GRADUATE SCHOOL OF MANAGEMENT
FACULTY OF APPLIED MATHEMATICS & CONTROL PROCESSES
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(Russian Chapter)

# CONTRIBUTIONS TO GAME THEORY AND MANAGEMENT

## Volume II

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Collected papers
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The collection contains papers accepted for the Second International Conference Game Theory and Management (June 26 – 27, 2008, St. Petersburg University, St. Petersburg, Russia). The presented papers belong to the field of game theory and its applications to management.

The volume may be recommended for researches and post-graduate students of management, economic and applied mathematics departments.



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**Успехи теории игр и менеджмента. Вып. 2.** Сб. статей второй международной конференции по теории игр и менеджменту / Под ред. Л.А. Петросяна, Н.А. Зенкевича. – СПб.: Высшая школа менеджмента СПбГУ, 2009. – 514 с.

Сборник статей содержит работы участников второй международной конференции «Теория игр и менеджмент» (26–27 июня 2008 года, Высшая школа менеджмента, Санкт-Петербургский государственный университет, Санкт-Петербург, Россия). Представленные статьи относятся к теории игр и ее приложениям в менеджменте.

Издание представляет интерес для научных работников, аспирантов и студентов старших курсов университетов, специализирующихся по менеджменту, экономике и прикладной математике.

# Content

# PREFACE

This edited volume contains a selection of papers that are an outgrowth of the Second International Conference on Game Theory and Management with a few additional contributed papers. These papers present an outlook of the current development of the theory of games and its applications to management and various domains, in particular, energy, the environment and economics.

The International Conference on Game Theory and Management, a two day conference, was held in St. Petersburg, Russia in June 26-27, 2008. The conference was organized by Graduate School of Management St. Petersburg University in collaboration with The International Society of Dynamic Games (Russian Chapter) and Faculty of Applied Mathematics and Control Processes SPU within the framework of a National Priority Project in Education. More than 100 participants from 21 countries had an opportunity to hear state-of-the-art presentations on a wide range of game-theoretic models, both theory and management applications.

Plenary lectures covered different areas of games and management applications. They had been delivered by Professor John F. Nash, Princeton University (USA), Nobel Prize Winner in Economics in 1994; Professor Tamer Basar, University of Illinois at Urbana-Champaign (USA); Professor Geert J. Olsder, Delft University of Technology (the Netherlands); Professor Leon A. Petrosyan, St. Petersburg University (Russia); Professor David W.K. Yeung, Hong Kong Baptist University (Hong-Kong). The importance of strategic behavior in the human and social world is increasingly recognized in theory and practice. As a result, game theory has emerged as a fundamental instrument in pure and applied research. The discipline of game theory studies decision making in an interactive environment. It draws on mathematics, statistics, operations research, engineering, biology, economics, political science and other subjects. In canonical form, a game obtains when an individual pursues an objective(s) in a situation in which other individuals concurrently pursue other (possibly conflicting, possibly overlapping) objectives and in the same time the objectives cannot be reached by individual actions of one decision maker. The problem is then to determine each individual's optimal decision, how these decisions interact to produce equilibria, and the properties of such outcomes. The foundations of game theory were laid some sixty years ago by von Neumann and Morgenstern (1944).

Theoretical research and applications in games are proceeding apace, in areas ranging from aircraft and missile control to inventory management, market development, natural resources extraction, competition policy, negotiation techniques, macroeconomic and environmental planning, capital accumulation and investment. In all these areas, game theory is perhaps the most sophisticated and fertile paradigm applied mathematics can offer to study and analyze decision making under real world conditions.

The papers presented at this Second International Conference on Game Theory and Management certainly reflect both the maturity and the vitality of modern day game theory and management science in general, and of dynamic games, in particular. The maturity can be seen from the sophistication of the theorems, proofs, methods and numerical algorithms contained in the most of the papers in these contributions. The vitality is manifested by the range of new ideas, new applications, the growing number of young researchers and the expanding world wide coverage of research centers and institutes from whence the contributions originated.

The contributions demonstrate that GTM2008 offers an interactive program on a wide range of latest developments in game theory and management. It includes recent advances in topics with high future potential and exiting developments in classical fields.

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Editors, Leon A. Petrosyan and Nikolay A. Zenkevich

# Generalized 'Lion & Man' Game of R.Rado

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**Abstract** In the present work we study the game of degree for generalized "Lion and Man" game where "Lion" L moves on the plane while "Man" M must move along the given curve $\Gamma$ . The case when $\Gamma$ is circumference the problem was formulated by R.Rado and can be considered as the first example of dynamic games. By elementary but refined arguments R.Rado (Littlewood, 1957, Rado, 1973) proved that $L$ can capture $M$ if speeds of both points are equal. Further interesting results on "Lion & Man" game concerning the case when both points moved inside a circle, was obtained by J.O.Flynn (1973, 1974).

**Keywords:** differential game, pursuit, evasion, strategy, "Lion and Man".

## 1. Introduction

We study the generalized game "Lion and Man" in the case of R.Rado, whose work can be considered as the first meaningful example of the Theory of differential games.

In this generalized game the Lion $L$ moves on the plane and tries to capture Man $M$ who can't leave the given curve $\Gamma$. Maximum speeds of this points are equal to $\rho$ and $\sigma$ respectively, $\rho > 0$, $\sigma > 0$. Rado analyzed the case when $\Gamma$ is circumference of the radius $R$, $R > 0$, and $\rho = \sigma$. By refined and quite elementary arguments he showed that the point $L$ can capture the point $L$. Moreover, if the initial position of $L$ is at the center $O$ of the circumference $\Gamma$, the capture can be completed for the time $T = \frac{\pi R}{2\rho}$. Later J.Flint investigated the case where $\Gamma$ is a circumference and $\sigma > \rho$. He proved that there exists number $l$, $l > 0$, such that $L$ can do $|LM| = l$ approach, while $M$ is able to ensure the inequality $|LM| > l$ (Flynn, 1973, 1974).

If $M$ moves in the circle $G$ bounded by the circumference $\Gamma$, then S. Bezikovich proved that $M$ can avoid of encounter with $L$, i.e. can ensure the relation $M \neq L$, thou $L$ can approach the point $M$ to any positive distance.

It should be noted that cases of the game "Lion and Man" studied by R.Rado and S.Bezikovich are quite different in point of view of the theory of differential games: in the case of Rado states of the points can be defined by two coordinates in the polar system, while in the case of Bezicovich the phase space exactly tree dimensional (Lewin, 1986).Moreover, if we assume that $L$ can move all over the plane (as we do in this paper), then the case of Rado relates to the games without phase constraints, while in the case of Bezicovich the condition $M \in G$ is essential and makes a phase constraint.

Note also, arguments of Rado relayed on symmetry of circumference and they lose validity for curves different from the circumference.

In the present work the quantity problem for generalized "Lion and Man" game of R.Rado is studied when L moves along the given curve $\Gamma$. Let $\Gamma$ be given by the absolutely continuous map $\gamma : R \to R^n$ and parameterized by arc length. Motions of the points $L$ and $M$ are described respectively by the equations $\dot{x} = u,\ \dot{y} = v$ with the phase constraint $y(t) \in \Gamma$ and control constraints $|u| \leq \rho,\ \ |v| \leq \sigma\ (x,\ y,\ u,\ v \in R^n)$. The problem is interesting if $\Gamma$ is embedded into $R^n$ isometrically to $R$ or $S$. (If $\Gamma$ is isometric to half line $[0, \infty)$ then problem can be reduced to the first case.)

In dependent on $\Gamma$, $\rho$ and $\sigma$ we'll prove either the pursuit problem or the evasion problem has a solution. Any of the existing approaches being applied to the pursuit-evasion problems can be taken as the basis of the formalization (Friedman, 1971, Krasovskiy and Subbotin, 1988, Petrosyan, 1977). Exact statement of the problem requires definition of the concept of strategy $U$ of the pursuer and that of the evader $V$. The strategy $U$ relays on current information with discrimination of the evader: at each time $t,\ t \geq 0$, the value $u(t)$ will be found by using information about $x(s)$, $y(s),\ 0 \leq s \leq t$ and $v(t)$. According to the strategy $V$ the value $v(t)$ will be found by using information about $x(s)$, $y(s)$, $0 \leq s \leq t$.

There is no necessity for description this concepts in detail, since each time we prove the solvability of the problems constructively. When we consider the pursuit problem, we construct a concrete strategy $U^*$, such that the trajectories $x(t)$, $y(t)$, are generated uniquely by arbitrary measurable control $v(\cdot)$ of the evader and initial positions $x(0) = x_0,\ y(0) = y_0$ and prove that $x(t) = y(t)$ for some $t \in [0, T]$ . In this case we'll say that for the initial position $(x_0,\, y_0)$ the pursuit problem is solvable on the time interval $[0, T]$ .

Similarly, for the evasion problem we construct a concrete strategy $V_*$ such that the trajectories $x(t)$, $y(t)$ are generated uniquely by arbitrary control of the pursuer $u(\cdot)$ and the initial position $(x_0,\, y_0)$ and prove that $x(t) \neq y(t)$ for all $t \in [0, +\infty)$ . In this case we say that for the initial position $(x_0,\, y_0)$ evasion problem is solvable.

Since relation $x \neq y$ doesn't exclude possibility of approaching the point $x$ to $y$ to any small distance, it is required in the evasion problem (Pontryagin, 1970) that strategy $V_*$ has the following properties: there exists positive quantity $\delta = \delta(x_0,\, y_0)$ such that $|x(t) - y(t)| \geq \delta$, for all $t,\ t \geq 0$ independent on control $u(\cdot)$.

We list the main results.

**Theorem 1.** *Let $\rho = \sigma$ and $\Gamma$ be closed. Then there exists a strategy of the pursuer that guaranties the capture, i.e. there exists $T$, $T > 0$, such that for any admissible control function $v(\cdot)$ of $M$ $x(t) = y(t)$ at some $t \in [0,\, T]$ for the corresponding trajectories.*

Now suppose $\Gamma$ is not closed, so $\Gamma \backslash \gamma(0)$ consists of two components $\Gamma_+$, $\Gamma_-$.

Condition $A$ : There exists a point $\gamma(s_\pm) \in \Gamma_\pm$ such that $|x(0) - \gamma(s_\pm)| = |s_\pm|$ respectively.

Obviously if Condition $A$ is not occurred for $\Gamma_+$ or $\Gamma_-$ then $M$ easily avoids a capture. It turns out the converse also is true.

**Theorem 2.** *Suppose $\rho = \sigma$ and $\Gamma$ is not necessarily closed and Condition $A$ holds for both arcs $\Gamma_\pm$. Then there exists a strategy of the pursuer that guaranties the capture, i.e. there exists $T$, $T > 0$, such that for any admissible control function $v(\cdot)$ of $M$ $x(t) = y(t)$ at some $t \in [0, T]$ for the corresponding trajectories.*

For the proof of this theorem we refer to (Kuchkarov, 2008). The following theorem improves and extends the main result of (Azamov, 1986), where was assumed $\Gamma$ to be plane curve with class of smoothness $C^2$.

**Theorem 3.** *Let $\rho < \sigma$ and the function $\gamma'$ satisfies Lipschitz condition i.e.*

$$|\gamma'(s_1) - \gamma'(s_2)| \leq \lambda |s_1 - s_2|, \ \lambda > 0.$$

*Then the evasion problem is solvable.*

## 2. Proof of the Theorems

### 2.1. The radial strategy of pursuit

It can be supposed

$$x(0) = 0, \ y(0) = \gamma(0) \neq 0. \tag{1}$$

Let $v = v_\nu + v_\tau$ (respectively $u = u_\nu + u_\tau$) be the decomposition such that $v_\nu$ $(u_\nu)$ is directed to $y$ (respectively $x$) and $v_\tau$ (respectively $u_\tau$) is orthogonal to $v_\nu$ $(u_\nu)$.

**Definition 1.** The function

$$U_R(x, \ y, \ v) = \ \xi v_\tau + y(\rho^2 - \xi^2 |v_\tau|^2)^{1/2} \Big/ |y| \tag{2}$$

defined on the region $|v| \leq \sigma$, $|x| \leq |y|$, $y \neq 0$ in $R^{6\mathrm{n}}$ *will be called Rado's radial strategy of pursuit (briefly R-strategy)*, where $\xi = |x|/|y|$.

**Lemma 1.** *The formula (2) representing the decomposition of the vector $u = U_R$ has the following properties:*

*a) $|u| = \rho$(speed of the point $L$ is maximal);*

*b) angular speeds of $L$ and $M$ equal:*

$$\frac{|u_\tau|}{|x|} = \frac{|v_\tau|}{|y|};$$

*c) $|u_\nu| > |v_\nu|$ (velocity of $L$ is directed to the side of $M$ ).*

Proof is straightforward.

Here $R-$strategy of Rado will be used in the game in case of $\rho = \sigma$. If this is the case, without any loss of generality we can assume that $\rho = \sigma = 1$, since if it is necessary, we can change the scale.

### 2.2. Proof of Theorem 1.

Let the closed smooth curve $\Gamma$ be given by the natural equation $y = \gamma(s)$ where $s$ is arc-length and $\gamma(\cdot)$ is a continuously differentiable periodical function with the period equal to the length of $\gamma$ of the curve $\Gamma$. Hence we can suppose $|d\gamma(s)/ds| = 1$ and the point $\gamma(s)$ moves counter-clockwise along $\Gamma$ when parameter $s$ grows. Notice that $\Gamma$ may have self-crossings.

We can write the equation of the motion of $M$ as

$$ds(t)/dt = w(t), \ s(0) = 0 \tag{3}$$

(see (1)) where $w(\cdot)$ is the control function of $M$ satisfying the conditions:

a) $-1 \leq w(t) \leq 1$ a.e.; b) $w(\cdot)$ is measurable.

Let $W$ be the class of all control functions of $M$. In order to avoid cumbersome calculations we restrict ourselves considering the subclass $W^0$ consisting of all functions $w(\cdot) \in W$ satisfying the additionally condition

c) $w(t) = 1$ or $w(t) = -1$ a.e.

It should be noted that the last assumption will not restrict generality because of possibility to approximate any function $w_*(\cdot) \in W$ by sequence $w_n(\cdot) \in W^0$ so that the corresponding trajectories $x_n(t) \to x_*(t)$ uniformly on every finite interval.

The function $y(t) = \gamma(s(t))$ describes the motion of $M$.

If $dy/dt = v$ is its velocity, then $w\langle t\rangle = \langle \gamma'(s(t)),\ v(t)\rangle$ and $v(t) = w(t)\gamma'(s(t))$.

If $\Gamma$ is a Jordanian curve then there exists the natural one-to-one correspondence between $W$ (respectively $W^0$ ) and the class of all admissible control functions $v(\cdot) : [0,\ \infty) \to \mathrm{R}^2$ such that $|v(t)| \leq 1\ \ (|v(t)| = 1)$ a.e. and

$$y(t) = y_0 + \int_0^t v(\tau)d\tau \in ,\ t \geq 0. \tag{4}$$

**Lemma 2.** *For any control function $w(\cdot) \in W^0$ the Cauchy problem*

$$\frac{dx}{dt} = u_R(x,\ y(t),\ v(t)),\ x(0) = 0 \tag{5}$$

*has a solution. It is unique on the interval $J$, where $J = [0, +\infty)$, if $|x(t)| < |y(t)|$ for all $t > 0$, in other case $J = [0, \theta]$, where $\theta = \min\{t\,|x(t) = y(t)\}$.*

Correctness of the statement follows from Caratheodory's existence and uniqueness theorem for the Cauchy problem by virtue of condition $|x(0)| = 0 < |y(0)|$ (see (1)).

For the exact formulation it is required to reveal some numerical characteristics of the curve $\Gamma$.

Let

$$S_+ = \{s\,|\langle\gamma(s),\ \gamma'(s)\rangle > 0\},\ S_- = \{s\,|\langle\gamma(s),\ \gamma'(s)\rangle < 0\},$$

$$S_0 = \{s\,|\langle\gamma(s),\ \gamma'(s)\rangle = 0\}$$

be the partition of some fixed segment $I$ of the length $\gamma$ into three parts according to rising, descending and circular parts of $\Gamma$ with respect to the initial position 0 of the point $L$. The segment $I$ will be defined concretely below, in the Main Lemma. Notice that $S_+$ and $S_-$ are open. Obviously if $mS_0 = g$, then $\langle\gamma(s),\ \gamma'(s)\rangle = 0$ a.e., which implies that $\Gamma$ is a circumference, that is the case considered by R.Rado ($mX$ denotes the Lebegue measure of $X$.)

So one can suppose that $mS_0 < g$. Then periodicity of $\gamma(\cdot)$ implies the inequalities $\gamma_+ = mS_+ > 0$ and $\gamma_- = mS- > 0$. And the desired characteristics will be

$$\delta = \min\{g_+,\ g_-\},\ K = \max|\gamma(s)|.$$

It's clear $\delta > 0,\ M > 0$.

**Lemma 3.** *The solution of the Cauchy problem (5) may be represented in the form $x(t) = \lambda(t)y(t)$, where $\lambda(\cdot)$ is a nonnegative scalar function satisfying the conditions*

$$\frac{d\lambda}{dt} = \frac{\sqrt{1-(v_\nu\lambda)^2} - v_\tau\lambda}{|y(t)|},\ \lambda(0) = 0. \tag{6}$$

*Proof.* If the function $\lambda = \lambda(t)$ satisfies the conditions (6) then it can be seen by Lemma 1 that the product $\lambda(t)y(t)$ appears in the solution of the Cauchy problem (5). That's why Lemma 1 is a corollary of uniqueness theorem for differential equations. □

**Lemma 4.** $\lambda'(t) > 0$ *a.e. whenever* $\lambda(t) < 1$. *Moreover*

$$\frac{d\lambda(t)}{dt} \geq \frac{\sqrt{1-\lambda(t)}}{K}$$

*as soon as* $\langle y(t),\, v(t)\rangle \leq 0$ $(t \in J)$.

*Proof.* The first part of the statement follows from (6). The second part is the corollary of the inequality

$$\sqrt{1-\lambda^2 q^2} - \lambda p \geq \sqrt{1-\lambda}, \tag{7}$$

where the parameters $\lambda,\ p,\ q$ satisfy the conditions $p \leq 0,\ p^2 + q^2 = 1,\ 0 \leq \lambda \leq 1$. The inequality (7) can easily be checked by raising to square. □

Thus $\lambda(t)$ increases. The task is to prove that $\lambda(t)$ reaches the value 1 meaning $L = M$.

Properties of the function $s(\cdot)$. In Lemma 5 and 6 the time parameter t varies on the time interval $[\alpha,\ \beta]$. Further we put $Y = s([\alpha,\ \beta])$. Notice that $Y \subset [s(\alpha) - \beta + \alpha,\ s(\alpha) + \beta - \alpha]$ in virtue of $|s'(t)| \leq 1$.

**Lemma 5.** *If $X$ is a measurable subset of $Y$ then $mX \leq m\left(s^{-1}(X)\right)$ (where $s^{-1}(X)$ is preimage of the set $X$).*

Proof easily follows from the condition $|s'(t)| \leq 1$.

The following notations are important for the future: if $X \subset Y$, then

$$\begin{aligned} T_+(X) &= \{t | s(t) \in X,\ s'(t) = 1\}, \\ T_-(X) &= \{t | s(t) \in X,\ s'(t) = -1\}, \\ T_\pm(X) &= \{t | s(t) \in X\}. \end{aligned} \tag{8}$$

**Lemma 6.** *If $X$ is open in the topology of $Y$ and $s(\alpha) = s(\beta)$ then*

$$mT_+(X) - mT_-(X) = 0. \tag{9}$$

*Proof.* If $X = Y$, then equality (9) follows from the property $w(t) = +1$ or $w(t) = -1$ and the equality $s(\alpha) = s(\beta)$. Now consider the case $X = (c,\ +\infty) \cap Y$. Here the set $\{t | s(t) \in X\}$ will be union of a countable family of open intervals. The equality (9) is true for each of the components of the union as it has been noticed just above - as in the situation $X = Y$. That's why it is true for the entire union as well. □

For the further notice $m\{t|s(t)=c\}=0$ because otherwise we would get

$$m\{t|s'(t)=0\}>0$$

notwithstanding with the preposition $w(t)=\pm 1$.

This implies that (9) holds for $X=(c_1,\ c_2)\cap Y$ and $X=[c_1,\ c_2)\cap Y$. Finally (9) arises for any open set $X$ in relative topology of $Y$ from the properties above.

**Corollary 1.** *If $X$ is open in the topology of $Y$, then*

$$mT_+(X)-mT_-(X)=\operatorname{sgn}\left(s(\beta)-s(\alpha)\right)m\left(X\cap[s',\ s'']\right) \tag{10}$$

*where $[s',\ s'']=[s(\alpha),\ s(\beta)]$ if $s(\alpha)\le s(\beta)$ and $[s',\ s'']=[s(\beta),\ s(\alpha)]$ in the other case.*

*Proof.* Suppose $s(\alpha)<s(\beta)$. Set $\tilde{\beta}=\beta+s(\beta)-s(\alpha)$ and $\tilde{w}=w(t)$ when $\alpha\le t\le\beta$ and $\tilde{w}(t)=-1$ when $\beta\le t\le\tilde{\beta}$.

Further let

$$\tilde{s}(t)=s(\alpha)+\int_{\alpha}^{t}\tilde{w}(t)dt$$

and $\tilde{T}_+(X),\ \tilde{T}_-(X)$ be a partition of the segment $[\alpha,\ \tilde{\beta}]$ for the function $\tilde{s}\,(\cdot)$ analogical to the partition (8) for $s\,(\cdot)$ . Notice $s(\alpha)=\tilde{s}\,(\alpha)=\tilde{s}(\tilde{\beta})$ . Hence we have $m\tilde{T}_+(X)=m\tilde{T}_+(X)$ and

$$m\tilde{T}_-(X)=mT_-(X)+m\{t\in[\beta,\ \tilde{\beta}\,]|\tilde{s}(t)\in X\}=$$
$$=mT_-(X)+m\{s|s\in[s(\tilde{\beta}),\ s(\beta)],\ s\in X\}=$$
$$=mT_-(X)+m(X\cap[s',\ s'']).$$

The second equality is based on the property $s'(t)=-1$ for $t\in[\beta,\ \tilde{\beta}]$ in according with the definition.

The proof can be provided in the same way if $s(\alpha)>s(\beta)$. Q.E.D. □

During the following lemma t changes in the time interval $[0,\,g]$. Let $t^*$ ( respectively $t_*$ ) be a point of the maximum (minimum) of $s(t)$ on the interval $[0,\,g]$. Notice that $s\left([0,\ g]\right)=[s(t^*),\ s(t_*)]\subset[s(t^*)-g,\ s(t^*)]$.

Below we take the segment $[s(t^*)-\gamma,\ s(t^*)]$ as the interval $I$ and deal with the sets: $T_+(S_+),\ T_-(S_+),\ T_+(S_-),\ T_-(S_-),\ T_\pm(S_0)$.

**The main Lemma 7.**

$$mT_-(S_+)+mT_+(S_-)+mT_\pm(S_0)\ge\frac{\delta}{4}.$$

*Proof.* By virtue of analogy it's enough to consider the situation $s(g)\ge 0$.

Thus $s\left([0,\ g]\right)\subset[s(t_*),\ s(t^*)]\subset[s(t^*)-g,\ s(t^*)]$.

It will be considered two cases separately.

The $1^{st}$ case: $s(t^*)\le g-g_-/2$. Let

$$\tilde{S}_\varepsilon=S_\varepsilon\cap[s(t^*),g] \tag{10}$$

where $\varepsilon$ replaces one of the signs $+$, $-$ or 0. Here we have $mT_+(\tilde{S}_-) = mT_-(\tilde{S}_-)$ because the intersection $\tilde{S}_- \cap [s',\ s'']$ has the measure equal 0 (see, (10)).

Similarly $mT_+(\tilde{S}_-) = mT_-(\tilde{S}_-)$.

Applying this two equalities, the notation (10) and the assumption $s(t^*) \leq g - g_-/2$ we get

$$mT_-(S_+) + mT_+(S_-) + mT_\pm(S_0) \geq$$

$$\geq \frac{1}{2}[mT_+(\tilde{S}_+) + mT_-(\tilde{S}_+) + mT_+(\tilde{S}_-) + mT_-(\tilde{S}_-) + mT_\pm(\tilde{S}_0)] \geq$$

$$\geq \frac{1}{2}m[s(t^*),\ g] = \frac{1}{2}(g - s(t^*)) \geq \frac{g_-}{4} \geq \frac{\delta}{4}.$$

The $2^{nd}$ case: $s(t^*) \leq g - g_-/2$. Here in virtue of Lemma 6 $mT_+(S_-) = mT_-(S_-)$. Further using Lemma 5 and the condition for $s(t^*) \leq g - g_-/2$ we get

$$2mT_+(S_-) \geq mT_+(S_-) + mT_-(S_-) = m\{t|s(t) \in S_-\} \geq$$

$$\geq m\{t|s(t) \in S_- \cap [0,\ s(t^*)]\} =$$

$$= ms^{-1}\{S_- \cap [0,\ s(t^*)]\} \geq m\{S_- \cap [0,\ s(t^*)]\} =$$

$$mS_- + m[0,\ s(t^*)] - m\{S_- \cup [0,\ s(t^*)]\} \geq$$

$$g_- + s(t^*) - g > g_-/2.$$

Here the relation $S_- \cap [0,\ s(t^*)] \subset S_- \cap [s(t_*),\ s(t^*)] \cap ([0,\ g])$ is used in order to apply Lemma 5.

Consequently, here also

$$mT_-(S_+) + mT_+(S_-) + mT_0 \geq g_-/4.$$

Q.E.D. □

**Ending of the proof of theorem 1.** Let $N = [8K/\delta] + 1$. It should be shown for any $w(\cdot) \in W_0$ there exists $t \in [0,\ Ng]$ such that $\lambda(t) = 1$. Assume the contrary i.e. $\lambda(t) < 1$ for all $t \in [0,\ Ng]$. Then Lemma 4 implies

$$\sqrt{1-\lambda(g)} - 1 = -\frac{1}{2}\int\limits_0^g \frac{d\lambda(t)}{dt}\frac{1}{\sqrt{1-\lambda(t)}}dt \leq$$

$$\leq -\frac{1}{2}\left(\int_{T_-(S_+)} \frac{d\lambda(t)}{dt}\frac{1}{\sqrt{1-\lambda(t)}}dt + \int_{T_+(S_-)} \frac{d\lambda(t)}{dt}\frac{1}{\sqrt{1-\lambda(t)}}dt\right) -$$

$$-\frac{1}{2}\int_{T_\pm(S_0)} \frac{d\lambda(t)}{dt}\frac{1}{\sqrt{1-\lambda(t)}}dt$$

Let each integral in the last expression be estimated. If $t \in T_-(S_+)$ i.e. $s(t) \in S_+$ and $w(t) = -1$ then

$$\langle \gamma'(s(t)),\ \gamma(s(t))\rangle > 0,\ v(t) = -\gamma'(s(t)).$$

These relations and $y(t) = \gamma(s(t))$ imply $\langle v(t),\ z(t)\rangle < 0$.

Similarly it $t \in T_+(S_-)$ i.e. $s(t) \in S_-$ and $w(t) = +1$ then $v(t) = \gamma'(s(t))$ that's why $\langle v(t),\ z(t)\rangle < 0$ again.

At last it $t \in T_\pm(S_0)$ then $\langle v(t),\ z(t)\rangle = 0$. Thus $\lambda'(t) \geq \sqrt{1-\lambda}\big/K$ in all cases in virtue of Lemma 4. Applying this inequality and the main Lemma we get

$$\sqrt{1-\lambda(g)} - 1 \leq -\frac{mT_{+-} + mT_{-+} + mT_0}{2K} \leq -\frac{\delta}{8K}.$$

All reasonings done above for the interval $[0,\ g]$ are correct for each of the time intervals $[(k-1)g,\ kg],\ k = 2,\ 3,\ ...,\ N$ of the length $g$. Obtained in this way estimations being summarized will lead to the following

$$\sqrt{1-\lambda(N\lambda)} \leq 1 - \frac{\delta N}{8K} < 0$$

the last inequality contradicts to the assumption has been made in the beginning of the proof. Q.E.D.

Thus, in the generalized Rado's game R-strategy is winning on the time interval $[0,\ \theta]$ where $\theta = 8Kg/\delta$. □

**Remark 1.** As obvious from the proof, Theorem is also true for curves $\mathit{\Gamma}$ given by the absolutely continuous function $\gamma(\cdot)$ if piecewise constant functions $w(\cdot)$ are taken as admissible.

**Remark 2.** Another more simple proof of Theorem 1 follows from the proof of the Theorem 2. The last will be published in [9].

### 2.3. Proof of the theorem 3.

Let $\rho < \sigma$ and the function $\gamma'$ satisfies Lipschitz condition i.e.

$$|\gamma'(s_1) - \gamma'(s_2)| \leq \lambda |s_1 - s_2|,\ \lambda > 0. \tag{11}$$

This condition allows localize the evasion problem.

Let $s_0$ and $\alpha_0$ are positive roots of the equations

$$\frac{\sigma-\rho}{\rho} - \frac{\lambda^2 s^2}{6} - \lambda s = 0, \tag{12}$$

$$\frac{2\alpha}{1-\alpha^2} = \frac{3\lambda s_0}{6-\lambda^2 s_0^2}, \tag{13}$$

respectively ( their existence and uniqueness are clear).

**Lemma 8.** *Let* $\langle x_0, \gamma'(0)\rangle \leq 0$ $(\langle x_0, \gamma'(0)\rangle > 0)$. *Then*

$$|x_0 - \gamma(s_0)| - \rho s_0/\sigma \geq \lambda s_0^2 \qquad \left(|x_0 - \gamma(-s_0)| - \rho s_0/\sigma \geq \lambda s_0^2\right). \tag{14}$$

*Furthermore, if* $|x_0| \geq \lambda s_0^2$, *then*

$$|x_0 - \gamma(s)| - \rho s/\sigma \geq \alpha_0\, |x_0| \quad \left(|x_0 - \gamma(-s)| - \rho s/\sigma \geq \alpha_0\, |x_0|\right),\ 0 \leq s < s_0. \tag{15}$$

*Proof.* From (11) we obtain $|\gamma'(0) - \gamma'(s)| \leq \lambda s,\ s \geq 0$. Squaring this inequality we have $2 - 2\gamma_1'(s) \leq (\lambda s)^2,\ s \geq 0$. Here and in the sequel we denote

$$\gamma_1(s) = \langle \gamma(0), \gamma(s) \rangle,\ \gamma_2(s) = \gamma(s) - \langle \gamma(0), \gamma(s) \rangle \gamma(0).$$

From here according to $\gamma_1(0) = 0$ we obtain $\gamma_1(s) \geq s - \lambda^2 s^3/6,\ s \geq 0$. Then

$$|\gamma_2(s)| \leq \sqrt{s^2 - (\gamma_1(s))^2} \leq \sqrt{s^2 - \left(s - \lambda^2 s^3/6\right)^2} = \lambda s^2 \sqrt{3^{-1} - \lambda^2 s^2/36} \leq \lambda s^2$$

at least on the interval $s \in [0, s_0]$. Hence, by definition of $s_0$ (see, 12), we have

$$\gamma_1(s) - |\gamma_2(s)| - \rho s/\sigma \geq s - \lambda^2 s^3/6 - \lambda s^2 - \rho s/\sigma \geq$$

$$\geq s\left(\frac{\sigma - \rho}{\rho} - \frac{\lambda^2 s^2}{6} - \lambda s\right) \geq 0,\ 0 \leq s \leq s_0. \tag{16}$$

First, we prove the inequality (15). It is not difficult to verify that using (12) and (13) yields

$$|x_0 - \gamma(s)| \geq \left(\gamma_1^2(s) + (|x_0| - |\gamma_2(s)|)^2\right)^{1/2} \geq \left(|x_0|^2 + \gamma_1^2(s)\right)^{1/2} -$$

$$- |\gamma_2(s)| \geq \alpha_0 |x_0| + |\gamma_1(s)| - |\gamma_2(s)|.$$

Therefore, 16) implies (15). We proceed to show the inequality (14). In accordance with (12), we have

$$|x_0 - \gamma(s_0)| - \rho s_0/\sigma \geq |\gamma_1(s_0)| - \rho s_0/\sigma \geq$$

$$\geq s_0 - \lambda^2 s_0^3/6 - \rho s_0/\sigma = s_0 - s_0\left(\lambda^2 s_0^2/6 - \rho/\sigma\right) =$$

$$= s_0 - s_0(1 - \lambda s_0) = \lambda s_0^2$$

This completes the proof of the lemma. □

**Proof of the theorem 3.** Let $u(\cdot)$ be a control of the pursuer $L$ and $x(\cdot)$ is the solution of the Cauchy problem $\dot{x} = u(t)$, $x(0) = x_0$. We define the strategy $V_*$ and corresponding trajectory $y(\cdot)$ of the evader by stepwise process. Let $\Delta = \{t_0,\ t_1,\ t_2,\ ...\}$ is the partition of the interval $[0, \infty)$, $t_i = is_0/\sigma,\ i = 0,\ 1,\ ....$

We assume that the trajectory $y(\cdot)$ is defined up to the instant of time $t_i$. We set $x(t_i) = x_i,\ y(t_i) = y_i,\ (i = 1, 2,\ ....\ )$ We continue the trajectory $y(\cdot)$ on the time interval $[t_i, t_{i+1}]$ in the following way: if $\langle x_i - y_i, \gamma'(s(y_i)) \rangle \leq 0$, then $y(\cdot)$ is defined as the solution of the Cauchy problem $\dot{y} = \sigma\gamma'(s(y))$, , $y(t_i) = y_i$, and if $\langle x - y, \gamma'(s(y)) \rangle > 0$, then $y(\cdot)$ is defined as the solution of the Cauchy problem $\dot{y} = -\sigma\gamma'(s(y))$ , $y(t_i) = y_i$. Note that in both cases $|\dot{y}(t)| \equiv 1$.

Let $\langle x_0 - y_0, \gamma'(s(y_0)) \rangle \leq 0$. Then at $0 \leq t \leq s_0/\sigma$ we have $s(t) = \sigma t$ and

$$|x(t) - y(t)| \geq \left| x_0 - \gamma(s(t)) + \int_0^t u(t)dt \right| \geq |x_0 - \gamma(s(t))| -$$

$$-\rho t = |x_0 - \gamma(s(t))| - \rho s(t)/\sigma.$$

Hence,

$$|x(t) - y(t)| \geq \min_{0 \leq t \leq t_1} (|x_0 - \gamma(s(t))| - \rho s(t)/\sigma) > 0,\ 0 \leq t \leq t_1. \qquad (17)$$

Moreover, by using the inequality (14) we have

$$|x(t_1) - y(t_1)| \geq |x_0 - \gamma(s(t_1))| - \rho s(t_1)/\sigma = |x_0 - \gamma(s_0)| - \rho s_0/\sigma \geq \lambda s_0^2.$$

(In case of $\langle x_0 - y_0, \gamma'(s(y_0)) \rangle > 0$ arguments are the same.)

The last relation allows us to continue the evasion process to the next time interval $[t_1, t_2] = [s_0/\sigma,\ 2s_0/\sigma\ ]$,

while preserving the estimate

$$|x(t) - y(t)| \geq \alpha_0 \lambda s_0^2,\ \ |x(t_2) - y(t_2)| \geq \lambda s_0^2.$$

This follows from lemma 8 if we take $y(t_1)$, $x(t_1)$ as the initial position of players and apply the reasoning above. This process can be continued infinitely. As the evader each time passes the path of length $s_0$, then the estimate $|x(t) - y(t)| \geq \alpha_0 \lambda s_0^2$ is true for all $t \geq t_1$.

Thus, the point $M$ can act in the game in a such way that the distance between $M$ and $L$ estimates by

$$|x(t) - y(t)| \geq \min_{0 \leq t \leq s_0/\sigma} (|x_0 - \gamma(s(t))| - \rho s(t)/\sigma) > 0,\ \ 0 \leq t \leq s_0/\sigma,$$

$$|x(t) - y(t)| \geq \alpha_0 \lambda s_0^2,\ \ t \geq s_0/\sigma.$$

The proof of the theorem is complete. □

**Remark 3.** Let $\Gamma$ be an arbitrary smooth curve with unbounded curvature. As the curvature $K(s)$ of the curve $\Gamma$ is a continuous function, then it is bounded on every interval $[s_1, s_2]$ and, hence, the function $\gamma\prime(s)$ satisfies the Lifschitz condition on each bounded interval. By using this fact, it can be shown as it was done in proof of theorem 3 that the evasion problem is solvable. However, the evader not always can ensure the estimate for the distance below by positive constant (for example, if $\Gamma$ is graph of the function $y = xsin(1/x)$ ).

## 3. Evasion from Many Pursuers Along the Smooth Surface

We consider the differential game with many pursuers moving all over the space and evader moving along given surface. We'll construct a positional evasion strategy ensuring the estimate for the distance between players below by positive constant, provided the number of pursuers doesn't exceed dimension of the surface. We are given $n$ dimensional surface $\Gamma$ from the class of smoothness $C^2$ in the space $R^{n+1}$. At each point principal curvature of the surface $\Gamma$ is bounded in absolute value by number $1/r$, $r > 0$ and any geodesic on the surface $\Gamma$ is either closed or from each its point in both direction has infinite length.

Movement of the pursuers $L_i$ and the evader $M$ are described by

$$L_i:\ \ \dot{x}_i = u_i,\ \ x_i(0) = x_{i0}, \quad M:\ \ \dot{y} = v,\ \ y(0) = y_0,$$

in the space $R^{n+1}$, where $x_{io} \neq y_0$; $u_i$, $v$ are control vectors that satisfy conditions $|u_i| \leq \rho_i$, $|v| \leq \sigma$, $i = 1, 2, ..., k$. During the game the evader $M$ can't leave the surface $\Gamma$: $y(t) \in \Gamma$, $t \geq 0$.

Aim of the present section is to construct the strategy of the evader ensuring estimate for the distance between evader and closest to pursuer. Control and strategy of players are defined as well as in section 1.

**Theorem 4.** *Let $n \geq 2$, $k \leq n$. Then evasion problem is solvable. Furthermore, there exist a positive constant $\mu = \mu(x_0, y_0, \rho, \sigma, r)$ and a positional strategy of $M$ such that for any admissible control of $L = \{L_1, L_2, ..., L_k\}$ the inequality*

$$\min_{1\leq i\leq k} \left|x_i(t) - y_(t)\right| \geq \mu$$

*holds for all $t$, $t \geq 0$.*

*Proof.* Let $K(z, m)$ be the normal curvature of the surface $\Gamma$ in direction of the unit tangent vector $m$ at $z$, and $\gamma_z(\cdot, m)$ be a geodesic (parameterized by the length of the arc) on $\Gamma$ such that $\gamma_z(0, m) = z$, $\gamma_z'(0, m) = m$ $|\gamma_z'(z, m)| \equiv 1$. Then $\gamma_z''(s, m) = K_{zm}(s)N_{zx}(s)$, where $N$ is unit oriented normal vector field in $\Gamma$, and $N_{zm}(s)$ is its value at $\gamma_z(s, m)$. Hence, $|\gamma_z''(s, m)| \leq 1/r$ and all the functions of the form $\gamma_z'(\cdot, m)$ satisfy the Lifschitz condition with the same constant $\lambda > 0$ :

$$|\gamma_z'(s_1, m) - \gamma_z'(s_2, m)| \leq \lambda |s_1 - s_2| . \tag{18}$$

At $t = 0$ we define the unit vector $v_0$ from the tangent space $T_{y(0)}$ so that

$$\langle v_0, x_{io} - y_0 \rangle = \min_{|v|=1} \max_{1\leq i\leq k} \langle v, x_{io} - y_0 \rangle ,$$

where $v \in T_{y(0)}$ and construct the geodesic $\gamma_{y(0)}(s, v_0)$, $s \geq 0$. It is not difficult to verify by using $k \leq n$ that $\langle v_0, x_{io} - y_0 \rangle \leq 0$. As (18) holds, then according to Theorem 3 the evader $M$ preserves the inequality

$$|x_i(t) - y(t)| \geq \min_{0\leq t\leq s_0} (|\gamma(s) - x_{i0}| - \rho s/\sigma) > 0$$

at $0 \leq s \leq s_0/\sigma$. while moving along geodesic $\gamma_{y(0)}(s, v_0)$, $s \geq 0$, with the speed $\sigma$ (see,(17))

Moreover, as in proof of theorem 3 applying lemma 7 at $t = s_0/\sigma$ we have

$$|x_i(s_0/\sigma) - y(s_0/\sigma)| \geq \lambda s_0^2.$$

his allows to repeat the process infinitely. Every time starting from the second, applying lemma 7 as in proof of theorem 3, it can be shown preservation of the inequality $|x_i(t) - y(t)| \geq \alpha_0 \lambda s_0^2$ for all $t \geq s_0/\sigma$ and $i = 1, 2, ..., k$. The proof of the theorem is complete. □

**Remark 4.** Let $\Gamma$ be any smooth surface of unbounded curvature. As the principal curvatures $K(s)$ of the surface $\Gamma$ is continuous function, then it is bounded on each interval $[s_1, s_2]$. By using this fact it can be shown as in proof of theorem 4 that evasion problem is solvable. If the surface $\Gamma$ is not smooth (for example, $\Gamma$ is a cone), then theorem 4, in general, is not valid (see, appendix 2)

## Appendix

### 1. Evasion along the Graph

Let $\Gamma$ is a finite graph. Finite graph is a set consisting of finite number of points (called vertexes) and arcs of curves (called edges) connecting pairs of these points, each point being connected at least one another point.

