$

where $y \in X$, is called the characteristic function. Here $\theta_{ij}(y)$ is the value of the utility function $\theta(y)$ in a network game $G_y = (N, \theta(y))$.

Having the set of players N and the function $v(y, \cdot)$, defined by (1), one can construct a game in characteristic function form. In this game a player has only "worths" from the connection with other players. Define a payment to a player in the network. For this purpose we select an optimality principle from the cooperative game theory (in our case we select a Shapley value (Shapley, 1953) because of its uniqueness), and calculate an imputation $\gamma(y) = (\gamma_1(y), \ldots, \gamma_n(y))$ based on this principle. The components of the imputation are calculated as follows:

$$\gamma_k(y) = \sum_{\{S:\ S \subseteq N,\ k \in S\}} \frac{(n-s)!(s-1)!}{n!} \left[v(y, S) - v(y, S \setminus k) \right]. \tag{2}$$

Here s is the cardinal number of a set S, and $v(y, S)$ is the characteristic function defined by (1).

Convert the expression in a square brackets in the right side of the equality (2). Using (1) for each $y \in X$ and $k \in N$, we get:

$$\begin{aligned} v(y, S) - v(y, S \setminus k) &= \sum_{(i,j) \in g^y:\ i,j \in S} \theta_{ij}(y) - \sum_{(i,j) \in g^y:\ i,j \in S \setminus k} \theta_{ij}(y) = \\ &= \sum_{(i,k) \in g^y:\ i \in S \setminus k} \theta_{ik}(y) + \sum_{(k,j) \in g^y:\ j \in S \setminus k} \theta_{kj}(y). \end{aligned} \tag{3}$$

Subject to (3) the components of the Shapley value have the form:

$$\gamma_k(y) = \sum_{\{S:\ S\subseteq N,\ k\in S\}} \frac{(n-s)!(s-1)!}{n!} \left[\sum_{(i,k)\in g^y:\ i\in S\setminus k} \theta_{ik}(y) + \right.$$
$$\left. + \sum_{(k,j)\in g^y:\ j\in S\setminus k} \theta_{kj}(y) \right], \qquad (4)$$

where $y \in X$, $k \in N$.

The value $\sum_{(i,k)\in g^y:\ i\in S\setminus k} \theta_{ik}(y) + \sum_{(k,j)\in g^y:\ j\in S\setminus k} \theta_{kj}(y)$ is a contribution of player k, if he joints to a coalition $S \setminus k$ and forms a coalition S. Here the first summand $\sum_{(i,k)\in g^y:\ i\in S\setminus k} \theta_{ik}(y)$ is an additional utility of the coalition $S\setminus k$, which the player k brings in. The second summand $\sum_{(k,j)\in g^y:\ j\in S\setminus k} \theta_{kj}(y)$ is an additional utility of the player k, which he obtains after joining the coalition $S \setminus k$.

Suppose in the game a path $\{x_0, \ldots, x_l\}$ is realized. The total payoff of a player $i \in N$ along this path is defined in the follow way:

$$\sum_{x\in\{x_0,\ldots,x_l\}} \gamma_i(x), \quad i \in N,$$

where $\gamma_i(x)$ is computed by formula (4) in the network game $G_x = (N, \theta(x))$.

2.3. The definition of a multistage network game with perfect information

Definition 2. The n person multistage network game with perfect information is a graph tree K with the following properties:

- the set of vertices X is divided into $n+1$ disjunctive sets $P_1, P_2, \ldots, P_n, P_{n+1}$. Here P_i, $i \in N$ is a set of personal moves of a player i, and $P_{n+1} = \{x : F_x = \emptyset\}$ is a set of terminal vertices;
- in each vertex $x \in X$ a network $G_x = (N, \theta(x))$ is uniquely defined, where N is a set of nodes (a set of players), and $\theta : g^x \mapsto R$ is a utility function.

Definition 3. A strategy $u_i(\cdot)$ of a player $i \in N$ is a mapping which to each vertex $x \in P_i$ corresponds a vertex $y \in F_x$.

For each n-tuple of strategies (strategy profile) $u(\cdot) = (u_1(\cdot), \ldots, u_n(\cdot))$ in the game on the graph tree K define a *payoff function* in the following way. Let a strategy profile $u(\cdot) = (u_1(\cdot), \ldots, u_n(\cdot))$ generate a path $\{x_0, x_1, \ldots, x_l\}$ from the initial vertex x_0 to a terminal one x_l. Then the payoff function of a player i has the form:

$$H_i(u(\cdot)) = \sum_{x\in\{x_0,\ldots,x_l\}} \gamma_i(x), \quad i \in N.$$

Here $\gamma_i(x)$ is a payment to the player i. The payment corresponds to i-th component of the Shapley value, which is computed for $v(x, \cdot)$ defined for the network game $G_x = (N, \theta(x))$ in the vertex x (see Sec. 2.2.).

Definition 4. A strategy profile $u^*(\cdot) = (u_1^*(\cdot), \ldots, u_i^*(\cdot), \ldots, u_n^*(\cdot))$ is called a Nash equilibrium in a multistage network game on a graph tree K with the initial vertex x_0 if

$$H_i(u^*(\cdot)||u_i(\cdot)) \leq H_i(u^*(\cdot))$$

for each $i \in N$ and each admissible u_i.

3. The construction of Nash equilibrium in the multistage network game

Suppose that the game length is equal to $l+1$. To define an optimal behavior of each player we use the concept of Nash equilibrium in a finite multistage game with perfect information.

Introduce the Bellman's function φ_i^t as a payoff of the player i in a Nash equilibrium (Nash, 1951) in the $l-t$ stage game (suppose $\varphi_i^{l+1} = 0$). Values of the Bellman's function φ are defined in a common way using a backward induction (solving the Bellman's equation with boundary conditions at the terminal vertices) in each vertices of the graph tree K.

In this case the boundary condition has the form:

$$\varphi_i^l(x_l) = \gamma_i(x_l), \quad i \in N$$

for each terminal vertex $x_l \in P_{n+1}$

In an intermediate vertex x_t of the graph tree K Bellman's function satisfies the following functional equation:

$$\begin{aligned}\varphi_{i(x_t)}^t(x_t) &= \max_{y \in F_{x_t}} \left(\gamma_{i(x_t)}(x_t) + \varphi_{i(x_t)}^{t+1}(y)\right) = \\ &= \gamma_{i(x_t)}(x_t) + \max_{y \in F_{x_t}} \left(\varphi_{i(x_t)}^{t+1}(y)\right) = \\ &= \gamma_{i(x_t)}(x_t) + \varphi_{i(x_t)}^{t+1}(\bar{y}). \end{aligned} \tag{5}$$

For a player $j \neq i(x_t)$ the values of Bellman's function are obtained from the condition:

$$\varphi_j^t(x_t) = \gamma_j(x_t) + \varphi_j^{t+1}(\bar{y}). \tag{6}$$

Solving Bellman's equation we obtain values φ_i^t, $t = 0, \ldots, l$, $i \in N$. At $t = 0$ the equation has completely solved. An n-dimensional profile $(\varphi_1^0(x_0), \ldots, \varphi_n^0(x_0))$ we call the *value of the multistage network game*.

Together with the value of the multistage network game we obtain the *optimal players strategies*, which constitute the subgame perfect equilibrium: in each vertex $x \in X$ of the graph tree K a player $i(x)$ chooses a vertex $y \in F_x$ in accordance with the rule (4). In the equilibrium the path from the initial vertex to a terminal one is realized. Such path we call *the optimal path in the multistage network game*.

Theorem 1. *An n-tuple of strategies $u^*(\cdot) = (u_1^*(\cdot), \ldots, u_1^*(\cdot))$, where for each vertex $x \notin P_{n+1}$ a strategy $u_i^*(\cdot)$, $i \in N$ is defined as $u_i^*(\cdot) = \bar{y}$, and $\bar{y}$ can be found using (4), constitutes a subgame perfect equilibrium.*

Remark Let a vertex $\bar{y} \in F_{x_t}$ maximize the function $\varphi_{i(x_t)}^{t+1}(y)$ in (4). Suppose that a vertex $\tilde{y} \in F_{x_t}$ $(\tilde{y} \neq \bar{y})$ is also a maximum point of this function. Obviously, the following equality holds:

$$\varphi_{i(x_t)}^{t+1}(\bar{y}) = \varphi_{i(x_t)}^{t+1}(\tilde{y}),$$

which lead us to the same value $\varphi^t_{i(x_t)}(x_t)$. Therefore, a player who makes a decision in the vertex x_t (a player $i(x_t)$) may choose any vertex $y \in F_{x_t}$ which maximize the function $\varphi^{t+1}_{i(x_t)}(y)$ in (4).

But, generally, in vertices $\bar{y}$ and $\tilde{y}$ for each player $j \in N$, $j \neq i(x_t)$ the following inequality is true:

$$\varphi^{t+1}_j(\bar{y}) \neq \varphi^{t+1}_j(\tilde{y}).$$

This means that the player $i(x_t)$ choosing a point from the set

$$I(x_t) = \arg\max_{y \in F_{x_t}} \varphi^{t+1}_{i(x_t)}(y) \tag{7}$$

influences the decision of the following player (because of the difference between values of the Bellman's functions of a players followed by $i(x_t)$ in points of the set $I(x_t)$). This implies that in general in the multistage network game the optimal path is not unique.

Nonuniqueness of the optimal trajectory can be avoided by introducing the notion of *indifferent Nash equilibrium* in the multistage game with perfect information (Petrosjan and Mamkina, 2005).

Since in generally $|I(x_t)| \geq 1$, the player $i(x_t)$ is supposed to choose each vertex from the set $I(x_t)$ with equal probabilities, i. e. $p^{x_t}(y) = 1/|I(x_t)|$, for each $y \in I(x_t)$. Then for an intermediate vertex x_t of the graph tree K, φ^t_i satisfies the following equation (similar to (4)):

$$\varphi^t_{i(x_t)}(x_t) = \gamma_{i(x_t)}(x_t) + \frac{1}{|I(x_t)|} \cdot \sum_{y \in I(x_t)} \varphi^{t+1}_{i(x_t)}(y). \tag{8}$$

For the player $j \neq i(x_t)$, a value of φ can be calculated as follows:

$$\varphi^t_j(x_t) = \gamma_j(x_t) + \frac{1}{|I(x_t)|} \cdot \sum_{y \in I(x_t)} \varphi^{t+1}_j(y). \tag{9}$$

Solving Bellman's equation we obtain values φ^t_i, $t = 0, \ldots, l$, $i \in N$. At $t = 0$ the equation has completely solved. An n-dimensional profile $(\varphi^0_1(x_0), \ldots, \varphi^0_n(x_0))$ we also call the *value of the multistage network game*.

Similar to the Theorem 1, the following theorem is true.

Theorem 2. *An n-tuple of strategies $u^{IE}(\cdot) = (u^{IE}_1(\cdot), \ldots, u^{IE}_n(\cdot))$, where for each vertex $x \notin P_{n+1}$ a strategy*

$$u^{IE}_i(x) = \{p^x(y)\}, y \in I(x), p^x(y) = \frac{1}{|I(x)|}, \quad i \in N,$$

constitutes a subgame perfect equilibrium. Here y can be found using (7)–(8).

4. Numerical example

To illustrate the Nash equilibrium constraction algorithm in the network game we give a numerical example.

Consider a 3-stage network game. Let $N = \{1, 2, 3\}$ be the set of players. Construct a graph tree K with the initial vertex x_0.

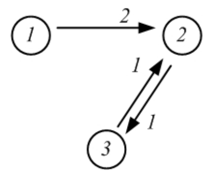

Fig. 1. Network G_{x_0}

Let in x_0 the network shown in the Fig. 1 is given.

The set of edges is $g^{x_0} = \{(1,2),(2,3),(3,2)\}$. The utility matrix $\Theta(x_0)$ has the form:

$$\Theta(x_0) = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Suppose that in the initial vertex the Pl. 1 makes a move. He has 3 alternatives: (1) not to take any actions (the game process moves to the vertex x_1); (2) break the connection with Pl. 2 (the game process moves to the vertex x_2); (3) propose a connection to Pl. 3 (the game process moves to the vertex x_3). We get:

$$\begin{aligned} &g^{x_1} = g^{x_0}, && \text{if Pl. 1 chooses the first alternative in } x_0;\\ &g^{x_2} = g^{x_0} \setminus (1,2), && \text{if Pl. 1 chooses the second alternative in } x_0;\\ &g^{x_3} = g^{x_0} \cup (1,3), && \text{if Pl. 1 chooses the third alternative in } x_0. \end{aligned}$$

Suppose that in x_1, x_2, x_3 utility matrices are:

$$\Theta(x_1) = \begin{pmatrix} 0 & -3 & -1 \\ 2 & 0 & 2 \\ 5 & 1 & 0 \end{pmatrix}, \Theta(x_2) = \Theta(x_3) = \begin{pmatrix} 0 & 3 & -2 \\ -1 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}.$$

Let x_1 and x_3 be terminal vertices, and x_2 is a personal position of Pl. 2.

In x_2 Pl. 2 has 3 alternatives: (1) not to take any actions (the game process moves to the vertex x_4); (2) propose a connection to Pl. 1 (the game process moves to the vertex x_5); (3) break the connection with Pl. 3 (the game process moves to the vertex x_6). We get:

$$\begin{aligned} &g^{x_4} = g^{x_2}, && \text{if Pl. 2 chooses the first alternative in } x_2;\\ &g^{x_5} = g^{x_2} \cup (2,1), && \text{if Pl. 2 chooses the second alternative in } x_2;\\ &g^{x_6} = g^{x_2} \setminus (2,3), && \text{if Pl. 2 chooses the third alternative in } x_2. \end{aligned}$$

Suppose that in x_4, x_5, x_6 utility matrices are:

$$\Theta(x_4) = \begin{pmatrix} 0 & -3 & -1 \\ 2 & 0 & 2 \\ 5 & 1 & 0 \end{pmatrix}, \Theta(x_5) = \Theta(x_6) = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 2 \\ 2 & 4 & 0 \end{pmatrix}.$$

Let x_4 and x_6 be terminal vertices, and x_5 is a personal position of Pl. 3.

In x_5 Pl. 3 has 3 alternatives: (1) not to take any actions (the game process moves to the vertex x_7); (2) propose a connection to Pl. 1 (the game process moves to the vertex x_8); (3) break the connection with Pl. 2 (the game process moves to the vertex x_9). We get:

$$\begin{aligned}
g^{x_7} &= g^{x_5}, && \text{if Pl. 3 chooses the first alternative in } x_5;\\
g^{x_8} &= g^{x_5} \cup (3,1), && \text{if Pl. 3 chooses the second alternative in } x_5;\\
g^{x_9} &= g^{x_5} \setminus (3,2), && \text{if Pl. 3 chooses the third alternative in } x_5.
\end{aligned}$$

Suppose that in x_7, x_8, x_9 utility matrices are:

$$\Theta(x_7) = \Theta(x_8) = \Theta(x_9) = \begin{pmatrix} 0 & -3 & -1 \\ 2 & 0 & 2 \\ 5 & 1 & 0 \end{pmatrix}.$$

Let x_7, x_8, x_9 be terminal vertices.

Then sets of personal positions P_1, P_2, P_3 and the set of terminal ones P_4 have the form:

$$\begin{aligned}
P_1 &= \{x_0\},\\
P_2 &= \{x_2\},\\
P_3 &= \{x_5\},\\
P_4 &= \{x_1, x_3, x_4, x_6, x_7, x_8, x_9\},
\end{aligned}$$

and the graph tree K is shown in Fig. 2.

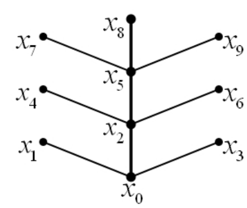

Fig. 2. Graph tree K

First, compute individual payments to each player in each vertex of the graph tree K.

Consider x_0. Construct a characteristic function by the rule (1):

$$\begin{aligned}
&v(x_0, \{1,2,3\}) = 4,\\
&v(x_0, \{1,2\}) = 2,\\
&v(x_0, \{1,3\}) = 0,\\
&v(x_0, \{2,3\}) = 2,\\
&v(x_0, \{1\}) = v(x_0, \{2\}) = v(x_0, \{3\}) = 0.
\end{aligned}$$

Individual payments to players in x_0 are computed in accordance with the Shapley value (4). We get:

$$\gamma(x_0) = (1, 2, 1).$$

Individual payments to players in others vertices of the graph tree K are computed similarly. We give the final values:

$$\begin{aligned}
&\gamma(x_1) = (-1.5, 0, 1.5), && \gamma(x_6) = (0, 2, 2),\\
&\gamma(x_2) = (0, 1, 1), && \gamma(x_7) = (1, 2.5, 1.5),\\
&\gamma(x_3) = (0.5, 2.5, 0), && \gamma(x_8) = (3.5, 2.5, 4),\\
&\gamma(x_4) = (0, 1.5, 1.5), && \gamma(x_9) = (1, 2, 1).\\
&\gamma(x_5) = (-0.5, 2.5, 3), &&
\end{aligned}$$

After computing payments to players in each vertex of the graph tree K, the computation of the equilibrium in the multistage network game does not present any difficulties. This procedure is fully similar to the construction of the Nash equilibrium in a multistage game with perfect information with the difference that in the classical setting players payoffs are defined in terminal vertices, and in intermediate they are equal to zero. Desired equilibrium in the multistage network game is found by using (4)-(6).

Equilibrium players strategies are:

$$\begin{aligned} u_1^*(x_0) &= x_2, \\ u_2^*(x_2) &= x_5, \\ u_3^*(x_5) &= x_8. \end{aligned}$$

In the equilibrium (u_1^*, u_2^*, u_3^*) the optimal path $\{x_0, x_2, x_5, x_8\}$ from the initial vertex x_0 to the terminal one x_8 is realized.

Along the optimal path the game evolves in the following way. At the initial stage the network G_{x_0} shown in Fig. 1 is given. Then Pl. 1 breaks the connection with Pl. 2. This leads to the network G_{x_2} shown in Fig. 3. Then Pl. 2 makes a move

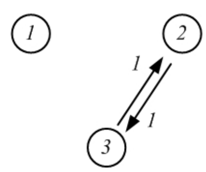

Fig. 3. Network G_{x_2}

and proposes a connection to Pl. 1. This leads to the network G_{x_5} shown in Fig. 4. And finally Pl. 3 ends the game proposing the connection to Pl. 1. This leads to

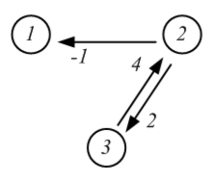

Fig. 4. Network G_{x_5}

the network G_{x_8} shown in Fig. 5.

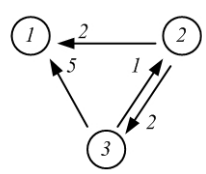

Fig. 5. Network G_{x_8}

The value of the multistage network game equals to $(4, 8, 9)$ and stage payments to players are as follows:

$$\gamma(x_0) = (1, 2, 1),$$
$$\gamma(x_2) = (0, 1, 1),$$
$$\gamma(x_5) = (-0.5, 2.5, 3),$$
$$\gamma(x_8) = (3.5, 2.5, 4).$$

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Best Response Digraphs for Two Location Games on Graphs

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Abstract. We investigate two classes of location games on undirected graphs, where two players simultaneously place one facility each on a vertex. In the first class, called 'Voronoi games', the payoff for a player is the number of vertices closer to that player's facility than to the other one, plus half of the number of vertices with equal distance. For the other class, called 'restaurant location games', the payoff for a player equals 1 plus κ times the number of private neighbors plus $\kappa/2$ times the number of common neighbors, if both locations are different, and 1/2 plus $\kappa/2$ times the number of common neighbors provided both locations are identical, for some constant κ. For both classes the question of the existence of pure Nash equilibria is investigated. Although Voronoi games, which are obviously constant-sum games, do not need to have pure Nash equilibria, Nash equilibria exist if the play graphs are trees. Restaurant location games have always at least one pure Nash equilibrium. We also try to express these Nash equilibria in graph-theoretical terms, and investigate the structure of so-called best response digraphs for the games in relation to the structure of the underlying play graph.

Keywords: simultaneous games, graphs, best response digraph, pure Nash equilibria.

We consider simultaneous 2-player location games played on weighted graphs. The games are defined by two ingredients: There is an underlying undirected graph $G = (V, E)$—the play graph— which may have positive vertex weights $w(x), x \in V$ and positive edge weights $w(xy), xy \in E$. The second ingredient is a nonincreasing function f defined on nonnegative real numbers. Players A and B simultaneously select one vertex, let's say vertices a and b. Let V_A, V_B, and V_0 denote the sets of those vertices that are closer to a than to b, closer to b than to a, or with equal distance to a and b, respectively. In other words, $V_A = \{x|d_G(x,a) < d_G(x,b)\}$, $V_B = \{x|d_G(x,a) > d_G(x,b)\}$, and $V_0 = \{x|d_G(x,a) = d_G(x,b)\}$. Of course the distance is defined using the edge weights, $d_G(x,y)$ is the minimum sum of edge weights, taken over all paths from x to y. For such choice of moves, the payoffs for players A and B are defined by

$$A(a,b) = \sum_{x \in V_A} f(d(x,a))w(x) + \frac{1}{2} \sum_{x \in V_0} f(d(x,a))w(x)$$
$$B(a,b) = \sum_{x \in V_B} f(d(x,b))w(x) + \frac{1}{2} \sum_{x \in V_0} f(d(x,b))w(x)$$

The graph could model a physical street network with villages as vertices. The players simultaneously place one facility on some vertex. People may or may not visit such a facility, depending on the distance they would have to travel, but if they do, they will always choose the closest one, no matter to which one of the players it belongs. The numbers $f(d)$ would model the fraction of the population that would visit if the distance to travel would be d. The payoff for each player is the total number of customers visiting his or her facility.

It is possible to modify the games by allowing each player to place k facilities. In this paper we will concentrate on the single-facility case $k = 1$.

The games are obviously symmetric. They are ususally not zero-sum, except in the case of $f(x) = 1$, where all customers always visit one of the locations.

The paper concentrates on two special cases for the function f. The case of a constant function $f(x) = 1$ is usually called **Voronoi game**. We sometimes call it **supermarket location game**, since every possible customer *has* to visit one of the locations. The other special case, in a sense on the opposite side, is the case where $f(2) = 0$ and all edge-weights are greater or equal to 1. This case we name **restaurant location game**, assuming that customers are not willing to go very far for a (possibly fast-food) restaurant. In the Voronoi game case, we will focus on special graphs, namely trees and chordal graphs.

For these two special special classes of games, we will discuss the question of how the best response digraph looks like for these games, in particular in relation to the underlying undirected play graph. The best response digraph, which will be defined in the next section, captures the notion of 'best responses'. We will also see that every restaurant location game, but not every Voronoi game, has some pure Nash equilibria. For Voronoi games, pure Nash equilibria exist under the condition that the play graph is a tree.

Location games have been initiated by H. Hotelling (Hotelling, 1929), considering the case of location on a straight line. Later, location games for other geometrical patterns have been investigated. Location games on graphs have been investigated by V. Knoblauch in two papers in 1991 and 1995. However, the models she considered differ from ours. In Knoblauch, 1991, location everywhere on the edges is allowed. In Knoblauch, 1995, location is allowed only on some specified vertices of the graph, and only functions f of the form $f(x) = 1$ for $x \leq m$, and $f(x) = 0$ otherwise, are considered. In this second model, Knoblauch could show that every simultaneous 2-person symmetric game is equivalent to such a simultaneous location game on a graph.

There is a large literature on Voronoi games played in rectangular or square spaces, but mostly the games are played sequentially, and each player has several facilities, compare Cheong et al., 2004. Teramoto et al., 2006, discuss multiple-facility Voronoi games on graphs, but also sequential ones. Dürr and Nguyen Kim, 2007, also discuss Voronoi games, but with emphasis on more than two players.

1. Best Response Digraphs

Before discussing these location games, let us introduce the best response digraphs, which will be used later. For Normal Form games, the notion of a *best response* is cruical for the definition of pure Nash equilibria. In the case of two players, this notion could be modeled by a bipartite digraph, or a general digraph in case of symmetric games. Recall that a game is symmetric if both players have the same strategy sets V, and $A(x, y) = B(y, x)$ for every $x, y \in V$, where $A(x, y)$ and $B(x, y)$

denote the payoffs of player A or B if player A plays strategy x and player B chooses strategy y. Then the *(condensed) best response digraph* of the game has V as vertex set, and an arc from x to y, for $x, y \in V$, if y is among the best responses for one player to move x for the other player.

In this (condensed) best response digraph, pure Nash equilibria appear either as pairs of vertices (x, y) with both (x, y) and (y, x) arcs in the best response digraph—such antidirected arcs are called *digons*—, or pairs (x, x) with (x, x) an arc in the digraph—a so-called *loop*. Thus pure Nash equilibria can be recognized very easily in the best response digraph.

It is easy to see that every digraph where every vertex has at least one out-going arc is the best response digraph of some symmetric two-player game. We can even construct such a game where all payoffs are just the numbers 0 or 1. We define $B(x, y) = 1$ if (x, y) is an arc in D, and $B(x, y) = 0$ otherwise. By defining the payoffs for player A by $A(x, y) = B(y, x)$ we get a symmmetric game with our digraph as best response digraph.

2. Constant-sum symmetric games

A two-player game is *constant-sum* if there is some constant c such that $A(x, y) + B(x, y) = c$ for every strategy x of player A and every strategy y of player B. By concentrating on constant-sum games, the shape of the occuring best response digraphs is further restricted.

Theorem 1. *A digraph is the best response digraph of a two-player symmetric constant-sum game if and only if*

(1) every vertex has at least one outgoing arc, and
(2) For every $x \neq y \in V$, there is a digon between x and y if and only if there are loops both at x and y.

Proof. Given a digraph D obeying the two conditions above, we show how to construct a zero-sum two-player game with all payoffs equal to -2, -1, 0, 1, 2, whose best response digraph equals D. Since the game is supposed to be zero-sum, it suffices to define the payoffs $A(x, y)$ for player A.

Let V_0 be the set of all loop vertices. We will construct a game where best responses to strategies in V_0 have a payoff of 0, and best responses to strategies not in V_0 have a payoff of 2.

For every $x \in V$, we must define $A(x, x) = 0$. For the definition of the other values $A(x, y)$ we distinguish six cases:

- Case $x, y \in V_0$: In this case, according to condition (2) above, both (x, z) and (y, x) are arcs, and we define $A(x, y) = A(y, x) = 0$.
- The case where exactly one of x, y lies in V_0, for instance $x \in V_0, y \notin V_0$: Since by condition (2) not both (x, y) and (y, x) can be arcs, we get three subcases:
  - (x, y) is an arc but (y, x) not. We define $A(x, y) = A(y, x) = 0$.
  - (y, x) is an arc but (x, y) not. We define $A(x, y) = 2$ and $A(y, x) = -2$.
  - None of $(x, y), (y, x)$ is an arc. Then we define $A(x, y) = 1$ and $A(y, x) = -1$.
- Case $x, y \notin V_0$: Again by condition (2) not both (x, y) and (y, x) can be arcs, so we get two subcases:
  - One of $(x, y), (y, x)$ is an arc, let's say (x, y). We define $A(x, y) = -2$ and $A(y, x) = 2$.

- None of $(x, y), (y, x)$ is an arc. We define $A(x, y) = A(y, x) = 0$.

That we constructed a symmetric game is obvious, since in each of the cases we defined payoffs fulfilling $A(x, y) = -A(y, x)$. That exactly all out-neighbors y of a vertex x achieve the maximum payoff $A(y, x)$ can also be seen by just checking these cases.

Conversely we show that the best response digraph of a two-player constant-sum game obeys the two properties above. The first one is obvious. Note that $A(x, x) = B(x, x) = \frac{c}{2}$ for all strategies x, if c is the constant sum of payoffs. Therefore the best response to every strategy must have a payoff of at least $\frac{c}{2}$, and those strategies x where $\frac{c}{2}$ is best possible are exactly those where the loop (x, x) is an arc in the best response digraph—the 'loop vertices'.

Now choose two loop vertices x and y, implying $A(y, x) \leq \frac{c}{2}$ and $A(x, y) \leq \frac{c}{2}$. Using the second inequality, and exploiting symmetry and the c-sum property, we obtain $A(y, x) = B(x, y) = c - A(x, y) \geq c - \frac{c}{2} = \frac{c}{2}$. Thus $A(y, x) = \frac{c}{2}$, meaning that y is a best response to x, and in the same way x is also a best response to y, so we have a digon in the best response digraph.

On the other hand, if we have a digon between two vertices x and y, then, since $A(x, y) \geq \frac{c}{2}$ and $A(y, x) \geq \frac{c}{2}$, but $A(y, x) = B(x, y) = c - A(x, y) \leq c - \frac{c}{2} = \frac{c}{2}$, thus equality $A(x, y) = \frac{c}{2}$ and in the same way also $A(y, x) = \frac{c}{2}$, thus x and y must both be loop vertices then. □

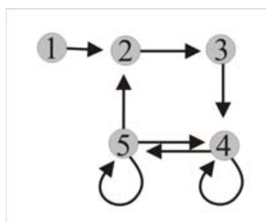

Fig. 1. A digraph

Take as an example the digraph shown in Figure 1. This digraph obeys both conditions of Theorem 1, therefore it must be the best response digraph of a two-player symmetric constant-sum game. Using the construction described in the proof, we get the game shown in Table 1.

Table 1. Normal Form whose best response digraph equals the digraph in Figure 1

	1	2	3	4	5
1	0,0	-2,2	0,0	-1,1	-1,1
2	**2**,-2	0,0	-2,2	-1,1	**0**,0
3	0,0	**2**,-2	0,0	-2,2	-1,1
4	1,-1	1,-1	**2**,-2	**0**,0	**0**,0
5	1,-1	0,0	1,-1	**0**,0	**0**,0

Player A's highest payoffs in each column are highlighted in bold. The corresponding row strategy is player A's best response to the corresponding column strategy of player B.

3. Voronoi Games (Supermarket Location)

Recall that this class of location games is defined by the condition $f(x) = 1$ for all $x \geq 0$. That implies that we get a constant-sum game, since all possible customers will visit one of the locations. Therefore the sum of the payoffs equals the sum $\sum_{x \in V} w(x)$ of all vertex weights.

The best response digraphs of these Voronoi games have to obey the conditions described in Theorem 1. However, the class of best response digraphs of these games is strictly smaller than that, since we have

Proposition 1. *The directed cycles C_3 and C_4 are not best response digraph of any Voronoi game.*

This can be shown by looking on all 3- or 4-vertex graphs. It certainly would be interesting to find a characterization of best response digraphs of these games.

For Voronoi games played on the continuous square, best responses are not really defined, but responses get better and better if the answering move gets closer and closer to the other move, in the direction of the center of the square. The only pure Nash equilibrium is where both players place their facility into the center of the square.

If we consider grids $P_n \times P_m$ as discrete versions of squares, we get best responses that have always distance at most 2 from the original move vertex, and three pattern of pure Nash equilibria: If both n and m are odd, the only Nash equilibrum is the pair of both locations in the center. If one of n and m is odd, then a Nash equilibrium is any combination of the two central vertices of the grid. If both n and m are even, then the center consists of four vertices, and any combination of two of them forms a Nash equilibrium. See Figure 2 for an example.

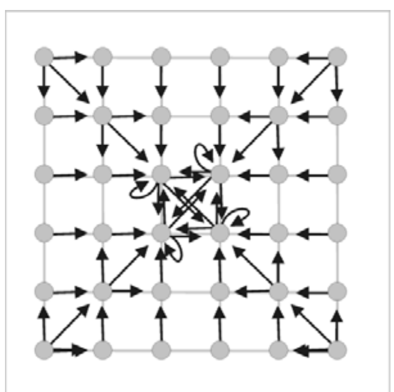

Fig. 2. The best response digraph for the 6×6 grid

Thus for variants of the original geometric Voronoi game, we always have Nash equilibria with pairs of vertices in the 'center' of the graph, and best response arcs do not span large distances in the underlying play graph. However, all this can be different for other graphs, as can be seen by a family of examples generalizing an example given in Dürr and Nguyen Kim, 2007: Construct a graph G_r, $r \geq 2$, from a cycle $x_0, x_1, \ldots, x_{3r-1}$, by adding three vertices y_0, y_1, y_2 and making y_0 adjacent to both x_0 and x_1, y_1 adjacent to both x_r and x_{r+1}, and y_2 adjacent to both x_{2r} and x_{2r+1}. See Figure 3 for G_3. Then none of the Voronoi games on these graphs has a pure Nash equilibrium, and some of the best response arcs of the best response digraph cover diametral vertices in the graph G_r. For instance, for even r, $x_{3r/2}$ is a best response to x_0, and for odd r, both $x_{3r+1/2}$ and $x_{3r-1/2}$ are best responses to x_0.

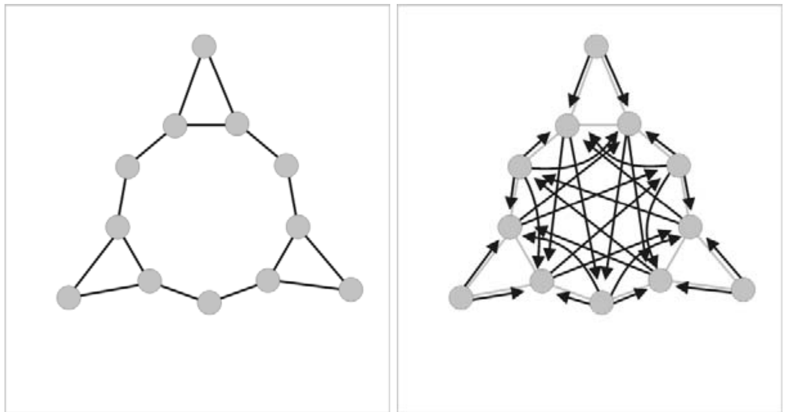

Fig. 3. The best response digraph for supermarket location on G_3

3.1. Trees

We will see that the features of Voronoi games in grids can also be observed trees. First, best responses are always very close to the original vertex:

Lemma 1. *If the play graph of a single-location Voronoi game is a tree T (with any positive vertex- and edge weights), every best response vertex to a vertex y is always the vertex y itself, or a vertex T-adjacent to y. Therefore all arcs of the best response digraph are either loops or span edges of the tree.*

Proof. Take nonadjacent vertices x and y. Let z be the neighbor of y on the unique y-x path in T. Then $B(y,z) \geq B(y,x)$, since every vertex closer to x than to y is also closer to z than to x. If $z \neq x$, then we get strict inequality. Therefore, in that case, x is not the best response to y. $\square$

The *branch weight* $bw_T(x)$ of a vertex x in a vertex-weighted tree T is the maximum sum of weights of any connected component of $T \setminus x$. The *w-centroid* of a tree consists of all vertices minimizing this branch-weight. It can be shown that a vertex x lies in the w-centroid of a tree if and only if $bw_T(x) \leq \frac{1}{2}\sum_{w \in V} w(v)$, compare Kariv and Hakimi 1979. Using this definition, one could be a little more precise to tell exactly which neighbor is the best response to a vertex x: It is the neighbor in the branch with weight $bw_T(x)$, as long as this number is greater or equal to $\frac{1}{2}\sum_{w \in V} w(v)$, and $bw_T(x)$ is the payoff for this best response. Since one can always get a payoff of $\frac{1}{2}\sum_{w \in V} w(v)$ by placing on the same vertex, if $bw_T(x) = \frac{1}{2}\sum_{w \in V} w(v)$, then both x itself and also the neighbor y in this branch are best responses to x. But in this case, x is also a best response to x. If $bw_T(x) < \frac{1}{2}\sum_{w \in V} w(v)$, then the only best response to x is x itself. We have

Theorem 2. *The best response digraph of the Voronoi game played on a weighted tree has loops in all vertices of the w-centroid. Thus the Nash equilibria consist of each player placing their facility on such a vertex in the w-centroid.*

Note that edge weights play no role in the analysis so far.

There are questions of how well the resulting pattern of facilities serves the population as a whole. For Voronoi games, a measure would be the sum of the shortest distances to the closest facility, summed over all vertices. There are many papers in the graph-theory literature about how to place facilities to minimize this measure. For 1 or 2 facilities, the corresponsing solution is called *1-median* respectively *2-median*. Kariv and Hakimi 1979 showed that for weighted trees, a vertex is in the w-centroid if and only if it is a 1-median. Therefore both players will place their facility as if they were supposed to minimize the distance sums to all vertices, and as if the other player would not exist. Of course the locations achieved playing the game will have a much higher distance sum than the 2-medians.

3.2. Chordal Graphs

Trees do not contain cycles. Chordal graphs are the graphs without induced (chordless) cycles of length of four or more. They are also generalizations of trees since they can be characterized as intersection graphs of trees: A graph $G = V, E)$ is chordal if and only there is some tree T and subtrees $T_x, x \in V$ of T such that vertices x and y of G are adjacent in G if and only if the corresponding subtrees T_x and T_y have nonempty intersection (Buneman, 1974, Gavril, 1974, Walter, 1978). Another concept is also useful: Given such a tree representation $(T_x | x \in V)$ of a chordal graph $G = (V, E)$, let us define T_x^p as the union of all trees T_y where $d_G(x, y) \leq p - 1$. Then we have $T_x = T_x^1$, but also the fact that $d_G(x, y) \leq p$ if and only if T_x and T_y^p have nonempty intersection.

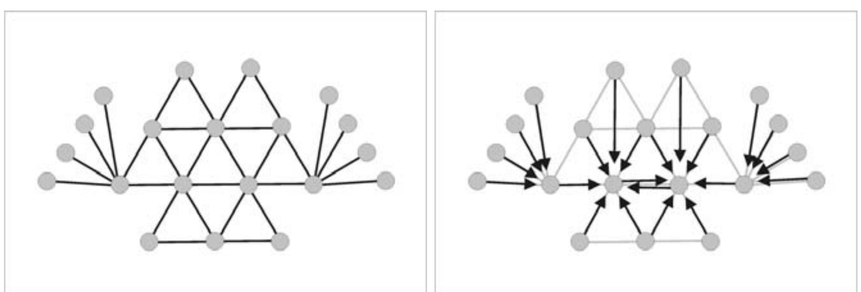

Fig. 4. A chordal graph and its best response digraph for supermarket location.

Lemma 1 is not longer true for chordal graphs. However, arcs of the best response digraph will not have a distance larger than 2 in the chordal graph.

Theorem 3. *For the Voronoi game on a connected chordal graph G with all edge-weights equal to 1, every best response vertex x to a vertex y obeys $d_G(x, y) \leq 2$.*

Proof. Assume to the contrary that x is a best response to vertex y on a connected chordal graph $G = (V, E)$ with $d_G(x, y) > 2$. Let G be the intersection graph of the family $(T_x / x \in V)$ of subtrees of some tree T. Then T_x and T_y are disjoint. Let $s_0, s_1, \ldots s_m$ be the path connecting T_x and T_y, with $s_0 \in V(T_x)$, $s_m \in V(T_y)$, but all other vertices $s_1, \ldots s_{m-1}$ outside of T_x and T_y. Choose that vertex $z \in V$ where T_z contains s_0 and s_k, with k maximum among all such subtrees. By our assumption $d_G(x, y) > 2$ we know that $k < m$. We claim that z is a better response to y, which gives the desired contradiction.

Let V_x denote those vertices closer to x than to y, V_y those vertices closer to y than to x, and V_0 those vertices having the same distance. All neighbors of x belong to V_x, all neighbors of y belong to V_y. Moreover, every vertex v where T_x separates T_v and T_y belongs to V_x, and every vertex v where T_y separates T_v and T_x belongs to V_y. These latter vertices, and also all neighbors of y, are closer to z than to x. Thus, for the vertices considered so far, moving from y to z does not make a difference.

Now we look at the remaining vertices v. For them, the path between T_v and T_x, as well as the path between T_v and T_y meet the path $s_0, s_1, \ldots s_m$ first. Let p be the highest integer for which T_v^p is still disjoint to T_x and T_z. Then, if T_v^{p+1} intersects T_y, it also intersects T_z. Thus, all these v are not further away to z than to y.

We have seen that z is as good a response to x than y. Now look at the shortest path $x = x_0, x_1, \ldots, x_m = z$ between x and z. If m is even, $x_{m/2}$ has the same distance to x and z, but is closer to x than to y by the choice of z, therefore z is a better response to x than y. If m is odd, then $x_{(m+1)/2}$ is closer to z than to x, but has the same distance to x and y, therefore z is a better response to x than y also in this case. □

We conjecture that actually every Voronoi game on every chordal graph has some pure Nash equilibria.

4. Restaurant Location Games

We first need a general theorem for games of a special shape:

Theorem 4. *Assume a symmetric two-player game has the properties that*

1. *for every option x there is a value $\kappa(x)$, and*
2. *for every two options x, y there is a value $\gamma(x, y) = \gamma(y, x)$, such that*
3. *if player A chooses option x and player B option y, then A's payoff equals $A(x, y) = \kappa(x) - \gamma(x, y)$, and B's payoff equals $B(x, y) = \kappa(y) - \gamma(x, y)$.*

Then all arcs in every directed cycle of length at least 2 in the (condensed) best response digraph are digons, and in particular the game has at least one Nash equilibrium in pure strategies.

Proof. Let $x_0, x_1, \ldots x_m, x_0$ be a directed cycle in the best response digraph of the game. In case $m = 1$ we have a loop, and in case $m = 2$ we just have a digon, so let's assume that the length m of the directed cycle is greater or equal to 3.

Then x_1 is best response to x_0, x_2 is best response to x_1, and so on, and x_0 is best response to x_m. We get

$$\begin{aligned} B(x_0, x_1) &\geq B(x_0, x_m), \\ A(x_2, x_1) &\geq A(x_0, x_1), \\ B(x_2, x_3) &\geq B(x_2, x_1), \\ A(x_4, x_3) &\geq A(x_2, x_3), \\ &\cdots \end{aligned}$$

Proceeding in this way, we either arrive at $A(x_0, x_m) \geq A(x_{m-1}, x_m)$ in case of an even cycle length m, or we arrive at $B(x_m, x_0) \geq B(x_m, x_{m-1})$, and proceed with $A(x_1, x_0) \geq A(x_m, x_0), \ldots$ until we eventually arrive at $A(x_0, x_m) \geq A(x_{m-1}, x_m)$

In other words, we get

$$\begin{aligned}
\kappa(x_1) - \gamma(x_0, x_1) &\geq \kappa(x_m) - \gamma(x_0, x_m),\\
\kappa(x_2) - \gamma(x_2, x_1) &\geq \kappa(x_0) - \gamma(x_0, x_1),\\
\kappa(x_3) - \gamma(x_2, x_3) &\geq \kappa(x_1) - \gamma(x_2, x_1),\\
\kappa(x_4) - \gamma(x_4, x_3) &\geq \kappa(x_2) - \gamma(x_2, x_3),\\
\cdots &\geq \cdots\\
\kappa(x_0) - \gamma(x_0, x_m) &\geq \kappa(x_{m-1}) - \gamma(x_{m-1}, x_m)
\end{aligned}$$

where either every $\kappa(x_k)$ and every $\gamma(x_k, x_{k+1}) = \gamma(x_{k+1}, x_k)$ occurs once (in case of even m) or twice (otherwise) on each side.

The sums of the expressions on the left and on the right are equal, therefore we must have equality in each one of the inequalities. That means that x_0 is also a best response to x_1, x_1 is also a best response to x_2, and so on. Therefore each of the pairs x_i and x_{i+1} forms a Nash equilibrium in pure strategies.

Now, since every vertex in a best response digraph is the start of at least one arc, starting with any vertex and moving along arcs, we eventually find either a loop or a cycle of length ≥ 2. Since Nash equilibria correspond to loops or digons in the best response digraph, the existence of pure Nash equilibria follows from the first statement. □

Theorem 5. *In every simultaneous two-player restaurant location game with all vertex and edge weights equal to 1, there is at least one Nash equilibrium in pure strategies.*

Proof. For every vertex x of G, let $N(x)$ be the set of all neighbors of x in G, We define

$$\kappa(x) = f(0) + f(1)|N(x)|.$$

$\kappa(x)$ is the payoff a player would get placing the location at x if there would not be another player. From this value, some value is subtracted because there is another player. The part subtracted, however, is identical for both players. It is

$$\begin{aligned}
\gamma(x, x) &= \frac{f(0)}{2} + \frac{f(1)|N(x)|}{2}\\
\gamma(x, y) &= \frac{f(1)|N(x) \cap N(y)|}{2} \text{ if } x \text{ and } y \text{ are distinct and nonadjacent,}\\
\gamma(x, y) &= f(0) + \frac{f(1)|N(x) \cap N(y)|}{2} \text{ if } x \text{ and } y \text{ are distinct and adjacent,}
\end{aligned}$$

Then the payoffs $A(x, y)$ and $B(x, y)$ for player A and B are $A(x, y) = \kappa(x) - \gamma(x, y)$, and $B(x, y) = \kappa(y) - \gamma(x, y)$, therefore, by Theorem 4 the result follows. □

However, location games with $f(2) > 0$ do not have to have a pure Nash equilibrium. An example of a restaurant location game with no pure Nash equilibrium is given in Figure 5 for $f(0) = 1, f(1) = 0.5, f(2) = 0.3, f(4) = 0$.

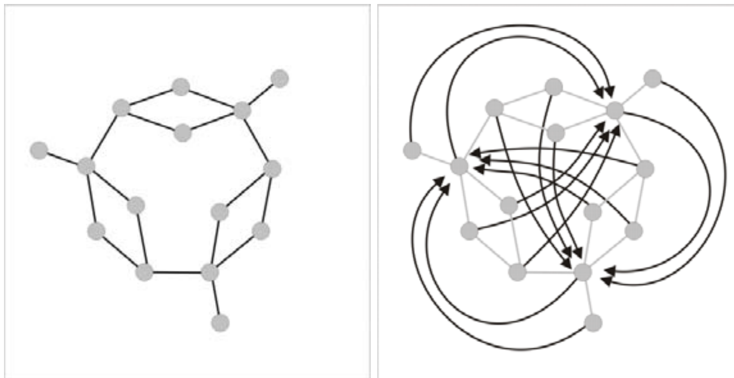

Fig. 5. On the right is the best response digraph for restaurant location with $f(0) = 1, f(1) = 0.5, f(2) = 0.3, f(4) = 0$ on the graph to the left.

The example in Figure 6 shows that, in our restaurant location games, the Nash equilibrium outcomes are not always best-possible for society, or even for a monopoly that would place two restaurants. Different to the Voronoi games, the measure of quality of such a placement may be the number of people visiting one of the facility, i.e. the sum of both payoffs. In the example in Figure 6, 1 versus 5, and also 5 versus 6 are the Nash equilibria, giving payoffs of 2 and 2.5. However, if player A builds in 1 and player B in 8, A gets a payoff of 3 and B a payoff of 2. Still it seems that restaurant location games are closer to the optimum for society than the Vornoi (supermarket location) games.

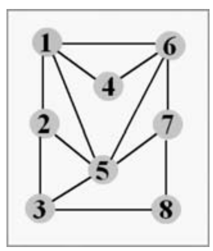

Fig. 6. 1 versus 8 is best for society

Let me close by presenting an example in Figure 7 that shows that if each player places two restaurants, then we do no longer necessarily have pure Nash equilibria.

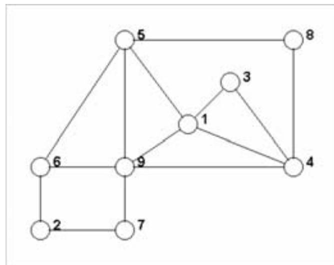

Fig. 7. This graph has no pure Nash equilibrium in the restaurant location model with two restaurants for each player

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Uncertainty Aversion and Equilibrium in Extensive Games*

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Abstract. This paper formulates a rationality concept for extensive games in which deviations from rational play are interpreted as evidence of irrationality. Instead of confirming some prior belief about the nature of non-rational play, we assume that such a deviation leads to genuine uncertainty. Assuming complete ignorance about the nature of non-rational play and extreme uncertainty aversion of the rational players, we formulate an equilibrium concept on the basis of Choquet expected utility theory. Equilibrium reasoning is thus only applied on the equilibrium path, maximin reasoning applies off the equilibrium path. The equilibrium path itself is endogenously determined. In general this leads to strategy profiles differ qualitatively from sequential equilibria, but still satisfy equilibrium and perfection requirements. In the centipede game and the finitely repeated prisoners' dilemma this approach can also resolve the backward induction paradox.

Keywords: rationality, extensive game, uncertainty aversion, perfect equilibrium, backward induction, maximin, Choquet expected utility theory.

1. Introduction

According to the principle of sequential rationality, a rational player of an extensive form game regards his opponents as rational even after a deviation from rational play. The internal consistency of this principle is subject of much debate (see, e.g., Aumann (1995, 1996, 1998), Binmore (1996), Reny (1993)).