We assume that $\Gamma$ is a singly connected and each edge of the graph is image of a mapping $\gamma : [0, \alpha] \to R^n$ from the class of smoothness $C^2$. Evader moves along the edges of the graph, and pursuer moves all over the space.

Let $A$ be a vertex of the graph $\Gamma$ and $\gamma_i(\cdot)$, $i = 1, 2, ..., m$ is the set of all edges of the graph $\Gamma$ with one end at $A$ and $A = \gamma_i(0)$ for all $i = 1, 2, ..., m$. We say that vertex $A$ is convenient for evasion, if convex hull of all vectors $\gamma'_(0)$ contains $A$. By using lemma 7 it can be shown as in proof of theorem 3 that if $\rho < \sigma$ and all vertices of the graph $\Gamma$ are convenient for evasion, then in this version of the game it is solvable the evasion problem by preserving estimates for distances between players below. If $\rho = \sigma$, then arguing as in proof of theorem 1 one can verified that point $L$ by using $R$-strategy will be able to complete pursuit from any initial position.

### 2. The Case when the evader moves along piecewise smooth curve

**Theorem 5.** *Let $\Gamma$ be a closed piecewise smooth Jordan curve and $2\alpha$ is the least angle between tangent rays at breaking points, $\alpha \in [0, \pi/2]$. If $\rho > \sigma sin\alpha$, then there exists an initial point $x_0, y_0$, for that pursuit problem is solvable. If here $\Gamma$ is convex or $\rho = \sigma$, then pursuit problem is solvable for any initial positions.*

Proof of both parts of theorem are derived from the following statement. Let $\Delta$ be isosceles triangle with the base $BC$ and vertex $A$, $\angle A = 2\alpha, \alpha \in (0, \pi/2)$. Let $x_0 \in BC$, $y_0 \in AB \bigcup AC$ and segment with ends $x_0, y_0$ are perpendicular to $BC$. Then strategy of parallel approach (Petrosyan, 1977, Azamov and Samatov, 2000) ensures solvability of the pursuit problem for the time $h/\sqrt{\rho^2 - \sigma^2 sin^2\alpha}$, where $h$ is hight of $\Delta$.

The following statement can be also proved on the basis of the same consideration.

**Theorem 6.** *Let $\Gamma$ be an arbitrary finite graph in the space $R^n$ with straight edges. Then there exists $k \in (0, 1)$ such that at $\rho > k\sigma$ pursuit problem is solvable for any initial position.*

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# On the Computation of Semivalues for TU Games

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**Abstract** In an earlier paper, (Dragan, 2006a) , we proved that every Least Square Value is the Shapley Value of a game obtained by rescaling from the given game. In the paper where the Least Square Values were introduced by Ruiz, Valenciano and Zarzuelo, the authors have shown that the efficient normalization of a Semivalue is a Least Square Value. In the present paper, we develop the idea suggested by these two results, and we obtain a direct relationship between the Efficient normalization of a Semivalue and the Shapley Value (see also Dragan, 2006b). The main tools for proofs were the so called Average per capita formulas we proved earlier for the Shapley Value (Dragan, 1992) and for a Semivalue (Dragan and Martinez-Legas, 2001). Note that the connection between the Efficient normalization of a Semivalue and the Shapley Value has been used for computing a Semivalue, via a rescaling of the worth of coalitions in the given game. In this paper, the main purpose is to offer two other alternatives for the computation of a Semivalue, via the Shapley Value; beside the rescaling done before the computation, we consider rescalings within the computation. As seen above, this paper contains results from different sources, so that to make the paper self contained we shall be proving below our results together with the new results appearing here for the first time. Our proofs are algebraic, in opposition to those found in Ruiz et al., which are axiomatic. The direct connection between a Semivalue and the Shapley Value does not need any reference to the Least Square Values, which may well be unknown to the reader of the present paper.
In the first section, we prove the Average per capita formula for Semivalues, (Theorem 1), from which we derive our earlier Average per capita formula for the Shapley Value, to be used later. In the second section , we derive an Average per capita formula for the Efficient normalization of a Semivalue, by computing the efficiency term, (Theorem 2), as well as the main results, showing the connection between the Efficient normalization and the Shapley Value, (Theorems 3 and 4). In the third section, we discuss a first alternative for the computation of a Semivalue via the Shapley Value, illustrated in Example 1: we compute the needed ratios from the Average per capita formula for the Shapley Value and rescaling is done only $n-1$ times, over these ratios. In the last section, we discuss a second alternative method for computing Shapley Values; the algorithm has been invented for computing Weighted Shapley Values and it will be adapted to the computation of the Semivalues. Some formulas are derived from this new method for computing the Weighted Shapley Values, based upon the null space of the Weighted Shapley Value operator, (Dragan, 2008). Note that the Semivalues are not Weighted Shapley Values. The Example 2 is illustrating the algorithm on the same 4-person game as before. The motivation for the present work was the fact that we know other works for computing the Shapley Value, but no computational work for Semivalues is known to us.

**Keywords:** Shapley Value, Semivalue, Average per capita formula, Least Square Value, Weighted Shapley Value.

## 1. Average per Capita Formula for Semivalues

Let $G^N$ be the vector space of cooperative TU games with a fixed set of players $N$, $n = |N|$. Consider a nonnegative weight vector $p^n \in R^n$, satisfying the normalization condition

$$\sum_{s=1}^{n} \binom{n-1}{s-1} p_s^n = 1. \tag{1}$$

The Semivalues associated with $p^n$ have been introduced axiomatically by P. Dubey, A. Neyman and R. J. Weber, (1981), as values on $G^N$ and even on more general structures, uniquely defined by a group of axioms suggested by the axioms of the Shapley Value. For $G^N$ they proved that a Semivalue associated with a weight vector $p^n$ is given by the formula

$$SE_i(N, v) = \sum_{S:\ i \in S \subseteq N} p_s^n [v(S) - v(S - \{i\})], \quad \forall i \in N, \tag{2}$$

where $s = |S|$, and $p_s^n$ is the common weight of the marginal contributions for all coalitions of size $s$. We take this formula as the definition of Semivalues on $G^N$. To define the Semivalues on the union of all spaces $G^N$ when $N$ is arbitrary, we mean for player sets of different sizes, we need a sequence of nonnegative weight vectors $p^1, p^2, \ldots, p^n, \ldots$, all satisfying normalization conditions similar to (1), that is $p_1^1 = 1, \quad p_1^2 + p_2^2 = 1, \quad p_1^3 + 2p_2^3 + p_3^3 = 1, \ldots$ and so on. The definition of a Semivalue on $G^T$ is given by a formula similar to (2), where $N$ is replaced by $T$, $n$ by $t$, and $p^n$ by $p^t$. However, the sequence of weight vectors are supposed to satisfy what we call "inverse Pascal triangle" conditions

$$p_s^{t-1} = p_s^t + p_{s+1}^t, \quad s = 1, ..., t-1. \tag{3}$$

It easy to see that if $p^n$ is given, and (3) hold, then all weight vectors $p^t, t \leq n$, are uniquely determined and they satisfy the corresponding normalization conditions (1).

Note the important fact that among the Semivalues, for $p_s^n = \frac{(s-1)!(n-s)!}{n!}$, we get the Shapley value, for $p_s^n = 2^{1-n}$, we get the Banzhaf value, and many other values are also Semivalues. Therefore, if we prove an Average per capita formula for a Semivalue, then we get such a formula for the Shapley Value, for the Banzhaf Value, and for other values, by chosing particular expressions of the weight vectors.

We call an Average per capita formula, any formula in which occur only the average worth of various coalitions, defined as follows:

$$v_s = \frac{\sum_{|S|=s} v(S)}{\binom{n}{s}}, \quad v_s^i = \frac{\sum_{|S|=s, i \notin S} v(S)}{\binom{n-1}{s}}, \quad \forall i \in N, \quad s = 1, ..., n-1. \tag{4}$$

Clearly, $v_s$ is the average worth of coalitions of size $s$, while $v_s^i$ is the average worth of coalitions of size $s$ which do not contain player $i$. If we denote $v_n = v(N)$, then there are $n$ averages $v_s$, and $n(n-1)$ averages $v_s^i$, hence all together there are $n^2$ numbers associated with a given game. Let us introduce also new weights, defined for all $t \leq n$ by

$$q_s^t = \frac{p_s^t}{\gamma_s^t}, \quad s = 1, ..., t, \tag{5}$$

where $\gamma_s^t = (t!)^{-1}(s-1)!(t-s)!$, that is the weights for the Shapley Value on $G^T$.

**Theorem 1 (Dragan, and Martinez-Legas, 2001).** *Let $SE : G^N \to R^n$ be a Semivalue associated with a nonnegative weight vector $p^n$ satisfying the normalization condition (1). Let $q^n$ be the nonnegative weight vector defined by (5). Then, SE defined by (2), may be expressed in terms of the averages (4) and the weights (5) as*

$$SE_i(N, v) = q_n^n \frac{v_n}{n} + \sum_{s=1}^{n-1} \frac{q_s^n v_s - q_s^{n-1} v_s^i}{s}, \quad \forall i \in N. \tag{6}$$

For $q_s^n = 1$, $s = 1, \ldots, n$, that is $p^n = \gamma^n$, we obtain:

**Corollary 1.** *The Shapley Value of the game $(N, v)$, is given by*

$$SH_i(N, v) = \frac{v_n}{n} + \sum_{s=1}^{n-1} \frac{v_s - v_s^i}{s}, \quad \forall i \in N. \tag{7}$$

*Proof.* For $i \in N$ fixed, rewrite (2) as

$$SE_i(N, v) = p_n^n v(N) + \sum_{S:i\in S\subset N} p_s^n v(S) - \sum_{S:i\in S\subseteq N} p_s^n v(S - \{i\}); \tag{8}$$

now, write the two sums separately as

$$\sum_{S:i\in S\subset N} p_s^n v(S) = \sum_{s=1}^{n-1} p_s^n (\sum_{|S|=s, i\in S\subset N} v(S)) = \sum_{s=1}^{n-1} p_s^n (\sum_{|S|=s} v(S) - \\ - \sum_{|S|=s, i\notin S} v(S)), \tag{9}$$

and

$$\sum_{S:i\in S\subseteq N} p_s^n v(S - \{i\}) = \sum_{s=1}^{n-1} p_{s+1}^n (\sum_{|S|=s} v(S)). \tag{10}$$

From (8), (9) and (10), with notations (4), we obtain

$$SE_i(N, v) = p_n^n v_n + \sum_{s=1}^{n-1} [p_s^n \binom{n}{s} v_s - p_s^{n-1} \binom{n-1}{s} v_s^i], \tag{11}$$

where we have used (3) for $t = n$. If in (11) we introduce the new weights (5) by noticing that $p_s^n \binom{n}{s} = s^{-1} q_s^n$, $s = 1, \ldots, n$, we get (6).

Note that the new weights should satisfy the normalization condition $\sum_{s=1}^{n} q_s^n = n$, derived from (1) and (5), and the inverse Pascal triangle conditions (3) become

$$q_s^{t-1} = (1 - st^{-1})q_s^t + st^{-1}q_{s+1}^t, \quad s = 1, \ldots, t-1. \tag{12}$$

In the next section we derive a new Average per capita formula to express the normalizing term for the Efficient normalization of the Semivalue, and we shall derive also all needed results allowing the statement of the algorithm for its computation.

## 2. Average per Capita Formula for the Efficiency Term

In the paper where the Least Square Values have been introduced by Ruiz, Valenciano and Zarzuelo (1998), the authors defined what they called the Efficient normalization of a Semivalue $SE$, associated with a nonnegative weight vector $p^n = (p_s^n)$. This is the value $ESE : G^N \to R^n$ written as

$$ESE_i(N, v) = SE_i(N, v) + \alpha, \quad \forall i \in N, \tag{13}$$

with $\alpha$ such that $ESE$ is efficient, that is

$$\alpha = \frac{1}{n}[v(N) - \sum_{j \in N} SE_j(N, v)]. \tag{14}$$

We call $\alpha$ the efficiency term and we intend to derive an Average per capita formula for $\alpha$, a fact which will be useful below theoretically and practically in the algorithm for computing the Semivalue. Of course, the normalization can be done in other ways, too. From (6), we obtain

$$\begin{aligned} \sum_{j \in N} SE_j(N, v) &= q_n^n v_n + \sum_{s=1}^{n-1} \frac{n q_s^n v_s - q_s^{n-1} \sum_{j \in N} v_s^j}{s} \\ &= q_n^n v_n + n \sum_{s=1}^{n-1} \frac{(q_s^n - q_s^{n-1}) v_s}{s}, \end{aligned} \tag{15}$$

where we have used the equality $\sum_{j \in N} v_s^j = n v_s$, holding for all $s = 1, \ldots, n-1$. In this way, from (13) and (14) we proved the result:

**Theorem 2.** *The efficiency term for the additive normalization of a Semivalue is given by the Average per capita formula*

$$\alpha = (1 - q_n^n)\frac{v_n}{n} - \sum_{s=1}^{n-1} \frac{(q_s^n - q_s^{n-1}) v_s}{s}. \tag{16}$$

Notice that this formula is expressing $\alpha$ in terms of the averages $v_s$ only, so that it is easy to compute the efficiency term. Putting together the Average per capita formulas (6) and (16) of the Theorems 1 and 2, we proved algebraically the main result for the Efficient normalization of a Semivalue:

**Theorem 3.** *The Efficient normalization of a Semivalue associated with a nonnegative weight vector $p^n = (p_s^n)$ is given by*

$$ESE_i(N, v) = \frac{v_n}{n} + \sum_{s=1}^{n-1} q_s^{n-1} \frac{v_s - v_s^i}{s}, \quad \forall i \in N, \tag{17}$$

*where $q_s^{n-1}$ are expressed in terms of $p^n$ and $\gamma^n$ as*

$$q_s^{n-1} = \frac{p_s^n + p_{s+1}^n}{\gamma_s^n + \gamma_{s+1}^n}, \quad s = 1, \ldots, n-1. \tag{18}$$

Note that (18) is derived from (5) for $t = n-1$ and (3) for $t = n$, taking into account that The Shapley weights satisfy also these conditions. Of course, for the Shapley Value we have $\alpha = 0$ and $q_s^{n-1} = 1$, so that we get (7). Instead, for the Banzhaf Value we get a new formula, where $q_s^{n-1} = 2^{2-n}(\gamma_s^{n-1})^{-1}$.

Note that Theorem 3 can be derived from the relationship axiomatically proved by Ruiz et al. between the Efficient normalization of a Semivalue and the Least Square Values and our relationship between the Least Square Values and the Shapley Values proved earlier.
In the present paper, it was no need of the Least Square Values, and therefore we have chosen the above proof.

Consider a game $v \in G^N$ and rescale it by introducing the new game $w \in G^N$ :

$$w(N) = v(N), \quad w(S) = q_s^{n-1} v(S), \quad \forall S \subset N. \tag{19}$$

By formulas similar to (4), we have

$$w_s = q_s^{n-1} v_s, \quad w_s^i = q_s^{n-1} v_s^i, \quad \forall i \in N, \quad s = 1, \ldots, n-1. \tag{20}$$

Therefore, from (17) and (20) we obtain the right hand side in (7), for the new game $(N, w)$; we proved:

**Theorem 4.** *The Efficient normalization of the Semivalue of a game $v \in G^N$, associated with the weight vector $p^n \in R_+^n$, is the Shapley Value of a new game $w \in G^N$, obtained by a rescaling of the given game with factors $q_s^{n-1}$ for the worth of coalitions of size $s$, $s = 1, \ldots, n-1$, derived from the weight vector $p^n$ and the Shapley weights by formulas (18).*

Of course, if we subtract $\alpha(S) = \alpha s$, $\forall S \subseteq N$, from the worth $v(S)$, of the given game and use the linearity of the Shapley Value, we get also that the Semivalue is a Shapley Value. Notice that formula (17) of Theorem 3, as well as Theorem 4, are helpful in computing a Semivalue of a TU game via the Shapley Value, as it will be discussed in the next section.

## 3. The Average per Capita Formulas and the Computation of Semivalues

As suggested by the formula (17), we can compute first the ratios needed in the computation of the Shapley Value of the given game, and rescale the ratios; obviously, we may rescale the given game first and further compute the Shapley Value as justified by Theorem 4, as done in (Dragan, 2006b, p. 1547). In both cases, we compute the Efficient normalization of the Semivalue and from each component, we subtract the number $\alpha$. We give here an example based upon the above remark

*Example 1.* Consider a four person simple game in which the winning coalitions are {1}, {2}, {1,2}, {1,3}, {2,3}, (1,2,3}, {1,2,4} and {1,2,3,4} and the weight vector $p^4 \in R^4$ is given by $p^4 = (\frac{1}{8}, \frac{1}{8}, \frac{1}{18}, \frac{1}{3})$. From (3) for $t = 4$, we get the weight vector

$p^3 = (\frac{1}{4}, \frac{13}{72}, \frac{7}{18})$; both $p^4$and $p^3$ satisfy the corresponding normalization condition (1). Taking into account that $\gamma^3 = (\frac{1}{3}, \frac{1}{6}, \frac{1}{3})$, we use (5) for $t = 3$, to obtain $q^3 = (\frac{3}{4}, \frac{13}{12}, \frac{7}{6})$, the vector which is weighing the ratios in (17), to get the Efficient normalization of the Semivalues. Now, the usual computation of ratios in the Shapley Value Average per capita formula for the given game provides

$$v_1 = \frac{1}{2}, \quad v_1^1 = v_1^2 = \frac{1}{3}, \quad v_1^3 = v_1^4 = \frac{2}{3}, \rightarrow (v_1 - v_1^s) = (\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}),$$

$$v_2 = \frac{1}{2}, \quad v_2^1 = v_2^2 = v_2^3 = \frac{1}{3}, \quad v_2^4 = 1, \rightarrow \frac{1}{2}(v_2 - v_2^s) = (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, -\frac{1}{4}), \tag{21}$$

$$v_3 = \frac{1}{2}, \quad v_3^1 = v_3^2 = 0, \quad v_3^3 = v_3^4 = 1, \rightarrow \frac{1}{3}(v_3 - v_3^s) = (\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}).$$

By using $q^3$ and formula (17), we obtain

$$q_1^3(v_1 - v_1^s) + q_2^3\frac{v_2 - v_2^s}{2} + q_3^3\frac{v_3 - v_3^s}{3} = (\frac{59}{144}, \frac{59}{144}, -\frac{11}{48}, -\frac{85}{144}), \tag{22}$$

so that by adding $\frac{1}{4}$ to each component, we have the Efficient normalization

$$ESE(N, v) = (\frac{95}{144}, \frac{95}{144}, \frac{1}{48}, -\frac{49}{144}). \tag{23}$$

Now, we compute $\alpha$ by means of (16); we need $q^4 = (\frac{1}{2}, \frac{3}{2}, \frac{2}{3}, \frac{4}{3})$ and $q^3$ computed above, to use in (16) together with the averages $v_1, v_2, v_3$, to get $\alpha = \frac{1}{48}$. We got

$$SE(N, v) = ESE(N, v) - \alpha e = (\frac{23}{36}, \frac{23}{36}, 0, -\frac{13}{36}). \tag{24}$$

Of course, we may verify the answer by means of formula (2).

As shown by the work in the Example 1, the algorithm has the operations:
a) compute $q^{n-1}$, b) compute the ratios appearing in the Average per capita formula for the Shapley Value, of the given game, that is $\frac{v_s - v_s^i}{s}$, for all $s = 1, \ldots, n-1$, followed by the weighted sum of these ratios; c) for all $i \in N$, add $\frac{v_n}{n}$, and we got the Efficient normalization of the Semivalue. d) Compute $\alpha$ and subtract it from each component, to get the Semivalue. Note that above we have used our Average per capita formula (7).

A different approach is offered by computing the Shapley Value via a recent algorithm, (Dragan, 2008), based upon the null space of the Shapley Value, computed for the first time in (Dragan et.al, 1989). In fact the algorithm has been proved for the Weighted Shapley Value, which is not a Semivalue, and applied to the Shapley Value. To make the paper self contained let us describe instead the null space for the Shapley Value and the algorithm, applied to Semivalues, in the next section.

## 4. The Null Space of the Shapley Value and the Computation of Semivalues

In $G^N$, consider the set of games

$$W = \{W_S \in G^N : S \subseteq N, S \neq \emptyset\}, \forall S \subset N, \tag{25}$$

defined by

$$\begin{aligned} W_S(T) &= s, \quad \text{if} \quad T = S,\\ W_S(T) &= -1, \quad \text{if} \quad T = S \cup \{j\}, j \notin S,\\ W_S(T) &= 0, \quad \text{otherwise.} \end{aligned} \tag{26}$$

For $S = N$ the middle case can not occur.

For example, when $N = \{1, 2, 3\}$, this set of games is

| | $W_1$ | $W_2$ | $W_3$ | $W_{12}$ | $W_{13}$ | $W_{23}$ | $W_{123}$ |
|---|---|---|---|---|---|---|---|
| $\{1\}$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\{2\}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $\{3\}$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $\{1,2\}$ | $-1$ | $-1$ | 0 | 2 | 0 | 0 | 0 |
| $\{1,3\}$ | $-1$ | 0 | $-1$ | 0 | 2 | 0 | 0 |
| $\{2,3\}$ | 0 | $-1$ | $-1$ | 0 | 0 | 2 | 0 |
| $\{1,2,3\}$ | 0 | 0 | 0 | $-1$ | $-1$ | $-1$ | 3 |

It is obvious that this is a linearly independent vector set of 7 vectors in $G^{\{1,2,3\}} = R^7$, hence this is a basis for the space, and the same thing is true in general for $W$ in $G^N = R^{2^n-1}$. The similar basis of 15 vectors in $G^{\{1,2,3,4\}} = R^{15}$, will be used later in an example.

The computation of the Shapley Value for the games in basis W will be further needed and we shall do it now by means of the concept of Potential introduced by Hart and Mas Colell (1987). Recall that if the potential function is known, $P : G^N \to R$, then we have

$$SH_i(N, v) = P(N, v) - P(N - \{i\}, v), \forall i \in N. \tag{27}$$

An useful result for our purpose is the explicit expression of the potential in term of the coalitional form of the game offered also in the cited paper:

$$P(N, v) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} v(S). \tag{28}$$

By (28), the potentials for the games (26) are

$$P(N, W_S) = 0, \forall S \subset N, \quad P(N, W_N) = 1. \tag{29}$$

We can prove the following

**Lemma 1.** *The Shapley Value of the basic vectors in W are*

$$SH(N, W_S) = 0, \quad \text{if} \quad |S| \leq n-2, \quad SH(N, W_{N-\{i\}}) = -e_i, \quad i = 1, \ldots, n, \tag{30}$$

*where $e_i$ is a vector in $R^n$ with the $i-th$ component equal one, and the others equal zero, and*

$$SH(N, W_N) = e, \tag{31}$$

*where $e$ has all components equal one.*

*Proof.* Now, by using the potentials (29) in (27) written for $v = W_S$, and performing some elementary operations, we get that in the right hand side of (27) we have the first term equal one when $S = N$, while the second term in the restricted game $(N - \{i\}, v)$ equals zero, or the second term equal $-1$ when $S = N - \{i\}$, while the first term is zero; both are zero when $S = N - \{j\}, j \neq i$. In all the other cases the Shapley Value equals zero. From the Lemma follows the following:

**Corollary 2.** *The null space of the Shapley Value is generated by the basic games in the set*

$$\{W_S \in G^N : S \subset N, |S| \leq n - 2\} \cup \{W_N + \sum_{i \in N} W_{N-\{i\}}\}. \tag{32}$$

*Proof.* This set has $2^n - n - 1$ linearly independent games from the null space. As the range of the operator is $R^n$, by a well known theorem of linear algebra, the entire null space is generated by the games in the set (32).

The algorithm that we intend to state is borrowing an idea from Maschler's algorithm for computing a Shapley Value, namely the given game can be transformed into a new game in which the worth of the characteristic function are vanishing one at a time in each step. However, while Maschler's algorithm is using sequential allocations of the worth of coalitions, until the entire worth is allocated, in our algorithm we are transporting the worth to coalitions of higher sizes, until the new game has zero worth for all coalitions of sizes smaller than or equal to $n - 2$. As it will be seen below, the Shapley Value can now be easily computed. Of course, we shall apply this to the rescaling of the given game, in order to compute the Efficient normalization of a Semivalue. The rescaling is easier because all worth we are rescaling belong to coalitions of the same size. Now, the transformation of the given game into a new game with the same Shapley Value is made by using combinations of the games from the null space of the operator, which will not change the value. A similar approach has been used earlier for the computation of the Weighted Shapley Values and the Kalai-Samet Values (Dragan, 2008).

Let $s$ be an integer, $1 \leq s \leq n - 2$, such that either $s = 1$, or if $s \geq 2$, suppose that a game $v^{s-1} \in G^N$ derived from $v^o = v$ is available, satisfying

$$v^{s-1}(T) = 0, \forall T \subset N, |T| \leq s - 1, \tag{33}$$

and

$$SH(N, v^{s-1}) = SH(N, v). \tag{34}$$

Suppose that $v^{s-1}(T) \neq 0$ for some coalition $T \subset N$ with $|T| = s$, and $s \leq n - 2$. Then, the derivation of the game $v^s$ satisfying conditions similar to (33) and (34), is explained by means of the result:

**Theorem 5.** *Let $v^{s-1} \in G^N$ be a game satisfying (33) and (34) and $s \leq n-2$. Then, the game*

$$v^s = v^{s-1} - \sum_{T:|T|=s} \frac{v^{s-1}(T)}{|T|}.W_T, \tag{35}$$

*where $W_T$ are the games (26), satisfy the conditions obtained from (33) and (34) by changing $s$ into $s+1$.*

*Proof.* As the games $W_T$ with $|T| \leq n-2$ belong to the null space of the Shapley Value, the equalities similar to (34) when $s$ is replaced by $s+1$, hold; it remains to show that the conditions similar to (33) still hold. The equality (35) can be written by components

$$v^s(U) = v^{s-1}(U) - \frac{1}{s} \sum_{T:|T|=s} v^{s-1}(T).W_T(U), \forall U \subseteq N. \tag{36}$$

If $|U| \leq s-1$, then $W_T(U) = 0$ for all $U \subseteq N$, when $|T| = s$, hence from (33) and (36) we get $v^s(U) = 0$. If $|U| = s$, as $|T| = s$ in the sum, then we have $W_T(U) \neq 0$ only when $U = T$, and in this case we get $W_T(U) = W_T(T) = |T|$, so that from (36), taking into account (33), we have $v^s(U) = 0$. Hence, for all coalitions $U$ with $|U| \leq s$ we got $v^s(U) = 0$.

Note that in (35), or (36), taking into account the expressions (26) of basic games, we have in the sum non zero terms only for some coalitions $U$ with $|U| = s+1$; namely, this happens when $T = U - \{j\}, j \in U$, and in this case $W_T(T \cup \{j\}) = -1$. Therefore, we obtain from Theorem 5, by means of (26), the formula which allows the computation of the characteristic function for the game obtained at the end of step $s$; as proved in Theorem 5, all worth for coalitions of sizes at most $s$ are zero, all worth for coalitions of sizes at least $s+2$ are unchanged, and the worth for coalitions of size $s+1$ are provided by the following:

**Corollary 3.** *For any coalition $U$ of size $s+1$, the linear transformation (35), which makes the worth for coalitions of size $s$ equal to zero, gives*

$$v^s(U) = v^{s-1}(U) + \frac{1}{s} \sum_{j \in U} v^{s-1}(U - \{j\}). \tag{37}$$

*The worth of coalitions of higher sizes remain the same.*

Now, Theorem 4 and Corollary 3, will show how should we compute the Efficient normalization of a Semivalue: formula (37) should be applied to the game $(N, w)$ obtained after a rescaling, with the factor $q_s^{n-1}$, of the worth for coalitions of size $s$; as $|U| = s+1$, formula (37) for $w \in G^N$, after dividing by $q_{s+1}^{n-1}$, becomes

$$v^s(U) = v^{s-1}(U) + \frac{q_s^{n-1}}{s q_{s+1}^{n-1}} \sum_{j \in U} v^{s-1}(U - \{j\}), \quad \forall U, |U| = s+1. \tag{38}$$

Hence, the step $s \leq n-2$ is done as follows:

- the worth of coalitions of sizes 1 to $s-1$, (if $s \geq 1$), and $s+2$ to $n$, (if $s \leq n-2$), are unchanged;

· the worth of coalitions of size $s$ become zero;
· the worth of coalitions of sizes $s+1$ are computed by formula (37).

The algorithm ends when all coalitions of sizes at most $n-2$ have worth zero, the worth of coalitions of size $n-1$ have been computed in the last step, and the grand coalition has the initial worth. Now, after multiplying the worth of coalitions of size $n-1$ by $q_{n-1}^{n-1}$,to get the corresponding worth of $w$, we can easily compute the Shapley Value, or even derive a special formula applicable in this case. Note that the factor multiplying the sum in (38) may be expressed in terms of the weights of the Semivalue and the Shapley coefficients. Of course, to compute the Semivalue from each component of the Efficient normalization we should subtract $\alpha$, given by formula (16).

*Example 2.* Let us return to the game considered in Example 1, and compute the Semivalue defined by the weight vector $p^4 = (\frac{1}{8}, \frac{1}{8}, \frac{1}{18}, \frac{1}{3})$; the weight vector needed in computations is $q^3 = (\frac{3}{4}, \frac{13}{12}, \frac{7}{6})$. In the first step, $s = 1$, the factor in front of the sum in formula (38) is $\frac{9}{13}$, and we can compute the worth of coalitions of size two

$$v^1(1,2) = \frac{31}{13}, \quad v^1(1,3) = \frac{22}{13}, \quad v^1(1,4) = \frac{9}{13}, \quad v^1(2,3) = \frac{22}{13},$$

$$v^1(2,4) = \frac{9}{13}, \quad v^1(3,4) = 0.$$

In the second step, the factor in front of the sum in (38) is $\frac{13}{28}$, and we can compute the worth of coalitions of size three

$$v^2(1,2,3) = \frac{103}{28}, \quad v^2(1,2,4) = \frac{77}{28}, \quad v^2(1,3,4) = \frac{31}{28}, \quad v^2(2,3,4) = \frac{31}{28}.$$

Now, we have $s = n-2$, and we get the worth of coalitions of the game $w$ obtained by rescaling, and the application of the algorithm, as

$$w(1) = w(2) = w(3) = w(4) = 0, \quad w(1,2) = w(1,3) = w(1,4)$$
$$= w(2,3) = w(2,4) = w(3,4) = 0,$$

$$w(1,2,3) = \frac{103}{24}, \quad w(1,2,4) = \frac{77}{24}, \quad w(1,3,4) = \frac{31}{24},$$

$$w(2,3,4) = \frac{31}{24}, \quad w(1,2,3,4) = 1.$$

We obtain the Efficient normalization of the Semivalue by computing the Shapley Value $ESE(N,v) = (\frac{95}{144}, \frac{95}{144}, \frac{1}{48}, -\frac{49}{144})$, like in Example 1; the Semivalue follows from the computation of $\alpha = \frac{1}{48}$. Obviously, the Semivalue is the same.

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# Stochastic Reaction Strategies and a Zero Inflation Equilibrium in a Barro-Gordon Model

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**Abstract** We study a game theoretic model of the conflict which arises between a monetary authority and the private sector with regard to the inflation-rate. Building on the simple static Barro and Gordon (Barro and Gordon, 1983a) model we assume that rather than playing a one shot game the monetary authority and private sector react to each other repeatedly for an infinite number of times. Both, the monetary authority's and the private sector's reactions are assumed to be stochastic in the form of fixed behavioral transition probabilities. These probabilities are interpreted as strategies in a new game. We study the set of Nash-equilibira of this new game and how these correspond to the classical discretionary Nash-equilibrium identified by Barro and Gordon as well as the non-Nash low inflationary state. In contrast to Barro and Gordon we show that the low-inflationary state can be realized as a Nash-equilibrium in our model.

**Keywords:** Monetary Policy; Game Theory; Stochastic Reactive Strategies.

## 1. Introduction

Monetary policy has been discussed in many macroeconomic investigations by use of various techniques, models and assumptions. One direction of research leads into game-theoretic models and has been started by influential works such as *Barro* and *Gordon* (Barro and Gordon, 1983a) and *Kydland* and *Prescott* (Kydland and Prescott, 1977). These game theoretic models have at their center the conflict between a monetary authority and the private sector. The monetary authority is assumed to have at least some sort of control over the level of inflation, from which it may try to exploit the Phillips curve, while the private sector's objective is to predict the inflation rate correctly in order to make the right decision with regards to current employment. Specifically the model by Barro and Gordon (1983a) is set up in the following way: First the private agents choose their expected rate of inflation $\pi^e \in \mathbb{R}_+$ and *announce* it. Then the monetary authority private can choose *actual* inflation $\pi \in \mathbb{R}_+$. The cost to the monetary authority is given by

$$\tilde{Z} := \frac{a}{2}\pi^2 - b(\pi - \pi^e). \tag{1}$$

This cost illustrates the tradeoff between aversion toward inflation on the one hand, and benefits due to a lower level of unemployment caused by a surprise inflation. These effect are controlled by the parameters $a$ and $b$. The private sector's losses are given by

$$(\pi - \pi^e)^2 \tag{2}$$

which means that whatever the level $\pi$ of inflation the monetary authority sets, it is optimal for the private sector to have expected exactly this level, i.e. $\pi^e = \pi$. Assuming optimal behavior of the private sector, the game then essentially becomes a single player game and in fact a standard quadratic optimization problem which can be easily solved. In fact the equilibrium level of inflation is given by

$$\pi^D = \frac{b}{a}. \tag{3}$$

The result is a single strict Nash-equilibrium $\left(\pi^D, \pi^D\right)$ where the monetary authority delivers and the private sector correctly anticipates high inflation. The inflation level $\pi^D$ in equation (3) is referred to as discretionary inflation. Though the private agents' optimal behavior is dependent on the bank's choice of $\pi$, the opposite is *not* true for the bank at all - equation (1) is always optimized not taking into account any value for $\pi^e$. Therefore the original character of the game (private agents choose $\pi^e$ and announce it) is technically equivalent to a simultaneously played game. We only have to ensure that the private sector cannot see the $\pi$ when choosing the corresponding $\pi^e$. The beneficial effect of high inflation for the monetary authority is however eradicated because the private sector correctly anticipates discretionary inflation, while the negative effect caused by the part $\frac{a}{2}\pi^2$ remains in the payoff to the monetary authority.

The equilibrium outcome of the classical Barro Gordon game must therefore be regarded as inefficient. In fact, both monetary authority and private sector would be better off choosing $\pi = \pi^e = 0$. While not applicable, inefficiencies like this can arise in the economic context. On the other hand there is also empirical evidence that the inflation rate (3) predicted by the classical Barro and Gordon model appears to be too high. In practice there should be mechanisms that lower the equilibrium inflation rate, which are not present in this model. Within the context of efficiency many of these mechanisms have to do with reputation and trust and are generally studied within the framework of repeated games. In a repeated game players have the opportunity to punish their opponents if they divert from a particular strategy. A general problem however is the credibility of these punishment strategies. Barro and Gordon introduced the so called *loss of reputation* framework in (Barro and Gordon, 1983b). However it has been argued by al-Nowaihi and Levine (1994) that under the assumption that punishments only hurt the central bank but not the private sector, the only remaining equilibrium remains the high inflation discretionary one. In this article we argue that the low- or zero-inflationary state can be realized as a Nash-equilibrium within a different sort of repeated game setting in which players are assumed to react stochastically to each other according to fixed behavioral transition probabilities. In this way we show that low inflation should not a priori be ruled out based on evidence coming from the classic and repeated Barro and Gordon models. In our model low inflation and high inflation are equally reasonable outcomes, and in furthermore under a simple behavioral assumption the low-inflation level becomes more realistic. The type of strategies which we are using in our model have been introduced by Hofbauer in (Hofbauer and Sigmund) to explain positive levels of cooperation in the context of the prisoners' dilemma. Our adaptation of this idea to the Barro Gordon game diverts from Hofbauer in the way that we apply it to asymmetric games and in addition to that give an interpretation of the realized payoffs as accumulated payoffs, while taking discounting into account. Even though we assume that the game is repeated infinitely often, our model is finite time, which means that the time between two periods converges to zero. Effectively our model is then a continuous time model. The time horizon is intended to correspond to a time unit of economic significance, possibly a financial year or a business cycle and it is assume that agents can not change or update their behavioral strategies in this period. In a Companion paper we investigate the change of behavioral strategies over a long term time scale under an adaptive dynamic and in this way investigate dynamic stability aspects of the Nash-equilibria presented in this article. The remainder is organized as follows. In section 2 introduce stochastic reactive strategies and adapt Hofbauer's approach to asymmetric games. In section three we give an interpretation of payoffs as discounted payoffs which are accumulated over time. We apply this framework to the Barro-Gordon game in section 4. In section 5 we investigate how the discretionary Nash-equilibrium of the original Barro Gordon game can be realized as a Nash equilibrium in our game. In section 7, following a similar analysis as in section 6, we demonstrate that zero-inflation can also be realized as a Nash-equilibrium of our game. Section 8 contains the conclusions.

## 2. Stochastic Reactive Strategies : Classical Setup

Stochastic reactive strategies have been introduced by Hofbauer in (Hofbauer and Sigmund) to explain a certain degree of cooperation within the prisoners dilemma game. The general idea is that, while the original game is repeated infinitely many times, players act according to fixed behavioral strategies which are represented by certain transition probabilities. If the original game has two pure strategies for each player these transition probabilities are given by the conditional probabilities of playing pure strategy 1 after the opponent played his pure strategy 1 in the previous round and playing pure strategy 1 after the opponent played his pure strategy 2 in the previous round. The unconditional probabilities can be computed using Bayes theorem and it can be shown that these unconditional probabilities converge and can in fact be interpreted as mixed strategies in the original game, leading to a well defined payoff. It has to be said clearly however that the strategies in the new game are the stochastic reaction strategies, consisting of a pair of conditional probabilities $(p, q)^\top$ and that the space of pure strategies in the new game is therefore given by $[0, 1] \times [0, 1]$. We do not consider mixed strategies, which would be probability distributions on $[0, 1] \times [0, 1]$ in what follows, but focus on pure Nash-equilibria of the new game. The setup has originally been used for symmetric games, but it is not hard to adapt it to asymmetric games. To see this assume that the payoffs for player one and two are given respectively by the two matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Pure strategies can be represented by $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. A stochastic reaction multi strategy then consists of a pair $(v, v') \in ([0, 1] \times [0, 1])^2$ where

$$v = \begin{pmatrix} p \\ q \end{pmatrix} \quad \text{and} \quad v' = \begin{pmatrix} p' \\ q' \end{pmatrix}$$

Here $p, p', q, q'$ denote the following conditional probabilities:

$p =$ probability player 1 plays $e_1$ given player 2 played $e_1$ in the previous round
$q =$ probability. player 1 plays $e_1$ given player 2 played $e_2$ in the previous round
$p' =$ probability player 2 plays $e_1$ given player 1 played $e_1$ in the previous round
$q' =$ probability player 2 plays $e_1$ given player 1 played $e_2$ in the previous round

The game is now repeated over and over again and players react stochastically according to the probabilities identified above. If we define

$$s_n, s_{n'} \in \{e_1, e_2\} \tag{4}$$

as the action chosen in period $n$ by players 1 and 2 respectively and define

$$X_n = \begin{pmatrix} s_n \\ s'_n \end{pmatrix}$$

then $(X_n)$ can be formally considered as a discrete time Markov chain with four states, for which transition probabilities can be computed in terms of $p, p'q, q'$. In principal the following construction can be realized assuming an arbitrary number of strategies $e_i$. However, we stick to the case of two strategies and assume a finite time horizon $T$, which could possibly correspond to a financial year, a business cycle or a different unit of time, but will essentially assume that within this period the game is repeated infinitely many times. This interpretation is inessential for the classical setup, but crucial in our adaptation in the next section. Hofbauer now proceeds by introducing the unconditional probabilities

$$\begin{aligned} c_n &= Prob\left(s_n = e^1\right) \\ c'_n &= Prob\left(s'_n = e^1\right). \end{aligned}$$

and considering their limits for $n \to \infty$. These limits can be computed as follows. Note that it follows from Bayes formula that

$$\begin{aligned} c'_{n+1} =& p'c_n + q'(1 - c_n) \\ c_{n+2} =& pc'_{n+1} + q(1 - c'_{n+1}) \end{aligned}$$

and by cross-substitution

$$\begin{aligned} c_{n+2} =& p(p'c_n + q'(1 - c_n)) + q(1 - p'c_n - q'(1 - c_n)) \\ =& q + p(q' + (p' - q')c_n) - q(q' + (p' - q')c_n) \\ =& q + (p - q)(q' + (p' - q')c_n) \end{aligned}$$

and similar for $c'_{n+2}$. Under the assumption $-1 < p - q < 1$ or $-1 < p' - q' < 1$ convergence of these sequences is guaranteed. Substitution of

$$\begin{aligned} c &= \lim_n c_n \\ c' &= \lim_n c'_n \end{aligned}$$

above gives

$$\begin{aligned} c(1-(p-q)(p'-q')) =& q+q'(p-q) \\ \Leftrightarrow c =& \frac{q+q'(p-q)}{1-(p-q)(p'-q')} \end{aligned}$$

In a similar way we get an expression for $c'$. In summary we obtain the following expressions for $c$ and $c'$ as functions of $p, q, p'$ and $q'$:

$$\begin{aligned} c =& \tfrac{q+q'(p-q)}{1-(p-q)(p'-q')} \\ c' =& \tfrac{q'+q(p'-q')}{1-(p-q)(p'-q')}. \end{aligned} \tag{5}$$

The vectors $(c, 1-c)^t op$ and $(c', 1-c')^t op$ can be interpreted as (long run) mixed strategies, and Hofbauer argues, that if the payoff in the first round is rather insignificant, a payoff for the whole period can be defined as

$$\begin{aligned} \mathcal{P}_1^H\left(\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} p' \\ q' \end{pmatrix}\right) &= (c, 1-c)A\begin{pmatrix} c' \\ 1-c' \end{pmatrix} \\ \mathcal{P}_2^H\left(\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} p' \\ q' \end{pmatrix}\right) &= (c, 1-c)B\begin{pmatrix} c' \\ 1-c' \end{pmatrix} \end{aligned}$$

where the superscript $H$ indicates Hofbauer-payoffs. Strategies spaces in this setup are given by $[0, 1] \times [0, 1]$. While not expressed in Hofbauer, it is natural to think of the strategies in terms of $(p, q)$ and $(p', q')$ as geno-typic strategies, while $e_1$ and $e_2$or alternatively on the level of mixed strategies $c$ and $c'$, are their pheno-typic realizations.

## 3. The Model

In this section we apply the the formal setup developed in the previous sections to the original *Barro Gordon game* as it is stated as a one shot game in (Barro and Gordon, 1983b). In contrast to (Barro and Gordon, 1983b) we assume that this particular game is played repeatedly for infinitely many times within the original period $[0, T]$ according to some fixed stochastic reaction strategies. Whenever the game was played a payoff for each player arises. These infinitely many payoffs are discounted and accumulate over time to produce a payoff for the whole period as described in the previous section. If one only believe in finitely many sub periods (e.g. four) our concept is still reasonable by the fast convergence as shown in the previous table. We now have to specify the matrices $A$ and $B$ that correspond to the original Barro and Gordon game. To be consistent with game theoretic standards we would like to deal with welfare functions rather than cost functions. Furthermore, we set the parameters $a$ and $b$ equal to 1. Hence by (1) and (2) we get

$$Z_t := (\pi_t - \pi_t^e) - \frac{1}{2}\pi_t^2 \tag{6}$$

for the central Bank and

$$-(\pi_t - \pi_t^e)^2 \tag{7}$$

for the private agents. Under these assumptions, the discretionary rate, obtained when optimizing equation (6) is given by 1. The payoff the monetary authority gets then equals $\frac{1}{2}$ if the private agents choose $\pi_t^e = 0$ respectively $-\frac{1}{2}$ if the private sector chooses $\pi_t^e = 1$. If the central bank chooses $\pi = 0$ it gets 0 (for $\pi_t^e = 0$) respectively $-1$ given $\pi_t^e = 1$. The private agents on the other hand get 0 if they were right and $-1$ if not. In particular we have:

| Strategy | | Payoff | |
|---|---|---|---|
| Bank | Private Agents | Bank | Private Agents |
| zero | zero | 0 | 0 |
| zero | discretion | $-1$ | $-1$ |
| discretion | zero | $\frac{1}{2}$ | $-1$ |
| discretion | discretion | $-\frac{1}{2}$ | 0 |

For convenience we multiply the payoffs of the monetary authority by two. As this is a positive affine transformation it does not affect the Nash-equlibria at all. Hence suitable payoff matrices for the bank and the private agents are given by

$$A := \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{8}$$

Though the game is started by the private agents, in the following we think of the bank to be player one and the private agents to represent player two (i.e. the monetary authority's strategy is denoted by $v$, whilst $v'$ is the strategy of the private agents). Let us re-emphasize however, that for the original one-shot game it does in fact not matter whether the game is started by the private agents or played simultaneously. The concrete interpretation of $p, q, p', q'$ in this case is given as follows: (MA=monetary authority, PA= private agents)

$p =$ MA sets inflation low given PA expected low inflation in the previous round

$q =$ MA sets inflation low given PA expected high inflation in the previous round

$p' =$ PA expect low inflation given MA delivered low inflation in the previous round

$q' =$ PA expect low inflation given MA delivered high inflation in the previous round

For a fixed set of probabilities, i.e. fixed strategies $v = \begin{pmatrix} p \\ q \end{pmatrix}$ and $v' = \begin{pmatrix} p' \\ q' \end{pmatrix}$, the payoffs are given by

$$\mathcal{P}_1(v, v') = (c, 1-c) \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c' \\ 1-c' \end{pmatrix} = 2c' - c - 1 \tag{9}$$

and

$$\mathcal{P}_2(v, v') = (c, 1-c) \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c' \\ 1-c' \end{pmatrix} = 2cc' - c - c' \tag{10}$$

where $c$ and $c'$ are given as functions of $p, q, p'$ and $q'$ as in (5). It is worth noting that the private agents' optimal payoff is given by 0.

**Lemma 1.** *The payoff for the private agents never exceeds* 0. *It will be equal to* 0 *if and only if either* $c = c' = 0$ *or* $c = c' = 1$ *holds.*

*Proof.* Clearly $c, c' \in [0,1]$ implies that

$$2cc' - c - c' = c(\underbrace{c' - 1}_{\leq 0}) + c'(\underbrace{c - 1}_{\leq 0})$$

with equality if and only if $c = c' = 0$ or $c = c' = 1$. The statement therefore follows from equation 10.

So far we have motivated the use of stochastic reaction strategies and have developed a concrete model based on the original Barro and Gordon game. We will now focus on particular Nash-equilibria of this game.

## 4. The Discretionary Nash Equilibrium

We indicated before that we can think of the strategies $v = (p, q)^\top$ and $v' = (p', q')^\top$ as being on the genetic level, while they relate to pheno-typic strategies $c, c'$, which in fact relate to mixed strategies in the original Barro-Gordon game, i.e. mixed in discretion and zero-inflation. In this section we study under which conditions the discretionary phenotype, i.e. high inflation delivered by the monetary authority and high inflation expected by the private agents, can be realized as a Nash-equilibrium on the geno-typic level. We will show that this poses rather restrictive assumption on the behavior of the private agents. First of all we note that (pure) discretion is the case if and only if $c = c' = 0$. To start with, we need to identify the genotypes which belong to the phenotype discretion.

**Lemma 2.** *Discretionary phenotypes always require* $q = q' = 0$. *More precisely we state that* $c = c' = 0$ *implies* $q = q' = 0$ *and furthermore if* $p < 1$ *or* $p' < 1$ *then* $q = q' = 0$ *implies* $c = c' = 0$.

*Proof.* Looking at the formulas for $c$ and $c'$ (5) the second statement is obvious. Now suppose $c$, $c'$ and $q$ are equal to zero. Then we conclude from $pp' < 1$ and (5)

$$0 = q' + q(p' - q') = q'..$$

Similarly. assuming that $q' = 0$ we find that $q = 0$. Hence suppose $q \neq 0$ and $q' \neq 0$. Then

$$\begin{aligned} 0 = q + q'(p - q) \quad &\text{and} \quad 0 = q' + q(p' - q') \\ \Rightarrow p - q = -\frac{q}{q'} \quad &\text{and} \quad p' - q' = -\frac{q'}{q} \\ \Rightarrow p - q = \frac{1}{p' - q'}. & \end{aligned}$$

The only two strategies satisfying this relation are $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$. As can be easily verified, $(0, 1, 0, 1)$ never delivers $c = c' = 0$. Hence $q = q' = 0$.

The following Corollary states that each players payoff under discretionary behavior is independent of the values $p$ and $p'$. This fact will play an important part in the proof of our first main theorem.

**Corollary 1.** *Assume that* $0 \leq p + \tilde{p}, p' + \tilde{p}'$ *and either* $p + \tilde{p} < 1$ *or* $p' + \tilde{p}' < 1$. *Then we have*

$$\mathcal{P}_1(p, 0, p', 0) = \mathcal{P}_1(p + \tilde{p}, 0, p' + \tilde{p}', 0) = -1$$
$$\mathcal{P}_2(p, 0, p', 0) = \mathcal{P}_2(p + \tilde{p}, 0, p' + \tilde{p}', 0) = 0.$$

*Proof.* By the previous Lemma $c$ and $c'$ are zero for any choice of $p$ and $p'$ as long as $q, q' = 0$ and $p, p' < 1$ holds. The result then follows directly from (9) and (10).

An immediate consequence is that in our setup, discretion can no longer be realized as a strict Nash-equilibrium. The next theorem shows that there are in fact geno-typic strategies which lead to discretion, but do not represent a Nash-equilibrium. This in turn gives scope for evolutionary drift away from a high inflation state on the pheno-typic level. The following theorem contains our first main result.

**Theorem 1.** *Among all the strategies leading to discretionary behavior only those represent Nash equilibria, which satisfy that* $p' \leq \frac{1}{2}$.

*Proof.* We have to show the following:

$$\mathcal{P}_1(p, 0, p', 0) \geq \mathcal{P}_1(\tilde{p}, \tilde{q}, p', 0) \quad \forall \tilde{p}, \tilde{q} \in [0, 1] \tag{11}$$
$$\mathcal{P}_2(p, 0, p', 0) \geq \mathcal{P}_2(p, 0, \tilde{p}', \tilde{q}') \quad \forall \tilde{p}', \tilde{q}' \in [0, 1] \tag{12}$$

if and only if $p' \leq \frac{1}{2}$. Equation (14) is a direct consequence of Lemma 6.1 and Lemma 5.1. Let us therefore turn to equation (13). It follows from Corollary 6.2 that

$$\mathcal{P}_1(p, 0, p', 0) = \mathcal{P}_1(\tilde{p}, 0, p', 0).$$

In order to establish equation (13) it therefore suffices to show that the function

$$\tilde{q} \mapsto \mathcal{P}_1(\tilde{p}, \tilde{q}, p', 0)$$

is monotonic decreasing in $\tilde{q} \in [0, 1]$. Using the formulas (18) and (24) from the appendix it is easy to verify that

$$\frac{\partial}{\partial \tilde{q}} \mathcal{P}_1(\tilde{p}, \tilde{q}, p', 0) = (2p' - 1) \frac{1 - \tilde{p}p'}{(1 - (\tilde{p} - \tilde{q})p')^2}.$$

Note that neither $1 - \tilde{p}p'$ nor $(1 - (\tilde{p} - \tilde{q})p')^2$ can ever be negative. Under the assumption that $p' \leq \frac{1}{2}$ the derivative above is therefore negative for all $\tilde{q} \in [0, 1]$ and equation (13) holds. On the other side if $p' > \frac{1}{2}$, a positive derivative at $\tilde{q} = 0$ implies that (13) can not hold, which concludes the proof.

Let us remind ourselves for the moment that $p'$ denotes the conditional probability that private agents expect low inflation given that the monetary authority delivered low inflation in the previous round. The level of $p'$ can therefore be interpreted as a trust parameter, which on a different time scale than considered here, may have been arisen from an effect of reputation of the monetary authority. The condition in Theorem 6.3 that $p' \leq \frac{1}{2}$ can then be interpreted, that discretion can only persist as a Nash-equilibrium in our setup, if the reputation of the monetary authority and hence the trust of the private agents in the monetary authority has been significantly damaged.

## 5. The Zero Inflation Equilibrium

In this section we study how low inflation can arise as a Nash-equilibrium on the pheno-typic level. It will turn out, that the analysis is very similar to the one carried out in the previous section. For matter of completeness and illustration though, we include all necessary arguments. First of all note that the zero inflation phenotype corresponds to the case $c = c' = 1$. In analogy to Lemma 2 the following Lemma helps us to identify those geno-typic strategies which correspond to the phenotype zero inflation and can be achieved if and only if $p = p' = 1$:

**Lemma 3.** *Zero-inflation phenotypes require $p = p' = 1$. Furthermore if $q > 0$ or $q' > 0$ then $p = p' = 1$ implies $c = c' = 1$.*

*Proof.* The second statement follows directly from the formulas for $c$ and $c'$ (see (5)). In order to see that the first implication holds, let us assume for the moment that $q > 0$ or $q' > 0$ holds. Then $c = 1$ implies

$$\begin{aligned} & 1 - (p-q)(p'-q') = q + q'(p-q) \\ \Leftrightarrow\quad & 1 - p'(p-q) = q \\ \Leftrightarrow\quad & 1 - pp' = q(1-p') \\ \Leftrightarrow\quad & 1 - q = p'(p-q). \end{aligned}$$

For $p < q$ and $p' > 0$ the last expression has no solution, whereas $p > q$ implies $p' \geq 1$ with equality if and only if $p = 1$. For $p = q$ the above expression can only be fulfilled if $p = q = 1$. On the other hand $c' = 1$ yields

$$1 - q' = p(p' - q'),$$

which reduces to $1 = p'$ if $p = 1$ as required. To cover the case $q = q' = 0$ note that $c$ and $c'$ are equal to zero whenever $pp' < 1$. Hence $p = p' = 1$ must hold.

We will later need the following result which is proved in complete analogy to Corollary 6.2.

**Corollary 2.** *Assume that $q + \tilde{q}, q' + \tilde{q}' \leq 1$ and either $q + \tilde{q} > 0$ or $q' + \tilde{q}' > 0$, then the following holds*

$$\begin{aligned} \mathcal{P}_1(1, q, 1, q') &= \mathcal{P}_1(1, q + \tilde{q}, 1, q' + \tilde{q}') \\ \mathcal{P}_2(1, q, 1, q') &= \mathcal{P}_2(1, q + \tilde{q}, 1, q' + \tilde{q}'). \end{aligned}$$

The following Theorem includes our second main result.

**Theorem 2.** *Among all strategies leading to zero inflation only those are a Nash equilibriums, which satisfy $q' \leq \frac{1}{2}$.*

*Proof.* Similar as in the proof of Theorem 6.3 we have to show that

$$\mathcal{P}_1(1, q, 1, q') \geq \mathcal{P}_1(\tilde{p}, \tilde{q}, 1, q') \quad \forall \tilde{p}, \tilde{q} \in [0, 1] \tag{13}$$

$$\mathcal{P}_2(1, q, 1, q') \geq \mathcal{P}_2(1, q, \tilde{p}', \tilde{q}') \quad \forall \tilde{p}', \tilde{q}' \in [0, 1] \tag{14}$$

if and only if $q' \leq \frac{1}{2}$. Let us note that $\mathcal{P}_2(\cdot)$ equals zero whenever $p = p' = 1$ and that this is optimal for the private agents. Therefore the inequality (16) holds in any case. Now, considering inequality (15), we note that from Corollary 7.2. we can conclude that

$$\mathcal{P}_1(1, q, 1, q') = \mathcal{P}_1(1, \tilde{q}, 1, q')$$

and that (15) would therefore hold, if the function

$$\tilde{p} \mapsto \mathcal{P}_1(\tilde{p}, \tilde{q}, 1, q')$$

is monotonic increasing in $\tilde{p} \in [0, 1]$. Now using equations (17) and (23) in the appendix it can be easily verified that

$$\frac{\partial}{\partial \tilde{p}} \mathcal{P}_1(\tilde{p}, \tilde{q}, 1, q') = (1 - 2q') \frac{q' + \tilde{q}(1 - q')}{(1 - (\tilde{p} - \tilde{q})(1 - q'))^2}.$$

Note that nominator and denominator of the fraction are both positive and therefore that the derivative is positive as long as $q' \leq \frac{1}{2}$, which establishes inequality (15). On the other side if $q > \frac{1}{2}$ a negative derivative at $\tilde{p} = 1$ implies that (15) can not hold, which concludes the proof.

As in the previous section let us briefly elaborate on the meaning of the condition in Theorem 7.3. We have that $q'$ is given by the conditional probability that private agents expect low inflation given that the monetary authority delivered high inflation in the previous round. Therefore $q'$ can be interpreted as some kind of ignorance of the private agents with respect to observed behavior of the monetary authority. The interpretation of Theorem 7.3 is therefore, that as long as the level of ignorance is sufficiently low $q' \leq \frac{1}{2}$, zero-inflation can very well be realized as a Nash-equilibrium. Combining this result with Theorem 6.3, we can state that under the assumption that the reputation of the monetary authority resp. the private agents trust in the monetary authority is not significantly damaged while on the other side, the private agents are not blind and ignorant toward the monetary authorities action, zero inflation is realized as a Nash equilibrium, while the classical Barro-Gordon high inflation, discretionary policy is not.

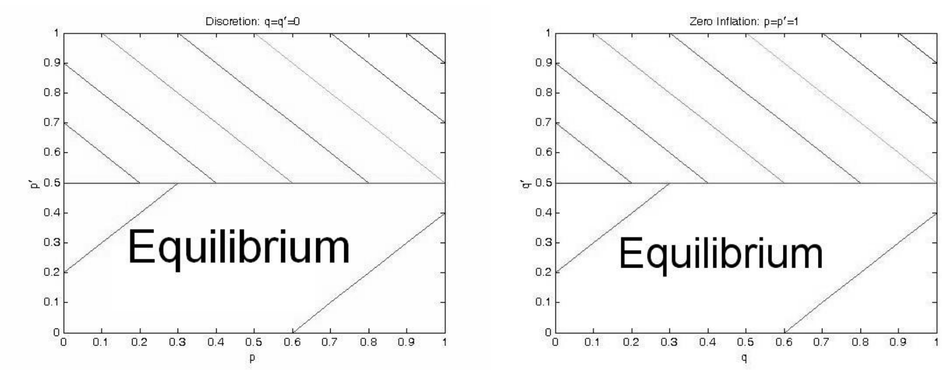


**Figure1.** On the left the discretionary equilibrium: $q = q' = 0$ and $p' \leq \frac{1}{2}$! On the right the zero inflation equilibrium: $p = p' = 1$ and $q' \leq \frac{1}{2}$!

Finally we remark, that we can not exclude further equilibria on the genotypic level, which in fact correspond to mixed strategies in the original Barro-Gordon game. The analysis in this article was mainly motivated by establishing the impossibility of discretion and the possibility of zero-inflation as a Nash equilibrium in a (modified) Barro-Gordon framework. Further studies which include the dynamic aspect and stability properties will follow.

## 6. Conclusion

In the original *Barro Gordon game* (the one shot game version) discretion is the only one *Nash equilibrium*. Furthermore it is a *strict Nash equilibrium*. As we have seen in the last two sections this is not longer the case for our continuous time version of the game. Furthermore we have seen that there are strategies $p, q$ and $p', q'$ that lead to discretion resp. to zero inflation. Among those there are strategies such that discretion (resp. zero inflation) are a *Nash equilibrium* (for $p' \leq \frac{1}{2}$ resp. $q' \leq \frac{1}{2}$). In this sense discretion and zero inflation are now of the same quality (see Figure 1).

Discretion as a *Nash equilibrium* becomes even more unlikely if one assumes a strategy more likely to Tit For Tat for the private agents. In particular the private agents tend not to believe in zero inflation again when they just *see* discretion. On the other hand they are more likely to believe in zero inflation if zero inflation is the case right now. In other words the private agents would always have a strategy such that $q' < \frac{1}{2}$ and $p' > \frac{1}{2}$. Under this restriction by Theorem 1 we know that discretion is never a *Nash equilibrium*, whilst by Theorem 2 we know that zero inflation is always a *Nash equilibrium*.

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## Appendix

### C,C'

$$c = \frac{q + q'(p-q)}{1-(p-q)(p'-q')} \tag{15}$$

$$c' = \frac{q' + q(p'-q')}{1-(p-q)(p'-q')} \tag{16}$$

### Derivatives 1

$$\frac{\partial c}{\partial p} = \frac{q' + q(p'-q')}{(1-(p-q)(p'-q'))^2} \tag{17}$$

$$\frac{\partial c}{\partial q} = \frac{1 - q' - p(p'-q')}{(1-(p-q)(p'-q'))^2} \tag{18}$$

$$\frac{\partial c'}{\partial p'} = \frac{q + q'(p-q)}{(1-(p-q)(p'-q'))^2} \tag{19}$$

$$\frac{\partial c'}{\partial q'} = \frac{1 - q - p'(p-q)}{(1-(p-q)(p'-q'))^2} \tag{20}$$

**Derivatives 2**

$$\frac{\partial c}{\partial p'} = (p-q)\frac{q + q'(p-q)}{(1-(p-q)(p'-q'))^2} = (p-q)\frac{\partial c'}{\partial p'} \tag{21}$$

$$\frac{\partial c}{\partial q'} = (p-q)\frac{1 - q - p'(p-q)}{(1-(p-q)(p'-q'))^2} = (p-q)\frac{\partial c'}{\partial q'} \tag{22}$$

$$\frac{\partial c'}{\partial p} = (p'-q')\frac{q' + q(p'-q')}{(1-(p-q)(p'-q'))^2} = (p'-q')\frac{\partial c}{\partial p} \tag{23}$$

$$\frac{\partial c'}{\partial q} = (p'-q')\frac{1 - q' - p(p'-q')}{(1-(p-q)(p'-q'))^2} = (p'-q')\frac{\partial c}{\partial q} \tag{24}$$

**Proofs**

*Proof (or equation (5)).* We have seen that

$$c_{n+2} = q + (p-q)(q' + (p'-q')c_n)$$

Hence for $k \in \mathbb{N}$ we get

$$\begin{aligned}
c_{n+2k} =& q + (p-q)(q' + (p'-q')[q + (p-q)(q' + (p'-q')[\\
&\dots(q' + (p'-q')c_n)\dots])])\\
=& q\sum_{i=0}^{k-1}((p-q)(p'-q'))^i\\
&+ (p-q)q'\sum_{i=0}^{k-1}((p-q)(p'-q'))^i + ((p-q)(p'-q'))^k c_n\\
=& (q + q'(p-q))\sum_{i=0}^{k-1}((p-q)(p'-q'))^i + ((p-q)(p'-q'))^k\\
\overset{*}{=}& (q + q'(p-q))\frac{1 - ((p-q)(p'-q'))^k}{1-(p-q)(p'-q')} + ((p-q)(p'-q'))^k c_n\\
\overset{k\to\infty}{\longrightarrow}& \frac{q + q'(p-q)}{1-(p-q)(p'-q')}
\end{aligned}$$

where $*$ such as the last convergence is true by the assumption that $-1 < (p-q)(p'-q') < 1$ holds. It it is easy to check that the very same can be done for $c'$. Hence we are done.

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# Tracing the Modern Concept of Convexity*

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**Abstract** Several manifestations of convexity were studied already in the antiquity. The modern concept emerged, however, only around 1900, notably in the works of Peano and Minkowski. This motivates us to review here some main contributions of the latter. In doing so, we attempt to offer a friendly invitation to the history and concepts of convex analysis. Our emphasis is on convex bodies, extreme points, separation, Minkowski functionals, supports and polarity.

## 1. Background and Motivation

The notion of convexity dates back to Archimedes at least, but receded, during the birth and early growth of calculus, into the background. The last half century has seen a surge of interest though. Convexity, and the analysis of its many manifestations, now holds much prestige and offers wide applicability; see e.g. (Borwein and Lewis, 2000), (Hiriart-Urruty and Lemaréchal, 1993), (Hörmander, 1994), (Rockafellar, 1970), (Tikhomirov, 1980).

The rapid development of theory motivates us to reconsider briefly the historical introduction of some key concepts. In doing so, we limit attention here to some chief contributions made by the pioneer Hermann Minkowski (1864-1909), referred to as M for short. We shall review none of M's many results *outside* convex analysis - be it in number theory or physics[1]. And even *inside* that field we select just a few, namely those that concern the interplay between geometry and optimization, excluding here discrete structures. Of great interest to such interplay - and to extensions of convex analysis (Aubin and Frankowska, 1990), (Clarke, 1983), (Clarke et al.), (Rockafellar and Wets, 1998) - are several interwoven facts and features. Most of them relate to:

• the algebraic and topological properties of convex sets - and especially of polyhedral instances;
• the characterization of such sets by functional counterparts such as gauges or supports; and finally,

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* "From Germany, after an enquiry through the Commissariat of Enlightenment, Persikov was sent three parcels containing mirrors, *convexo-convex*, *concavo-concave* and even some sort of *convexo-concave* ground glasses"(Bulgakov, 2003).

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[1] The exclusion of number theoretic results is warranted, we feel, given the treatise (Hancock,1939).

• the dual description and polarity of these objects that stems from separation theorems.

These various faces of convexity, already identified by M, are the ones we shall emphasize. Thus, M's pioneering studies of volumes, areas, curvatures, and closest packings of convex sets will get no mention. In particular, we keep silent on the Brunn-Minkowski theory (and related isoperimetric or differential-geometric properties) of convex bodies.

It seems fair to claim that convex analysis stems mainly from mechanics and geometry; see (Fenchel, 1983).[2] However, since M, certainly a founding father, took little part in these fields, the said claim is not quite correct.[3] In fact, he came indirectly, repeatedly - and rather surprisingly - to convex analysis from number theory and by a geometric approach to the latter.[4] In reviewing some features of that approach we try not to regard results with hindsight. Rather we attempt to appreciate key theorems - and sometimes their proofs - within the mathematical culture that reigned, say around 1895. At places it appears natural though, to mention some later developments briefly. Also, we record how the notion of convexity evolved in the hands of M, a particular feature being that he started out from a most important class of convex functions - and not from corresponding sets.

Throughout we seek to stress the originality of selected results - and to make clear how modern M was in his approach. We address several sorts of readers. Expertise in convex analysis is not needed. In fact, the subsequent material should interest diverse people be they students or scholars of analysis, optimization, geometry, economics, or the history of mathematics.

Our reading of Minkowski's works were focused mainly on *Geometrie der Zahlen* (1896) - GZ for short - and the two chief chapters XXII and XXV in *Gesammelte Abhandlungen* (1911) - henceforth called GA.[5] Unless stated otherwise each subsequent reference to pages concern GA vol. II. All theorems singled out with bold letters in the main text below are attributed to M.

For a start, we found it inviting to read M's abstract of GZ in a letter to Hermite (GA, chap. XI):

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[2] In their excellent book Borwein and Lewis (2000) say that "*convex analysis developed historically from the calculus of variations.*." This assertion seems justifiable only up to the extent that isoperimetric problems reside at the heart of the two fields. Giaquinta and Hildebrandt (Giaquinta and Hildebrandt, 1996) opinion though, that: "*Amazingly, convex functionals do not play any role in the classical calculus of variations before the turn of the century, although they seemed predestinated to assume a central place. One has to realize that only Minkowski recognized the notions of convex set and convex function to be central concepts in mathematics, although the concepts of a convex curve and a convex surface were well-known and often used in ancient times; then convexity meant "locally eggshaped."*"

[3] It may mirror a natural tendency, in any field, to appropriate non-intentional contributors provided they be great.

[4] For example, concerning approximation of numbers by rationals, M says in the preface to (Minkowski, (1896) ): "*Ich bin zu meinen Sätzen durch räumliche Anschauung gekommen.*" In a letter December 1887 to Hilbert he says: "*Ich bin auch ganz Geometer geworden, und bedaurere aus diesem Grunde doppelt, nicht in Ihrem Kreise weilen zu können.*" (Minkowski, 1970).

[5] GA were published posthumously. Omitted there is GZ and another number theoretic book (Minkowski, 1927) - as well as studies of the relativity principle, written by M alone or with Lorenz and Einstein.

## 2. Minkowki's Partial Résumé

"The largest part of the book (GZ) deals with functions $\varphi$ in $n$ variables $x_1, x_2, ..., x_n$, which, like the square root of a positive quadratic form, satisfy the conditions

$$\text{(A)} \begin{cases} \varphi(x_1, x_2, ..., x_n) > 0, \text{ unless } x_1 = 0, x_2 = 0, ..., x_n = 0, \\ \varphi(0, 0, ..., 0) = 0, \\ \varphi(tx_1, tx_2, ..., tx_n) = t\varphi(x_1, x_2, ..., x_n) \text{ si } t > 0, \end{cases}$$
$$\text{(B) } \varphi(x_1 + y_1, x_2 + y_2, ..., x_n + y_n) \leq \varphi(x_1, x_2, ..., x_n) + \varphi(y_1, y_2, ..., y_n),$$
$$\text{(C) } \varphi(-x_1, -x_2, ..., -x_n) = \varphi(x_1, x_2, ..., x_n).$$

Let $\xi_1, \xi_2, ..., \xi_\nu$ be linear forms with real coefficients in variables $x_1, x_2, ..., x_n$, and among these let $n$ forms have determinant different from zero. Denote by $\Phi(x_1, x_2, ..., x_n)$ the maximum of the absolute values of $\xi_1, \xi_2, ..., \xi_\nu$. Evidently, such a function $\Phi$ satisfies the conditions of a function $\varphi$. I establish the theorem: When $\varphi$ solves (A), (B), (C), and $\delta$ is any positive quantity, one may always find a function $\Phi$, as just characterized, such that, for all possible $x_1, x_2, ..., x_n$ one has

$$1 \leq \varphi/\Phi < 1 + \delta. \tag{1}$$

As a result, the intgral $\int\int ... \int dx_1 dx_2 ... dx_n$ over the domain $\varphi(x_1, x_2, ..., x_n) \leq 1$ has always a definite value. Denote by $J$ that value. I show that *one can always find integers* $x_1, x_2, ..., x_n$ *such that* $0 < \varphi(x_1, x_2, ..., x_n) \leq 2/J^{1/n}$."

Prominent already here are *positive sublinear functionals*, *finitely generated convex sets*, *polyhedral approximation*, *convex bodies*, and *integer solutions.* It all started, however, with

## 3. Quadratic Forms and Integer Lattices

Positive quadratic forms were central in M's first number theoretic studies. Brief mention of two early works suffices to bring this out. One work - his very first: GA, chapt. I, which he wrote at the age of 17 - more than solved the award-winning problem, stated by the French Academy of Sciences, to represent any natural number as the sum of five squares. As vehicle M used *quadratic forms* with integer coefficients - and undertook to classify these. The geometric properties of positive quadratic forms, and notably their convex lower level-sets (so-called *ellipsoids*), then occupied center stage as smooth instances of *convex bodies.* The other early work: GA, chapt. XIX - on closest packing of such bodies - brought M to study arithmetic equivalence and reduced versions of positive quadratic forms (see also GA, chapt. XXI). In that enterprise he showed that the latter objects constitute a *convex cone* generated by finitely many *extreme rays.*

For our review we must first fix some notations. Although M frequently took the three-dimensional space as the ambient setting, equipped with its customary Grassmannian inner product, the results considered below extend verbatim to finite-dimensional spaces - and often beyond (Holmes, 1975).[6] For this reason, and to facilitate reading, we prefer the modern (coordinate-free) setting of a Euclidean space $\mathbb{E}$, endowed with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$. Regarding that space our symbols are standard: Let $affC$, $bdC$, $clC$, $convC$, $\dim C$, $intC$, $vol_{(d)}C$

[6] Köthe (Köthe, 1969) says : "*Provided that necessary care is taken, methods that go back to Minkowski can also be applied to convex sets in vector spaces of infinite dimension.*"

denote the *affine hull, boundary, closure, convex envelope, dimension, interior* and ($d$-dimensional) *volume* respectively of $C \subseteq \mathbb{E}$. When $C, D \subseteq \mathbb{E}$ and $\mathcal{R} \subseteq \mathbb{R}$, we write $C + \mathcal{R}D$ for the *Minkowski combination* $\{c + rd : c \in C, d \in D, r \in \mathcal{R}\}$. In particular, given two points $x, y \in \mathbb{E}$ we shall deal with closed segments $[x, y] := \{x\} + [0, 1]\{y - x\}$ and half-open versions $[x, y) := \{x\} + [0, 1)\{y - x\}$. $I$ always denotes a nonempty finite set of indices, and $\mathbb{R}^I$ is the set of all functions $x : I \to \mathbb{R}$.

As said, positive quadratic forms were main objects in M's first papers. The mapping $x \mapsto \langle x, Qx\rangle^{1/2}$ is then positively homogeneous, subadditive, non-negative, and vanishes only at the origin. M calls functionals $\gamma : \mathbb{R}^I \to \mathbb{R}$ that satisfy precisely these properties, *einhellige Strahldistanzen.* This special class of convex functions, now named *Minkowski functionals*, constitute his point of departure in GZ. He shows there that each of them is continuous, bounded below (and above) by a suitably scaled version of $\max_{i \in I} |x_i|$, the relevant factors being called *Distanzcoefficienten*; see also (3).

Much of M's motivation for considering such functionals stems from Hermite's preceding concern with the smallest number $r$ for which $\langle x, Qx\rangle^{1/2} = r$ is solvable in integers, that is, the minimal level $r$ such that the *ellipsoid* $\left\{x : \langle x, Qx\rangle^{1/2} \le r\right\}$ intersects the integer *lattice* $\mathbb{Z}^I$. This problem led M to translate a symmetric convex ground set $C = \{x : \gamma(x) \le r\}$ by every point in $\mathbb{Z}^I$. More generally, any basis $\mathcal{B}$ in (the finite-dimensional Euclidean) space $\mathbb{E}$ generates a *complete lattice* $L := \sum_{b \in \mathcal{B}} \mathbb{Z}b$ whose *fundamental mesh,* the parallelepiped $\sum_{b \in \mathcal{B}} [0, 1]\, b$, has volume $\mu$. Then it holds

**Theorem** (on lattice points)[7]. *Let $C$ be symmetric convex and $L$ a complete lattice with mesh-volume $\mu$.* Suppose $volC > 2^{\dim \mathbb{E}}\mu$. *Then $C$ contains at least one nonzero lattice point.* □

**Corollary**. *For each $j \in I$ let $a_{.j}x := \sum_{i \in I} a_{ij}x_i$ be a linear form and $c_j$ a positive number such that $\Pi_j c_j > |\det(a_{ij})| > 0$. Then there exists $\mathbf{z} = (z_i) \in \mathbb{Z}^I$ such that $|a_{.j}\mathbf{z}| < c_j$ for all $j$.* □

Only some properties of ellipsoids were crucial here, namely those that led him to identify the notion of convex surfaces and bodies as fundamental:

## 4. Convex Surfaces, Bodies and Sets

Given a positive, sublinear $\gamma$, M singles out the corresponding *surface* (Aichfläche) $\mathcal{S}_\gamma := \{x : \gamma(x) = 1\}$ together with the associated *body* (*Aichkörper*):

$$C_\gamma := \{x : \gamma(x) \le 1\}, \tag{2}$$

and shows that the latter is stable with respect to formation of segments; that is, $x, y \in C_\gamma \Rightarrow [x, y] \subseteq C_\gamma$. He announces in GZ §8 - and proves in §16 - that $\mathcal{S}_\gamma$ is supported in each point by a hyperplane.

Thus, M starts from positive sublinear functions $\gamma$ and proceeds to consider their representative level sets $\mathcal{S}_\gamma$ and bodies $C_\gamma$. Only thereafter does he follow a

[7] This theorem resides at the foundation of essential parts of algebraic number theory (Ribenboim, 2001).

long geometric tradition - one that dates back to Archimedes at least - according to which *convexity* is a property displayed by certain curves and surfaces. Intuitively, a convex surface should never dent inward; that is, any chord should reside on its "inner" side. Equivalently, any "tangent" should be "external." So, quite naturally, when M introduces a convex surface - encompassing an "interior point", say 0 - as primitive object, he states the

**Definition (**convex surfaces GZ §17). A set $\mathcal{S} \subset \mathbb{E}$ is declared a **nowhere concave surface** around 0 iff first, each ray from 0 contains at least one point in $\mathcal{S} \setminus 0$; and second, through each point in $\mathcal{S}$ there must pass at least one supporting hyperplane (Stützebene), this meaning that $\mathcal{S}$ lies fully on one side (in one halfspace). □

M argues that $0 \notin \mathcal{S}$; otherwise any line, not contained in a supporting hyperplane through 0, would intersect $\mathcal{S}$ on both sides of that plane - a contradiction. Similarly, each ray from 0 has exactly one point in common with $\mathcal{S}$.

Letting $\gamma_{\mathcal{S}}(x) := \inf \{r \geq 0 : x \in r\mathcal{S}\}$ M establishes a one-to-one correspondence between surfaces of the said sort and positive sublinear functionals via $\mathcal{S} = \{x \in \mathbb{E} : \gamma_{\mathcal{S}}(x) = 1\}$.[8]

Such a surface $\mathcal{S}$ generates a *convex body* $C := [0,1]\,\mathcal{S}$ with $bdC = \mathcal{S}$.[9] Thus emerges also a bijection $C \leftrightarrow \mathcal{S}$ between convex bodies and their surfaces. This approach, making convex bodies enter second - either as a lower level $\{x : \gamma_{\mathcal{S}}(x) \leq 1\}$ or as an envelope *conv* $\{0 \cup \mathcal{S}\}$ - was turned around in M's subsequent, more modern definition. Convexity is tested there not by means of barycenters - as did Archimedes, or by supporting hyperplanes as did M first - but simply by one-dimensional sections:

**Definition.** A set $C \subset \mathbb{E}$ is called a **convex body** when primo, every line has a closed interval, a point, or nothing in common with $C$, and secondo, $C$ does not belong to a proper subspace.[10] □

In passing, it deserves notice that *sets*, and especially Cantor's set theory, around 1900 still received less than full respect within the mathematical community. Notably Kronecker, one of M's prominent teachers, conveyed lack of enthusiasm for Cantor's work. Hilbert (Hilbert and Minkowski, 1911) exaggerates somewhat though, in claiming that M *"was the first mathematician of our generation..who recognized the great significance of Cantor's theory.."*[11] Also worth mention is that the basic notions of topology and linear algebra were recently conceived, or in their early infancy, at M's time and neither commonly known nor widely used.

[8] GZ p. 36. Of particular interest is the $p$-norm $\gamma(x) = \left(\sum_{i\in I} |x_i|^p\right)^{\frac{1}{p}}$; see GZ §18.

[9] The notion *nowhere concave surface*, used in GZ, is replaced by *convex surface* in GA vol.II. p. 123.

[10] See GA p. 131 and 103. M proves that the barycenter $\int_C xdx/volC$ of a convex body belongs to its interior (GA 144); on this see also (Bonnesen and Fenchel, 1934).

[11] Hilbert's assertion seems appropriate for the two of them. But it hardly fits even his own professional environment. Indeed, A. Schönflies, a collegue of Hilbert in Göttingen, wrote on set theory in (Encyklopädie der Mathematischen Wissenschaften), vol.I, part 1, Ch.5. And Cantor's influence had long been great in France and Italy. M and Hilbert were collegues from 1902-1908 in Göttingen; see (Reid, 1962). Hilbert says in his Gedächtnisrede: "*Er war mir ein Geschenk des Himmels...*" (GA, vol. I, p.XXXI).

Regarding the definition above, M immediately brings out that convexity - and thereby also closure and boundedness - derives from the corresponding property being valid along any line. His argument goes as follows: Since $C$ has full dimension, it contains a simplex whence an interior point (GA p. 131).[12] Fix any point of that sort as the origin; employ orthogonal coordinates to have $\mathbb{E} = \mathbb{R}^I$ for some finite set $I$; and define an associated *gauge function* this way: Along an arbitrary ray, emanating from the chosen origin, let $[0, x^0]$ denote the largest closed line segment contained in $C$. For any other point $x = (x_i)$ on that same ray, each ratio $x_i/x_i^0, i \in I$, must equal a common number $\gamma_C(x) \geq 0$.[13]

Clearly, the function $\gamma_C$ so defined satisfies $C = \{x : \gamma_C(x) \leq 1\}$. For an alternative definition of $\gamma_C$, also due to M and now standard, see (12). Upon verifying that $\gamma_C$ is positively homogeneous, subadditive - and thus continuous - M concludes that $C$ must be closed. Letting $M$ and $m > 0$ denote the attained maximum and minimum respectively of $\gamma_C$ on the unit sphere, the positive homogeneity yields

$$m\,\|x\| \leq \gamma_C(x) \leq M\,\|x\| \text{ for all } x. \tag{3}$$

Consequently, $C$, being contained in a ball of radius $M$, must be bounded as well. Clearly, in current jargon, $\gamma_C$ defines a *norm* - and an associated geometry where the standard body $\{x : \gamma_C(x) \leq 1\}$ takes the place of the customary closed unit ball. M also mentions almost explicitly (GA p. 136) that $x \in intC, y \in C \Rightarrow [x, y) \subset intC$, a property now named *linear accessibility.*[14] His arguments in GA p. 161-2 bring out that $C = cl(intC)$, and conversely, if $C$ equals the closure of an open convex set $\mathcal{O}$, then $\mathcal{O} = intC$.