Attributing non-rational deviations to 'trembles' of otherwise perfectly rational players (Selten (1975)) is logically consistent, but raises the second concern with the principle of sequential rationality, that is its empirical plausibility. Quite independently of the question whether there exists a rationality concept that implies, or is at least consistent with sequential rationality, the question arises if there is room for an alternative rationality concept, in which deviations from the solution concept are interpreted as evidence of non-rationality. In this paper, we attempt to formulate such a rationality concept on the basis of Choquet expected utility theory.

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In a seminal series of papers, Kreps, Milgrom, Roberts, and Wilson (1982) (henceforth KMRW) developed the methodology for analysing games with possibly non-rational opponents. In their models, there is some a priori uncertainty about the rationality of the opponent. Under subjective expected utility, players act as if they possess a probability distribution over the 'type' of the opponents' non-rationality. They maximise utility given their beliefs, and in sequential equilibrium their beliefs are consistent with the play of rational opponents. KMRW have shown how even small degrees of uncertainty about rationality can have large equilibrium effects. They showed that this can explain both intuitive strategic phenomena, particularly in industrial organization, and, at least to some degree, experimental evidence.

One problem in this approach, however, is for an outside observer to specify the probability distribution over the types of the non-rational opponents before experimental or field data are available. A second problem is that analysing the strategic interaction as a game with incomplete information implies that the other players, whether rational or not, can be modelled as 'types', who possess a consistent infinite hierarchy of beliefs about the strategic interaction. Thus, the players in this methodology are not really non-rational; rather, they are rational but have preferences that differ from those that the game attributes to 'rational' players.

In this paper, we argue that a consistency argument addresses both of these problems. A game-theoretic solution concept that singles out rational strategies implicitly defines all other strategies as non-rational. Thus, consistency requires that beliefs about non-rational players should not exclude any of these non-rational strategies. In other words, if the rationality concept is point-valued, the beliefs about non-rational play should include all deviations, and thus must be set-valued. So in this sense, the rationality concept itself pins down beliefs about non-rational play, but excludes subjective expected utility theory (henceforth SEU) as the adequate model of these beliefs. Thus, SEU is not an appropriate framework for beliefs about non-rationality when rationality is endogenous.

Thus, this paper argues that, after an opponent deviates from rational play, a rational player faces genuine uncertainty. What matters, then, is the rational player's attitude towards uncertainty. This paper formulates the equilibrium concept for the case in which rational players are completely uncertainty averse. It is this case that has led to the development of decision theories with set-valued and non-additive beliefs as an explanation of the Ellsberg (1961) paradox. Consequently, we base the equilibrium concept on Choquet expected utility theory (henceforth CEU) developed by Schmeidler (1989).

This paper joins a growing literature that applies CEU to games. The first of these were Dow and Werlang (1994) and Klibanoff (1993). Dow and Werlang (1994) consider normal form games in which players are CEU maximisers. Klibanoff (1993) similarly considers normal form games in which players follow maxmin-expected utility theory (Gilboa and Schmeidler (1989)), which is closely related to CEU. In Hendon et al. (1995) players have belief functions, which amounts to a special case of CEU. Extensions and refinements have been proposed by Eichberger and Kelsey (1994), Lo (1995*a*), Marinacci (1994) and Ryan (1997). Epstein (1997*a*) analysed rationalizability in normal form games. These authors consider normal form games and do not distinguish between rational and non-rational players. The paper closest to ours is Mukerji (1994), who considers normal form games only but argues that the distinction between rational and non-rational players is necessary to reconcile

CEU with the equilibrium concept. For normal form games our concepts differ only in motivation and technical detail. The present paper mainly concerns extensive form games. Extensive games have been studied by Lo (1995*b*) and Eichberger and Kelsey (1995). Lo (1995*b*) extends Klibanoff's approach to extensive games, Eichberger and Kelsey (1995) are the first to use the Dempster-Shafer updating rule (see section 3) in extensive games. They do not distinguish between rational and non-rational players.

This paper is organized as follows: The next section discusses an example. Section 3 presents Choquet expected utility theory and discusses the problem of updating non-additive beliefs. In section 4 we formulate the equilibrium concept for two player games with perfect information. In section 5 we discuss the centipede game and the finitely repeated prisoners' dilemma in order to relate the equilibrium concept to the foundations of game theory. Section 6 elaborates on the extension of the solution concept to general extensive games. Section 7 concludes. There is one appendix on details of updating non-additive beliefs.

2. An Example

Consider the following extensive form game, in which payoffs are given in von Neumann - Morgenstern utilities[1]:

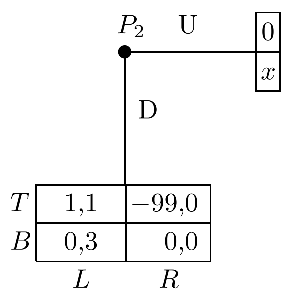

Fig. 1. A simultaneous game preceded by an outside option

First, consider the case that $x = 4$. Then D cannot be rational for P_2, because it is strictly dominated. Therefore, P_1 knows at the beginning of the subgame that P_2 is not rational and, consequently, has no reason to assume that P_2 will play his strictly dominant strategy L in the subgame. P_1's best reply to L is T, but, intuitively, it is very risky.

In the absence of a theory of rationality, P_1 faces true uncertainty about P_2's play in the subgame. Therefore, if P_1 is sufficiently uncertainty averse, it becomes rational for him to play B. Thus, under these assumptions the rational strategies are U, L (because it is strictly dominant in the subgame) and B. This strategy combination is not a Nash equilibrium, yet no player has an incentive to deviate unilaterally from these rational strategies. Moreover, if there is some initial doubt

[1] After D, both players P_1 and P_2 know that P_2 chose D and they play the normal form subgame, i.e. choose simultaneously between T and B, respectively L and R.

$\epsilon > 0$ about P_2's rationality[2], then D is also not a probability zero event, because nothing is known about a non-rational player, who might therefore well play D[3]

All that it takes to reach these conclusion formally is a calculus that allows non-additive, or set-valued, beliefs, and that captures P_1's uncertainty as well as his uncertainty aversion. In addition, in order to conclude that that P_2 must be non-rational after D we need an updating rule for non-additive beliefs.

Secondly, consider the case that $x = 2$. The above criticism of subgame perfection still applies: The equilibrium (T, L) in the subgame makes U rational, but once U is designated as rational, P_1 faces true uncertainty after D and, if uncertainty averse, will rationally deviate to B. So (U, L, T) is not a rational solution. However, neither is (D, L, B), because if D is rational then P_1 is justified in anticipating strategy L, and should play his best reply T. Now P_2 has an incentive to deviate.

So suppose it is rational for P_2 to play D with probability p. Suppose further that there is a probability $\epsilon > 0$ that P_2 is not rational at the beginning of the game. Then P_1's optimal strategy in the subgame will depend on his belief about the rationality of P_2, given p and ϵ. The same updating rule that for $x = 4$ allows the natural conclusion that P_2 is rational after D gives the result[4] that

$$v(P_2 \text{ rational} \mid D) = \frac{(1-\epsilon)p}{1-(1-\epsilon)(1-p)}.$$

Note that $v(P_2 \text{ rational} \mid D) = (1-\epsilon)\frac{p}{p+\epsilon(1-p)} < 1-\epsilon$. Thus, in line with his uncertainty aversion P_1 considers the worst case when he updates his belief ϵ. This worst case is that a non-rational player will play D with probability 1, because this makes it most likely that his behavior in the subgame is unpredictable, and, again due to uncertainty aversion, should be evaluated with the worst outcome.

Since a rational P_2 will play L, P_1 knows that T will give utility 1 with probability $\frac{(1-\epsilon)p}{1-(1-\epsilon)(1-p)}$. With the complementary probability, P_2 is non-rational and the theory is silent about what this means. Again, P_1 faces true uncertainty, and if he is extremely uncertainty averse, he will allocate the complementary probability to the worst outcome -99. So P_1's expected utility from T is

$$\frac{(1-\epsilon)p}{1-(1-\epsilon)(1-p)} \cdot 1 - 99 \cdot \frac{\epsilon}{1-(1-\epsilon)(1-p)}.$$

In a mixed strategy equilibrium P_1 must be willing to randomize, so we must have

$$\frac{(1-\epsilon)p - 99\epsilon}{1-(1-\epsilon)(1-p)} = 0,$$

[2] For simplicity, assume in this example that there is no doubt about the rationality of P_1.

[3] If rationality is common knowledge at the beginning of the game, then D is indeed a probability zero event. In general, we take the view that there is a difference between probability zero events in decision theory and probability zero events in games, where an event for one player is an act for another. Here, D is an act that might destroy this common knowledge. It is still intuitive that P_1 should consider P_2 as non-rational. This conclusion could be formally reached by taking limits as $\epsilon \to 0$. However, in this paper we concentrate on the case $\epsilon > 0$.

[4] See the next section and the appendix.

i.e. $p = 99\frac{\epsilon}{1-\epsilon}$. Note, first, that P_1 is willing to mix only if $\epsilon < \bar{\epsilon} = \frac{1}{100}$, i.e. if the initial doubt about rationality is small enough, otherwise T will be too risky. Secondly, as ϵ goes to zero, p goes to zero, i.e. it is less and less rational for P_2 to play D. Both aspects are quite intuitive.

Further, P_2 must be willing to randomize as well, so that we must have $2 = q \cdot 1 + (1-q) \cdot 3$, i.e. $q = \frac{1}{2}$, where q is the probability that P_1 plays T. Overall, the rational strategies for given $\epsilon > 0$ are given by $(p^* = 99\frac{\epsilon}{1-\epsilon}, L, q^* = \frac{1}{2})$ if $\epsilon < \bar{\epsilon}$ and (D, L, B) if $\epsilon \geq \bar{\epsilon}$. Again, no player has an incentive to deviate.

Finally, we can consider the case $\epsilon \longrightarrow 0$. This gives the strategy profile $(U, L, q^* = \frac{1}{2})$. Note, however, that if $\epsilon = 0$ (as opposed to $\epsilon \searrow 0$), P_1 has an incentive to deviate from $q^* = \frac{1}{2}$.

3. Choquet Expected Utility and Updating

Under SEU, a player has preferences over acts that map a set of states of nature S into a set of consequences Z. Under consistency assumptions about this preference ordering, the player acts as if he possesses a utility function u over consequences (cardinal, i.e. unique up to affine transformations), and a probability distribution p over states that represents subjective beliefs, and maximises expected utility. This axiomatisation of SEU is due to Savage (1954). Anscombe and Aumann (1963) have simplified this approach by assuming that acts map states into lotteries (probability distributions) over states.

Ellsberg's paradox (Ellsberg (1961)) provides evidence, however, that players do not necessarily act as if their beliefs are probability distributions. On the contrary, these experiments provide evidence for the hypothesis that beliefs are non-additive, and that players are uncertainty averse.

CEU also considers a preference relation over acts. Under weaker consistency assumptions, a player still acts as if he possesses a cardinal utility function u and subjective beliefs v, and maximises 'expected utility'. The difference to SEU is that beliefs no longer have to be additive. Non-additive beliefs are given by a set function v that maps events (sets of states) into $\mathbb{R}$ such that

(i) $v(\emptyset) = 0$,
(ii) $v(\Omega) = 1$,
(iii) $E \subseteq F \implies v(E) \leq v(F)$.

CEU was first axiomatised by Schmeidler (1989)[5].

The expectation of a utility function with respect to non-additive beliefs v is defined by Choquet (1953). For $u \geq 0$ the Choquet integral is given by the extended Riemann integral

$$\int u \; dv := \int_0^\infty v(u \geq t) dt,$$

where $v(u \geq t)$ is short for $v(\{s \in S | u(s) \geq t\})$. For arbitrary $u = u^+ - u^-$, where $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$ denote the positive and the negative part, the Choquet integral is defined as $\int u \; dv := \int u^+ \; dv - \int u^- \; d\tilde{v}$, where $\tilde{v}$ is the dual of v, i.e. $\tilde{v}(E) := 1 - v(\overline{E})$ and $\overline{E}$ is the complement of E.

Non-additive beliefs express uncertainty aversion[6] if v is supermodular, i.e. $v(E \cup E') + v(E \cap E') \geq v(E) + v(E')$. If v is supermodular, then its core $Core(v) :=$

[5] See also Gilboa (1987), Wakker (1989) and Sarin and Wakker (1992).

[6] Note that we only claim that supermodularity is sufficient for uncertainty aversion, not that it is also necessary. Necessity is a controversial question (see Epstein (1997*b*) and

$\{p|p(E) \geq v(E)\}$ of additive set functions p that eventwise dominate v is non-empty (Shapley, 1971). In that case, we can equivalently think of the players as possessing the set of additive beliefs $Core(v)$. The Choquet integral of u is then given by $\int u \; dv = \min_{p \in Core(v)} \int u \; dp$ (Schmeidler (1986), Schmeidler (1989), Gilboa and Schmeidler (1989)). .

Finally, we have to specify how players update beliefs. There is no universally agreed upon updating rule for non-additive beliefs. Instead, we take the view that different updating rules are appropriate for different circumstances. In line with the assumption that players are uncertainty averse, we use the Dempster-Shafer rule (Dempster (1967), Shafer (1976)), which is given by

$$v(A|B) := \frac{v(A \cup \overline{B}) - v(\overline{B})}{1 - v(\overline{B})}$$

The Dempster-Shafer rule reduces to Bayes' Rule if the capacity v is additive. Gilboa and Schmeidler (1993) show that the Dempster-Shafer rule corresponds to pessimistic updating. The Dempster-Shafer rule is not dynamically consistent, but there is no dynamically consistent updating rule for non-additive beliefs (see, e.g., Epstein and Breton (1993) and Eichberger and Kelsey (1996)). Thus the Ellsberg paradox implies that updating must be dynamically inconsistent. The Dempster-Shafer rule preserves supermodularity (Fagin and Halpern (1990)), and is commutative (Gilboa and Schmeidler (1993)).

Finally, we note that our approach does not rely on the details of the Dempster-Shafer rule. Any updating rule that takes into account that there are no probability zero events when non-rational play is unrestricted is admissible. Which updating rule will eventually prove to be the correct one is an issue that will have to be settled experimentally, for a first step in this direction see Cohen et al. (1999).

4. Perfect Choquet Equilibria

We use the following notation for extensive form games as defined in Selten (1975) and Kreps and Wilson (1982*a*): Let Γ be an extensive game, finite and with perfect recall. Let V be the set of vertices, with decision nodes X and endnodes Z. Let $\emptyset$ be the origin (empty history). Let $\preceq$ be the precedence relation, i.e. $v \preceq v'$ means that there is a path from v to v'. The relation $\preceq$ is an arborescence, i.e. a partial ordering in which different nodes have disjoint successor sets. Let I be the player set. Let X_i be the decision nodes of player $i \in I$. Let H_i be the set of player i's information sets $h_i \in H_i$. Let $A(h_i)$ be the set of actions that are available to player i at his information set h_i, similarly let $A(x_i)$ be the set of actions that are available to player i at his decision node x_i. Let A_i be the set of actions available to player i at some information set. Let X_0 be the set of all nodes at which there is a random move, and for $x_0 \in X_0$ let $\pi(x_0)$ be the probability distribution over $A(x_0)$.

Let S_i be the set of pure strategies $s_i : H_i \longrightarrow A_i$ of player i, $s_i(h_i) \in A(h_i)$. Let Σ_i be the set of behavior strategies of player i, i.e. $\sigma_i(h_i)$ is a probability distribution over $A(h_i)$. The sets S and Σ are the sets of pure and behavior strategy profiles $s \in S = \times_{i \in I} S_i$, $\sigma \in \Sigma = \times_{i \in I} \Sigma_i$. As usual, s_{-i} and σ_{-i} denote i-incomplete

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Ghirardato and Marinacci (1997)). The reason why we associate uncertainty aversion with supermodularity is that we can then think interchangeably of non-additive and set-valued beliefs.

strategy combinations. Similarly, $s_{i,-h_i}$ and $\sigma_{i,-h_i}$ denote h_i-incomplete strategies of player i, i.e. strategies that do not specify an action at information set h_i.

Let $u_i : Z \longrightarrow \mathbb{R}$ be the von Neumann - Morgenstern utility function of player i. For $s \in S$, let $u_i(s)$ be the expected utility of player i if the pure strategy combination s is played and random moves are drawn according to the distributions $\pi(x_0)$. For $\sigma \in \Sigma$, let $u_i(\sigma)$ be the expected utility of player i if the behavior strategy combination σ is played. For a decision node $x \in X$, let $u_i(\sigma|x)$ be the conditional expected utility of player i, if the game starts at decision node x and the behavior strategy combination σ is played.

The definition of a perfect Choquet equilibrium will become quite involved for general extensive games. For this reason, we first restrict attention to two-player games with perfect information.

Since in extensive games lack of mutual knowledge of rationality arises endogenously whenever a player deviates from his rational strategy, we consider a situation in which rationality is in general not mutual knowledge. We aim to define what rational strategies are. We assume that rational players maximise Choquet - expected utility, i.e. possess a utility function u and maximise utility given their beliefs. Since the opponent may be rational or not, their beliefs can be expressed as two capacities v_R and $v_{\overline{R}}$, where v_R is the belief about the play of rational opponents and $v_{\overline{R}}$ the belief about the play of non-rational opponents. Let $\epsilon_{ij}(x_i)$ be player i's belief that player j is not rational at decision node x_i.

So for given beliefs the rational player chooses his action at decision node x_i by maximising

$$\max_{a \in A(x_i)} \; [1 - \epsilon_{ij}(x_i)] \int u_i(a, \sigma^*_{i,-x_i}, s_j \mid x_i) \; d \; v_R$$
$$+ \epsilon_{ij}(x_i) \int u_i(a, \sigma^*_{i,-x_i}, s_j \mid x_i) \; d \; v_{\overline{R}},$$

where $\sigma^*_{i,-x_i}$ is player i's plan how to continue playing.

The strategy of a rational opponent has to be determined endogenously. So, in equilibrium beliefs v_R gave to coincide with the opponent's rational strategy σ^*_j. In particular, v_R is an additive belief and the Choquet integral reduces to the usual integral, i.e.

$$\int u_i(a, \sigma^*_{i,-x_i}, s_j \mid x_i) \; d \; v_R = \int u_i(a, \sigma^*_{i,-x_i}, \sigma^*_j \mid x_i).$$

It remains to specify the beliefs $v_{\overline{R}}$ about play of non-rational opponents. Since the solution concept specifies rational strategies only, every deviation has to be considered non-rational. Thus $v_{\overline{R}}$ should not impose any restriction on the play of a non-rational player, so that the rational player faces non-additive uncertainty. What matters then is the rational players attitude towards uncertainty. We define the solution concept for the case in which rational players are uncertainty-averse[7] Consequently, we assume that $v_{\overline{R}}$ is the basic capacity that assigns belief

$$v_{\overline{R}}(S'_j) = \begin{cases} 1, \, S'_j = S_j \\ 0, \, else \end{cases}$$

[7] The Ellsberg paradox seems to point towards uncertainty aversion, and this has been the main motivation for developing CEU. Smithson (1997) reports that uncertainty aversion is a robust phenomenon in the Ellsberg experiment. However, Smithson (1997) also draws attention to the fact that uncertainty aversion is not a universal empirical fact.

to the event that a non-rational player's strategy is in the set S_j'.

Modelling complete uncertainty as a basic capacity as opposed to a uniform probability distribution also has the practical advantage that the expected value of the utility function does not depend on the description of the state space. For instance, if a superfluous move, i.e. a copy one of the opponent's strategies, is added to the opponent's strategy set, Choquet expected utility under a basic capacity is the same, whereas the expected utility under a uniform probability distribution would, in general, change.

For this capacity, the Choquet integral reduces to[8]

$$\int u_i(a, \sigma^*_{i,-x_i}, s_j \mid x_i) \; d \, v_{\overline{R}} = \min_{s_j \in S_j} u_i(a, \sigma^*_{i,-x_i}, s_j \mid x_i).$$

Overall, a rational player thus maximises his expected utility, given his beliefs ϵ_{ij}, v_R and $v_{\overline{R}}$. The perfection requirement now means that a player maximises his utility at each decision node in the game, conditional on that node being reached. Moreover, as the game progresses he updates his beliefs, and since his beliefs about non-rational opponents are non-additive he does so on the basis of the Dempster-Shafer rule. In a perfect Choquet equilibrium, a rational player correctly anticipates the play of a rational opponent and has no incentive to deviate. Formally:

Definition 1. Let Γ be a finite extensive two-player game with perfect information. Then σ^* is a perfect Choquet equilibrium iff (if and only if) for each players i, each of his decision nodes x_i, and each pure action $a^*(x_i)$ in the support of $\sigma^*_i(x_i)$

$$\begin{aligned} a^*(x_i) \; \in \; \arg \max_{a \in A(x_i)} \; & [1 - \epsilon_{ij}(x_i)] \; u_i(a, \sigma^*_{i,-x_i}, \sigma^*_j \mid x_i) \\ & + \epsilon_{ij}(x_i) \; [\min_{s_j \in S_j} \; u_i(a, \sigma^*_{i,-x_i}, s_j \mid x_i)], \end{aligned}$$

$$\epsilon_{ij}(x_i) := \frac{\epsilon_{ij}(x_i')}{1 - [1 - \epsilon_{ij}(x_i')][1 - \prod_{x_i' \prec x_j \prec x_i} \sigma^*_j(x_j)]},$$

where x_i' is player i's decision node that precedes x_i, the product is taken over all decision nodes of player j that lie between x_i' and x_i, and $\sigma^*_j(x_j)$ is the probability that player j takes the action that leads from x_i' to x_i[9].

Thus, a perfect Choquet equilibrium resolves the infinite regress that arises in a situation in which rationality is not mutual knowledge: Rationality means to maximise utility given beliefs; these beliefs take into account that a rational opponent will do the same, and that the rationality concept does not restrict the play of a non-rational opponent.

[8] Note that in contrast to some of the literature on CEU in games we do not restrict rational players' beliefs to 'simple capacities', i.e. distorted probability distributions. In principle, the players may have arbitrary beliefs about non-rational play. Here, we assume instead that a rational player distinguishes between rational and non-rational opponents.

[9] The updating rule takes into account that the opponent may move more than once between x_i' and x_i. See remark (6) in the appendix.

Note that in a perfect Choquet-equilibrium the equilibrium path is supported by a different solution concept, i.e. maximin play, off the equilibrium path. Consequently, the solution concept does not suffer from the logical deficiency of subgame-perfection, where the equilibrium path is supported by equilibrium reasoning off the equilibrium path.

Note also the important difference between subjective expected utility theory and Choquet expected utility in justifying the maximin-strategy against non-rational opponents. Under subjective expected utility the maximin-strategy is rational only if the rational player believes that the non-rational opponent minimaxes him. This belief seems difficult to justify. Under CEU the maximin-strategy is rational because the rational player cannot exclude the possibility that the non-rational opponent plays, perhaps by chance, a minimax-strategy and because he reacts aversely towards the uncertainty created by the lack of possibility to forecast a non-rational opponent's play.

This solution concept generalizes immediately to repeated normal form games, i.e. multi-stage games with observed actions (Fudenberg and Tirole (1991)), in which the players move simultaneously in each stage, and learn the (pure) actions after each stage.

In section 6 we discuss the extension of this equilibrium concept to general extensive games and to more than two players. In the next section we relate the solution concept to the foundations of game theory.

5. Subgame-Perfection

The aim of this section is to relate our solution concept to the discussion on the foundations of game theory. The example in section 2 already shows how a perfect Choquet equilibrium differs from subgame-perfection. The following discussion of the centipede game shows that backward induction need not be based on common knowledge of rationality. Thus, we argue that our solution concept provides a robustness criterion for subgame-perfect equilibria.

5.1. The Centipede Game

The logical consistency of subgame-perfection has been controversial for a long time (Binmore (1987-88), Reny (1993)). Selten's (1975) concept of trembling-hand perfection circumvents these difficulties by explaining deviations from rationality as unsystematic trembles of otherwise rational players, so that deviations are not evidence of non-rationality. Rationality is then defined as a limiting case of non-rationality where the probability of mistakes approaches zero. Though this approach is empirically implausible, it is logically consistent.

The logical status of subgame-perfection was further clarified by Aumann (1995, 1998) (see also Binmore (1996), Aumann (1996)). Aumann (1995) shows that common knowledge of 'ex ante substantive rationality' implies the backward induction outcome in perfect information games. Here, a player is 'rational' if there is no other strategy that the player knows to give him higher expected utility than the one he chooses.

The distinction between 'ex ante' and 'ex post' rationality refers to the point in the game when his knowledge matters. 'Ex ante rationality' at some decision node v means that at the beginning of the game he knows of no better action at v, 'ex post' rationality means that when v is reached he knows of no better strategy. Consequently, 'ex ante' rationality is weaker than 'ex post' rationality.

The distinction between 'substantive' and 'material' rationality refers to the decision nodes where the player is assumed to be rational. Thus 'substantive' rationality means that a player is rational at all decision nodes, whether they are reached by rational play or not. 'Material' rationality, on the other hand, means that players are only assumed to be rational at reached decision nodes. 'Material' rationality is weaker than substantive rationality, and Aumann shows that 'material' rationality does not imply the backward induction outcome.

Aumann (1998) notices that his result can be sharpened for the centipede game. The centipede game (Rosenthal (1981)) has become a cornerstone for the discussion of the foundations of game theory. Aumann shows that, due to its specific payoff structure, common knowledge of 'ex post material rationality' implies the backward induction outcome in the centipede game. The rationality concept cannot be weakened to 'ex ante material rationality'. Note that common knowledge of rationality implies the backward induction outcome, not the backward induction strategy profile.

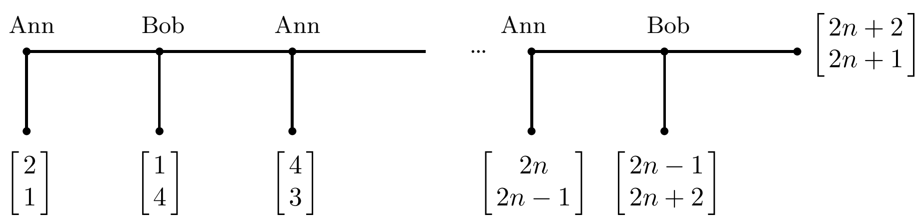

Fig. 2. A Centipede Game

It is immediate that the perfect Choquet equilibrium in the centipede game is to play down everywhere: At the last node 'down' is strictly dominant, at the penultimate node the player knows that a rational opponent will go down at the last node, or will consider the non-rational opponent unpredictable and assume the worst. In either case the continuation payoff is less than that from going 'down', so again 'down' is optimal. The same reasoning applies at every earlier node.

This conclusion is interesting for the following reasons: First, not only do we get the backward induction outcome, but also the backward induction strategy profile. Moreover, this profile is achieved using the same logic as subgame-perfection. Secondly, the backward induction solution arises without mutual knowledge of rationality. Finally, the original objection to backward induction no longer applies: players are not assumed to be rational off the equilibrium path.

5.2. The Finitely Repeated Prisoners' Dilemma

One of the first papers to apply CEU to normal form games was Dow and Werlang (1994). In particular, Dow and Werlang develop an equilibrium concept for players who have non-additive beliefs and analyse the once-repeated prisoners' dilemma. They show that players with non-additive beliefs may not backward induct, and contrast their result with that of Kreps et al. (1982).

One of the differences between a perfect Choquet equilibrium and the Nash equilibrium under uncertainty of Dow and Werlang (1994) is the way in which uncertainty arises in the game. Dow and Werlang (1994) do not distinguish between rational and non-rational players, in their equilibrium concept players are CEU maximisers who lack, to some degree, logical omniscience. They anticipate that their opponents maximise CEU, but do not draw the conclusion about the strategy choice. In other words, theirs is an equilibrium in beliefs. In our model, it is the lack of mutual knowledge of rationality that gives rise to uncertainty. The rational players can anticipate how rational opponents will act, but not how non-rational opponents will

The main difference, however, that Nash equilibrium under uncertainty is a normal form concept. Thus when players may have non-additive beliefs, cooperation in the first period and defection in the second can be an equilibrium: If the players believe that the opponent cooperates in the second period if, and only if they do so at the first stage, they have an incentive to cooperate early, and for lack of logical omniscience both players may think so[10].

Under complete uncertainty aversion, in a perfect Choquet equilibrium both players defect at all stages: In the second stage defection is strictly dominant, in the first stage a player can anticipate that a rational opponent will defect in the next stage, and a non-rational opponent causes uncertainty that is evaluated by its worst outcome, i.e. defection. So the second stage action is independent of the first stage action, and again defection is strictly dominant in the first stage.

Note that Aumann's (1995) justification of the backward induction outcome does not apply to the repeated prisoners' dilemma, since it is not a perfect information game. Thus, the perfect Choquet equilibrium concept sheds some light on the robustness of subgame-perfect equilibria. It is perhaps surprising that backward induction is robust in games like the centipede game or the finitely repeated prisoners' dilemma, in which it is most counterintuitive.

However, we can also extend the result of Dow and Werlang (1994) in the following way: Instead of assuming that players are completely uncertainty averse, assume that they believe that if the opponent is non-rational, then he will defect in the first period and cooperate in the second period if, and only if, there was cooperation in the first period.

Now, if the probability that the opponent is non-rational is sufficiently high, then again it is rational to cooperate in the first period (and to defect in the second period), because first-period play influences the second-period play of the non-rational opponents. This result is interesting because it shows that backward induction may break down even if the strategies of the rational opponent can be correctly anticipated and subgame perfection is required. Yet, a basic shortcoming of this result is that it rests on this specific belief about non-rational play, that is as difficult to justify as the 'crazy type' of Kreps et al. (1982).

[10] In fact this phenomenon is also related to another aspect of the Nash equilibrium under uncertainty in Dow and Werlang (1994), i.e. their definition of support of a non-additive measure. The support implicitly defines the knowledge of the players. What the correct support notion is for non-additive beliefs is controversial. This issue does not arise in a perfect Choquet equilibrium, in which players know the rational strategies in the usual sense.

6. Extension

The aim of this section is to discuss the extension of the solution concept to general extensive games. This extension is not straightforward, as the following example shows[11]:

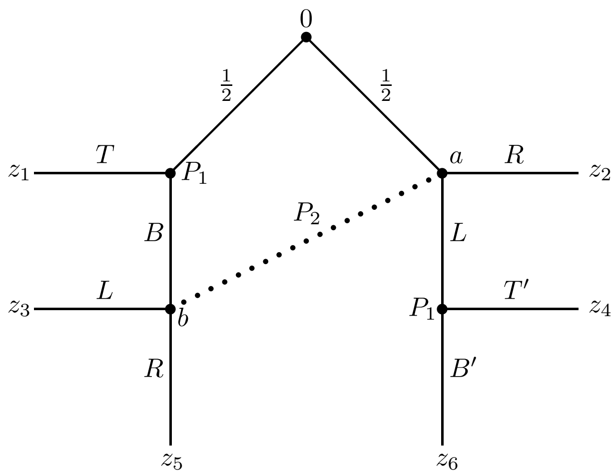

Fig. 3. Extensive form game I

Note, first, that in a general extensive game a player should hold different beliefs about the degree of his opponent's rationality at different decision nodes in the same information sets. In the above example, at note a player P_2 should have belief $\epsilon(\emptyset)$, i.e. the prior with which he started the game. However, if T strictly dominates B, then at decision node b P_2 should hold the updated belief that the opponent is non-rational, i.e. $\epsilon(b) = 1$.

The second problem, also illustrated in the above example, is that not all decision nodes of an information set matter equally for the player who moves there. Above, P_2's belief $\epsilon(b)$ does not matter at all, because P_1 does not move after b. Only at decision node a is P_2's belief about P_1's rationality relevant for his decision.

Overall, in the example the intuitively correct belief for P_2 to have at his information set is therefore his prior $\epsilon(\emptyset)$. We suggest to generalize this observation in the following way:

Let h_i be an information set of player i. Call a decision node $x_i \in h_i$ relevant for player i if his opponent has a decision node in the subtree that starts at x_i. For each relevant decision node x_i, calculate the (in general non-additive) belief $\mu'(x_i)$ that the node is reached given the optimal strategies and beliefs $\epsilon(h_i')$ at preceding information sets h_i'. Form an updated belief $\epsilon(x_i)$ for each relevant decision node, where $\epsilon(x_i)$ is the Dempster-Shafer update from the preceding information sets and the equilibrium strategies. Finally, define $\epsilon(h_i)$ as the expected value of $\epsilon(x_i)$ given $\mu(x_i) := \frac{\mu'(x_i)}{\sum_{x_i \in h_i} \mu'(x_i)}$. Formally:

Definition 2. Let Γ be a finite two-player game in extensive form.

[11] Nature gives the move to player 1 or 2 with probability $\frac{1}{2}$. Player 1 has full information, player 2 does not know if he moves first or second. The outcomes are denoted $z_{1,\ldots,6}$.

Then σ^* is a perfect Choquet equilibrium iff (if and only if) for each players i, each of his information sets h_i, and each pure action $a^*(h_i)$ in the support of $\sigma_i^*(h_i)$

$$a^*(h_i) \in \arg\max_{a \in A(h_i)} [1 - \epsilon_{ij}(h_i)]\ u_i(a, \sigma^*_{i,-h_i}, \sigma^*_j \mid h_i) \\ + \epsilon_{ij}(h_i)\ [\min_{s_j \in S_j} u_i(a, \sigma^*_{i,-x_i}, s_j \mid h_i)],$$

$$\mu'(x_i) := \prod_{x'_i \prec x_i} \sigma^*(x'_i) \prod_{x'_j \prec x_i} (\ [1 - \epsilon_{ij}(x'_j)]\sigma^*_j(x'_j) + \epsilon_{ij}(x'_i)\ )$$

$$\mu'(h_i) := \textstyle\sum_{x_i \in h_i} \mu'(x_i)$$

$$\mu(x_i) := \tfrac{\mu'(x_i)}{\mu'(h_i)}$$

$$u_i(a, \sigma^*_{i,-h_i}, \sigma^*_j \mid h_i) = \sum_{x_i \in h_i} \mu(x_i) u_i(a, \sigma^*_{i,-h_i}, \sigma^*_j \mid x_i),$$

$$\epsilon_{ij}(x_i) := \frac{\epsilon_{ij}(x'_i)}{1 - [1 - \epsilon_{ij}(x'_i)][1 - \prod_{x'_i \prec x_j \prec x_i} \sigma^*_j(x_j)]},$$

$$\epsilon_{ij}(h_i) := \sum_{x_i \in h_i} \mu(x_i)\epsilon_{ij}(x_i)$$

where x'_i (x'_j) are player i's (j's) decision nodes that precedes x_i, $\sigma^*_i(x'_i)$ ($\sigma^*_j(x'_j)$) is the probability that player i (j) takes the action that leads from x'_i (x_j') toward x_i.

Consider the following example:

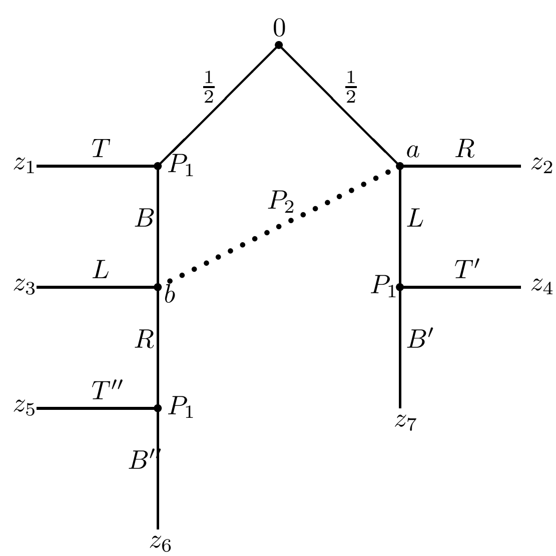

Fig. 4. Extensive form game II

Again, assume that T is strictly dominant, so that $\epsilon(b) = 1$. Then $\mu'(a) = \frac{1}{2}$ and $\mu'(b) = \frac{1}{2}\epsilon(\emptyset)$. Note that the calculation of $\mu'(b)$ again reflects uncertainty aversion, because the worst case is that non-rational players would play B, since this would give most weight to the worst outcome as P_2 weighs his decision at his information set.

The extension of the equilibrium concept to more than two players is conceptually straightforward, but computationally demanding. For two opponents P_j and P_k of player i, and beliefs ϵ_j and ϵ_k about their lack of rationality, the rational player has to take all four cases into account: that either of the players is non-rational, that both are rational and that neither is. So at decision node x_i he should maximise

$$\begin{aligned}\max_{a\in A(x_i)} \; & [(1-\epsilon_j)(1-\epsilon_k)]\; u_i(a, \sigma^*_{i,-x_i}, \sigma^*_j, \sigma^*_k) \\ & +[(1-\epsilon_j)\epsilon_k]\; \min_{s_k\in S_k}\; u_i(a, \sigma^*_{i,-x_i}, \sigma^*_j, s_k) \\ & +[\epsilon_j(1-\epsilon_k)]\; \min_{s_j\in S_j}\; u_i(a, \sigma^*_{i,-x_i}, s_j, \sigma^*_k) \\ & +[\epsilon_j\epsilon_k]\; \min_{(s_j,s_k)\in S_j\times S_k}\; u_i(a, \sigma^*_{i,-x_i}, s_j, s_k).\end{aligned}$$

In particular, taking into account that both players may be non-rational means that in the worst case their actions may be correlated. Beliefs ϵ_j and ϵ_k are then updated separately on the basis of the Dempster-Shafer rule.

7. Conclusion

The paper has suggested a solution concept that combines equilibrium logic with maximin play off the equilibrium path. The solution concept is natural if rationality is not mutual knowledge, no restriction is imposed on deviations from rationality, and if players are uncertainty-averse.

Nevertheless, the solution concept also has some limitations. First, the computational effort of calculating equilibria may be quite high. Secondly, experiments show that players sometimes systematically deviate from rational play, so that it may be possible to formulate more restrictive assumptions on non-rational play after all. Note that this would give rise to a difference between a descriptive solution concept with such restrictions, and a normative concept like ours where we based the lack of restrictions on the consistency argument that a rationality concept alone does not restrict non-rational play.

Thirdly, the assumption that players are completely uncertainty-averse is extreme. Yet, at the current stage of the development of Choquet expected utility theory there is no ideal alternative[12].

Finally, it seems a drastic consequence that strategic interaction comes to a complete halt after a deviation from rational play. Note, however, that the solution concept applies to one-shot games, and therefore leaves no room for real-world strategies to deal with doubts about rationality, e.g. experimentation and communication. Note also that trembling-hand perfection makes an equally extreme assumption to ensure logical consistency by postulating that otherwise fully ra-

[12] It would be possible to assume that players take a weighted average of the best and the worst outcome if they are certain to face a non-rational opponent. This is Hurwicz's optimism-pessimism index (see Arrow and Hurwicz (1972). However, by introducing another free parameter the model would lose predictive power.

tional players 'tremble', and that deviations therefore provide no evidence against rationality.

Needless to say, our solution concept provides nothing but a first step that may be a basis for a more refined analysis.

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Appendix

Let v be a capacity and consider the events $E, F \in \Sigma$. The Dempster-Shafer rule specifies that the posterior capacity of event E is given by (if $v(\overline{F}) < 1$)

$$v(E|F) := \frac{v(E \cup \overline{F}) - v(\overline{F})}{1 - v(\overline{F})}.$$

Let ϵ be the prior probability that a player is not rational with $0 < \epsilon < 1$. Assume that a rational player chooses an action A with probability p. Then the posterior belief about the opponent's rationality after action A is given by

$$v(R|A) := \frac{(1-\epsilon)p}{1 - (1-\epsilon)(1-p)}.$$

This is derived as follows:

Let R be the event that the player is rational, let $\overline{R}$ be the event that he is non-rational.

We want to calculate

$$v(R|A) := \frac{v(R \cup \overline{A}) - v(\overline{A})}{1 - v(\overline{A})}. \tag{1}$$

Since a player is either rational or not, we have

$$v(R|A) + v(\overline{R}|A) = 1. \tag{2}$$

Consequently,

(3) $v(R|A) = \frac{v(R \cup \overline{A}) - v(\overline{A})}{1 - v(\overline{A})}$,

(4) $v(\overline{R}|A) = \frac{v(\overline{R} \cup \overline{A}) - v(\overline{A})}{1 - v(\overline{A})}$

imply

$$v(\overline{A}) = v(R \cup \overline{A}) + v(\overline{R} \cup \overline{A}) - 1.$$

Further,

(5) $v(\overline{A}|R) = \frac{v(\overline{A} \cup \overline{R}) - v(\overline{R})}{1 - v(\overline{R})}$, and

(6) $v(\overline{A}|\overline{R}) = \frac{v(\overline{A} \cup R) - v(R)}{1 - v(R)}$.

We know that
(7) $v(R) = 1 - \epsilon$,
(8) $v(\overline{R}) = \epsilon$,
(9) $v(\overline{A}|R) = 1 - p$, and
(10) $v(\overline{A}|\overline{R}) = 0$.
so that
(11) $v(\overline{A} \cup \overline{R}) = (1-\epsilon)(1-p) + \epsilon$, and
(12) $v(\overline{A} \cup R) = 1 - \epsilon$.
Thus

$$v(\overline{A}) = (1-\epsilon) + (1-\epsilon)(1-p) + \epsilon - 1 = (1-\epsilon)(1-p). \tag{13}$$

Consequently,

$$\begin{aligned} v(R|A) &:= \frac{(1-\epsilon) - (1-\epsilon)(1-p)}{1-(1-\epsilon)(1-p)} \\ &= \frac{(1-\epsilon)p}{1-(1-\epsilon)(1-p)}, \\ v(\overline{R}|A) &= \frac{\epsilon}{1-(1-\epsilon)(1-p)} \end{aligned}$$

Remarks:
(1) The derivation is only valid under lack of mutual knowledge of rationality, i.e. for $\epsilon > 0$ and $\epsilon < 1$, otherwise $v(\overline{A}|R)$ or $v(\overline{A}|\overline{R})$ are not well-defined.
(2) With $0 < \epsilon < 1$ there are no probability zero events, since

$$v(\overline{A}) = (1-\epsilon)(1-p) < 1.$$

This holds for any $p \in [0, 1]$, including the boundaries.
(3) In particular, if $\epsilon > 0$ then $\epsilon' > 0$, independently of p. However, if $p = 0$, then $\epsilon' = 1$. Thus we also need to be able to update the belief $\epsilon = 1$. Intuitively, if the prior belief about the opponent is that he is non-rational and beliefs about his behavior are uncertainty averse, then there are no probability zero events, and the posterior belief should also be that the opponent is non-rational.This can be justified directly from the Dempster-Shafer rule (1): From monotonicity, $v(\overline{R}) \leq v(\overline{R} \cup olA)$, therefore $v(\overline{R} \cup \overline{A}) = 1$. Also, (6) implies $v(\overline{A}|\overline{R}) = v(\overline{A} \cup R)$, so again by monotonicity, $v(\overline{A}) \leq v(\overline{A} \cup R) = 0$. Since this result also follows if we substitute $\epsilon = 1$ into (13), we do not have to explicitly track this special case.
(4) The reason why $\epsilon = 0$ has to be excluded is that there is no parallel argument that $\epsilon = 0$ and $p = 0$ should give $\epsilon' = 1$. (3) implies $v(\overline{A} \cup \overline{R}) = v(\overline{A}|R) = 1$ and (1) gives $\epsilon' = \frac{1-v(\overline{A})}{1-v(\overline{A})}$, but $v(\overline{A}) \not< 1$.
(5) Note that action A is always interpreted as evidence of non-rationality: $v(\overline{R}|A) = \epsilon \frac{1}{1-(1-\epsilon)(1-p)} > \epsilon$. Thus updating is in line with uncertainty aversion, which gives rise to non-additive beliefs in the first place. For the player, the worst case is that the non-rational opponent chose action A with probability 1, because this makes it more likely, under uncertainty aversion, to receive the worst outcome in the next stage.
(6) Note that if $\epsilon' = \frac{\epsilon}{1-(1-\epsilon)(1-p)}$ and $\epsilon'' = \frac{\epsilon'}{1-(1-\epsilon')(1-p')}$ then $\epsilon'' = \frac{\epsilon}{1-(1-\epsilon)(1-p \cdot p')}$.
(7) Finally, note that the argument rests heavily on (2), i.e. the the requirement about beliefs that an opponent is either rational or non-rational, so that these beliefs have to be additive.

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Nash Equilibrium in Games with Ordered Outcomes

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Abstract. We study Nash equilibrium in games with ordered outcomes. Given game with ordered outcomes, we can construct its mixed extension. For it the preference relations of players are to be extended to the set of probability measures. In this work we use the canonical extension of an order to the set of probability measures.
It is shown that a finding of Nash equilibrium points in mixed extension of a game with ordered outcomes can be reduced to search so called balanced matrices, which was introduced by the author. The necessary condition for existence of Nash equilibrium points in mixed extension of a game with ordered outcomes is a presence of balanced submatrices for the matrix of its realization function. We construct a certain method for searching of all balanced submatrices of given matrix using the concept of extreme balanced matrix. Necessary and sufficient conditions for Nash equilibrium point in mixed extension of a game with ordered outcomes are given also.

Keywords: Game with ordered outcomes, Nash equilibrium, Mixed extension of a game with ordered outcomes, Balanced matrix, Extreme balanced matrix, Balanced collection.

1. Introduction

A game G with ordered outcomes is a game in which preferences of players by partial ordered relations are given (all preliminaries see in Rozen (2010)).

Definition 1. A situation x^0 is called *Nash equilibrium point* in game G if for any player i and its strategy x_i the condition

$$F(x^0 \parallel x_i) \overset{\omega_i}{\leqq} F(x^0) \tag{1}$$

holds, where $\overset{\omega_i}{\leqq}$ is a preference relation of player i.

We construct a mixed extension of a finite game G with ordered outcomes in the following way. Mixed strategies of players in game G is defined as usually that is probability measures on the sets of their pure strategies. A realization function F of a game G can be extended to a mapping $\widetilde{F}$ of mixed situations of game G into the set $\widetilde{A}$ consisting of probability measures on the set outcomes A. Now we need to extend the preference relations of players on $\widetilde{A}$. We define the *canonical extension* of order $\omega \subseteq A^2$ to the set $\widetilde{A}$ by formula

$$\mu \overset{\widetilde{\omega}}{\leqq} \nu \Longleftrightarrow (\forall f \in C(\omega))\, \overline{f}(\mu) \leqslant \overline{f}(\nu), \tag{2}$$

where $\mu, \nu \in \widetilde{A}$, $C(\omega)$ is the set of all isotonic functions from A to $\mathbb{R}$ under order ω and $\overline{f}(\mu) = (f,\, \mu)$ is the standard scalar product.

Remark 1. In evident form the condition $\mu \overset{\widetilde{\omega}}{\leqq} \nu$ means that $\mu(B) \leqslant \nu(B)$ for any subset $B \subseteq A$ which is majorant stable under order ω (see Rozen (1976)). Recall that a subset $B \subseteq A$ is called majorant stable one if the conditions $a \in B$ and $a' \overset{\widetilde{\omega}}{\geqq} a$ imply $a' \in B$.

Corollary 1 (convexity of the canonical extension). *Conditions* $\mu_k \overset{\widetilde{\omega}}{\leqq} \nu_k$, *where* $\alpha_k > 0$ $(k = 1, \ldots, p)$, $\sum_{k=1}^{p} \alpha_k = 1$, *imply*

$$\sum_{k=1}^{p} \alpha_k \mu_k \overset{\widetilde{\omega}}{\leqq} \sum_{k=1}^{p} \alpha_k \nu_k. \tag{3}$$

Moreover, if at least one of these conditions is strict then condition (3) is strict also.

In this article we consider games of two players with ordered outcomes in the form

$$G = \langle X, Y, A, \omega_1, \omega_2, F \rangle$$

where X is a set of strategies of player 1, Y is a set of strategies of player 2, A is a set of outcomes, ω_k is an order relation on A which represents preferences of player $k = 1, 2$ and $F\colon X \times Y \to A$ is a realization function.

According to a general definition a situation $\left(x^0, y^0\right) \in \widetilde{X} \times \widetilde{Y}$ is Nash equilibrium point in game G if for any $x \in \widetilde{X}$ and $y \in \widetilde{Y}$ conditions

$$\widetilde{F}_{(x,\, y^0)} \overset{\widetilde{\omega}_1}{\leqq} \widetilde{F}_{(x^0,\, y^0)}, \widetilde{F}_{(x^0,\, y)} \overset{\widetilde{\omega}_2}{\leqq} \widetilde{F}_{(x^0,\, y^0)}$$

hold.

In section 3 it is shown that, for game G, the existence of Nash equilibrium points in mixed strategies is connected with so called balanced submatrices its realization function. We study some basic properties of balanced matrices in section 2. Theorem 1 shows a simple connection between balanced matrices and well known in game theory balanced collections. Note that up to now, a concept of balanced collection was used for systems of coalitions in cooperative games only (see, for example, Bondareva (1968), Shapley (1965), Peleg (1965)). The main result of our work is Theorem 4 which states a necessary and sufficient conditions for Nash equilibrium point in mixed extension of game G with ordered outcomes. To illustrate these results we consider some examples.

2. Balanced matrices and balanced collections of subsets

We consider an arbitrary matrix M of size $m \times n$ over a set A as a mapping $F\colon I \times J \to A$ where $I = \{1, \ldots, m\}$, $J = \{1, \ldots, n\}$. Put $M = \|F(i, j)\|$.

We denote by S_m the standard simplex consisting of m–components vectors:

$$S_m = \left\{ x = (x_1, \ldots, x_m) : x_i \geq 0, \sum_{i=1}^{m} = 1 \right\}.$$

For any $x \in S_m, y \in S_n$ and $a \in A$ put

$$\widetilde{F}_{(x,y)}(a) = \sum_{\substack{F(i,j)=a \\ (i,j) \in I \times J}} x_i \cdot y_j. \tag{4}$$

In particular for any $i = 1, \ldots, m$ and $j = 1, \ldots, n$ we have

$$\widetilde{F}_{(x,j)}(a) = \sum_{F(i,j)=a} x_i,\ \widetilde{F}_{(i,y)}(a) = \sum_{F(i,j)=a} y_j. \tag{5}$$

Remark 2. We identify any index $i \in I$ with unit vector $e_i = (0, \ldots, 1, \ldots, 0)$ and also for any index $j \in J$.

Lemma 1 (linearity property). *The function $\widetilde{F}$ is linear under all arguments.*

Proof (of lemma 1). Verify a linearity under first argument. Put $x^k \in S_m, y \in S_n, \alpha_k \geq 0\ (k = 1, \ldots, p)$, $\sum_{k=1}^{p} \alpha_k = 1$. We have

$$\widetilde{F}_{\left(\sum_{k=1}^{p} \alpha_k x^k, y\right)}(a) = \sum_{\substack{F(i,j)=a \\ (i,j) \in I \times J}} \left(\sum_{k=1}^{p} \alpha_k x_i^k\right) y_j = \sum_{\substack{F(i,j)=a \\ (i,j) \in I \times J}} \sum_{k=1}^{p} \alpha_k x_i^k y_j =$$

$$= \sum_{k=1}^{p} \sum_{\substack{F(i,j)=a \\ (i,j) \in I \times J}} \alpha_k x_i^k y_j = \sum_{k=1}^{p} \alpha_k \sum_{\substack{F(i,j)=a \\ (i,j) \in I \times J}} x_i^k y_j = \sum_{k=1}^{p} \alpha_k \widetilde{F}_{(x^k,y)}(a)$$

which was to be proved. □

Definition 2. A matrix $\|F(i,j)\|$ is called *balanced* one if there exists such a pair of vectors $\left(x^0, y^0\right) \in S_m \times S_n$ with positive components that for any $x \in S_m, y \in S_n$ and $a \in A$ the equalities

$$\widetilde{F}_{(x,\, y^0)}(a) = \widetilde{F}_{(x^0,\, y^0)}(a),\ \widetilde{F}_{(x^0,\, y)}(a) = \widetilde{F}_{(x^0,\, y^0)}(a) \tag{6}$$

hold. In this case the pair $\left(x^0, y^0\right)$ is called a *balance pair of vectors.*

Note that (4) means the equality of corresponding measures:

$$\widetilde{F}_{(x,\, y^0)} = \widetilde{F}_{(x^0,\, y^0)},\ \widetilde{F}_{(x^0,\, y)} = \widetilde{F}_{(x^0,\, y^0)}.$$

Definition 3. A vector $x^0 \in S_m$ with positive components is called a *row balance vector* for matrix $\|F(i,j)\|$ $(i \in I, j \in J)$ if for any $j_1, j_2 \in J$ the equality $\widetilde{F}_{(x^0,\, j_1)} = \widetilde{F}_{(x^0,\, j_2)}$ holds. In other words, in this case, the measure $\widetilde{F}_{(x^0,\, j)}$ does not depend on $j \in J$. A *column balance vector* is defined dually that is a vector $y^0 \in S_n$ with positive components such that the measure $\widetilde{F}_{(i,\, y^0)}$ does not depend on $i \in I$.

Lemma 2. *A pair of vectors $\left(x^0, y^0\right)$ is a balance pair for matrix $\|F(i,j)\|$ if and only if x^0 is a row balance vector and y^0 is a column balance vector.*

Indeed let $\left(x^0, y^0\right)$ be a balance pair of vectors. Setting in (4) $y = j_1$ and then $y = j_2$ we have $\widetilde{F}_{(x^0,\, j_1)} = \widetilde{F}_{(x^0,\, y^0)}$, $\widetilde{F}_{(x^0,\, j_2)} = \widetilde{F}_{(x^0,\, y^0)}$ hence x^0 is a row balance vector. Dually, y^0 is a column balance vector. Conversely let x^0 be a row balance vector and y^0 a column balance vector. For arbitrary $a \in A$ put $p\,(a) = \widetilde{F}_{(x^0,\, j)}\,(a)\ \ (j \in J)$. Fix a vector $y = (y_1, \ldots, y_n) \in S_n$. By using a linearity for $\widetilde{F}$ (see Lemma 1), we get

$$\widetilde{F}_{(x^0,\, y)}\,(a) = F_{\left(x^0, \sum_{j\in J} y_j e_j\right)}\,(a) = \sum_{j\in J} y_j \widetilde{F}_{(x^0, j)}\,(a) =$$
$$= \sum_{j\in J} y_j p\,(a) = p\,(a) \sum_{j\in J} y_j = p\,(a)\,,$$

i.e. the measure $\widetilde{F}_{(x^0, y)}$ does not depend on $y \in S_n$. Then, for any $y \in S_n$ we have $\widetilde{F}_{(x^0,\, y)} = \widetilde{F}_{(x^0,\, y^0)}$ and dually $\widetilde{F}_{(x,\, y^0)} = \widetilde{F}_{(x^0,\, y^0)}$, hence $\left(x^0, y^0\right)$ is a balance pair of vectors.

Corollary 2. *A balanced matrix is homogeneous (that is, any its row and any column contains a same set of elements).*

For the proof it is sufficiently to remark that the set of elements consisting i-th row of matrix $\|F\,(i,j)\|$ coincides with spectrum of the measure $\widetilde{F}_{(i,\, y^0)}$ which, in our case, does not depend on $i \in I$.

Recall the concept of balanced collection.

Definition 4. Let E be an arbitrary set and $(E_1, \ldots, E_p)$ be a covering system of its subsets. The collection $(E_1, \ldots, E_p)$ is called *balanced one* if there exists a representative vector for it, i.e. a vector $\lambda = (\lambda_1, \ldots, \lambda_p)$ with positive components such that for any $e \in E$ the equality

$$\sum_{E_k \ni e} \lambda_k = 1$$

holds.

Definition 5. Two balanced collections $\left(E_1^1, \ldots, E_p^1\right)$ and $\left(E_1^2, \ldots, E_p^2\right)$ are said to be *collinearly balanced* if for them there exist collinear representative vectors λ^1 and λ^2 (in notation $\lambda^1 \| \lambda^2$).

Given a matrix $F\colon I \times J \to A$, we define for arbitrary $a \in A$ two collections of *characteristic subsets* $\left(U_i^a\right)_{i\in I}$ and $\left(V_j^a\right)_{j\in J}$ by setting

$$j \in U_i^a \Leftrightarrow F\,(i,j) = a,\ i \in V_j^a \Leftrightarrow F\,(i,j) = a. \tag{7}$$

Theorem 1. *A matrix $F\colon I \times J \to A$ is balanced if and only if for any $a_1, a_2 \in pr_2 F$*

1. *Collections $\left(U_i^{a_1}\right)_{i\in I}$ and $\left(U_i^{a_2}\right)_{i\in I}$ are collinearly balanced ones;*
2. *Collections $\left(V_j^{a_1}\right)_{j\in J}$ and $\left(V_j^{a_2}\right)_{j\in J}$ are collinearly balanced ones.*

Lemma 3. *A matrix $F\colon I \times J \to A$ has a row balanced vector if and only if*

a) for any $a \in pr_2 F$, the collection of its characteristic subsets $\left(U_i^a\right)_{i\in I}$ is balanced one;

b) for any $a_1, a_2 \in pr_2 F$, collections $\left(U_i^{a_1}\right)_{i\in I}$ and $\left(U_i^{a_2}\right)_{i\in I}$ are collinearly balanced ones.

Proof (of lemma 3). Necessity. Let $x = (x_1, \dots, x_m)$ be a row balance vector. For arbitrary $a \in pr_2F$ put $\lambda_i^a = x_i / p(a)$ where $p(a) = \widetilde{F}_{(x,j)}(a)$ (the right part of this equality does not depend on $j \in J$, see Definition 3). Check that the vector $\lambda^a = (\lambda_i^a)_{i \in I}$ is a representative one for the collection $(U_i^a)_{i \in I}$. Fix $j \in J$. By using (5), we have

$$\sum_{U_i^a \ni j} \lambda_i^a = \sum_{F(i,j)=a} \lambda_i^a = \sum_{F(i,j)=a} \frac{x_i}{p(a)} = \frac{1}{p(a)} \sum_{F(i,j)=a} x_i = \frac{1}{p(a)} p(a) = 1.$$

Furthermore, for any $a_1, a_2 \in pr_2F$, we get

$$\lambda_i^{a_1} : \lambda_i^{a_2} = \frac{x_i}{p(a_1)} : \frac{x_i}{p(a_2)} = \frac{p(a_2)}{p(a_1)} = \text{ const under } i \in I,$$

i.e. representative vectors λ^{a_1} and λ^{a_2} are collinear ones.

Sufficiency. Let $\lambda^a = (\lambda_i^a)_{i \in I}$ be a representative vector for balanced collection $(U_i^a)_{i \in I}$ and moreover, for any $a_1, a_2 \in pr_2F$ corresponding vectors are collinear, $\lambda^{a_1} \| \lambda^{a_2}$. Fix $a \in pr_2F$ and put

$$x_i = \frac{\lambda_i^a}{\sum_{i' \in I} \lambda_{i'}^a}, x = (x_i)_{i \in I}.$$

Because representative vectors are collinear, right parts of these equalities do not depend on element $a \in A$. It is evident that $x_i > 0$ and $\sum_{i \in I} x_i = 1$. By using that vector λ^a is a representative one, we have for any $j \in J$

$$\widetilde{F}_{(x,\,j)}(a) = \sum_{F(i,j)=a} x_i = \sum_{U_i^a \ni j} \frac{\lambda_i^a}{\sum_{i' \in I} \lambda_{i'}^a} = \frac{\sum_{U_i^a \ni j} \lambda_i^a}{\sum_{i' \in I} \lambda_{i'}^a} = \frac{1}{\sum_{i' \in I} \lambda_{i'}^a} = \text{const}.$$

Thus, the measure $\widetilde{F}_{(x,\,j)}$ does not depend on $j \in J$, i.e. x is a row balance vector for matrix $\|F(i,j)\|$.

Now, using Lemma 2, we obtain the proof of Theorem 1 from Lemma 3 and its dual. $\square$

For matrix M over a set A and $a \in A$ we denote by $\chi(M^a)$ a matrix with elements m_{ij}^a, where

$$\begin{cases} m_{ij}^a = 1 & \text{if } F(i,j) = a \\ m_{ij}^a = 0 & \text{if } F(i,j) \neq a. \end{cases}$$

Theorem 2. *Suppose $M = \|F(i,j)\|$ is a balanced matrix. Then the following conditions are equivalent:*

1. *Matrix M has an unique pair of balance vectors;*
2. *Matrix M is a square one and $\mathrm{Det}\,\chi(M^a) \neq 0$ for any $a \in A$;*
3. *Matrix M is a square one and $\mathrm{Det}\,\chi(M^a) \neq 0$ for some $a \in A$.*

The proof of Theorem 2 is based on the following lemmas.

Lemma 4. *Let E be an arbitrary finite set, $|E| = m$ and $(E_1, \dots, E_n)$ be a balanced collections of its subsets. Then the collection $(E_1, \dots, E_n)$ has an unique representative vector if and only if the collection of characteristic vectors $(\chi(E_1), \dots, \chi(E_n))$ is linearly independent.*

Proof (of lemma 4). A vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a representative one for the collection $(E_1, \ldots, E_n)$ if and only if it is a positive solution for the following system of linear equations:

$$(\lambda_1, \ldots, \lambda_n)\ \chi_M = (1, \ldots, 1)\,, \tag{8}$$

where by χ_M denoted $n \times m$ matrix whose i-th row $\chi(E_i)$ is the characteristic function of the subset $E_i \subseteq E$ $(i = 1, \ldots, n)$ and $(1, \ldots, 1)$ is m-component vector consisting of 1. Because the collection $(E_1, \ldots, E_n)$ is balanced, (8) is a compatible system consisting of m linear equations under $\lambda_1, \ldots, \lambda_n$. It is well known that such a system has an unique solution if and only if $r = n$ where $r = rank\ \chi_M$, hence, in this case, the system of its rows is linearly independent.

Conversely, assume $r < n$. In this case, the solution of (8) can be represented in the form $\lambda^* + L$ where λ^* is a partial solution of (8) and L is a general solution of a corresponding homogeneous system (note that L is a linear space with dimension $n - r$). Let λ^* be a representative vector for collection $(E_1, \ldots, E_n)$. Then all its components are positive. Fix some $l^* \in L$ with $l^* \neq \mathbf{0}$. Then there exists such positive $\varepsilon > 0$ that all components of vector $\lambda^* + \varepsilon l^*$ remain be positive. Hence $(\lambda^* + \varepsilon l^*)$ is some positive solution of (8) with $\lambda^* + \varepsilon l^* \neq \lambda^*$, then a representative vector for the collection $(E_1, \ldots, E_n)$ is not unique. □

Lemma 5. *Let M be a matrix over set A. Then the following conditions are equivalent:*

1) Matrix M has an unique row balance vector;

2) For any $a \in pr_2M$, the collection of characteristic vectors $(\chi(U_1^a), \ldots, \chi(U_m^a))$ is linearly independent;

3) For some $a \in pr_2M$, the collection of characteristic vectors $(\chi(U_1^a), \ldots, \chi(U_m^a))$ is linearly independent.

Proof (of lemma 5). 1) ⇒ 2). Fix $a \in pr_2M$. According to Lemma 3 a), the system of characteristic subsets $(U_1^a, \ldots, U_m^a)$ forms a balanced collection on the set J.

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ be two its representative vectors. Then vectors $x = (x_1, \ldots, x_m)$ and $x' = (x'_1, \ldots, x'_m)$ defined by

$$x_i = \frac{\lambda_i}{\sum_{i' \in I} \lambda_{i'}},\ x'_i = \frac{\lambda'_i}{\sum_{i' \in I} \lambda'_{i'}}$$

are row balance vectors for matrix M (see the proof of Lemma 3). According to assumption 1) we have $x = x'$, i.e. for any $i \in I$ holds the equality

$$\frac{\lambda_i}{\sum_{i' \in I} \lambda_{i'}} = \frac{\lambda'_i}{\sum_{i' \in I} \lambda'_{i'}},$$

hence $\lambda_i = k\lambda'_i$ where $k = \text{const}$. Fix $j_0 \in J$. Since λ and λ' are representative vectors, we get

$$1 = \sum_{j_0 \in U_i^a} \lambda_i = \sum_{j_0 \in U_i^a} k\lambda'_i = k \sum_{j_0 \in U_i^a} \lambda'_i = k,$$

hence $k = 1$ and $\lambda' = \lambda$. Thus, the balanced collection $(U_1^a, \ldots, U_m^a)$ has an unique representative vector and, according with Lemma 4, the collection $(\chi(U_1^a), \ldots, \chi(U_m^a))$ is linearly independent.

It is evident that 2) $\Rightarrow$ 3). Prove 3) $\Rightarrow$ 1). Assume that the collection of characteristic vectors $(\chi(U_1^a), \ldots, \chi(U_m^a))$ is linearly independent for some $a \in pr_2 M$. Let $x' = (x_1', \ldots, x_m')$ and $x'' = (x_1'', \ldots, x_m'')$ be two row balance vectors for matrix M. Then vectors λ' and λ'' defined by $\lambda_i' = x_i'/p(a)$, $\lambda_i'' = x_i''/p(a)$ $(i \in I)$ are representative vectors for balanced collection $(U_1^a, \ldots, U_m^a)$ (see the proof of Lemma 3). Because the collection $(\chi(U_1^a), \ldots, \chi(U_m^a))$ is linearly independent it follows from Lemma 4 that $\lambda' = \lambda''$ hence $x' = x''$ which was to be proved. □

Let us prove Theorem 2. Assume that matrix M has an unique pair of balance vectors. Then, according with Lemma 2, matrix M has an unique row balance vector. Fix an element $a \in pr_2 M$ and consider the matrix $\chi(M^a)$ defined above. Since rows of matrix $\chi(M^a)$ are vectors $(\chi(U_1^a), \ldots, \chi(U_m^a))$ and according to Lemma 5, we get that the collection of these vectors is linearly independent, then $m = r$ where $r = rank\, \chi(M^a)$. Dually the equality $n = r$ holds, hence we obtain $m = n$ and $Det\, \chi(M^a) \neq 0$ which completes the proof of Theorem 2.

Now we consider a problem for finding of all balanced submatrix for given matrix. Given a matrix $\|F(i,j)\|$ $(i \in I, j \in J)$, fix two subsets $I_0 \subseteq I, J_0 \subseteq J$. We denote by $B_{J_0}^1$ a convex polyhedron which consists of all vectors $x \in S_m$ satisfying the condition

$$(\forall j_1, j_2 \in J_0)\, \widetilde{F}_{(x,\, j_1)} = \widetilde{F}_{(x,\, j_2)}. \tag{9}$$

Dually is defined a convex polyhedron $B_{I_0}^2$ which consists of all vectors $y \in S_n$ satisfying the condition

$$(\forall i_1, i_2 \in I_0)\, \widetilde{F}_{(i_1,\, y)} = \widetilde{F}_{(i_2,\, y)}. \tag{10}$$

Definition 6. A submatrix $F(I_0 \times J_0) = \|F(i,j)\|$ $(i \in I_0, j \in J_0)$ is called an *extreme balanced submatrix* if it is balanced one and then there exists a balance pair of vectors for it which is an extreme point in the convex polyhedron $B_{J_0}^1 \times B_{I_0}^2$.

For arbitrary polyhedron P, we denote by $Ext\, P$ the set of extreme point in P. It is easy to show that $Ext\, \left(B_{J_0}^1 \times B_{I_0}^2\right) = Ext\, \left(B_{J_0}^1\right) \times Ext\, \left(B_{I_0}^2\right)$.

Lemma 6. *A vector $x^0 \in B_{J_0}^1$ is an extreme point in $B_{J_0}^1$ if and only if it is minimal under spectrum, i.e. for any $x \in B_{J_0}^1$ the condition*

$$Sp\, x \subseteq Sp\, x^0 \Longrightarrow x = x^0 \tag{11}$$

holds (we denote by $Sp\, x$ the spectrum of vector x : $Sp\, x = \{i \in I : x_i \neq 0\}$).

Proof (of lemma 6). Let x^0 be an extreme point in $B_{J_0}^1$. Consider a vector $x \in B_{J_0}^1$ satisfying the condition $Sp\, x \subseteq Sp\, x^0$. Assume that $x \neq x^0$. Because

$$\sum_{i \in Sp\, x} x_i = \sum_{i \in Sp\, x^0} x_i^0 = 1$$

then between non-zero components of vector x there exists such $x_{i'}$ that $0 < x_{i'}^0 < x_{i'}$ hence

$$0 < \varepsilon = \min_{i \in Sp\, x} \frac{x_i^0}{x_i} < 1. \tag{12}$$

It follows from (13) that $|\varepsilon x_i| \leq x_i^0$ for all $i \in Sp\, x$. Hence $x_i^0 \pm \varepsilon x_i \geq 0$ i.e. all components of both vectors $x^0 \pm \varepsilon x$ are non-negative. Put

$$x^1 = \frac{x^0 + \varepsilon x}{1+\varepsilon},\ x^2 = \frac{x^0 - \varepsilon x}{1-\varepsilon}.$$

Check that $x^1, x^2 \in B^1_{J_0}$. We have

$$\sum_{i\in I} x_i^1 = \sum_{i\in I} \frac{x_i^0 + \varepsilon x_i}{1+\varepsilon} = \frac{1}{1+\varepsilon}\sum_{i\in I} x_i^0 + \frac{\varepsilon}{1+\varepsilon}\sum_{i\in I} x_i = \frac{1}{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon} = 1;$$
$$\sum_{i\in I} x_i^2 = \sum_{i\in I} \frac{x_i^0 - \varepsilon x_i}{1-\varepsilon} = \frac{1}{1-\varepsilon}\sum_{i\in I} x_i^0 - \frac{\varepsilon}{1-\varepsilon}\sum_{i\in I} x_i = \frac{1}{1-\varepsilon} - \frac{\varepsilon}{1-\varepsilon} = 1.$$

Because $x^0, x \in B^1_{J_0}$ and by using the linearity of $\widetilde{F}$ (see Lemma 1), we get for any $j_1, j_2 \in J_0$:

$$\widetilde{F}_{(x^1,j_1)} = \widetilde{F}_{\left(\frac{x^0}{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon}x, j_1\right)} = \frac{1}{1+\varepsilon}\widetilde{F}_{(x^0,j_1)} + \frac{\varepsilon}{1+\varepsilon}\widetilde{F}_{(x,j_1)} =$$
$$= \frac{1}{1+\varepsilon}\widetilde{F}_{(x^0,j_2)} + \frac{\varepsilon}{1+\varepsilon}\widetilde{F}_{(x,j_2)} = \widetilde{F}_{\left(\frac{1}{1+\varepsilon}x^0 + \frac{\varepsilon}{1+\varepsilon}x, j_2\right)} = \widetilde{F}_{(x^1,j_2)}$$

We obtain the equality $\widetilde{F}_{(x^1,j_1)} = \widetilde{F}_{(x^1,j_2)}$; the equality $\widetilde{F}_{(x^2,j_1)} = \widetilde{F}_{(x^2,j_2)}$ can be shown analogously. Thus, $x^1, x^2 \in B^1_{J_0}$. Since $\frac{1+\varepsilon}{2}x^1 + \frac{1-\varepsilon}{2}x^2 = x^0$ and $\frac{1+\varepsilon}{2} + \frac{1-\varepsilon}{2} = 1$ it follows from the definition of extreme point that $x^1 = x^2$. Then we get

$$x^0 = \frac{1+\varepsilon}{2}x^1 + \frac{1-\varepsilon}{2}x^2 = \frac{1+\varepsilon}{2}x^1 + \frac{1-\varepsilon}{2}x^1 = x^1,$$

hence $\frac{x^0+\varepsilon x}{1+\varepsilon} = x^0$ and $x = x^0$. The last equality is in contradiction with our assumption which completes the proof of necessary condition.

Let us verify the sufficient condition. Assume that for vector x^0 the condition (12) holds. Let x^1 and x^2 be two vectors such that $x^0 = \alpha_1 x^1 + \alpha_2 x^2$ where $x^1, x^2 \in B^1_{J_0},\ \alpha_1, \alpha_2 > 0,\ \alpha_1 + \alpha_2 = 1$. Then $Sp\, x^1 \subseteq Sp\, x^0, Sp\, x^2 \subseteq Sp\, x^0$, and according with (12) we obtain $x^1 = x^0$ and $x^2 = x^0$ hence $x^1 = x^2$. Thus x^0 is an extreme point in $B^1_{J_0}$ which was to be proved. □

Corollary 3. *Let M be an extreme balanced submatrix. Then*

1. *Submatrix M has an unique row balance vector;*
2. *Submatrix M has an unique collumn balance vector;*
3. *Submatrix M is a square one;*
4. *$Det\, \chi\,(M^a) \neq 0$ for arbitrary a which belongs to M.*

Proof (of corollary 3). Let $\left(x^0, y^0\right)$ be a balance pair of vectors for submatrix $M = F\,(I_0 \times J_0)$ provided $x^0 \in Ext\, B^1_{J_0}$ and $y^0 \in Ext\, B^2_{I_0}$. Consider a row balance vector x for $F\,(I_0 \times J_0)$, i.e. $x \in B^1_{J_0}, Sp\, x = I_0$. Because $Sp\, x = Sp\, x^0$, it follows from Lemma (6) that $x = x^0$. Thus, the assertion 1) is proved. Dually, the assertion 2) is true. Both assertions 3) and 4) follow from Theorem 2. □

The main result of this section states

Theorem 3. *Let $\|F(i,j)\|$ $(i \in I, j \in J)$ be an arbitrary matrix over $A, I_0 \subseteq I, J_0 \subseteq J$ and $M = F(I_0 \times J_0)$ be its submatrix. A submatrix M is an extreme balance submatrix if and only if for any $a, a_1, a_2 \in pr_2 M$ the following conditions hold:*

1) Submatrix M is a square one;

2) Det $\chi(M^a) \neq 0$;

3) All components of vector $\lambda^a = (1, \ldots, 1)[\chi(M^a)]^{-1}$ are positive;

4) Vectors λ^{a_1} and λ^{a_2} are collinear;

5) All components of vector $\delta^a = [\chi(M^a)]^{-1}(1, \ldots, 1)^T$ are positive;

6) Vectors δ^{a_1} and δ^{a_2} are collinear.

Proof (of Theorem 3). Necessity of conditions 1) and 2) is shown in Corollary 3. Put $|I_0| = |J_0| = k$. According to Theorem 1 for balanced submatrix M, the collection of characteristic subsets $(U_i^a)_{i \in I_0}$ is balanced hence the following system of linear equations

$$(\lambda_1^a, \ldots, \lambda_k^a)\,\chi(M^a) = (1, \ldots, 1) \tag{13}$$

has a positive solution for any $a \in pr_2 M$. Setting $\lambda^a = (\lambda_1^a, \ldots, \lambda_k^a)$ we obtain from (14) $\lambda^a = (1, \ldots, 1)[\chi(M^a)]^{-1}$ and 3) is shown. Assertion 4) follows from Theorem 1. Dually we get 5) and 6).

Sufficiency. Assume the conditions 1)-6) hold. From 3) we obtain the equality

$$\lambda^a \chi(M^a) = (1, \ldots, 1)$$

i.e. λ^a is a representative vector for collection of characteristic subsets $(U_i^a)_{i \in I_0}$. It follows from 4) that collections $(U_i^{a_1})_{i \in I_0}$ and $(U_i^{a_2})_{i \in I_0}$ are collineary balanced ones. According with Lemma 3, a submatrix M has a row balance vector x^0 and according to Lemma 5 this vector is an unique. Dually, submatrix M has an unique column balance vector y^0. Because vector x^0 is minimal under spectrum, from Lemma 6 we obtain that x^0 is an extreme point in $B_{J_0}^1$ and dually y^0 is an extreme point in $B_{I_0}^2$. Thus, (x^0, y^0) is an extreme point in $B_{J_0}^1 \times B_{I_0}^2$, i.e. submatrix M is an extreme balanced submatrix which completes the prove of Theorem 3. □

Combining Theorem 3 with Krein-Millman Theorem, we get a certain method for searching of all balanced submatrices of given matrix $\|F(i,j)\|$ $(i \in I, j \in J)$.

3. A finding of Nash equilibrium points in mixed extension of games with ordered outcomes

Our method for finding of Nash equilibrium points is based on searching of balanced submatrices of realization function. Consider a game G of two players with ordered outcomes in the form

$$G = \langle I, J, A, \omega_1, \omega_2 \rangle, \tag{14}$$

where $I = \{1, \ldots, m\}$ is a set of strategies of player 1, $J = \{1, \ldots, n\}$ is a set of strategies of player 2, A is a set of outcomes, ω_k is an order relation which represents preferences of player $k = 1, 2, F\colon I \times J \to A$ a realization function.

Lemma 7. *A situation (x^0, y^0) in mixed strategies is Nash equilibrium point in mixed extension of game G if and only if for any $i \in I$ and $j \in J$ the following conditions hold:*

$$\widetilde{F}_{(i,\,y^0)} \overset{\widetilde{\omega}_1}{\leqq} \widetilde{F}_{(x^0,\,y^0)}, \widetilde{F}_{(x^0,\,j)} \overset{\widetilde{\omega}_2}{\leqq} \widetilde{F}_{(x^0,\,y^0)}.$$

Proof (of lemma 7). The necessity is obvious. Conversely, put $x = (x_1, \ldots, x_m) \in S_m$. Summarizing the conditions $\widetilde{F}_{(i,\, y^0)} \overset{\widetilde{\omega}_1}{\leqq} \widetilde{F}_{(x^0,\, y^0)}$ with weights x_i and using the linearity property (see Lemma 1) we obtain $\widetilde{F}_{(x,\, y^0)} \overset{\widetilde{\omega}_1}{\leqq} \widetilde{F}_{(x^0,\, y^0)}$. The condition $\widetilde{F}_{(x^0,\, y)} \overset{\widetilde{\omega}_2}{\leqq} \widetilde{F}_{(x^0,\, y^0)}$ is proved analogously. □

Lemma 8. *Let $\left(x^0, y^0\right)$ be Nash equilibrium point in mixed extension of game G. Then the matrix which is a restriction of matrix of outcomes $\|F\left(i, j\right)\|$ under the pair of spectra $\left(Sp\, x^0, Sp\, y^0\right)$ is balanced one.*

Proof (of lemma 8). For any $i \in Sp\, x^0$ we have $\widetilde{F}_{(i,\, y^0)} \overset{\widetilde{\omega}_1}{\leqq} \widetilde{F}_{(x^0,\, y^0)}$. Assume that at least one of these conditions is strict. Then by using the convexity property (see Corollary 1) we get

$$\sum_{i \in Sp\, x^0} x_i^0 \widetilde{F}_{(i,\, y^0)} \overset{\widetilde{\omega}_1}{<} \sum_{i \in Sp\, x^0} x_i^0 \widetilde{F}_{(x^0,\, y^0)}. \tag{15}$$

Using the linearity property (Lemma 1) we obtain from (16) inequality $\widetilde{F}_{(x^0,\, y^0)} \overset{\widetilde{\omega}_1}{<} \widetilde{F}_{(x^0,\, y^0)}$ which is false. Thus, the equality $\widetilde{F}_{(i,\, y^0)} = \widetilde{F}_{(x^0,\, y^0)}$ holds for any $i \in Sp\, x^0$, i.e. y^0 is a column balance vector for submatrix $F\left(Sp\, x^0 \times Sp\, y^0\right)$. Analogously we can prove that x^0 is a row balance vector for this submatrix. According to Lemma 2, $\left(x^0, y^0\right)$ is a balance pair for submatrix $F\left(Sp\, x^0 \times Sp\, y^0\right)$. □

Corollary 4. *Suppose that there exists a Nash equilibrium point in mixed extension of game G. Then the matrix of outcomes of game G contains a balanced submatrix.*

Now we state the main result of our work.

Theorem 4. *A situation in mixed strategies $\left(x^0, y^0\right)$ is Nash equilibrium point in mixed extension of game G if and only if the following conditions hold:*

1. The restriction of matrix of outcomes under the pair of spectra $\left(Sp\, x^0, Sp\, y^0\right)$ is balanced matrix;

2. $\widetilde{F}_{(i,\, y^0)} \overset{\widetilde{\omega}_1}{\leqq} \widetilde{F}_{(x^0,\, y^0)}$ for all $i \notin Sp\, x^0$;

3. $\widetilde{F}_{(x^0,\, j)} \overset{\widetilde{\omega}_2}{\leqq} \widetilde{F}_{(x^0,\, y^0)}$ for all $j \notin Sp\, y^0$.

The proof of Theorem 4 follows directly from Lemma 7 and Lemma 8.

Corollary 5. *A game G have a quite Nash equilibrium point in mixed strategies if and only if its matrix of outcomes is balanced. In this case, any pair of balance vector $\left(x^0, y^0\right)$ is Nash equilibrium point with $Sp\, x^0 = I$ and $Sp\, y^0 = J$.*

Let us note that the last condition is defined only by realization function of game G and does not depend on order relations $\omega_k,\ k = 1, 2$.

Remark one special case of Theorem 4.

Corollary 6. *Suppose $I_0 \subseteq I, J_0 \subseteq J$ are two subsets and corresponding submatrix $F(I_0 \times J_0)$ is balanced. Assume for any $i' \notin I_0, j' \notin J_0$ there exist $i_0 \in I_0, j_0 \in J_0$ such that for all $i \in I_0$ and $j \in J_0$ hold the conditions:*

$$\begin{gathered} F(i',\, j) \overset{\omega_1}{\leqq} F\left(i^0,\, j\right), \\ F(i,\, j') \overset{\omega_2}{\leqq} F\left(i,\, j^0\right) \end{gathered} \tag{16}$$

hold. Then any balance pair for submatrix $\left(x^0, y^0\right)$, complemented by zero components for $i \in I \setminus I_0$ and $j \in J \setminus J_0$, is Nash equilibrium point in mixed extension of game G.

Appendix

1. Balanced matrices

Examples of balanced matrices together with their balance vectors by Table 1-3 are given.

Table 1.

I \ J	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{3}$	a	b	c
$\frac{1}{3}$	b	c	a
$\frac{1}{3}$	c	a	b

Table 2.

I \ J	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$
$\frac{1}{6}$	a	a	b	b
$\frac{1}{3}$	a	b	b	a
$\frac{1}{6}$	b	b	a	a
$\frac{1}{3}$	b	a	a	b

Remark that any Latin square forms a balanced matrix and moreover its balance pair is a pair of vectors whose components coincide (see Table 1).

To verify that a given pair of vectors $\left(x^0, y^0\right)$ is a balance one for matrix $\|F(i,j)\| \quad (i \in I, j \in J)$ we can check equalities $\widetilde{F}_{(i_1,\, y^0)} = \widetilde{F}_{(i_2,\, y^0)}$ and $\widetilde{F}_{(x^0,\, j_1)} = \widetilde{F}_{(x^0,\, j_2)}$ for any $i_1, i_2 \in I$ and $j_1, j_2 \in J$.

2. A finding of extreme balanced matrices

Consider a matrix of size 4×4 over set $A = \{a, b, c, d, e\}$ given by Table 4

$$M = \begin{pmatrix} a & a & b \\ a & b & a \\ b & a & a \end{pmatrix}$$

Table 3.

I \ J	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$\frac{1}{6}$	a	b	c	b	a	c
$\frac{1}{6}$	c	b	c	a	b	a
$\frac{1}{3}$	b	a	a	c	c	b
$\frac{1}{6}$	c	c	b	a	b	a
$\frac{1}{6}$	a	c	b	b	a	c

Table 4.

I \ J	1	2	3	4
1	a	a	b	a
2	a	b	a	b
3	b	a	a	e
4	c	a	d	e

Let us verify that its submatrix M defined by the pair (I_0, J_0), where $I_0 = \{1, 2, 3\}$ and $J_0 = \{1, 2, 3\}$ is an extreme balanced submatrix. We have

$$\chi(M^a) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and

$$\chi\left(M^b\right) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We compute $Det\,(\chi(M^a)) = -2$ and $Det\,\left(\chi\left(M^b\right)\right) = -1$. Then we find inverse matrices:

$$(\chi(M^a))^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and

$$\left(\chi\left(M^b\right)\right)^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We now compute

$$\lambda^a = (1,1,1)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right),$$

$$\lambda^b = (1,1,1)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (1,1,1).$$

Thus, all components of vectors λ^a and λ^b are positive and $\lambda^a \| \lambda^b$.

Also

$$\delta^a = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} (1,1,1)^T = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T,$$

$$\delta^b = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} (1,1,1)^T = (1,1,1)^T.$$

We see that all components of vectors δ^a and δ^b are positive and $\delta^a \| \delta^b$. It is shown that all assumptions of Theorem 3 are satisfied, consequently a submatrix M is an extreme balance one. To find a balance pair of vectors for this submatrix, we use Theorem 1. We have $x_1 = \frac{1}{3}, x_2 = \frac{1}{3}, x_3 = \frac{1}{3}, y_1 = \frac{1}{3}, y_2 = \frac{1}{3}, y_3 = \frac{1}{3}$. Thus the balance pair for submatrix M is $x^0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $y^0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

3. A finding of Nash equilibrium points in mixed strategies

Consider an antagonistic game with ordered outcomes given by realization function F (see Table 5) and order relation ω for player 1 (see Diagram 1); an order relation for player 2 is defined as inverse order relation ω^{-1}.

Table 5.

F	1	2	3	4
1	a	a	b	a
2	a	b	a	b
3	b	a	a	e
4	c	a	d	e

It is shown above that submatrix M of matrix given by Table 5 is balanced and its balance pair is $x^0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and $y^0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ (see Appendix 2).

Because hold the following conditions:

$$c \overset{\omega}{\leqq} b, \quad a \overset{\omega}{\leqq} a, \quad d \overset{\omega}{\leqq} a;$$

$$a \overset{\omega}{\geqq} a, \quad b \overset{\omega}{\geqq} b, \quad e \overset{\omega}{\geqq} a,$$

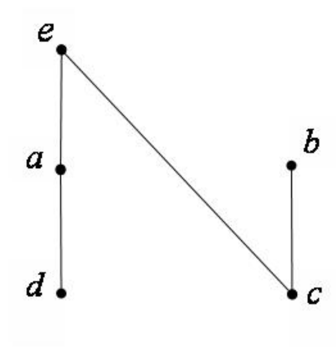

Fig. 1. Diagram 1

then by using Corollary 6, we get that the pair of vectors $\left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)\right)$ is Nash equilibrium point in mixed extension of considered game.

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Cooperative Optimality Concepts for Games with Preference Relations

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Abstract. In this paper we consider games with preference relations. The cooperative aspect of a game is connected with its coalitions. The main optimality concepts for such games are concepts of equilibrium and acceptance. We introduce a notion of coalition homomorphism for cooperative games with preference relations and study a problem concerning connections between equilibrium points (acceptable outcomes) of games which are in a homomorphic relation. The main results of our work are connected with finding of covariant and contrvariant homomorphisms.

Keywords: Nash equilibrium, Equilibrium, Acceptable outcome, Coalition homomorphism

1. Introduction

We consider a n-person game with preference relations in the form

$$G = \langle (X_i)_{i\in N},\, A,\, F,\, (\rho_i)_{i\in N}\rangle \tag{1}$$

where $N = \{1, \ldots, n\}$ is a set of players, X_i is a set of *strategies* of player $i\,(i \in N)$, A is a set of *outcomes*, realization function F is a mapping of set of *situations* $X = X_1 \times \ldots \times X_n$ in the set of outcomes A and $\rho_i \subseteq A^2$ is a preference relation of player i. In general case each ρ_i is an arbitrary reflexive binary relation on A. Assertion $a_1 \overset{\rho_i}{\lesssim} a_2$ means that outcome a_1 is less preference than a_2 for player i. Given a preference relation $\rho_i \subseteq A^2$, we denote by $\rho_i^s = \rho_i \cap \rho_i^{-1}$ its symmetric part and $\rho_i^* = \rho_i \backslash \rho_i^s$ its strict part (see Savina, 2010).

The cooperative aspect of a game is connected with its coalitions. In our case we can define for any coalition $T \subseteq N$ its set of strategies X_T in the form

$$X_T = \prod_{i\in T} X_i. \tag{2}$$

We construct a preference relation of coalition T with help of preference relations of players which form the coalition. We denote a preference relation for coalition T by ρ_T. The following condition is minimum requirement for preference of coalition T:

$$a_1 \overset{\rho_T}{\lesssim} a_2 \;\Rightarrow\; (\forall i \in T)\, a_1 \overset{\rho_i}{\lesssim} a_2. \tag{3}$$

In section 2 we consider some important concordance rules. Let $\mathcal{K}$ be a fix collection of coalitions. In section 3 we introduce the following cooperative optimality concepts: Nash $\mathcal{K}$–equilibrium, $\mathcal{K}$–equilibrium, quite $\mathcal{K}$–acceptance, $\mathcal{K}$–acceptance and connections between these concepts are established in Theorem 1. In next section we consider coalition homomorphisms. The main results of our paper are presented in section 5.

2. Concordance rules for preferences of players

To construct a preference relation for coalition T we need to have preference relations of all players its coalition and also certain rule for concordance of preferences of players. Such set of rules is called *concordance rule.* It is known that important concordance rules are the following.

2.1. Pareto concordance

Definition 1. Outcome a_2 is said to (non strict) dominate by Pareto outcome a_1 for coalition T if a_2 is better (not worse) than a_1 for each $i \in T$, i.e.

$$a_1 \overset{\rho_T}{\lesssim} a_2 \;\Leftrightarrow\; (\forall i \in T)\, a_1 \overset{\rho_i}{\lesssim} a_2. \tag{4}$$

In this case symmetric part of preference relation for coalition T is defined by the formula

$$a_1 \overset{\rho_T}{\sim} a_2 \Leftrightarrow (\forall i \in T)\, a_1 \overset{\rho_i}{\sim} a_2 \tag{5}$$

and strict part is defined by the formula

$$a_1 \overset{\rho_T}{<} a_2 \;\Leftrightarrow\; \begin{cases} (\forall i \in T)\; a_1 \overset{\rho_i}{\lesssim} a_2, \\ (\exists j \in T)\; a_1 \overset{\rho_j}{<} a_2 \end{cases} \tag{6}$$

Thus, outcome a_2 dominate a_1 if and only if a_2 is better than a_1 for all players of coalition T and strictly better at least for one player $j \in T$.

2.2. Modified Pareto concordance

In this case strict part of preference relation ρ_T is defined by the equivalence

$$a_1 \overset{\rho_T}{<} a_2 \;\Leftrightarrow\; (\forall i \in T)\, a_1 \overset{\rho_i}{<} a_2, \tag{7}$$

and symmetric part is given by

$$a_1 \overset{\rho_T}{\sim} a_2 \;\Leftrightarrow\; (\forall i \in T)\, a_1 \overset{\rho_i}{\sim} a_2. \tag{8}$$

2.3. Concordance by majority rule

Outcome a_2 is strictly better than outcome a_1 for coalition T if and only if a_2 is strictly better than a_1 for majority of players of coalition T, i.e.

$$a_2 \overset{\rho_T}{>} a_1 \Leftrightarrow \left|\left\{ i \in T \colon a_2 \overset{\rho_i}{>} a_1 \right\}\right| > \left|\frac{T}{2}\right|.$$

For this rule, symmetric part of preference relation ρ_T is given by the equivalence

$$a_1 \overset{\rho_T}{\sim} a_2 \Leftrightarrow \left|\left\{ i \in T \colon a_1 \overset{\rho_i}{\sim} a_2 \right\}\right| > \left|\frac{T}{2}\right|.$$

2.4. Concordance under summation of payoffs

For games with payoff functions in the form $H = \left\langle (X_i)_{i\in N}, (u_i)_{i\in N} \right\rangle$, the following concordance rule of preferences for coalition T is used

$$x^1 \overset{\rho_T}{\lesssim} x^2 \Leftrightarrow \sum_{i\in T} u_i(x^1) \leq \sum_{i\in T} u_i(x^2) \tag{9}$$

and the strict part of ρ_T is given by:

$$x^1 \overset{\rho_T}{<} x^2 \Leftrightarrow \sum_{i\in T} u_i(x^1) < \sum_{i\in T} u_i(x^2).$$

In this case preference relation ρ_T and its strict part are transitive.

Remark 1. Let $\{T_1, \ldots, T_m\}$ be partition of set N. Then collection of strategies of these coalitions $(x_{T_1}, \ldots, x_{T_m})$ define a single situation $x \in X$ in game G. Namely, the situation x is such a situation that its projection on T_k is x_{T_k} $(k = 1, \ldots, m)$. Hence we can define a realization function F by the rule: $F\left(x_{T_1}, \ldots, x_{T_m}\right) \overset{df}{=} F\left(x\right).$ In particular if T is one fix coalition then the function $F\left(x_T, x_{N\setminus T}\right)$ is defined.

Remark 2. Consider a game with payoff functions $H = \left\langle (X_i)_{i\in N}, (u_i)_{i\in N} \right\rangle$ where $u_i\colon \prod_{i\in N} X_i \to \mathbb{R}$ is a payoff function for players i. Then we can define the preference relation of player i by the formula

$$x^1 \overset{\rho_i}{\lesssim} x^2 \Leftrightarrow u_i(x^1) \leq u_i(x^2).$$

Let the preference relation of coalition T be Pareto dominance, i.e.

$$x^1 \overset{\rho_T}{\lesssim} x^2 \Leftrightarrow (\forall i \in T)\ u_i(x^1) \leq u_i(x^2).$$

Then considered above concordance rules is becoming well known rules for cooperative games with payoff functions. (see Moulin, 1981).

3. Coalitions optimality concepts

In this part we consider games with preference relations of the form (1). For games of this class two types of optimality concepts are introduced and connections between these concepts are established.

Let $\mathcal{K}$ be an arbitrary fixed family of coalitions of players N.

3.1. Equilibrium concepts

Definition 2. A situation $x^0 = (x_i^0)_{i\in N} \in X$ is called *Nash $\mathcal{K}$–equilibrium* (Nash $\mathcal{K}$–equilibrium point) if for any coalition $T \in \mathcal{K}$ and any strategy $x_T \in X_T$ the condition

$$F(x^0 \parallel x_T) \overset{\rho_T}{\lesssim} F(x^0) \tag{10}$$

holds.

Remark 3. 1. In the case $\mathcal{K} = \{\{1\}, \ldots, \{n\}\}$, Nash $\mathcal{K}$–equilibrium is Nash equilibrium in the usual sense.

2. In the case $\mathcal{K} = \{N\}$, a situation x^0 is Nash $\{N\}$–equilibrium means $F\left(x^0\right)$ is greatest element under preference ρ_T.

We now define some generalization of Nash equilibrium.

A strategy $x_T^0 \in X_T$ is called a *refutation of the situation $x \in X$ by coalition T* if the condition

$$F(x \parallel x_T^0) \overset{\rho_T}{>} F(x) \tag{11}$$

holds.

Definition 3. A situation $x^0 = (x_i^0)_{i \in N} \in X$ is called $\mathcal{K}$*–equilibrium point* if any coalition $T \in \mathcal{K}$ does not have a refutation of this situation, i.e. for any coalition $T \in \mathcal{K}$ and any strategy $x_T \in X_T$ the condition

$$F(x^0 \parallel x_T) \overset{\rho_T}{\not>} F(x^0)$$

holds.

Remark 4. 1. In the case $\mathcal{K} = \{\{1\}, \ldots, \{n\}\}$, $\mathcal{K}$–equilibrium is equilibrium in the usual sense.
2. In the case $\mathcal{K} = \{N\}$, $\mathcal{K}$–equilibrium point is Pareto optimal.
3. In the case $\mathcal{K} = 2^N$, $\mathcal{K}$–equilibrium point is called *strong equilibrium* one.

3.2. Acceptable outcomes and acceptable situations

A strategy $x_T^0 \in X_T$ is called a *objection* of coalition T *against outcome* $a \in A$ if for any strategy of complementary coalition $x_{N \setminus T} \in X_{N \setminus T}$ the condition

$$F(x_T^0, x_{N \setminus T}) \overset{\rho_T}{>} a \tag{12}$$

holds.

Definition 4. An outcome $a \in A$ is called *acceptable* for coalition T if this coalition does not have objections against this outcome.

An outcome $a \in A$ is said to be $\mathcal{K}$*–acceptable* if it is acceptable for all coalitions $T \in \mathcal{K}$, that is

$$(\forall T \in \mathcal{K})(\forall x_T \in X_T)(\exists x_{N \setminus T} \in X_{N \setminus T}) F(x_T, x_{N \setminus T}) \overset{\rho_T}{\not>} a. \tag{13}$$

A strategy $x_T^0 \in X_T$ is called a *objection* of coalition T *against situation* $x^* \in X$ if this strategy is an objection against outcome $F\left(x^*\right)$.

We define also a quite acceptable concept by changing quantifiers: $\forall x_T\ \exists x_{N \setminus T} \to \exists x_{N \setminus T}\ \forall x_T$.

Definition 5. An outcome a is called *quite* $\mathcal{K}$*-acceptable* for family of coalitions $\mathcal{K}$ if the condition

$$(\forall T \in \mathcal{K})(\exists x_{N \setminus T} \in X_{N \setminus T})(\forall x_T \in X_T) F(x_{N \setminus T}, x_T) \overset{\rho_T}{\not>} a \tag{14}$$

holds.

A situation $x^0 \in X$ is called *quite* $\mathcal{K}$*-acceptable* if outcome $F\left(x^0\right)$ is quite $\mathcal{K}$-acceptable one.