The priority M assigns to convex bodies reflects, of course, the convenience of situating his geometric objects within a minimal topological context, one which makes for coincidence between relative and absolute interiors.[15] That is, he prefers to arrange situations so that the *algebraic interior*

$$coreC := \{c \in C : \mathbb{R}_+(C - c) = \mathbb{E}\} \tag{4}$$

equals the topological interior. In finite dimensions, when $C$ is convex, $intC = coreC$. More generally, $intC \subseteq coreC$, and what imports in manifold settings is

[12] In modern notation, M brings out here that for any simplex in finite dimesions, and more generally for any convex body $C$, $relintC = coreC = intC$.

[13] See GA p. 132. More generally, for any two vectors $x = (x_i), y = (y_i)$ on the same ray must satisfy $x_i/y_i = \gamma_C(x)/\gamma_C(y)$ provided $y_i \neq 0$. Then, if $\gamma_C(x) = 1$ and $\|y\| = 1, 1/\gamma_C(y)$ is the distance of $x$ to the origin; see GA p. 134.

[14] As said, the avenue just described runs opposite to M's first approach to convex sets in GZ §1-8. There he introduced positively homogeneous "radial distances" (*Strahldistanzen*) $S(x - x') \geq 0$ along rays. These should satisfy $S(x - x') = 0 \Leftrightarrow x = x'$, but not necessarily the triangle inequality. A function of this sort is fully determined by any of its lower level sets $\{x : S(x - x') \leq r\}$, the reference point $x'$ being arbitrary and $r > 0$. When $S(\cdot)$ is continuous, M calls such sets *starshaped bodies* (*Strahlenkörper*). A vector collection $C$, having nonempty interior, is a convex body iff starshaped with respected to any interior point. Thus, starshaped bodies are convex iff $S(\cdot)$ is a norm (*ein einhelliger Strahldistanz*; see (Minkowski, (1896) , § 24)). In addition, $S$ is symmetric (wechselseitig) iff $C$ is so; that is, $S(x) = S(-x)$ iff $x \in C \Rightarrow -x \in C$.

[15] When $intC = \varnothing$, one may invoke relative interior as described in (Rockafellar, 1970) and (Rockafellar and Wets, 1998).

to have at least one of these sets nonempty. Anyhow, M certainly accommodates convex sets which have empty interior or are unbounded, saying that those he mostly considers have three defining properties: First, $x, y \in C \Rightarrow [x, y] \subseteq C$; second, $C$ is bounded; and third, it must be closed (GA p. 106 and 134 ). Examples of such objects, says M, include (the convex polytope (Grünbaum, 1967))

$$P := \left\{ \sum_{i \in I} \lambda_i x^i : \sum_{i \in I} \lambda_i = 1, \forall \lambda_i \geq 0 \right\}, \tag{5}$$

with $I$ finite. The set $P$ so defined must be part of any convex set that comprises $\{x^i\}$ (GA p. 135). The last observation amounts, of course, to say that the *convex hull of a set, formed by convex combinations of its members, equals the intersection of all convex supersets.* In particular, $P = conv \left\{ x^i : i \in I \right\}$. Thus M provides the customary direct, primal, algebraic-geometric description of a convex set in terms of its habitat. For a minimal description of polytopes M notes that if $x^j = \sum_{i \in I} \lambda_i x^i \in P$ with $\lambda_j < 1$, then $P = conv \left\{ x^i : i \in I \diagdown j \right\}$; see GA p. 135-136.[16]. Repeated elimination leaves $P$ finally as the convex hull of

## 5. Extreme Points

Regarding the set $P$ in (5) M declares $x^i$ a *corner point* (Eckpunkt) if it does not belong to the convex hull of $x_j, j \neq i$, and he adds:

**Definition.** "A line through a **corner point** cannot contain points from $P$ on both sides"(GA p. 136). □

Corner points are precisely those he later calls *extreme,* that is, those elements $x$ in a convex set $C$ for which $C \diagdown x$ remains convex. As a prelude to his characterization of compact convex sets, M notes that $P$ equals the convex hull of its corner points (GA p. 135-136).[17] M devotes an entire section to this general notion (GA §12, p. 157-161). Established there is the "reconstruction" of a compact convex set from its subset $extC$ of extreme points:

**Theorem (**extreme points GA p. 160). *Any compact convex subset of a finite-dimensional vector space equals the convex hull of its extreme points:* $C = conv(extC)$. [18] □

In essence, M's proof of this important result has become the standard, inductive one which invokes that any point extreme in the intersection with a supporting hyperplane must also be extreme in the original set. (Clearly, the assumption that $intC$ be non-empty, is superfluous.)

M demonstrates that any convex body $C$, containing 0 in its interior, can be closely approximated, from inside and outside, by two homothetic polytopes (GA

[16] In other words: A compact convex set $P$ is a polytope if $extP$ is finite.

[17] If $P$, as defined above (5), contains interior points, it is named a *polyeder.* That notion differs from the modern *polyhedron*, meaning an intersection of finitely many closed half-spaces. Put differently: M's polyeder is a *bounded* polyhedron with nonempty interior.

[18] Carathéodory's sharper form says that at most $\dim \mathbb{E} + 1$ extreme points are needed.

p. 138-139). More precisely, given any $\delta > 0$ there exists a set $P$ of type (5) such that

$$P \subseteq C \subseteq (1+\delta)P. \tag{6}$$

This result, used for his study of volumes (GA p. 124), he later refines, showing that $P$ can be generated exclusively by extreme points of $C$ (GA p. 160-161). Approximation (6) relates closely to his number theoretic studies. To wit, (1) is a functional counterpart to set-theoretic version (6) - and also similar to (3). (Corresponding smooth approximations $P$ and $\Phi$ are displayed in GA p. 233-235.)

M's extreme point theorem has become a fundamental tool of functional analysis. There it is referred to as a the *Krein-Milman theorem* (Krein and Milman, 1940), saying that *a compact convex subset of a locally convex separated space equals the closed convex hull of its extreme points.* (For a proof and historical references see (Valentine, 1964).) Also intimately related is Choquet's theorem: *Any element of a metrizable compact convex set in a locally convex separated space is the barycenter of a Borel probability measure supported by the extreme points of the said set;* see Phelps (Phelps, 2001).

Returning to finite dimensions, and accommodating the set $rextC$ of *extreme rays*, the compactness assumption may be dropped to have Klee's theorem (Klee, 1958): *For a line-free closed convex set $C$ it holds that $C = conv(extC \cup rextC)$.* Of particular importance are instances where both sets $extC$ and $rextC$ are finite:

## 6. Finitely Generated Sets

The finitely generated, bounded set $P$, defined in (5), is nowadays called a *convex polytope.* The combinatorial properties of these objects have long attracted much interest. Fundamental in that regard is the

**Theorem (**on polytopes**,** GA, chapt. XXII; see (Grünbaum, 1967)). (i) *Let $P$ be a d-dimensional polytope. Then, each maximal proper face (ie, facet) $F$ of $P$ has a unique outward normal vector $v_F$ with $\|v_F\| = vol_{d-1}F$, such that*

$$F \neq F' \Rightarrow v_F \notin \mathbb{R}_+ v_{F'}; \quad \dim aff\,\{v_F\} = d, \quad \textit{and} \quad \sum v_F = 0. \tag{7}$$

(ii) *Conversely, given a finite system of vectors $\{v_F\}$ satisfying* (7)*, there exists a d-dimensional polytope $P$ - unique up to translation - having outward normals $v_F$ to facets $F$ with $\|v_F\| = vol_{d-1}F$.*

We find M's proof of (ii) particularly interesting. An outline goes as follows: Clearly, for appropriate $\mathbf{r} = (r_F) \geq 0$, the polytope $P(\mathbf{r}) := \cap_F \left\{ x : \left\langle \frac{v_F}{\|v_F\|}, x \right\rangle \leq r_F \right\}$ is *the* natural candidate. Since, by the Brunn-Minkowski theorem,[19] $0 \leq \mathbf{r} \mapsto \psi(\mathbf{r}) := [volP(\mathbf{r})]^{1/d}$ is concave, continuous and homogenous, the (hypographical) set $\{(\mathbf{r}, w) \geq 0 : \psi(\mathbf{r}) \geq w\}$ must be a closed convex cone. That cone has a compact convex "base" $\mathbb{B} = \{\mathbf{r} \geq 0 : \psi(\mathbf{r}) = 1\}$ over which the linear form $\mathbf{r} \mapsto \sum_F \|v_F\| r_F$

[19] **The Brunn-Minkowski theorem**: *If $C_0, C_1 \subseteq E$ are closed convex and $C_r := (1-r)C_0 + rC_1$, then the function $[0,1] \ni r \mapsto (volC_r)^{1/\dim \mathbb{E}}$ is concave, and strictly so unless $C_0, C_1$ are homothetic or lie in parallel hyperplanes.* For proofs see (Bonnesen and Fenchel, 1934).

attains its minimum at some $\mathbf{r}^*$. Then $P(\mathbf{r}^*)$ has a vector of facet volumes proportional to $(\|v_F\|)$ so, after suitable scaling the desired conclusion obtains.

One "evident" but important feature comes up in connection with polytopes, namely: The topological closure derives from purely algebraic assumptions. In that regard M, while dealing almost exclusively with closed sets, notes that if $P$ has non-empty interior, then each convex combination in (5), having strictly positive coefficients, must be interior (ein innerer Punkt). Otherwise, when $intP = \emptyset$, such a combination is *inwendig,* i.e. it belongs to the relative interior (GA p. 136).

As said, M mostly dealt with bounded closed convex sets, (5) being one example. Important unbounded instances include linear inequality systems $Ax \leq 0$, featuring a matrix $A$ of size $I \times J$, the solution set

$$K := \left\{x \in \mathbb{R}^J : Ax \in \mathbb{R}^I_-\right\} \tag{8}$$

of which must be not only closed convex but also a *cone*, that is to say, $rK \subseteq K$ for all real $r \geq 0$; see GZ §19.[20] The geometric theory of linear inequalities thus begins with M (and was later developed systematically by Motzkin (Motzkin, 1936)). M finds that $K$ is formed by all non-negative linear combinations of finitely many so-called *extreme directions (äusserste Lösung)*:

**Theorem (**on polyhedral cones GZ §19): *A polyhedral cone, as defined in (8) is generated by finitely many rays.* $\square$

Conversely, Weyl (Weyl, 1935), in adding to this "elementary" characterization, showed that a finitely generated cone must be polyhedral.[21] As above, a topological proposition thus derives from purely algebraic assumptions, namely: *Any finitely generated cone must be closed.*

Noteworthy is the key role assigned by M and Weyl to *extreme supports* (extreme Stütze/ Stützebenen).[22] M defines these objects as supporting hyperplanes that correspond (via polarity) to extreme points of the polar set (see GA p. 164 and Section 8 below). So, in the present context, the normal vectors to extremal supports of the *polar cone*

$$K^- := K^0 := \{x^* : \langle x^*, x\rangle \leq 0, \forall x \in K\} \tag{9}$$

constitute the extreme directions that generate $K$. Anyway, if the linear homogeneous inequality system $Ax \leq 0$ admits non-zero solutions, then, geometrically speaking, these constitute an unbounded pyramid with apex at 0. This result dictates minimal descriptions of polyhedral cones - either internal-primal in terms of extreme directions or external-dual by means of extreme supports.

These results on cones point to *Farkas' Lemma* (Farkas, 1902): If for some $c \in \mathbb{R}^J$, the system $-x \in K,\ \langle c, -x\rangle > 0$ is inconsistent; that is, if the inequalities

$$Ax \geq 0, \quad \langle c, x\rangle < 0 \tag{10}$$

[20] M made inhomogeneous systems $Ax \leq b, b \neq 0$, homogenous by adjoining a new variable - a technique which later has become standard.

[21] For a new proof of both statements see (Rockafellar and Wets, 1998) 3.52.

[22] In §14 p.166-168 M characterizes an extreme support as what he names a *Tangentialeben* (a tangent plane). The latter object is a supporting plane $H_\varepsilon := \{x : \langle x^*, x\rangle = \delta^*_C(x^*) - \varepsilon\}$, with $\varepsilon = 0$, such that the parallel intersection $C \cap H_\varepsilon$, for $\varepsilon > 0$ sufficiently small, contains a ball (of dimension one less than $C$) with radius $\rho(\varepsilon)$ such that $\rho(\varepsilon)/\varepsilon \to +\infty$ as $\varepsilon \to 0$.

have no solution, then $c$ must belong to the *polar cone* $K^0$ generated by the rows $a_{i\cdot}$ of $A$ (see (9) and Section 8). This means that the system

$$\sum y_i a_{i\cdot} = c,\ \forall y_i \geq 0, \tag{11}$$

admits a solution.

Farkas' Lemma helps to emphasize the key role of convexity in combinatorial optimization (Bachem,1983 ), (Schrijver, 2003). It also serves as the point of departure in modern *theory of finance* (Duffie, 1992). Specifically, suppose security $j \in J$ is available right now, in any quantity $x_j \in \mathbb{R}$, at unit price $c_j$, and promises payoff $a_{ij}$ tomorrow if state $i \in I$ then comes about. A portfolio or investment strategy $x \in \mathbb{R}^J$ in securities which solves (10), is called an *arbitrage* (Ellerman, 1984). That investment requires negative purchase cost (by offering an immediate bonus $|\langle c, x\rangle| > 0$), and it yields at least 0 payoff whatever happens tomorrow. Such opportunities cannot reasonably persist in an equilibrated market. Thus, an equilibrium price system $c$ must solve (11).

The Minkowski-Weyl algebraic-geometric characterization of polyhedral cones fits to a large, now classic field concerned with the solvability and geometry of linear/affine inequality/equality systems (Motzkin, 1936). Incorporated there is a long list of *theorems on alternatives* (De Giuli et al. ), including that of Gordan (Gordan, 1873), frequently used to prove Fritz-John type optimality conditions for constrained programs (Borwein and Lewis, 2000). Linear programming aroused interest in those theorems because they bear on the feasible set. Conversely, they are often proved via linear programming duality (Chvátal, 1983).

A subset of an Euclidean space is nowadays called *finitely generated* if it equals the Minkowski sum $P + K$ of a polytope $P$ and a finitely generated cone $K$. Concerning these objects there is a theorem attributed to Minkowski-Weyl**:** *A set is finitely generated iff polyhedral.*

So, a minimal description of finitely generated sets comes either via extreme points/directions or supports; see (Bazaraa and Shetty, 1979). Defining a function $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ to be polyhedral (or finitely generated) iff its *epigraph* $epi f := \{(x, r) : f(x) \leq r\}$ is of that special sort, we arrive at most important objects of mathematical programming: namely, functions $f$ whose effective domains $\{x : f(x) < +\infty\}$ are polyhedral and, on that set, equal the pointwise maximum of finitely many affine functions.

Convex sets that cannot be finitely generated are often best described by

## 7. Gauges, Supports and Tangent Cones

When 0 is interior to the convex body $C$, M shows that the associated *gauge* - also called the *Minkowski (gauge) functional* -

$$x \mapsto \gamma_C(x) := \min\{r \geq 0 : x \in rC\} \ldots (= \min\{r \geq 0 : x \in r bdC\}) \tag{12}$$

vanishes only at the origin, is well defined, finite-valued, positively homogenous, and subadditive whence continuous.[23]

[23] When $\mathbb{E} = \mathbb{R}^I$ with finite $I$, and $x = (x_i)$, the subadditivity entails $\gamma_C(x) \leq \sum_{i\in I} \gamma_C(x_i e^i)$, see GA p. 132-133. Letting $\Gamma := \max_{i\in I} \gamma_C(\pm e^i)$, it follows that $\gamma_C(x) \leq \Gamma \sum_{i\in I} |x_i|$ and thereby $\left\{x : \|x\|_\infty \leq \frac{1}{\Gamma \# I}\right\} \subset \left\{x : \|x\|_1 \leq \frac{1}{\Gamma}\right\} \subseteq C$.

Function (12) has a constructive, clear-cut, geometric meaning, namely: $\gamma_C(x)$ tells exactly how much must $C$ be inflated/deflated in order to just contain $x$. It was not common practice at M's time though, to define or use functions, like (12), that assume no explicit, closed form. The originality of M's approach is underlined by his alternative, more axiomatic definition of gauge (12): *$\gamma_C$ is the positively homogeneous function that equals* 1 *on the boundary of a convex body $C$ with* $0 \in$ *$intC$*.

When $C$ is symmetric, (12) defines a *norm* $x \mapsto \gamma_C(x)$ on $\mathbb{E}$, with associated metric (distance) $d(x, y) := \gamma_C(x - y)$ and closed unit ball $C$. A vista then opens up for other geometries than that of Euclid. Finite-dimensional vector spaces normed in this manner are often called Minkowski spaces (Valentine, 1964). More generally, construction (12) makes it natural to declare a topological vector space *normable* if it contains a convex body (that is, a bounded closed convex set with nonempty interior).

Conversely, given a function $x \mapsto \gamma(x)$ satisfying the properties just mentioned, its associated lower level set (2) must be a convex body, containing 0 in its interior, and satisfy $\gamma_C = \gamma$ :

**Theorem (**on gauges). *Via (12) and (2) there is a one-to-one correspondence $C \leftrightarrow \gamma_C$ between two special classes of quite different nature: on one side compact convex sets $C \subset \mathbb{E}$ which contain* 0 *as interior point; on the other side non-negative, positively homogenous, subadditive, continuous functions $\gamma = \gamma_C : \mathbb{E} \to \mathbb{R}$ that vanish only at the origin.* $\square$

Within the said classes a monotonicity property evidently holds: $C \subseteq C' \Leftrightarrow \gamma_C \geq \gamma_{C'}$. Also, $\gamma_{rC} = \gamma_C / r$ whenever $r > 0$, and for any finite family of convex bodies $C_i, i \in I$, each containing 0 as interior point, we have $\gamma_{conv \cup_i C_i} = \max_i \gamma_{C_i}$; see GA p. 157.

The gauge of a convex body $C$ with $0 \in intC$, provides a unit of measurement along every 1-dimensional linear subspace. Orthogonal to such a subspace stands a so-called *hyperplane.* Therefore, instead of asking how $C$ fares on lines, one may inquire how it relates to hyperplanes. Pursuing this dual perspective M introduces the *support function* (Stützebenenfunktion) $\delta^*_C$ of $C$; see GA p. 4-6 and p. 144-147. The requirement that $0 \in intC$ is later dropped, GA p. 150-153. Quite generally, he posits

$$\delta^*_C(x^*) := \sup_{x \in C} \langle x^*, x \rangle = \sup \{ \langle x^*, x \rangle - \delta_C(x) : x \in \mathbb{E} \} \tag{13}$$

albeit without introducing the more modern, extended indicator $x \mapsto \delta_C(x)$ which equals 0 on $C$ and $+\infty$ elsewhere. He observes that $\delta^*_C(\cdot)$ so defined becomes positively homogeneous, subadditive. In particular, $0 \in C \Leftrightarrow \delta^*_C \geq 0$, and $\delta^*_C$ vanishes only at the origin iff $0 \in intC$. Also, $\delta^*_C(x^*) + \delta^*_C(-x^*) > 0$ for all nonzero $x^*$ iff $intC$ is nonempty. Otherwise, if $\delta^*_C(x^*) + \delta^*_C(-x^*) = 0$ for some nonzero $x^*$, then $C$ is part of the hyperplane $\{x : \langle x^*, x \rangle = \delta^*_C(x^*)\}$ . A symmetric (balanced) set $K := \frac{C - C}{2}$, with 0 as center (Mittelpunkt GA p. 4), has symmetric support $\delta^*_K(\cdot) = \delta^*_K(-\cdot)$.

More generally, its suffices for the sublinearity of $\gamma_C$ that $C$ be convex and $0 \in coreC$. Then $coreC = \{x : \gamma_C(x) < 1\} \subseteq C \subseteq \{x : \gamma_C(x) \leq 1\}$ . If moreover, $C$ is symmetric, then $\gamma_C$ becomes a *seminorm.* These are instrumental for the definition of topologies in linear spaces (Holmes, 1975). Note that $\gamma_C(x) = 0$ iff $\mathbb{R}_+ x \subseteq C$.

Most importantly, M records that the original set can be recovered, namely

$$C = \{x : \langle x^*, x\rangle \leq \delta_C^*(x^*), \forall x^*\}, \tag{14}$$

see GA p. 145. Thus, once again M identifies two sides of the same convex coin:

**Theorem** (on support functions). *Via (13) and (14) there is a one-one correspondence $C \leftrightarrow \delta_C^*$, this time between larger classes: on one side closed convex sets, on the other side positively homogeneous, subadditive functions.* □

(14) provides a dual description of a convex set by means of objects from the conjugate space $\mathbb{E}^*$ $(= \mathbb{E})$, consisting of all continuous linear functionals $x \mapsto x^*(x) = \langle x^*, x\rangle \in \mathbb{R}$. Clearly, monotonicity again prevails: $C \subseteq C' \Leftrightarrow \delta_C^* \leq \delta_{C'}^*$. Also, $\delta_{rC}^* = r\delta_C^*$ whenever $r > 0$, and for any finite family of compact sets $C_i, i \in I$, one has $\delta^*_{conv \cup_i C_i} = \max_i \delta^*_{C_i}$; see GA p. 156.

The possibility to codify a closed convex set by its support function is often very useful. For an economic example, let a price-taking (also called perfectly competitive) firm be fully described by a closed nonempty convex set $C$, representing its technology and comprising all feasible input-output vectors. That firm can equally well - and uniquely - be depicted by its profit function $x^* \mapsto \delta_C^*(x^*)$, reporting the maximal profit obtainable under diverse price regimes $x^* \in \mathbb{E}^*$; see (Diewert, 1981). Clearly, given a finite set of observations $\left[x^{*i}, \delta_C^*(x^{*i})\right], i \in I$, then econometrically speaking, $C$ is "overestimated" by the possibly larger set $\cap_{i\in I} \left\{x : \left\langle x^{*i}, x\right\rangle \leq \delta_C^*(x^{*i})\right\}$.

Other examples come from nonsmooth analysis, dealing with generalized derivatives and convex-valued correspondences. Specifically, consider the two *directional derivatives*

$$f^0(x; d) := \lim \sup_{\Delta x \to 0, t \to 0^+} \frac{f(x + \Delta x + td) - f(x + \Delta x)}{t} \tag{15}$$

and

$$f^\diamond(x; d) := \sup_{\Delta x} \left( \lim \sup_{t \to 0^+} \frac{f(x + t\Delta x + td) - f(x + t\Delta x)}{t} \right), \tag{16}$$

introduced by Clarke (Clarke, 1983) and Michel-Penot (Michel and Penot, 1984) respectively. These definitions were motivated by the need to extend differential calculus beyond smooth or convex/concave functions. When $f : \mathbb{E} \to \mathbb{R}$ is merely Lipschitz around $x$, the said derivatives are sublinear in the direction $d$, and $f^\diamond(x; \cdot) \leq f^0(x; \cdot)$. Therefore, using M's apparatus (14), one may define associated *subdifferentials*

$$\partial f^0(x) := \left\{x^* : \langle x^*, d\rangle \leq f^0(x; d), \forall d\right\}, \ \partial f^\diamond(x) := \{x^* : \langle x^*, d\rangle \leq f^\diamond(x; d), \forall d\}$$

to obtain w*-compact sets $\partial f^\diamond(x) \subseteq \partial^0 f(x)$ that figure in non-smooth differential calculus and reproduce (15), (16) via (13):

$$f^0(x; d) = \max\left\{\langle x^*, d\rangle : x^* \in \partial f^0(x)\right\}, \ f^\diamond(x; d) = \max\left\{\langle x^*, d\rangle : x^* \in \partial f^\diamond(x)\right\}$$

The connections $f^0(x; \cdot) \leftrightarrow \partial f^0(x)$, $f^\diamond(x; \cdot) \leftrightarrow \partial f^\diamond(x)$ are characteristic of the role played by convexity - and notably by Minkowski's constructions - in modern analysis; they are instrumental in the passage from linear/ smooth/ uni-valued relations to corresponding nonlinear/ nonsmooth/ multi-valued objects. In short, M was first to build bridges between convex functions and sets, between convex geometry and analysis.

Functions (15), (16) provide conical (sublinear) approximations of $f$ at the point $x$ of reference. Basic in this regard is the conical approximation of a convex body $C$ at any boundary point $x$. Specifically, M defines the (shifted) *tangent cone* (*Projektionsraum* GA p. 161)

$$x + T_C(x) := cl\,\{x + \mathbb{R}_{++}(intC - x)\}\,,$$

and he identifies it with the intersection of all supporting half-planes passing through $x$; that is,

$$x + T_C(x) = \{e \in \mathbb{E} : \langle x^*, e\rangle \leq \delta^*(x^*) \text{ whenever } \delta^*(x^*) = \langle x^*, x\rangle\}\,.$$

A full circle of constructions closes here, namely: If $C = \{\gamma \leq 1\}$ for a positive, sublinear $\gamma$, and $x \in bdC = \{\gamma = 1\}$ , then

$$\gamma^0(x;\cdot) = \gamma^\diamond(x;\cdot) = \lim_{r\to 0^+} \frac{\gamma(x + r\cdot) - \gamma(x)}{r} =: \gamma'(x,\cdot)$$

and $T_C(x) = \{\gamma'(x,\cdot) \leq 0\}$ . Vectors $x^*$ for which $\delta^*(x^*) = \langle x^*, x\rangle$ are called *normals* to $C$ at $x$; they constitute a closed convex cone $N_C(x)$, (composed of exactly those points $x^* - x$ such that $x^*$ has $x$ as its best approximation in $C$.) M classifies boundary points of $C$ according to the dimension of their normal cones. In particular, when $intN_C(x)$ is nonempty, he declares $x$ a *corner point (Eckpunkt).* Such points, already encountered in connections with polytopes, are extreme and at most countably many.

A half-space $\{a \leq 0\}$ , containing $C$, is called *extreme* if for any two different, bounding half-spaces $\{a_i \leq 0\} \supseteq C$ and positive numbers $r_i, i = 1, 2$, it holds that $a \neq r_1a_1 + r_2a_2$. M shows in GA p. 166 that (6) obtains with a polytope $P$ delineated only by extreme half-spaces of $C$. Preceding this notion, of course, is the more elementary concept of separation:

## 8. Separation, Polarity and Duality

M introduced the concept of a *supporting hyperplane* (with prescribed outward normal) to a closed set (GA p. 106), and shows - as a preliminary for later results - that every boundary point of a polytope (5) admits a supporting hyperplane. Any polytope is fully characterized by a finite family of supporting planes (GA p. 137). M went on to demonstrate, in two different ways, the following

**Theorem** (on supporting hyperplanes GA p.139). *Through every boundary point of a convex body there passes a supporting hyperplane.* □

M's first proof uses the already obtained result for polytopes by invoking the two-sided approximation (6). His second demonstration is distinctly modern in flavor, employing the *orthogonal projection* $P_C(x) := \arg\min_{c\in C} \|x - c\|$ , shown to be attained. If $x \notin C$, then the hyperplane through $x$, having unit normal vector directed outwards along $x - P_C(x)$, does not intersect $C$. (This argument carries verbatim over to Hilbert spaces.) When $x \in bdC$, he supposes without loss of generality that $0 \in intC$ and notes that $x \notin rC$ for any $r \in (0,1)$. Now M replaces $C$ by $rC$, repeats the preceding argument, and finally he let $r \nearrow 1$ to achieve the desired result. M proceeded to prove a first

**Theorem** (on separation, geometric form). *Two convex bodies $C, C'$ with $C \cap intC' = \emptyset$, can be separated by a hyperplane.*[24] $\square$

For the argument he again used compactness, continuity, and strict convexity as instruments. The proof goes broadly as follows: First assume $C \cap C' = \emptyset$. The minimum distance $\inf_{c \in C, c' \in C'} \|c - c'\|$ is then positive and uniquely realized by two points $c \in C, c' \in C'$. In arguing this his use of closure and boundedness (i.e. compactness) has become standard, but was at that time quite modern. A plane with unit normal vector $(c - c')/\|c - c'\|$, passing through any point strictly between $c, c'$, will ensure separation. When $C, C'$ have boundary points in common, he assumes $0 \in C$, replaces $C$ by $rC$, $0 < r < 1$, and proceeds as described above by letting $r \nearrow 1$.

In GZ §18 M calls a convex body *everywhere convex* (überall convex) iff every supporting hyperplane touches merely in one point. Evidently, this amounts to strict convexity. Prime examples include lower level-sets $C = \{x : Q(x) \leq r\}, r > 0$, of positive quadratic forms $Q$; see GZ §49.[25] Geometric considerations, akin to the classical inversion with respect to a circle (Hartshorne, 1997), lead M (GA p. 146) to call the lower level set $C^0 := \left\{x^0 : \delta_C^*(x^0) \leq 1\right\}$ the *polar* of $C$. Thus $x^0 \in C^0$ iff $\left\langle x^0, x\right\rangle \leq 1$ for all $x \in C$. The symmetry of this relation he uses to show that for a convex body $C$ with $0 \in intC$ it holds that $C^{00} := (C^0)^0 = C$. This result is a first version of the so-called bipolar theorem*: For any set $C \subseteq E$ one has* $C^{00} = cl(conv(C \cup 0))$. Included, as special case, is the *bipolar cone theorem*, namely: for any set $K \subseteq \mathbb{R}$ one has $(K^-)^- = cl(conv(\mathbb{R}_+ K))$, see (9).

Polarity now serves particularly well in the study of so-called *convex processes,* these being set-valued maps whose graphs are closed convex cones (Borwein and Lewis, 2000), (Borwein, 1983). They provide a unifying format for linear maps, convex cones, and linear programming. M emphasizes the importance of duality by pointing out that the polar of a polytope $P$, containing 0 as interior point, is a set of the same sort; see GA p. 146-147. Specifically*: Suppose $P = \cap_{i \in I} \left\{x : \left\langle x^{io}, x\right\rangle \leq 1\right\}$ for some finite set $I$. Then $P^0 = conv\left\{x^{i0}\right\}_{i \in I}$.*

The fundamental significance of M's separation theorem came later to sharp light in functional analysis - on infinite-dimensional vector spaces - with various formulations of Hahn, Banach, Ascoli and Mazur (Dieudonné, 1981).[26] Important

[24] For an extension see Theorem 2.39 in (Rockafellar and Wets, 1998).

[25] Another instance comes via **Minkowski's theorem on mixed volumes**: *Let $C_j \subset \mathbb{E}$ be a convex body and $r_j \geq 0$ for $j = 1, ..., n$. Then $vol \sum_j r_j C_j = \sum v(C_{j_1}, ..., C_{j_n}) r_{j_1} ... r_{j_n}$, the sum extending independently over all $j_1, ..., j_n$.* $\square$ (The polynomial coefficients are called *mixed volumes*; see (Ewald, 1996).) In GA, Chap. XIII M takes two non-homothetic convex bodies $C_1, C_2$ in $\mathbb{R}^3$ and develops $vol(C_1 + rC_2)$ as a polynomial of degree 3. By the Brunn-Minkowski theorem $0 \leq r \mapsto vol(C_1 + rC_2)^{1/3}$ is *strictly concave.* This yields an isoperimetric inequality which, for the special instance $vol(C_1) = 4\pi R^3/3$, $C_2 =$ unit ball, tells that, among convex bodies, *for given volume the sphere has minimal surface.*

[26] (Holmes, 1975) contains ten different but equivalent formulations.

in such spaces are algebraic and topological versions of the corresponding so-called Hahn-Banach theorem, requiring that either the algebraic interior (4) or topological interior of some ground set be nonempty; see (Lassonde, 1988) and references therein.

Often, the said theorem comes in *analytic* (as opposed to the above geometric) form, saying that if a linear function on a linear subspace is bounded above by a sublinear function, then the first can be extended to the whole space while preserving the same upper bound. However, after representing the linear function by its *graph* and the sublinear function $S$ by its epigraph $epiS := \{(e, r) \in \mathbb{E} \times \mathbb{R} : S(e) \leq r\}$, one may separate those two sets to have the desired linear extension. Thus, in essence, M's geometric form remains as potent as the analytical versions. Also, in infinite dimensions things are turned somewhat around in that gauges and support functions become crucial for defining and understanding the (locally convex) topologies.

We shall not review this but rather return to (13). We insert there any convex function $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ which is *proper* (i.e. $f$ attains finite values somewhere) and lower semicontinuous (or *closed* for short; i.e. $\liminf_{x' \to x} f(x') \geq f(x), \forall x$) to produce the so-called *Fenchel convex conjugate*

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : x \in \mathbb{E}\}$$

The operator $f \mapsto f^*$ thus associates to a proper closed convex function $f$ on $\mathbb{E}$ another proper closed convex function $f^*$ defined on the dual space $\mathbb{E}^*$. Moreover, using M's separation theorem, Fenchel (1949) showed that $f^{**} := (f^*)^* = f$ (and thereby that convex conjugation is bijective). Convex conjugacy thus gives a involutory dual description of proper closed functions. M's merit was to initiate that description for the special case of proper closed sets (that is, for extended indicators) - as done in (13), (14). Various forms of duality, say of Langrangian or Fenchel type, are intimately related to convex conjugates. These manifestations of convexity have become central in theory as well as in computation (Borwein and Lewis, 2000), (Hiriart-Urruty and Lemaréchal, 1993).

## 9. Concluding Remarks

To record and appreciate all contributions of M that invoke convexity, in one form or another, the ambitious historian must follow his impressive routes ¿from number theory to physics, from quadratic forms to relativity theory. Most readers, including us, lack sufficient preparation, motivation, or time to fully undertake such a long, all-comprising journey.[27] It is, however, certainly well worth quite a while to see what and how much M contributed to modern analysis. His studies, often motivated by number theoretic issues, show a rich interplay between analytic, geometric and topological ideas - organized or culminating around the concept of

[27] Hancock (1939) finds that M "*as an expositor was very poor.*" We disagree out of *three* reasons. *First*, for fairness one should bear in mind that linear algebra and topology, both masterly used by M, were in their infancy around 1895 (and still not quite "modern" in Hancock (Hancock,1939)). *Second*, since M was a pioneer, offering manifold novelties, one can hardly expect didatics to keep full pace with his contributions. *Third*, M died "*im Vollbesitz seiner Lebenskraft, aus der Mitte freudigstens Wirkens, von der Höhe seines wissenschaftlichen Schaffens*," (GA vol. I, p.V), before some works in GA had found their final form.

convex sets. This was a major novelty in number theory. It took however, about thirty years before convexity made its way into game theory (Ville, 1938)[28] and functional analysis (Ascoli, 1932), ( Mazur, 1933). Two more decades passed before it became central in optimization theory (Fenchel, 1951). And only recently has the same interplay come to full fruition in nonsmooth (variational) analysis (Clarke, 1983), (Rockafellar and Wets, 1998). In these subjects, beginning with (Minkowski, (1896) ), the *geometric* nature of various issues have been unifying; see for example (Holmes, 1975).

Instrumental for all this development were M's separation theorem and two classes of convex functions: gauges (12) and supports (13) - both sublinear, both introduced by him. While these constitute the most important special case of convex functions, the general concept came first with Jensen (1906). Later, the unifying epigraphical, "geometric" perspective of Fenchel (1951) brought convex sets and functions on equal footing. His optic made it possible to "identify" tangent planes (cones) with generalized (directional) derivatives.

Interesting in this regard is that two main, early users of convex analysis - noncooperative game theory (Nash 1951) and mathematical economics (Debreu 1959) - let convexity substitute for uni-valuedness and smoothness. Equally interesting, in those fields, is the lapse of almost two hundred years - and the role of convexity in fixed point theorems - needed to consolidate the first, pioneering economic insights of A. Smith (1776) and A. Cournot (1838).

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# Weightedness for Simple Games with Less than 9 Voters*

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**Abstract** In voting systems, game theory, switching functions, threshold logic, circuits and electrical engineering, coherent structures and systems' reliability and cryptography, among other fields, there is an important problem that consists in determining the weightedness of a binary voting system by means of *trades among voters in sets of coalitions.* (Taylor and Zwicker, 1995) construct a sequence of games $G_k$ based on magic squares which are $(k-1)$-trade robust but not $k$-trade robust, each one of these games $G_k$ has $k^2$ players. (Freixas and Molinero, 2008) propose a refinement on trade robustness called *invariant*-trade robustness and prove that as few as $2k+1$ voters are needed to find games being $k$-invariant trade robust but not $(k+1)$-invariant trade robust.
In this work we classify all simple games with less than nine players according to the two criteria: invariant-trade robustness and trade robustness. The classification obtained in this work with eight players is new.
Moreover, some new experiments establish new conjectures about the trade robustness of complete simple games.

**Keywords:** Simple games, Asummability, Trade robustness, Invariant-trade robustness.

## 1. Introduction and Preliminaries

In this section we sketch some notions about simple games necessary for the pursuit of the work. We refer the interested reader to (Freixas and Molinero, 2008) for more details.

**Definition 1.** A simple game $G$ is a pair $(N, \mathcal{W})$ in which $N = \{1, 2, \ldots, n\}$ and $\mathcal{W}$ is a collection of subsets of $N$ that satisfies: $N \in \mathcal{W}$, $\emptyset \notin \mathcal{W}$ and (monotonicity) $S \in \mathcal{W}$ and $S \subseteq T \subseteq N$ then $T \in \mathcal{W}$.

Any set of voters is called a *coalition*, and the set $N$ is called the *grand coalition*. Members of $N$ are called *players* or *voters*, and the subsets of $N$ that are in $\mathcal{W}$ are called *winning coalitions*. A subset of $N$ that is not in $\mathcal{W}$ is called a *losing coalition*. A *minimal winning coalition* is a winning coalition all of whose proper subsets are losing. Because of monotonicity, any simple game is completely determined by its set of minimal winning coalitions. Some real–world examples of simple games are given in (Taylor, 1995).

Of fundamental importance to simple games are the subclasses of weighted simple games and complete simple games.

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**Definition 2.** A simple game $G = (N, \mathcal{W})$ is said to be weighted if there exists a "weight function" $w : N \to \mathbb{R}$ and a real number "quota" $q \in \mathbb{R}$ such that a coalition $S$ is winning precisely when the sum of the weights of the players in $S$ meets or exceeds the quota.

The *associated weight vector* is $(w_1, \ldots, w_n)$. Any specific example of such a weight function $w : N \to \mathbb{R}$ and quota $q$ as in Definition 2 are said to *realize* $G$ as a weighted game. A particular realization of a weighted simple game is denoted as $[q; w_1, \ldots, w_n]$.

One of the most important problems for simple games is determining whether a simple game can be realized as a weighted simple game. The only results giving necessary and sufficient conditions can be found under one of the next three topics: *geometric approach based on separating hyperplanes*; *algebraic approach based on systems of linear inequalities*; *approach based on trading transforms.* The approach based on trades is the most natural and suggests several interpretations that will be tackled here from a computational viewpoint.

**Definition 3.** Suppose $G = (N, \mathcal{W})$ is a simple game. Then a trading transform (for G) is a coalition sequence $\mathcal{J} = \langle S_1, \ldots, S_j, T_1, \ldots, T_j \rangle$ (from $G$) of even length satisfying the following condition: $|\{i : p \in S_i\}| = |\{i : p \in T_i\}|$ for all $p \in N$. $S_i$ are called the pre–trade coalitions and the $T_i$ the post–trade coalitions, and we will say that $\langle S_1, \ldots, S_j \rangle$ has been converted by a trade to $\langle T_1, \ldots, T_j \rangle$.

**Definition 4.** A $k$-trade for a simple game $G$ is a trading transform $\mathcal{J} = \langle S_1, \ldots, S_j, T_1, \ldots, T_j \rangle$ in which $j \leq k$. The simple game $G$ is $k$-trade robust if there is no such $\mathcal{J}$ for which all the $S$s are winning in $G$ and all the $T$s are losing in $G$. If $G$ is $k$-trade robust for all $k$, then $G$ is said to be trade robust.

Loosely speaking, $G$ is $k$-trade robust if a sequence of $k$ or fewer (not necessarily distinct) winning coalitions can never be rendered losing by a trade.

**Theorem 1.** *(Taylor and Zwicker, 1992) For a simple game $G$, the following are equivalent:*

$(i)$ *$G$ is weighted.* $(ii)$ *$G$ is trade robust.* $(iii)$ *$G$ is $2^{2^{|N|}}$-trade robust.*

**Theorem 2.** *(Taylor and Zwicker, 1995) For each integer $k \geq 3$, there exists a simple game $G_k$ with $k^2$ players, that is $(k-1)$-trade robust, but not $k$-trade robust.*

Now, it will be of interest determining the minimum number of voters needed to reach games within these categories. In this paper we will make experiments in order to solve this problem for some small values. Unfortunately the number of simple games is too large to be tackled straightforwardly. So, we will introduce some definitions to deal with another significant class of simple games (complete simple games).