These optimality concepts are analogous to well known optimality concepts of games with payoff functions (see Moulin, 1981).

Now we consider connections between these optimality concepts.

Lemma 1. *Nash $\mathcal{K}$–equilibrium point is also a $\mathcal{K}$–equilibrium point but converse is false.*

Proof (of lemma). Let $x^0 = (x_i^0)_{i\in N}$ be Nash $\mathcal{K}$–equilibrium point then for any coalition $T \in \mathcal{K}$ and any strategy $x_T \in X_T$ the condition $F(x^0 \parallel x_T) \overset{\rho_T}{\lesssim} F(x^0)$ holds. Suppose $F(x^0 \parallel x_T) \overset{\rho_T}{>} F(x^0)$. The system of conditions

$$\begin{cases} F(x^0 \parallel x_T) \overset{\rho_T}{\lesssim} F(x^0) \\ F(x^0 \parallel x_T) \overset{\rho_T}{>} F(x^0) \end{cases}$$

is false. Hence, $F(x^0 \parallel x_T) \overset{\rho_T}{\ngtr} F(x^0)$. □

Thus, Nash $\mathcal{K}$–equilibrium is $\mathcal{K}$–equilibrium. But the converse is false. Indeed, consider

Example 1. Consider an antagonistic game G whose realization function F is given by Table 1 and preference relation for player 1 by Diagram 1; preference relation of player 2 is given by inverse order, $\mathcal{K} = \{\{1\}, \{2\}\}$.

Table 1. Realization function

F	t_1	t_2
s_1	a	b
s_2	c	d

Situation (s_1, t_1) is $\mathcal{K}$–equilibrium. Since $F(s_1, t_1) = a$ and $a\|b, a\|c$ (i.e. a and b is incomparable, a and c is incomparable) then (s_1, t_1) is not Nash $\mathcal{K}$–equilibrium.

Remark 5. If all preference relations $(\rho_T)_{T\in\mathcal{K}}$ is linear then Nash $\mathcal{K}$–equilibrium and $\mathcal{K}$–equilibrium are equivalent.

Proposition 1. *An objection of coalition T against situation x^* is also a refutation of this situation.*

Proof (of proposition). Let x_T^0 be an objection of coalition T against situation x^*. Then according to definition of objection the strategy x_T^0 is an objection of coalition T against outcome $F(x^*)$, i.e. for any strategy of complementary coalition $x_{N\setminus T} \in X_{N\setminus T}$ the condition $F(x_T^0, x_{N\setminus T}) \overset{\rho_T}{>} F(x^*)$ holds.

Let us take $x_{N\setminus T} = x^*_{N\setminus T}$ as a strategy of complementary coalition, we have $F(x_T^0, x^*_{N\setminus T}) \overset{\rho_T}{>} F(x^*)$.

Since strategy $x_{N\setminus T}$ is an arbitrary one then we get strategy x_T^0 is a refutation of this situation. □

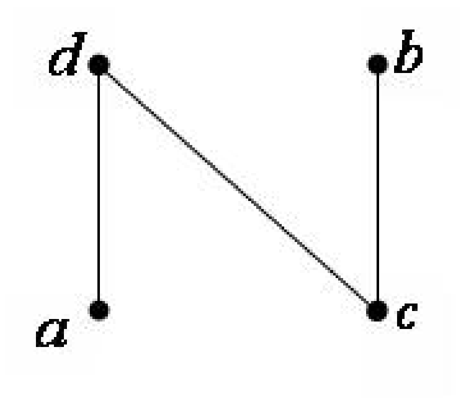

Fig. 1. Diagram 1

Corollary 1. *Any $\mathcal{K}$–equilibrium point is also $\mathcal{K}$–acceptable.*

We have to prove the more strong assertion.

Lemma 2. *Any $\mathcal{K}$–equilibrium point is also quite $\mathcal{K}$–acceptable.*

Proof (of lemma). Let x^0 be $\mathcal{K}$–equilibrium point. Suppose $x_{N\setminus T} = x^0_{N\setminus T}$ for all coalitions $T \in \mathcal{K}$. Then for any coalition $T \in \mathcal{K}$ we have $F\left(x_{N\setminus T}, x_T\right) = F\left(x^0_{N\setminus T}, x_T\right) = F(x^0 \parallel x_T) \overset{\rho_T}{\not>} F(x^0)$. Hence, x^0 is quite $\mathcal{K}$–acceptable. □

Lemma 3. *Any quite $\mathcal{K}$–acceptable outcome is $\mathcal{K}$–acceptable.*

The proof of Lemma 3 is obvious.

The main result of the part 3 is the following theorem.

Theorem 1. *Consider introduced above coalitions optimality concepts: Nash $\mathcal{K}$–equilibrium, $\mathcal{K}$–equilibrium, quite $\mathcal{K}$–acceptance, $\mathcal{K}$–acceptance. Then each consequent condition is more weak than preceding, i.e.*

Nash $\mathcal{K}$–equilibrium $\Rightarrow$ $\mathcal{K}$–equilibrium $\Rightarrow$ quite $\mathcal{K}$–acceptance $\Rightarrow$ $\mathcal{K}$–acceptance.

The proof of Theorem 1 follows from Lemmas 1, 2, 3.

4. Coalition homomorphisms for games with preference relations

Let

$$G = \langle (X_i)_{i\in N}, A, F, (\rho_i)_{i\in N}\rangle$$

and

$$\Gamma = \langle (Y_i)_{i\in N}, B, \Phi, (\sigma_i)_{i\in N}\rangle$$

be two games with preference relations of the players N.

Any $(n+1)$–system consisting of mappings $f = (\varphi_1, \ldots, \varphi_n, \psi)$ where for any $i = 1, \ldots, n$, $\varphi_i \colon X_i \to Y_i$ and $\psi \colon A \to B$, is called *a homomorphism* from game G into game Γ if for any $i = 1, \ldots, n$ and any $a_1, a_2 \in A$ the following two conditions

$$a_1 \overset{\rho_i}{\lesssim} a_2 \;\Rightarrow\; \psi(a_1) \overset{\sigma_i}{\lesssim} \psi(a_2), \tag{15}$$

$$\psi(F(x_1, \ldots, x_n)) = \Phi(\varphi_1(x_1), \varphi_2(x_2), \ldots, \varphi_n(x_n)) \tag{16}$$

are satisfied.

A homomorphism f is said to be *strict homomorphism* if system of the conditions

$$a_1 \overset{\rho_i}{<} a_2 \;\Rightarrow\; \psi(a_1) \overset{\sigma_i}{<} \psi(a_2), \quad (i = 1, \ldots, n) \tag{17}$$

$$a_1 \overset{\rho_i}{\sim} a_2 \;\Rightarrow\; \psi(a_1) \overset{\sigma_i}{\sim} \psi(a_2) \quad (i = 1, \ldots, n) \tag{18}$$

holds instead of condition (15).

A homomorphism f is said to be *regular homomorphism* if the conditions

$$\psi(a_1) \overset{\sigma_i}{<} \psi(a_2) \;\Rightarrow\; a_1 \overset{\rho_i}{<} a_2, \tag{19}$$

$$\psi(a_1) \overset{\sigma_i}{\sim} \psi(a_2) \;\Rightarrow\; \psi(a_1) = \psi(a_2) \tag{20}$$

hold.

A homomorphism f is said to be *homomorphism "onto"*, if each $\varphi_i\,(i = 1, \ldots, n)$ is a mapping "onto".

Now we introduce a concept of coalition homomorphism.

For the first step, we need to fix some rule for concordance of preferences; recall that the preference relation for coalition T denoted by ρ_T.

Definition 6. A homomorphism f is said to be:

– *a coalition homomorphism* if it preserves preference relations for all coalitions, i.e. for any coalition $T \subseteq N$ the condition

$$a_1 \overset{\rho_T}{\lesssim} a_2 \;\Rightarrow\; \psi(a_1) \overset{\sigma_T}{\lesssim} \psi(a_2) \tag{21}$$

holds;

– *a strict coalition homomorphism* if for any coalition $T \subseteq N$ the system of the conditions

$$\begin{cases} a_1 \overset{\rho_T}{<} a_2 \Rightarrow \psi(a_1) \overset{\sigma_T}{<} \psi(a_2), \\ a_1 \overset{\rho_T}{\sim} a_2 \Rightarrow \psi(a_1) \overset{\sigma_T}{\sim} \psi(a_2) \end{cases} \tag{22}$$

is satisfied;

– *a regular coalition homomorphism* if for any coalition $T \subseteq N$ the system of the conditions

$$\begin{cases} \psi(a_1) \overset{\sigma_T}{<} \psi(a_2) \Rightarrow\ a_1 \overset{\rho_T}{<} a_2, \\ \psi(a_1) \overset{\sigma_T}{\sim} \psi(a_2) \Rightarrow\ \psi(a_1) = \psi(a_2) \end{cases} \tag{23}$$

is satisfied.

It is easy to see that the following assertion is true.

Lemma 4. *For Pareto concordance (and also for modified Pareto concordance), any surjective homomorphism from G into Γ is a surjective coalition homomorphism.*

Lemma 5. *For Pareto concordance (and also for modified Pareto concordance), any strict homomorphism from G into Γ is a strict coalition homomorphism.*

Proof (of lemma 5). We consider Pareto concordance for preferences as a concordance rule. Verify the conditions of system (22) for preference relation ρ_T. According to defition of Pareto concordance the condition $a_1 \overset{\rho_T}{<} a_2$ is equivalent system

$$\begin{cases} (\forall i \in T)\ a_1 \overset{\rho_i}{\lesssim} a_2, \\ (\exists j \in T)\ a_1 \overset{\rho_j}{<} a_2. \end{cases}$$

Since strict homomorphism is homomorphism then from the first condition of system it follows that $(\forall i \in T)\, \psi\left(a_1\right) \overset{\sigma_i}{\lesssim} \psi\left(a_2\right)$. Since homomorphism f is strict then $(\exists j \in T)\, \psi\left(a_1\right) \overset{\sigma_i}{<} \psi\left(a_2\right).$

From last two conditions we get $\psi\left(a_1\right) \overset{\sigma_T}{<} \psi\left(a_2\right).$

Now according to definition of symmetric part of relation ρ_T we have $a_1 \overset{\rho_T}{\sim} a_2 \Leftrightarrow (\forall i \in T)\, a_1 \overset{\rho_i}{\sim} a_2$. Since homomorphism f is strict then we get $(\forall i \in T)\, \psi\left(a_1\right) \overset{\sigma_i}{\sim} \psi\left(a_2\right)$, i.e. $\psi\left(a_1\right) \overset{\sigma_T}{\sim} \psi\left(a_2\right).$ □

Now we consider modified Pareto concordance for preferences of players as a concordance rule.

Lemma 6. *For modified Pareto concordance, any regular homomorphism from G into Γ is a regular coalition homomorphism.*

Proof (of lemma 6). Verify the condition (23) for strict part of preference relation σ_T. According to definition of modified Pareto concordance for preferences the condition $\psi\left(a_1\right) \overset{\sigma_T}{<} \psi\left(a_2\right)$ is equivalent $(\forall i \in T)\, \psi\left(a_1\right) \overset{\sigma_i}{<} \psi\left(a_2\right)$. Since homomorphism f is regular then we have $(\forall i \in T)\, a_1 \overset{\rho_i}{<} a_2$, i.e. $a_1 \overset{\rho_T}{<} a_2$.

Verify the condition (23) for symmetric part of σ_T. According to definition of modified Pareto concordance we have

$$\psi\left(a_1\right) \overset{\sigma_T}{\sim} \psi\left(a_2\right) \Leftrightarrow (\forall i \in T)\, \psi\left(a_1\right) \overset{\sigma_i}{\sim} \psi\left(a_2\right).$$

Since homomorphism f is regular then from the last condition it follows that $(\forall i \in T)\, \psi\left(a_1\right) = \psi\left(a_2\right)$, i.e. $\psi\left(a_1\right) = \psi\left(a_2\right)$. □

5. The main results

The main result states a correspondence between sets of $\mathcal{K}$-acceptable outcomes and $\mathcal{K}$-equilibrium situations of games which are in homomorphic relations under indicated types.

A homomorphism f is said to be *covariant* if f–image of any optimal solution in game G is an optimal solution in $\varGamma$.

A homomorphism f is said to be *contrvariant* if f–preimage of any optimal solution in game $\varGamma$ is an optimal solution in G.

Theorem 2. *For Nash $\mathcal{K}$–equilibrium, any surjective homomorphism is covariant under Pareto concordance and under modified Pareto concordance also.*

Proof (of theorem 2). We consider Pareto concordance for preferences as a concordance rule. Let x^0 be Nash $\mathcal{K}$–equilibrium point in game G. We have to prove that $\varphi\left(x^0\right)$ is Nash $\mathcal{K}$–equilibrium point in game $\varGamma$.

We fix arbitrary strategy $y_T \in Y_T$. Since f is homomorphism "onto" then according to Lemma 4 we obtain $\left(\exists x_T^* \in X_T\right) \varphi_T\left(x_T^*\right) = y_T$. For any strategy x_T the condition $F(x_T, x^0_{N\setminus T}) \overset{\rho_T}{\lesssim} F(x^0)$ holds. Hence, for strategy x_T^* the condition $F(x_T^*, x^0_{N\setminus T}) \overset{\rho_T}{\lesssim} F(x^0)$ is satisfied. Since f is homomorphism then $\psi\left(F(x_T^*, x^0_{N\setminus T})\right) \overset{\sigma_T}{\lesssim} \psi\left(F(x^0)\right)$. By condition (16): $\varPhi\left(\varphi_T\left(x_T^*\right), \varphi_{N\setminus T}\left(x^0_{N\setminus T}\right)\right) \overset{\sigma_T}{\lesssim} \varPhi\left(\varphi\left(x^0\right)\right)$, i.e. $\varPhi\left(y_T, \varphi_{N\setminus T}\left(x^0_{N\setminus T}\right)\right) \overset{\sigma_T}{\lesssim} \varPhi\left(\varphi\left(x^0\right)\right)$.

Since strategy $y_T \in Y_T$ is arbitrary one then $\varphi\left(x^0\right)$ is Nash $\mathcal{K}$–equilibrium. $\square$

Theorem 3. *For $\mathcal{K}$–equilibrium, any strict surjective homomorphism is contrvariant under Pareto concordance and under modified Pareto concordance also.*

Proof (of theorem 3). Consider Pareto concordance for preferences as a concordance rule. Let y^0 be $\mathcal{K}$–equilibrium point. We have to prove that situation x^0 with $\varphi\left(x^0\right) = y^0$ is $\mathcal{K}$–equilibrium point.

Suppose $x^0 = \left(x_i^0\right)_{i\in N}$ is not $\mathcal{K}$–equilibrium then there exists coalition $T \in \mathcal{K}$ and strategy $x_T^* \in X_T$ such that $F\left(x_T^*, x^0_{N\setminus T}\right) \overset{\rho_T}{>} F\left(x^0\right)$. Since homomorphism f is strict then according to Lemma 5 we get $\psi\left(F\left(x_T^*, x^0_{N\setminus T}\right)\right) \overset{\sigma_T}{>} \psi\left(F\left(x^0\right)\right)$. According to condition (16) we obtain $\varPhi\left(\varphi_T\left(x_T^*\right), \varphi_{N\setminus T}\left(x^0_{N\setminus T}\right)\right) \overset{\sigma_T}{>} \varPhi\left(\varphi\left(x^0\right)\right)$. The last condition means $\varPhi\left(\varphi_T\left(x_T^*\right), y^0_{N\setminus T}\right) \overset{\sigma_T}{>} \varPhi\left(y^0\right)$. Thus, strategy $\varphi_T\left(x_T^*\right)$ is refutation of situation y^0 by coalition T, which is contradictory with y^0 is $\mathcal{K}$–equilibrium point.

Hence, x^0 is $\mathcal{K}$–equilibrium point. $\square$

Theorem 4. *For $\mathcal{K}$–acceptance, any strict surjective homomorphism is contrvariant under Pareto concordance and under modified Pareto concordance also.*

Proof (of theorem 4). Consider Pareto concordance for preferences as a concordance rule. Let outcome b with $\psi\left(a\right) = b$ be $\mathcal{K}$–acceptable one in game $\varGamma$. Assume that outcome a is not acceptable for all coalitions $T \in \mathcal{K}$, i.e. there exists such strategy $x_T^0 \in X_T$ that for any strategy $x_{N\setminus T} \in X_{N\setminus T}$ the condition

$$F(x_T^0, x_{N\setminus T}) \overset{\rho_T}{>} a \tag{24}$$

holds.

Let $y_{N\setminus T} = (y_j)_{j\in N\setminus T}$ be arbitrary strategy of complementary coalition $N \setminus T$ in game Γ. Since f is homomorphism "onto" then according to Lemma 4 we have $\left(\exists x^*_{N\setminus T} \in X_{N\setminus T}\right) \varphi_{N\setminus T}\left(x^*_{N\setminus T}\right) = y_{N\setminus T}$. By (24) the condition $F(x_T^0, x^*_{N\setminus T}) \overset{\rho_T}{>} a$ holds. According to Lemma 5 we get $\psi\left(F(x_T^0, x^*_{N\setminus T})\right) \overset{\sigma_T}{>} \psi(a)$. By (16) we have $\psi\left(F(x_T^0, x^*_{N\setminus T})\right) = \Phi\left(\varphi_T\left(x_T^0\right), \varphi_{N\setminus T}\left(x^*_{N\setminus T}\right)\right)$. Thus, the condition $\Phi\left(\varphi_T\left(x_T^0\right), y_{N\setminus T}\right) \overset{\sigma_T}{>} \psi(a)$ is satisfied. Hence, strategy $\varphi_T\left(x_T^0\right)$ is objection of coalition T against outcome b which is contadictory with b is $\mathcal{K}$–acceptable outcome.

Hence, outcome a is $\mathcal{K}$–acceptable. □

Theorem 5. *For $\mathcal{K}$–equilibrium, any regular surjective homomorphism is covariant under modified Pareto concordance.*

Proof (of theorem 5). Let x^0 be $\mathcal{K}$–equilibrium. We have to prove that situation $\varphi\left(x^0\right)$ is $\mathcal{K}$–equilibrium.

Suppose $\varphi\left(x^0\right)$ is not $\mathcal{K}$–equilibrium, i.e.

$$(\exists T \in \mathcal{K})\ (\exists y_T \in Y_T)\ \Phi\left(\varphi\left(x^0\right)\|y_T\right) \overset{\sigma_T}{>} \Phi\left(\varphi\left(x^0\right)\right) \tag{25}$$

Since homomorphism f is surjective then according to Lemma 4 we have $(\exists x^*_T \in X_T)\,\varphi_T\left(x^*_T\right) = y_T$. Hence, the condition $\Phi\left(\varphi_T\left(x^*_T\right), \varphi_{N\setminus T}\left(x^0_{N\setminus T}\right)\right) \overset{\sigma_T}{>} \Phi\left(\varphi\left(x^0\right)\right)$ holds. By (16) we get $\Phi\left(\varphi_T\left(x^*_T\right), \varphi_{N\setminus T}\left(x^0_{N\setminus T}\right)\right) = \psi\left(F\left(x^*_T, x^0_{N\setminus T}\right)\right)$. Thus, $\psi\left(F\left(x^0\|x^*_T\right)\right) \overset{\sigma_T}{>} \psi\left(F\left(x^0\right)\right)$. Because homomorphism f is regular then according to Lemma 6 we obtain $F\left(x^0\|x^*_T\right) \overset{\rho_T}{>} F\left(x^0\right)$. Thus, strategy x^*_T is refutation of situation x^0 by coalition T, which is contradictory with x^0 is $\mathcal{K}$–equilibrium.

Hence, $\varphi\left(x^0\right)$ is $\mathcal{K}$–equilibrium in game Γ. □

Appendix

Consider the example conserning of concordance rules.

Let G be a game of three players with set of outcomes $A = \{a, b, c, d, e\}$. Preference relations for each player are given by Diagrams 2,3,4.

Using Diagrams 2 – 4 we can define preference relations in the following form:

$$\begin{aligned} &\rho_1 : a < b, b \sim c, c \sim d, b < e \\ &\rho_2 : a \sim b, b \sim c, c < d, e < d \\ &\rho_3 : a < c, b \sim c, c < d, b \sim e, d \sim e. \end{aligned}$$

Then according to Pareto concordance (see 2.1) for coalition $T = \{1, 2\}$ we have $\rho_T : a \lesssim b, b \lesssim c, c \lesssim d$ where strict part consists of two conditions $a \overset{\rho_T}{<} b, c \overset{\rho_T}{<} d$ and symmetric part is $b \overset{\rho_T}{\sim} c$.

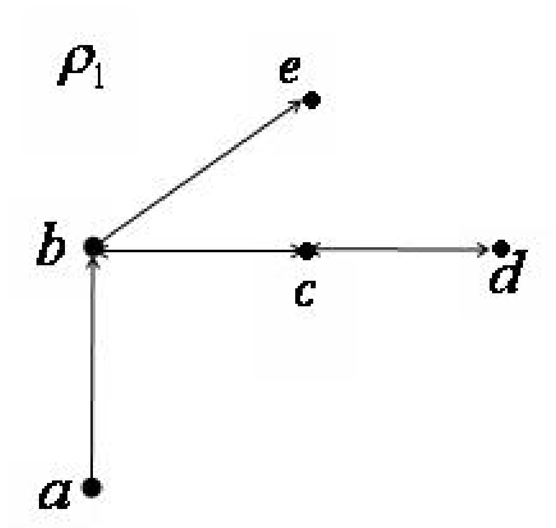

Fig. 2. Diagram 2

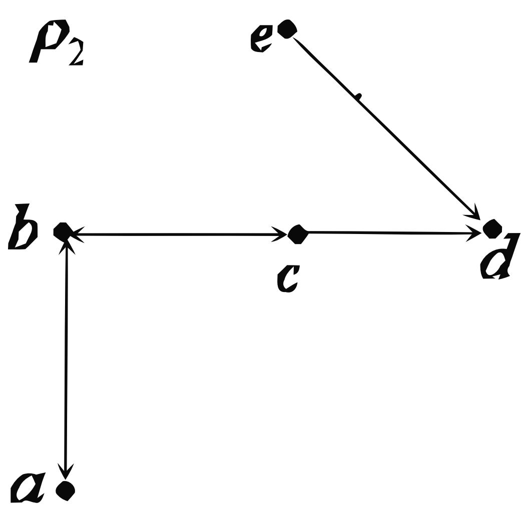

Fig. 3. Diagram 3

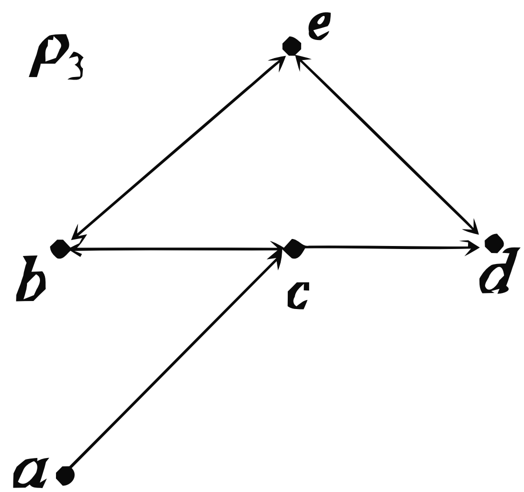

Fig. 4. Diagram 4

For $T = \{1,3\}$ a preference relation ρ_T is defined by $b \lesssim c, c \lesssim d, b \lesssim e$ where strict part is $c \overset{\rho_T}{<} d, b \overset{\rho_T}{<} e$ and symmetric part is $b \overset{\rho_T}{\sim} c$.

For $T = \{2,3\}$ relation ρ_T is $b \lesssim c, c \lesssim d, e \lesssim d$ where $c \overset{\rho_T}{<} d, e \overset{\rho_T}{<} d, b \overset{\rho_T}{\sim} c$.

For $T = \{1,2,3\}$ relation ρ_T is $b \lesssim c, c \lesssim d$ where $c \overset{\rho_T}{<} d, b \overset{\rho_T}{\sim} c$.

According to modified Pareto concordance (see 2.2) for coalition $T = \{1,2\}$ strict part ρ_T is empty set and symmetric part consists of one condition $b \overset{\rho_T}{\sim} c$.

For $T = \{2,3\}$ strict part of preference relation ρ_T is defined by $c \overset{\rho_T}{<} d$ and symmetric part is $b \overset{\rho_T}{\sim} c$.

Preference relation ρ_T for coalition $T = \{1,2,3\}$ in the game with majority rule (see 2.3): $a \overset{\rho_T}{\lesssim} b, b \overset{\rho_T}{\sim} c, c \overset{\rho_T}{<} d, b \overset{\rho_T}{\lesssim} e, e \overset{\rho_T}{\lesssim} d$.

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A Fuzzy Cooperative Game Model for Configuration Management of Open Supply Networks*

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Abstract. The paper considers the problem of open supply networks (OSNs) configuring in a highly dynamic economic environment. A novel coalition formation mechanism is proposed, which helps to resolve conflictions between the objectives of the OSN participants and to agree upon effective solutions. This mechanism is based on a generalized model of a fuzzy cooperative game with core. The model was applied for configuring of an automotive OSN. Simulation results are considered.

Keywords: cooperative game, core, supply network, configuring.

1. Introduction

In a todays highly competitive market manufacturers face the challenge of reducing manufacturing cycle time, delivery lead-time and inventory. As a consequence, new organizational forms of enterprise integration emerge to address these challenges resulting in more agile structures of federated enterprises known as adaptive, agile and open supply chains and networks. These organizations are based on the principles of partnership between the enterprises, agile network structures instead of linear chains and are driven by novel business strategies based on the product demand (Fig. 1.). In contrast to conventional chains and supply networks, such organizations can be called open supply networks (OSNs), which are characterized by:

- Availability of alternative providers.
- Availability of alternative configurations meeting orders specifications.
- Expediency of dynamic configuration and reconfiguration of the network depending on the order stream and economic benefit of every enterprise.
- Conflicting objectives of each organization and non-integrated decision making processes.

An OSN belongs to the class of systems with dynamically changing structures, which means that once a new order comes, a new configuration emerges. Thus OSN configuring can be considered one of the main supply chain management

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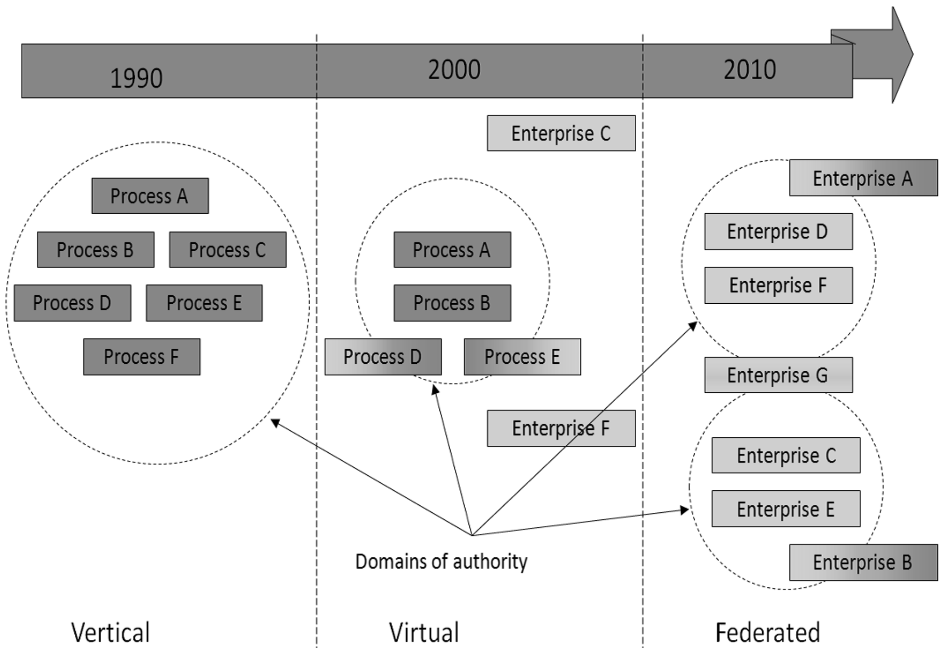

Policy:	Do it all	Outsource	Partnership
Process:	Push	Partial outsourcing	Orchestration
Enablers:	ERP	Integration	Network/Sensors/RFID

Fig. 1. Organizational forms of enterprise integration (adopted from (McBeath et al., 2010))

tasks (Garavelli, 2003; Chandra and Grabis, 2007). Traditionally, configuring has been solved in a two-stage fashion: i) a structure of a network is formed at a strategic level and ii) its behavior is optimized at tactical and operational levels based on demand forecast. Being suitable for vertical and even virtual enterprises (Fig. 1.), such practices, unfortunately, do not meet the requirements of a highly dynamic environment (Smirnov, 1999). One of the consequences is a so-called *bullwhip effect*, when even small demand fluctuations in a loosely balanced forecast-driven distribution systems lead to increased inventories, and, as a consequence, spatial constraints, unused capital, obsolete inventories and so on (Suckya, 2009).

Federated enterprises composed of self-interested entities with probably conflicting goals require flexible dynamic configurations. Unfortunately, adoption of more flexible and dynamic practices, like constraint satisfaction, auctions and knowledge-based approaches, which offer the prospect of better matches between suppliers and customers as market conditions change, has faced difficulties, due to the complexity of many supply chain relationships and the difficulty of effectively supporting dynamic trading practices (Hosam and Khaldoun, 2006; Sandkuhl et al., 2007; Smirnov et al., 2006). Due to the conflictions among the objectives of each organization and non-integrated decision making processes, there has been a need for new mechanisms, which help to resolve those conflictions and to agree upon effective solutions.

For advanced business strategies of OSN like demand-driven or build-to-order SN, a task of configuring of virtual production channels can be defined as a coalition formation task. The benefit distribution among the OSN members has proved to be fuzzy, uncertain, and ambiguous (Roth, 1995). Using the theory of fuzzy cooperative games (FCGs), we can process the uncertainty and pass from the introduction of a fuzzy benefit concept through the bargaining process to the conclusion about the corresponding fuzzy distribution of individual benefits among the coalition members.

In this paper, a game-theoretic approach is used to form coalitions: a class of FCG with core is considered. The basic definition of the fuzzy core was proposed by (Mareŝ, 2001) and extended to the area of multiagent systems by (Sheremetov and Romero-Cortes, 2003). In this paper, an extended definition of the core is considered. The definition introduces fuzzy individual payments and binary values φ_{ij} added to the fuzzy core to form the structure of effective coalitions.

The rest of the paper is structured as follows. In the next section, a OSN configuring task is defined as a problem of coalition formation. In section 3, different approaches to coalition formation in cooperative games theory are analyzed. In section 4, the mathematical structure of the model is described; it is shown that the model represents an extension of the model proposed in (Mareŝ, 2001). A case study applying the proposed approach is discussed in the context of OSNs in section 5. A prototype consisting of seven enterprises and generating a structure of three coalitions is considered. Finally, the results of the paper are discussed in Conclusions section.

2. Supply network configuring as a coalition formation task

The task of configuring can be defined as a selection of those agents (enterprises), which have available competencies to complete the demand/order, and joining them together in the most efficient structure according to the selected criteria. The main components of the configuring task are: order, resource and configuration. Let us consider that to fulfill the order T, I tasks should be executed: T={T_1, T_2,..., T_I}. Each task $T_i (i = 1, 2, \ldots, I)$, is defined by a tuple $\langle\{B_{T_i}\}, \{Pref_{T_i}\}\rangle$, where $\{B_{T_i}\}$ – is a vector of numerical values of the dimension r: $\{B_{T_i}\} = (b^1_{T_i}, b^2_{T_i}, \ldots,\ b^r_{T_i})$, $b^k_{T_i} \geq 0$, characterizing a capacity on each competency $k = 1, 2, \ldots, r$, required to perform a task T_i. If the tasks are ordered, then $T_1 \preceq T_2 \preceq \ldots \preceq T_l \preceq T_m \preceq \ldots \preceq T_I$, where $T_l \preceq T_m$ means that T_l precedes the task T_m. The preferences vector $Pref_{T_i}$ may include additional parameters like the preferred lot size, penalties for backorders, etc. Fulfillment of each order and each task T_i implies a payoff: *Payoff* (T_i).

Example 1. Suppose that the order is to produce 100 products (cars of a specific model) per week. A car consists of four basic components: 1) the body T_1 (14 external tubes $b^1_{T_1}$ and 5 exterior sheets $b^2_{T_2}$); 2) the interior T_2 (a dash board $b^1_{T_2}$, 3 seats $b^2_{T_2}$: two front and one rear); 3) the chassis T_3 (4 wheels $b^1_{T_3}$, 2 axles $b^2_{T_3}$: a front and a rear, 4 dampers $b^3_{T_3}$: two front and two rear); 4) the power train T_4 (a motor $b^1_{T_4}$ and a transmission $b^2_{T_4}$). In other words: $T = \{T_1, T_2, T_3, T_4\}$, $B_{T_1} = (1400, 500)$, $B_{T_2} = (100, 300)$, $B_{T_3} = (400, 200, 400)$, $B_{T_4} = (100, 100)$.

The enterprises of the supply network represent resources. Depending upon their role in the OSN, these resources can be suppliers of raw materials and components, assembly plants or warehouses. They are modeled as active autonomous entities with purposeful actions and, thus, may be called agents. Let us consider a finite set of agents $Agent = \{A_1, A_2, \ldots, A_N\}$. Then each agent $A_j \in Agent (j = 1, 2, \ldots, N)$ is defined as a tuple $\langle\{B_{A_j}\}, \{Pref_{A_j}\}\rangle$. For simplicity lets designate A_j as j. Then B_j – is a vector of numerical values of the dimension r: $B_j = (b^1_j, b^2_j, \ldots,\ b^r_j)$, $b^{k'}_j \geq 0$, characterizing agents available capacity on each competency $k' = 1, 2, \ldots, r$. The preferences vector $Pref_k$ denote agents preferences on the lot size, orders time lag, etc.

Finally, a configuration is such a set of agents (resources) $C_T \subseteq Agent$ that their joint capacity of competencies satisfies the requirements of an order T. To solve the configuring task for the case when the agents and tasks competencies coincide ($b^k_{T_i}$ and $b^{k'}_j$ mean the capacities on the same competency) means to assign resources to the tasks in such a way that the order T is fulfilled. Each agent $A_j \in Agent$ may be assigned to a task T_i *iff* it has available capacities $\exists k \in \{1, 2, \ldots, r\}$, $b^k_j \geq 0$. Being self-interested, each agent will try to optimize this assignment according to one of the following criteria:

– Maximize the use of his capacities $\sum\limits_{k=1}^{r} (b^k_j - b^k_{T_i}) \to \min$;
– To get the most profitable task (to increase the payoff) $\sum\limits_{k=1}^{r} g(b^k_{T_i}) - \sum\limits_{k=1}^{r} f_{T_i}(b^k_j) \to \max$, where $g(b^k_{T_i})$ – is a reward function associated with the payoff *Payoff* (T_i), $f_{T_i}(b^k_j)$ – a cost of agents $j \in Agent$ capacity b^k_j required to fulfill the task T_i;
– Reduce the task T_i fulfillment time: $\sum\limits_{k=1}^{r} t_{T_i}\left(b^k_j\right) \to \min$, where $t_{T_i}\left(b^k_j\right)$ – time of fulfillment of the task T_i by agent $k \in Agent$ using his capacity b^k_j.

Agents can form coalitions to execute tasks. The notion of coalition is widely used in organizational systems. A coalition can be defined as a group of self-interested agents that by means of negotiation protocols decide to cooperate in order to solve a problem or to achieve a goal (Gasser, 1991). Within the context of this paper, a coalition is defined as a group of agents joining their capacities for task T_i fulfillment. A coalition is described by a tuple: $\langle K_{T_i}, alloc_{T_i}, u_{T_i} \rangle$, where $K_{T_i} \subseteq Agent$ and $K_{T_i} \neq \emptyset$; $alloc_{T_i}$ – allocation function assigning each task i a group of m agents such that $alloc_{T_i} = K_{T_i}$, if $\sum\limits_m b_m^k \geq b_{T_i}^k$. If for each competency k, $b_j^k \geq b_{T_i}^k$, K_{T_i} may consist of a single agent $j \in K_{T_i}$, then $alloc_{T_i} = j$. The coalition of all agents involved in the orders T execution is called grand coalition K_T. The utility of a coalition is defined by a characteristic function: $v(K_{T_i}) = Payoff(T_i) - \sum\limits_k \sum\limits_j f_{T_i}(b_j^k) \cdot \varphi(T_i, k, j)$, where φ – a binary variable, defining the fact that an agent participates in the task execution with his capacity b_j^k:

$$\varphi(T_i, k, j) = \begin{cases} -1, & \text{if an agent executes } b_j^k \\ 0, & \text{in the opposite case} \end{cases}$$

The coalitions utility $v(K_{T_i})$ is distributed between the coalition members according to the vector of payoff distribution $u_{T_i} = \left\{ u_{T_i}^1, u_{T_i}^2, \ldots, u_{T_i}^{|K_{T_i}|} \right\}$, where $u_{T_i}^j$ – payoff of agent $j \in Agent$, and $u_{T_i}^{|K_{T_i}|}$ – coalition payoff. If within a coalition K_{T_i}, an agent j executes different competencies then $u_{T_i}^j = \sum\limits_k g_j(b_{T_i}^k)$,

$g_j(b_{T_i}^k) = \frac{b_{T_i}^k}{\sum_{l=1}^{r} b_{T_i}^l} v(K_{T_i}) \cdot \varphi(T_i, k, j)$ is fulfilled.

The grand coalition K_T, joining together all the agents participating in the orders fulfillment corresponds to the configuration of the supply network C_T. Thus to form a coalition means to find the appropriate coalition structure which permits to maximize the payoff for all agents belonging to this structure.

3. Coalition formation in cooperative game theory

Taxonomy of coalition formation algorithms includes both distributed and centralized algorithms. In this paper we restrict this analysis by those represented by the theory of games (Aubin, 1981). Until recently, in the domain of supply chains and networks management, non-cooperative game theory was usually used for modeling of the competing enterprises as a game with zero-sum. In that context, all the players are considered being self-interested trying to optimize their own profits without taking into account their effects on the other players. The main purpose of such a game is to find the optimal strategy for each player and determine if the obtained strategy coordinates the supply chain, i.e. maximizes the global profit. Nevertheless, the cooperative nature of federated enterprises causes necessity of considering OSN within the context of cooperative game theory in order to model and understand the behavior of cooperating network partners. The principal difference between both approaches lies in different assumptions about the nature of the game and of the rational behavior of the players. In other words, cooperative games are considered in those cases when the players can form coalitions.

In the context of supply networks configuring, the theory of cooperative games offers results that show the structure of possible interaction between partners and the conditions required for it. The main questions to be answered are: what coalitions will be formed, how the common wealth will be distributed and if the obtained coalition structure is stable. Cooperative game theory represents a variety of models and the selection of the appropriate approach for OSN configuring is a challenging task. The models of coalition formation are usually classified based upon the type of the environment and the principles of the payoff distribution (Fig. 3.). The environment can de superadditive and subadditive. Usually, coalitions joining together can increase the wealth of their players. If they form a single coalition (grand coalition), the only question is to find acceptable distributions of the payoff of the grand coalition. But in the latter case, at least one coalition does not meet this condition. The payoff distribution should guarantee the stability of the coalition structure when no one player has an intention to leave a coalition because of the expectation to increase its payoff. Moreover, profit distribution can be fuzzy, uncertain, and ambiguous (Mareŝ, 2001). Using the theory of fuzzy cooperative games (FCGs), we can process the uncertainty and pass from the introduction of a fuzzy profit concept through the bargaining process to the conclusion about the corresponding fuzzy distribution of individual payoffs.

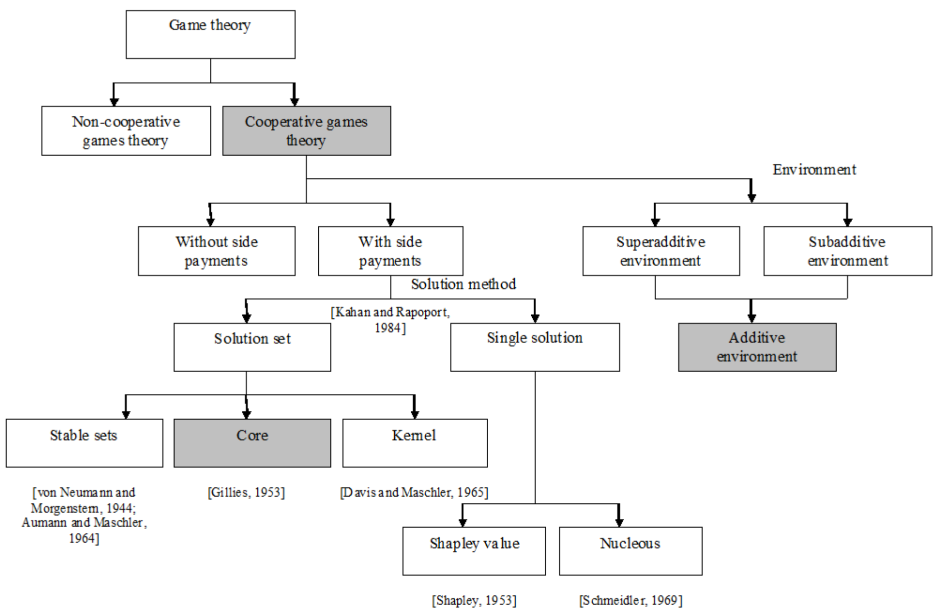

Fig. 2. Cooperative games' taxonomy

Due to the model complexity, most of the models of cooperative games have been developed for superadditive environments and for fuzzy settings allow us to consider only linear membership functions. Nevertheless, for realistic applications additive environments and the absence of the restrictions on the type of membership functions is a time challenge.

According to (Kahan and Rapoport, 1984), cooperative games can be divided into two classes based on the way a solution of the game is obtained: games with a solution set and games with a single solution. To the former class belong the approaches of the stable sets (Von Neumann and Morgenstern, 1944), the core (Gillies, 1953), the kernel (Davis and Maschler, 1965) and bargaining set (Aumann and Maschler, 1964). To the latter – Shapley value (Shapley, 1953),

τ value in the TU-games (Tijs, 1981) and the nucleolus (Schmeidler, 1969). Core and Stable sets are two widely used mechanisms for analyzing the possible set of stable outcomes of cooperative games with transferable utilities. The concept of a core is attractive since it tends to maximize the so called social wealth, i.e. the sum of coalition utilities in the particular coalition structure. Such imputations are called C-stable. The core of a game with respect to a given coalition structure is defined as a set of such imputations that prevent the players from forming small coalitions by paying off all the subsets an amount which is at least as much they would get if they form a coalition. Thus the core of a game is a set of imputations which are stable. The problem of the core is that, on the one hand, the computational complexity of finding the optimal structure is high since for the game with n players at least $2^{|n|-1}$ of the total $|n|^{|n|/2}$ coalition structures should be tested. On the other hand, for particular classes of the game a core can be empty. Because of these problems, using the C-stable coalition structures has been quite unpopular so far (Klusch and Gerber, 2002). In this paper we show that these problems can be solved in a proposed generalized model of a FCG.

4. Generalized model of a fuzzy cooperative game with core

A FCG is defined as a pair $(Agent, w)$, where $Agent$ is nonempty and finite set of players, subsets of $Agent$ joining together to fulfil some task T_i are called coalitions K, and w is called a characteristic function of the game, being $w : 2^n \to \Re^+$ a mapping connecting every coalition $K \subset Agent$ with a fuzzy quantity $w(K) \in \Re^+$, with a membership function $\mu_K : R \to [0,1]$. A modal value of $w(K)$ corresponds to the characteristic function of the crisp game $v(K)$: max $\mu_K(w(K)) = \mu_K(v(K))$. For an empty coalition $w(\emptyset) = 0$. A fuzzy core for the game $(Agent, w)$ with the imputation $X = (x_{ij})_{i\in I, j\in Agent} \in \Re^+$ as a fuzzy subset C_F of $\Re^+$:

$$C_F = \left\{ x_{ij} \in \Re^+ : \nu \succeq (w(Agent), \sum_{\substack{i \in I,\\ j \in Agent}} x_{ij}\varphi_{ij}), \min_{\substack{K_i \in \bar{k}\\ j \in Agent}} (\nu \succeq (\sum_{j \in K_i} x_{ij}\varphi_{ij}, w(K_i))) \right\}, \tag{1}$$

where x_{ij} is the fuzzy payment of an agent j participating in a coalition i, $i = 1, 2, \ldots, I$, $j = 1, 2, \ldots, N$, $\bar{k} = [K_1, K_2, ..., K_l]$ is the ordered structure of effective coalitions; $\succeq$ – is a fuzzy partial order relation with a membership function $\nu \succeq : R \times R \to [0,1]$, and φ_{ij} is a binary variable such that:

$$\varphi_{ij} = \begin{cases} 1, & \text{if an agent } j \text{ participates in a coalition } i; \\ 0, & \text{otherwise.} \end{cases}$$

This variable can be considered as a result of some agents strategy on joining a coalition. A fuzzy partial order relation is defined as follows.

Definition 1. Let a, b be fuzzy numbers with membership functions μ_a and μ_b respectively, then *the possibility of partial order* $a \succeq b$ is defined as $v \succeq (a,b) \in [0,1]$ as follows: $v \succeq (a,b) = \sup_{\substack{x, y \in R\\ x \geq y}} (\min(\mu_a(x), \mu_b(y)))$.

The core C_F is the set of possible distributions of the total payment achievable by the coalitions, and none of coalitions can offer to its members more than they can obtain accepting some imputation from the core. The first argument of the core C_F indicates that the payments for the grand coalition are less than the characteristic function of the game. The second argument reflects the property of group rationality of the players, that there is no other payoff vector, which yields more to each player. The membership function $\mu_{C_F} : R \to [0,1]$, is defined as:

$$\mu_{C_F} = \min \left\{ \nu \succeq (w(Agent), \sum_{\substack{i \in I, \\ j \in Agent}} x_{ij}\varphi_{ij}), \min_{\substack{K_i \in \bar{k} \\ j \in Agent}} (\nu \succeq (\sum_{j \in K_i} x_{ij}\varphi_{ij}, w(K_i))) \right\}, \quad (2)$$

With the possibility that a non-empty core C_F of the game $(Agent, w)$ exists:

$$\gamma_{C_F}(Agent, w) = \sup(\mu_{C_F}(x) : x \in \Re^n) \quad (3)$$

The solution of a cooperative game is a coalition configuration (S, x) which consists of (i) a partition S of *Agent*, the so-called coalition structure, and (ii) an efficient payoff distribution x which assigns each agent in *Agent* its payoff out of the utility of the coalition it is member of in a given coalition structure S. A coalition configuration (S, x) is called stable if no agent has an incentive to leave its coalition in S due to its assigned payoff x_i.

A game $(Agent, w)$ is defined as superadditive, subadditive, or simply additive for any two coalitions $K, L \subset Agent$, $K \cap L = \emptyset$ as follows:

$$\begin{array}{ll} w(K \cup L) \succeq w(K) \oplus w(L) & -superadditive, \\ w^*(K \cup L) \preceq w^*(K) \oplus w^*(L) & -subadditive, \\ w^*(K \cup L) = w^*(K) \oplus w^*(L) & -additive, \end{array} \quad (4)$$

where $\oplus$ – is a sum of fuzzy numbers with a membership function defined as: $\mu_{a \oplus b}(x) = \sup_{x,y \in R}(\min(\mu_a(y), \mu_b(x-y)))$, * defines superoptimal values of the corresponding coalitions (Marês, 2001).

The properties of the game are defined in three lemmas and two theorems (Sheremetov and Romero-Cortes, 2003). One of them proves that the fuzzy set of coalition structures forming the game core represents a subset of the fuzzy set formed by the structure of effective coalitions. In turn, this inference allows us to specify the upper possibility bound for the core, which is a very important condition for the process of solution searching, because in this case, the presence of a solution that meets the efficiency condition may serve as the signal to terminate the search algorithm.

Definition 2. *A coalition K is called effective* if it can't be eliminated from the coalition structure by a subcoalition $L \subset K$. A set of effective coalitions is called a coalition structure. A possibility that a coalition K is effective is defined as follows: $\sup_{x \in R^n}(\min(\mu_k(x), \mu_l^*(x) : L \subset K))$.

Theorem 1. *Let $(Agent, w)$ be a fuzzy coalition game. Then for some structure of effective coalitions $\bar{k}$, its possibility is at least equal to the possibility of forming the core.*

Proof. From formula 2, if all φ_{ij} are equal to 1, then we obtain the structure of coalitions that belong to the core; otherwise, the coalition structure corresponds to the generalized model. In addition, the inequality $\nu \succeq (\sum_{j \in K_i} x_{ij}, \sum_{j \in K_i} x_{ij}\varphi_{ij}, \ i \in I)$ holds with positive possibility and, consequently, the possibility of the structure is higher for the generalized model than for the basic one. □

It should be noted that the above statements take into account only the characteristics of the game $(Agent, w)$; therefore, any real argument can be introduced into the fuzzy core. For example, such restrictions as a number of agents in each coalition and those defining coalitions to be overlapping or not or regulating the tasks order are admissible. This feature is very important for the application of the model for OSN configuration management.

To find the analytical (exact) solution of the FCG, it is necessary to determine the fuzzy super-optimum and the fuzzy relation of domination (Mareŝ, 2001), which is extremely difficult in real applications. Therefore, it is proposed to use a heuristic technique of finding solutions that are close to the optimal one. In the considered case, the techniques of soft computing using genetic algorithms (GA) in the context of fuzzy logic are applied. It is equivalent to binary encoding of the fuzzy core with the fitness function equal to the supremum of all minimums of the membership function. Application of GA allows one to obtain an approximate solution for the games with a large number of players and a membership function of any type. Being an anytime algorithm that steadily improves the solution, the GA can find the best solution under the time constraints.

5. Case study: a cooperative game for 3-echelons automotive OSN

The developed model of a cooperative game was used for configuring of an OSN's production channel for a specific car's model. The demand is represented by a uniform distribution around the linear trend:

$$d_t = a + b \cdot t + \sigma \cdot \mu, \tag{5}$$

where t – time, d – demand (d_t corresponds to time interval $[t-1, t]$), a – basis value, b – trend (equals 0 for a demand without trend), μ – random noise uniformly distributed within [0,1], and σ – distribution amplitude. For the demand forecasting *Simple Moving Average (SMA)* is used:

$$f_{t+2} = f_{t+1} = \frac{\sum_{i=t-n+1}^{t} d_i}{n}, \tag{6}$$

where f – forecast, n – forecast base.

Suppose that the OSN contains several enterprises capable of satisfying the demand both in components' production and vehicle's assembly. The configuring task can be defined as follows: to select an effective configuration of a production channel (both the enterprises and the demand's distribution between them) such that an ordered quantity of vehicles ($a = 100$) can be produced on five consecutive week intervals ($n = 5$) with a low noise ($\sigma = 5$) and without fluctuations associated with storing and delivery of the final and intermediate products. The enterprises pursuit a goal of maximizing their payoffs. The following parameters are considered:

production capacity (units per week), production cost (per unit), stocking costs (per unit per week) and penalties for backorders (per unit per week). Stocks are unlimited. Payoffs for each component are fuzzy variables defined, for simplicity, by a uniform positive ramp membership function. The forecasting model for the demand is the following:

$$100 + 5t + 5\mu, \quad \mathit{for}\ t = 1, \ldots, 5, \tag{7}$$

Component production can be performed by 6 enterprises, each with different competencies (Table 1). For simplicity, the competencies are restricted to the task level. The payoff for the assembled car is *$20000*.

Table 1. Input data for the fuzzy cooperative game for automotive OSN configuring

Enterprise	capacity (units per week)	Competency	Membership function (MF)	Component's price (parameters of the MF)	Production cost	Stocking cost (per unit per week)	Penalties for backorders (per unit per week)	Associated variable
1	100	Body	Positive ramp (+)	*$(6500-7000)*	*$4500*	*$250*	*$400*	x_{11}, φ_{11}
2	100	Motor	Positive ramp (+)	*$(4500-5000)*	*$3500*	*$150*	*$300*	x_{22}, φ_{22}
3	100	Transmission	Positive ramp (+)	*$(3800-4000)*	*$2500*	*$50*	*$250*	x_{33}, φ_{33}
4	300	Body	Positive ramp (+)	*$(6500-7000)*	*$4900*	*$300*	*$400*	x_{14}, φ_{14}
		Motor	Positive ramp (+)	*$(4500-5000)*	*$3800*	*$200*	*$300*	x_{24}, φ_{24}
		Transmission	Positive ramp (+)	*$(3800-4000)*	*$2700*	*$80*	*$250*	x_{34}, φ_{34}
5	100	Motor	Positive ramp (+)	*$(4500-5000)*	*$3600*	*$170*	*$300*	x_{15}, φ_{15}
		Transmission	Positive ramp (+)	*$(3800-4000)*	*$2600*	*$60*	*$250*	x_{25}, φ_{25}
6	200	Body	Positive ramp (+)	*$(6500-7000)*	*$4700*	*$270*	*$400*	x_{16}, φ_{16}
		Motor	Positive ramp (+)	*$(4500-5000)*	*$3600*	*$170*	*$300*	x_{26}, φ_{26}
7	150	Assembly	Positive ramp (+)	*$(2000-4000)*	*$1500*	*$750*	*$1500*	x_{47}, φ_{47}

The order is decomposed into tasks which correspond to each car component's assembly. As a result, an effective structure of three coalitions (according to the number of the components) is to be formed considering capacity constraints. The structure of the core of the cooperative game is shown in Table 2. Additional constraints define the viability of the obtained solution.

The following notation is used: x_{ijt} – the quantity of the i component to be produced by agent j in time t, $w(Agent)$ – fuzzy payoff per unit for car production,

Table 2. The structure of the core of the cooperative game

Core's component	Definition
$C = \{2500x_{11t} + 2100x_{14t} + 2300x_{16t} + 1500x_{22t} + 1200x_{24t} + 1400x_{25t} + 1400x_{26t} + 1500x_{33t} + 2500x_{47t} + 1300x_{34t} + 1400x_{35t} \geq (100 + 5t + 5\mu)\ w\,(Agent)$,	Constraint on the grand coalition
$2500x_{11t} + 2100x_{14t} + 2300x_{16t} \leq (100 + 5t + 5\mu)\ w\,(k_1)$ $1500x_{22t} + 1200x_{24t} + 1400x_{25t} + 1400x_{26t} \leq (100 + 5t + 5\mu)\ w\,(k_2)$ $1500x_{33t} + 1300x_{34t} + 1400x_{35t} \leq (100 + 5t + 5\mu)\ w\,(k_3)$ $2500x_{47t} \leq (100 + 5t + 5\mu)\ w\,(k_4)$	Constraints on the components' coalitions
$x_{11t} + x_{14t} + x_{16t} \leq 100 + 5t + 5\mu$ $x_{22t} + x_{24t} + x_{25t} + x_{26t} \leq 100 + 5t + 5\mu$ $x_{33t} + x_{34t} + x_{35t} \leq 100 + 5t + 5\mu$ $x_{47t} \leq 100 + 5t + 5\mu$	Constraints on the forecasted demand for each component
$x_{11t} \leq 100 \quad x_{14t} \leq 300 \quad x_{16t} \leq 200$ $x_{22t} \leq 100 \quad x_{24t} \leq 300 \quad x_{25t} \leq 100 \quad x_{26t} \leq 200$ $x_{33t} \leq 100 \quad x_{34t} \leq 300 \quad x_{35t} \leq 100$ $x_{47t} \leq 150$	Capacity constraints on the payoffs
$x_{ijt} \in R^+,\ i = 1, ..., 4;\ \ j = 1, ..., 7\ \ t = 1, ..., 5$	

$w(k_1)$ - fuzzy payoff per unit for Body Production, $w(k_2)$ - fuzzy payoff per unit for Motor Production, $w(k_3)$ - fuzzy payoff per unit for Transmission Production, $w(k_4)$ - fuzzy payoff per unit for car assembly, and μ - uniform random variable in [0,1]. The solution of the game obtained using Evolver package and genetic algorithms is shown in Table 3.

Table 3. The coalition structure and the number of produced components for five time intervals

t	x_{11t}	x_{22t}	x_{33t}	x_{14t}	x_{24t}	x_{34t}	x_{25t}	x_{35t}	x_{16t}	x_{26t}	x_{47t}
1	100	100	100	0	0	0	5,299	5,3	5,3	0,001	105,3
2	100	100	100	0	0	0	8,399	11,5	11,5	3,101	111,5
3	100	100	100	0	0	0	10,25	15,2	15,2	4,95	115,2
4	100	100	100	0	0	0	14,50	23,7	23,7	9,20	123,7
5	100	100	100	0	0	0	16,75	28,2	28,2	11,45	128,2

The common network payoffs per car obtained for each time interval are equal to 7,578.46, 7,578.22, 7,578.22, 7,577.84, 7,577.84 respectively. The payoffs (p) of the participating enterprises per car/component are as follows: $p_1 = 2500$; $p_2 = 1500$; $p_3 = 1500$; $p_4 = 0$; $p_5 = 1400$ (motor and transmission); $p_6 = 2300$ (body); $p_6 = 1400$ (motor); $p_7 = 2500$. The same gross payoffs per enterprise were obtained for each time interval for each component: $w(Agent) = 20{,}000$; $w(k_1)$= 7,000; $w(k_2)$=5,000; $w(k_3)$=4,000 $w(k_4)$=4,000. The possibility of the fuzzy game $\gamma_c(Agent, w) = 1.00$ (because of the simplicity of the case study), though the imputation obtained took into account the subjective estimations of the players defined by their fuzzy payments.

The analysis of the obtained solution shows the following. The constraint capacity of the first 3 units though having minimal production costs, does not permit them to satisfy all the demand. That is why, while demand is increasing, other enterprises are involved in the production. In the case of Motor Production (k_2), the incrementing production of this component is assigned to both enterprises 5 and 6 (Table 3). If we compare the parameters of these enterprises (Table 2), it can be seen that they are the same both for the production cost (*$3600* per motor) and for stocking (*$170* per motor /week). That means that the solution strategy looks for a balanced final solution.

In the conducted experiments on model complexity the number of iterations needed to approach the optimal solution served as the investigated variable with the following factors: the number of agents and coalitions, the accuracy, and the order of fuzzy payments. Results show that the number of iterations (computation time) decreases or remains constant when the number of agents increases. In other words, it takes less time to form coalitions. On the other hand, the results demonstrate almost linear relation between the numbers of coalitions and agents. On the whole, the experiments justified that all factors are highly significant; the only surprise was that the order of fuzzy payments substantially influence the number of iterations (the convergence time).

6. Conclusions

In this paper, the approach to OSN configuration based on formation of enterprise coalitions as a result of a fuzzy cooperative game was considered. Uncertainty in realistic cooperation models occurs in two cases: when players participate in several coalitions, and when there exist fuzzy expectations of player and coalition benefits. The presented approach is mostly aimed at the latter case. This uncertainty of the agent payments may be caused by such dynamic events as production failures, changes in confidence estimations and reputations of potential coalition partners, and receiving unclear or even incomplete information and data during the task performance and negotiation.

The proposed model considers the coalitions' efficiency by introducing binary variables φ_{ij} into the fuzzy core. This permits not only to increase individual benefits for players but also the possibility to find an effective and stable agreement. Using the constraints of the application domain the number of viable coalitions can be significantly decreased, thus reducing the algorithmic complexity of the problem. Though in the case study a positive ramp membership function was used (to be able to use also conventional Excel solver), the general solution method (applying genetic algorithms) permits the use of function of any type (linear or nonlinear, universal or not). Obviously, there is no guaranty that the obtained solution corresponds to a global optimum, but for a game with side payments, there is no algorithm to obtain the optimal solution.

The fields of FCGs and dynamic coalition formation are still in their infancy and require further research efforts. For example, the notions of a superadditive FCG and a "stable" distribution of fuzzy payments in the games using fuzzy extension of the core and Shapley values were examined in (Mareŝ, 2001). Some aspects of application of the coalition game models to the development of dynamic coalition formation schemes were considered in (Klusch and Gerber, 2002). Nevertheless, sub-additive fuzzy games and the notions of "uncertain" stability and effective algorithms for FCGs represent the subjects for current research. In the future work, the development of algorithms for dynamic formation of fuzzy coalitions seems to be the promising and challenging problem in the field of self-organizing system research.

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Modeling of Environmental Projects under Condition of a Random Time Horizon

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Abstract. One game-theoretical model of environmental project is considered under condition of a random game duration. The game ends at the random moment in time with Weibull distribution. According to Weibull distribution form parameter, the game can be in one of 3 scenarios such as "an infant", "an adult" and "an aged" scenario. The solutions obtained with a help of Pontryagin maximum principle both for non-cooperative and cooperative forms of the game are analyzed for each stage.
Keywords:: differential games, environment, pollution control, random duration.

Introduction

The game-theoretical model of environmental project with 2 non-identical players had been considered in the paper (Breton, Zaccour, Zahaf, 2005) under condition that the game has a fixed terminal time T. We consider the modification of this model with elements of stochastic framework, in the sense that the terminal time T is a random value (see (Petrosjan, Murzov, 1966; Petrosjan, Shevkoplyas, 2003; Shevkoplyas, 2009)) in order to increase the realness of the modeling.

It turns out that this formulation of the game can be simplified to standard formulation for dynamic programming under some restrictions for instantaneous utility functions of the players.

With the help of Pontryagin maximum principle we find the solutions both for non-cooperative and cooperative forms of the game under condition of Weibull distribution for the random value T. According to Weibull distribution form parameter, the game can be in one of 3 scenarios such as "an infant", "an adult" and "an aged" scenario. The solutions are analyzed for each stage of the game and the interpretation in the context of the environmental economics is given.

The paper is structured as follows. Section 1 contains a general formulation of differential games with random time horizon. Conditions for instanteneous utility functions of the players to simplify this problem are investigated.

Section 2 is devoted to the modified differential game of pollution control.

1. Game Formulation

There are n players which participate in differential game $\Gamma(x_0)$. The game $\Gamma(x_0)$ with dynamics

$$\dot{x} = g(x, u_1, \dots, u_n), \quad x \in R^n,\ u_i \in U \subseteq \operatorname{comp} R^l, \tag{1}$$
$$x(t_0) = x_0.$$

starts from initial state x_0 at the time instant t_0. But here we suppose that the terminal time of the game is the random variable T with known probability distribution function $F(t)$, $t \in [t_0, \infty)$ (Petrosjan, Murzov, 1966; Petrosjan, Shevkoplyas, 2003).

Suppose, that for all feasible controls of players, participating the game, there exists a continuous at least piecewise differentiable and extensible on $[t_0, \infty)$ solution of (1).

Denote the instantaneous payoff of player i at the time τ, $\tau \in [t_0, \infty)$ by $h_i(\tau, x(\tau), u_1, \ldots, u_n)$, or briefly $h_i(\tau)$. Suppose, that for all feasible controls of players which participate the game, the instantaneous payoff function of each player is bounded, piecewise continuous function of time τ (piecewise continuity is treated as following: function $h_i(\tau)$ could have only finitely many point of discontinuity on each interval $[t_0, t]$ and bounded on this interval).

Thereby, the function $h_i(\tau)$ is Riemann integrable on every interval $[t_0, t]$, in other words for every $t \in [t_0, \infty)$ there exists an integral $\int_{t_0}^{t} h_i(\tau) d\tau$.

So, we have that the expected integral payoff of the player i can be represented as the following Lebesgue-Stieltjes integral:

$$K_i(x_0, t_0, u_1, \ldots, u_n) = \int_{t_0}^{\infty} \int_{t_0}^{t} h_i(\tau) d\tau dF(t), \quad i = 1, \ldots, n. \tag{2}$$

1.1. Transformation of integral functional

The transformation of integral functional in the form of double integral (2) to standard for dynamic programming form had been obtained in the paper (Burness, 1976) without details and in the papers (Boukas, Haurie and Michel, 1990, Chang, 2004) with the help of integration by parts. Now we obtain this result by interchanging the variables of integration. Here we point some restrictions for utility function which were not stated before.

Nonnegative instantaneous payoff Suppose, that for any admissible strategies (controls) of each player instantaneous payoff is nonnegative function:

$$h_i(\tau, x(\tau), u_1, \ldots, u_2) \geq 0, \qquad \forall\ \tau \in [t_0, \infty). \tag{3}$$

We denote $A \subset R^2$ as given below:

$$A = \{(t, \tau) | t \in [t_0, \infty], \tau \in [t_0, t]\}.$$

Denote

$$A_t = \{\tau | (t, \tau) \in A\},$$
$$A_\tau = \{t | (t, \tau) \in A\}.$$

Consider Lebesgue-Stieltjes measure μ_F, corresponding to the function F (Gelbaum, 1967, Zorich, 2002), and ordinary Lebesgue measure μ_t on $[t_0, \infty]$. Rewrite (2) in a new form:

$$K_i(t_0, x_0, u_1, \ldots, u_n) = \int_{[t_0, \infty]} \left[\int_{A_t} h_i(\tau) d\mu_\tau \right] d\mu_F, \quad i = 1, \ldots, n. \tag{4}$$

Let us assume that integrals in the right side of (4) exist and condition (3) is satisfied, then Fubini's theorem holds (Gelbaum, 1967, Zorich, 2002):

$$\int\limits_{[t_0,\infty]} \left[\int\limits_{A_t} h_i(\tau) d\mu_\tau \right] d\mu_F = \int\limits_{[t_0,\infty]} \left[\int\limits_{A_\tau} h_i(\tau) d\mu_F \right] d\mu_\tau, \quad i = 1, \ldots, n. \tag{5}$$

Let us take the interior integral:

$$\int\limits_{[t_0,\infty]} \left[\int\limits_{A_\tau} h_i(\tau) d\mu_F \right] d\mu_\tau = \int\limits_{[t_0,\infty]} [\, h_i(\tau)\mu_F(A_\tau)]\, d\mu_\tau, \quad i = 1, \ldots, n.$$

Finally, we obtain

$$\int\limits_{[t_0,\infty]} [\, h_i(\tau)\mu_F(A_\tau)]\, d\mu_\tau = \int\limits_{t_0}^{\infty} h_i(\tau)(1 - F(\tau)) d\tau, \quad i = 1, \ldots, n.$$

As it stated in (Gelbaum, 1967, Zorich, 2002), the existence of the integral above implies the existence of the integral in the right side of (2).

Thus, the following proposition is proved.

Proposition 1. Let the instantaneous payoff $h_i(t)$ be a bounded, piecewise continuous function of time and satisfy the condition of nonnegativity (3). Then the expectation of integral payoff of player i (2) can be expressed in a simple form:

$$K_i(t_0, x_0, u_1, \ldots, u_n) = \int\limits_{t_0}^{\infty} h_i(\tau)(1 - F(\tau)) d\tau, \quad i = 1, \ldots, n. \tag{6}$$

Moreover, integrals in (2) and (6) exist or do not exist simultaneously.

Then we have the following formula for total expected payoff

General case Now we remove the condition (3) of nonnegativity of the function of instantaneous payoff. In this section we treat the integrals in the right side of (2) as Riemann integrals (or as nonintrinsic Riemann integrals). In this treatment the right side (2) represents the expected payoff of player i in case of absolute convergence of the outer integral. In other words, for the existence of the expectations in (2), it is necessary and sufficient that the following integrals exist in the sense of Riemann nonintrinsic integrals:

$$\int\limits_{t_0}^{\infty} \left| \int\limits_{t_0}^{t} h_i(\tau) d\tau \right| dF(t) < +\infty, \quad i = 1, \ldots, n. \tag{7}$$

Let us define $H_i(t) = \int_0^t h_i(\tau) d\tau$. Here $H_i(t)$, $i = 1, \ldots, n$ are piecewise differentiable functions, because the functions $h_i(t)$ are assumed to be piecewise continuous function. Assume the existence of a continuous density of distribution of the terminal time of the game $f(t) = F'(t)$ on the interval $[t_0, \infty)$. If (7) holds, then the expected payoffs is given by:

$$K_i(t_0, x_0, u_1, \ldots, u_n) = \lim_{T \to \infty} \int\limits_{t_0}^{T} H_i(t) dF(t), \quad i = 1, \ldots, n. \tag{8}$$

Separately consider the integral in the (8), divided into a sum of integrals on the intervals of continuity of a function $h_i(t)$:

$$\int_{t_0}^{T} H_i(t)dF(t) = \sum_{k=0}^{N_T} \int_{\theta_k}^{\theta_{k+1}} H_i(t)dF(t), \tag{9}$$

there $\theta_0 = t_0, \theta_{N_T+1} = T,\ \ \theta_j, j = 1, \ldots, N_T$ — point of discontinuity of $h_i(t)$ on the interval (t_0, T). On each interval, the integration by parts is used:

$$\sum_{k=0}^{N_T} \int_{\theta_k}^{\theta_{k+1}} H_i(t)dF(t) = \sum_{k=0}^{N_T} \left[H_i(\theta_{k+1})F(\theta_{k+1}) - H_i(\theta_k)F(\theta_k) - \int_{\theta_k}^{\theta_{k+1}} h_i(t)F(t)dt \right].$$

Substitute this representation into equation (9) and after transformations we obtain:

$$\int_{t_0}^{T} H_i(t)dF(t) = H_i(T)F(T) - \int_{t_0}^{T} h_i(t)F(t)dt = \int_{t_0}^{T} h_i(t)\left[F(T) - F(t)\right]dt,$$

and further

$$\int_{t_0}^{T} h_i(t)\left[F(T) - F(t)\right]dt = \int_{t_0}^{T} h_i(t)\left[F(T) - 1\right]dt + \int_{t_0}^{T} h_i(t)\left[1 - F(t)\right]dt.$$

Thus, we have:

$$\lim_{T\to\infty} \int_{t_0}^{T} H_i(t)dF(t) = \lim_{T\to\infty} \left(\int_{t_0}^{T} h_i(t)\left[F(T) - 1\right]dt + \int_{t_0}^{T} h_i(t)\left[1 - F(t)\right]dt \right).$$

So, we can formulate the following proposition.

Proposition 2. Under conditions (7) we can represent the expected payoff in form (6), if the following condition holds:

$$\lim_{T\to\infty} (F(T) - 1) \int_{t_0}^{T} h_i(t)dt = 0. \tag{10}$$

Proof. Indeed, if the conditions (7) hold, then the limit in (8) exists. In this case, the fullfilment of conditions (10) implies the existence of

$$\lim_{T\to\infty} \int_{t_0}^{T} h_i(t)\left[1 - F(t)\right]dt.$$

And this limit is equal to the limit in (8).

This means that the expected payoff could be obtained by formula (6).

Thus, if calculation of the expected payoff by formula (2) causes some difficulties, but one can guarantee fulfillment of conditions (7) and (10), then the expectation of payoff could be find by more simply formula (6).

1.2. Subgame

Let the game $\Gamma(x_0)$ develops along the trajectory $x(t)$. Then at the each time instant ϑ, $\vartheta \in (t_0;\infty)$ players enter new game (subgame) $\Gamma(x(\vartheta))$ with initial state $x(\vartheta) = x$. Clearly, there is a probability $F(\vartheta)$ that the game $\Gamma(x_0)$ will be finished before ϑ. Then the probability to start the subgame $\Gamma(x(\vartheta))$ equals to $(1 - F(\vartheta))$.

Then the expected total payoff of the player i is calculated by following formula:

$$K_i(x, \vartheta, u_1, \ldots, u_n) = \int_{\vartheta}^{\infty} \left[\int_{\vartheta}^{t} h_i(\tau, x(\tau), u_1, \ldots, u_n) d\tau \right] dF_{\vartheta}(t), \qquad (11)$$

here $F_{\vartheta}(t)$, $t \geq \vartheta$ — conditionally probability distribution function of the random terminal time in the game $\Gamma(x(\vartheta))$. In this paper we cosider only stationary processes, so we have the following expression for $F_{\vartheta}(t)$:

$$F_{\vartheta}(t) = \frac{F(t) - F(\vartheta)}{1 - F(\vartheta)}, \quad t \in [\vartheta, \infty). \qquad (12)$$

Further we assume an existence of a density function $f(t) = F'(t)$. As above we get the formula for conditional density function:

$$f_{\vartheta}(t) = \frac{f(t)}{1 - F(\vartheta)}. \qquad (13)$$

Using (13), we get the total payoff for player i in the subgame $\Gamma(x(\vartheta))$:

$$K_i(x, \vartheta, u_1, \ldots, u_n) = \frac{1}{1 - F(\vartheta)} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} h_i(\tau, x(\tau), u_1, \ldots, u_n) d\tau\, f(t) dt. \qquad (14)$$

Using the transformation of the integral payoff as above, within the the framework of restrictions on h_i from Proposition 2, under condition of existence of the density function, we can rewrite (14) in the following form

$$K_i(x, \vartheta, u_1, \ldots, u_n) = \frac{1}{1 - F(\vartheta)} \int_{\vartheta}^{\infty} (1 - F(\tau)) h_i(\tau, x(\tau), u_1, \ldots, u_n) d\tau. \qquad (15)$$

Let us remark, that using the Hazard function $\lambda(\vartheta)$ which is given by the following definition:

$$\lambda(t) = \frac{f(t)}{1 - F(t)}, \qquad (16)$$

we can also rewrite the term $(1 - F(\tau))$ from subintegral function of (15) as

$$1 - F(\tau) = e^{-\int_{t_0}^{\tau} \lambda(t) dt}.$$

Then we can use another form of the expected payoff (15) of the player i:

$$K_i(x, \vartheta, u_1, \ldots, u_n) = e^{\int_{t_0}^{\vartheta} \lambda(t) dt} \int_{\vartheta}^{\infty} e^{-\int_{t_0}^{\tau} \lambda(t) dt} h_i(\tau, x(\tau), u_1, \ldots, u_n) d\tau. \qquad (17)$$

It is obviously, that for exponential distribution of the random value T we have $\lambda(t) = \lambda$. That is why the problem with random duration under exponential distribution of T equivalents to well-known deterministic problem with constant discounting of payoffs. This fact was marked in the paper (Haurie, 2005) for another concepts of random game duration (a multigenerational game model).

1.3. Weibull distribution case

In mathematical reliability theory the Hazard function $\lambda(t)$ (16) describing life circle of the system usually has the following characteristics: it is decreasing function for "burn-in" period, it is near constant for "adult" period (or regime of normal exploitation) and it is increasing function for "wear-out" period. One of the most important probability distribution describing three periods of life circles is Weibull Law. Using Weibull distribution allows to consider three "scenarios" of the game in the sense of behaviour of the random variable T.

For Weibull distribution we have the following characteristics:

$$
\begin{aligned}
f(t) &= \lambda\delta(t-t_0)^{\delta-1}e^{-\lambda(t-t_0)^\delta}; \\
\lambda(t) &= \lambda\delta(t-t_0)^{\delta-1}; \\
t &\geq t_0;\ \lambda > 0;\ \delta > 0.
\end{aligned} \tag{18}
$$

Here λ and δ are two parametres. $\delta < 1$ corresponds to "burn-in" period, $\delta = 1$ corresponds to "adult" period and $\delta > 1$ corresponds to "wear-out" period. It is a well–known fact that the Weibull distribution for adult stage ($\delta = 1$, $\lambda(t) = \lambda = const$) is equivalent to exponential distribution. Thus if we use exponential distribution for random final time instant T then we indeed consider the game in "adult" scenario.

2. Differential game of pollution control

2.1. Model

As an example we consider differential game with environmental context based on the model (Breton, Zaccour, Zahaf, 2005).

There are 2 players (firms, countries) involved into the game of pollution control. Each player manage his emissions $e_i \in [0; b_i]$, $i = 1, 2$.

Tne net revenue of player i at time instant t is given by quadratic functional form:

$$R(e_i(t)) = e_i(t)(b_i - 1/2e_i(t)).$$

Denote the stock of accumulated net emissions by $P(t)$. The dynamics of the stock is given by the following equation with initial condition:

$$\dot{P} = \sum_{i=1}^{n} e_i(t), \quad P(t_0) = P_0.$$

Costs of the player depend on the stock of pollution. Then, the instanteneous utility function of the player i is equal to $R(e_i(t)) - d_iP(t), d_i > 0$.

Further we assume that the game of pollution control starts at the time instant $t_0 = 0$ and then the game ends at the random moment in time T which is decsribed by Weibull distribution (18).

Then we have the following integral payoff for player i:

$$K_i(P_0, 0, e_1, \ldots, e_n) = \int_0^\infty \int_0^t \left(R_i(e_i(\tau)) - d_iP(\tau)\right) \lambda\delta t^{\delta-1}e^{-\lambda t^\delta} d\tau dt. \tag{19}$$

2.2. Simplification of the problem

Let us prove that the integral payoff (19) can be rewriten in the simple form accordingly to (15). Without loss of generality we assume $\lambda = 1$, because λ is a parameter of scale. Moreover, we consider case of $\delta = 2$ (Ralegh distribution, wear-out scenario of the game).

Then from (19) we obtain the following form of payoff for player i:

$$K_i(0, P_0, e_1, \dots, e_n) = \int_0^\infty \int_0^t (R_i(e_i(\tau)) - d_i P(\tau)) d\tau \ 2te^{-t^2} dt, \tag{20}$$

under condition of integral convergence

$$\int_0^\infty \left| \int_0^t (R_i(e_i(\tau)) - d_i P(\tau)) d\tau \right| \ 2te^{-t^2} dt. \tag{21}$$

To prove the existence (21) we make the following estimation:

$$P(\tau) \le P_0 + \sum_{i=1}^n b_i \tau = P_0 + B\tau,$$
$$R_i(e_i(\tau)) \le \frac{b_i^2}{2},$$
$$\text{where} B = \sum_{i=1}^n b_i.$$

We get the following estimation for (21):

$$\int_0^\infty \left| \int_0^t (R_i(e_i(\tau)) - d_i P(\tau)) d\tau \right| \ 2te^{-t^2} dt \le \int_0^\infty \int_0^t |(R_i(e_i(\tau)) - d_i P(\tau))| \, d\tau \, 2te^{-t^2} dt \le$$
$$\le \int_0^\infty \int_0^t (|(R_i(e_i(\tau))| + |d_i P(\tau))|) d\tau \ 2te^{-t^2} dt \le \int_0^\infty \left(\int_0^t R_i(e_i(\tau)) d\tau + \int_0^t d_i P(\tau) d\tau \right) \ 2te^{-t^2} dt.$$

Finally, we have the estimation for (21):

$$\int_0^\infty \left| \int_0^t (R_i(e_i(\tau)) - d_i P(\tau)) d\tau \right| \ 2te^{-t^2} dt \le \int_0^\infty \left(\frac{b_i^2}{2} t + d_i \left(P_0 t + \frac{Bt^2}{2} \right) \right) \ 2te^{-t^2} dt. \tag{22}$$

The integral in the right-hand side of the inequality (22) is absolutely convergent integral. Then integral (21) is convergent integral (see (Zorich, 2002)).

So, we proved that for all controls the expression (20) is the mathematical expectation of the integral payoff of the player i.

To prove the satisfaction of condition (10) let us rewrite the left-hand side for Weibull distribution with $\lambda = 1$, $\delta = 2$:

$$\lim_{T \to \infty} (F(T) - 1) \int_0^T h_i(t) dt = \lim_{T \to \infty} e^{-T^2} \int_0^T (R_i(e_i(\tau)) - d_i P(\tau)) d\tau.$$

Using above obtained estimations, we get:

$$\left| e^{-T^2} \int_0^T (R_i(e_i(\tau)) - d_i P(\tau)) d\tau \right| \leq e^{-T^2} \left(\int_0^T \frac{b_i^2}{2} d\tau + \int_0^T d_i (P_0 + B\tau) d\tau \right) =$$
$$= e^{-T^2} \left(\frac{b_i^2}{2} T + d_i \left(P_0 T + \frac{BT^2}{2} \right) \right).$$

It is clear that

$$\lim_{T \to \infty} e^{-T^2} \left(\frac{b_i^2}{2} T + d_i \left(P_0 T + \frac{BT^2}{2} \right) \right) = 0.$$

Then we get

$$\lim_{T \to \infty} e^{-T^2} \int_0^T (R_i(e_i(\tau)) - d_i P(\tau)) d\tau = 0.$$

That is why the condition (10) holds and the payoff (20) can be rewritten in the following simple form:

$$K_i(0, P_0, e_1, \ldots, e_n) = \int_0^\infty (R_i(e_i(t)) - d_i P(t))\ e^{-t^2} dt. \tag{23}$$

Similar, we can proove that reordering of the integrals can be also used for the problem with $\delta = 1$, $\delta = 1/2$.

2.3. Open–loop Nash equilibrium

To find the open–loop Nash equilibrium we use Pontryagin maximum principle.

Each player $i = 1, 2$ solves the maximization problem:

$$\max_{e_i \in [0; b_i]} K_i(P_0, 0, e_1, e_2) = \int_0^\infty \left(R_i(e_i^N(t)) - d_i P(t) \right) e^{-\lambda s^\delta} ds.$$

The Hamiltonian for this problem is as follows:

$$H_i(\Lambda(t), P(t), e_i(t)) = \left(e_i(t) \left(b_i - \frac{1}{2} e_i(t) \right) - d_i P(t) \right) e^{-\lambda t^\delta} + \Lambda \sum_{i=1}^2 e_i(t).$$

We have to find the maximum of Hamiltonian: $\max\limits_{e_i \in [0; b_i]} H_i(\Lambda(t), P(t), e_i)$.

Further we use short notations e_i, λ_i, Λ for $e_i(t), \lambda_i(t), \Lambda(t)$.

We consider functional:

$$L(e_i) = -H_i + \lambda_1(-e_i) + \lambda_2(e_i - b_i).$$

For optimality of the control e_i^N it is necessary the fulfillment of the Karush – Kuhn – Tucker conditions:

1. Stationarity: $\min\limits_{e_i} L_i(e_i) = L_i(e_i^N)$.
2. Complementary slackness: $\lambda_1(-e_i^N) = 0$; $\lambda_2(e_i^N - b_i) = 0$.
3. Dual feasibility: $\lambda_j \geq 0,\ j = 1, 2$.

We have $\frac{dL}{de_i} = -\Big((b_i - e_i)e^{-\lambda t^\delta} + \Lambda\Big) - \lambda_1 + \lambda_2$, then from stationarity condition we get:

$$-\Big((b_i - e_i^N)e^{-\lambda t^\delta} + \Lambda\Big) - \lambda_1 + \lambda_2 = 0. \tag{24}$$

Let us consider the following cases:

1. $\lambda_2(t) \neq 0$ (from nonnegativity we can rewrite $\lambda_2(t) > 0$.) Then from complementary slackness condition $(e_i^N(t) - b_i) = 0$ we get $e_i^N(t) = b_i$ and $\lambda_1 = 0$. Then $\lambda_2(t) = \Lambda(t)$. As we will see below, $\Lambda(t) \le 0$, which is in contrast to $\lambda_2(t)$. That is why this assumption is wrong.
2. $\lambda_2(t) = 0$. Then we get

$$-\Big((b_i - e_i^N(t))e^{-\lambda t^\delta} + \Lambda(t)\Big) - \lambda_1(t) = 0.$$

Let us assume $\lambda_1(t) \neq 0$, then from complementary slackness condition we get $e_i^N(t) = 0$. Then we have $\lambda_1(t) = -\Big(b_i e^{-\lambda t^\delta} + \Lambda(t)\Big)$. If $\lambda_1(t) = 0$ then $e_i^N(t) = \Big(b_i e^{-\lambda t^\delta} + \Lambda(t)\Big)e^{t^2}$. That is why following formula is true:

$$e_i^N(t) = \begin{cases} b_i + \Lambda(t)e^{\lambda t^\delta}, & \text{if } \Big(b_i e^{-\lambda t^\delta} + \Lambda(t)\Big) \ge 0, \\ 0, & \text{if } \Big(b_i e^{-\lambda t^\delta} + \Lambda(t)\Big) < 0. \end{cases}$$

Adjoint function $\Lambda(t)$ we get from adjoint equation

$$\dot{\Lambda} = -\frac{\partial H_i}{\partial P}.$$

Then we get differential equation $\dot{\Lambda}(t) = d_i e^{-\lambda t^\delta}$, and the solution for adjoint function:

$$\Lambda(t) = d_i \int_0^t e^{-\lambda s^\delta} ds + c.$$

We consider problem with time $t \in [0, \infty)$ and the condition for $\Lambda(t)$ has a form:

$$\lim_{t\to\infty} \Lambda(t) = 0.$$

Let us take the Weibull distribution for the Hazard function $\lambda(t)$ (18). Then we obtain equillibrium emissions for three scenarios of the game ($\delta = 1$, $\delta = 2$, $\delta = 1/2$).

For regim of normal exploitation of the environmental equipments ($\delta = 1$) we obtain $\Lambda(t) = d_i \int_0^t e^{-\lambda s^\delta} ds + c = -\frac{d_i}{\lambda}e^{-\lambda t} + \frac{d_i}{\lambda} + c$. From condition $\lim_{t\to\infty} \Lambda(t) = \frac{d_i}{\lambda} + c = 0$ we get $c = -\frac{d_i}{\lambda}$. Then we have

$$\Lambda(t) = -\frac{d_i}{\lambda}e^{-\lambda t}.$$

Then the open–loop Nash emissions for the problem have form:

$$e_i^N(t) = b_i + \Lambda(t)e^{\lambda t} = b_i - \frac{d_i}{\lambda}, \quad i = 1, 2,$$

and finally we obtain

$$e_i^N(t) = b_i - \frac{d_i}{\lambda}, \ \text{if} \ \ b_i - \frac{d_i}{\lambda} > 0, \tag{25}$$
$$e_i^N(t) = 0, \ \ \text{if} \ \ b_i - \frac{d_i}{\lambda} \leq 0.$$

For wear-out scenario ($\delta = 2$) we obtain

$$e_i^N(t) = \frac{d_i}{\sqrt{\lambda}} \frac{\sqrt{\pi}}{2} (\mathrm{erf}(\sqrt{\lambda}\, t) - 1) e^{\lambda t^2} + b_i, \tag{26}$$

if the right side of this equation is positive and $e_i^N(t) = 0$ otherwise.

For initial scenario of management ($\delta = 1/2$) we get

$$e_i^N(t) = \frac{d_i}{\lambda^2} \left(-2 - 2\lambda\sqrt{t} \right) + b_i, \tag{27}$$

if the right side of this equation is positive and $e_i^N(t) = 0$ otherwise.

Then the grafic representation of the equillibrium emissions are as at the Fig. 1.

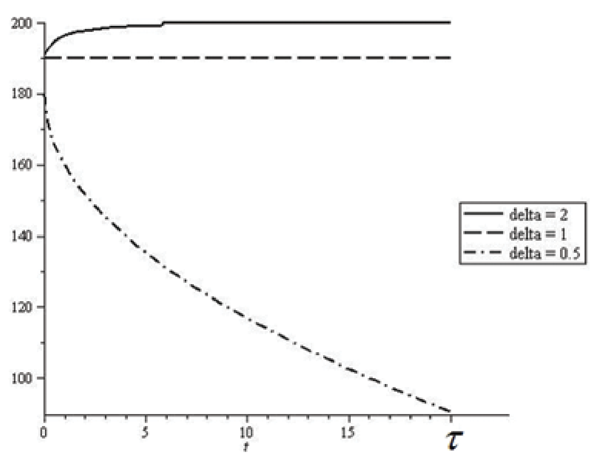

Fig. 1. Equillibrium emissions for three scenarios of the game

The time instant when the emissions of the player for the initial scenario become zero is equal to

$$\tau = \left(b_i \frac{\lambda}{2d} - \frac{1}{\lambda} \right)^2. \tag{28}$$

2.4. Cooperative solution

Suppose that players are agree to cooperate and maximize the total payoff:

$$\max_{(e_1;e_2)\in[0;b_1]\times[0;b_2]}(K_1(P_0,0,e_1,e_2)+K_2(P_0,0,e_1,e_2)) = \int_0^\infty \left(R_1(e_1^C)+R_2(e_2^C)-(d_1+d_2)P\right)e^{-\lambda s^\delta}ds.$$

Then by the similar way as for non–cooperative case we obtain optimal controls for players $i = 1,2$.

Optimal emissions for cooperative version of "adult" game:

$$e_i^C(t) = b_i - \frac{d}{\lambda},\ \text{if}\, b_i - \frac{d}{\lambda} > 0,$$
$$e_i^C(t) = 0,\ \text{if}\ \ b_i - \frac{d}{\lambda} \leq 0,$$

where $d = d_1 + d_2$.

Optimal emissions for wear-out scenario of the game (Raleigh distribution, $\delta = 2$) are given by:

$$e_i^C(t) = \frac{d}{\sqrt{\lambda}}\frac{\sqrt{\pi}}{2}(\operatorname{erf}(\sqrt{\lambda}\,t)-1)e^{\lambda t^2}+b_i,$$

if the right side of this equation is positive and $e_i^C(t) = 0$ otherwise.

We have the following form for optimal emissions for burn-in scenario ($\delta = \frac{1}{2}$):

$$e_i^C(t) = \frac{d}{\lambda^2}\left(-2-2\lambda\sqrt{t}\right)+b_i,$$

if the right side of this equation is positive and $e_i^C(t) = 0$ otherwise.

One could compare the optimal emissions for cooperative and non-cooperative form of the game. It can be easily proved that for each of considered shape parameters $\delta = 1/2; 1; 2$ the optimal emissions for cooperation are less then for non-cooperative version of the game:

$$e_i^C(t) < e_i^N(t).$$

The graphic representation of this fact is given for $\delta = 2$ (see Fig. 2).

Consider the example of "adult" game with two asymetric agents. Let use take the parameteres $d_1 = 2$, $b_1 = 50$, $d_2 = 1$, $b_2 = 20$. Let $\delta = 1$ (exponential distribution case, regime of normal work). Then we obtain the following numerical results.

i	d_i	b_i	e_i^N	e_i^C	K_i^N	K_i^C
1	2	50	48	47	1114	1117.5
2	1	20	19	17	132.5	131.5
$\sum$					1246.5	1249

This example demonstrates that the joint payoff for cooperative version is greater than for non-cooperative, but the optimal emissions are less.

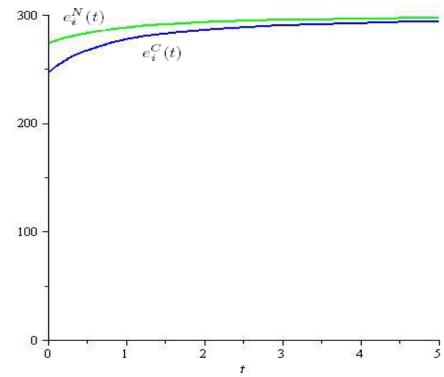

Fig. 2. Comparison of cooperative and non–cooperative emissions for $\delta = 2$

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A Data Transmission Game in OFDM Wireless Networks Taking into Account Power Unit Cost

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Abstract. The goal of this work is to extend the model from Altman E., Avrachenkov K., Garnaev A. "Closed form solutions for water-filling problems in optimization and game frameworks" for the case with different fading channel gains and also taking into account cost of power unit which can produce essential impact on behaviour of users. The equilibrium strategies for the extended model are found in closed form.
Keywords: equilibrium strategies, resource allocation, OFDM.

1. Introduction

In wireless networks and DSL access networks the total available power for signal transmission has to be distributed among several resources. In the context of wireless networks, the resources may correspond to frequency bands (e.g. as in OFDM). This spectrum of problems can be considered in game-theoretical multiusers scenario which leads to "Water Filling Game" or "Gaussian Interference Game"(Lai and Gamal, 2005; Popescu and Rose, 2003, 2004; Yu, 2002; Altman et al., 2007b, 2010), where each user perceives the signals of the other users as interference and maximizes a concave function of the noise to interference ratio. A natural approach in the non-cooperative setting is the application of the Iterative Water Filling Algorithm (IWFA) (Yu et al., 2002). Recently, Luo and Pang (2006) proved the convergence of IWFA under fairly general conditions. An interested reader can find more references on non-cooperative power control in (Lai and Gamal, 2005). We would like to mention that the water filling problem and jamming games with transmission costs have been analyzed in (Altman et al., 2007a). In (Altman et al., 2007b, 2010) a closed form approach to Nash equilibrium was developed for symmetric water filling game. The goal of this work is to extend this approach for scenario with different fading channel gains and also taking into account cost of power unit which can produce essential impact on behaviour of users.

2. Two Players Game

In this section we consider game-theoretical formulation of the situation where two users (transmitters) transmit signals through n sub-carriers taking into account tariff assigned by provider. It is worth to note that this tariff can impact on the user's behavior essentially, namely, for big tariff users can reject from using provider at all, and of course for small tariff users will employ the network facilities in the full range. A strategy of user j $(j = 1, 2)$ is vector $T^j = (T^j_1, \ldots, T^j_n)$, where $T^j_i \geqslant 0$ and

$$\sum_{i=1}^{n} T^j_i \leqslant \bar{T}^j,$$

where $\bar{T}^j > 0$ is the total signal which user j has to transmit. The payoff to users are quality of service (in our case it is Shannon capacities) minus transmission expenses. Thus, the payoffs are given as follows:

$$v^1(T^1, T^2) = \sum_{i=1}^{n} \ln\left(1 + \frac{T_i^1}{g_i T_i^2 + N_i^0}\right) - C\sum_{i=1}^{n} T_i^1,$$

$$v^2(T^1, T^2) = \sum_{i=1}^{n} \ln\left(1 + \frac{T_i^2}{g_i T_i^1 + N_i^0}\right) - C\sum_{i=1}^{n} T_i^2,$$

where $N_i^0 > 0$ is the uncontrolled noise, g_i is fading sub-carrier gains for sub-carrier i. C is a cost per transmitted power unit.

Our goal is to find Nash equilibrium for this game. The strategies T^{1*}, T^{2*} are the Nash equilibrium if for any strategies T^1, T^2 the following inequalities hold:

$$v^1(T^1, T^{2*}) \leqslant v^1(T^{1*}, T^{2*}),$$

$$v^2(T^{1*}, T^2) \leqslant v^2(T^{1*}, T^{2*}),$$

Note that the payoffs v^1 and v^2 are concave by T^1 and T^2 respectively since

$$\frac{\partial^2 v^1(T^1, T^2)}{\partial (T_i^1)^2} = -\frac{1}{(N_i^0 + g_i T_i^2 + T_i^1)^2} < 0,$$

$$\frac{\partial^2 v^2(T^1, T^2)}{\partial (T_i^2)^2} = -\frac{1}{(N_i^0 + g_i T_i^1 + T_i^2)^2} < 0.$$

Thus, Kuhn–Tucker Theorem from non-linear programming allows to show how equilibrium strategies depend on Lagrange multipliers.

Theorem 1. (T^1, T^2) *is a Nash equilibrium if and only if there are non-negative* ω^1 *and* ω^2 *(Lagrange multipliers) such that*

$$\begin{aligned} &\frac{1}{g_i T_i^2 + T_i^1 + N_i^0} - C \begin{cases} = \omega^1 & \text{for } T_i^1 > 0, \\ \leqslant \omega^1 & \text{for } T_i^1 = 0, \end{cases} \\ &\frac{1}{g_i T_i^1 + T_i^2 + N_i^0} - C \begin{cases} = \omega^2 & \text{for } T_i^2 > 0, \\ \leqslant \omega^2 & \text{for } T_i^2 = 0, \end{cases} \end{aligned} \tag{1}$$

where

$$\omega^1 \begin{cases} \geqslant 0 & \text{for } \sum_{i=1}^n T_i^1 = \bar{T}^1, \\ = 0 & \text{otherwise}, \end{cases}$$

$$\omega^2 \begin{cases} \geqslant 0 & \text{for } \sum_{i=1}^n T_i^2 = \bar{T}^2, \\ = 0 & \text{otherwise}. \end{cases}$$

In the next theorem we specify the structure of equilibrium strategies in details assuming that $g_i < 1$.

Theorem 2. *Let* $g_i < 1$ *and* (T^1, T^2) *be a Nash equilibrium, then*

(a) *if* $T_i^1 = T_i^2 = 0$ *then* $i \in I_{00}(\omega^1, \omega^2)$, *where*

$$I_{00}(\omega^1, \omega^2) = \left\{ i \in [1, n] : \frac{1}{\omega^1 + C} \leqslant N_i^0, \frac{1}{\omega^2 + C} \leqslant N_i^0 \right\},$$

(b) *if* $T_i^{1*} > 0$ *and* $T_i^{2*} = 0$ *then*

$$T_i^1 = \frac{1}{\omega^1 + C} - N_i^0$$

and $i \in I_{10}(\omega^1, \omega^2)$, *where*

$$I_{10}(\omega^1, \omega^2) = \left\{ i \in [1, n] : \frac{1}{\omega^1 + C} > N_i^0, \quad g_i \left(\frac{1}{\omega^1 + C} - N_i^0 \right) \geqslant \frac{1}{\omega^2 + C} - N_i^0 \right\},$$

(c) *if* $T_i^2 > 0$ *and* $T_i^1 = 0$ *then*

$$T_i^2 = \frac{1}{\omega^2 + C} - N_i^0$$

and $i \in I_{01}(\omega^1, \omega^2)$, *where*

$$I_{01}(\omega^1, \omega^2) = \left\{ i \in [1, n] : g_i \left(\frac{1}{\omega^2 + C} - N_i^0 \right) \geqslant \frac{1}{\omega^1 + C} - N_i^0, \quad \frac{1}{\omega^2 + C} > N_i^0 \right\},$$

(d) *if* $T_i^1 > 0$ *and* $T_i^2 > 0$ *then*

$$\begin{aligned} T_i^1 &= \frac{\left(\frac{1}{\omega^1 + C} - N_i^0 \right) - g_i \left(\frac{1}{\omega^2 + C} - N_i^0 \right)}{1 - g_i^2}, \\ T_i^2 &= \frac{\left(\frac{1}{\omega^2 + C} - N_i^0 \right) - g_i \left(\frac{1}{\omega^1 + C} - N_i^0 \right)}{1 - g_i^2}, \end{aligned} \tag{2}$$

and $i \in I_{11}(\omega^1, \omega^2)$, *where*

$$I_{11}(\omega^1, \omega^2) = \left\{ i \in [1, n] : 0 < g_i \left(\frac{1}{\omega^2 + C} - N_i^0 \right) < \frac{1}{\omega^1 + C} - N_i^0 \right. \\ \left. \frac{1}{\omega^1 + C} - N_i^0 < \frac{1}{g_i} \left(\frac{1}{\omega^2 + C} - N_i^0 \right) \right\}$$

Proof. (a) Let $T_i^1 = T_i^2 = 0$. Then, by (1),

$$\frac{1}{g_i T_i^2 + T_i^1 + N_i^0} = \frac{1}{N_i^0} \leqslant \omega^1 + C,$$

$$\frac{1}{g_i T_i^1 + T_i^2 + N_i^0} = \frac{1}{N_i^0} \leqslant \omega^2 + C.$$

Thus, $i \in I_{00}$ and (a) follows.