**Definition 5.** Let $(N, \mathcal{W})$ be a simple game, $i$ and $j$ be two voters. Players $i$ and $j$ are said to be equally desirable, denoted by $i \approx j$ if: for any coalition $S$ such that $i \notin S$ and $j \notin S$, $S \cup \{i\} \in \mathcal{W} \Leftrightarrow S \cup \{j\} \in \mathcal{W}$.

**Definition 6.** ((Isbell, 1958)) Let $(N, \mathcal{W})$ be a simple game, $i$ and $j$ be two voters. Player $i$ is said to be more desirable than $j$, denoted by $i \succ j$ if the following two conditions are fulfilled:

1. For every coalition $S$ such that $i \notin S$ and $j \notin S$, $S \cup \{j\} \in \mathcal{W} \Rightarrow S \cup \{i\} \in \mathcal{W}$.
2. There exists a coalition $T$ such that $i \notin T$ and $j \notin T$, $T \cup \{i\} \in \mathcal{W}$ and $T \cup \{j\} \notin \mathcal{W}$.

The *desirability relation* denoted by $\succeq$ is defined in $N$ as follows: $i \succeq j$ if $i \succ j$ or $i \approx j$, we say that $i$ is at least as desirable as $j$ as coalitional partner. It is straightforward to check that $\approx$ is an equivalence relation, and that $\succeq$ is a partial ordering of the resulting equivalence classes.

**Definition 7.** A simple game $(N, \mathcal{W})$ is complete or linear if the desirability relation is a complete preordering.

If $w_i \geq w_j$ in a weighted game then $i \succeq j$. Hence, if $(N, \mathcal{W})$ is weighted then $(N, \mathcal{W})$ is complete. This result is significant to our purpose because the first step to study the weightedness of a given game $G$ consists in determining whether it is complete. If it fails to be complete then the game is not weighted and the problem is solved.

Thus, hereafter we will be confined to consider only complete simple games.

In a complete simple game we may decompose $N$ in a collection of subsets, called classes, $N_1 > N_2 > \cdots > N_t$ forming a partition of $N$ and understanding that if $i \in N_p$ and $j \in N_q$ then: $p = q$ if and only if $i \approx j$ and $p < q$ if $i \succ j$. The following is a characterization of complete simple games.

(Carreras and Freixas, 1996) provide a classification theorem for complete simple games that allow to enumerate all these games up to isomorphism by listing the possible values of certain invariants. Previously to state it we need some definitions.

**Definition 8.** Given $\overline{n} = (n_1, \ldots, n_t) \in \mathbb{N}^t$, then $\Lambda(\overline{n}) = \{\overline{m} \in (\mathbb{N} \cup \{0\})^t : \overline{m} \preceq \overline{n}\}$ is the set of all vectors $\overline{m} = (m_1, \ldots, m_t)$ whose components satisfy $0 \leq m_k \leq n_k$ for $1 \leq k \leq t$. In $\Lambda(\overline{n})$ the $\delta$-ordering given by the comparison of partial sums is:

$$\overline{m}\,\delta\,\overline{p} \text{ if and only if } \sum_{i=1}^{k} m_i \geq \sum_{i=1}^{k} p_i \quad \text{for } 1 \leq k \leq t.$$

Moreover,

1. If $\overline{m}\,\delta\,\overline{p}$ we will say that vector $\overline{m}$ $\delta$-*dominates* vector $\overline{p}$.
2. If $\overline{m} \neq \overline{p}$ and $\overline{m}\,\delta\,\overline{p}$ we will say that $\overline{m}$ strictly $\delta$-*dominates* $\overline{p}$.
3. If $\overline{m}\,\not\delta\,\overline{p}$ and $\overline{p}\,\not\delta\,\overline{m}$ we will say that $\overline{m}$ and $\overline{p}$ are *not $\delta$-comparable.*

From now on, we shall write $\Sigma_k(\overline{m}) = \sum_{i=1}^{k} m_i$ for $1 \leq k \leq t$ and $\Sigma(\overline{m}) = (\Sigma_1(\overline{m}), \ldots, \Sigma_t(\overline{m}))$ so that $\overline{m}\,\delta\,\overline{p}$ if and only if $\Sigma(\overline{m}) \geq \Sigma(\overline{p})$. It is not difficult to check that the couple $(\Lambda(\overline{n}), \delta)$ is a distributive lattice.

**Definition 9.** Let $\overline{m}_i,\ \overline{m}_h \in \Lambda(\overline{n})$, the notation $i < h$ will mean that there exists some $l$ such that $\Sigma_k(\overline{m}_i) = \Sigma_k(\overline{m}_h)$ for $k < l$ and $\Sigma_l(\overline{m}_i) > \Sigma_l(\overline{m}_h)$.

**Definition 10.** Two simple games $(N, \mathcal{W})$ and $(N', \mathcal{W}')$ are said to be isomorphic if there is a bijective map $f : N \to N'$ such that $S \in \mathcal{W}$ if and only if $f(S) \in \mathcal{W}'$; $f$ is called an isomorphism of simple games.

The following theorem has three parts. The first part shows how to associate a vector $\overline{n}$ and a matrix $\mathcal{M}$ to a complete simple game $(N, \mathcal{W})$ and describes the restrictions that these parameters need to fulfill. The second part establishes that isomorphic complete simple games $(N, \mathcal{W})$ and $(N', \mathcal{W}')$ correspond to the same associated vector $\overline{n}$ and matrix $\mathcal{M}$ (*uniqueness*). The third part shows that a vector $\overline{n}$ and a matrix $\mathcal{M}$ fulfilling the conditions in Part A correspond to a complete simple game $(N, \mathcal{W})$ (*existence*).

**Theorem 3.** *(Carreras and Freixas, 1996)*

**Part A** *Let $(N, \mathcal{W})$ be a complete simple game with nonempty classes $N_1 > N_2 > \ldots > N_t$, let $\overline{n}$ be the vector defined by their cardinalities, and let $\mathcal{M} = (m_{i,j})$, with $1 \le i \le r$ and $1 \le j \le t$, be the matrix satisfying the four conditions below:*

*(1) $m_{i,j} \in \mathbb{N} \cup \{0\}$ and $0 \le m_{i,j} \le n_i$ for all $i, j$ with $1 \le i \le r$ and $1 \le j \le t$;*

*(2) every pair of rows of $\mathcal{M}$, $\overline{m}_h$ and $\overline{m}_{h'}$ are not $\delta$-comparable if $h \neq h'$;*

*(3) if $t = 1$ then $m_{1,1} > 0$; if $t > 1$ then for every $k < t$ there exists some $h$ such that $m_{h,k} > 0$ and $m_{h,(k+1)} < n_{k+1}$; and*

*(4) the rows of $\mathcal{M}$ are lexicographically ordered by partial sums.*

**Part B** *(Uniqueness) Two complete simple games $(N, \mathcal{W})$ and $(N', \mathcal{W}')$ are isomorphic if and only if $\overline{n} = \overline{n}'$ and $\mathcal{M} = \mathcal{M}'$.*

**Part C** *(Existence) Given a vector $\overline{n}$ and a matrix $\mathcal{M}$ satisfying the conditions of Part A, there exists a complete simple game $(N, \mathcal{W})$ the characteristic invariants of which are $\overline{n}$ and $\mathcal{M}$.*

In the next example we illustrate how to obtain $(N, \mathcal{W}^m)$, where $\mathcal{W}^m$ denotes the minimal winning coalitions, from a pair $(\overline{n}, \mathcal{M})$ and conversely.

*Example 1.*

(a) Consider the complete simple game (see the first simple game given in Table 4) defined by

$$\overline{n} = (1, 2, 2, 3) \quad \text{and} \quad \mathcal{M} = \begin{pmatrix} 1\,2\,1\,0 \\ 1\,0\,2\,3 \\ 0\,2\,0\,3 \end{pmatrix}.$$

The pair $(\overline{n}, \mathcal{M})$ obviously satisfies conditions $(1)-(4)$ in Part A of Theorem 3; $\sum_{i=1}^{4} n_i = 8$ gives the number of voters so that $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $N$ decomposes in $N_1 \cup N_2 \cup N_3 \cup N_4$ where $|N_i| = n_i$ for all $1 \le i \le 4$. Hence,

$$N_1 = \{1\} > N_2 = \{2, 3\} > N_3 = \{4, 5\} > N_4 = \{6, 7, 8\}.$$

The minimal winning coalitions are given by the models: $(1, 2, 1, 0)$, $(1, 1, 1, 3)$, $(1, 0, 2, 3)$, $(0, 2, 2, 1)$, $(0, 2, 1, 2)$ and $(0, 2, 0, 3)$. We determine how many coalitions are associated with each model in Table 1. Thus, we should need $2 + 4 + 1 + 3 + 6 + 1 = 17$ minimal winning coalitions to describe the game in classical form $(N, \mathcal{W})$.

| Model | Number of coalitions | Minimal winning coalitions |
|---|---|---|
| (1,2,1,0) | 2 | {1,2,3,4}, {1,2,3,5} |
| (1,1,1,3) | 4 | {1,2,4,6,7,8}, {1,2,5,6,7,8}, {1,3,4,6,7,8}, {1,3,5,6,7,8} |
| (1,0,2,3) | 1 | {1,4,5,6,7,8} |
| (0,2,2,1) | 3 | {2,3,4,5,6}, {2,3,4,5,7}, {2,3,4,5,8} |
| (0,2,1,2) | 6 | {2,3,4,6,7}, {2,3,4,6,8}, {2,3,4,7,8}, {2,3,5,6,7}, {2,3,5,6,8}, {2,3,5,7,8} |
| (0,2,0,3) | 1 | {2,3,6,7,8} |

**Table1.** Models and minimal winning coalitions

(b) Assume, conversely, that a simple game is given in the classical form $(N, \mathcal{W}^m)$, where $N = \{1,2,3,4,5,6,7,8\}$ and

$$\begin{aligned}\mathcal{W}^m = \{ \; & \{1,2,3,4\}, \{1,2,3,5\},\\ & \{1,2,4,6,7,8\}, \{1,2,5,6,7,8\}, \{1,3,4,6,7,8\}, \{1,3,5,6,7,8\},\\ & \{1,4,5,6,7,8\},\\ & \{2,3,4,5,6\}, \{2,3,4,5,7\}, \{2,3,4,5,8\},\\ & \{2,3,4,6,7\}, \{2,3,4,6,8\}, \{2,3,4,7,8\}, \{2,3,5,6,7\}, \{2,3,5,6,8\}, \{2,3,5,7,8\},\\ & \{2,3,6,7,8\} \; \}.\end{aligned}$$

One may easily check that $1 \succ 2 \approx 3 \succ 4 \approx 5 \succ 6 \approx 7 \approx 8$ so that $\overline{n} = (1,2,2,3)$. Next, we consider all vectors $\Lambda(\overline{n})$ and list out all $\delta$-dominated minimal winning vectors:

$$(1,2,1,0), (1,0,2,3) \text{ and } (0,2,0,3).$$

Thus,

$$\overline{n} = (1,2,2,3) \quad \text{and} \quad \mathcal{M} = \begin{pmatrix} 1\,2\,1\,0 \\ 1\,0\,2\,3 \\ 0\,2\,0\,3 \end{pmatrix}$$

which satisfy conditions $(1)-(4)$ in Part A of Theorem 3; and therefore defines a complete simple game.

We firstly reformulate the developed theory in the preceding section to deal with it from a more efficient computational viewpoint. We just recall here the main definitions and results given in (Freixas and Molinero, 2008). The basic idea is that it simplifies the description of the algorithms as well as it meaningfully improves the performance of the experiments.

**Definition 11.** *(cf. Definition 3)* Suppose $G = (N, \mathcal{W})$ is a simple game. Then a $\delta$-trading transform (for G) is a coalition sequence $\mathcal{J} = \langle S_1, \ldots, S_j, T_1, \ldots, T_j \rangle$ (from $G$) of even length satisfying condition $|\{i : p \in S_i\}| = |\{i : p \in T_i\}|$ for all $p \in N$, where $S_1, \ldots, S_j$ are $\delta$-minimal winning coalitions.

**Definition 12.** *(cf. Definition 4)* A $k$-$\delta$-trade for a simple game $G$ is a $\delta$-trading transform $\mathcal{J} = \langle S_1, \ldots, S_j, T_1, \ldots, T_j \rangle$ in which $j \leq k$. The simple game $G$ is $k$-$\delta$-trade robust if there is no such $\mathcal{J}$ for which all the $S$s are $\delta$-minimal winning coalitions in $G$ and all the $T$s are losing in $G$. If $G$ is $k$-$\delta$-trade robust for all $k$, then $G$ is said to be $\delta$-trade robust.

In the following, we provide a trading version applied to indices of columns of $\mathcal{M}$ and vectors instead of players and coalitions.

**Definition 13.** Let $G = (N, \mathcal{W})$ be a complete simple game with characteristic invariants $(\overline{n}, \mathcal{M})$. A vectorial trading transform for $G$ is a vectorial sequence $\mathcal{J}' = \langle \overline{x}_1, \ldots, \overline{x}_j, \overline{y}_1, \ldots, \overline{y}_j \rangle$ of even length satisfying the following conditions:

$$\sum_{i=1}^{j} x_{i,k} = \sum_{i=1}^{j} y_{i,k} \quad \forall\, k \in [1, t] \tag{1}$$

where $\overline{x}_1, \ldots, \overline{x}_j$ are rows of $\mathcal{M}$ with repetitions allowed, and $\overline{y}_1, \ldots, \overline{y}_j$ belong to $\Lambda(\overline{n})$.

**Definition 14.** Let $G = (N, \mathcal{W})$ be a complete simple game with characteristic invariants $(\overline{n}, \mathcal{M})$. Then, $G$ is $k$-invariant-trade robust ($k$-`I-T-R`, for short) if there is no a vectorial trading transform $\mathcal{J}' = \langle \overline{x}_1, \ldots, \overline{x}_j, \overline{y}_1, \ldots, \overline{y}_j \rangle$ such that each $\overline{x}_i$ is a row of $\mathcal{M}$ and each $\overline{y}_k \in \Lambda(\overline{n})$ for $1 \leq k \leq j$ satisfies $\overline{y}_k \not\delta\, \overline{m}_i$ for every row $\overline{m}_i$ of $\mathcal{M}$. If $G$ is $k$-`I-T-R` for all positive integer $k$, then $(N, \mathcal{W})$ is invariant-trade robust (`I-T-R`, for short).

Theorem 1 by (Taylor and Zwicker, 1992) for simple games can be adapted to complete simple games.

**Theorem 4.** *(Freixas and Molinero, 2008) For a simple game G, the following are equivalent:*
*(i) G is weighted.* *(ii) G is invarint-trade robust.* *(iii) G is* $2^{2^{|N|}}$*-I-T-R.*

Now, we are going to classify all complete simple games with less than nine voters from our experimental results. We have used an Algorithm that is a direct application of Theorems 3 and 4, as it was done in (Freixas and Molinero, 2008), but with some improvements about how to store the information and using an specific procedure. Our programs have been written for C++ and run under Linux in Pentium 4 at 1.7 GHz with 512 Mb of RAM.

Unfortunately, the number of matrices associated to a fixed number $n$ of voters is huge for $n > 8$ and it spends too much time. However, now we have been able to classify all complete simple games with $n = 8$ voters as follows: We have computed all weighted games with just eight voters following the same idea given in (Muroga et al., 1962) for *non-isomorphic* weighted games (2730164); then we have computed the non 2-`I-T-R` (13134200) with the remaining games; i.e., for each remaining game we check if it is 2-`I-T-R` from its characteristic invariant $(\overline{n}, \mathcal{M})$ and finding a failure matrix $\mathcal{Y}$ (the rows of it correspond to vectors that represent losing coalitions that are maximal with respect to the $\delta$-odering given in Definition 8). Then we have computed the non 3-`I-T-R` (308257) with the remaining games; and, so on with the remaining complete games (2497 + 70). Table 2 provides a detailed classification of all complete simple games: the number of complete games (briefly *CG*), the number of weighted games (briefly *WG*), and the number of non $k$-`I-T-R` but $(k-1)$-`I-T-R` games. Finally, the number of complete games being non weighted is gathered in non-`I-T-R`.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| *CG* | 1 | 3 | 8 | 25 | 117 | 1171 | 44313 | 16175188 |
| *WG* | 1 | 3 | 8 | 25 | 117 | 1111 | 29373 | 2730164 |
| *non I-T-R* | 0 | 0 | 0 | 0 | 0 | 60 | 14940 | 13445024 |
| *non* 2-*I-T-R* | 0 | 0 | 0 | 0 | 0 | 57 | 13915 | 13134200 |
| *non* 3-*I-T-R* | 0 | 0 | 0 | 0 | 0 | 3 | 1011 | 308257 |
| *non* 4-*I-T-R* | 0 | 0 | 0 | 0 | 0 | 0 | 14 | 2497 |
| *non* 5-*I-T-R* | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 70 |

**Table2.** Full classification of simple games for $n < 9$ by invariant-trade robustness

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| *CG* | 1 | 3 | 8 | 25 | 117 | 1171 | 44313 | 16175188 |
| *WG* | 1 | 3 | 8 | 25 | 117 | 1111 | 29373 | 2730164 |
| *non trade robust* | 0 | 0 | 0 | 0 | 0 | 60 | 14940 | 13445024 |
| *non* 2-*trade robust* | 0 | 0 | 0 | 0 | 0 | 60 | 14940 | 13445024 |
| *non* 3-*trade robust* | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Table3.** Full classification of simple games for $n < 9$ by trade robustness

In particular, $n = 6$ is the minimum number of voters required to achieve simple games which are 2-`I-T-R` but not 3-`I-T-R`; $n = 7$ is the minimum number of voters required to achieve simple games which are 3-`I-T-R` but not 4-`I-T-R`; $n = 8$ is the minimum number of voters required to achieve simple games which are 4-`I-T-R` but not 5-`I-T-R`.

In (Freixas and Molinero, 2008) it is enumerated all these extreme cases for $n < 8$ giving vector $\overline{n}$, matrix $\mathcal{M}$ and matrix $\mathcal{Y}$ which fulfills equation in Definition 13 and showed a failure to be $k$-`I-T-R`: $k = 3$ for $n = 6$, and $k = 4$ for $n = 7$. Here we have new results for $n = 8$, which show a failure to be 5-`I-T-R`. Table 4 enumerates some of these 70 extreme cases giving for them the vector $\overline{n}$, matrix $\mathcal{M}$ and matrix $\mathcal{Y}$. We note that Table 4 shows some 4-`I-T-R` but non 5-`I-T-R` simple games with the minimum number of columns (i.e., 4), and with the maximum number of columns (i.e., 8).

Concerning to the trade robustness, we have done similar experiments (using an analogous algorithm) but with trade robustness to get the data of Table 3. This table provides a detailed classification of all complete simple games with less than 9 voters: the number of complete games, the number of weighted games, and the number of non $k$-trade robust but $(k-1)$-trade robust games.

The relevant obtained result proves the following claim.

*Claim.* Let $n < 9$ be the number of voters of a given game $G$. Then

$$G \text{ is weighted} \iff G \text{ is 2-trade robust.}$$

Note that $n = 9$ is the smallest number of voters required for a game to be 2-trade robust but not 3-trade robust. This follows from the previous Claim and that Taylor and Zwicker (Taylor and Zwicker, 1995) construct a "*magic square*" game being 2-trade robust but non 3-trade robust.

## 2. Conclusions and Future Work

In this paper we have made experiments that allow:

| *Vector* $\overline{n}$ | *Matrix* $\mathcal{M}$ | *Matrix* $\mathcal{Y}$ | *Vector* $\overline{n}$ | *Matrix* $\mathcal{M}$ | *Matrix* $\mathcal{Y}$ |
|---|---|---|---|---|---|
| ( 1, 2, 2, 3 ) | $\begin{pmatrix} 1&2&1&0 \\ 1&0&2&3 \\ 0&2&0&3 \end{pmatrix}$ | $\begin{pmatrix} 1&2&0&1 \\ 1&2&0&1 \\ 1&2&0&1 \\ 0&1&2&3 \\ 0&1&2&3 \end{pmatrix}$ | ( 1, 2, 3, 2 ) | $\begin{pmatrix} 1&0&2&2 \\ 0&2&0&2 \\ 0&1&3&1 \end{pmatrix}$ | $\begin{pmatrix} 1&2&0&0 \\ 0&1&2&2 \\ 0&1&2&2 \\ 0&1&2&2 \\ 0&1&2&2 \end{pmatrix}$ |
| ( 2, 1, 2, 3 ) | $\begin{pmatrix} 2&0&0&1 \\ 1&1&1&0 \\ 0&1&0&3 \end{pmatrix}$ | $\begin{pmatrix} 1&1&0&1 \\ 1&1&0&1 \\ 1&1&0&1 \\ 1&1&0&1 \\ 1&1&2&3 \end{pmatrix}$ | ( 2, 1, 3, 2 ) | $\begin{pmatrix} 2&0&0&0 \\ 0&1&0&2 \\ 0&0&3&1 \end{pmatrix}$ | $\begin{pmatrix} 1&1&0&0 \\ 1&1&0&0 \\ 0&0&2&2 \\ 0&0&2&2 \\ 0&0&2&2 \end{pmatrix}$ |
| ( 2, 2, 1, 3 ) | $\begin{pmatrix} 2&0&1&0 \\ 1&0&0&3 \\ 0&2&0&3 \end{pmatrix}$ | $\begin{pmatrix} 2&0&0&1 \\ 2&0&0&1 \\ 2&0&0&1 \\ 0&1&1&3 \\ 0&1&1&3 \end{pmatrix}$ | ( 2, 3, 1, 2 ) | $\begin{pmatrix} 2&0&1&0 \\ 1&0&1&2 \\ 0&3&0&2 \end{pmatrix}$ | $\begin{pmatrix} 2&0&0&1 \\ 2&0&0&1 \\ 0&2&1&2 \\ 0&2&1&2 \\ 0&2&1&2 \end{pmatrix}$ |

| *Vector* $\overline{n}$ | *Matrix* $\mathcal{M}$ | *Matrix* $\mathcal{Y}$ |
|---|---|---|
| ( 1, 1, 1, 1, 1, 1, 1, 1 ) | $\begin{pmatrix} 1&1&0&0&0&0&1&0 \\ 1&0&1&0&0&1&0&1 \\ 1&0&0&1&1&0&1&1 \\ 0&1&1&1&0&0&1&1 \end{pmatrix}$ | $\begin{pmatrix} 1&1&0&0&0&0&0&1 \\ 1&0&1&1&0&0&0&0 \\ 1&0&1&0&0&0&1&1 \\ 1&0&0&1&0&1&1&1 \\ 0&1&1&0&1&1&1&1 \end{pmatrix}$ |
| ( 1, 1, 1, 1, 1, 1, 1, 1 ) | $\begin{pmatrix} 1&1&0&0&0&0&1&0 \\ 1&0&1&0&0&0&1&1 \\ 1&0&0&1&1&0&1&1 \\ 0&1&1&1&0&1&0&1 \end{pmatrix}$ | $\begin{pmatrix} 1&1&0&0&0&0&0&1 \\ 1&0&1&1&0&0&0&0 \\ 1&0&0&1&0&1&1&1 \\ 0&1&1&1&0&0&1&1 \\ 0&1&1&0&1&1&1&1 \end{pmatrix}$ |
| ( 1, 1, 1, 1, 1, 1, 1, 1 ) | $\begin{pmatrix} 1&1&0&0&0&0&1&0 \\ 1&0&1&0&0&0&1&1 \\ 1&0&0&1&0&1&0&1 \\ 0&1&1&1&1&0&1&1 \end{pmatrix}$ | $\begin{pmatrix} 1&1&0&0&0&0&0&1 \\ 1&0&1&1&0&0&0&0 \\ 1&0&0&1&0&0&1&1 \\ 1&0&0&0&1&1&1&1 \\ 0&1&1&1&0&1&1&1 \end{pmatrix}$ |
| ( 1, 1, 1, 1, 1, 1, 1, 1 ) | $\begin{pmatrix} 1&0&0&0&0&0&1&0 \\ 0&1&1&0&0&0&1&1 \\ 0&1&0&1&0&1&0&1 \\ 0&0&1&1&1&0&1&1 \end{pmatrix}$ | $\begin{pmatrix} 1&0&0&0&0&0&0&1 \\ 0&1&1&1&0&0&0&0 \\ 0&1&0&1&0&0&1&1 \\ 0&1&0&0&1&1&1&1 \\ 0&0&1&1&0&1&1&1 \end{pmatrix}$ |
| ( 1, 1, 1, 1, 1, 1, 1, 1 ) | $\begin{pmatrix} 1&0&1&0&0&0&0&0 \\ 1&0&0&0&0&1&0&1 \\ 0&1&1&0&0&0&1&0 \\ 0&1&0&1&0&0&1&1 \\ 0&0&1&1&1&0&1&1 \end{pmatrix}$ | $\begin{pmatrix} 1&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&1&1 \\ 0&1&1&0&0&0&0&1 \\ 0&1&0&0&1&1&1&1 \\ 0&0&1&1&0&1&1&1 \end{pmatrix}$ |
| ( 1, 1, 1, 1, 1, 1, 1, 1 ) | $\begin{pmatrix} 1&0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&1&1 \\ 0&1&1&0&0&0&1&0 \\ 0&1&0&1&0&1&0&1 \\ 0&0&1&1&1&0&1&1 \end{pmatrix}$ | $\begin{pmatrix} 1&0&0&1&0&0&0&0 \\ 0&1&1&0&0&0&0&1 \\ 0&1&0&1&0&0&1&1 \\ 0&1&0&0&1&1&1&1 \\ 0&0&1&1&0&1&1&1 \end{pmatrix}$ |

**Table4.** Some 4-`I-T-R` but non 5-`I-T-R` simple games for 8 voters (with the minimum number of columns, 4, and with the maximum number of columns, 8)

(*i*) To classify all complete simple games for $n < 9$ voters according to whether they are: weighted games, non 2-`I-T-R`, non 3-`I-T-R`, non 4-`I-T-R` or non 5-`I-T-R`.

(*ii*) To check if a particular complete simple game with $n$ voters is $k$-`I-T-R` for each positive integer $k$.

(*iii*) To classify all complete simple games for $n < 9$ voters according to whether they are: weighted games, non 2-trade robust or non 3-trade robust.

(*iv*) To check if a particular complete simple game with $n$ voters is $k$-trade robust for each positive integer $k$.

In (Freixas and Molinero, 2008) it is proved that as few as $2k + 1$ voters are needed to find games being $k$-`I-T-R` but non $(k + 1)$-`I-T-R`. Our experiments with less than 9 voters suggest that the number of considered voters, $2k + 1$, might be decreased to $k + 4$ (see Table 4).

*Conjecture 1.* It is possible to find a sequence of complete simple game $G_k$ which is $k$-`I-T-R` but non $(k + 1)$-`I-T-R` for all $k > 1$, where the number of voters is exactly $k + 4$ instead of $2k + 1$ (cf. Theorem 5.1 in (Freixas and Molinero, 2008) for $k \geq 4$).

Another future work is, for a fixed number of voters $n$ $(n > 8)$, to generate a random game $(\overline{n}, \mathcal{M})$ and study trade robustness and invariant-trade robustness. If the game is weighted then it is needed finding appropriate weights for voters and a quota (a realization), for the class of weighted games.

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# Hierarchies in Voting Simple Games*

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**Abstract** A work by Friedman, McGrath and Parker introduced the concept of a hierarchy of a simple voting game and characterized which hierarchies, induced by the desirability relation, are achievable in weighted games. They proved that no more hierarchies are obtainable if weighted games are replaced by the larger class of linear games.
In a subsequent paper by Freixas and Pons, it was proved that only four hierarchies, conserving the ordinal equivalence between the Shapley–Shubik and the Penrose–Banzhaf–Coleman power indices, are non–achievable in simple games. It was also proved that all achievable hierarchies are obtainable in the class of weakly linear games.
In this paper, we define a new class of totally pre–ordered games, the almost linear games, smaller than the class of weakly linear games, and prove that all hierarchies achievable in simple games are already achievable in almost linear games.

*Key words*: simple game, power index, desirability, weak desirability, almost desirability, linear game, weakly linear game, almost linear game.

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*JEL Class.*: C71, D71.

## 1. Introduction

The concept of a *hierarchy* of a simple game, introduced in (Friedman et al., 2006), captures the ordering of the influence held by the voters (or players) in the game. For example, writing that a five-player game $G$ has hierarchy $>==>$ means that there is one player which has the maximum influence, another one that has the minimum influence and the other three have all the same intermediate influence. A situation where each player has a different amount of influence will be called a strict hierarchy.

Any power index considered in a simple game induces a total ordering on the set of voters, and thus a hierarchy. Two power indices which induce the same hierarchy are said to be *ordinally equivalent.* In (Diffo Lambo and Moulen, 2002) and (Felsenthal and Machover, 1998) it was shown that the Penrose–Banzhaf–Coleman (PBC, henceforth) and the Shapley–Shubik (SS, henceforth) power indices are ordinally equivalent for linear games, i.e., games for which the desirability relation is complete, and that the common induced hierarchy is the one given by the desirability relation.

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In (Carreras and Freixas, 2008) weakly linear games, i.e., games for which the weak desirability relation is complete, were introduced and it was proved that all regular semivalues (see (Carreras and Freixas, 1999) and (Carreras and Freixas, 2000) for references on regular semivalues) are ordinally equivalent for this kind of games, and that the common induced hierarchy is the one given by the weak desirability relation. As linear games form a subclass of weakly linear games, and both the PBC and the SS power indices are regular semivalues, this work extended and generalized the former ones.

In (Friedman et al., 2006) all achievable hierarchies in linear games (induced by the desirability relation) were characterized. Precisely, it was proved that all hierarchies are achievable in linear games except the types $== \cdots ==>>$ and $== \cdots ==>>>$. Furthermore, it was proved that all hierarchies achievable in linear games are also achievable in weighted games.

In (Freixas and Pons, 2008) all achievable hierarchies, induced by the weak desirability relation, were characterized in weakly linear games and it was proved that all hierarchies are achievable in weakly linear games provided that the number of voters is greater than 5. More precisely, it was proved that the only non achievable hierarchies are: $>>$, $>>>$, $=>>$ and $=>>>$ .

In this paper, we define a new pre–ordering, called *the almost desirability relation*, which lays between the desirability and the weakly desirability relations. We prove that all hierarchies achievable in simple games are already achievable in almost linear games, i.e., games in which the almost desirability relation is complete. The class of almost linear games is strictly larger than the class of linear games but strictly smaller than the class of weakly linear games.

The paper is organized as follows. Basic definitions and preliminary results are included in Section 2. Section 3 contains the definition and properties of the almost desirability relation. In Section 4 we prove that all hierarchies obtainable in simple games are obtainable in almost linear games. Some Conclusions end the paper in Section 5.

## 2. Definitions and Preliminaries

In the sequel, $N = \{1, 2, \ldots, n\}$ denote a fixed but otherwise arbitrary finite set of *players*. Any subset $S \subseteq N$ is a *coalition*. A cooperative game $v$ (in $N$, omitted hereafter) is a *simple game* (SG, henceforth) if (a) $v(S) = 0$ or 1 for all $S$,[1] (b) is monotonic, i.e. $v(S) \leq v(T)$ whenever $S \subset T$, and (c) $v(N) = 1$. Either the family of *winning* coalitions $\mathcal{W} = \mathcal{W}(v) = \{S \subseteq N : v(S) = 1\}$ or the subfamily of *minimal* winning coalitions $\mathcal{W}^m = \mathcal{W}^m(v) = \{S \in \mathcal{W} : T \subset S \Rightarrow T \notin \mathcal{W}\}$ determines the game. A simple game is *proper* if for any winning coalition, its complement is not winning. A voter $i \in N$ is null in $\mathcal{W}$ if $i \notin S$ for all $S \in \mathcal{W}^m$. $\mathcal{W}_i$ denotes the set of winning coalitions which contain $i$. Finally, the *null extension* of game $\mathcal{W}$ for a voter $n+1$ outside $N$ is the game $\mathcal{W}'$ whose voters belong to $N \cup \{n+1\}$ and $(\mathcal{W}')^m = \mathcal{W}^m$.

[1] For a detailed discussion of some issues raised by allowing abstentions, see (Felsenthal and Machover, 1998) and for several levels of approval in input and output, see (Freixas and Zwicker, 2003).

**The desirability relation**

**Definition 1.** (Isbell, 1958) Let $v$ be a simple game and $i, j \in N$. Then

$$\begin{array}{llll} i \succsim_D j & \text{iff} & S \cup \{j\} \in \mathcal{W} \Rightarrow S \cup \{i\} \in \mathcal{W} & \text{for all} \quad S \subseteq N \backslash \{i, j\}, \\ i \succ_D j & \text{iff} & i \succsim_D j \quad \text{and} \quad j \not\succsim_D i, & \\ i \approx_D j & \text{iff} & i \succsim_D j \quad \text{and} \quad j \succsim_D i. & \end{array}$$

It is not difficult to check that $\succsim_D$ is a preordering. The relation $\succsim_D$ (resp., $\succ_D$) is called the *desirability* (resp., *strict desirability*) relation, and $\approx_D$ is the *equi–desirability* relation.

**Definition 2.** A simple game $v$ is *linear*[2] whenever the desirability relation $\succsim_D$ is complete.

In a linear game, the hierarchy given by the desirability relation coincides with the hierarchy induced by SS and PBC power indices.

**The weak desirability relation**

Given a simple game $v$, let us define, for each $i \in N$ and $1 \leq k \leq n$,

$$\mathcal{C}_i = \{S \in \mathcal{W} : S \backslash \{i\} \notin \mathcal{W}\} \quad \text{and} \quad \mathcal{C}_i(k) = \{S \in \mathcal{C}_i : |S| = k\}.$$

$\mathcal{C}_i$ is the set of winning coalitions $S$ for which $i$ is *crucial*, while $\mathcal{C}_i(k)$ is the subset of such coalitions having cardinality $k$.

**Definition 3.** (Carreras and Freixas, 2008) Let $v$ be a simple game and $i, j \in N$. Then

$$\begin{array}{llll} i \succsim_d j & \text{iff} & |C_i(k)| \geq |C_j(k)| & \text{for all} \quad k = 1, 2, \ldots, n, \\ i \succ_d j & \text{iff} & i \succsim_d j \quad \text{and} \quad j \not\succsim_d i, & \\ i \approx_d j & \text{iff} & i \succsim_d j \quad \text{and} \quad j \succsim_d i. & \end{array}$$

Then $\succsim_d$ is a preordering called the *weak desirability* relation. The relation $\succ_d$ is the *strict weak desirability* relation and $\approx_d$ is the *weak equi–desirability* relation.

In (Diffo Lambo and Moulen, 2002) it is proved that the desirability relation is a sub–preordering of the weak desirability relation, that is to say, for any $i, j \in N$, $i \succsim_D j$ implies $i \succsim_d j$ and $i \succ_D j$ implies $i \succ_d j$.

**Definition 4.** (Carreras and Freixas, 2008) A simple game $v$ is *weakly linear* (WLSG, henceforth) whenever the weak desirability relation $\succsim_d$ is complete.

In a weakly linear game, the hierarchy given by the weak desirability relation coincides with the hierarchy induced by SS and PBC power indices.

As stated in (Carreras and Freixas, 2008), the completeness of the desirability relation $\succsim_D$ implies the completeness of the weak desirability relation $\succsim_d$ so that all linear games are also weakly linear. Moreover, if $v$ is a linear game then $v$ is weakly linear and the desirability relation $\succsim_D$ and the weak desirability relation $\succsim_d$ coincide.

There are weakly linear games that are not linear, as the one in the following example.

[2] linear games are also called complete, ordered or directed games in the literature, see (Taylor and Zwicker, 1999) (henceforth, LSG) for references on these names.

*Example 1.* Let $N = \{1, 2, 3, 4\}$ and let $v$ be the game defined by

$$\mathcal{W}^m = \{\{1, 2\}, \{1, 3\}, \{2, 4\}\}.$$

This game is not linear because the desirability relation only gives:

$$1 \succ_D 3 \quad \text{and} \quad 2 \succ_D 4,$$

while the weak desirability relation gives:

$$1 \approx_d 2 \succ_d 3 \approx_d 4.$$

Thus, the game is weakly linear and induces the hierarchy $=>=$.

## 3. The Almost Desirability Relation

In this section we introduce a new binary relation on the set $N$. This relation will be proved to be a pre–ordering, and it will be used to characterize all hierarchies achievable in simple games.

**Definition 5.** Let $v$ be a simple game and $i, j \in N$.

$$\begin{aligned} i \succsim_a j \quad &\Leftrightarrow \begin{cases} |C_i(h)| \geq |C_j(h)| \quad \text{for all} \quad h > n-3, \\ \text{and} \\ S \cup \{j\} \in \mathcal{W} \Rightarrow S \cup \{i\} \in \mathcal{W}, \text{ for all } S \subseteq N \backslash \{i, j\} \text{ with } |S| < n-3, \end{cases} \\ i \succ_a j \quad &\Leftrightarrow \quad i \succsim_a j \quad \text{and} \quad j \not\succsim_a i, \\ i \approx_a j \quad &\Leftrightarrow \quad i \succsim_a j \quad \text{and} \quad j \succsim_a i. \end{aligned}$$

Then $\succsim_a$ is called the *almost desirability* relation. The relation $\succ_a$ is the *strict almost desirability* relation and $\approx_a$ is the *almost equi–desirability* relation.

**Proposition 1.** *Let $v$ be a simple game. Then, the almost desirability relation is a pre–ordering in $N$.*

*Proof.* To prove that the *almost desirability relation* is transitive, assume that $i, j, k$ are different elements in $N$ such that $i \succsim_a j \succsim_a k$.
It is clear that $|C_i(h)| \geq |C_j(h)| \geq |C_k(h)|$ for any $h > n-3$, and thus $|C_i(h)| \geq |C_k(h)|$ for all $h > n-3$. Suppose now that $S \subseteq N \setminus \{i, k\}$ is such that $|S| < n-3$ and $S \cup \{k\} \in \mathcal{W}$, and we will prove that $S \cup \{i\} \in \mathcal{W}$. There are two possibilities:

If $j \notin S$ it is $S \subseteq N \setminus \{j, k\}$ and, since $j \succsim_a k$, $|S| < n-3$ and $S \cup \{k\} \in \mathcal{W}$, we have $S \cup \{j\} \in \mathcal{W}$. But it is also true that $S \subseteq N \setminus \{i, j\}$ and, since $i \succsim_a j$, we have $S \cup \{i\} \in \mathcal{W}$.

If $j \in S$, let $S' = S \setminus \{j\}$. Then $S' \cup \{k\} \subseteq N \setminus \{i, j\}$ and $S' \cup \{k\} \cup \{j\} = S \cup \{k\} \in \mathcal{W}$. Since $i \succsim_a j$ and $|S' \cup \{k\}| = |S| < n-3$, it is $S' \cup \{k\} \cup \{i\} \in \mathcal{W}$. But $S' \cup \{i\} \subseteq N \setminus \{j, k\}$ and, since $j \succsim_a k$ and $|S' \cup \{i\}| = |S| < n-3$, we have $S' \cup \{i\} \cup \{j\} = S \cup \{i\} \in \mathcal{W}$. □

Proposition 2 shows that the desirability relation is a sub–preordering of the almost desirability relation, and that this one is a sub–preordering of the weak desirability relation. To prove this proposition we need two lemmas, whose proofs are adapted from Lemma 3.1 in (Diffo Lambo and Moulen, 2002).

**Lemma 1.** *Let $v$ be a simple game and $i,j \in N$. Let $h$ be an integer with $1 \leq h < n$.*

$$\left.\begin{array}{l} S \cup \{j\} \in \mathcal{W} \Rightarrow S \cup \{i\} \in \mathcal{W} \\ \textit{for all } S \subseteq N \backslash \{i,j\} \textit{ with } h-3 < |S| < h \end{array}\right\} \Rightarrow |C_i(h)| \geq |C_j(h)|.$$

*Proof.* Assume the hypothesis and consider $S \in C_j(h)$. There are two possibilities:

If $i \in S$ then $S \in C_i(h)$; otherwise $S \setminus \{i\} \in \mathcal{W}$ and, taking $S' = S \setminus \{i,j\} \subseteq N \setminus \{i,j\}$, we would have $S' \cup \{j\} = S \setminus \{i\} \in \mathcal{W}$, $S' \cup \{i\} = S \setminus \{j\} \notin \mathcal{W}$ and $|S'| = h-2$, in contradiction with the hypothesis.

If $i \notin S$, let $S' = S \backslash \{j\}$. Since $S' \subseteq N \backslash \{i,j\}$ , $|S'| = h-1$ and $S' \cup \{j\} = S \in \mathcal{W}$, then, we have $S' \cup \{i\} \in \mathcal{W}$. This proves that $S' \cup \{i\} \in C_i(h)$, because it is clear that $|S' \cup \{i\}| = |S| = h$ and that $S' \notin \mathcal{W}$.

In either case, for any $S \in C_j(h)$ we obtain a set $\varphi(S) \in C_i(h)$, and it is easy to see that this mapping $\varphi$ is injective. This proves that $|C_i(h)| \geq |C_j(h)|$. □

**Lemma 2.** *Let $v$ be a simple game and $i,j \in N$.*

$$\left.\begin{array}{l} a) \begin{cases} T \cup \{i\} \in \mathcal{W} \textit{ and } T \cup \{j\} \notin \mathcal{W} \\ \textit{for some } T \subseteq N \backslash \{i,j\} \end{cases} \\ \\ \textit{and} \\ \\ b) \begin{cases} S \cup \{j\} \in \mathcal{W} \Rightarrow S \cup \{i\} \in \mathcal{W}, \textit{ for all} \\ S \subseteq N \backslash \{i,j\} \textit{ with } |T|-2 < |S| < |T|+1 \end{cases} \end{array}\right\} \Rightarrow |C_i(h)| > |C_j(h)| \textit{ for } h = |T|+1.$$

*Proof.* Assume the hypothesis *a*) and *b*). From Lemma 1, by taking into account hypothesis *b*), it is $|C_i(h)| \geq |C_j(h)|$ for $h = |T|+1$. This fact was proved there by seeing that, in this case, the function $\varphi : C_j(h) \to C_i(h)$ defined by

$$\varphi(S) = \begin{cases} S & \text{if } i \in S \\ (S \setminus \{j\}) \cup \{i\} & \text{if } i \notin S \end{cases}$$

is injective. We will prove now that, by adding hypothesis *a*), the function $\varphi$ is not onto, that is to say, there exist some $S' \in C_i(h)$ such that $\varphi(S) \neq S'$ for any $S \in C_j(h)$.
Let $T$ be the set in hypothesis *a*) and consider $S' = T \cup \{i\}$. It is clear that $i \in S'$, $S' \in \mathcal{W}$ and $S' \setminus \{i\} = T \notin \mathcal{W}$ (because $T \cup \{j\} \notin \mathcal{W}$). Thus, $S' \in C_i(h)$ for $h = |T|+1$. And it is not difficult to see that $S'$ is not image of any $S \in C_j(h)$ by the function $\varphi$. □

**Proposition 2.** *Let $v$ be a simple game and $i,j \in N$. Then,*

$$\begin{array}{lll} i \succsim_D j & \Rightarrow i \succsim_a j & \Rightarrow i \succsim_d j \\ i \succ_D j & \Rightarrow i \succ_a j & \Rightarrow i \succ_d j \end{array}$$

*Proof.* Assume that $i \succsim_D j$. To prove that $i \succsim_a j$, we need to see that $|C_i(h)| \geq |C_j(h)|$ when $h > n-3$. By applying Lemma 1 we deduce that $|C_i(h)| \geq |C_j(h)|$ for all $h < n$. Thus, it only remains the case $h = n$. If $|C_j(h)| = 1$ then $N \in C_j(n)$ and it is $N \setminus \{j\} \notin \mathcal{W}$. This leads to $N \setminus \{i\} \notin \mathcal{W}$, because if $N \setminus \{i\} \in \mathcal{W}$, taking $S = N \setminus \{i,j\}$ we would have $S \cup \{j\} \in \mathcal{W}$ but $S \cup \{i\} \in \mathcal{W}$, in contradiction with

$i \succsim_D j$. Thus, $N \in C_i(n)$ and this is equivalent to $|C_i(h)| = 1$. We have proved that $|C_i(n)| \geq |C_j(n)|$.

Assume now that $i \succsim_a j$. To prove that $i \succsim_d j$, we need to see that $|C_i(h)| \geq |C_j(h)|$ when $h \leq n-3$, and this is an immediate consequence of Lemma 1.

Assume that $i \succ_D j$, that is to say, $i \succsim_D j$ and $j \not\succsim_D i$, and we will prove that $j \not\succsim_a i$. Since $j \not\succsim_D i$ then there is some $T \subseteq N\backslash\{i,j\}$ such that $T \cup \{i\} \in \mathcal{W}$ and $T \cup \{j\} \notin \mathcal{W}$. If $|T| < n-3$ then it is clear that $j \not\succsim_a i$, so that we can assume $|T| \geq n-3$. Since hypothesis $b$) of Lemma 2 is satisfied for any $S \subseteq N\backslash\{i,j\}$, we deduce $|C_i(h)| > |C_j(h)|$ for $h = |T|+1$. But $|T|+1 > n-3$ and thus $j \not\succsim_a i$.

Finally, assume that $i \succ_a j$, that is to say, $i \succsim_a j$ and $j \not\succsim_a i$, and we will prove that $j \not\succsim_d i$. The fact that $j \not\succsim_a i$ includes two possibilities:
(a) There is some $h > n-3$ such that $|C_i(h)| > |C_j(h)|$. This clearly implies $j \not\succsim_d i$.
(b) There is some $T \subseteq N\backslash\{i,j\}$, with $|T| < n-3$ , such that $T \cup \{i\} \in \mathcal{W}$ and $T \cup \{j\} \notin \mathcal{W}$. By applying Lemma 2 we deduce that $|C_i(h)| > |C_j(h)|$ for $h = |T|+1$, and this leads to $j \not\succsim_d i$. □

**Definition 6.** A simple game $v$ is *almost linear* (ALSG, henceforth) whenever the almost desirability relation $\succsim_a$ is complete.

It is clear that the completeness of the desirability relation $\succsim_D$ implies the completeness of the almost desirability relation $\succsim_a$, and also that the completeness of the almost desirability relation implies the completeness of the weakly desirability relation $\succsim_d$.

If $v$ is a linear game then $v$ is almost linear and the desirability relation $\succsim_D$ and the almost desirability relation $\succsim_a$ coincide. Similarly, if $v$ is an almost linear game then $v$ is weakly linear and the almost desirability relation $\succsim_a$ coincides with the weak desirability relation $\succsim_d$.

In an almost linear game, the hierarchy given by the almost desirability relation coincides with the hierarchy induced by SS and PBC power indices.

There are almost linear games that are not linear. For instance, the game in Example 1. Indeed, we have seen that the game in Example 1 is not linear, but it is clearly almost linear because $1 \approx_a 2 \succ_a 3 \approx_a 4$.

There are also weakly linear games that are not almost linear, as the following example.

*Example 2.* Let $N = \{1,2,3,4,5\}$ and let $v$ be the null extension for voter 5 of the game considered in example 1. Clearly, $v$ is weakly linear with

$$1 \approx_d 2 \succ_d 3 \approx_d 4 \succ_d 5,$$

but it is not almost linear because $\{1,3\} \in \mathcal{W}$ and $\{2,3\} \notin \mathcal{W}$ implies $2 \not\succsim_a 1$, and, similarly, $\{2,4\} \in \mathcal{W}$ and $\{1,4\} \notin \mathcal{W}$ implies $1 \not\succsim_a 2$.

## 4. Hierarchies Achievable in Almost Linear Games

The comments in previous sections allow us to state that for linear games the hierarchy induced by the desirability relation and the ones induced by the almost and by the weak desirability relation coincide. But almost linear games form a larger class in which other hierarchies are possible. Similarly, in almost linear games, the hierarchy induced by the almost desirability relation and the one induced by the weak desirability relation coincide. But weakly linear games form a larger class in which other hierarchies are possible. The notation to describe the different possible hierarchies is stated in the following.

**Definition 7.** An ALSG with $1 \succsim_a 2 \succsim_a \cdots \succsim_a n$ is said to have the hierarchy $r_1 r_2 \dots r_{n-1}$ if each $r_i$ is either $>$ or $=$ depending on whether $i \succ_a i+1$ or $i \approx_a i+1$ respectively.

It is straightforward to see that if an ALSG does not satisfy condition $1 \succsim_a 2 \succsim_a \cdots \succsim_a n$ then there is an isomorphic ALSG with this ordering. Thus we only need to consider, hereafter, ALSGs with ordering $1 \succsim_a 2 \succsim_a \cdots \succsim_a n$ as is assumed in Definition 7.

A hierarchy is said to be achievable in an almost linear game if there exist a game of this type which has this hierarchy. The aim of this section is to show that for $n > 5$ all hierarchies are achievable. The proof of the following theorem is adapted from (Freixas and Pons, 2008).

**Theorem 1.** *All hierarchies are achievable in an ALSG except:*

$$>>\,, \quad >>>\,, \quad =>> \quad \textit{and} \quad =>>>\,.$$

*Proof.* Theorem 3 in (Friedman et al., 2006) guarantees that all hierarchies except $== \cdots ==>>$ and $== \cdots ==>>>$ are achievable in linear games and because every linear game is almost linear, all hierarchies achievable in linear games are also achievable in almost linear games. Hence, we only need to show now the existence of almost linear games with hierarchies: $== \cdots ==>>$ and $== \cdots ==>>>$.

Note that if the hierarchy $== \cdots ==>>$ is achieved in an ALSG with $n$ players without null voters then the hierarchy $== \cdots ==>>>$ is achieved in the *null extension* of this game for a voter $n+1$, because that voter $n+1$ is strictly smaller than any other player by the almost desirability relation.

Now, let us construct a (proper) game $G_n$ for every $n > 4$ with hierarchy $== \cdots ==>>$ and without null voters. The minimal winning coalitions for $G_n$ are defined as follows:

$$\begin{array}{ll} S_i = N \setminus \{n-i, n\} & \text{for } i = 1, 2, \dots, \lceil \frac{n}{2} \rceil + 1, \\ S_i = N \setminus \{n-i, n-1\} & \text{for } i = \lceil \frac{n}{2} \rceil + 2, \dots, n-1. \end{array}$$

To show that $G_n$ is almost linear we need to prove that, for $1 \le i < n$:

a) $|C_i(h)| \ge |C_{i+1}(h)| \quad$ for all $\quad h > n-3$,
b) $S \cup \{i+1\} \in \mathcal{W} \Rightarrow S \cup \{i\} \in \mathcal{W}$, for all $S \subseteq N \setminus \{i, j\}$ with $|S| < n-3$.

In Theorem 3.1 of (Freixas and Pons, 2008) it was proved that:

- $|\mathcal{C}_i(n)| = 0$ for any voter $i$,
- $|\mathcal{C}_1(n-1)| = |\mathcal{C}_2(n-1)| = \cdots = |\mathcal{C}_{n-2}(n-1)| > |\mathcal{C}_{n-1}(n-1)| > |\mathcal{C}_n(n-1)|$,
- $|\mathcal{C}_1(n-2)| = |\mathcal{C}_2(n-2)| = \cdots = |\mathcal{C}_{n-2}(n-2)| > |\mathcal{C}_{n-1}(n-2)| > |\mathcal{C}_n(n-2)|$,
- If $|S| < n-2$ then $S \notin \mathcal{W}$.

The first three results prove *a*), and the last one proves *b*). Thus, it has been proved that $1 \approx_a 2 \approx_a \cdots \approx_a n-2 \succ_a n-1 \succ_a n$, that is to say, the game is almost linear and has the hierarchy $== \cdots =>>$.

To conclude the proof we need to show that the four hierarchies:

$$>>, \quad >>>, \quad =>> \quad \text{and} \quad =>>>$$

are not achievable in almost linear games.

But it was proved in (Freixas and Pons, 2008) that these hierarchies are not achievable in weakly linear games. Then, since the class of weakly linear games is larger than the class of almost linear games, these hierarchies are neither achievable in almost linear games.

## 5. Conclusion

Our paper complements the hierarchy theory initiated in (Friedman et al., 2006) and continued in (Bean et al. 2008). Indeed, in (Friedman et al., 2006) it is proved that linear games show many different hierarchies, although two sequences of hierarchies are never available. It is also proved that all hierarchies achievable in linear games are also achievable in weighted games. Moreover, in (Bean et al. 2008), it is proved that simple majority weighted games are not enough to get all the achievable hierarchies.

In this paper we have introduced a new preordering, the almost desirability relation ( $\succsim_a$), for simple games and a new class of games, the almost linear simple games. We proved that the class of almost linear games is larger than the class of linear games but smaller than the class of weakly linear games.

We have proved that *all* hierarchies are achieved in almost linear games as long as the number of voters is greater or equal than 6. For less than 6 voters, only four hierarchies are not achieved in this class of games, and they are:

$$>>\,, \quad >>>\,, \quad =>> \quad \text{and} \quad =>>>\,.$$

But, since it was proved in (Freixas and Pons, 2008) that these four hierarchies are not achieved either in weakly linear games, we assert that all hierarchies achievable in simple games are already achievable in almost linear games.

As a consequence of these results we can assert that, given any complete pre–ordering defined on a finite set (with more than five elements), it is possible to construct a simple game such that the pre–orderings induced by the Shapley–Shubik and the Penrose–Banzhaf–Coleman power indices coincide with the given pre–ordering.

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# The Dynamic Game with State Payoff Vector on Connected Graph*

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**Abstract** By introducing point (state) payoff vector to every point node on connected graph in this paper, dynamic game is researched on finite graph. The concept of strategy about games on graph defined by C.Berge is introduced to prove the existence theorem of absolute equilibrium about games on connected graph with point payoff vector. The complete algorithm and an example in three-dimensional connected mesh-like graph are given in this paper.

**Keywords:** connected graph, point payoff vector, simply strategy, absolute equilibrium, three-dimensional mesh-like graph.

## 1. Introduction

These guidelines include complete descriptions of the fonts, spacing, and related information for producing your proceedings manuscripts. Please follow them as closely as possible.
The game tree which mainly describes the process of dynamic games is a kind of simple structure graph. Therefore, the research results of game on graph could generally popularize the category of dynamic game. The original type of games on graph, whose definition was given by C. Berge, is the games on finite tree (E.Zemelo, 1961). Literature (Rozen, 2005) discusses games on graph whose target structure is defined by coherent relation of terminal point set. The author extends C. Berge's concept, namely, on every given point of graph, the choice for the next point is determined by the former experienced points, rather than only determined by the last point that the player had just reached. The results on literature (Berge and Ghouila-Houri, 1965; Rozen, 2005)are both given on the two- dimensional graph for games with terminal payoff. The point payoff vector is introduced to every point point on finite graph is expected in this paper. The absolute equilibrium of dynamic games is researched by applying the concept of strategy of games on graph defined by C. Berge. The related tasks finished by the author contain the following:
(1) By establishing the corresponding relations between situation and game tree on two- dimensional directed graph, games on directed graph is transformed to game tree. Also, the algorithm is given, and the Shapley vector is chosen as the cooperative solution of the two-dimensional directed graph.
(2) Partial cooperative dynamic games is studied on two-dimensional mesh-like finite graph. Players adopt partial cooperative behaviors rather than completed cooperative behaviors. The main feature of partial cooperation is that behaviors of each player are the combination of cooperative behaviors and individual behaviors.

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Also, algorithm of the solution of partial cooperative games and the optimal path are given on two-dimensional directed graph.
Symbol system, strategy of plays and point payoff of games on graph are given on the second part of this paper. For the third, the existence theorem of absolute equilibrium about games on connected graph with point payoff vector is proved. On the fourth part, the complete algorithm of absolute equilibrium is constructed. For the last part, the author gives an example of absolute equilibrium about games on three-dimensional connected mesh-like graph with point payoff vector.

## 2. Symbol and Definition

Connected graph with point set $A$ is written as $< A, \gamma >$, where $\gamma \subseteq A^2$ ( $\gamma$ is the arc set of the connected graph) and point set is $A = \{a_0, a_1, \dots\}$. $\gamma < a >$ denotes all the point sets after point $a$, and $\gamma' < a >$ is the immediate subsequent point set of point $a$. $A_f$ is written as the point set which has not subsequent points. $< A, \gamma >$ is called $n$ point graph, if subdivision of $A \backslash A_f$ is given, which is $\{A_1, \dots, A_n\}$. Using the term of games, set of players is $N = \{1, 2, \dots, n\}$, $A_i$ is the point set of player $i \in N$, the set of decision-making nodes is written as $\tilde{A} = \{A_1, A_2, \dots, A_n\}$ and $A_f$ is the set of terminal points. The path (orbit) of graph $< A, \gamma >$ is the sequence of points $a_0, a_1, \dots, a_t$. For $k = 1, \dots, t, \dots$ we have $a_k \in \gamma < a_{k-1} >$. An orbit is called a situation, if it is infinite or it contains terminal point $a_l \in A_f$. All of the rest orbits are called opening situation. Every mapping $s_i : A_i \to A$ satisfying the condition $s_i \subseteq \gamma$ is called the simple strategy of player $i$. The sets of all the simple strategies of player $i$ is written as $S_i$. Situation $a \in A$ and simple strategy $(s_1, \dots, s_n) \in S_1 \times \dots \times S_n$ define the situation $< a, s_1, \dots, s_n >= a, s(a), s^2(a), \dots,$ where $s = s_1 \bigcup \dots \bigcup s_n$. If each path on graph $< A, \gamma >$ is finite, then every point $a \in A$ has a relation with a mapping $F_a : S_1 \times \dots \times S_n \to A_f$, which maps the situation$(s_1, \dots, s_n)$ under a simple strategy $(s_1, \dots, s_n)$ to a terminal point of the situation $< a, s_1, \dots, s_n >$.

**Definition 1.** Giving every point $a$ an $n$-dimensional real vector $f_a = (f_a^1, \dots, f_a^n)^T$, it is called a point payoff vector of point $a$ and the $i$-th component $f_a^i$ is called player $i$ 's point payoff of point $a$.

**Definition 2.** The $n$-dimensional vector $h(a_r, \dots, a_l) = \sum_{k=r}^{l} f_{a_k} = (h_1(a_r, \dots, a_l), \dots, h_n(a_r, \dots, a_l))^T$ is called situation payoff vector corresponding to situation $a_r, \dots, a_l (a_l \in A_f)$ on graph $< A, \gamma >$. The $i$ -th component $h_i(a_r, \dots, a_l), i = 1, \dots, n$ is called player $i$'s situation payoff corresponding to situation $a_r, \dots, a_l$.

According to the definition of simple strategy, different situations may lead to different situations all from initial point $a_r$. Suppose that the situation corresponding with situation $(s_1, \dots, s_i, \dots, s_n)$ is $a_r, \dots, a_l (a_l \in A_f)$, and the situation corresponding with $(s_1, \dots, s_i', \dots, s_n)$ is $a_r, \dots, a_k (a_k \in A_f)$. Notation $\leq^j$is defined as following:
$h_j(a_r, \dots, a_l) \leq h_j(a_r, \dots, a_k) \Leftrightarrow F_{a_r}(s_1, \dots, s_i, \dots, s_n) \leq^j F_{a_r}(s_1, \dots, s_i', \dots, s_n)$
where $i = 1, \dots, n, j = 1, \dots, n$.
Choosing $a_0 \in A$ as an initial point, non-cooperative simple games $\varGamma_{a_0}(T)$ is achieved on graph with point payoff vector $T =< A, \gamma; A_1, \dots, A_n, A_f; f_{a \in A} >$ where the strategy set of player $i$ is $S_i$, set of terminal point is $A_f$, and $f_a$ is point payoff vector of point $a$ on graph $< A, \gamma >$.

## 3. Existence Theorem of Absolute Equilibrium about Games on Connected Graph with Point Payoff Vector

**Theorem 1.** *Nash equilibrium situation exists under simple strategy for each simple game $\Gamma_{a_0}(T)$, where $a_0 \in A$.*

*Proof.* First, the subset of point set is defined for $\alpha$ by induction method as follow:
a) $C_0 = A_f$,
b) Suppose that for all the $\beta < \alpha$, subset $C_\beta \subseteq A$ have been defined.
If $\alpha$ is finite, then $C_\alpha = C_{\alpha-1} \bigcup \{a \in A \backslash A_f : \gamma < a > \subseteq C_{\alpha-1}\}$ is defined. If $\alpha$ is infinite, then $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$. The smallest ordinal number $p(a)$ is called the rank of point $a \in A$, we have $a \in C_{p(a)}$. If there isn't infinite path on graph $< A, \gamma >$, then every point has rank and the rank is finite.
Second, mapping $s^* : A \to A$ is defined according to the rank of points by induction method as following. For each 0 rank point $a (a \in A_f)$, we define $s^*(a) = a$. If the rank of point $a \in A \backslash A_f$ is $p(a) = \alpha$, and mapping $S^*(a')$ is defined by point $a' \in A$ with rank $p(a') < \alpha$. If $\gamma < a' > \neq \emptyset$, then $s^*(a') \in \gamma < a' >$. Therefore, $s^*$ has been defined in $C_{\alpha-1}$, and it satisfies condition $s^* \subseteq \gamma$. $s_j^{*\alpha-1}$ is noted as $s^*$'s restriction which is established in the subset $C_{\alpha-1} \bigcap A_j (j = 1, \ldots, n)$. Since $C_{\alpha-1}$ is $\gamma$-steady, which is to say that the situation whose initial point has emerged into $C_{\alpha-1}$ is completely located in $C_{\alpha-1}$, and $s_j^{*\alpha-1}$ is regarded as a simple strategy of player $j$ in the subgame of $C_{\alpha-1}$. Considering $a \in C_\alpha \backslash C_{\alpha-1}$, we have $\emptyset \neq \gamma < a > \subseteq C_{\alpha-1}$. As pointd above, function $F_x(s_1^{*\alpha-1}, \ldots, s_n^{*\alpha-1})$ is defined on each $x \in \gamma < a >$. We denote $T_a = \{F_x(s_1^{*\alpha-1}, \ldots, s_n^{*\alpha-1}) : x \in \gamma < a >\}$, namely, $T_a$ is the terminal point set of situation whose initial point passes set $\gamma < a >$ by player $j = 1, \ldots, n$ adopting strategy $s_j^{*\alpha-1}$. Obviously, $T_a$ is the nonempty subset of the set of terminal points which can be reached from point $a$. If $a \in A_i$, we can calculate player $i$ 's situation payoff, the maximum of which is $h_i^*$, in every situation of subgame $T_a$. When player $i$ chooses maximal situation payoff $h_i^*$ on point $a$, and the chosen point is $x^*$. We denote $s^*(a) = x^*$. Hence, the following relation is satisfied for each $x \in \gamma < a >$:

$$F_x(s_1^{*\alpha-1}, \ldots, s_n^{*\alpha-1}) \leq^j F_{x^*}(s_1^{*\alpha-1}, \ldots, s_n^{*\alpha-1}) \tag{1}$$

By induction method, the mapping $s^*$ has been defined for each $a \in A$. If $\gamma < a > \neq \emptyset$, then $s^*(a) \in \gamma < a >$. Namely, $s^* \subseteq \gamma$. If $s_j^*$ is the restriction of mapping $s^*$ in $A_j$, then $s_j^*$ is a simple strategy of player $j$.
Finally, it is proved by induction method for the rank of $a_0$ that situation $s^* = (s_1^*, \ldots, s_n^*)$ is Nash equilibrium of every game $\Gamma_{a_0}(T)$, where $a_0 \in A$. Situation $s^* = (s_1^*, \ldots, s_n^*)$ is called absolute equilibrium of games on graph $< A, \gamma >$.
**Step 1**. Suppose $p(a_0) = 0$ , that is to say $a_0 \in C_0 = A_f$ . Now the value $a_0$ of function $F_{a_0}$ is independent of the situation. Therefore, every situation is Nash equilibrium.
**Step 2**. If $\alpha$ is given, we suppose $p(a_0) = \alpha$ and situation $(s_1^*, \ldots, s_n^*)$ is Nash equilibrium of every game $\Gamma_x(T)$, where $p(x) < \alpha$. The initial point $a_0 \in A$ is chosen, we will prove that $(s_1^*, \ldots, s_n^*)$ is Nash equilibrium of game $\Gamma_{a_0}(T)$. In fact, if $a_0 \in A_i$, suppose that player $j (j = 1, \ldots, n)$ adopts the simple strategy $s_j'$ instead of $s_j^*$. If $j \neq i$, considering $\gamma < a_0 > \subseteq C_{\alpha-1}$, we get the result. We only need to consider $j = i$. So note $s_i^*(a_0) = a_1, s_i'(a_0) = a_1'$. By formula (1), we have

$$F_{a_1'}(s_1^*, \ldots, s_n^*) \leq^i F_{a_1}(s_1^*, \ldots, s_n^*)$$

Because $a'_1 \in \gamma < a_0 >\subseteq C_{\alpha-1}$ , then according to assumption mentioned above,

$$F_{a'_1}(s^*_1, \dots, s^*_{i-1}, s'_i, s^*_{i+1}, \dots, s^*_n) \leq^i F_{a'_1}(s^*_1, \dots, s^*_n)$$

By the relations between the two formulas above, we have

$$F_{a'_1}(s^*_1, \dots, s^*_{i-1}, s'_i, s^*_{i+1}, \dots, s^*_n) \leq^i F_{a_1}(s^*_1, \dots, s^*_n) \tag{2}$$

According to the definition of $F_a$ and $s^*$, the following equations are satisfied,

$$F_{a_0}(s^*_1, \dots, s^*_n) = F_{a_1}(s^*_1, \dots, s^*_n)$$

$$F_{a_0}(s^*_1, \dots, s^*_{i-1}, s'_i, s^*_{i+1}, \dots, s^*_n) = F_{a'_1}(s^*_1, \dots, s^*_{i-1}, s'_i, s^*_{i+1}, \dots, s^*_n)$$

Considering formula (2), we have

$$F_{a_0}(s^*_1, \dots, s^*_{i-1}, s'_i, s^*_{i+1}, \dots, s^*_n) \leq^i F_{a_0}(s^*_1, \dots, s^*_n)$$

Namely, situation $(s^*_1, \dots, s^*_n)$ is Nash equilibrium in game $F_{a_0}(T)$. The proof of the theorem is finished.

## 4. Algorithm about Absolute Equilibrium in Games with Point Payoff Vector on Connected Graph

First, calculate the rank $p(a)$ of the point $a \in A$ on graph $< A, \gamma >$ according to the definition of rank. Assume $\max_{a\in A} p(a) = T$. The set of points $A$ on graph $< A, \gamma >$ is split up into $T + 1$ subsets $P_0, P_1, \dots, P_T$, where $P_k$ is the set of points whose rank equal to $k$, $\bigcup_{k=0}^{T} P_k = A, P_l \bigcap P_m = \emptyset, l \neq m$. In the following, we will use backward induction method according to the ranks of the points.

**Step 0**: Consider each point $a_0$ whose rank equals to 0, namely, $a_0 \in P_0 = C_0 = A_f$. Since nobody makes move here, by definition 2, we have $h(a_0) = f_{a_0}$. Denote function by $r^0_i : P_0 \to R$, where $r^0_i(a_0) = h_i(a_0)$, let $r^0(a_0) = (r^0_1(a_0), \dots, r^0_n(a_0))^T = h(a_0)$, we denote $s^*(a_0) = a_0$.

**Step 1**: Consider each point $a_1 \in P_1$ . Since $\gamma' < a_1 >\subseteq P_0$ for $a_1$, by definition 2, we have $h(a_1, a_0) = f_{a_1} + r^0(a_0)$. Assume $a_1 \in A_i$, then player $i$ chooses $\bar{a_0} \in \gamma' < a_1 >$ satisfying $\max_{a_0\in\gamma'<a_1>} h_i(a_1, a_0) = h_i(a_1, \bar{a_0})$. Denote function by $r^1_i : P_1 \to R$, where $r^1_i(a_1) = h_i(a_1, \bar{a_0})$ on $a_1$, and let $r^1(a_1) = (r^1_1(a_1), \dots, r^1_n(a_1))^T = h(a_1, \bar{a_0})$. Now we get $s^*_i(a_1) = \bar{a_0}$ on $a_1 \in P_1$.

**Step 2**: Consider each point $a_2 \in P_2$. Denote $\gamma' < a_2 >= Z_0 < a_2 > \bigcup Z_1 < a_2 >$, where $Z_0 < a_2 >\subseteq P_0$ prescribes the set of points next to $a_2$ with 1 rank. In the following part, we use the similar prescription.

1) For the points in $Z_0 < a_2 >\subseteq P_0$, when $a_0 \in \gamma' < a_2 >$, by definition 2 we have $h(a_2, a_0) = f_{a_2} + r^0(a_0)$.

2) For the points in $Z_1 < a_2 >\subseteq P_1$, $\bar{a_0}$ has been chosen on step 1, by definition 2 we have

$$h(a_2, a_1, \bar{a_0}) = f_{a_2} + r^1(a_1)$$

When $a_0 \in \gamma' < a_2 >$, by definition 2 we have $h(a_2, a_0) = f_{a_2} + r^0(a_0)$. Assume that $a_2 \in A_i$, then player $i$ choose the point $\bar{a}_1 \in \gamma' < a_2 >$ which can reach

$\max\{\max_{a_0 \in Z_0(a_2)} h_i(a_2, a_0), \max_{a_1 \in Z_1(a_2)} h_i(a_2, a_1, \bar{a_0})\}$ . Also we denote function by $r_i^2 : P_2 \to R$, where

$$r_i^2(a_2) = \begin{cases} h_i(a_2, \bar{a_1}), & \text{if } \bar{a_1} \in Z_0\langle a_2 \rangle \subseteq p_0 \\ h_i(a_2, \bar{a_1}, \bar{a_0}), & \text{if } \bar{a_1} \in Z_1\langle a_2 \rangle \subseteq p_1 \end{cases}$$

at $a_2$, let $r^2(a_2) = (r_1^2(a_2), \ldots, r_n^2(a_2))^T$. Now we get $s_i^*(a_2) = \bar{a}_1$ at $a_2 \in P_2$ .
**Step t**: Consider the each point $a_t \in P_t, t \leq T$. Now $r_i^0(a_0), \ldots, r_i^{t-1}(a_{t-1})$ and $r^0(a_0), \ldots, r^{t-1}(a_{t-1})$, have been definite.
Denote

$$\gamma' < a_t >= Z_0 < a_t > \bigcup Z_1 < a_t > \bigcup \cdots \bigcup Z_{t-1} < a_t >$$

For the points in $Z_0 < a_t > \subseteq P_0$, when $a_0 \in \gamma' < a_t >$, by definition 2 we have

$$h(a_t, a_0) = f_{a_t} + r^0(a_0)$$

1)For the points in $Z_1 < a_t > \subseteq P_1$, when $a_1 \in \gamma' < a_t >$, $\bar{a}_0$ has been chosen on step 1. By definition 2, we have

$$h(a_t, a_1, \bar{a_0}) = f_{a_t} + r^1(a_1)$$

...
t) For the points in $Z_{t-1} < a_t > \subseteq P_{t-1}$ , when $a_{t-1} \in \gamma' < a_t >$ and $\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{t-2}$ has been chosen on step $t-1$, by definition 2 we have

$$h(a_t, a_{t-1}, \bar{a}_{t-2}, \ldots, \bar{a}_0) = f_{a_t} + r^{t-1}(a_{t-1})$$

Assume that $a_t \in A_i$, then player $i$ chooses $\bar{a}_{t-1} \in \gamma' < a_t >$ which can reach

$$\max \begin{Bmatrix} \max_{a_0 \in Z_0(a_t)} h_i(a_t, a_0), \max_{a_1 \in Z_1(a_t)} h(a_t, a_1, \bar{a}_0), \\ \ldots, \max_{a_{t-1} \in Z_{t-1}(a_t)} h(a_t, a_{t-1}, \bar{a}_{t-2}, \ldots, \bar{a}_0)\} \end{Bmatrix}$$

Denote function by $r_i^t : P_t \to R$, where

$$r_i^t(a_t) = \begin{cases} h_i(a_t, \bar{a}_{t-1}), & \text{if } \bar{a}_{t-1} \in Z_0 < a_t > \subseteq P_0 \\ h_i(a_t, \bar{a}_{t-1}, \bar{a}_0), & \text{if } \bar{a}_{t-1} \in Z_1 < a_t > \subseteq P_1 \\ \ldots\ldots & \\ h_i(a_t, \bar{a}_{t-1}, \bar{a}_{t-2}, \ldots, \bar{a}_0), & \text{if } \bar{a}_{t-1} \in Z_{t-1} < a_t > \subseteq P_{t-1} \end{cases}$$

at $a_t$, let $r^t(a_t) = (r_1^t(a_t), \ldots, r_n^t(a_t))^T$ . Now we get $s_i^*(a_t) = \bar{a}_{t-1}$ on $a_t \in P_t$.
Continue the process till $t = T$. Similarly, we get $s_i^*(a_T) = \bar{a}_{T-1}$.
Given all above, by the algorithm we get the player's choice for each point on connected graph with point payoff vector. According to the proof of theorem 3, when we choose arbitrarily point $a_0 \in A$ to be the initial point on connected graph $< A, \gamma >$ with point payoff vector, strategy $(s_1^*, \ldots, s_n^*)$ which is independent on $a_0$ is Nash equilibrium on simple game $\Gamma_{a_0}(T)$, that is, situation $s^* = (s_1^*, \ldots, s_n^*)$ is the absolute equilibrium of the game. The equilibrium route is related to the initial point. When we choose the point $a_0 \in P_L, 0 \leq L \leq T$ and $a_0 \in A_i$ as the initial point, the absolute equilibrium $s^*$ can define the equilibrium route of the simple game $\Gamma_{a_0}(T)$, and the payoff on equilibrium situation is:

$$r^L(a_0) = (r_1^L(a_0), \ldots, r_n^L(a_0))^T$$

We need to point out that, in the algorithm above, the definition of function $r^k(a)$ on some point $a \in P_k, 0 < k \leq T$ maybe more than one, so choose one of them randomly as defined.

## 5. The Calculation Model of Absolute Equilibrium about Games on Three-dimensional Connected Graph with Point Payoff Vector

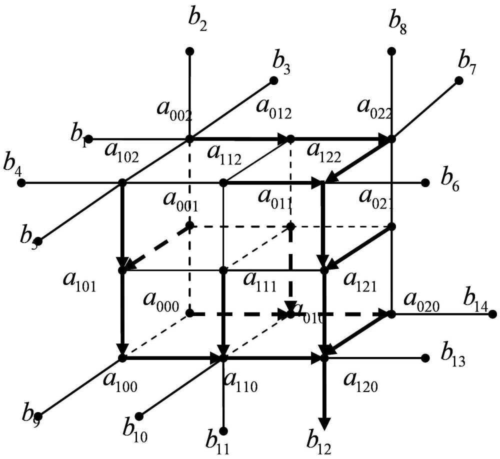


Figure 1

Consider the three-dimensional connected graph $< A, \gamma >$ (Fig 1), where the set of players is $N = \{1, 2, 3\}$. The terminal payoff is

$$A_f = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13}, b_{14}\}$$

The decision point sets of player 1, player 2, player 3 are respectively

$$A_1 = \{a_{000}, a_{110}, a_{020}, a_{121}, a_{011}, a_{002}\}, \quad A_2 = \{a_{100}, a_{012}, a_{021}, a_{111}, a_{122}, a_{102}\}$$

$$A_3 = \{a_{010}, a_{120}, a_{001}, a_{101}, a_{112}, a_{022}\}$$

**Players' strategy:** On graph $< A, \gamma >$ , we define the players' strategy in horizontal and two-dimensional points are along the mesh to the right, the left or terminal points; the strategy of players in right-and-left and two-dimensional points are along the mesh to downward ,the front, or terminal points; the strategy of players in fore-and-aft and two-dimensional points are along the mesh to the right, downward or terminal points.

By definition 1, we give the point payoff vector on every point on graph $< A, \gamma >$. Then we get the game on three-dimensional mesh-like and connected graph with point payoff vector $T =< A, \gamma; A_1, A_2, A_3, A_f; f_{a \in A} >$. This example gives the point payoff vector as following:

$$f_{a_{000}} = (1, 2, 2)^T, f_{a_{010}} = (2, 1, 3)^T, f_{a_{020}} = (4, 2, 1)^T, f_{a_{001}} = (5, 3, 2)^T$$

$$f_{a_{011}} = (2,2,2)^T, f_{a_{021}} = (1,3,2)^T, f_{a_{002}} = (2,4,5)^T, f_{a_{012}} = (6,5,2)^T$$
$$f_{a_{022}} = (1,5,3)^T, f_{a_{100}} = (3,6,7)^T, f_{a_{110}} = (2,5,1)^T, f_{a_{120}} = (5,7,4)^T$$
$$f_{a_{101}} = (2,5,4)^T, f_{a_{111}} = (4,0,2)^T, f_{a_{121}} = (3,3,4)^T, f_{a_{102}} = (4,6,3)^T$$
$$f_{a_{112}} = (0,5,3)^T, f_{a_{122}} = (3,2,4)^T, f_{b_1} = (4,3,2)^T, f_{b_2} = (2,5,4)^T$$
$$f_{b_3} = (3,4,5)^T, f_{b_4} = (4,6,3)^T, f_{b_5} = (2,5,3)^T, f_{b_6} = (4,2,3)^T, f_{b_7} = (3,2,6)^T$$
$$f_{b_8} = (4,3,1)^T, f_{b_9} = (3,5,7)^T, f_{b_{10}} = (4,3,5)^T, f_{b_{11}} = (2,6,5)^T, f_{b_{12}} = (3,2,4)^T$$
$$f_{b_{13}} = (2,4,1)^T, f_{b_{14}} = (5,3,2)^T$$

First, compute the ranks of all points. Get the rank-subdivision of the point set $A$ on the graph $< A, \gamma >$:

$$P_0 = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13}, b_{14}\}, P_1 = \{a_{120}\}$$
$$P_2 = \{a_{110}, a_{121}, a_{020}\}, P_3 = \{a_{100}, a_{111}, a_{010}, a_{122}, a_{021}\}$$
$$P_4 = \{a_{101}, a_{112}, a_{000}, a_{011}, a_{022}\}, P_5 = \{a_{102}, a_{001}, a_{012}\}, P_6 = \{a_{002}\}$$

**Step 0**: Consider every point in 0 rank point set $P_0$. According to algorithm, we note $r^0(b_j) = h(b_j), j = 1, \ldots, 14$, and define $s^*(b_j) = b_j, j = 1, \ldots, 14$.

**Step 1**: Consider every point in 1 rank point set $P_1 = \{a_{120}\}$. According to algorithm, we get $h(a_{120}, b_{12}) = f_{a_{120}} + r^0(b_{12}) = (8,9,8)^T$ $h(a_{120}, b_{13}) = (7,11,5)^T$. Since $a_{120} \in A_3$, $h_3(a_{120}, b_{12}) = 8 > 5 = h_3(a_{120}, b_{13})$, player 3 will choose $b_{12}$, noting $r^1(a_{120}) = (r_1^1(a_{120}), r_2^1(a_{120}), r_3^1(a_{120}))^T = h(a_{120}, b_{12}) = (8,9,8)^T$, similarly we get $s^*(a_{120}) = b_{12}$.

**Step 2**: Consider every point in 2 rank point set $P_2 = \{a_{110}, a_{121}, a_{020}\}$. For $a_{110} \in P_2$, we have $\gamma' < a_{110} >= Z_0 < a_{110} > \bigcup Z_1 < a_{110} >$, where $Z_0 < a_{110} >= \{b_{10}, b_{11}\}, Z_1 < a_{110} >= \{a_{120}\}$. We get $h(a_{110}, b_{10}) = (6,8,6)^T, h(a_{110}, b_{11}) = (4,11,6)^T, h(a_{110}, a_{120}, b_{12}) = f_{a_{110}} + r^1(a_{120}) = (10,14,9)^T$. For $a_{110} \in A_1$, and $\max\{h_1(a_{110}, b_{10}), h_1(a_{110}, b_{11}), h_1(a_{110}, b_{120}, b_{12})\} = 10 = h_1(a_{110}, a_{120}, b_{12})$, So player 1 will choose $a_{120}$ , noting $r^2(a_{110}) = (r_1^2(a_{110}), r_2^2(a_{110}), r_3^2(a_{110}))^T = h(a_{110}, a_{120}, b_{12}) = (10,14,9)^T$.

Then we get $s^*(a_{110}) = a_{120}$.

Similarly, for $a_{121} \in P_2$, since $a_{121} \in A_1$, and player 1 have the only choice $a_{120}$, we note $r^2(a_{121}) = (r_1^2(a_{121}), r_2^2(a_{121}), r_3^2(a_{121}))^T = h(a_{121}, a_{120}, b_{12}) = (11,12,12)^T$, and get $s^*(a_{121}) = a_{120}$. For $a_{020} \in P_2$, we have $\gamma' < a_{020} >= Z_0 < a_{020} > \bigcup Z_1 < a_{020} >$, where $Z_0 < a_{020} >= \{b_{14}\}$,$Z_1 < a_{020} >= \{a_{120}\}$. Since $a_{020} \in A_1, h_1(a_{020}, b_{14}) = 9 < 12 = h_1(a_{020}, a_{120}, b_{12})$, player 1 will also choose $a_{120}$, noting $r^2(a_{020}) = (r_1^2(a_{020}), r_2^2(a_{020}), r_3^2(a_{020}))^T = h(a_{020}, a_{120}, b_{12}) = (12,11,9)^T$, then we get $s^*(a_{020}) = a_{120}$.