(b) Let $T_i^1 > 0, T_i^2 = 0$. Then, by (1),

$$\frac{1}{g_i T_i^2 + T_i^1 + N_i^0} = \frac{1}{T_i^1 + N_i^0} = \omega^1 + C,$$

$$\frac{1}{g_i T_i^1 + T_i^2 + N_i^0} = \frac{1}{g_i T_i^1 + N_i^0} \leqslant \omega^2 + C.$$

The first relation implies

$$T_i^1 = \frac{1}{\omega^1 + C} - N_i^0 \tag{3}$$

and the second relation yields:

$$g_i T_i^1 \geqslant \frac{1}{\omega^2 + C} - N_i^0,$$

which jointly with (3) implies:

$$g_i(\frac{1}{\omega^1 + C} - N_i^0) \geqslant \frac{1}{\omega^2 + C} - N_i^0. \tag{4}$$

Besides, since $T_i^1 > 0$, (3) yields

$$\frac{1}{\omega^1 + C} > N_i^0$$

which jointly with (4) implies that $i \in I_{10}$ and (b) follows.

(c) follows from (b) by symmetry.

(d) Let $T_i^1 > 0, T_i^2 > 0$. Then

$$\begin{cases} g_i T_i^2 + T_i^1 + N_i^0 = \dfrac{1}{\omega^1 + C}, \\ g_i T_i^1 + T_i^2 + N_i^0 = \dfrac{1}{\omega^2 + C}. \end{cases}$$

Solving this system of equations yields (2).

The facts that T_i^1 and T_i^2 have to be positive and $g_i < 1$ jointly with (2) implies that

$$\left(\frac{1}{\omega^1 + C} - N_i^0\right) > g_i \left(\frac{1}{\omega^2 + C} - N_i^0\right),$$

$$g_i \left(\frac{1}{\omega^1 + C} - N_i^0\right) < \left(\frac{1}{\omega^2 + C} - N_i^0\right).$$

Thus, $i \in I_{11}$ and Theorem 2 follows. □

Now carry on investigation of the case $g_i < 1$ for any i. Note that it is possible to simplify the form of the sets I taking into account three possible combinations of relations between ω^1 and ω^2 as it is given in the following lemma.

Lemma 1. *Let $g_i < 1$.*

1. *If* $\omega^1 > \omega^2$ *then*

$$\begin{aligned}
&I_{00}(\omega^1,\omega^2) = \left\{ i \in [1,n] : \frac{1}{\omega^2 + C} \leqslant N_i^0 \right\},\\
&I_{10}(\omega^1,\omega^2) = \emptyset,\\
&I_{01}(\omega^1,\omega^2) = \left\{ i \in [1,n] : \frac{\frac{1}{\omega^1 + C} - g_i \frac{1}{\omega^2 + C}}{1 - g_i} \leqslant N_i^0 < \frac{1}{\omega^2 + C} \right\},\\
&I_{11}(\omega^1,\omega^2) = \left\{ i \in [1,n] : \frac{\frac{1}{\omega^1 + C} - g_i \frac{1}{\omega^2 + C}}{1 - g_i} > N_i^0 \right\}
\end{aligned}$$

2. *If* $\omega^1 < \omega^2$ *then*

$$\begin{aligned}
&I_{00}(\omega^1,\omega^2) = \left\{ i \in [1,n] : \frac{1}{\omega^1 + C} \leqslant N_i^0 \right\},\\
&I_{10}(\omega^1,\omega^2) = \left\{ i \in [1,n] : \frac{\frac{1}{\omega^2 + C} - g_i \frac{1}{\omega^1 + C}}{1 - g_i} \leqslant N_i^0 < \frac{1}{\omega^1 + C} \right\},\\
&I_{01}(\omega^1,\omega^2) = \emptyset,\\
&I_{11}(\omega^1,\omega^2) = \left\{ i \in [1,n] : \frac{\frac{1}{\omega^2 + C} - g_i \frac{1}{\omega^1 + C}}{1 - g_i} > N_i^0 \right\}
\end{aligned}$$

3. *If* $\omega^1 = \omega^2$ *then*

$$\begin{aligned}
&I_{00}(\omega^1,\omega^2) = \left\{ i \in [1,n] : \frac{1}{\omega^2 + C} \leqslant N_i^0 \right\},\\
&I_{10}(\omega^1,\omega^2) = \emptyset,\\
&I_{01}(\omega^1,\omega^2) = \emptyset,\\
&I_{11}(\omega^1,\omega^2) = \left\{ i \in [1,n] : \frac{1}{\omega^2 + C} > N_i^0 \right\}.
\end{aligned}$$

Lemma 1 and Theorem 2 imply that the equilibrium strategies have the form given in the following theorem.

Theorem 3. *Let* $g_i < 1$. *The equilibrium strategies*

$$(T^1, T^2) = (T^1(\omega^1,\omega^2), T^2(\omega^1,\omega^2))$$

as functions on Lagrange multipliers have to have the following form.

(a) Let $\omega^1 < \omega^2$. Then

$$T_i^1(\omega^1,\omega^2)=\begin{cases}\dfrac{1}{1+g_i}\left(\dfrac{\dfrac{1}{\omega^1+C}-\dfrac{g_i}{\omega^2+C}}{1-g_i}-N_i^0\right), & \text{if } \dfrac{\dfrac{1}{\omega^2+C}-\dfrac{g_i}{\omega^1+C}}{1-g_i}>N_i^0\\[2ex] \dfrac{1}{\omega^1+C}-N_i^0, & \text{if } \dfrac{\dfrac{1}{\omega^2+C}-\dfrac{g_i}{\omega^1+C}}{1-g_i}\leqslant N_i^0\\ & \text{and } N_i^0<\dfrac{1}{\omega^1+C}\\ 0, & \text{if } \dfrac{1}{\omega^1+C}\leqslant N_i^0\end{cases}$$

$$T_i^2(\omega^1,\omega^2)=\begin{cases}\dfrac{1}{1+g_i}\left(\dfrac{\dfrac{1}{\omega^2+C}-\dfrac{g_i}{\omega^1+C}}{1-g_i}-N_i^0\right), & \text{if } \dfrac{\dfrac{1}{\omega^2+C}-\dfrac{g_i}{\omega^1+C}}{1-g_i}>N_i^0\\[2ex] 0, & \text{if } \dfrac{\dfrac{1}{\omega^2+C}-\dfrac{g_i}{\omega^1+C}}{1-g_i}\leqslant N_i^0\end{cases}$$

(b) Let $\omega^1 > \omega^2$. Then

$$T_i^1(\omega^1,\omega^2)=\begin{cases}\dfrac{1}{1+g_i}\left(\dfrac{\dfrac{1}{\omega^1+C}-\dfrac{g_i}{\omega^2+C}}{1-g_i}-N_i^0\right), & \text{if } \dfrac{\dfrac{1}{\omega^1+C}-\dfrac{g_i}{\omega^2+C}}{1-g_i}>N_i^0\\[2ex] 0, & \text{if } \dfrac{\dfrac{1}{\omega^1+C}-\dfrac{g_i}{\omega^2+C}}{1-g_i}\leqslant N_i^0\end{cases}$$

$$T_i^2(\omega^1,\omega^2)=\begin{cases}\dfrac{1}{1+g_i}\left(\dfrac{\dfrac{1}{\omega^2+C}-\dfrac{g_i}{\omega^1+C}}{1-g_i}-N_i^0\right), & \text{if } \dfrac{\dfrac{1}{\omega^1+C}-\dfrac{g_i}{\omega^2+C}}{1-g_i}>N_i^0\\[2ex] \dfrac{1}{\omega^2+C}-N_i^0, & \text{if } \dfrac{\dfrac{1}{\omega^1+C}-\dfrac{g_i}{\omega^2+C}}{1-g_i}\leqslant N_i^0\\ & \text{and } N_i^0<\dfrac{1}{\omega^2+C}\\ 0, & \text{if } \dfrac{1}{\omega^2+C}\leqslant N_i^0\end{cases}$$

(c) Let $\omega^1 = \omega^2$. Then

$$T_i^1(\omega^1,\omega^2)=T_i^2(\omega^1,\omega^2)=\begin{cases}\dfrac{1}{1+g_i}\left(\dfrac{1}{\omega^1+C}-N_i^0\right), & \text{if } \dfrac{1}{\omega^1+C}>N_i^0\\ 0, & \text{if } \dfrac{1}{\omega^1+C}\leqslant N_i^0\end{cases}$$

Our next goal is to find the Lagrange multipliers. To do so we formulate some important monotonous and continuous properties of the equilibrium strategies and also their sums

$$H^1(\omega^1,\omega^2)=\sum_{i=1}^n T_i^1(\omega^1,\omega^2) \quad H^2(\omega^1,\omega^2)=\sum_{i=1}^n T_i^2(\omega^1,\omega^2),$$

which follow from explicit formulas of Theorem 4.

Theorem 4. *The equilibrium strategies $(T^1(\omega^1,\omega^2), T^2(\omega^1,\omega^2))$ have the following properties:*

1. *$T^1(\omega^1,\omega^2)$, $T^2(\omega^1,\omega^2)$ and $H^1(\omega^1,\omega^2)$, $H^2(\omega^1,\omega^2)$ are continuous on ω^1 and ω^2,*
2. *$T^1(\omega^1,\omega^2)$ and $H^1(\omega^1,\omega^2)$ are decreasing on ω^1 and increasing on ω^2,*
3. *$T^2(\omega^1,\omega^2)$ and $H^2(\omega^1,\omega^2)$ are increasing on ω^1 and decreasing on ω^2.*

By Theorems 1 and 3 each couple of Lagrange multipliers (ω^1,ω^2) satisfying the following conditions:

$$H^1(\omega^1,\omega^2) = \bar{T}^1, \quad H^2(\omega^1,\omega^2) = \bar{T}^2 \tag{5}$$

supplies a Nash equilibrium.

To find such pair (ω^1,ω^2) let introduce function $H(\omega^1,\omega^2)$:

$$\begin{aligned} H(\omega^1,\omega^2) &= H^1(\omega^1,\omega^2) + H^2(\omega^1,\omega^2) = \\ &\sum_{i=1}^n T_i^1(\omega^1,\omega^2) + \sum_{i=1}^n T_i^2(\omega^1,\omega^2) = \sum_{i=1}^n \left[T_i^1(\omega^1,\omega^2) + T_i^2(\omega^1,\omega^2)\right]. \end{aligned} \tag{6}$$

$T_i^1(\omega^1,\omega^2) + T_i^2(\omega^1,\omega^2)$ for $i \in [1,n]$ can have one of the following forms.
(a) Let $\omega^1 < \omega^2$. Then

$$T_i(\omega^1,\omega^2) = T_i^1(\omega^1,\omega^2) + T_i^2(\omega^1,\omega^2) =$$
$$\begin{cases} \dfrac{1}{1+g_i}\left(\dfrac{1}{\omega^1+C} + \dfrac{1}{\omega^2+C} - 2N_i^0\right), & \dfrac{\dfrac{1}{\omega^2+C} - \dfrac{g_i}{\omega^1+C}}{1-g_i} > N_i^0 \\ \dfrac{1}{\omega^1+C} - N_i^0, & \dfrac{\dfrac{1}{\omega^2+C} - \dfrac{g_i}{\omega^1+C}}{1-g_i} \leqslant N_i^0 < \dfrac{1}{\omega^1+C} \\ 0, & \dfrac{1}{\omega^1+C} \leqslant N_i^0. \end{cases}$$

(b) Let $\omega^1 > \omega^2$. Then

$$T_i(\omega^1,\omega^2) = T_i^1(\omega^1,\omega^2) + T_i^2(\omega^1,\omega^2) =$$
$$\begin{cases} \dfrac{1}{1+g_i}\left(\dfrac{1}{\omega^2+C} + \dfrac{1}{\omega^1+C} - 2N_i^0\right), & \dfrac{\dfrac{1}{\omega^1+C} - \dfrac{g_i}{\omega^2+C}}{1-g_i} > N_i^0 \\ \dfrac{1}{\omega^2+C} - N_i^0, & \dfrac{\dfrac{1}{\omega^1+C} - \dfrac{g_i}{\omega^2+C}}{1-g_i} \leqslant N_i^0 < \dfrac{1}{\omega^2+C} \\ 0, & \dfrac{1}{\omega^2+C} \leqslant N_i^0. \end{cases}$$

(c) Let $\omega^1 = \omega^2$. Then

$$T_i(\omega^1,\omega^2) = T_i^1(\omega^1,\omega^2) + T_i^2(\omega^1,\omega^2) =$$
$$\begin{cases} \dfrac{2}{1+g_i}\left(\dfrac{1}{\omega^1+C} - N_i^0\right), & \dfrac{1}{\omega^1+C} > N_i^0 \\ 0, & \dfrac{1}{\omega^1+C} \leqslant N_i^0. \end{cases}$$

Lemma 2. *$H(\omega^1,\omega^2)$ is continuous and decreasing on ω^1 and ω^2.*

In the next theorems we show such couple of Lagrange multipliers is unique as well as Nash equilibrium. The first theorem deals with the situation where tariff is big.

Theorem 5. *Let $g_i < 1$ for any i. If*

$$H(0,0) \leqslant \bar{T}^1 + \bar{T}^2 \tag{7}$$

then the game has unique Nash equilibrium (T^1, T^2), where
(a) If $H^1(0,0) \leqslant \bar{T}^1$ and $H^2(0,0) \leqslant \bar{T}^2$ then $(T^1, T^2) = (T^1(0,0), T^2(0,0))$.
(b) If $H^1(0,0) > \bar{T}^1$ and $H^2(0,0) < \bar{T}^2$ then $(T^1, T^2) = (T^1(\omega_{10}^{1*}, 0), T^2(\omega_{10}^{1*}, 0))$, where ω_{10}^{1*} is unique solution of equation $H^1(\omega_{10}^{1*}, 0) = \bar{T}^1$.
(c) If $H^1(0,0) < \bar{T}^1$ and $H^2(0,0) > \bar{T}^2$ then $(T^1, T^2) = (T^1(0, \omega_{01}^{2*}), T^2(0, \omega_{01}^{2*}))$, where ω_{01}^{2*} is unique solution of equation $H^2(0, \omega_{01}^{2*}) = \bar{T}^2$.

Proof. By Lemma 2 $H(\omega^1, \omega^2)$ is decreasing on ω^1 and ω^2. Thus, if $H(0,0) < \bar{T}^1 + \bar{T}^2$ then where is no solution of system 5.

Therefore there are three possible situations.

(a) $\omega^1 = \omega^2 = 0$. This holds only if $H^1(0,0) \leqslant \bar{T}^1$ and $H^2(0,0) \leqslant \bar{T}^2$.

(b) $\omega^1 > 0, \omega^2 = 0$. Let $\omega^1 = \omega_{10}^{1*}$. Then $H^1(\omega_{10}^{1*}, 0) = \bar{T}^1$ and $H^2(\omega_{10}^{1*}, 0) \leqslant \bar{T}^2$. $H^1(\omega^1, \omega^2)$ is strictly decreasing on ω^1, while it's positive. Thus $H^1(0,0) > H^1(\omega_{10}^{1*}, 0) = \bar{T}^1$. Which jointly with (7) yields that $H^2(0,0) < \bar{T}^2$.

(c) $\omega^1 = 0, \omega^2 > 0$. Let $\omega^2 = \omega_{01}^{2*}$. Then $H^1(0, \omega_{01}^{2*}) \leqslant \bar{T}^1$ and $H^2(0, \omega_{01}^{2*}) = \bar{T}^2$. $H^2(\omega^1, \omega^2)$ is strictly decreasing on ω^2, while it's positive. Thus $H^2(0,0) > H^2(0, \omega_{01}^{2*}) = \bar{T}^2$. Which jointly with (7) yields that $H^1(0,0) < \bar{T}^1$. □

Lemma 3. *Let $g < 1$ and*

$$H^1(\omega^1, \omega^2) = \bar{T}^1, \quad H^2(\omega^1, \omega^2) = \bar{T}^2.$$

If $\bar{T}^2 > \bar{T}^1$ then $\omega^1 > \omega^2$.

Proof. Assume the contrary. Let $\bar{T}^2 > \bar{T}^1$, but $\omega^1 < \omega^2$ (equality is impossible here because in this case $H^1(\omega^1, \omega^2) = H^2(\omega^1, \omega^2)$). Then, for any $i \in [1, n]$

$$T_i^1(\omega^1, \omega^2) - T_i^2(\omega^1, \omega^2) =$$
$$\begin{cases} \dfrac{1}{1 - g_i}\left(\dfrac{1}{\omega^1 + C} - \dfrac{1}{\omega^2 + C}\right), & \text{if } \dfrac{\dfrac{1}{\omega^2 + C} - \dfrac{g_i}{\omega^1 + C}}{1 - g_i} > N_i^0 \\ \dfrac{1}{\omega^1 + C} - N_i^0, & \text{if } \dfrac{\dfrac{1}{\omega^2 + C} - \dfrac{g_i}{\omega^1 + C}}{1 - g_i} \leqslant N_i^0 \\ & \text{and } N_i^0 < \dfrac{1}{\omega^1 + C} \\ 0, & \text{if } \dfrac{1}{\omega^1 + C} \leqslant N_i^0. \end{cases}$$

Thus, for any $i \in [1, n]$ $T_i^1(\omega^1, \omega^2) - T_i^2(\omega^1, \omega^2) \geqslant 0$. Therefore, $H^1(\omega^1, \omega^2) - H^2(\omega^1, \omega^2) \geqslant 0$. But this contradicts the condition that $\bar{T}^2 > \bar{T}^1$ and Lemma 3 follows. □

Without loss of generality further we assume that $\bar{T}^2 > \bar{T}^1$.

Theorem 6. *Let $g_i < 1$ for any i. Let also $\bar{T}^2 > \bar{T}^1$*
If

$$H(0,0) > \bar{T}^1 + \bar{T}^2. \tag{8}$$

Then, the game has unique Nash equilibrium (T^1, T^2) and it is given as follows:
(a) if

$$H^2(\omega_{10}^{1*}, 0) \leqslant \bar{T}^2,$$

where where ω_{10}^{1*} is unique solution of equation $H^1(\omega_{10}^{1*}, 0) = \bar{T}^1$, then

$$(T^1, T^2) = (T^1(\omega_{10}^{1*}, 0), T^2(\omega_{10}^{1*}, 0)),$$

(b) if

$$H^2(\omega_{10}^{1*}, 0) > \bar{T}^2,$$

then

$$(T^1, T^2) = (T^1(\omega_{11}^{1*}, \omega_{11}^{2*}), T^2(\omega_{11}^{1*}, \omega_{11}^{2*})),$$

where ω_{11}^{1*} and ω_{11}^{2*} is unique solution of system of equations

$$H^1(\omega_{11}^{1*}, \omega_{11}^{2*}) = \bar{T}^1, \quad H^2(\omega_{11}^{1*}, \omega_{11}^{2*}) = \bar{T}^2.$$

Proof. (a) Let $\omega^1 > 0, \omega^2 = 0$. Then

$$H^1(\omega_{10}^{1*}, 0) = \bar{T}^1 \text{ and } H^2(\omega_{10}^{1*}, 0) \leqslant \bar{T}^2. \tag{9}$$

$H^1(\omega^1, \omega^2)$ is strictly decreasing on ω^1, while it's positive. Thus

$$H^1(0,0) > H^1(\omega_{10}^{1*}, 0) = \bar{T}^1.$$

$H^2(\omega^1, \omega^2)$ is increasing on ω^1, therefore $H^2(0,0) \leqslant H^2(\omega_{10}^{1*}, 0) \leqslant \bar{T}^2$.

(b) Let $\omega^1 = 0, \omega^2 > 0$. Then $H^1(0, \omega_{01}^{2*}) \leqslant \bar{T}^1$ and $H^2(0, \omega_{01}^{2*}) = \bar{T}^2$. $H^2(\omega^1, \omega^2)$ is strictly decreasing on ω^2, while it's positive. Thus

$$H^2(0,0) > H^2(0, \omega_{01}^{2*}) = \bar{T}^2.$$

$H^1(\omega^1, \omega^2)$ is increasing on ω^2, therefore $H^1(0,0) \leqslant H^1(0, \omega_{01}^{2*}) \leqslant \bar{T}^1$. This contradicts to (8).

(c) Let $\omega^1 > 0, \omega^2 > 0$. Say, without loss of generality, that $\bar{T}^2 > \bar{T}^1$. Thus, by Lemma 3, further we assume that $\omega^1 > \omega^2$.

Consider function $H^1(\omega^2, \omega^2)$, which is decreasing on ω^2. Since

$$H(0,0) = 2H^1(0,0) > \bar{T}^1 + \bar{T}^2 \text{ and } \bar{T}^2 > \bar{T}^1$$

then $H^1(0,0) > T^1$. Therefore, for any ω^2, such that $H^1(\omega^2, \omega^2) > \bar{T}^1$ there exist $\omega^1 = \omega^1(\omega^2)$ such what $H^1(\omega^1(\omega^2), \omega^2) = \bar{T}^1$. Function $\omega^1 = \omega^1(\omega^2)$ is defined on the interval $[0, \omega^{2*}]$, where ω^{2*} is unique solution of equation $H^1(\omega^{2*}, \omega^{2*}) = \bar{T}^1$. $\omega^1 = \omega^1(\omega^2)$ is increasing on ω^2.

To find ω^2, supplying a Nash equilibrium, consider a function $H(\omega^1(\omega^2), \omega^2)$. This function $H(\omega^1(\omega^2), \omega^2)$ is decreasing on ω^2. Since $H(\omega^1(\omega^2), \omega^2)$ is continuous on ω^2 and

$$H(\omega^1(\omega^{2*}), \omega^{2*}) = 2H^1(\omega^{2*}, \omega^{2*}) = 2\bar{T}^1 < \bar{T}^1 + \bar{T}^2,$$

then solution of equation $H(\omega^1(\omega^2),\omega^2)=\bar{T}^1+\bar{T}^2$ exists only if

$$H^2(\omega^1(0),0)>T^2.$$

Solution of $H^2(\omega^1,\omega^2)=\bar{T}^2,\omega^1>\omega^2$ exist only for such ω^1 that

$$H^2(\omega^1,\omega^1)<\bar{T}^2 \text{ and } H^2(\omega^1,0)>\bar{T}^2.$$

Condition $H^2(\omega^1,0)>\bar{T}^2$ implies that $H^2(+\infty,0)>\bar{T}^2$. Let consider $H^2(0,0)$. If $H^2(0,0)>\bar{T}^2$ then the first condition yields that $\omega^1>\tilde{\omega^1}$, where $\tilde{\omega}^1$ is the unique root of equation $H^2(\tilde{\omega}^1,\tilde{\omega}^1)=\bar{T}^2$ and the second condition is always true. If $H^2(0,0)\leqslant\bar{T}^2$ then the first condition is always true and the second condition yields that $\omega^1>\bar{\omega}^1$, where $\bar{\omega}^1$ is the unique root of equation $H^2(\bar{\omega}^1,0)=\bar{T}^2$. Thus for any $\omega^1>\max\{\tilde{\omega}^1,\bar{\omega}^1\}$ there exists $\omega^2(\omega^1)$ such that $H^2(\omega^1,\omega^2(\omega^1))=\bar{T}^2$.

To find ω^1 supplying Nash equilibrium consider function $H(\omega^1,\omega^2(\omega^1))$. Since $\omega^2(\omega^1)$ is increasing on ω^1 $H^2(\omega^1,\omega^2(\omega^1))$ is decreasing on ω^1.

$$H(\omega^1,\omega^2(\omega^1))=\tilde{H}(\omega^1)=H^1(\omega^1,\omega^2(\omega^1))+\bar{T}^2$$

Note that $\tilde{H}(+\infty)=\bar{T}^2<\bar{T}^1+\bar{T}^2$. Thus the equation

$$\tilde{H}(\omega^1)=\bar{T}^1+\bar{T}^2$$

has a root if either

$$H^2(0,0)>\bar{T}^2$$

or

$$H^2(0,0)\leqslant\bar{T}^2 \text{ and } H^1(\omega^1,0)>\bar{T}^1. \tag{10}$$

To complete the proof we have to show that two conditions (9) and (10) can't hold simultaneously. Assume that both these conditions hold. Thus by (10) for some $\bar{\omega}^1$

$$H^2(\bar{\omega}^1,0)=\bar{T}^2 \text{ and } H^1(\bar{\omega}^1,0)>\bar{T}^1.$$

So

$$\omega_{10}^{1*}>\bar{\omega}^2. \tag{11}$$

Therefore

$$\bar{T}^2=H^2(\bar{\omega}^1,0)\leqslant H^2(\omega_{10}^{1*},0). \tag{12}$$

From (9) and (12) follows $\bar{T}^2=H^2(\omega_{10}^{1*},0)$. Thus $\omega_{10}^{1*}=\bar{\omega}^1$ which contradicts to (11). So two conditions can't hold simultaneously and uniqueness of the equilibrium follows. □

3. Numerical Results

In this section we supply a numerical example of finding equilibrium strategies for different power costs by an algorithm based on the bisection method and Theorems 5 and 6.

We assume that there are five channels ($n=5$) and the background noise is permanent for all them ($N_i^0=0.1$, $i\in[1,5]$). Let the fading channel gains are $g=[0.9,0.8,0.7,0.6,0.5]$ and the total signal user 1 (2) has to transmit is $\bar{T}^1=1$ ($\bar{T}^2=5$). On Figures 1, 2, 3 and 4 the equilibrium strategies of the users are given for the power costs $C=0.1,0.9,1.5$ and 2.3 respectively.

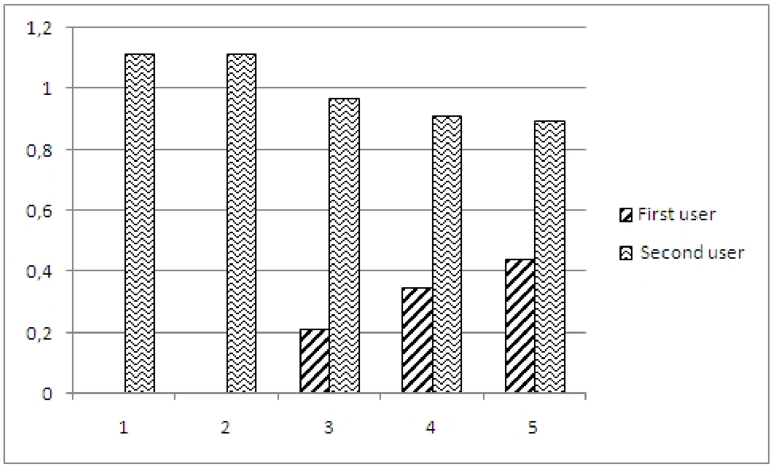

Fig. 1. The equilibrium strategies for $C = 0.1$

For the low power cost $C = 0.1$ (Figure 1) both users transmit all the signal they intend to ($\sum_{i=1}^{5} T_i^1 = 1$ and $\sum_{i=1}^{5} T_i^2 = 5$). In this situation user 2 employs all the five channels while user 1 employs only three channels with highest quality.

If the power cost grows up to $C = 0.9$ (Figure 2) user 2 decreases the transmitted signal ($\sum_{i=1}^{5} T_i^2 < 5$) and no longer send the total signal (namely, he send $\sum_{i=1}^{5} T_i^2 \approx 4.47$) and user 1 employs more channels (namely, four channels) while his transmitted signal remains $\sum_{i=1}^{5} T_i^1 = 1$.

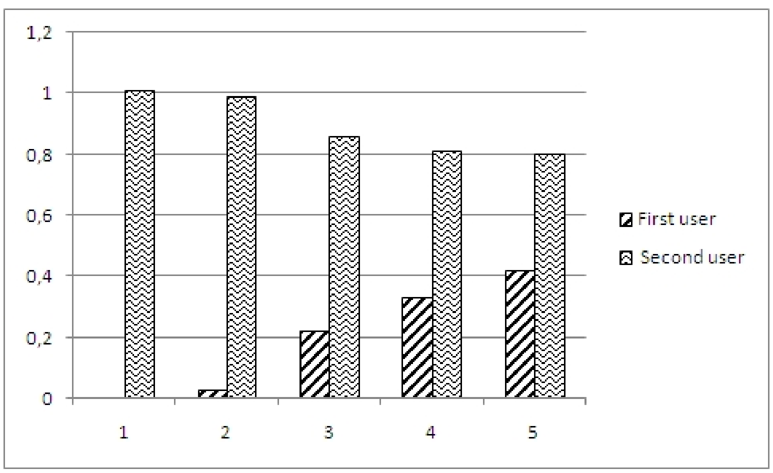

Fig. 2. The equilibrium strategies for $C = 0.9$

With further increasing of the power cost up to $C = 1.5$ (Figure 3) user 1 becomes to employ all the five channels ($\sum_{i=1}^{5} T_i^1 = 1$), while the signal transmitted by user 2 goes on to decrease ($\sum_{i=1}^{5} T_i^2 \approx 2.2$).

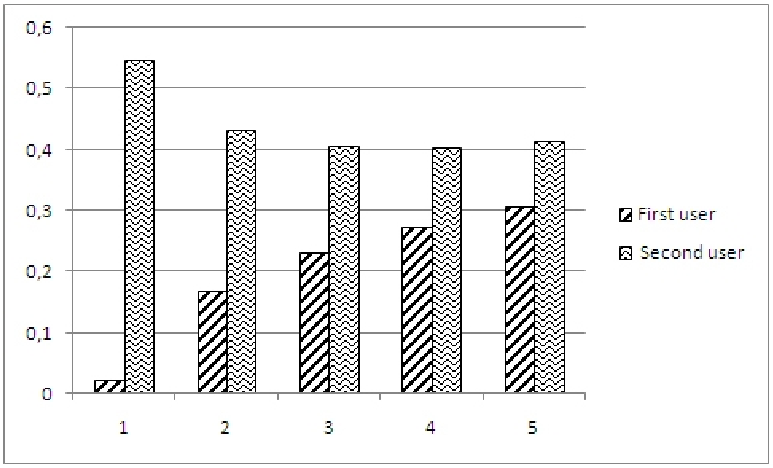

Fig. 3. The equilibrium strategies for $C = 1.5$

If the power cost gets to $C = 2.3$ (Figure 4) then the equilibrium strategies of users coincides to each other ($\sum_{i=1}^{5} T_i^1 = \sum_{i=1}^{5} T_i^2 = 0.99$).

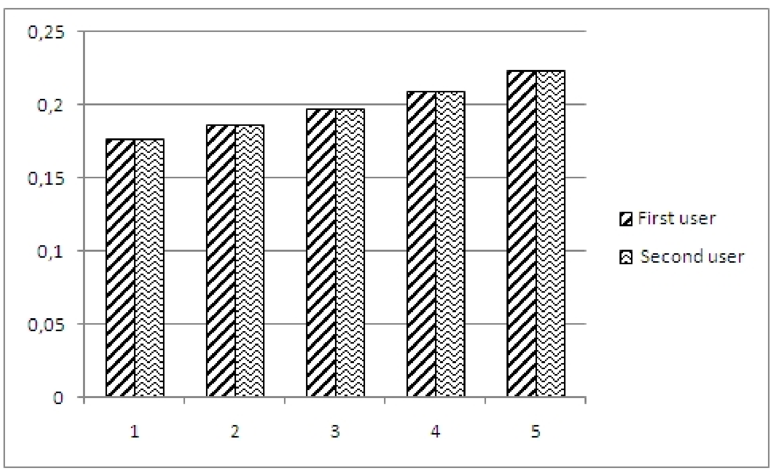

Fig. 4. The equilibrium strategies for $C = 2.3$

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Strict Proportional Power and Fair Voting Rules in Committees*

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Abstract. In simple weighted committees with a finite number n of members, fixed weights and changing quota, there exists a finite number r of different quota intervals of stable power ($r \leq 2^n - 1$) with the same sets of winning coalitions for all quotas from each of them. If in a committee the sets of winning coalitions for different quotas are the same, then the power indices based on pivots, swings, or minimal winning coalitions are also the same for those quotas. If the fair distribution of voting weights is defined, then the fair distribution of voting power means to find a quota that minimizes the distance between relative voting weights and relative voting power (optimal quota). The problem of the optimal quota has an exact solution via the finite number of quotas from different intervals of stable power.

Keywords: Fairness, optimal quota, simple weighted committee, strict proportional power, voting and power indices

AMS Classification: 91A12, 91A40, 05C65

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1. Introduction

Let us consider a committee with n members. Each member has some voting weight (number of votes, shares etc.) and a voting rule is defined by a minimal number of weights required for passing a proposal. Given a voting rule, voting weights provide committee members with voting power. Voting power means an ability to influence the outcome of voting. Voting power indices are used to quantify the voting power.

The concept of fairness is being discussed related to the distribution of voting power among different actors of voting. This problem was clearly formulated by Nurmi (1982, p. 204): "*If one aims at designing collective decision-making bodies which are democratic in the sense of reflecting the popular support in terms of the voting power, we need indices of the latter which enable us to calculate for any given distribution of support and for any decision rule the distribution of seats that is 'just'. Alternatively, we may want to design decision rules that – given the distribution of seats and support—lead to a distribution of voting power which is identical with the distribution of support.*"

Voting power is not directly observable: as a proxy for it voting weights are used. Therefore, fairness is usually defined in terms of voting weights (e.g. voting

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weights are proportional to the results of an election). Assuming that a principle of fair distribution of voting weights is selected, we are addressing the question of how to achieve equality of voting power (at least approximately) to relative voting weights. The concepts of strict proportional power and the randomized decision rule introduced by Holler (1982a, 1985, 1987), of optimal quota of Słomczyński and Życzkowski (2007), and of intervals of stable power (Turnovec, 2008b) are used to find, given voting weights, a voting rule minimizing the distance between actors' voting weights and their voting power.

Concept of fairness is frequently associated with so-called square root rule, attributed to British statistician Lionel Penrose (1946). The square root rule is closely related to indirect voting power measured by the Penrose-Banzhaf power index.[1] Different aspects of the square root rule have been analysed in Felsenthal and Machover (1998, 2004), Laruelle and Widgrén (1998), Baldwin and Widgrén (2004), Turnovec (2009). The square root rule of "fairness" in the EU Council of Ministers voting was discussed and evaluated in Felsenthal and Machover (2007), Słomczyński and Życzkowski (2006, 2007), Hosli (2008), Leech and Aziz (2008), Turnovec (2008a) and others. Nurmi (1997a) used it to evaluate the representation of voters' groups in the European Parliament.

In the second section basic definitions are introduced and the applied power indices methodology is shortly resumed. The third section introduces concepts of quota intervals of stable power, index of fairness and exact optimal quota. While the framework of the analysis of fairness is usually restricted to the Penrose-Banzhaf concept of power, we are treating it in a more general setting and our results are relevant for any power index based on pivots or swings and for any concept of fairness.

2. Committees and Voting Power

A simple weighted committee is a pair $[N, \mathbf{w}]$, where N be a finite set of n committee members $i = 1, 2, \ldots, n$, and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a nonnegative vector of committee members' voting weights (e.g. votes or shares). By 2^N we denote the power set of N (set of all subsets of N). By voting coalition we mean an element $S \in 2^N$, the subset of committee members voting uniformly (YES or NO), and $w(S) = \sum_{i \in S} w_i$ denotes the voting weight of coalition S. The voting rule is defined by quota q satisfying $0 < q \leq w(N)$, where q represents the minimal total weight necessary to approve the proposal. Triple $[N, q, \mathbf{w}]$ we call a *simple quota weighted*

[1] The square root rule is based on the following propositions: Let us assume n units with population $p_1, p_2, \ldots, p_n$, and the system of representation by a super-unit committee with voting weights $w_1, w_2, \ldots, w_n$ of units' representations. It can be rigorously proved that for sufficiently large p_i the absolute Penrose-Banzhaf power of individual citizen of unit i in unit's referendum is proportional to the square root of p_i. If the relative Penrose-Banzhaf voting power of unit i representation is proportional to its voting weight, then indirect voting power of each individual citizen of unit i is proportional to the product of voting weight w_i and square root of population p_i. Based on the conjecture (not rigorously proved) that for n large enough the relative voting power is proportional to the voting weights, the square root rule concludes that the voting weights of units' representations in the super-unit committee proportional to square roots of units' population lead to the same indirect voting power of each citizen independently of the unit she is affiliated with.

committee. The voting coalition S in committee $[N, q, \mathbf{w}]$ is called a winning one if $w(S) \geq q$ and a losing one in the opposite case. The winning voting coalition S is called critical if there exists at least one member $k \in S$ such that $w(S \backslash k) < q$ (we say that k is critical in S). The winning voting coalition S is called minimal if any of its members is critical in S.

A priori voting power analysis seeks an answer to the following question: Given a simple quota weighted committee $[N, q, \mathbf{w}]$, what is an influence of its members over the outcome of voting? The absolute voting power of a member i is defined as a probability $\Pi_i[N, q, \mathbf{w}]$ that i will be decisive in the sense that such a situation appears in which she would be able to decide the outcome of voting by her vote (Nurmi, 1997b and Turnovec, 1997), and a relative voting power as:

$$\pi_i[N, q, w] = \frac{\Pi_i[N, q, w]}{\sum\limits_{k \in N} \Pi_k[N, q, w]}.$$

Three basic concepts of decisiveness are used: swing position, pivotal position and membership in a minimal winning coalition (MWC position). The *swing position* is an ability of an individual voter to change the outcome of voting by a unilateral switch from YES to NO (if member j is critical with respect to a coalition S, we say that he has a swing in S). The *pivotal position is* such a position of an individual voter in a permutation of voters expressing a ranking of attitudes of members to the voted issue (from the most preferable to the least preferable) and the corresponding order of forming of the winning coalition, in which her vote YES means a YES outcome of voting and her vote NO means a NO outcome of voting (we say that j is pivotal in the permutation considered). The MWC position is an ability of an individual voter to contribute to a minimal winning coalition (membership in the minimal winning coalition).

Let us denote by $W(N, q, \mathbf{w})$ the set of all winning coalitions and by $W_i(N, q, \mathbf{w})$ the set of all winning coalitions with i, $C(N, q, \mathbf{w})$ as the set of all critical winning coalitions, and by $C_i(N, q, \mathbf{w})$ the set of all critical winning coalitions i has the swing in, by $P(N, q, \mathbf{w})$ the set of all permutations of N and $P_i(N, q, \mathbf{w})$, the set of all permutations i is pivotal in, $M(N, q, \mathbf{w})$ the set of all minimal winning coalitions, and $M_i(N, q, w)$ the set of all minimal winning coalitions with i. By $card(S)$ we denote the cardinality of S, $card(\emptyset) = 0$.

Assuming many voting acts and all coalitions equally likely, it makes sense to evaluate the a priori voting power of each member of the committee by the probability to have a swing, measured by the absolute Penrose-Banzhaf (PB) power index (Penrose, 1946; Banzhaf, 1965):

$$\Pi_i^{PB}(N, q, w) = \frac{card(C_i)}{2^{n-1}},$$

($card(C_i)$ is the number of all winning coalitions the member i has the swing in and 2^{n-1} is the number of all possible coalitions with i). To compare the relative power of different committee members, the relative form of the PB power index is used:

$$\pi_i^{PB}(N, q, w) = \frac{card(C_i)}{\sum\limits_{k \in N} card(C_k)}.$$

While the absolute PB is based on a well-established probability model (see e.g. Owen, 1972), its normalization (relative PB index) destroys this probabilistic interpretation, the relative PB index simply answers the question of what is the voter i's share in all possible swings.

Assuming many voting acts and all possible preference orderings equally likely, it makes sense to evaluate an a priori voting power of each committee member by the probability of being in pivotal situation, measured by the Shaply-Shubik (SS) power index (Shapley and Shubik, 1954):

$$\Pi_i^{SS}(N, q, w) = \frac{card(P_i)}{n!},$$

($card(P_i)$ is the number of all permutations in which the committee member i is pivotal, and $n!$ is the number of all possible permutations of committee members). Since $\sum_{i \in N} card(P_i) = n!$ it holds that:

$$\pi_i^{SS}(N, q, \mathbf{w}) = \frac{card(P_i)}{\sum_{k \in N} card(P_k)} = \frac{card(P_i)}{n!},$$

i.e. the absolute and relative form of the SS-power index is the same.[2]

Assuming many voting acts and all possible coalitions equally likely, it makes sense to evaluate the voting power of each committee member by the probability of membership in a minimal winning coalition, measured by the absolute Holler-Packel (HP) power index:

$$\Pi_i^{HP}(N, q, \mathbf{w}) = \frac{card(M_i)}{2^n},$$

($card(M_i)$ is the number of all minimal winning coalitions with i, and 2^n is the number of all possible coalitions).[3] Originally the HP index was defined and is usually being presented in its relative form (Holler, 1982b; Holler and Packel, 1983)

$$\pi_i^{HP}(N, q, \mathbf{w}) = \frac{card(M_i)}{\sum_{k \in N} card(M_k)}.$$

[2] Supporters of the Penrose-Banzhaf power concept sometimes reject the Shapley-Shubik index as a measure of voting power. Their objections to the Shapley-Shubik power concept are based on the classification of power measures on so-called I-power (voter's potential influence over the outcome of voting) and P-power (expected relative share in a fixed prize available to the winning group of committee members, based on cooperative game theory) introduced by Felsenthal et al. (1998). The Shapley-Shubik power index was declared to represent P-power and as such is unusable for measuring influence in voting. We tried to show in Turnovec (2007) and Turnovec et al. (2008) that objections against the Shapley-Shubik power index, based on its interpretation as a P-power concept, are not sufficiently justified. Both Shapley-Shubik and Penrose-Banzhaf measure could be successfully derived as cooperative game values, and at the same time both of them can be interpreted as probabilities of being in some decisive position (pivot, swing) without using cooperative game theory at all.

[3] The definition of an absolute HP power index is provided by the author (a similar definition of absolute PB power can be found in Brueckner (2001), the only difference is that we relate the number of MWC positions of member i to the total number of coalitions, not to the number of coalitions of which i is a member).

Above definition of the absolute HP index allows a clear probabilistic interpretation. Multiplying and dividing it by the $card(M)$, we obtain:

$$\frac{card(M_i)}{card(M)}\frac{card(M)}{2^n}.$$

In this breakdown the first term gives the probability of being a member of a minimal winning coalition, provided the MWC is formed, and the second term the probability of forming a minimal winning coalition assuming that all voting coalitions are equally likely. The relative HP index has the same problem with a probabilistic interpretation as the relative PB index.[4]

In the literature there are still two other concepts of power indices: the Johnston (J) power index based on swings, and the Deagan-Packel (DP) power index, based on membership in minimal winning coalitions.

The Johnston power index (Johnston, 1978) measures the power of a member of a committee as a normalized weighted average of the number of her swings, using as weights the reciprocals of the total number of swings in each critical winning coalition (the swing members of the same winning coalition have the same power, which is equal to 1/[number of swing members]).

The Deegan-Packel power index (Deegan and Packel, 1978) measures the power of a member of a committee as a normalized weighted average of the number of minimal critical winning coalitions he is a member of, using as weights the reciprocals of the size of each MCWC

It is difficult to provide some intuitively acceptable probabilistic interpretation for relative J and DP power indices. They provide a normative scheme of the division of rents in the committee rather than a measure of an a priori power (and in the sense of Felsenthal and Machover (1998) classification they can be considered as measures of P power).

It can be easily seen that for any $\alpha > 0$ and any power index based on swings, pivots or MWC positions it holds that $\Pi_i[N, \alpha q, \alpha \mathbf{w}] = \Pi_i[N, q, \mathbf{w}]$. Therefore, without the loss of generality, we shall assume throughout the text that $\sum_{i \in N} w_i = 1$ and $0 < q \leq 1$, using only relative weights and relative quotas in the analysis.

3. Quota Intervals of Stable Power, Fairness and Strict Proportional Power

Let us formally define a few concepts we shall use later in this paper:

Definition 1. A simple weighted committee $[N, \mathbf{w}]$ has a property of *strict proportional power* with respect to a power index $\boldsymbol{\pi}$, if there exists a voting rule q^* such that $\pi[N, q^*, \mathbf{w}] = \mathbf{w}$, i.e. the relative voting power of committee members is equal to their relative voting weights.

In general, there is no reason to expect that such a voting rule exists. Holler and Berg extended the concept of a strict proportional power in the model with randomized voting rule.

[4] For a discussion about the possible probabilistic interpretation of the relative PB and HP see Widgrén (2001).

Definition 2. Let $[N, \mathbf{w}]$ be a simple weighted committee, $\mathbf{q} = (q_1, q_2, \ldots, q_m)$ be a vector of different quotas, $\boldsymbol{\pi}^k$ be a relative power index for quota q_k, and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a probability distribution over elements of $\mathbf{q}$. The *randomized voting rule* $(\mathbf{q}, \boldsymbol{\lambda})$ selects within different voting acts by random mechanism quotas from $\mathbf{q}$ by the probability distribution $\boldsymbol{\lambda}$. Then $[N, \mathbf{w}]$ has a property of *strict proportional expected power* with respect to a relative power index $\boldsymbol{\pi}$, if there exists a randomized voting rule $(\mathbf{q}^*, \boldsymbol{\lambda}^*)$ such that the vector of the mathematical expectations of relative voting power is equal to the vector of voting weights:

$$\boldsymbol{\pi}(N, (\mathbf{q}, \boldsymbol{\lambda}), \mathbf{w}) = \sum_{k=1}^{m} \lambda_k \boldsymbol{\pi}^k(N, q_k, \mathbf{w}) = \mathbf{w}.$$

The concept of randomized voting rule and strict proportional expected power was introduced by Holler (1982a, 1985), and studied by Berg and Holler (1986).

Definition 3. Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a fair distribution of voting weights (with whatever principle is used to justify it) in a simple weighted committee $[N, \mathbf{w}]$, $\boldsymbol{\pi}$ is a relative power index, ($\boldsymbol{\pi}[N, q, \mathbf{w}]$ is a vector valued function of q), and d is a distance function, then the voting rule q_1 is said to be *at least as fair* as voting rule q_2 with respect to the selected $\boldsymbol{\pi}$ if $d(\mathbf{w}, \boldsymbol{\pi}(N, q_1, \mathbf{w})) \leq d(\mathbf{w}, \boldsymbol{\pi}(N, q_2, \mathbf{w}))$.

Intuitively, given $\mathbf{w}$, the voting rule q_1 is preferred to the voting rule q_2 if q_1 generates a distribution of power closer to the distribution of weights than q_2.

Definition 4. The voting rule q^* that minimizes a distance d between $\boldsymbol{\pi}[N, q, \mathbf{w}]$ and $\mathbf{w}$ is called an optimal voting rule (*optimal quota*) for the power index π.

Proposition 1. *Let $[N, q, \mathbf{w}]$ be a simple weighted quota committee and C_{is} be the set of critical winning coalitions of the size s in which i has a swing, then*

$$card(P_i) = \sum_{s \in N} card(C_{is})(s-1)!(n-s)!$$

is the number of permutations with the pivotal position of i in $[N, q, \mathbf{w}]$.

Proof. Proof follows directly from Shapley and Shubik (1954). □

From Prop. 1 it follows that the number of pivotal positions corresponds to the number and structure of swings. If in two different committees sets of swing coalitions are identical, then the sets of pivotal positions are also the same.

Proposition 2. *Let $[N, q_1, \mathbf{w}]$ and $[N, q_2, \mathbf{w}]$, $q_1 \neq q_2$, be two simple quota-weighted committees such that $W[N, q_1, \mathbf{w}] = W[N, q_2, \mathbf{w}]$, then*

$$C_i(N, q_1, \mathbf{w}) = C_i(N, q_2, \mathbf{w})$$

$$P_i(N, q_1, \mathbf{w}) = P_i(N, q_2, \mathbf{w}),$$

and

$$M_i(N, q_1, \mathbf{w}) = M_i(N, q_2, \mathbf{w})$$

for all $i \in N$.

Proof. Without a loss of generality suppose $q_1 > q_2$. Assuming that the statement of the proposition is not true, i.e. a member k from S has a swing in S for quota q_2 and does not have a swing for quota q_1, we obtain $w(S\backslash k) - q_2 < 0$ and $w(S\backslash k) - q_1 \geq 0$, hence $S\backslash k \notin W(N, q_2, w)$ and $S\backslash k \in W(N, q_1, w)$, which contradicts the assumption of Prop. 2 that the sets of winning coalitions are equal. From Prop. 1 it follows that $C_i(N, q_1, \mathbf{w}) = C_i(N, q_2, \mathbf{w})$ implies $P_i(N, q_1, w) = P_i(N, q_2, \mathbf{w})$. In a similar way we can prove identity for sets of minimal winning coalitions. Assuming that M is the minimal winning coalition for q_2 and is not a minimal winning coalition for q_1, we have $w(S) - q_2 \geq 0$ and for any $i \in S$ $w(S\backslash i) - q_2 < 0$, and there exists at least one $k \in S$ such that $w(S\backslash k) - q_1 \geq 0$. But in this case $S\backslash k$ is the losing coalition for q_2 and the winning coalition for q_1 what contradicts the assumption of Prop. 2 that the sets of winning coalitions are the same, hence $M_i(N, q_1, \mathbf{w}) = M_i(N, q_2, \mathbf{w})$. □

From Prop. 2 it follows that in two different committees with the same set of members, the same weights and the same sets of winning coalitions, the PB-power indices, SS-power indices and HP-power indices are the same in both committees, independently of quotas. Moreover, since the J-index is based on the concept of swing and the DP power index is based on membership in minimal winning coalitions, the J and DP indices are also the same.

Proposition 3. *Let $[N, q, \mathbf{w}]$ be a simple quota weighted committee with a quota q,*

$$\mu^+(q) = \min_{S \in W[N,q,w]} (w(S) - q)$$

and

$$\mu^-(q) = \min_{S \in 2^N \backslash W(N,q,w)} (q - w(S)).$$

Then for any particular quota q we have $W[N, q, \mathbf{w}] = W[N, \gamma, \mathbf{w}]$ for all $\gamma \in (q - \mu^-(q), q + \mu^+(q)]$.

Proof. (a) Let $S \in W[N, q, \mathbf{w}]$, then from the definition of $\mu^+(q)$

$$w(S) - q \geq \mu^+(q) \geq 0 \Rightarrow w(S) - q - \mu^+(q) \geq 0 \Rightarrow S \in W(N, q + \mu^+, w),$$

hence S is winning for quota $q + \mu^+(q)$. If S is winning for $q + \mu^+(q)$, then it is winning for any quota $\gamma \leq q + \mu^+(q)$.

(b) Let $S \in 2^N \backslash W[N, q, \mathbf{w}]$, then from the definition of $\mu^-(q)$

$$q - w(S) \geq \mu^-(q) \geq 0 \Rightarrow q - \mu^-(q) - w(s) \geq 0 \Rightarrow S \in 2^N \backslash W(N, q - \mu^-(q), w),$$

hence S is losing for quota $q - \mu^-(q)$. If S is losing for $q - \mu^-(q)$, then it is losing for any quota $\gamma \geq q - \mu^-(q)$.

From (a) and (b) it follows that for any $\gamma \in (q - \mu^-(q), (q - \mu^+(q)]$

$$\begin{array}{ll} S \in W(N, q, \mathbf{w}) & \Rightarrow S \in W(N, \gamma, \mathbf{w}) \\ S \in \{2^N \backslash W(N, \gamma, \mathbf{w})\} & \Rightarrow S \in \{2^N \backslash W(N, q, \mathbf{w})\} \end{array}$$

which implies that $W(N, q, \mathbf{w}) = W(N, \gamma, \mathbf{w})$. □

From Prop. 2 and 3 it follows that swing, pivot and MWC-based power indices are the same for all quotas $\gamma \in (q - \mu^-(q), q + \mu^+(q)]$. Therefore the interval of quotas $(q - \mu^-(q), q + \mu^+(q)]$ we call an *interval of stable power* for quota q. Quota $\gamma^* \in (q - \mu^-(q), q + \mu^+(q)]$ is called the marginal quota for q if $\mu^+(\gamma^*) = 0$.

Example 1. Consider a committee $[N, q, \mathbf{w}]$ where $n = 3, w_1 = 0.1, w_2 = 0.4, w_3 = 0.5$. Then $2^N = (\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\})$. Consider a simple majority quota $q = 0.51$. Then:

$$\begin{aligned}
&W[N,q,w] &&= (\{1,3\},\{2,3\},\{1,2,3\})\\
&2^N \backslash W[N,q,w] &&= (\emptyset,\{1\},\{2\},\{3\},\{1,2\})\\
&\mu^+(q) &&= \min\{w_1 + w_3 - 0.51 = 0.09, w_2 + w_3 - 0.51 = 0.39,\\
& && \quad w_1 + w_2 + w_3 - 0.51 = 0.49\} = 0.09\\
&\mu^-(q) &&= \min\{0.51 - w_1 = 0.41, 0.51 - w_2 = 0.11, 0.51 - w_3 = 0.01,\\
& && \quad 0.51 - w_1 - w_2 = 0.01\} = 0.01
\end{aligned}$$

The quota interval of stable power for quota $q = 0.51$ is $(0.5, 0.6]$. The marginal quota for $q = 0.51$ is $\gamma^* = 0.6$.

Now we define a partition of the power set 2^N into equal weight classes $\Omega_0, \Omega_1, \ldots, \Omega_r$ (such that the weight of different coalitions from the same class is the same and the weights of different coalitions from different classes are different). For the completeness set $w(\emptyset) = 0$. Consider the weight-increasing ordering of equal weight classes $\Omega^{(0)}, \Omega^(1), \ldots, \Omega^{(r)}$ such that for any $t < k$ and $S \in \Omega(t)$, $R \in \Omega(k)$ it holds that $w(S) < w(R)$. Denote $q_t = w(S)$ for any $S \in \Omega^{(t)}$, $t = 1, 2, \ldots, r$.

Proposition 4. *Let $\Omega^{(0)}, \Omega^{(1)}, \ldots, \Omega^{(r)}$ be the weight-increasing ordering of the equal weight partition of 2^N. Set $q_t = w(S)$ for any $S \in \Omega^{(t)}$, $t = 0, 1, 2, \ldots, r$. Then there is a finite number $r \le 2^n - 1$ of marginal quotas q_t and corresponding intervals of stable power $(q_{t-1}, q_t]$ such that $W[N, q_t, \mathbf{w}] \subset W[N, q_{t-1}, \mathbf{w}]$.*

Proof. Proof follows from the fact that $card(2^N) = 2^n$ and an increasing series of k real numbers $a_1, \ldots, a_k$ subdivides interval $(a_1, a_k]$ into $k-1$ segments. An analysis of voting power as a function of the quota (given voting weights) can be substituted by an analysis of voting power in a finite number of marginal quotas. □

From Prop. 4 it follows that there exists at most r distinct voting situations generating r vectors of power indices.

Example 2. In the committee from Example 1 enumerate equal weight classes of voting coalitions in the weight-increasing order (see Table 1):

Table 1. Equal weight classes of coalitions

t	$W(t)$	$w(S)$	Intervals	Marginal quotas
0	$\emptyset$	0		
1	{1}	0.1	(0,0.1]	0.1
2	{2}	0.4	(0.1,0.4]	0.4
3	{1,2},{3}	0.5	(0.4,0.5]	0.5
4	{1,3}	0.6	(0.5,0.6]	0.6
5	{2,3}	0.9	(0.6,0.9]	0.9
6	{1,2,3}	1	(0.9,1]	1

In this committee we have seven classes of equal weight voting coalitions ordered by weights and $r = 6$ intervals of stable power and corresponding marginal quotas.

Proposition 5. *Let $[N, q, w]$ be a simple quota weighted committee and $(q_{t-1}, q_t]$ is the interval of stable power for quota q. Then for any $\gamma = 1 - q_t + \varepsilon$, where $\varepsilon \in (0, q_t - q_{t-1}]$ and for all $i \in N$*

$$card(C_i(N, q, \mathbf{w})) = card(C_i(N, \gamma, \mathbf{w}))$$

and

$$card(P_i(N, q, \mathbf{w})) = card(P_i(N, \gamma, \mathbf{w})).$$

Proof. Let S be a winning coalition, k has the swing in S and $(q_{t-1}, q_t]$ is an interval of stable power for q. Then it is easy to show that $N \backslash S \cup k$ is a winning coalition, k has a swing in $N \backslash S \cup k$ and $(1 - q_t, 1 - q_{t-1}]$ is an interval of stable power for any quota $\gamma = 1 - q_t + \varepsilon$ $(0 < \varepsilon \leq q_t - q_{t-1})$. Let R be a winning coalition, j has a swing in R, and $(1 - q_t, 1 - q_{t-1}]$ is an interval of stable power for quota $\gamma = 1 - q_t + \varepsilon$ $(0 < \varepsilon \leq q_t - q_{t-1})$. Then $N \backslash R \cup j)$ is a winning coalition, j has a swing in $N \backslash R \cup j$ and $(q_{t-1}, q_t]$ is an interval of stable power for any quota $q = q_{t-1} + \tau$ where $0 < \tau \leq q_t - q_{t-1}$. □

While in $[N, q, \mathbf{w}]$ the quota q means the total weight necessary to pass a proposal (and therefore we can call it a *winning quota*), the *blocking quota* means the total weight necessary to block a proposal. If q is a winning quota and $(q_{t-1}, q_t]$ is a quota interval of stable power for q, then any voting quota $1 - q_{t-1} + \varepsilon$ (where $0 < \varepsilon \leq q_t - q_{t-1}$), is a blocking quota. From Prop. 4 it follows that the blocking power of the committee members, measured by swing and pivot-based power indices, is equal to their voting power. It is easy to show that voting power and blocking power might not be the same for power indices based on membership in minimal winning coalitions (HP and DP power indices). Let r be the number of marginal quotas, then from Prop. 5 it follows that for power indices based on swings and pivots the number of majority power indices does not exceed $int(r/2) + 1$.

Example 3. In Table 2 we provide the numbers of the decisive positions of committee members for different intervals of stable power in the committee from Example 1 (swings, pivots and MWC memberships), in Table 3 we show the corresponding relative PB, SS and HP power indices.

Proposition 6. *Let $\mathbf{q} = (q_1, q_2, \ldots, q_m)$ be the set of all majority marginal quotas in a simple weighted committee $[N, \mathbf{w}]$, and $\boldsymbol{\pi}^k$ be a vector of Shapley-Shubik relative power indices corresponding to a marginal quota q_k, then there exists a vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that:*

$$\sum_{k=1}^{m} \lambda_k = 1, \lambda_k \geq 0, \sum_{k=1}^{m} \lambda_k \pi^k = \mathbf{w}$$

Proof. Proof follows from Berg and Holler (1986, p. 426), who, using geometrical arguments, showed that there exists a finite number of different relative power indices corresponding to different majority quotas such that the vector of relative voting weights is interior with respect to the simplex generated by them. The randomized voting rule $(\mathbf{q}, \boldsymbol{\lambda})$ leads to strict proportional expected SS power. Clearly, if there exists an exact quota q^* such that $\pi_i(N, q^*, \mathbf{w}) = w_i$, we can find it among finite number of marginal majority quotas. □

Table 2. Swings, pivots and MWC memberships in quota intervals of stable power

Quota interval	Members			
	1	2	3	S
	SWINGS			
(0, 0.1]	1	1	1	3
(0.1, 0.4]	0	2	2	4
(0.4, 0.5]	1	1	3	5
(0.5, 0.6]	1	1	3	5
(0.6, 0.9]	0	2	2	4
(0.9, 1]	1	1	1	3
	PIVOTS			
(0, 0.1]	2	2	2	6
(0.1, 0.4]	0	3	3	6
(0.4, 0.5]	1	1	4	6
(0.5, 0.6]	1	1	4	6
(0.6, 0.9]	0	3	3	6
(0.9, 1]	2	2	2	6
	MWC memberships			
(0, 0.1]	1	1	1	3
(0.1, 0.4]	0	1	1	2
(04, 0.5]	1	1	1	3
(0.5, 0.6]	0	1	1	2
(0.6, 0.9]	0	1	1	2
(0.9, 1]	1	1	1	3

Table 3. SS, PB and HP relative power indices

t	Interval	Marginal quota	SS-power	PB-power	HP-power
1	(0, 0.1]	0.1	(1/3,1/3,1/3)	(1/3,1/3,1/3)	(1/3,1/3,1/3)
2	(0.1,0.4]	0.4	(0,1/2,1/2)	(0,1/2,1/2)	(0,1/2,1/2)
3	(0.4,0.5]	0.5	(1/6,1/6,4/6)	(1/5,1/5,3/5)	(1/3,1/3,1/3)
4	(0.5,0.6]	0.6	(1/6,1/6,4/6)	(1/5,1/5,3/5)	(1/4,1/4,1/2)
5	(0.6,0.9]	0.9	(0,1/2,1/2)	(0,1/2,1/2)	(0,1/2,1/2)
6	(0.9,1]	1	(1/3,1/3,1/3)	(1/3,1/3,1/3)	(1/3,1/3,1/3)

Example 4. The randomized voting rule in the committee from Example 1 applied to the SS power index:

$$\begin{array}{llll} \frac{1}{6}\lambda_1 & & +\frac{1}{3}\lambda_3 & = \frac{1}{10} \\ \frac{1}{6}\lambda_1 & +\frac{1}{2}\lambda_2 & +\frac{1}{3}\lambda_3 & = \frac{4}{10} \\ \frac{4}{6}\lambda_1 & +\frac{1}{2}\lambda_2 & +\frac{1}{3}\lambda_3 & = \frac{5}{10} \\ \lambda_1 & +\lambda_2 & +\lambda_3 & = 1 \\ & & \lambda_j & \geq 0 \end{array}$$

The system has a unique solution $\lambda_1 = \frac{1}{5}$, $\lambda_2 = \frac{3}{5}$, $\lambda_3 = \frac{1}{5}$. If there is a random mechanism selecting marginal quotas $q_1 = 0.6$, $q_2 = 0.8$, $q_3 = 1$ with probabilities$\lambda_1 = \frac{1}{5}, \lambda_2 = \frac{3}{5}, \lambda_3 = \frac{1}{5}$, then the mathematical expectation of the SS-power of the members of the committee will be equal to their relative weights (we obtain the case of strict proportional power).

Although we can apply a randomized voting rule to any relative power index, based on pivots and swings, the problem is with the interpretation of what we get For instance, the relative PB index has no probabilistic interpretation, and the randomized voting rule calculated for it by Prop. 6 does not provide the mathematical expectation of the number of swings, leading to a relative PB power equal to weights.

Example 5. We can easily verify that

$$\lambda_1 = \frac{5}{20}, \lambda_2 = \frac{12}{20}, \lambda_3 = \frac{3}{20}$$

are the probabilities for a randomized voting rule equalizing relative PB power to relative voting weights in the committee from Example 1. The swings for majority quotas and PB absolute power see in Table 4.

Table 4. Absolute PB power

	SWINGS			
Majority quotas	Member 1	Member 2	Member 3	S
(0,5, 0,6]	1	1	3	5
(0,6, 0,9]	0	2	2	4
(0,9, 1]	1	1	1	3
	ABSOLUTE PB POWER			
(0,5, 0,6]	1/4	1/4	3/4	5/4
(0,6, 0,9]	0	2/4	2/4	4/4
(0,9, 1]	1/4	1/4	1/4	3/4

Applying a randomized voting rule equalizing the relative PB power to voting weights we obtain the mathematical expectation of absolute power

$$\begin{pmatrix} 1/4 \\ 1/4 \\ 3/4 \end{pmatrix} \frac{5}{20} + \begin{pmatrix} 0 \\ 2/4 \\ 2/4 \end{pmatrix} \frac{12}{20} + \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \end{pmatrix} \frac{3}{20} = \begin{pmatrix} 8/80 \\ 32/80 \\ 42/80 \end{pmatrix},$$

which yields an expected relative power

$$\begin{pmatrix} 8/80 \\ 32/80 \\ 42/80 \end{pmatrix} \frac{1}{82/80} = \begin{pmatrix} 8/82 \\ 32/82 \\ 42/82 \end{pmatrix} \neq \begin{pmatrix} 1/10 \\ 4/10 \\ 5/10 \end{pmatrix}$$

different from voting weights.

In general, the number of majority power indices can be greater than the number of committee members, and the system

$$\sum_{k=1}^{r} \lambda_k = 1, \lambda_k \geq 0, \sum_{k=1}^{r} \lambda_k \boldsymbol{\pi}^k = \mathbf{w}$$

might not have the unique solution. To find the randomized voting rule leading to strict proportional power we can use the following optimization problem:

$$\min \sum_{i=1}^{n} abs(\sum_{k=1}^{r} \pi^k \lambda_k - w_i)$$

$$\text{s.t. } \sum_{k=1}^{r} \lambda_k = 1, \lambda_k \geq 0$$

that can be transformed into the equivalent linear programming problem (Gale, 1960):

$$\min \sum_{i=1}^{n} y_i$$

$$\begin{aligned} \text{s.t. } & \sum_{k=1}^{r} \pi_i^k \lambda_k - y_i \leq w_i \quad \text{for } i = 1, \ldots, n \\ & \sum_{k=1}^{r} \pi_i^k \lambda_k + y_i \geq w_i \quad \text{for } i = 1, \ldots, n \\ & \sum_{k=1}^{r} \lambda_k = 1 \\ & \lambda_k, y_i \geq 0 \quad \text{for } k = 1, \ldots, r, i = 1, \ldots, n \end{aligned}$$

This problem is easy to solve by standard linear programming simplex methods.

One can hardly expect that randomized voting rules leading to the strict proportional power would be adopted by actors in real voting systems. However, the design of a "fair" voting system can be based on an approximation provided by the quota generating the minimal distance between vectors of power indices and weights, which is called an *optimal quota*.

The optimal quota was introduced by Słomczyński and Życzkowski (2006, 2007) as a quota minimizing the sum of square residuals between the power indices and the voting weights by $q \in (0.5, 1]$

$$\sigma^2(q) = \sum_{i \in N} (\pi_i[N, q, \mathbf{w}] - w_i)^2.$$

Słomczyński and Życzkowski introduced the optimal quota concept within the framework of the so-called Penrose voting system as a principle of fairness in the EU Council of Ministers voting. Here power is measured by the Penrose-Banzhaf power index. The system consists of two rules (Słomczyński and Życzkowski, 2007, p. 393):

a) *The voting weight attributed to each member of the voting body of size n is proportional to the square root of the population he or she represents;*
b) *The decision of the voting body is taken if the sum of the weights of members supporting it is not less than the optimal quota.*

Looking for a quota providing a priori voting power "as close as possible" to the normalized voting weights, Słomczyński and Życzkowski (2007) are minimizing the sum of square residuals between the power indices and voting power for $q \in (0.5, 1]$. Using heuristic minimization of the sum of square residuals based on simulation (with the data obtained numerically by averaging quotas over a sample of random weights distributions), they propose the following heuristic approximation of the solution for relative PB index:

$$\underline{q} = \frac{1}{2}\left(1+\frac{1}{\sqrt{n}}\right) \leq q \leq \frac{1}{2}\left(1+\sqrt{\sum_{i\in N} w_i^2}\right) = \overline{q}.$$

Clearly $\underline{q} = \overline{q}$ if and only if all the weights are equal, but in this case any majority quota is optimal.

By the index of the fairness of a voting rule q in $[N, q, \mathbf{w}]$ we call:

$$\varphi(N, q, \mathbf{w}) = 1 - \sqrt{\frac{1}{2}\sum_i (\pi_i[N, q, \mathbf{w}] - w_i)^2}.$$

It is easy to see that $0 \leq \sqrt{\frac{1}{2}\sum_i (\pi_i[N, q, \mathbf{w}] - w_i)^2} \leq 1$ (zero in the case of the equality of weights and power, e.g. $w_1 = 1/2$, $w_2 = 1/2$, $\pi_1 = 1/2$, $\pi_2 = 1/2$, and 1 in the case of an extreme inequality of weights and power, e.g. $w_1 = 1$, $w_2 = 0$, $\pi_1 = 0$, $\pi_2 = 1$), hence $0 \leq \varphi(N, q, w) \leq 1$. We say that a voting rule q_1 is "at least as fair" as a voting rule q_2 if $\varphi(N, q_1, \mathbf{w}) \geq \varphi(N, q_2, \mathbf{w})$.[5]

Looking for a "fair" voting rule we can maximize φ, which is the same as to minimize $\sigma^2(q)$. Using marginal quotas and intervals of stable power we do not need any simulation.

Proposition 7. *Let $[N, q, w]$ be a simple quota-weighted committee and $\pi_i(N, q_t, w)$ be relative power indices for marginal quotas q_t, and q_t^* be the majority marginal quota minimizing*

$$\sum_{i\in N} (\pi_i(N, q_j, \mathbf{w}) - w_i)^2$$

($j = 1, 2, \ldots, r$, r is the number of intervals of stable power such that q_j are marginal majority quotas), then the exact solution of Słomczyński and Życzkowski's optimal quota (SZ optimal quota) problem for a particular power index used is any $\gamma \in (q_{t-1}^*, q_t^*]$ from the quota interval of stable power for q_t^*.

Proof. Proof follows from the finite number of quota intervals of stable power (Prop. 5). The quota q^* provides the best approximation of strict proportional power, that is related neither to a particular power measure nor to a specific principle of fairness. □

Example 6. In Table 5 we provide the values of the index of fairness for the majority marginal quotas in the committee from Example 1.

The exact optimal marginal quota for the PB and SS power index is the same: $q^* = q_5 = 0.9$ (and all quotas $q \in (0.6, 0.9]$).

[5] The index of fairness follows the same logic as measures of deviation from proportionality used in political science to evaluate the difference between results of an election and the composition of an elected body—e.g. Loosemore and Hanby (1971) is based on the absolute values of the deviation metric, or Gallagher (1991) using a square roots metric.

Table 5. Index of fairness

Shapley-Shubik relative power				
Member	Weight	SS for q_4=0.6	SS for q_5=0.9	SS for q_6=1
1	0.1	0.166667	0	0.333333
2	0.4	0.166667	0.5	0.333333
3	0.5	0.666667	0.5	0.333333
$\sum$	1	1	1	1
Index of fairness φ		0.791833	0.9	0.791833
Penrose-Banzhaf relative power				
Member	Weight	PB for q_4=0.6	PB for q_5=0.9	PB for q_6=1
1	0.1	0.2	0	0.333333
2	0.4	0.2	0.5	0.333333
3	0.5	0.6	0.5	0.333333
$\sum$	1	1	1	1
Index of fairness φ		0.826795	0.9	0.791833

The Słomczyński and Życzkowski heuristic approximation $\underline{q} = 0.79 \leq q \leq 0.82 = \overline{q}$ is a part of the exact interval of the optimal quota, but the next example illustrates that it might not be the case.

Example 7. Consider a committee of 4 members with weights (100/183, 49/183, 25/183, 9/183) and majority marginal quotas $q_1 = 100/183$, $q_2 = 109/183$, $q_3 = 125/183$, $q_4 = 135/183$, $q_5 = 149/183$, $q_6 = 158/183$, $q_7 = 174/183$, $q_8 = 183/183$. In Table 6 we provide the PB power indices for majority marginal quotas and corresponding values of index of fairness φ.

Table 6. Heuristic optimal quota might not work

PB relative power indices for marginal quotas								
Members	$q_1 = \frac{100}{183}$	$q_2 = \frac{109}{183}$	$q_3 = \frac{125}{183}$	$q_4 = \frac{134}{183}$	$q_5 = \frac{149}{183}$	$q_6 = \frac{158}{183}$	$q_7 = \frac{174}{183}$	$q_8 = \frac{183}{183}$
1	1	0.7	0.6	0.5	0.5	0.375	0.333	0.25
2	0	0.1	0.2	0.3	0.5	0.375	0.333	0.25
3	0	0.1	0.2	0.1	0	0.125	0.333	0.25
4	0	0.1	0	0.1	0	0.125	0	0.25
φ	0.726831	0.88205	0.94106	0.95781	0.8611	0.89186	0.84933	0.812

Comparing the values of index of fairness, we can see that optimal marginal quota in this case is $q_4 = 134/183 \approx 0.3224$, and the interval of stable power (optimal quotas) is $(125/183, 134/183] \approx (0.68306, 0.73224)$, while the Słomczyński and Życzkowski heuristic estimation gives an interval [0.75, 0.81283] that is completely outside of the exact solution.

4. Concluding Remarks

In simple quota weighted committees with a fixed number of members and voting weights there exists a finite number r of different quota intervals of stable power ($r \leq 2^n - 1$) generating a finite number of power indices vectors. For power indices with a voting power equal to blocking power the number of different power indices vectors corresponding to majority quotas is equal at most to int $(r/2) + 1$.

If the fair distribution of voting weights is defined, then the fair distribution of voting power is achieved by the quota that maximizes the index of fairness (minimizes the distance between relative voting weights and relative voting power). The index of fairness is not a monotonic function of the quota.

The problem of optimal quota has an exact solution via the finite number of majority marginal quotas. Słomczyński and Życzkowski introduced an optimal quota concept within the framework of the so called Penrose voting system as a principle of fairness in the EU Council of Ministers voting and related it exclusively to the Penrose-Banzhaf power index and the square root rule. However, the fairness in voting systems and approximation of strict proportional power is not exclusively related to the Penrose square-root rule and the Penrose-Banzhaf definition of power, as it is usually done in discussions about EU voting rules. In this paper it is treated in a more general setting as a property of any simple quota weighted committee and any well-defined power measure. Fairness and its approximation by optimal quota are not specific properties of the Penrose-Banzhaf power index and square root rule.

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Subgame Consistent Solution for Random-Horizon Cooperative Dynamic Games

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Abstract. In cooperative dynamic games a stringent condition – that of *subgame consistency* – is required for a dynamically stable cooperative solution. In particular, a cooperative solution is subgame consistent if an extension of the solution policy to a subgame starting at a later time with a state brought about by prior optimal behavior would remain optimal. This paper extends subgame consistent solutions to dynamic (discrete-time) cooperative games with random horizon. In the analysis new forms of the Bellman equation and the Isaacs-Bellman equation in discrete-time are derived. Subgame consistent cooperative solutions are obtained for this class of dynamic games. Analytically tractable payoff distribution mechanisms which lead to the realization of these solutions are developed. This is the first time that subgame consistent solutions for cooperative dynamic games with random horizon are presented.

Keywords: Cooperative dynamic games, random horizon, subgame consistency.
AMS Subject Classifications. Primary 91A12, Secondary 91A25.

1. Introduction

Cooperative games suggest the possibility of socially optimal and group efficient solutions to decision problems involving strategic action. In cooperative dynamic games a stringent condition – that of *subgame consistency* – is required for a dynamically stable cooperative solution. In particular, a cooperative solution is subgame consistent if an extension of the solution policy to a subgame starting at a later time with a state brought about by prior optimal behavior would remain optimal. In particular dynamic consistency ensures that as the game proceeds players are guided by the same optimality principle at each instant of time, and hence do not possess incentives to deviate from the previously adopted optimal behavior. A rigorous framework for the study of subgame-consistent solutions in cooperative stochastic differential games (which are in continuous-time) was established in Yeung and Petrosyan (2004 and 2006). A generalized theorem was developed for the derivation of an analytically tractable "payoff distribution procedure" leading to dynamically consistent solutions. Cooperative games with subgame consistent solutions are presented in Petrosyan and Yeung (2007), Yeung (2007 and 2008), and Yeung and Petrsoyan (2006).

In discrete-time dynamic games, Basar and Ho (1974) examined informational properties of the Nash solutions of stochastic nonzero-sum games and Basar (1978) developed equilibrium solution of linear-quadratic stochastic games with noisy observation. Krawczyk and Tidball (2006) considered a dynamic game of water allocation. Nie et al. (2006) considered dynamic programming approach to discrete time dynamic Stackelberg games. Dockner and Nishimura (1999) and Rubio and Ulph (2007) presented discrete-time dynamic game for pollution management. Dutta and Radner (2006) presented a discrete-time dynamic game used to study global warming. Ehtamo and Hamalainen (1993) examined cooperative incentive equilibrium for a dynamic resource game. Yeung and Petrosyan (2010) had developed a generalized theorem for the derivation of an analytically tractable "payoff distribution procedure" leading to dynamically consistent solutions for cooperative stochastic dynamic games.

In this paper, we extend subgame consistent solutions to discrete-time dynamic cooperative games with random horizon. In many game situations, the terminal time of the game is not known with certainty. Examples of this kind of problems include uncertainty in the renewal of lease, the terms of offices of elected authorities/directorships, contract renewal and continuation of agreements subjected to periodic negotiations. Petrosyan and Murzov (1966) first developed the Bellman Isaacs equations under random horizon for zero-sum differential games. Petrosyan and Shevkoplyas (2003) gave the first analysis of dynamically consistent solutions for cooperative differential with random duration. Shevkoplyas (2011) considered Shapley value in cooperative differential games with random horizon. In this paper, a class of discrete-time dynamic games with random duration is formulated. A dynamic programming technique for solving inter-temporal problems with random horizon is developed to serve as the foundation of solving the game problem. To characterize a noncooperative equilibrium, a set of random duration discrete-time Isaacs-Bellman equations is derived.

Moreover, subgame consistent cooperative solutions are derived for dynamic games with random horizon. Analytically tractable payoff distribution mechanisms which lead to the realization of these solutions are derived. It represents the first attempt to seek subgame consistent solution for cooperative dynamic games with random horizons. This analysis widens the application of cooperative dynamic game theory to problems where the game horizon is random. The organization of the paper is as follows. Game formulation and derivation of dynamic programming techniques for random horizon problems are provided in Section 2. A feedback Nash equilibrium is characterized for dynamic games with random horizon in Section 3. Dynamic cooperation under random horizon is presented in Section 4. Subgame consistent solutions and payment mechanism leading to these solutions are obtained in Section 5. Concluding remarks are given in Section 6.

2. Game Formulation and Dynamic Programming with Random Horizon

In this section, we first formulate a class of dynamic games with random duration. Then we develop a dynamic programming technique for solving inter-temporal problems with random horizon which will serve as the foundation of solving the game problem.

2.1. Game Formulation

The $n-$person dynamic game to be considered is a $\hat{T}$ stage game where $\hat{T}$ is random variable with range $\{1, 2, \cdots, T\}$ and corresponding probabilities $\{\theta_1, \theta_2, \cdots, \theta_T\}$. Conditional upon the reaching of stage τ, the probability of the game would last up to stages $\tau, \tau+1, \cdots, T$ becomes respectively

$$\frac{\theta_\tau}{\sum\limits_{\zeta=\tau}^{T} \theta_\zeta}, \frac{\theta_{\tau+1}}{\sum\limits_{\zeta=\tau}^{T} \theta_\zeta}, \cdots, \frac{\theta_T}{\sum\limits_{\zeta=\tau}^{T} \theta_\zeta}.$$

The payoff of player i at stage $k \in \{1, 2, \cdots, T\}$ is $g_k^i[x_k, u_k^1, u_k^2, \cdots, u_k^n, x_{k+1}]$. When the game ends after stage $\hat{T}$, player i will receive a terminal payment $q^i_{\hat{T}+1}(x_{\hat{T}+1})$ in stage $\hat{T}+1$.

The state space of the game is $X \in R^m$ and the state dynamics of the game is characterized by the difference equation:

$$x_{k+1} = f_k(x_k, u_k^1, u_k^2, \cdots, u_k^n) \tag{1}$$

,

for $k \in \{1, 2, \cdots, T\} \equiv \mathrm{T}$ and $x_1 = x^0$,

where $u_k^i \in R^{m_i}$ is the control vector of player i at stage k and $x_k \in X$ is the state.

The objective of player i is

$$E\left\{\sum_{k=1}^{\hat{T}} g_k^i[x_k, u_k^1, u_k^2, \cdots, u_k^n, x_{k+1}] + q^i_{\hat{T}+1}(x_{\hat{T}+1})\right\}$$
$$= \sum_{\hat{T}=1}^{T} \theta_{\hat{T}} \left\{\sum_{k=1}^{\hat{T}} g_k^i[x_k, u_k^1, u_k^2, \cdots, u_k^n, x_{k+1}] + q^i_{\hat{T}+1}(x_{\hat{T}+1})\right\}, \tag{2}$$

for $i \in \{1, 2, \cdots, n\} \equiv N$.

In the continuous time Petrosyan and Shevkoplyas (2003) game the game horizon can reach infinity. In the discrete-time game (1)-(2), the game horizon is random but finite. To solve the game (1)-(2), we first have to derive a dynamic programming technique for solving a random horizon problem.

2.2. Dynamic Programming for Random Horizon Problem

Consider the case when $n = 1$ in the system (1)-(2). The payoff at stage $k \in \{1, 2, \cdots, T\}$ is $g_k[x_k, u_k, x_{k+1}]$. If the game ends after stage $\hat{T}$, the decision maker will receive a terminal payment $q_{\hat{T}+1}(x_{\hat{T}+1})$ in stage $\hat{T}+1$.

The problem can be formulized as the maximization of the expected payoff:

$$E\left\{\sum_{k=1}^{\hat{T}} g_k[x_k,u_k,x_{k+1}] + q_{\hat{T}+1}(x_{\hat{T}+1})\right\}$$
$$= \sum_{\hat{T}=1}^{T} \theta_{\hat{T}} \left\{\sum_{k=1}^{\hat{T}} g_k[x_k,u_k,x_{k+1}] + q_{\hat{T}+1}(x_{\hat{T}+1})\right\}, \quad (3)$$

subject to the dynamics

$$x_{k+1} = f_k(x_k,u_k), \quad x_1 = \bar{x}_0. \quad (4)$$

Now consider the case when stage τ has arrived and the state is $\bar{x}_\tau$. The problem can formulized as the maximization of the payoff:

$$E\left\{\sum_{k=\tau}^{\hat{T}} g_k[x_k,u_k,x_{k+1}] + q_{\hat{T}+1}(x_{\hat{T}+1})\right\}$$
$$= \sum_{\hat{T}=\tau}^{T} \frac{\theta_{\hat{T}}}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta} \left\{\sum_{k=\tau}^{\hat{T}} g_k[x_k,u_k,x_{k+1}] + q_{\hat{T}+1}(x_{\hat{T}+1})\right\}, \quad (5)$$

subject to the dynamics

$$x_{k+1} = f_k(x_k,u_k), \quad x_\tau = \bar{x}_\tau. \quad (6)$$

We define the value function $V(\tau,x)$ and the set of strategies { $u_k = \psi_k(x)$, for $k \in \{\tau, \tau+1, \cdots, \mathrm{T}\}$ } which provides an optimal solution as follows:

$$V(\tau,x) = \max_{u_\tau,u_{\tau+1},\cdots,u_{\hat{T}}} E\left\{\sum_{k=\tau}^{\hat{T}} g_k[x_k,u_k,x_{k+1}] + q_{\hat{T}+1}(x_{\hat{T}+1}) \,|\, x_\tau = x\right\}$$
$$= \max_{u_\tau,u_{\tau+1},\cdots,u_{\hat{T}}} \sum_{\hat{T}=\tau}^{T} \frac{\theta_{\hat{T}}}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta} \left\{\sum_{k=\tau}^{\hat{T}} g_k[x_k,u_k,x_{k+1}] + q_{\hat{T}+1}(x_{\hat{T}+1}) \,|\, x_\tau = x\right\}$$
$$= \sum_{\hat{T}=\tau}^{T} \frac{\theta_{\hat{T}}}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta} \left\{\sum_{k=\tau}^{\hat{T}} g_k[x_k^*,\psi_k(x_k),x_{k+1}^*] + q_{\hat{T}+1}(x_{\hat{T}+1}^*) \,|\, x_\tau^* = x\right\}, \quad (7)$$

for $\tau \in \mathrm{T}$, where $x_{k+1}^* = f_k[x_k^*, \psi_k(x_k^*)]$, $x_1^* = \bar{x}_0$.

A Theorem characterizing an optimal solution to the random-horizon problem (3)-(4) is provided in the theorem below.

Theorem 1. *A set of strategies* $\{u_k = \psi_k(x)$, *for* $k \in \mathrm{T}\}$ *provides an optimal solution to the problem (3)-(4) if there exist functions* $V(k,x)$, *for* $k \in \mathrm{T}$, *such that the following recursive relations are satisfied:*

$$
\begin{aligned}
&V(T+1,x) = q_{T+1}(x),\\
&V(T,x) = \max_{u_T}\{g_T[x,u_T,f_T(x,u_T)] + V[T+1,f_T(x,u_T)]\},\\
&V(\tau,x) = \max_{u_\tau}\left\{ g_\tau[x,u_\tau,f_\tau(x,u_\tau)] + \frac{\theta_\tau}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta} q_{\tau+1}[f_\tau(x,u_\tau)] \right.\\
&\left. + \frac{\sum\limits_{\zeta=\tau+1}^{T}\theta_\zeta}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta} V[\tau+1,f_\tau(x,u_\tau)] \right\}, \quad \textit{for} \quad \tau \in \{1,2,\cdots,T-1\}.
\end{aligned}
\tag{8}
$$

Proof. By definition, the value function at stage $T+1$ is

$$V(T+1,x) = q_{T+1}(x).$$

We first consider the case when the last stage T has arrived.
The problem then becomes

$$\max_{u_T}\left\{ \ g_T[x_T,u_T,x_{T+1}] + q_{T+1}(x_{T+1}) \ \right\}$$

subject to

$$x_{T+1} = f_T(x_T,u_T), \quad x_T = \bar{x}_T. \tag{9}$$

Using $V(T+1,x)$=$q_{T+1}(x)$, the problem in (9) can be formulated as a single stage problem

$$\max_{u_T}\left\{ \ g_T[x_T,u_T,f_T(x,u_T)] + V(T+1,f_T(x,u_T) \ \right\},$$

with $x_T = x$.

Hence we have $V(T,x)$= $\max_{u_T}\left\{ \ g_T[x,u_T,f_T(x,u_T)] + V[T+1,f_T(x,u_T)] \ \right\}$.

Now consider the problem in stage $\tau\in\{1,2,\cdots,T-1\}$ in which one have to maximize

$$\sum_{\hat{T}=\tau}^{T} \frac{\theta_{\hat{T}}}{\sum\limits_{\zeta=\tau}^{T} \theta_\zeta} \left\{ \sum_{k=\tau}^{\hat{T}} g_k[x_k, u_k, x_{k+1}] + q_{\hat{T}+1}(x_{\hat{T}+1}) \right\}$$

$$= g_\tau[x_\tau, u_\tau, x_{\tau+1}] + \frac{\theta_\tau}{\sum\limits_{\zeta=\tau}^{T} \theta_\zeta} q_{\tau+1}(x_{\tau+1})$$

$$+ \frac{\sum\limits_{\zeta=\tau+1}^{T} \theta_{\hat{T}}}{\sum\limits_{\zeta=\tau}^{T} \theta_\zeta} \left\{ \sum_{k=\tau+1}^{\hat{T}} g_k[x_k, u_k, x_{k+1}] + q_{\hat{T}+1}(x_{\hat{T}+1}) \right\}$$

$$= g_\tau[x_\tau, u_\tau, x_{\tau+1}] + \frac{\theta_\tau}{\sum\limits_{\zeta=\tau}^{T} \theta_\zeta} q_{\tau+1}(x_{\tau+1})$$

$$+ \frac{\sum\limits_{\zeta=\tau+1}^{T} \theta_\zeta}{\sum\limits_{\zeta=\tau}^{T} \theta_\zeta} \frac{\sum\limits_{\zeta=\tau+1}^{T} \theta_{\hat{T}}}{\sum\limits_{\zeta=\tau+1}^{T} \theta_\zeta} \left\{ \sum_{k=\tau+1}^{\hat{T}} g_k[x_k, u_k, x_{k+1}] + q_{\hat{T}+1}(x_{\hat{T}+1}) \right\}. \quad (10)$$

Using $V(\tau+1, x)$ characterized, the problem (10) can be formulated as a single stage problem

$$\max_{u_\tau} \left\{ g_\tau[x, u_\tau, f_\tau(x, u_\tau)] + \frac{\theta_\tau}{\sum\limits_{\zeta=\tau}^{T} \theta_\zeta} q_{\tau+1}[f_\tau(x, u_\tau)] + \frac{\sum\limits_{\zeta=\tau+1}^{T} \theta_\zeta}{\sum\limits_{\zeta=\tau}^{T} \theta_\zeta} V[\tau+1, f_\tau(x, u_\tau)] \right\}, \quad (11)$$

with $x_\tau = x$.

Hence we have

$$V(\tau, x) = \max_{u_\tau} \left\{ g_\tau[x, u_\tau, f_\tau(x, u_\tau)] + \frac{\theta_\tau}{\sum\limits_{\zeta=\tau}^{T} \theta_\zeta} q_{\tau+1}[f_\tau(x, u_\tau)] \right.$$

$$\left. + \frac{\sum\limits_{\zeta=\tau+1}^{T} \theta_\zeta}{\sum\limits_{\zeta=\tau}^{T} \theta_\zeta} V[\tau+1, f_\tau(x, u_\tau)] \right\}, \quad \text{for} \quad \tau \in \{1, 2, \cdots, T-2\}. \quad (12)$$

and Theorem 1 follows.

Theorem 1 yields a set of Bellman equations for random horizon problems (3)-(4).

3. Random Horizon Feedback Nash Equilibrium

In this subsection, we investigate the noncooperative outcome of the discrete-time game (1)-(2). In particular, a feedback Nash equilibrium of the game can be characterized by the following theorem.

Theorem 2. *A set of strategies $\{\phi_k^i(x)$, for $k \in \mathrm{T}$ and $i \in N\}$ provides a feedback Nash equilibrium solution to the game (1)-(2) if there exist functions $V^i(k,x)$, for $k \in \mathrm{T}$ and $i \in N$, such that the following recursive relations are satisfied:*

$$\begin{aligned}
&V^i(T+1,x) = q_{T+1}^i(x),\\
&V^i(T,x) = \max_{u_T^i}\{g_T^i[x,\phi_T^1(x),\phi_T^2(x),\cdots,\phi_T^{i-1}(x),u_T^i,\phi_T^{i+1}(x),\cdots\\
&\qquad\cdots,\phi_T^n(x),\tilde f_T^i(x,u_T^i)] + q_{T+1}^i[\tilde f_T^i(x,u_T^i)]\},\\
&V^i(\tau,x) = \max_{u_\tau^i}\{g_\tau^i[x,\phi_\tau^1(x),\phi_\tau^2(x),\cdots,\phi_\tau^{i-1}(x),u_\tau^i,\phi_\tau^{i+1}(x),\cdots\\
&\qquad\cdots,\phi_\tau^n(x),\tilde f_\tau^i(x,u_\tau^i)] + \frac{\theta_\tau}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta} q_{\tau+1}^i[\tilde f_\tau^i(x,u_\tau^i)] + \frac{\sum\limits_{\zeta=\tau+1}^{T}\theta_\zeta}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta} V^i[\tau+1,\tilde f_\tau^i(x,u_\tau^i)]\},\\
&\textit{for}\quad \tau \in \{1,2,\cdots,T-1\},
\end{aligned} \tag{13}$$

where $\tilde f_k^i(x,u_k^i) = f_k[x,\phi_k^1(x),\phi_k^2(x),\cdots,\phi_k^{i-1}(x),u_k^i,\phi_k^{i+1}(x),\cdots,\phi_k^n(x)]$.

Proof. The conditions in (13) shows that the random horizon dynamic programming result in Theorem 1 holds for each player given other players' equilibrium strategies. Hence the conditions of a Nash (1951) equilibrium are satisfied and Theorem 2 follows.

The set of equations in (13) represents the discrete analogue of the Isaacs-Bellman equations under random horizon.

Substituting the set of feedback Nash equilibrium strategies $\{\phi_k^i(x)$, for $k \in \mathrm{T}$ and $i \in N\}$ into the players' payoff yields

$$V^i(\tau,x) = E\left\{\sum_{k=\tau}^{\hat T} g_k^i[x_k,\phi_k^1(x_k),\phi_k^2(x_k),\cdots,\phi_k^n(x_k),x_{k+1}] + q_{\hat T+1}^i(x_{\hat T+1})\right\}$$

$$= \sum_{\hat T=\tau}^{T}\frac{\theta_{\hat T}}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta}\left\{\sum_{k=\tau}^{\hat T} g_k^i[x_k,\phi_k^1(x_k),\phi_k^2(x_k),\cdots,\phi_k^n(x_k),x_{k+1}] + q_{\hat T+1}^i(x_{\hat T+1})\right\},\ i \in N,$$

where $x_\tau = x$. The $V^i(\tau,x)$ value function gives the expected game equilibrium payoff to player i from stage τ to the end of the game.

4. Dynamic Cooperation under Random Horizon

Now consider the case when the players agree to cooperate and distribute the payoff among themselves according to an optimality principle. Two essential properties that a cooperative scheme has to satisfy are group optimality and individual rationality.

4.1. Group Optimality

Maximizing the players' expected joint payoff guarantees group optimality in a game where payoffs are transferable. To maximize their expected joint payoff the players have to solve the discrete-time dynamic programming problem of maximizing

$$
\begin{aligned}
&E\left\{\sum_{j=1}^{n}\left[\sum_{k=1}^{\hat{T}} g_k^j[x_k, u_k^1, u_k^2, \cdots, u_k^n, x_{k+1}] + q_{\hat{T}+1}^j(x_{\hat{T}+1})\right]\right\} \\
&\qquad = \sum_{j=1}^{n}\sum_{\hat{T}=1}^{T}\theta_{\hat{T}}\left\{\sum_{k=1}^{\hat{T}} g_k^j[x_k, u_k^1, u_k^2, \cdots, u_k^n, x_{k+1}] + q_{\hat{T}+1}^j(x_{\hat{T}+1})\right\} \qquad (14)
\end{aligned}
$$

subject to (1).

Invoking the random horizon dynamic programming method in Theorem 1 we can characterize an optimal solution to the problem (14)-(1) as

Corollary 1. *A set of strategies $\{u_k^{i*} = \psi_k^i(x)$, for $k \in \kappa$ and $i \in N\}$ provides an optimal solution to the problem (14)-(1) if there exist functions $W(k, x)$, for $k \in \mathrm{T}$, such that the following recursive relations are satisfied:*

$$
\begin{aligned}
W(T+1, x) &= \sum_{j=1}^{n} q_{T+1}^j(x), \\
W(T, x) &= \max_{u_T^1, u_T^2, \cdots, u_T^n}\{\sum_{j=1}^{n} g_T^j[x, u_T^1, u_T^2, \cdots, u_T^n, f_T(x, u_T, u_T^1, u_T^2, \cdots, u_T^n)] \\
&\quad + q_{T+1}[f_T(x, u_T, u_T^1, u_T^2, \cdots, u_T^n)]\}, \\
W(\tau, x) &= \max_{u_\tau^1, u_\tau^2, \cdots, u_\tau^n}\{\sum_{j=1}^{n}\left[g_\tau^j[x, u_\tau^1, u_\tau^2, \cdots, u_\tau^n, f_\tau(x, u_\tau^1, u_\tau^2, \cdots, u_\tau^n)]\right. \\
&\quad + \frac{\theta_\tau}{\sum_{\zeta=\tau}^{T}\theta_\zeta} q_{\tau+1}^j[f_\tau(x, u_\tau^1, u_\tau^2, \cdots, u_\tau^n)]] \qquad (15) \\
&+ \frac{\sum_{\zeta=\tau+1}^{T}\theta_\zeta}{\sum_{\zeta=\tau}^{T}\theta_\zeta} W[\tau+1, f_\tau(x, u_\tau^1, u_\tau^2, \cdots, u_\tau^n)]\}, \quad \mathit{for} \quad \tau \in \{1, 2, \cdots, T-1\}.
\end{aligned}
$$

Substituting the optimal control $\{\psi_k^i(x)$, for $k \in \mathrm{T}$ and $i \in N\}$ into the state dynamics (1), one can obtain the dynamics of the cooperative trajectory as:

$$
x_{k+1} = f_k[x_k, \psi_k^1(x_k), \psi_k^2(x_k), \cdots, \psi_k^n(x_k)], \qquad (16)
$$

for $k \in \mathrm{T}$ and $x_1 = x^0$.