**Step 3**: Consider every point in 3 rank point set $P_3 = \{a_{100}, a_{111}, a_{010}, a_{122}, a_{021}\}$. The results are given as following: $r^3(a_{100}) = (13,20,16)^T$, $s^*(a_{100}) = a_{110}$, $r^3(a_{111}) = (14,14,11)^T$, $s^*(a_{111}) = a_{110}, r^3(a_{122}) = (14,14,16)^T$, $s^*(a_{122}) = a_{121}$; $r^3(a_{021}) = (12,15,14)^T$, $s^*(a_{021}) = a_{121}$. For $a_{010} \in P_3$, we have $\gamma' < a_{010} >= Z_0 < a_{010} > \bigcup Z_1 < a_{010} > \bigcup Z_2 < a_{010} >$ where $Z_0 < a_{010} >= Z_1 < a_{010} >= \emptyset$, $Z_2 < a_{010} >= \{a_{020}, a_{110}\}$. Since $a_{010} \in A_3$, and $h_3(a_{010}, a_{020}, a_{120}, b_{12}) = 12 = h_3(a_{010}, a_{110}, a_{120}, b_{12})$, by assumption, player 3 can arbitrarily choose $a_{110}$ or $a_{020}$. Then if player 3 choose point $a_{020}$, noting $r^3(a_{010}) = (14,12,12)^T$, then $s^*(a_{010}) = a_{020}$.

**Step 4**: Consider every point in 4 rank point set $P_4 = \{a_{101}, a_{112}, a_{000}, a_{011}, a_{022}\}$. The results are, $r^4(a_{101}) = (15, 25, 20)^T s^*(a_{101}) = a_{100}, r^4(a_{112}) = (14, 19, 19)^T$, $s^*(a_{112}) = a_{122}, r^4(a_{000}) = (15, 14, 14)^T$,$s^*(a_{000}) = a_{010}$;$r^4(a_{011}) = (16, 14, 14)^T$, $s^*(a_{011}) = a_{010}$,$r^4(a_{022}) = (15, 19, 19)^T$,$s^*(a_{022}) = a_{122}$.
**Step 5**: Consider every point in 5 rank point set $P_5 = \{a_{102}, a_{001}, a_{012}\}$. The results are, $r^5(a_{102}) = (19, 31, 23)^T$, $s^*(a_{102}) = a_{101}, r^5(a_{001}) = (20, 28, 22)^T$, $s^*(a_{001}) = a_{101}$; For $a_{012} \in P_5$, since $a_{012} \in A_2$, and $h_2(a_{012}, a_{022}, a_{122}, a_{121}, a_{120}, b_{12}) = 24 = h_2(a_{012}, a_{112}, a_{122}, a_{121}, a_{120}, b_{12})$, player 3 can arbitrarily choose $a_{112}$ or $a_{022}$. If player 3 chooses point $a_{022}$, noting $r^5(a_{012}) = (21, 24, 21)^T$, then $s^*(a_{012}) = a_{022}$.
**Step 6**: Now consider the unique 6 rank point $a_{002} \in P_6$. We have $\gamma' < a_{002} >= Z_0 < a_{002} > \bigcup Z_1 < a_{002} > \bigcup Z_2 < a_{002} > \bigcup Z_3 < a_{002} > \bigcup Z_4 < a_{002} > \bigcup Z_5 < a_{002} >$, where $Z_0 < a_{002} >= \{b_1, b_2, b_3\} Z_1 < a_{002} >= Z_2 < a_{002} >= Z_3 < a_{002} >= Z_4 < a_{002} >= \emptyset$, $Z_5 < a_{002} >= \{a_{102}, a_{001}, a_{012}\}$. For $a_{002} \in A_1$, noting $r^6(a_{002}) = (23, 28, 26)^T$, we get $s^*(a_{002}) = a_{012}$.
Finally we get the absolute equilibrium of the game, which is noted by bold black line in Fig.1.

## 6. Conclusion

The result of this paper is valid for random finite connected graph, no matter it is two-dimension or three-dimension. Of course the result includes the game tree we have known. We choose three-dimensional mesh-like graph in this paper only because initial inspiration comes from the exception for carrying out game's research on three-dimension space. Moreover, on usual dynamic games, usually the absolute equilibrium is perfect equilibrium. But from a different aspect, actually the absolute equilibrium is stronger than the perfect equilibrium.
The inductive method of the rank of point on the graph used in this paper will show its power on the research of game on complex graph. For simple structure graph, by listing all likely appeared situations, we can always transform game graph to the game tree. Then we can solve it by common method. However, considering games on three-dimensional connected graph, the complex computation leaded by the large amounts of situations can hinder and conceal some important research for the nature of game. We have reasons to believe that the arithmetic of the absolute equilibrium about the game for limited connected graph established by this paper can be popularized to partial cooperation game of finite three-dimensional graph with variable coalitional structure and so on.

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# The Scenario Bundle Method and the Security of Gas Supply for Greece

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**Abstract** The Liquefied Natural Gas (LNG) trade is without doubt one of the most interesting areas in energy shipping. Its role in the formulation of a national energy strategy, and in relation to the security of gas supply more specifically, is of concern in this paper. The "scenario bundle method" is applied to examine the Greek market. This approach is a semi-formal, rather than mathematical, game theoretic modelling approach. Scenario bundles are simple game structures and a systematic way of using qualitative judgments as a basis for the construction and evaluation of scenarios regarding possible future developments. This methodology allows strategy formulation by taking into account the commercial objectives of the involved players, and considering geopolitical interaction, regional conflicts and crisis situations. The scenario bundle approach helps in identifying critical parameters, as a result of concrete logical steps. The present analysis concludes with some interesting strategic suggestions for the security of gas supply for the Greek market, however the main purpose of the paper is to suggest a tool for national energy planning through a schematic illustration with considerable multi-level extensions.

**Keywords:** Game Theory, Scenario Bundle Method, National Energy Strategy, Security of Gas Supply, LNG

## 1. Introduction

Meeting the world's energy demands is one of the greatest challenges in the 21st century and, in many respects, natural gas is considered as the successor of oil. The Liquefied Natural Gas (LNG) trade is without doubt one of the most interesting areas in energy shipping, which dominates the world bulk maritime transport (Gkonis & Psaraftis, 2007b). Indeed, while for many decades natural gas markets were localized and isolated, the LNG trade (that is the transport of natural gas by sea) has contributed to the development of a "world gas market" (see for example Jensen, 2004, and Foss, 2005).

The present paper belongs to a broader research work that utilises the game theory methodology to analyse strategic decision-making for actors involved in LNG and the dynamics of this evolving energy shipping market (Gkonis & Psaraftis, 2007a). The game theoretic contributions of this research can be distinguished in "LNG business" - oriented and "national energy strategy" – oriented.

In the former category, the questions addressed involve: capacity and price competition among LNG shipping companies and the possibility of non-cooperative collusion in oligopolistic competition in the LNG shipping business (Gkonis &

Psaraftis, 2009); early commitment and entry deterrence in an LNG shipping market (Gkonis & Psaraftis, 2008).

In the latter category, national energy strategy issues are of concern in relation to gas supply and the role of LNG. This paper belongs to this research context and places a special focus on the "scenario bundle method". The method was developed by Selten (1999) to address international conflict situations. In this paper, it is applied to examine the security of natural gas supply for the Greek market. The purpose of the paper is to suggest a tool for national energy planning through a schematic illustration with considerable multi-level extensions.

The rest of the paper is organised as follows. In section 2, the scenario bundle method is reviewed and in section 3, the game theoretic definition of a scenario bundle is given. In section 4, the method is applied to model the security of gas supply for Greece. A reference scenario bundle is developed, as well as a sensitivity analysis scenario bundle. Other potential extensions are also discussed. In section 5, the general conclusions and the benefits of the method are assessed.

## 2. Review of the Scenario Bundle Method

The scenario bundle method was originally developed by Selten (1999) to address international conflict situations. As Selten explains, this approach is a semi-formal, rather than mathematical, game theoretic modelling approach. Scenario bundles are simple game structures and a systematic way of using qualitative judgments as a basis for the construction and evaluation of scenarios.

Selten worked together with A. Perlmutter, a political scientist, and they used for their modelling task a panel of experts who met at the "Research Conference on Strategic Decision Analysis focusing on the Persian Gulf" held in 1976 (Selten, 1999). Among the crisis scenarios examined using the scenario bundle method was an attack on Kuwait by Iraq. It is remarkable how close their analysis was to the actual facts that took place after almost 15 years.

The method was applied later on in different contexts, yet limited references can be found in the academic literature. Specifically, Jackson (1992) used the scenario bundle method to model the evolution of the Asia Pacific Economic Cooperation (APEC). Milovanovic (mimeo) used it to model endogenous corruption in privatised companies. Finally, Selten again used it with Reiter (Reiter & Selten, 2003) to analyse the Kosovo conflict. This work is cited in Avenhaus & Zartman (2007) in the book they edited on "Diplomacy Games - Formal Models and International Negotiations", where the scenario bundle method is referred to as a tool for such modelling purposes.

In this paper, the analysis framework of the scenario bundle method, as described by Selten, is adopted for the modelling purpose of the formulation of national energy strategy and planning, with specific reference to the security of gas supply and the role of LNG for the Greek market. Next, the parameters that the modelling task requires to be taken into account are examined and the general framework of the method is reviewed.

### Generalities

The scenario bundles are topical models as they relate to a specific region at a specific point in time. Scenario bundles indicate possible future developments. The method does not promise predictive reliability or moreover success. "No method exists to claim a certainty about future strategic decision-making of competing actors in any interaction. However, scientific speculations about future developments are not deterred by the lack of predictive reliability" (Selten, 1999). The method suggests that a systematic procedure for the integration of judgments may achieve better results than the unaided intuition of well-informed observers, market analysts or players.

**Actors and Goals**

The actors or players in the interaction, which is modelled as a game, need to be defined. The actors are assumed to be rational decision makers. Goals are defined for each of the players. The goal is a basic datum of an actor's rational decision making. Each player's objective is to maximize the expected value of his own objectives, or of his own payoff, measured in some utility scale (Myerson, 1991, also on utility theory see Luce & Raiffa, 1989).

**Influential factors**

These are influential factors (external to the players) that may play an important role in their decision-making and evolution of a scenario bundle.

**Initial options**

Applications of the scenario bundle method start from a situation in a specific geographical area at a specific point in time. *Initial options* are options which are open at the initial situation, before anything else has happened.

**Scenario bundle construction**

The graphical representation by a game tree is a natural way to describe a scenario bundle. Scenario bundles are actually extensive games. The initial situation corresponds to the origin of the tree, the starting point. The origin is a decision point for a player, as a result of an initial option which generates the scenario bundle. Possibilities are represented by branches of the tree leading to different nodes (as many branches as the options). Supposing that the initial option has been taken, it is examined which actors will need to react and make a decision. Accordingly the tree is continued. Choices of players which are "strategically" taken at the same time (not necessarily in real time terms) are graphically indicated.

**Stopping principles**

An end-point is a node beyond which the construction of a scenario bundle is not continued. The stopping principles put an end to the construction of a scenario bundle, which could otherwise continue indefinitely. According to Selten, the construction of a scenario bundle is continued until *a blind alley end-point*, *an inferiority end-point* or *a normal end-point*.

A scenario bundle ends at a blind-alley end-point, when no plausible options can be found after it. A scenario bundle ends at an inferiority end-point, when at that node an alternative option to a certain one will not be taken, no matter what reactions may be expected afterwards. It is the case of an inferior alternative dominated by a superior alternative. The construction of the scenario bundle does not continue after an inferior alternative.

A scenario bundle arrives at a normal end-point when a node without reactive pressure is reached. A node with reactive pressure on the contrary is a node where a player or a group of players are under pressure to make decision whether to react or not. Generally, a normal end-point could be seen as a new initial situation with a variety of new scenario bundles beginning there.

The scenario bundle method does not try to combine a set of consecutive bundles into a superstructure. Such an approach complicates things and does not serve the explanatory and modelling purpose.

**Plausibility**

The plausibility of initial options as well as of reaction options should be tested using certain criteria. In this sense, options should be realistic (realism criterion) and desirable for players (desirability criterion). The desirability criterion, unlike the realism criterion, takes into account possible side effects of an option.

**Consequences - payoffs**

The resulting consequences of special importance are indicated at the end nodes. They are associated with the payoffs of the players.

**Judgments and analysis**

The judgments on the preference of players are collected as the scenario bundle is analysed. The combined process of analysis and preference judgement begins at the end of the bundle and proceeds backwards. In this way an equilibrium solution is determined. During this backward process, choices which are judged not to be preferable are crossed out. The equilibrium solution is the collection of choices not crossed out.

## 3. Game Theoretical Definition

In this section, the formal game theoretical definition of the concept of a scenario bundle is adapted from Selten (1999). The introduction of a special treatment for coalitional players is avoided and certain weaker conditions apply as explained below.

**Elements of a scenario bundle**

The actors (players) are numbered from 1 to n.

A game tree $K$ is a tree in the sense of graph theory. It has a distinguished node o, the origin of $K$. A node $y$ follows a node $x$, if $x$ is on the path from o to $y$ and $y$ is different from $x$. A choice at $x$ is a branch, which connects $x$ with a node immediately following $x$. A node $y$ immediately follows a node $x$, if $y$ follows $x$ and $x$ and $y$ are connected by a branch of $K$.

The set of all nodes of $K$ is denoted by $X$. The set of all nodes $x \in X$, such that there are exactly m choices at $x$, is denoted by $X_m$. The nodes in $X_0$ are called end-points. The set of all choices at $x$ is denoted by $A(x)$.

The sub-tree $K_x$ of $K$ at $x$ is the game tree consisting of $x$ and all nodes of $K$ following $x$ (with the branches connecting such nodes).

**Definition of a scenario bundle**

A scenario bundle $B = (N, K, c, h)$ is defined as follows:

- $N = \{1, ..., n\}$ is the player set
- $K$ is the game tree
- $c$ is the decision point function which assigns a player $c(x)$ to every $x \in X \setminus X_0$. This means that a player $c(x) = \{i\}$ is expected to make a decision at every node (decision point) of the game tree. If $c(x) = 0$ then $x = \text{o}$ and $\text{o} \in X_1$ (i.e. this is the initial node where an event happens that triggers the scenario bundle).
- $h = (h_i, ..., h_n)$ is a system of payoff functions for the players $1, \ldots, n$. The function $h_i$ assigns a real number $h_i(z)$ to every end-point $z \in X_0$. The payoff

functions $h_i$ are ordinal utility indices representing preference rankings at the end-points.

Selten required the following condition to apply for $i = 1, \ldots, n$:

*Condition C1*: $h_i(y) \neq h_i(z)$ for $y \neq z : y,\, z \in X_0$
in order to exclude ambiguous preference rankings. However, in the approach adopted in the present work and for simplification purposes, the following weaker condition applies for $i = 1, \ldots, n$:

*Condition C2*: $h_i(y) \neq h_i(z)$ for $y \neq z : y, z \in X_0$ and $y$, $z$ immediately follow a node $x$ where $c(x) = \{i\}$

**Strategies**
A strategy combination is a function $s$ which assigns a decision (selection of choice) at $x$ to every $x \in X \setminus X_o$. The decision $s(x)$ is called the local strategy at $x$.

Consider a node $x \in X \setminus X_o$ and a strategy combination $s$. There is a uniquely determined end-point $z$ such that all branches on the path from $x$ to $z$ are selected by $s$. This end-point is denoted by $z(x, s)$ and it will be: $h_i(x, s) = h_i(z(x, s))$. The payoff $h_i(x, s)$ is called the local payoff of player $i$ at $x$ for $s$.

Consider a node $x \in X \setminus X_o$ with $c(x) = \{i\}$, i.e. a node where player $i$ is expected to make a decision. Then a decision $\alpha^* \in A(x)$ is optimal with respect to $s$, which is written as $s^*(x) = s/\alpha^*$, if it is:

$$h_i(x, s/\alpha^*) = \max h_i(x, s/\alpha) \quad \text{for all} \quad \alpha \in A(x)$$

For formal completeness the unique choice at o where $c(x) = 0$ is defined as optimal with respect to $s$.

If condition C1 above applied, then at every $x \in X \setminus X_o$ there would be only one optimal decision with respect to $s$. As the weaker condition C2 generally applies in the present work, this uniqueness of optimal decision is not guaranteed.

**Equilibrium**
A strategy combination $s^*$ is a perfect equilibrium strategy combination of scenario bundle $B = (N, K, c, h)$, if for every $x \in X \setminus X_o$ the decision $s^*(x)$ is optimal with respect to $s^*$.

Let $K_x$ be the subtree at $x$ of the game tree $K$ of the scenario bundle $B = (N, K, c, h)$. Let $c_x$ and $h_x$ be the restrictions of c and h to $K_x$. Then $B_x = (N, K_x, c_x, h_x)$ is called the subgame (sub-scenario bundle) of $B$ at $x$.

If $s^*$ is a perfect equilibrium strategy combination of scenario bundle $B = (N, K, c, h)$ and $s_x^*$ is the restriction of $s^*$ to $K_x$, then $s_x^*$ is a perfect equilibrium strategy combination of the subgame $B_x$. This is easily proven using backward induction: assuming that $s_x^*$ is not a perfect equilibrium strategy combination of the subgame $K_x$ and working backwards until the origin of $K$, leads to the conclusion that $s^*$ is not a perfect equilibrium strategy combination of the game tree $K$. This rationale justifies the use of backward induction for finding a perfect equilibrium strategy combination of a scenario bundle, which for simplicity reasons will be called *equilibrium* from now on.

If condition C1 above applied then it can easily be proven (see Selten, 1999) that every scenario bundle $B = (N, K, c, h)$ would have one and only one equilibrium.

Because condition C2 above applies (for simplification purposes), the uniqueness of the equilibrium is not guaranteed in the adopted approach.

## 4. Application: Security of Gas Supply for Greece

In this paper, the analysis framework of the scenario bundle method, as described above, is adopted for the modelling purpose of the formulation of national energy strategy and planning, with specific reference to the security of gas supply and the role of LNG. The schematic illustration presented next concerns the security of gas supply for the Greek market.

### 4.1. Scenario bundle 1 (Reference)

**Setting – players – goals**

The security of supply of the Greek gas market is considered within an indicative future period of 5 years. This time horizon is justified by the fact that no new infrastructure projects will be operational in the specific (and broader) area within this period. Moreover, the options considered can be materialised soon and without any infrastructure requirements.

As shown in Figure 1, the players involved in this game are Greece; its gas suppliers through pipelines i.e. Russia and Turkey; its LNG (Liquefied Natural Gas transported by ships) supplier Algeria; and a possible new LNG supplier. The transit countries Ukraine, Romania, Bulgaria (for the supply of Greece from Russia) are grouped in one transit player with insignificant role (assumption).

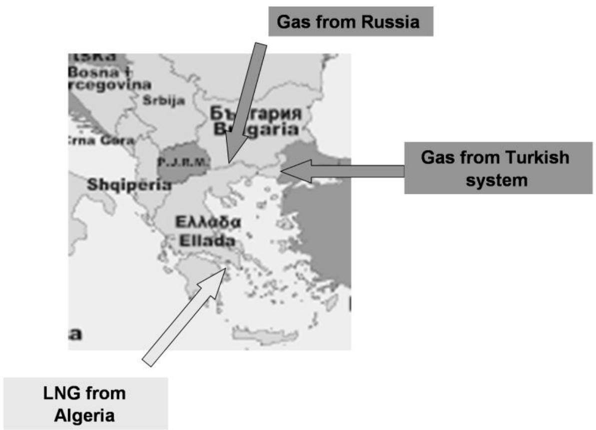


**Figure1.** Gas suppliers of the Greek market (map source: Europa website, 2008)

In this setting, primarily strategic commercial goals are considered (political/other goals will be avoided in this illustration). For Greece, the goal is to secure its continuous supply with gas. Russia's goal is to maintain or increase its market share in Greece and sustain its predominant role in the area as the major gas supplier. Algeria's goal is to maintain or increase its market share in the Greek gas market. Turkey initiates the examined scenario bundle and is not active after that point. The transit countries' role is assumed to be insignificant. Finally, an alternative LNG supplier is considered for the Greek gas market who is willing to supply gas once the opportunity arises, in order to secure short-term profits and possibly establish a longer-term presence in the area as a reliable gas source.

**Scenario bundle formulation**

The examined scenario bundle starts with a cut-off of gas supply to Greece from Turkey (see Figure 2). This could be the result of *force majeure,* e.g. a technical problem or a deficit in the gas imports of the Turkish system. Other reasons could lead to this event e.g. of political nature, which would further introduce reactions from the other players based on such reasoning (i.e. a broader geopolitical interaction, rather than a commercial energy supply interaction). Such parameters will however be left out of the present discussion.

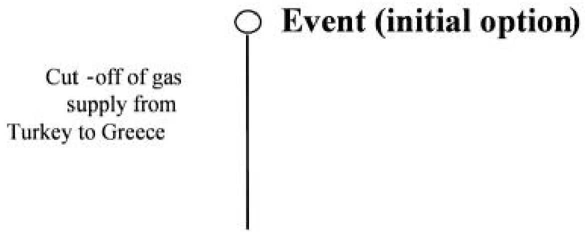


**Figure2.** Initial option – occurrence of an event

In the next phase (next node of the tree representing the game – interaction), it is examined which players will need to react and make a decision. As shown in Figure 3, Greece is the player that will have to respond. In case of no reaction, the result will be a gas deficit in the Greek market or, in other words, a crisis situation. This is indicated by a rectangle where this result is shown and the scenario ends there (an inferiority end-point).

The other immediate alternative for Greece is to request an increase in gas supply from Russia. An increase in LNG imports is not a first response in this strategic interaction (this option is considered later on), at least not before pipeline gas supplies have been considered. Indeed, in the examined setting of the Greek market and also in general, it is economically and technically preferable (at short notice) to increase the import volumes of pipeline gas rather than LNG.

Once increased gas supply has been requested from Russia, the latter is expected to respond next (Figure 3).

The construction of the scenario bundle continues in a similar fashion, as shown in Figure 4. Specifically, Russia either supplies the requested gas or does not supply it. In the latter case, Greece turns to Algeria and asks for increased supplies of LNG. If Algeria does not supply the requested gas, then the option is considered for Greece that it has the flexibility to seek an alternative supplier of LNG.

The choices of the players in this strategic interaction can be better understood if a utility index is used to represent their preferences. This utility index corresponds to each player's payoff by assigning a rough quantitative value to different outcomes of the interaction. These payoffs are shown in Table 1. The suggested index indicates only the ordinal preference (or indifference) of the player in relation to the outcome of the interaction.


**Figure3.** Greece's reaction after the occurrence of the event

Each outcome of the game is assigned with an array of payoffs $(a_1 \dots a_i \dots a_I)^T$, $i = 1..\mathrm{I}$ (in this scenario bundle I=4). When a player i is to make a choice at any node of the game, he will make it so as his resulting payoff is the maximum possible, i.e. the payoffs array is $(a_1 \dots a_i^* \dots a_I)^T$, where $a_i^*$ is the maximum of the possible payoffs $a_i$ at that stage of the game.

The payoff arrays according to the values given in Table 1 have been introduced in Figure 5.

**Scenario equilibrium**

The equilibrium solution of the previously formulated scenario bundle will be found using backward induction, i.e. working from the end of the tree to its start (origin).

So, initially the last node is considered where the alternative LNG supplier is to make a decision. As shown in Figure 6, he will consider his available payoffs at that point depending on his choice. Obviously, he will choose to supply LNG to the Greek market (which corresponds to a payoff of +1), instead of not exploiting this opportunity (a payoff of -1). The latter inferior choice is crossed out.

Next, the previous to last node is examined where Greece is to make a decision. As shown in Figure 7, Greece will consider turning to the alternative source of LNG (which corresponds to a payoff of +1), instead of not taking any action (a payoff of -1). The latter inferior choice is again crossed out.

In a similar fashion, the backward induction process continues up to the first node of the game three (after the initial event) where Greece is choosing an action (see Figure 8). The equilibrium solution consists of the branches of the game tree not crossed out during this process, and it is stressed in bold line in Figure 8.

**Observations**

The previously discussed scenario bundle is a rather schematic and basic one. Although it can be further perplexed with the introduction of more options and parameters (some of them are discussed later), the presented simplified analysis reaches some useful conclusions.

First of all, the Greek gas market is dependent on imports from Russia, Turkey and Algeria. In an event of supply cut-off from one of these sources (e.g. in the examined scenario from Turkey), Greece must compensate by increasing imports

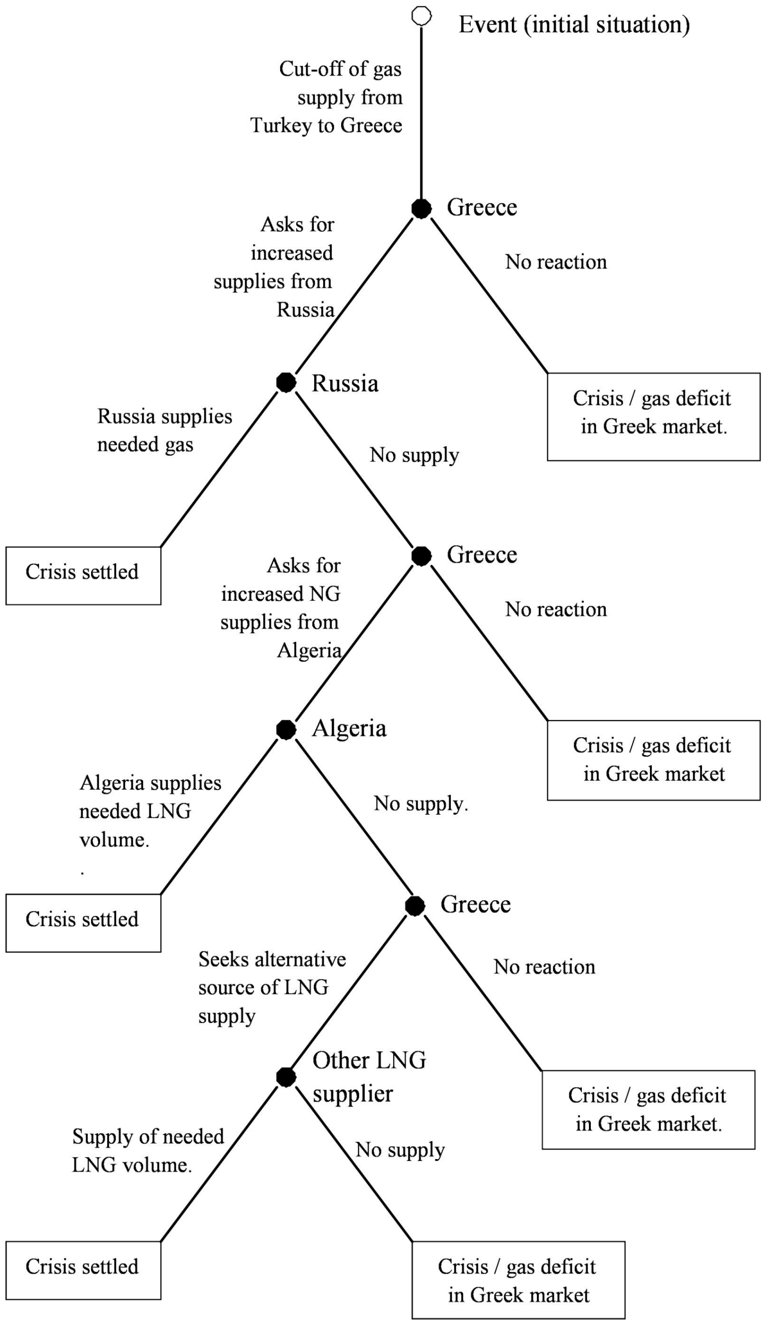


**Figure4.** Scenario bundle 1 complete game tree

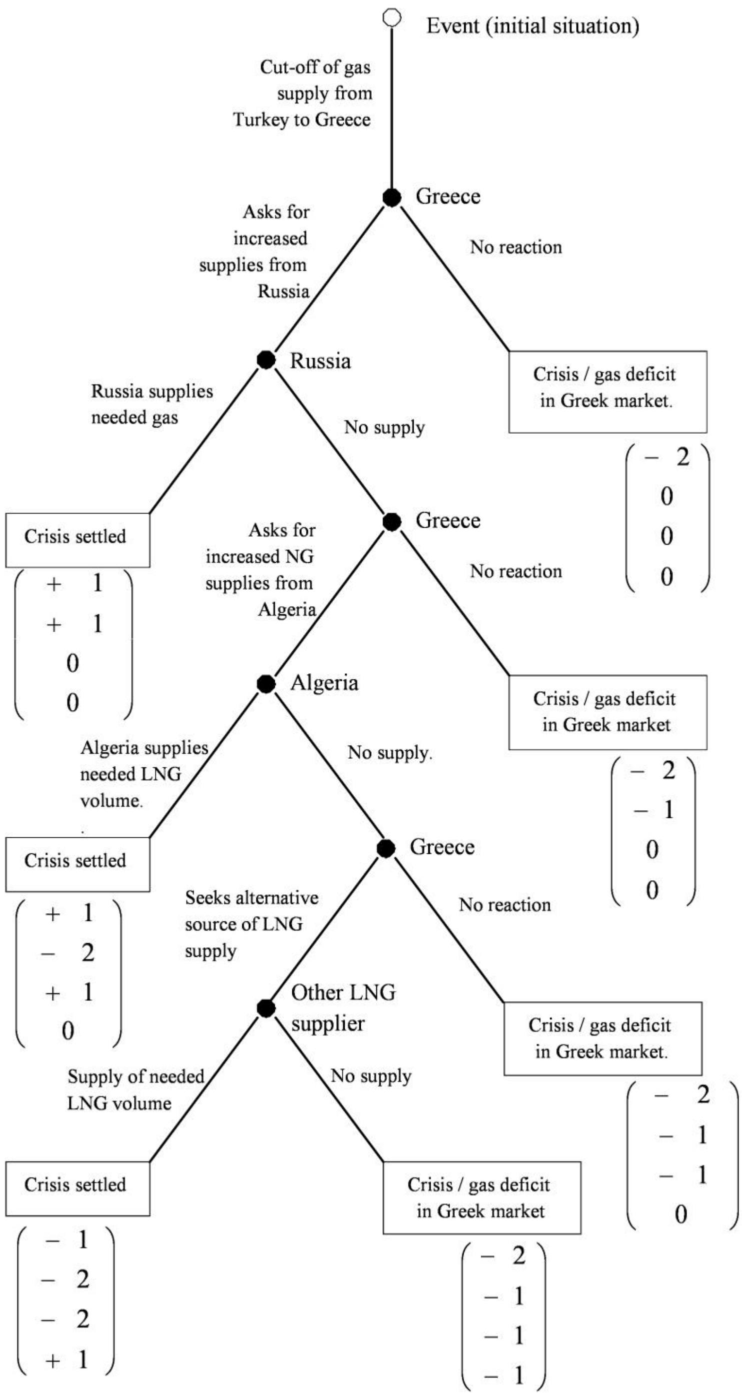


**Figure5.** Payoff arrays introduced to the Scenario bundle 1 game tree

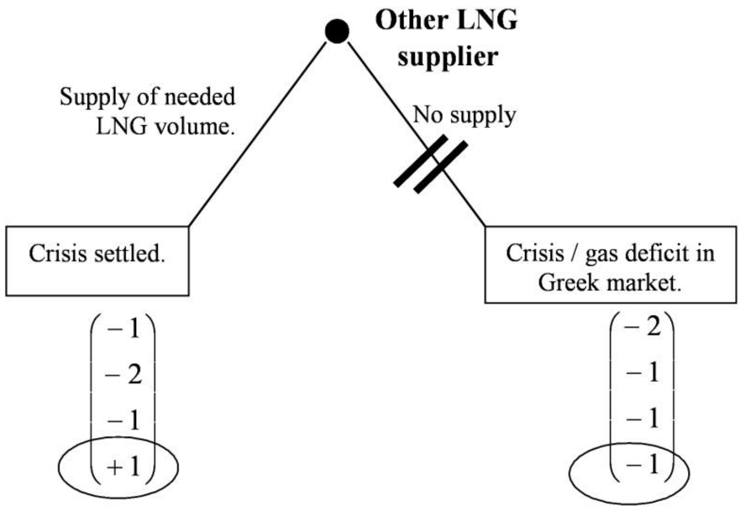


**Figure6.** Finding the solution using backward induction – stage 1

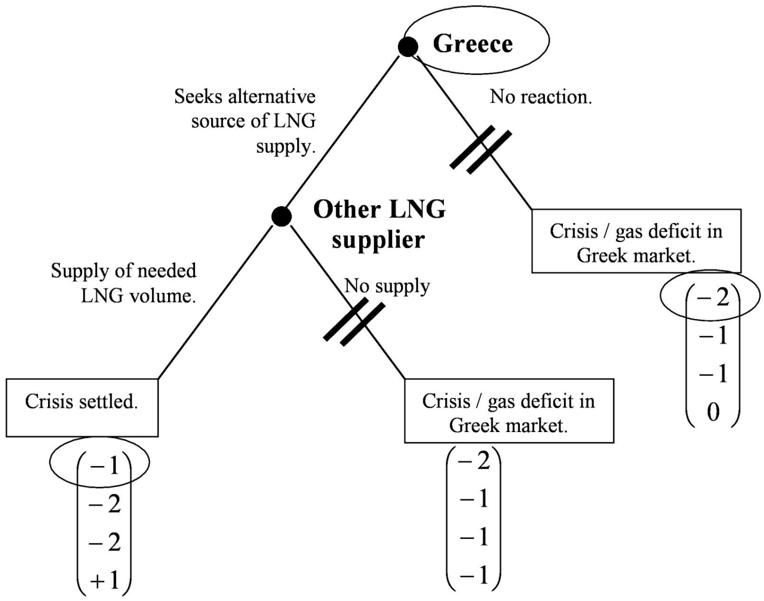


**Figure7.** Finding the solution using backward induction – stage 2

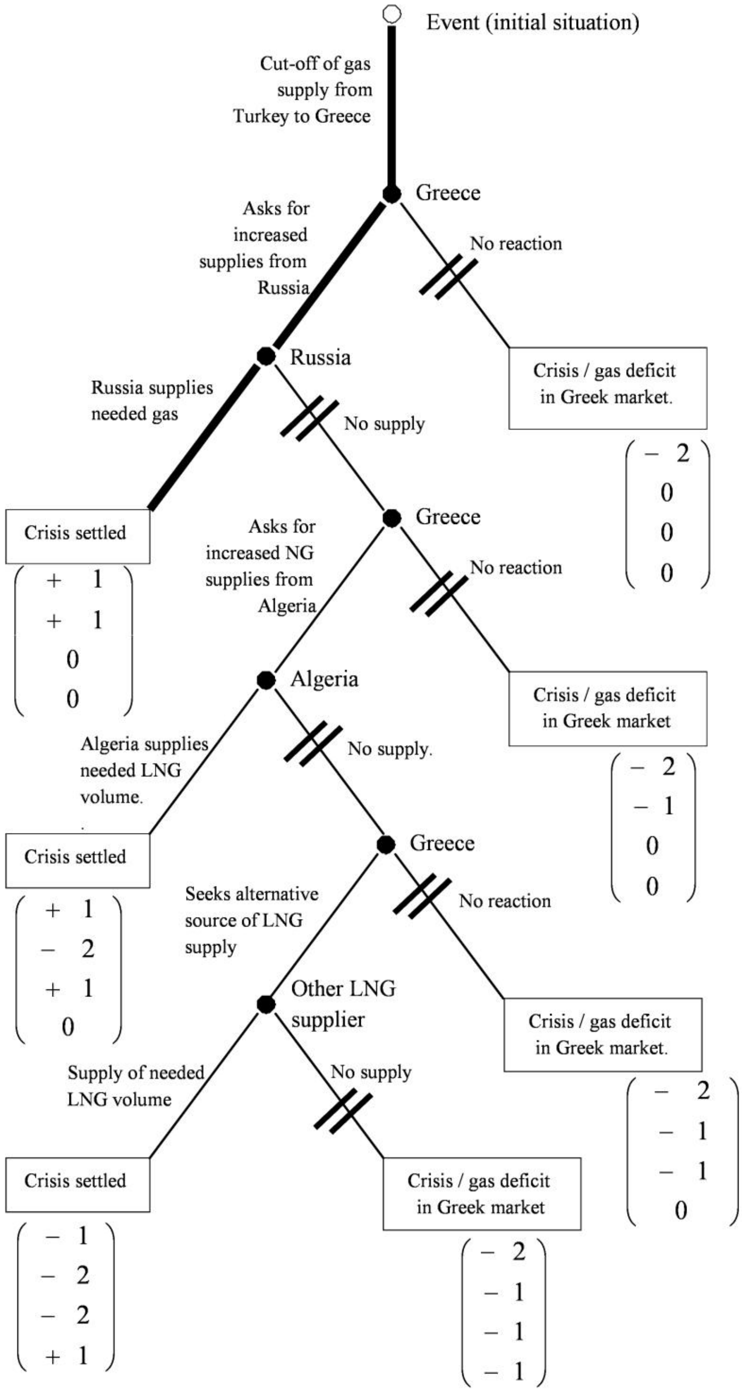


**Figure8.** Equilibrium solution – Scenario bundle 1

**Table1.** Players' payoffs - Scenario bundle 1.

| i= | Player | Outcome state | Payoff $a_i$ |
|---|---|---|---|
| 1 | Greece | Gas deficit / Market crisis | -2 |
| | | Crisis settled with supply of gas from emergency supplier | -1 |
| | | Crisis settled with supply of gas from existing suppliers | +1 |
| | | No event / no interaction | +2 |
| 2 / 3 | Russia / Algeria | No supply of extra gas to Greek market, another player takes the opportunity | -2 |
| | | No supply of extra gas to Greek market, no other player takes the opportunity | -1 |
| | | Supply of extra gas / increase of exports | +1 |
| | | No involvement / No interaction | 0 |
| 4 | Alternative LNG supplier | No supply of gas to market / misses the opportunity | -1 |
| | | Supply of gas / entering the Greek market | +1 |
| | | No involvement / No interaction | 0 |

from at least one of the other two gas suppliers (namely Russia and/or Algeria). Failure to increase gas supply from both of the other two sources leads to the need to secure supply from another LNG supplier, in order to avoid a gas deficit / crisis in the Greek market.

Greece by "securing" (this term is commented in the next paragraph) a 3$^{\text{rd}}$ alternative gas (LNG) supplier "motivates" both Russia and Algeria to supply gas to the Greek market, when asked to, in order to avoid a loss of market share (corresponding to a negative payoff of -2). Otherwise, they would not be motivated to this degree (-1 payoff). This observation is noted in Figure 9, where the presence of the alternative LNG supplier (willing to supply gas to the Greek market) is highlighted. The payoffs of the Russia and Algeria from -1 (in the case that they failed to supply the Greek market) further decrease to -2, as they face the entrance of a new competitor into "their market" and risk longer term loss of market share.

The strategic implications from the above analysis for Greece suggest that Greece should "pay a premium" for the "option" of gas supply from an alternative LNG supplier (the word "securing" was used above). The underlying meaning is that Greece should establish relations with an alternative LNG supplier who would be willing to supply gas in an emergency situation, from reserve capacity or even by redirecting cargoes from other destinations. Such an agreement / contract, apart from the definition of the (increased) level of tariffs, could be accompanied by special rights provisions allowing to the supplier preferential treatment after such a crisis e.g. by reserving a market share for the new supplier on a long term basis.

The exercise of the above-described "option" is not the purpose for the Greek security of supply planning, as much as it provides *additional security* by motivating existing suppliers to meet extra demand in crisis situations.

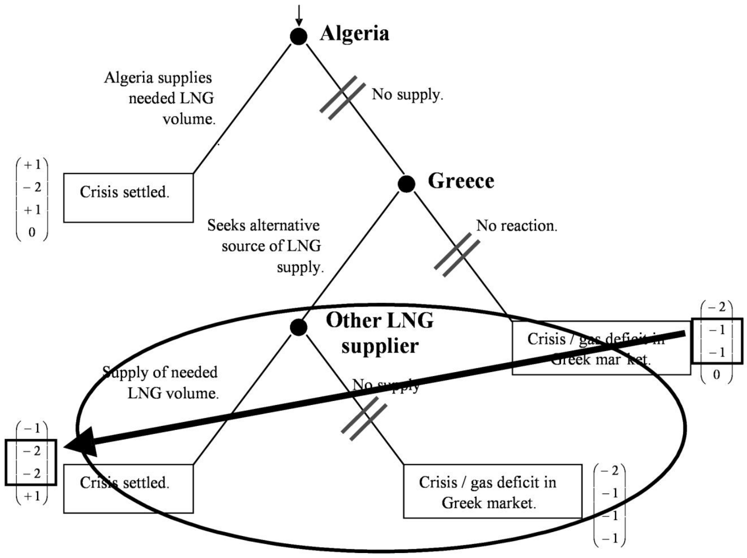


**Figure9.** The role of the alternative LNG supplier – Scenario bundle 1

The presented scenario has strategic implications also for Russia, the major gas supplier of the broader area, and although Greece is a relatively small market. If Russia wants to sustain its strategic role in the area, it should be ready to supply extra gas when needed, in order to avoid the intrusion of new competitors in this geopolitically and energy-wise sensitive crossroads.