We use x_k^* to denote the solution generated by (16).

Using the set of optimal strategies $\{\psi_k^i(x_k^*)$, for $k \in \mathrm{T}$ and $i \in N\}$ one can obtain the expected cooperative payoff as

$$W(\tau, x) = E\left\{\sum_{j=1}^{n}\left[\sum_{k=\tau}^{\hat{T}} g_k^j[x_k^*, \phi_k^1(x_k^*), \phi_k^2(x_k^*), \cdots, \phi_k^n(x_k^*), x_{k+1}^*] + q_{\hat{T}+1}^j(x_{\hat{T}+1}^*)\right]\right\}$$
$$= \sum_{j=1}^{n}\sum_{\hat{T}=\tau}^{T} \frac{\theta_{\hat{T}}}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta}\left\{\sum_{k=\tau}^{\hat{T}} g_k^j[x_k^*, \phi_k^1(x_k^*), \phi_k^2(x_k^*), \cdots, \phi_k^n(x_k^*), x_{k+1}^*] + q_{\hat{T}+1}^j(x_{\hat{T}+1}^*)\right\}. \tag{17}$$

4.2. Individual Rationality

The players then have to agree to an optimality principle in distributing the total cooperative payoff among themselves. For individual rationality to be upheld the imputation (see von Neumann and Morgenstern (1944)) a player receives under cooperation have to be no less than his expected noncooperative payoff along the cooperative state trajectory.

Let $\xi(\cdot,\cdot)$ denote the imputation vector guiding the distribution of the total cooperative payoff under the agreed-upon optimality principle along the cooperative trajectory $\{x_k^*\}_{k=1}^{T}$. At stage τ, the imputation vector according to $\xi(\cdot,\cdot)$ is $\xi(\tau, x_\tau^*) = [\xi^1(\tau, x_\tau^*), \xi^2(\tau, x_\tau^*), \cdots, \xi^n(\tau, x_\tau^*)]$, for $\tau \in \mathrm{T}$.

There is a variety of imputations that the players can agree upon. For examples, (i) the players may share the excess of the total cooperative payoff over the sum of individual noncooperative payoffs equally, or (ii) they may share the total cooperative payoff proportional to their noncooperative payoffs or a linear combination of (i) and (ii).

For individual rationality to be maintained throughout all the stages $\tau \in \mathrm{T}$, it is required that:

$$\xi^i(\tau, x_\tau^*) \geq V^i(\tau, x_\tau^*), \quad \text{for} \quad i \in N \quad \text{and} \quad \tau \in \mathrm{T}.$$

To satisfy group optimality, the imputation vector has to satisfy

$$W(\tau, x_\tau^*) = \sum_{j=1}^{n} \xi^j(\tau, x_\tau^*), \quad \text{for} \quad \tau \in \mathrm{T}.$$

5. Subgame Consistent Solutions and Payment Mechanism

To guarantee dynamical stability in a dynamic cooperation scheme, the solution has to satisfy the property of subgame consistency. A cooperative solution is subgame-consistent if an extension of the solution policy to a subgame starting at a later time with a state along the optimal cooperative trajectory would remain optimal. In particular, subgame consistency ensures that as the game proceeds players are guided by the same optimality principle at each stage of the game, and hence do not possess incentives to deviate from the previously adopted optimal behavior.

Therefore for subgame consistency to be satisfied, the imputation $\xi(\cdot,\cdot)$ according to the original optimality principle has to be maintained along the cooperative trajectory $\{\, x_k^* \,\}_{k=1}^{T}$. Let the imputation governed by the agreed upon optimality principle be

$$\xi(k, x_k^*) = [\xi^1(k, x_k^*), \xi^2(k, x_k^*), \cdots, \xi^n(k, x_k^*)] \quad \text{at stage} \quad k, \quad \text{for} \quad k \in \mathrm{T}. \tag{18}$$

Crucial to the analysis is the formulation of a payment mechanism so that the imputation in (18) can be realized as the game proceeds.

Following the analysis of Yeung and Petrosyan (2010), we formulate a discrete-time random-horizon Payoff Distribution Procedure (PDP) so that the agreed-upon imputations (18) can be realized. Let $B_k^i(x_k^*)$ denote the payment that player i will received at stage k under the cooperative agreement.

The payment scheme involving $B_k^i(x_k^*)$ constitutes a PDP in the sense that along the cooperative trajectory $\{x_k^*\}_{k=1}^{T}$ the imputation to player i over the stages from k to T can be expressed as:

$$\begin{aligned} \xi^i(\tau, x_\tau^*) &= E\left\{ \sum_{k=\tau}^{\hat{T}} B_k^i(x_k^*) + q_{\hat{T}+1}^i(x_{\hat{T}+1}^*) \right\} \\ &= \sum_{\hat{T}=\tau}^{T} \frac{\theta_{\hat{T}}}{\sum\limits_{\zeta=\tau}^{T} \theta_\zeta} \left\{ \sum_{k=\tau}^{\hat{T}} B_k^i(x_k^*) + q_{\hat{T}+1}^i(x_{\hat{T}+1}^*) \right\}, \quad i \in N \quad \text{and} \quad k \in \mathrm{T}. \end{aligned} \tag{19}$$

If the game lasts up to stage T, then at stage $T+1$, player i will receive a terminal payment $q_{T+1}^i(x_{T+1}^*)$ and $B_{T+1}^i(x_{T+1}^*) = 0$. Hence the imputation $\xi^i(T+1, x_{T+1}^*)$ equals $q_{T+1}^i(x_{T+1}^*)$.

Theorem 3. *A payment equaling*

$$B_\tau^i(x_\tau^*) = \xi^i(\tau, x_\tau^*) - \frac{\sum\limits_{\zeta=\tau+1}^{T} \theta_\zeta}{\sum\limits_{\zeta=\tau}^{T} \theta_\zeta} \xi^i(\tau+1, x_{\tau+1}^*) - \frac{\theta_\tau}{\sum\limits_{\zeta=\tau}^{T} \theta_\zeta} q_{\tau+1}^i(x_{\tau+1}^*), \quad \textit{for} \quad i \in N, \tag{20}$$

given to player i at stage $\tau \in \mathrm{T}$ would lead to the realization of the imputation $\xi(\tau, x_\tau^*)$in (18).

Proof. Using (19) we obtain

$$\xi^i(\tau,x_\tau^*)=B_\tau^i(x_\tau^*)+\frac{\theta_\tau}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta}q_{\tau+1}^i(x_{\tau+1}^*)+\frac{\sum\limits_{\zeta=\tau+1}^{T}\theta_{\hat{T}}}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta}\left\{\sum_{k=\tau+1}^{\hat{T}}B_k^i(x_k^*)+q_{\hat{T}+1}^i(x_{\hat{T}+1}^*)\right\}$$

$$=B_\tau^i(x_\tau^*)+\frac{\theta_\tau}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta}q_{\tau+1}^i(x_{\tau+1}^*)+\frac{\sum\limits_{\zeta=\tau+1}^{T}\theta_\zeta}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta}\frac{\sum\limits_{\zeta=\tau+1}^{T}\theta_{\hat{T}}}{\sum\limits_{\zeta=\tau+1}^{T}\theta_\zeta}\left\{\sum_{k=\tau+1}^{\hat{T}}B_k^i(x_k^*)+q_{\hat{T}+1}^i(x_{\hat{T}+1}^*)\right\}. \tag{21}$$

Invoking the definition of $\xi^i(\tau,x_\tau^*)$ in (19), we can express (21) as

$$\xi^i(\tau,x_\tau^*)=B_\tau^i(x_\tau^*)+\frac{\theta_\tau}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta}q_{\tau+1}^i(x_{\tau+1}^*)+\frac{\sum\limits_{\zeta=\tau+1}^{T}\theta_\zeta}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta}\xi^i(k+1,x_{k+1}^*). \tag{22}$$

Using (22) one can readily obtain (20). Hence Theorem 3 follows.

Note that the payoff distribution procedure $B_\tau^i(x_\tau^*)$ in (20) would give rise to the agreed-upon imputation

$$\xi(k,x_k^*)=[\xi^1(k,x_k^*),\xi^2(k,x_k^*),\cdots,\xi^n(k,x_k^*)] \text{ at stage } k, \text{ for } k\in \mathrm{T}.$$

Therefore subgame consistency is satisfied,

When all players are using the cooperative strategies, the payoff that player i will directly received at stage $k\in$ T is

$$g_k^i[x_k^*,\psi_k^1(x_k^*),\psi_k^2(x_k^*),\cdots,\psi_k^n(x_k^*),x_{k+1}^*].$$

However, according to the agreed upon imputation, player i is to received $B_k^i(x_k^*)$ at stage k. Therefore a side-payment

$$\varpi_k^i(x_k^*)=B_k^i(x_k^*)-g_k^i[x_k^*,\psi_k^1(x_k^*),\psi_k^2(x_k^*),\cdots,\psi_k^n(x_k^*),x_k^*], \text{ for } k\in \mathrm{T} \text{ and } i\in N, \tag{23}$$

will be given to player i.

6. Concluding Remarks

In this paper, we extend subgame consistent solutions to dynamic cooperative games with random horizon. Note that time consistency refers to the condition that the optimality principle agreed upon at the outset must remain effective throughout the game, at any instant of time along the optimal state trajectory. In the presence of stochastic elements, subgame consistency is required. In particular, subgame consistency requires that the optimality principle agreed upon at the outset must remain effective in any subgame with a later starting time and a realizable state brought about by prior optimal behavior along the optimal cooperative trajectory. However, the optimal cooperative trajectory (16) is a deterministic difference equation with a random stopping time $k+1 \in \{2, 3, \cdots T\}$. Hence the subgame consistency notion in this analysis is a form of optimal-trajectory-subgame consistency.

New forms of the Bellman equation and the Isaacs-Bellman equation are derived. Subgame consistent cooperative solutions are derived for dynamic games with random horizon. Analytically tractable payoff distribution mechanisms which lead to the realization of these solutions are derived. The analysis widens the application of cooperative dynamic game theory to problems where the game horizon is random. Finally, this is the first time that subgame consistent solutions are derived for cooperative dynamic games with random horizon further research along this line is expected.

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Efficient CS-Values Based on Consensus and Shapley Values

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Abstract. Two efficient values for transferable utility games with coalition structure are introduced and axiomatized by means of modified versions of null player property and four standard axioms (efficiency, additivity, external symmetry and internal symmetry). The first value uses the consensus value in game between coalitions and the Shapley value in games within coalitions. The second one uses the consensus and Shapley values in inverse order.

Keywords: coalition structure, coalition value, consensus value, Shapley value, axiomatization.

1. Introduction

In cooperative transferable utility games with coalition structure (CS-game) it is supposed that players are already partitioned into groups. As the allocation rule for such game a coalitional value (CS-value) can be chosen. If it is component efficient, the worth of every structural component is distributed among its members. The player's payoff does not depend on the coalitions formed by players outside his component. The efficient CS-values predict the grand coalition to form, i.e., the sum of individual payoffs equals to the grand coalition's worth. The components of coalition structure are interpreted as a priori unions (blocks, pressure groups, superplayers) which make decisions as a single player.

In this paper, we focus on efficient CS-values. The first one, proposed by Owen (Owen, 1977), is a generalization of the Shapley value (Shapley, 1953) to the coalitional context. Owen defined the CS-value by decomposing a CS-game into an external game played by the structural components (quotient game) and an internal game that is induced on the players within a component. The Owen value splits the grand coalition's worth among the components and then total reward of each component is shared among its members. Both payoffs are given by the Shapley value. In addition to efficiency, the Owen value is characterized by the additivity, external symmetry, internal symmetry and null player property (every null player gets nothing even though its coalition is in very strong position). But in real life the null player axiom does not seem good (Ju, et al., 2006; Kamijo, 2009; Hernandez-Lamoneda, et al., 2008). After the Owen value alternative CS-values have been presented in the literature (their description can be found, for example, in (Gomez-Rua and Vidal-Puga, 2008)), however they either satisfy the null player axiom or do not reflect the outside options of players within the same structural coalition.

Recently, Kamijo (Kamijo, 2007; Kamijo, 2009) introduced and axiomatized two efficient CS-values without null player property. They are called the two-step Shapley value and the collective value. An alternative characterization of the two-step Shapley value (Kamijo, 2009) is proposed in (Calvo and Gutierrez, 2010). The two-step Shapley value, as well as the Owen value, assumes that each structural component acts like a single player and receives its Shapley value in the quotient game for coalitions. Firstly, every player gets his Shapley value in the game restricted within his structural component. Secondly, the pure surplus received by the component in the quotient game is equally shared among the intra-coalition members. The collective value (Kamijo, 2007) differs from the two-step Shapley value that the Shapley value is replaced with weighed Shapley value. Both coalition values are insensitive to outside options. Let's assume that two players belong to the same structural component and one of them dominates another. If these players are symmetric in the restricted game inside component, then according to the two-step Shapley value and the collective value they receive equal payoffs.

The consensus CS-value proposed in (Zinchenko, et al., 2010) is Owen-type extension of the consensus value (Ju, et al., 2006), i.e. this solution concept applies the consensus value to games inter- and intra-coalitions. The consensus CS-value satisfies the axioms that traditionally are used to characterize the Owen value except for the null player axiom. This axiom has been replaced the modified dummy player property. In present paper, we introduce two new solution concepts: the consensus-Shapley CS-value and the Shapley-consensus CS-value. These CS-values are obtained by means of a composition of the consensus value with the Shapley value. Together with the Owen and consensus CS-values new concepts cover the possible variations of the application of the consensus and Shapley values to games with coalition structure. For their characterization we introduce two axioms which can be seen as modifications of the classical null player property. For similar characterization all CS-values using the consensus value, we present also a third modification of null player axiom.

The paper is divided into four sections. The next section contains notations and definitions which are needed in the paper. In section 3, we introduce new CS-values. Section 4 is devoted the axiomatic characterization of CS-values based on consensus and Shapley values.

2. Preliminaries

A *cooperative game with transferable utility* (TU-game) is a pair (N,ν) , where $N=\{1,\dots,n\}$ is a *player set* and $\nu\in G^N$, where

$$G^N=\{g:2^N\to R\mid g(\oslash)=0\},$$

is a *characteristic function*. A subset $S\subseteq N$ of player set is called a *coalition* and $\nu(S)$ is interpreted as the *worth* that is available to coalition S. The cardinality of set S is written as s. A vector $x\in R^N$ is called a *payoff vector* and x_i denotes the payoff of player i. Given $\emptyset\neq T\subseteq N$ a T*-unanimity game* is denoted by (N,u_T) and determined as: $u_T(S)=1$ if $T\subseteq S$ and $u_T(S)=0$ otherwise. Given $\alpha\in R$, let $(N,\alpha u_T)$ be the unanimity game multiplied by a scalar α, i.e. $(\alpha u_T)(S)=\alpha u_T(S)$ for all $S\subseteq N$. The game (N,ν), determined by $\nu(S)=0$ for all $S\subseteq N$, is a *zero-game*.

Two players $i, j \in N$ are *symmetric* in (N, ν) if $\nu(S \cup i) = \nu(S \cup j)$ for every $S \subseteq N \setminus \{i, j\}$. We say that players of coalition S with $s \geq 2$ are symmetric in (N, ν) if each pair of players of the coalition is symmetric in (N, ν). A player $i \in N$ is *dummy* in (N, ν) if he adds $\nu(i)$ to any coalition non-containing him. Denote by

$$Du(N, \nu) = \{i \in N \mid \nu(S \cup i) - \nu(S) = \nu(i),\ S \subseteq N \setminus i\}$$

a set of all dummy players in (N, ν). A player $i \in N$ is a *null* in (N, ν) if he adds nothing to any coalition non-containing him. Denote by

$$Nu(N, \nu) = \{i \in N \mid \nu(S \cup i) = \nu(S),\ S \subseteq N \setminus i\}$$

a set of all null players in (N, ν). For any set $G \subseteq G^N$ a *value* on G is a function $\phi : G \to R^N$ which assigns to every TU-game $\nu \in G$ a vector $\phi(N, \nu)$, where $\phi_i(N, \nu)$ represents the payoff to player i in (N, ν). Denote by $\Phi(N, \nu)$ a set of all values of game (N, ν). Consider the following properties of value $\phi \in \Phi(N, \nu)$.

Axiom 2.1 *(efficiency).* $\sum_{i \in N} \phi_i(N, \nu) = \nu(N)$ *for all* $\nu \in G$.

Axiom 2.2 *(additivity). For any two* $\nu, \omega \in G$, $\varphi(N, \nu + \omega) = \varphi(N, \nu) + \varphi(N, \omega)$, *where* $(\nu + \omega)(S) = \nu(S) + \omega(S)$ *for all* $S \subseteq N$.

Axiom 2.3 *(symmetry). For all* $\nu \in G$ *and every symmetric players* $i, j \in N$ *in* (N, ν), $\varphi_i(N, \nu) = \varphi_j(N, \nu)$.

Axiom 2.4 *(null player property). For all* $\nu \in G$ *and every* $i \in Nu(N, \nu)$, $\phi_i(N, \nu) = 0$.

Axiom 2.5 *(neutral dummy property) (Ju, et al., 2006). For all* $\nu \in G$ *and every* $i \in Du(N, \nu)$,

$$\phi_i(N, \nu) = \nu(i) + \frac{\nu(N) - \sum_{j \in N} \nu(j)}{2n}.$$

One of the most important values for G^N is the *Shapley value* (Shapley, 1953) which is denoted as Sh. The Shapley value is given by

$$Sh_i(N, \nu) = \sum_{S \subseteq N \setminus i} p_{ns}(\nu(S \cup i) - \nu(S)),\ i \in N, \tag{1}$$

where

$$\rho_{lk} = \frac{k!(l - k - 1)!}{l!},\quad k < l \ \ and \ \ k, l \in N.$$

The Shapley value is characterized by the axioms 2.1-2.4. For every $\nu \in G^N$ the *equal surplus division solution* $E \in \Phi(N, \nu)$ and the *consensus value* $K \in \Phi(N, \nu)$ (Ju, et al., 2006) are given by the formulas

$$E_i(N, \nu) = \nu(i) + \frac{\nu(N) - \sum_{j \in N} \nu(j)}{n},\ i \in N, \tag{2}$$

$$K(N, \nu) = \frac{E(N, \nu) + Sh(N, \nu)}{2},\ i \in N. \tag{3}$$

Since the consensus value equals the average of the Shapley value and the equal surplus solution, it: ″takes a neutral stand between the two polar opinions of utilitarianism and egalitarianism, and balances the tensions of the four fundamental principles of distributive justice″ (Ju, et al., 2006). The consensus value is characterized by the axioms 2.1-2.3 and 2.5.

3. New coalition values

We shall recall some facts which are useful later. A *coalition structure* $C = \{C_1, \dots, C_m\}$ on a player set is an exogenous partition of players into a set of groups, i.e. $\cup_{i=1}^{m} C_i = N$ and $C_i \cap C_j = \emptyset$ for $i \neq j$. The sets making up the partition are called *components*. We also assume $C_e \neq \emptyset$ for all $C_e \in C$. Denote by $\Im^N$ a set of all coalition structures on a fixed player set N. A *TU-game with coalition structure* (CS-game) (N, ν, C) consists of TU-game (N, ν), where $\nu \in G^N$, and the coalition structure $C \in \Im^N$. A family of all TU-games with coalition structure and player set N is denoted by U^N. For any set $G \subseteq G^N$ and any set $\Im \in \Im^N$ a *coalition value* (CS-value) on G is a function $f : G \times \Im \to R^N$ that associates with each game (N, ν, C) a vector $f(N, \nu, C) \in R^N$, where $f_i(N, \nu, C)$ represents the player i's payoff in game (N, ν) with coalition structure C. Denote by $F(N, \nu, C)$ a set of all CS-values of game (N, ν, C).

Let

$$M = \{e \mid C_e \in C\}$$

is a set of coalitional indices in C and let $p \in M$ be fixed. For every $S \in C_p \in C$ let

$$C_p^S = \{C_1, \dots, C_{p-1}, S, C_{p+1}, \dots, C_m\}$$

is a partition of set $N \setminus (C_p \setminus S)$. Similarly (Owen, 1977) we consider the following types of games.

1. Given a game $(N, \nu, C) \in U^N$ the *quotient game* (or the external game) is the TU-game (M, ν_C) determined by

$$\nu_C(Q) = \nu(\bigcup_{e \in Q} C_e), \quad Q \subseteq M. \tag{4}$$

This is interpreted as the game between the components of C in which each coalition $C_p \in C$ acts as a player.

2. Given a game $(N, \nu, C) \in U^N$ and a coalition $\emptyset \neq S \in C_p \in C$ the *modified quotient game* is the TU-game $(M, \nu_{C_p^S})$ determined by

$$\nu_{C_p^S}(Q) = \nu(S \cup \bigcup_{e \in Q \setminus p} C_e) \ for \ Q \ni p, \quad \nu_{C_p^S}(Q) = \nu_C(Q) \ for \ Q \subseteq M \setminus p. \tag{5}$$

A game $(M, \nu_{C_p^S})$ is played between the subcoalition S of component C_p and the remaining components of structure C. Assume that as the solution concept of this game the value

$$\psi \in \Phi(M, \nu_{C_p^S})$$

is chosen.

3. Given a game $(N, \nu, C) \in U^N$ and a component C_p the *reduced game* (or the internal game) is the TU-game (C_p, ν_p^ψ) within coalition C_p. Worth $\nu_p^\psi(S)$ of every nonempty coalition $S \subseteq C_p$ is equal to its payoff in the game $(M, \nu_{C_p^S})$ between components of structure C_p^S, i.e. $\nu_p^\psi(S) = \psi_p(M, \nu_{C_p^S})$, whenever $\emptyset \neq S \subseteq C_p$. Assume that as the solution concept of this game the value

$$\varphi \in \Phi(C_p, \nu_p^\psi)$$

is chosen. The corresponding CS-value $\varphi\psi \in F(N,\nu,C)$ is determined by

$$\varphi\psi_i(N,\nu,C) = \varphi(C_p, \nu_p^{\psi}),\ i \in C_p \in C. \tag{6}$$

Let $f \in F(N,\nu,C)$. We will use the following translations of axioms 2.1-2.4 to the coalitional framework.

Axiom 3.1 *(efficiency). For all* $(N,\nu,C) \in U^N$, $\sum_{i\in N} f_i(N,\nu,C) = \nu(N)$.

Axiom 3.2 *(additivity). For any two* (N,ν,C), $(N,\omega,C) \in U^N$, $f(N,\nu+\omega,C) = f(N,\nu,C) + f(N,\omega,C)$.

Axiom 3.3 *(external symmetry). For all* $(N,\nu,C) \in U^N$ *and any two symmetric in* (M,ν_C) *players* $r,e \in M$ *the total values for coalitions* C_r, C_e*, are equal, i.e.* $\sum_{i\in C_r} f_i(N,\nu,C) = \sum_{i\in C_e} f_i(N,\nu,C)$.

Axiom 3.4 *(internal symmetry). For all* $(N,\nu,C) \in U^N$*, any two players* i,j *who are symmetric in* (N,ν) *and belong to the same component in* C*, get the same payoffs, i.e.* $f_i(N,\nu,C) = f_j(N,\nu,C)$.

Axiom 3.5 *(null player property). For all* $(N,\nu,C) \in U^N$ *and every* $i \in Nu(N,\nu)$, $f_i(N,\nu,C) = 0$.

The consensus CS-value (Zinchenko, et al., 2010) is obtained by replacing the functions ψ and φ in (6) by the consensus value.

Definition 1. The *consensus CS-value* KK for $(N,\nu,C) \in U^N$ is determined by

$$KK_i(N,\nu,C) = K(C_p, \nu_p^K),\ i \in C_p \in C,$$

where

$$\nu_p^K(S) = K_p(M, \nu_{C_p^S}),\ S \subseteq C_p \in C. \tag{7}$$

So the consensus CS-value relates to the consensus value as the Owen value relates to the Shapley value. We can say that consensus CS-value is a composition of the consensus value with itself. It is easily verified that the consensus CS-value coincides with the consensus value in case that all unions are singletons and satisfies the quotient game property (the total payoff assigned to the players of an a component equals the payoff of this component in the quotient game). The consensus CS-value is the unique value on the class of CS-games satisfying the axioms 3.1-3.4 and modified dummy axiom (Zinchenko, et al., 2010). Similarly to consensus CS-value we will introduce two new value for games with coalition structure.

Definition 2. The *consensus-Shapley CS-value* KSh for $(N,\nu,C) \in U^N$ is determined by

$$KSh_i(N,\nu,C) = Sh_i(C_p, \nu_p^K),\ i \subseteq C_p \in C,$$

where function ν_p^K is determined by (7).

The consensus-Shapley CS-value reflects the result of a bargaining procedure by which, in the quotient game, each a component receives a payoff determined by the consensus value and, within each component, the members share this payoff in accordance with the Shapley value. In the grand coalition, there are no outside options, hence for trivial structure $C = \{N\}$ the consensus-Shapley CS-value coincides with the two-step Shapley value.

In order to axiomatize the consensus and the consensus-Shapley CS-values we need to express ν_p^K through characteristic functions of original game. For every coalition $\emptyset \neq S \subseteq C_p$ we have

$$\nu_p^K(S) \overset{(7),(3)}{=} \frac{E_p(M, \nu_{C_p^S}) + Sh_p(M, \nu_{C_p^S})}{2}$$

$$\overset{(1),(2)}{=} \frac{1}{2}[\nu_{C_p^S}(p) + \frac{\nu_{C_p^S}(M) - \sum_{e\in M} \nu_{C_p^S}(e)}{m} + \sum_{Q\subseteq M\setminus p} \rho_{mq}(\nu_{C_p^S}(p \cup Q) - \nu_{C_p^S}(Q))]$$

$$\overset{(5)}{=} \frac{1}{2}[\nu(S) + \frac{\nu(S \cup (N \setminus C_p)) - \nu(S) - \sum_{e\in M\setminus p} \nu(C_e)}{m} + \sum_{Q\subseteq M\setminus p} \rho_{mq}(\nu(S \cup \bigcup_{e\in Q} C_e) - \nu(\bigcup_{e\in Q} C_e))].$$

Because $\rho_{mq} = \frac{1}{m}$ for $q = 0$,

$$\nu_p^K(S) = \frac{m\nu(S) + \nu(S \cup (N \setminus C_p)) - \sum_{e\in M\setminus p} \nu(C_e)}{2m} + \sum_{\substack{Q\subseteq M\setminus p \\ Q\neq\emptyset}} \frac{\rho_{mq}}{2}(\nu(S \cup \bigcup_{e\in Q} C_e) - \nu(\bigcup_{e\in Q} C_e)), \quad \emptyset \neq S \subseteq C_p \in C. \tag{8}$$

Definition 3. The *Shapley-consensus CS-value* ShK for $(N, \nu, C) \in U^N$, is determined by

$$ShK_i(N, \nu, C) = K_i(C_p, \nu_p^{Sh}),\ i \subseteq C_p \in C,$$

where

$$\nu_p^{Sh}(S) = Sh_p(M, \nu_{C_p^S}),\ S \subseteq C_p \in C. \tag{9}$$

Hence, the Shapley-consensus CS-value uses Shapley value in the quotient game between components and the consensus value in reduced game within component.

4. Axiomatization

First, we introduce three modification of axiom 3.5.

Axiom 4.1 *(first modified null player property). Let $(N, \nu, C) \in U^N$, $C_p \in C$ and $i \in C_p \cap Nu(N, \nu) \neq \emptyset$. Then*

$$f_i(N, \nu, C) = \frac{\Delta_p(M, \nu_C)}{2c_p m}, \tag{10}$$

where

$$\Delta_p(M, \nu_C) = \nu(N \setminus C_p) - \sum_{e\in M\setminus p} \nu(C_e). \tag{11}$$

Axiom 4.2 *(second modified null player property). Let $(N,\nu,C) \in U^N$, $C_p \in C$ and $i \in C_p \cap Nu(N,\nu) \neq \emptyset$. Then*

$$f_i(N,\nu,C) = \frac{Sh_p(M,\nu_C) - \sum_{j\in C_p} Sh_p(M,\nu_{C_p^j})}{2c_p}.$$

Axiom 4.3 *(third modified null player property). Let $(N,\nu,C) \in U^N$ and $C_p \in C$. Then for every $i \in C_p \cap Nu(N,\nu) \neq \emptyset$ it holds that:*

(i) if $C_p \subseteq Nu(N,\nu)$, then $f_i(N,\nu,C)$ is determined by (10),

(ii) if $\Delta_p(M,\nu_C) = 0$ then

$$f_i(N,\nu,C) = \frac{K_p(M,\nu_C) - \sum_{j\in C_p} K_p(M,\nu_{C_p^j})}{2c_p}. \tag{12}$$

It is easily to prove that if both conditions $C_p \subseteq Nu(N,\nu)$ and $\Delta_p(M,\nu_C) = 0$ hold for some $p \in M$, then formulas (10) and (12) are reduced to $f_i(N,\nu,C) = 0$, $i \in C_p$. The axioms 4.1-4.3 show that even a null player can receive some portion of bargaining surplus if a coalition that he belongs to generates it. Notice, that axioms 4,1 and 4.2 determine the payoffs of all null players in (N,ν), but the axiom 4.3 determines payoffs of players which belong to null component $C_p \subseteq Nu(N,\nu)$ or are null (see Lemma 2 below) in reduced game within C_p. The following lemma shows that the null player $i \in C_p \cap Nu(N,\nu)$ always is null in reduced game (C_p,ν_p^{Sh}) inside his components C_p.

Lemma 1. *Let $(N,\nu,C) \in U^N$, $p \in M$ and $i \in C_p \cap Nu(N,\nu) \neq \emptyset$. Then $i \in Nu(C_p,\nu_p^{Sh})$.*

Proof. We have to prove that under maded assumptions $\nu_p^{Sh}(S\cup i) = \nu_p^{Sh}(S)$ for all $S \subseteq C_p \setminus i$. Take $S \subseteq C_p \in C$, then

$$v_p^{Sh}(S) \overset{(9),(1)}{=} \sum_{Q\subseteq M\setminus p} \rho_{mq}[\nu_{C_p^S}(Q\cup p) - \nu_{C_p^S}(Q)]$$

$$\overset{(5)}{=} \sum_{Q\subseteq M\setminus p} \rho_{mq}[\nu(S\cup \bigcup_{e\in Q} C_e) - \nu(\bigcup_{e\in Q} C_e)],$$

$$v_p^{Sh}(S\cup i) - v_p^{Sh}(S) = \sum_{Q\subseteq M\setminus p} \rho_{mq}[\nu(S\cup i\cup \bigcup_{e\in Q} C_e) - \nu(S\cup \bigcup_{e\in Q} C_e)] \overset{(i\in Nu(N,\nu))}{=} 0$$

for all $S \subseteq C_p \setminus i$. □

Next lemma gives the necessary and sufficient condition at which the null player in (N,ν) remains null in reduced game within his component.

Lemma 2. *Let $(N,\nu,C) \in U^N$, $p \in M$ and $i \in C_p \cap Nu(N,\nu) \neq \emptyset$. Then $i \in Nu(C_p,\nu_p^K)$ iff $\Delta_p(M,\nu_C) = 0$, where $\Delta_p(M,\nu_C)$ is determined by (11).*

Proof. It is sufficient to show that under maded assumptions $\nu_p^K(S\cup i) = \nu_p^K(S)$ for all $S \in C_p \setminus i$ iff $\Delta_p(M,\nu_C) = 0$. We consider two cases.

Case (a). $S \neq \emptyset$, $S \subseteq C_p \setminus i$. Then

$$\nu_p^K(S\cup i) - \nu_p^K(S) \overset{(8)}{=} \frac{m[\nu(S\cup i) + \nu(S)] + \nu(S\cup i\cup(N\setminus C_p)) - \nu(S\cup(N\setminus C_p))}{2m}$$

$$+\sum_{\substack{Q\subseteq M\setminus p\\ Q\neq\emptyset}}\frac{\rho_{mq}}{2}[\nu(S\cup i\cup\bigcup_{e\in Q}C_e)-\nu(S\cup\bigcup_{e\in Q}C_e)]\overset{(i\in Nu(N,\nu))}{=}0.$$

This gives

$$\nu_p^K(S\cup i)=\nu_p^K(S),\ p\in M,\ \ i\in C_p\cap Nu(N,\nu),\ \emptyset\neq S\in C_p\setminus i. \tag{13}$$

Case (b). $S=\emptyset$. Then

$$\nu_p^K(S\cup i)=\nu_p^K(i)\overset{(8)}{=}\frac{m\nu(i)+\nu(i\cup(N\setminus C_p))-\sum_{e\in M\setminus p}\nu(C_e)}{2m}$$

$$+\sum_{\substack{Q\subseteq M\setminus p\\ Q\neq\emptyset}}\frac{\rho_{mq}}{2}[\nu(i\cup\bigcup_{e\in Q}C_e)-\nu(\bigcup_{e\in Q}C_e)]\overset{(i\in Nu(N,\nu))}{=}\frac{\nu(N\setminus C_p)-\sum_{e\in M\setminus p}\nu(C_e)}{2m}.$$

Hence,

$$\nu_p^K(i)=\frac{\Delta_p(M,\nu_C)}{2m},\ \ i\in C_p\cap Nu(N,\nu),\ p\in M, \tag{14}$$

i.e. $\nu_p^K(i)=0$ iff $\Delta_p(M,\nu_C)=0$. □

Lemma 3. *For any $(N,\nu,C)\in U^N$, the consensus CS-value $KK\in F(N,\nu,C)$ satisfies the axioms 3.1 - 3.4 and 4.3.*

Proof. Because the consensus value satisfies the axioms 2.1 - 2.3, the consensus CS-value satisfies the axioms 3.1 - 3.4 (Gomez-Rua and Vidal-Puga, 2008). Let us see that KK satisfies the axiom 4.3. Assume $p\in M$ and $C_p\cap Nu(N,\nu)\neq\emptyset$. We consider two cases.

Case (a). $C_p\subseteq Nu(N,\nu)$. By Definition 1 and axiom 2.1, we have

$$\sum_{i\in C_p}KK_i(N,\nu,C)=\sum_{i\in C_p}K_i(C_p,\nu_p^K)=\nu_p^K(C_p)\overset{(7)}{=}K_p(M,\nu_C).$$

Since C_p contains the null players only, the player $p\in M$ is null in (M,ν_C). By axiom 2.5

$$K_p(M,\nu_C)=\nu_C(p)+\frac{\nu_C(M)-\sum_{e\in M}\nu_C(e)}{2m}\overset{(4)}{=}\nu(C_p)+\frac{\nu(N)-\sum_{e\in M}\nu(C_e)}{2m}$$

$$=\frac{\nu(N\setminus C_p)-\sum_{e\in M\setminus p}\nu(C_e)}{2m}\overset{(11)}{=}\frac{\Delta_p(M,\nu_C)}{2m}.$$

All players of coalition C_p are symmetric in (N,ν), therefore

$$KK_i(N,\nu,C)\overset{(axiom\ 3.4)}{=}\frac{K_p(M,\nu_C)}{c_p}=\frac{\Delta_p(M,\nu_C)}{2c_pm}$$

for all $i\in C_p$, i.e. KK satisfies (10).

Case (b). $\Delta_p(M,\nu_C)=0$. Take $i\in C_p\cap Nu(N,\nu)$. By Lemma 2 we have $i\in Nu(C_p,\nu_p^K)$. As the consensus value satisfies the neutral dummy property and $Nu(C_p,\nu_p^K)\subseteq Du(C_p,\nu_p^K)$, then

$$KK_i(N,\nu,C)\overset{(def.)}{=}K_i(C_p,\nu_p^K)\overset{(axiom\ 2.5)}{=}\nu_p^K(i)+\frac{\nu_p^K(C_p)-\sum_{j\in C_p}\nu_p^K(j)}{2c_p}$$

for each $i \in C_p \cap Nu(N,\nu)$. It is follows from $\nu_p^K(i) = 0$, $\nu_p^K(C_p) \overset{(7)}{=} K_p(M,\nu_C)$ and $\nu_p^K(j) \overset{(7)}{=} K_p(M,\nu_{C_p^j})$, $j \in C_p$, that KK satisfies (12). □

We are going to characterize the consensus CS-value by replacing the modified dummy axiom (Zinchenko, et al., 2010) by the third modified null player property (axiom 4.3).

Theorem 1. *KK is the unique value on the class of CS-games U^N satisfying the axioms 3.1 - 3.4 and 4.3.*

Proof. By Lemma 3 the consensus CS-value satisfies the axioms 3.1- 3.4 and 4.3. Using the standard scheme of characterization of CS-values through the additivity axiom, we will prove that these properties uniquely determine the payoffs for (multiplied) unanimity CS-game $(N,\alpha u_T,C)$, where $\emptyset \neq T \subseteq N$, $\alpha \in R$. The theorem statement will follows from that the set $\{u_T\}_{\emptyset \neq T \subseteq N}$ form a basis for G^N.

Let f be a CS-value on G^N satisfying the mentioned axioms. First, we calculate $\Delta_p(M,\nu_C)$, $K_p(M,\nu_C)$ and $K_p(M,\nu_{C_p^j})$, $j \in C_p$, for $(N,\alpha u_T,C)$. Denote

$$D = \{e \in M|\ C_e \cap T \neq \emptyset.\} \tag{15}$$

It is follows from

$$(\alpha u_T)(C_e) = \begin{cases} \alpha, & T \subseteq C_e, \\ 0, & otherwise, \end{cases} \qquad for\ all\ C_e \in C,$$

$$(\alpha u_T)(N \setminus C_p) = \begin{cases} \alpha, & p \in M \setminus D, \\ 0, & p \in D, \end{cases}$$

that

$$\Delta_p(M,(\alpha u_T)_C) \overset{(11)}{=} (\alpha u_T)(N \backslash C_p) - \sum_{e \in M \backslash p} (\alpha u_T)(C_e) = \begin{cases} \alpha, & p \in M \setminus D\ and\ d > 1, \\ 0, & otherwise. \end{cases}$$

Hemce, in game $(N,\alpha u_T,C)$ for everyone component $C_p \in C$ at least one of conditions of axiom 4.3 holds true: $C_p \subseteq Nu(N,\alpha u_T)$ or $\Delta_p(M,(\alpha u_T)_C) = 0$. From (3) and following formulas

$$Sh_i(N,\alpha u_T) = \begin{cases} 0, & i \in N \setminus T, \\ \frac{\alpha}{t}, & i \in T, \end{cases}$$

$$E_i(N,\alpha u_T) = \begin{cases} 0, & i \in N \setminus T,\ t = 1, \\ \alpha, & i \in T, \quad\ \ t = 1, \\ \frac{\alpha}{n}, & i \in N, \quad\ \ t > 1, \end{cases}$$

we have that

$$K_i(N,\alpha u_T) = \begin{cases} 0, & i \in N \setminus T,\ t = 1, \\ \alpha, & i \in T, \quad\ \ t = 1, \\ \frac{\alpha}{2n}, & i \in N \setminus T,\ t > 1, \\ \frac{\alpha(n+t)}{2nt}, & i \in T, \quad\ \ t > 1. \end{cases}$$

Clearly, the quotient game $(M,(\alpha u_T)_C)$ is (multiplied) D-unanimity game, i.e. the games $(M,(\alpha u_T)_C)$ and $(M,\alpha u_D)$ are equivalent. Therefore

$$K_p(M,(\alpha u_T)_C) = K_p(M,\alpha u_D) \quad for\ all\ p \in M.$$

Since $K(N,\alpha\nu) = \alpha K(N,\nu)$ for all $\nu \in G^N$ and $\alpha \in R$, we have

$$K_p(M,(\alpha u_T)_C) = \begin{cases} 0, & p \in M \setminus D,\ d = 1, \\ \alpha, & p \in D, \quad\quad\ \ d = 1, \\ \frac{\alpha}{2m}, & p \in M \setminus D,\ d > 1, \\ \frac{\alpha(m+d)}{2md}, & p \in D, \quad\quad\ \ d > 1. \end{cases}$$

If $j \in C_p \in C$ and structure $C_p^j = \{C_1,\ldots,C_{p-1},\ j,\ C_{p+1},\ldots,C_m\}$ is received from C by removal of the players belonging to coalition T (null players in $(N,\alpha u_T)$), then $(M,(\alpha u_T)_{C_p^j})$ coincides with $(M,(\alpha u_T)_C)$ and also with $(M,\alpha u_D)$. Otherwise $(M,(\alpha u_T)_{C_p^j})$ is zero-game. Thus

$$K_p(M,(\alpha u_T)_{C_p^j}) = \begin{cases} K_p(M,(\alpha u_T)_C), & (C_p \setminus j) \cap T = \emptyset, \\ 0, & otherwise, \end{cases} \quad for\ al\ j \in C_p \in C.$$

Let's show now that $f(N,\alpha u_T,C)$ is uniquely determined by the axioms enumerated in the theorem. By the axioms 3.3 and 3.1,

$$\sum_{p\in D}\sum_{i\in C_p} f_i(N,\alpha u_T,C) + \sum_{p\in M\setminus D}\sum_{i\in C_p} f_i(N,\alpha u_T,C)$$

$$= d\sum_{\substack{i\in C_p\\ p\in D}} f_i(N,\alpha u_T,C) + (m-d)\sum_{\substack{i\in C_p\\ p\in M\setminus D}} f_i(N,\alpha u_T,C) = \alpha.$$

If $p \in M \setminus D \neq \emptyset$, then $C_p \subseteq Nu(N,\alpha u_T)$ and by the axiom 4.3

$$f_i(N,\alpha u_T,C) = \frac{\Delta_p(M,(\alpha u_T)_C)}{2c_p m} = \begin{cases} 0, & d = 1, \\ \frac{\alpha}{2c_p m}, & d > 1, \end{cases} \quad for\ all\ i \in C_p,\ p \in M \setminus D.$$

Summing up $f_i(N,\alpha u_T,C)$ over $i \in C_p$ gives

$$\sum_{i\in C_p} f_i(N,\alpha u_T,C) = \begin{cases} 0, & d = 1, \\ \frac{\alpha}{2m}, & d > 1, \end{cases} \quad for\ all\ p \in M \setminus D.$$

If $p \in D$, then $\Delta_p(M,(\alpha u_T)_C) = 0$ and by the axiom 4.3

$$f_i(N,\alpha u_T,C) = \frac{K_p(M,(\alpha u_T)_C) - \sum_{j\in C_p} K_p(M,(\alpha u_T)_{C_p^j})}{2c_p}, \quad i \in C_p \setminus T,\ p \in D$$

Denote $t_p = |C_p \cap T|$. From formulas for $K_p(M,(\alpha u_T)_C)$ and $K_p(M,(\alpha u_T)_{C_p^j})$ we have that

$$f_i(N,\alpha u_T,C) = \begin{cases} 0, & t_p = 1, \\ \frac{\alpha}{2c_p}, & t_p > 1,\ d = 1, \\ \frac{\alpha(m+d)}{4c_p md}, & t_p > 1,\ d > 1, \end{cases} \quad for\ all\ \ i \in C_p \setminus T,\ p \in D.$$

All players of coalition $C_p \setminus T$ are symmetric in $(N,\alpha u_T)$. By the axiom 3.4

$$\sum_{\substack{i\in C_p\cap T\\ p\in D}} f_i(N,\alpha u_T,C)$$

$$= \frac{\alpha - (m-d)\sum_{i\in C_p: p\in M\setminus D} f_i(N,\alpha u_T,C)}{d} - \sum_{i\in C_p\setminus T: p\in D} f_i(N,\alpha u_T,C).$$

All players of coalition $C_p \cap T$ are also symmetric in $(N,\alpha u_T)$. From axioms 3.4 and simple calculations we have that

$$f_i(N,\alpha u_T,C) = \begin{cases} \alpha, & d = t_p = 1, \\ \frac{\alpha(c_p+t_p)}{2c_p t_p}, & d=1,\ t_p>1, \\ \frac{\alpha(m+d)(c_p+t_p)}{4c_p m d t_p}, & d>1, \end{cases} \quad \textit{for all } i\in C_p\cap T,\ p\in D.$$

The received formulas uniquely determine $f_i(N,\alpha u_T,C)$ for all $i\in N$. □

Theorem 2. *KSh is the unique value on the class of CS-games U^N satisfying the axioms 3.1 - 3.4 and 4.1.*

Proof. Because the consensus and Shapley values satisfy the axioms 2.1 - 2.3, KSh satisfies the axioms 3.1 - 3.4. Let $p\in M$ and $i\in C_p\cap Nu(N,\nu)\neq\emptyset$. From Definition 2, formulas (1), (13), (14) and the equality $\rho_{c_p s} = \frac{1}{c_p}$ for $s=0$, it follows that

$$KSh_i(N,\nu,C) = \sum_{S\subseteq C_p\setminus i} \rho_{c_p s}[\nu_p^K(S\cup i) - \nu_p^K(S)] \overset{(13),(14)}{=} \frac{\nu_p^K(i)}{2c_p} = \frac{\Delta_p(M,\nu_C)}{2c_p m},$$

i.e. KSh satisfies the axiom 4.1.

Let f be a CS-value on G^N satisfying the listed axioms. All members of coalition $N\setminus T$ are null players in $(N,\alpha u_T)$. By axiom 4.1

$$f_i(N,\alpha u_T,C) = \frac{\Delta_p(M,(\alpha u_T)_C)}{2c_p m} \quad \textit{for all } i\in N\setminus T.$$

Summing up last equality yields

$$\sum_{p\in M\setminus D}\sum_{i\in C_p} f_i(N,\alpha u_T,C) \overset{(axioms\ 3.3)}{=} \frac{m-d}{2m}\Delta_p(M,(\alpha u_T)_C),$$

$$\sum_{i\in C_p\setminus T} f_i(N,\alpha u_T,C) = \frac{c_p-t_p}{2c_p m}\Delta_p(M,(\alpha u_T)_C)) \quad \textit{for all } p\in M\setminus D,$$

where $t_P = |C_p\cap T|$ and D is determined by (15). All players of coalition D are symmetric in the quotient game $(M,(\alpha u_T)_C))$, hence

$$d[\sum_{\substack{i\in C_p\cap T\\ p\in D}} f_i(N,\alpha u_T,C) + \frac{c_p-t_p}{2c_p m}\Delta_p(M,(\alpha u_T)_C)] + \frac{m-d}{2m}\Delta_p(M,(\alpha u_T)_C) \overset{(axiom\ 3.1)}{=} \alpha$$

$$\Longrightarrow \sum_{\substack{i\in C_p\cap T\\ p\in D}} f_i(N,\alpha u_T,C) = \frac{\alpha}{d} + \frac{t_p d - c_p m}{2c_p m d}\Delta_p(M,(\alpha u_T)_C).$$

For $p\in D$ all players of coalition $C_p\cap T$ are symmetric in $(N,\alpha u_T)$. By axiom 3.4

$$f_i(N,\alpha u_T,C) = \frac{\alpha}{t_p d} - \frac{c_p m - t_p d}{2c_p t_p m d}\Delta_p(M,(\alpha u_T)_C) \quad \textit{for all } i\in C_p\cap T,\ p\in D.$$

Since $C_p \subseteq N \setminus T$ for $p \in M \setminus D$, and $\Delta_p(M, (\alpha u_T)_C)$ is determined uniquely for all $p \in M$ (see the proof of Theorem 1), CS-value $f_i(N, \alpha u_T, C)$ is also determine uniquely for all $i \in N$. □

Theorem 3. *ShK is the unique value on the class of CS-games U^N satisfying the axioms 3.1 - 3.4 and 4.2.*

Proof. The axiom 4.2, as well as the axiom 4.1, determines uniquely the payoffs of all null players in (N, ν), therefore the proof of this theorem is similar to that of Theorem 2. It only remains to show that ShK satisfies the axiom 4.2. Let $p \in M$ and $i \in C_p \cap Nu(N, \nu) \neq \emptyset$. By Lemma 1 we have $i \in Nu(C_p, \nu_p^{Sh})$. Hence

$$ShK_i(N, \nu, C) \overset{(def.)}{=} K_i(C_p, \nu_p^{Sh}) \overset{(axiom\ 2.5)}{=} \nu_p^{Sh}(i) + \frac{\nu_p^{Sh}(C_p) + \sum_{j \in C_p} \nu_p^{Sh}(j)}{2c_p}.$$

It is follows from $\nu_p^{Sh}(i) = 0$, $\nu_p^{Sh}(C_p) \overset{(9)}{=} Sh_p(M, \nu_C)$ and $\nu_p^{Sh}(j) \overset{(9)}{=} Sh_p(M, \nu_{C_p^j})$, $j \in C_p$, that ShK satisfies the axiom 4.2. □

Notice that KK and KSh do not satisfy the coalitional null player property introduced in (Kamijo, 2009) to characterize the two-step Shapley value. Therefore, even the member of null component $C_p \subseteq Nu(N, \nu)$ can get a positive KK and KSh CS-values.

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