The above observation is further supported in the sensitivity analysis (scenario bundle 2) discussed next.

### 4.2. Scenario bundle 2 (sensitivity analysis)

Scenario bundle 2 discussed in this section is actually a sensitivity analysis of the previously presented Reference scenario bundle (Scenario bundle 1). The sensitivity analysis concerns the payoffs of the involved players. The purpose of this sensitivity analysis is to examine the possible effects on the scenario equilibrium of the assumed payoffs associated with the players.

The formulation of the scenario bundle has not changed (initial event, options of players, succession of decisions), however the payoffs of players i=2 (Russia) and i=3 (Algeria) have changed according to new assumptions regarding their preferences. These new payoffs are shown in Table 2 (again the suggested payoff indicates only the ordinal preference - or indifference - of the player in relation to the outcome of the interaction).

Comparing with Table 1 of Scenario bundle 1, the following observations are made. The outcome "Supply of extra gas / increase of exports" corresponds to a payoff of -1 for Russia (it was +1 in Scenario bundle 1). It is assumed that the supply of extra gas to Greece from Russia (at list at short notice) is inconvenient e.g. because Russia considers contract obligations with other customers of priority importance. The outcome "No supply of extra gas to Greek market, no other player takes the opportunity" leaves Russia indifferent under the new circumstances. However, the outcome "No supply of extra gas to Greek market, another player takes the

**Table2.** Players' payoffs – Scenario bundle 2.

| i= | Player | Outcome state | Payoff $a_i$ |
|---|---|---|---|
| 1 | Greece | Gas deficit / Market crisis | -2 |
| | | Crisis settled with supply of gas from emergency supplier | -1 |
| | | Crisis settled with supply of gas from existing suppliers | +1 |
| | | No event / no interaction | +2 |
| 2 / 3 | Russia / Algeria | No supply of extra gas to Greek market, another player takes the opportunity | -2 / -1 |
| | | No supply of extra gas to Greek market, no other player takes the opportunity | 0 / 0 |
| | | Supply of extra gas / increase of exports | -1 / -2 |
| | | No involvement / No interaction | 0 |
| 4 | Alternative LNG supplier | No supply of gas to market / misses the opportunity | -1 |
| | | Supply of gas / entering the Greek market | +1 |
| | | No involvement / No interaction | 0 |

opportunity" still corresponds to the worst payoff for Russia for the same reasons explained in the previous scenario.

For Algeria, the outcome "No supply of extra gas to Greek market, another player takes the opportunity" corresponds to a payoff of -1 (it was -2 in Scenario bundle 1). The reason is that although this is an undesirable outcome for Algeria, under the new circumstances the worst outcome is "Supply of extra gas / increase of exports", which corresponds to a payoff of -2 (it was +1 in Scenario bundle 1). The assumed justification for this change is that Algeria is not in position to increase LNG exports to Greece (at list at short notice), as it has not available non-contracted LNG volumes and diverting cargoes from other destinations would mean breaching existing contracts. Finally, the outcome "No supply of extra gas to Greek market, no other player takes the opportunity" leaves also Algeria indifferent under the new circumstances.

The payoff arrays according to the new values given in Table 2 have been introduced in Figure 10.

**Scenario equilibrium**

The equilibrium solution is found using again backward induction, i.e. working from the end of the tree to its start (origin). The equilibrium solution consists of the branches of the game tree not crossed out during this process, and it is stressed in bold line in Figure 11. The equilibrium solution has not changed compared to Scenario bundle 1, i.e. the equilibrium outcome is stable in relation to the examined sensitivity analysis.

**Observations**

Scenario bundle 2 further supports the observation made in Scenario bundle 1 that Greece by "securing" a 3$^{rd}$ alternative gas (LNG) supplier "motivates" its existing suppliers to supply gas to the Greek market, when asked to, in order to avoid a loss of market share. This is exactly the reason why Russia chooses to supply gas to the Greek market in Scenario bundle 2, although this choice corresponds to a payoff -1 for Russia. Choosing not to supply gas would result to a payoff -2 because of the presence of the alternative LNG supplier. Figure 12 shows the hypothetical solution if Russia chose not to supply the Greek market, which is not equilibrium, as it corresponds to an inferior outcome for Russia.

Indeed, Figure 13 shows the equilibrium solution in Scenario bundle 2, if no alternative LNG supplier existed. In this case, not supplying the Greek market would be the preferred choice for Russia.

### 4.3. Other potential extensions

The above analysed scenario bundles are schematic illustrations that demonstrated the usefulness, functionality and most of all the potential of the method in analysing national energy strategies. These scenarios can be further refined and extended with the introduction of additional parameters. Moreover, the analysis can be taken to other levels of scenario formulation related to national strategic planning. Such refinements and extensions are discussed next.

First of all, further sensitivity analyses can be performed over the suggested payoffs of the involved players, i.e. changes in their values (within realistic ranges), so as to assess the potential impact on the optimal choices of the players and the scenario equilibrium. The assumed payoffs were suggested in accordance with the requirements for realism and desirability regarding the options available to the players and the examined outcomes of the interaction. In any case, they represent estimations of the analyst. Selten (1999) used a panel of experts in order to derive reliable payoff estimations in his research context.

Other possible initial events could be considered that lead to the construction of different scenario bundles. Such similar initial events (options) could include a cut of gas supply from Russia as a result of *force majeure* or a crisis situation in the transit countries. Or, on the other hand, similar events concerning the LNG cargoes coming from Algeria.

Other extensions of the model could concern the options available to the players. As an example, an option of Greece to switch to another fuel in case of gas shortage could be considered. However, choosing such an option could be treated as an end point of this scenario bundle and the initial event for another scenario bundle.

The introduction of other influential factors is another possible extension. The transit countries' could possibly be considered with an active role (introduction of a new player) or external uncertainty factors of potential influence could be considered, such as the oil price levels.

Extending the scenario bundle's horizon further into the future would necessitate taking into account the coming developments in the area of concern, such as the construction of new infrastructure projects. Referring to this specific context, after the considered time horizon of 5 years, the Italy – Greece gas pipeline interconnection (IGI) will be in place and the South Stream gas pipeline from Russia. Both these projects will turn Greece into (also) a transit country and introduce western Europe as a player with viable interests in the security of gas supply of the area.

Last but not least, the rationale of such an analysis can change level in order to introduce political or other goals for the involved players. Other reasons could

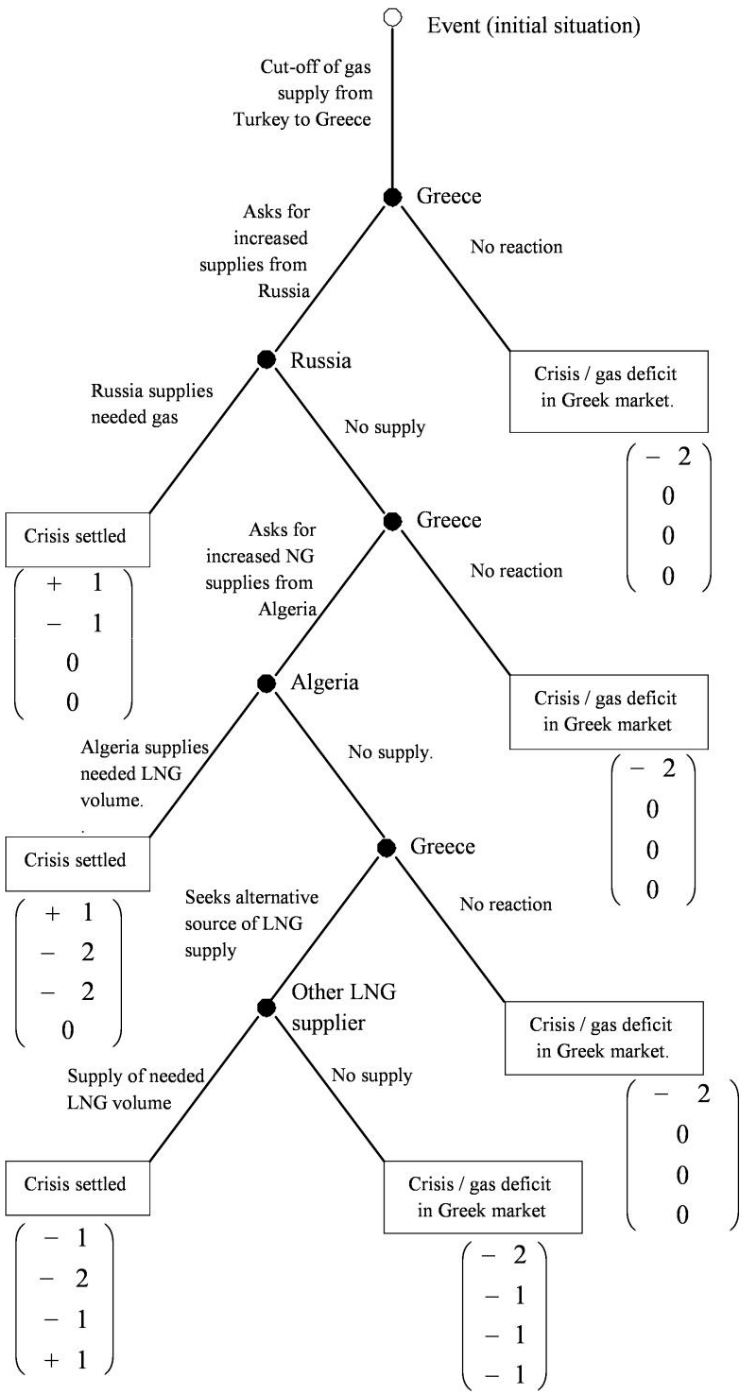


**Figure10.** Payoff arrays introduced to the Scenario bundle 2 game tree

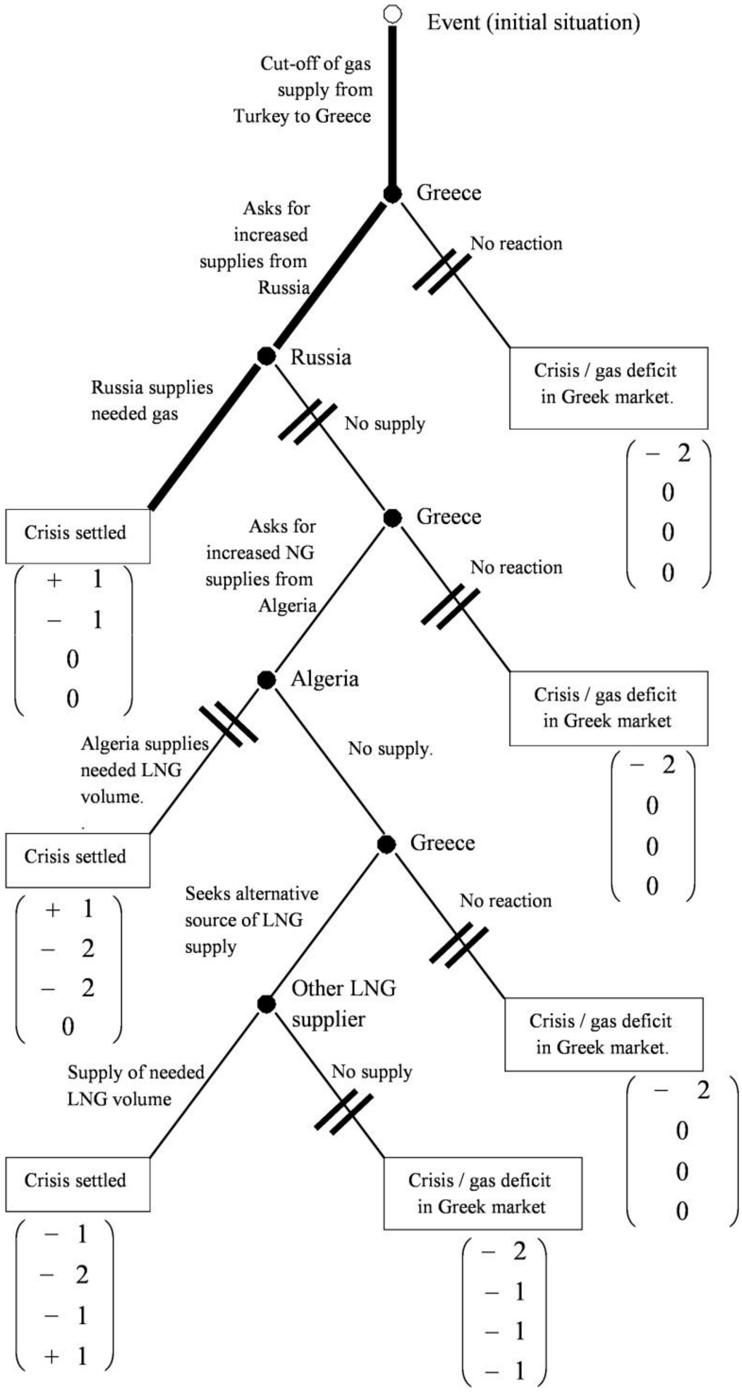


**Figure11.** Equilibrium solution – Scenario bundle 2

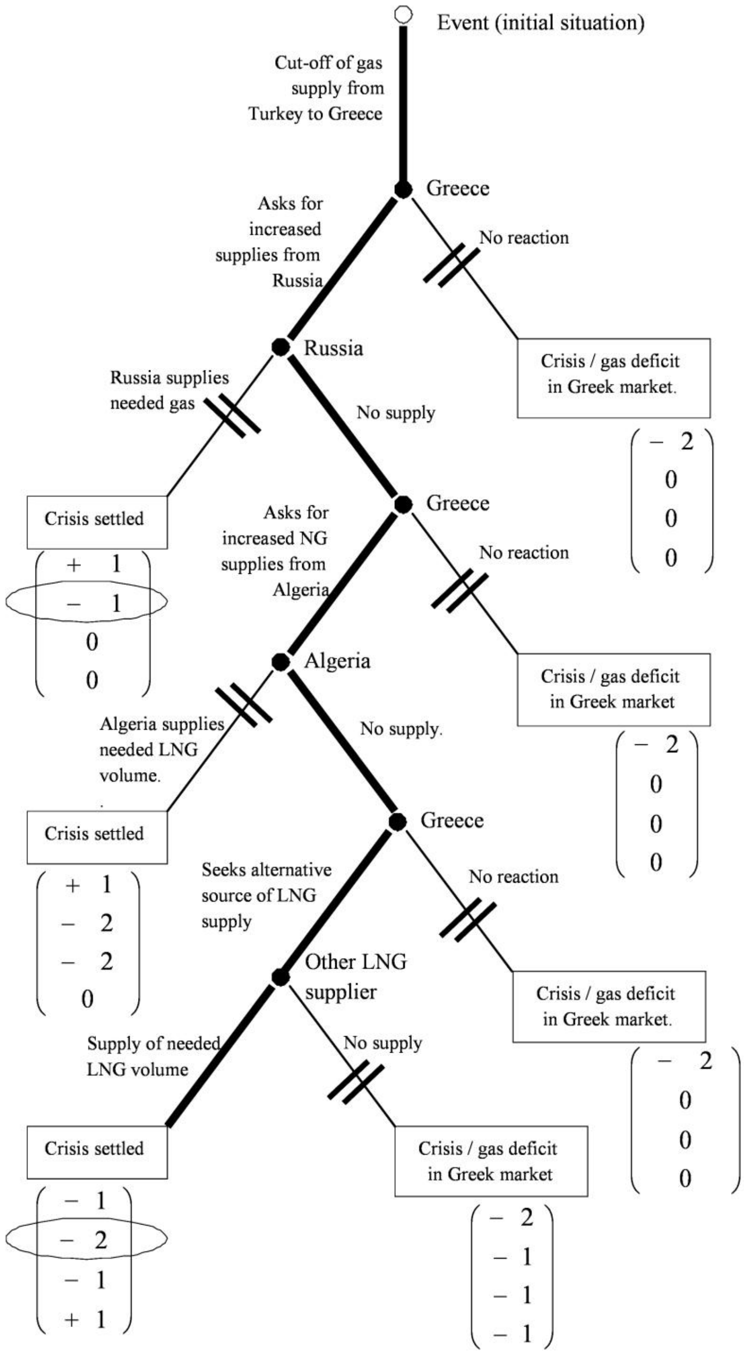


**Figure12.** Hypothetical solution (not an equilibrium solution)

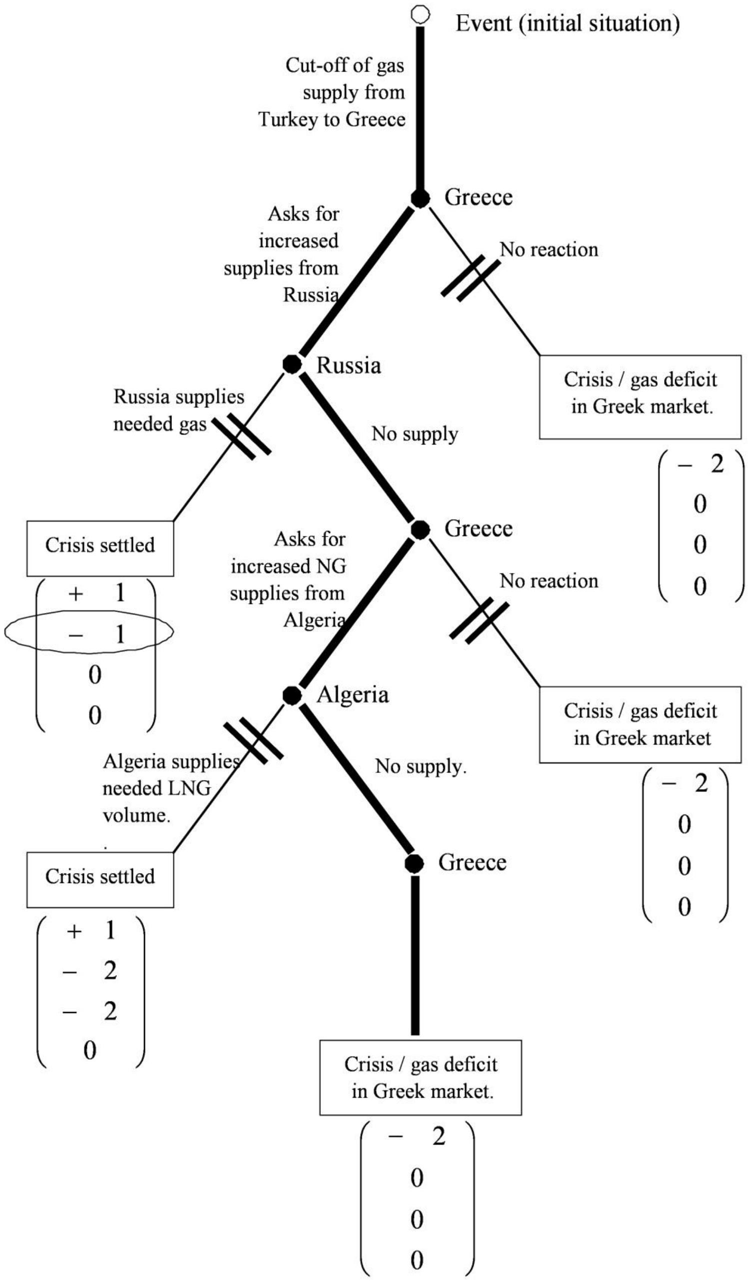


**Figure13.** Equilibrium solution – Scenario bundle 2 with no alternative LNG supplier

lead to the initial event e.g. of military or political nature, which would trigger reactions from the other players (i.e. a broader geopolitical interaction would be at play, rather than a commercial energy supply interaction).

## 5. General Conclusions and Benefits of the Method

The application of the scenario bundle method in the context of the security of gas supply for the Greek market demonstrated the potential usefulness of this tool in addressing national energy strategy questions. This methodology allows strategy formulation by taking into account the commercial objectives of the involved players, and considering geopolitical interaction, regional conflicts and crisis situations.

The conclusions reached using the scenario bundle method may seem straightforward in rather basic scenarios. However the scenario bundle approach helps in identifying critical parameters in a systematic way, as a result of concrete logical steps. This analysis gives confidence that even in more complex scenarios, an analyst would reach useful conclusions of practical value, that might not be obvious by simple observation of the strategic interaction.

According to Selten (1999), the benefits of the method can be understood by considering an analogy with a chess player. A chess player who tries to plan ahead cannot predict the future course of the game, however he approaches his decision problem with a "predictive spirit" and he explores the likely consequences of a selection of plausible moves. All possibilities cannot be examined, so a selection has to be made using criteria of plausibility. Stopping principles also need to be used to limit the depth of explorations. In the end, such an analysis is necessary in order to play in an efficient way and hope for success. In this sense, human decision making in chess seems to be analogous to the construction and evaluation of scenario bundles.

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# Quality Competition: Uniform vs. Non-uniform Consumer Distribution ⋆

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**Abstract** A two-stage game-theoretical model of duopoly and vertical product differentiation is examined. It is assumed, that there are two firms on some industrial market which produce homogeneous product differentiated by quality. The results of the research of a two-stage model of duopoly are presented. At the first stage of the model companies define quality level and at the second stage they compete in product price. It is supposed that consumers are uniformly distributed. This model was extended to the case when consumers are distributed non-uniformly. The research presents the comparative analysis of results in the case of uniform and non-uniform consumers' distribution.

**Keywords:** vertical product differentiation, duopoly, multi-stage game, non-uniform consumer distribution, sub-game perfect equilibrium.

## 1. Introduction

Investigation of quality problems is in the focus of interest of the industrial organization. It is caused by the fact that quality is the most powerful instrument of the company management and it is required to provide the competitiveness of the company.

There are different approaches to quantitative quality estimation. In this paper we follow the idea of product quality estimation using the solution of corresponding game-theoretical model.

Models that are investigated in this paper are dedicated to the duopoly modeling under condition of quality competition and vertical product differentiation. Quality competition allows companies to obtain some competitive advantages. Making investments in technological innovations companies can manage and control its product quality range and gain the leadership position on the observed market. It is necessary to mention that making investments in technological innovations helps companies manage expenditures connected with the production of goods of certain quality, and it can give competitive advantage to the firm as well (costs leadership). The main aspect of costs leadership strategy lies in the lower net costs in comparison to the competing firms.

In this paper models of vertical product differentiation are suggested. The problem of optimal product quality estimation is solved for all feasible values of initial

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parameters of the models. The aim of the work is not only to build mathematical models of product quality assessment but also to solve numerical estimation examples of the product characteristics we are interested in.

There were created two game-theoretical models of company operation which is to produce goods of some quality under condition of competition on some industrial market. The first model is the model of product quality estimation in case of duopoly and uniform consumer distribution. Then in the second model we extended the first one to the case of non-uniform consumer distribution.

So, two two-stage game-theoretical models of duopoly and vertical product differentiation are examined. The basis for our models is the model described by Jean Tirol ( Tirol, 1988). It is supposed however that investigated market is uncovered.

It is assumed, that there are two firms on some industrial market which produce homogeneous product differentiated by quality. Also, quality range is supposed to be given in our investigations. The models consist of two stages: at the first stage companies define quality level and at the second stage they compete in product price. In both models we propose that each firm's goal is the maximization of its profit while producing the goods of certain quality.

Using the backward induction the problem of optimal product quality estimation was solved.

To obtain the optimal product quality the sub-game perfect Nash equilibrium is found for the models described above. Methods of game theory and mathematical analysis are used to solve the problem. The suggested approach to the optimal product quality estimation for the described models of competition and vertical product differentiation can be used for recommendations on companies strategic planning.

Having the expert evaluation of market characteristics, the computation can be made using the models described in this paper. Thus, the analysis of quantitative results allow companies to formulate optimal production strategies.

## 2. Uniform Distribution

### 2.1. Problem Statement

This section represents the model of vertical differentiation - the differentiation by quality. When products are vertically differentiated all consumers agree about the most preferable set of product properties, so they agree that the higher is the product quality the better is the product. However, such natural ordering in the space of characteristics can be made only in case of equal product prices.

Two-stage game-theoretical model of duopoly under vertical product differentiation is investigated. It is assumed, that there are two firms on some industrial market which produce homogeneous product differentiated by quality. We will suppose that each consumer has unit demand and has different inclination to quality.

The utility function of the consumer with inclination to quality $\theta$ when buying the product of quality $s$ for price $p$ is:

$$U_\theta(p) = \begin{cases} \theta s - p, p \leq \theta s \\ 0, p > \theta s \end{cases} \tag{1}$$

We call parameter $\theta$ the "inclination to quality", which indicates customer's willingness to pay for quality. It is clear that a consumer with inclination to quality $\theta$ (from now on we will simply call him/her "consumer $\theta$") will consider the purchase of the product of quality $s$ for price $p$ if and only if its utility from this purchase is non-negative. Note that the first item in equation (1) can be interpreted as maximum price that consumer $\theta$ is ready to pay for the product of quality $s$ (the worth of the product for the consumer $\theta$).

As we have no information about the distribution of the parameter $\theta$ we suppose that it is uniformly distributed with unit density over the interval $[\underline{\theta}, \overline{\theta}]$, where $\overline{\theta} = \underline{\theta} + 1$, $\underline{\theta} > 0$. $\underline{\theta}$ and $\overline{\theta}$ are considered to be given. Otherwise we have to investigate separately the distribution of consumers' willingness to pay for quality.

Firm $i$ produces goods of quality $s_i$ and we assume that the product will be sold at the price $p > c(s_i)$, where $c(s_i)$ is firm's cost function which express the production costs for the product of quality $s$.

Firm $i$'s profit of producing a product of quality $s_i$ , where $s_i \in [\underline{s}, \overline{s}]$ , will be defined as following function:

$$\Pi_i(p_i, s_i) = p_i(s) D_i(p, s) - c(s_i), i = 1, 2 \tag{2}$$

where $p_i$ – product price of the firm $i$, $s_i$ – quality of the firm $i$'s product, $p = (p_1, p_2)$ – a vector of product prices of the competing firms, $s = (s_1, s_2)$ – a vector of product qualities, $D_i(p, s)$ – the demand function for the product of quality $s_i$ , $c(s_i)$ – production costs of firm $i$ for the product of quality $s_i$ .

The costs function is considered to be quadratic: $c(s_i) = ks_i^2, k > 0$.

In this model we propose that each firm's goal is the maximization of its profit function.

In such statement of the problem we have a two-stage model of duopoly, when:

- at the first stage companies define quality level;
- at the second stage they compete in product price.

### 2.2. Prices in Equilibrium

We assume that some regional industrial market is uncovered. In this case functions of the demand for the competing firms' goods can be presented in the following way:

$$\begin{cases} D_1(p_1, p_2) = \dfrac{p_2 - p_1}{\triangle s} - \dfrac{p_1}{s_1} \\ D_2(p_1, p_2) = \overline{\theta} - \dfrac{p_2 - p_1}{\Delta s} \end{cases} \tag{3}$$

To find optimal product price for firms-competitors, we use the first order condition of extremum:

$$\begin{cases} \dfrac{\partial \Pi_1}{\partial p_1} = \dfrac{p_2 s_1 - 2 p_1 s_2}{s_1 \Delta s} = 0 \\ \\ \dfrac{\partial \Pi_2}{\partial p_2} = \dfrac{\overline{\theta} \Delta s - 2 p_2 + p_1}{\Delta s} = 0 \end{cases}$$

Then the firms reaction functions are:

$$\begin{cases} p_1 = R_1(p_2) = \dfrac{s_1}{2s_2} p_2 \\ p_2 = R_2(p_1) = \dfrac{p_1 + \overline{\theta}\Delta s}{2} \end{cases} \tag{4}$$

Hence, we receive equilibrium prices:

$$\begin{cases} p_1^*(s) = \dfrac{\overline{\theta} s_1 \Delta s}{4s_2 - s_1} \\ p_2^*(s) = \dfrac{2\overline{\theta} s_2 \Delta s}{4s_2 - s_1} \end{cases} \tag{5}$$

Equilibrium demand for firms' products can be rewritten then such as:

$$\begin{cases} D_1^*(s) = \dfrac{\overline{\theta} s_2}{4s_2 - s_1} \\ D_2^*(s) = \dfrac{\overline{2\theta} s_2}{4s_2 - s_1} \end{cases} \tag{6}$$

And firms' profit functions of producing goods in equilibrium are defined as:

$$\begin{cases} \Pi_1^*(s) = \Pi_1(p^* s, s) = \dfrac{\overline{\theta}^2 s_1 s_2 \Delta s}{(4s_2 - s_1)^2} - c(s_1) \\ \Pi_2^*(s) = \Pi_2(p^* s, s) = \dfrac{4\overline{\theta}^2 s_2^2 \Delta s}{(4s_2 - s_1)^2} - c(s_2) \end{cases} \tag{7}$$

### 2.3. Optimal quality choice in case of quadratic costs

Taking into consideration the results obtained in the subsection 2.2. we receive that firms profit functions of producing goods and selling them for optimal price look in the following way:

$$\begin{cases} \Pi_1^*(s) = \dfrac{\overline{\theta}^2 s_1 s_2 \Delta s}{(4s_2 - s_1)^2} - c(s_1) \\ \Pi_2^*(s) = \dfrac{4\overline{\theta}^2 s_2^2 \Delta s}{(4s_2 - s_1)^2} - c(s_2) \end{cases} \tag{8}$$

Using the extremum conditions we can define the optimal product quality. Let us recall as well that production costs function is supposed to be quadratic, i.e. $c(s_i) = ks_i^2, k > 0$.

Then first order condition is:

$$\begin{cases} \dfrac{\partial \Pi_1^*}{\partial s_1} = \dfrac{\overline{\theta}^2 s_2^2}{(4s_2 - s_1)^3} (4s_2 - 7s_1) - 2ks_1 = 0 \\ \dfrac{\partial \Pi_2^*}{\partial s_2} = \dfrac{4\overline{\theta}^2 s_2}{(4s_2 - s_1)^3} \left(4s_2^2 - 3s_1 s_2 + 2s_1^2\right) - 2ks_2 = 0 \end{cases}$$

**Table1.** Initial data.

| $\underline{\theta s}$ | $\overline{\theta}\underline{s}$ | $\underline{s}$ |
|---|---|---|
| 350 | 420 | 70 |
| 300 | 367 | 67 |
| 320 | 388 | 68 |
| 290 | 355 | 65 |
| 350 | 410 | 60 |

To solve this system we can equate right parts of two equations. Then, we receive the following equation of the third degree:

$$4s_2^3 - 23s_2^2 s_1 + 12s_2 s_1^2 - 84s_1^3 = 0. \tag{9}$$

Let's make the following substitution: $s_2 = \mu s_1$. Then the solution of the cubic equation above is $\mu = 5.2512$ .

Then $s_2 = 5.2512s_1$. If we substitute this interdependence of product qualities in the extremum condition, we receive the following results:

1. If values of optimal product qualities from the system (12) belong to the interval $s_i^* \in [\underline{s}, \overline{s}]$ , then

$$\begin{cases} s_1^* = 0.0241\dfrac{\overline{\theta}^2}{k} \\ \\ s_2^* = 0.1266\dfrac{\overline{\theta}^2}{k} \end{cases} \tag{10}$$

Hence, firms profits in the equilibrium are:

$$\begin{cases} \Pi_1^* = \Pi_1^*(s^*) = 0.0125\dfrac{\overline{\theta}^4}{k} \\ \\ \Pi_2^* = \Pi_2^*(s^*) = 0.0123\dfrac{\overline{\theta}^4}{k} \end{cases} \tag{11}$$

2. If $s_1^* < \underline{s}, s_2^* \in [\underline{s}, \overline{s}]$ , then $s_1^{**} = \underline{s}$. From the first order condition we find the optimal product quality:

$$\frac{\partial \Pi_2^*}{\partial s_2} = \frac{4\overline{\theta}^2 s_2}{(4s_2 - \underline{s})^3}\left(4s_2^2 - 3s_2\underline{s} + 2\underline{s}^2\right) - 2ks_2 = 0. \tag{12}$$

Let's make the following substitution: $s_2 = \mu\underline{s}$. Then, we receive the following equation of the third degree in $\mu$:

$$-64k\underline{s}^3\mu^3 + \left(8\overline{\theta}^2\underline{s}^2 + 48k\underline{s}^3\right)\mu^2 - \left(6\overline{\theta}^2\underline{s}^2 + 12k\underline{s}^3\right)\mu + 4\overline{\theta}^2\underline{s}^2 + k\underline{s}^3 = 0. \tag{13}$$

We obtain the solution of this cubic equation numerically.

We assume that we know maximum prices that consumers with the lowest and highest inclination to quality are ready to pay for products of the lowest possible quality $\underline{s}$ . As well, we suppose that the coefficient $k = 0.02$.

Table 1 presents the initial data for this case.

**Table2.** Coefficients of equation and its solution.

| $\mu^3$ | $\mu^2$ | $\mu^1$ | $\mu^0$ | $\mu$ |
|---|---|---|---|---|
| -439040 | 1740480 | -1140720 | 712460 | 3,3304 |
| -384977 | 1366244 | -880317 | 544771,3 | 2,9339 |
| -402473 | 1506207 | -978728 | 608464,6 | 3,118 |
| -351520 | 1271840 | -822060 | 509592,5 | 2,9996 |
| -276480 | 1552160 | -1060440 | 676720 | 4,9376 |

**Table3.** Results.

| $s_1^{**}$ | $s_2^{**}$ | $\Pi_1^*$ | $\Pi_2^*$ |
|---|---|---|---|
| 70 | 233,128 | 30,82294 | 629,1544 |
| 67 | 196,5713 | 9,185541 | 388,6145 |
| 68 | 212,024 | 18,61086 | 486,4417 |
| 65 | 194,974 | 11,63705 | 393,1936 |
| 60 | 296,256 | 82,93263 | 1304,629 |

The coefficients at all powers of $\mu$ in the equation, that we receive using the initial data given above, are presented in the Table 2 below.
So, in the next table (see Table 3) we present the problem solution, i.e. the optimal product quality for our initial data.

3. If $s_1^* \in [\underline{s}, \overline{s}]$, $s_2^* > \overline{s}$ , then $s_2^{**} = \overline{s}$. The first order condition allows us to find the optimal product quality:

$$\frac{\partial \Pi_1^*}{\partial s_1} = \frac{\overline{\theta}^2 \overline{s}^2}{(4\overline{s} - s_1)^3} (4\overline{s} - 7s_1) - 2ks_1 = 0 \tag{14}$$

Using the following substitution: $s_1 = \mu\overline{s}$ , we obtain the equation of degree four in $\mu$:

$$2k\overline{s}^3\mu^4 - 24k\overline{s}^3\mu^3 + 96k\overline{s}^3\mu^2 - \left(7\overline{\theta}^2\overline{s}^2 + 128k\overline{s}^3\right)\mu + 4\overline{\theta}^2\overline{s}^2 = 0 \tag{15}$$

This quadric equation is solved numerically. We assume that we know maximum prices that consumers with the lowest and highest inclination to quality are ready to pay for products of the highest quality – $\overline{s}$ .
Thus, we receive the highest possible quality that both firms can produce. Table 4 presents the initial data for our problem.

**Table4.** Initial data.

| $\underline{\theta}\overline{s}$ | $\overline{\theta}\overline{s}$ | $\overline{s}$ |
|---|---|---|
| 600 | 720 | 120 |
| 655 | 752 | 97 |
| 700 | 800 | 100 |
| 755 | 930 | 175 |
| 780 | 930 | 150 |

**Table5.** Coefficients of equation and its solution.

| $\mu^4$ | $\mu^3$ | $\mu^2$ | $\mu^1$ | $\mu^0$ | $\mu$ |
|---|---|---|---|---|---|
| 69120 | -829440 | 3317760 | -8052480 | 2073600 | 0,2896 |
| 36507 | -438083 | 1752332 | -6294971 | 2262016 | 0,3995 |
| 40000 | -480000 | 1920000 | -7040000 | 2560000 | 0,4038 |
| 214375 | -2572500 | 10290000 | -19774300 | 3459600 | 0,1935 |
| 135000 | -1620000 | 6480000 | -14694300 | 3459600 | 0,2642 |

**Table6.** Results.

| $s_1^{**}$ | $s_2^{**}$ | $\Pi_1^*$ | $\Pi_2^*$ |
|---|---|---|---|
| 34,752 | 120 | 40,40304 | 603,6722 |
| 38,7515 | 97 | 77,85318 | 892,0378 |
| 40,38 | 100 | 86,52719 | 980,1692 |
| 33,8625 | 175 | 30,29724 | 487,8746 |
| 39,63 | 150 | 48,90488 | 765,9821 |

The coefficients at all powers of $\mu$ in the equation, that we receive using the initial data given above, are presented in the Table 5 below.
So, Table 6 presents the problem solution, i.e. the optimal product quality for our initial data.

4. If both values of optimal product qualities of the firms-competitors don't belong to to the quality range interval, i.e. $s_1^* < \underline{s}, s_2^* > \overline{s}$ then

$$\begin{cases} s_1^{**} = \underline{s} \\ s_2^* = \overline{s} \end{cases} \tag{16}$$

Thus we have the case of maximum differentiation. The profit functions in equilibrium are:

$$\begin{cases} \Pi_1^*(s) = \Pi_1^*(s^*) = \dfrac{\overline{\theta}^2 \overline{s}\underline{s}\,(\overline{s} - \underline{s})}{(4\overline{s} - \underline{s})^2} - k\underline{s}^2 \\ \\ \Pi_2^* = \Pi_2^*(s^*) = \dfrac{4\overline{\theta}^2 \overline{s}^2\,(\overline{s} - \underline{s})}{(4\overline{s} - \underline{s})^2} - k\overline{s}^2 \end{cases} \tag{17}$$

5. If $s_1^*, s_2^* < \underline{s}$, then $\Pi_i^* < 0$ , which is infeasible.

The results of the modeling shows us that company's decision on product quality strongly depends on the initial market conditions, which defines the model's parameters.

## 3. Non-Uniform Consumer Distribution

In this section we consider a model for a vertically differentiated product when consumers are distributed non-uniformly over investigated market.

Assume that there are two firms $i = 1, 2$ which produce products of quality $s_i \in [\underline{s}, \overline{s}]$ at a cost independent of $s_i$, i.e. $c(s_i) = C = const$. Suppose, that $s_2 > s_1$.

As in the section 2 the utility function of the consumer with inclination to quality $\theta$ when buying the product of quality $s$ for price $p$ is:

$$U_\theta(p) = \begin{cases} \theta s - p, p \le \theta s \\ 0, p > \theta s \end{cases} \tag{18}$$

But here it is assumed that parameter $\theta \in [0,1]$ which indicates the willingness to pay for quality is distributed over the population according to a continuous symmetric triangular density $f(\theta)$.

Generally the density $f(\theta)$ is defined as follows (see also Benassi et al., 2006):

$$f(\theta) = \begin{cases} 4\theta, & \text{for } \theta \in A = [0, \frac{1}{2}) \\ 2, & \text{for } \theta = \frac{1}{2} \\ 4(1-\theta), & \text{for } \theta \in B = (\frac{1}{2}, 1] \end{cases}$$

Then distribution function can be presented as follows:

$$F(\theta) = \begin{cases} 2\theta^2, & \text{for } \theta \in A = [0, \frac{1}{2}) \\ 2\theta, & \text{for } \theta = \frac{1}{2} \\ 4\theta - 2\theta^2, & \text{for } \theta \in B = (\frac{1}{2}, 1] \end{cases}$$

Figure 1 represents the view of the density function $f(\theta)$.

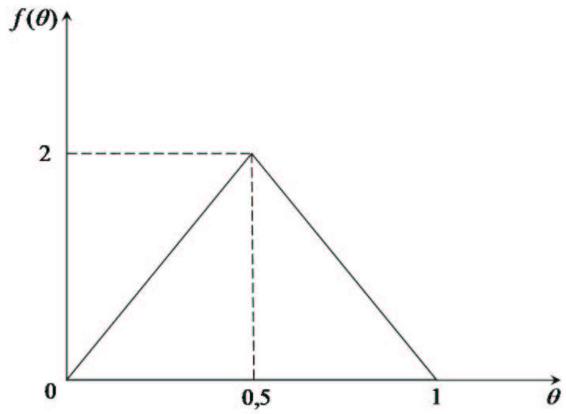


**Figure1.** The density function $f(\theta)$

First, let us introduce the following standard notations:

$$\theta_1 = \frac{p_1}{s_1};$$

- the marginal consumer who is indifferent between purchasing the lower-quality product, the first firm's product, and nothing at all.

$$\theta_2 = \frac{p_2 - p_1}{s_2 - s_1}.$$

-the marginal consumer who is indifferent between purchasing higher-quality product, the second firm's product, and lower-quality product, the first firm's product.

Each firm's demand can be written as follows:

$$D_1(p, s) = D_1(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} f(\theta) d\theta = F(\theta_2) - F(\theta_1); \tag{19}$$

$$D_2(p, s) = D_2(\theta_2) = \int_{\theta_2}^{1} f(\theta) d\theta = 1 - F(\theta_2). \tag{20}$$

where $F(\theta) = \int_0^\theta f(z)dz$ is the distribution corresponding to the density $f(z)$.

Each firm's profit of producing a product of quality $s_i$ , where $s_i \in [\underline{s}, \overline{s}]$ , is defined as follows:

$$\Pi_i(p, s) = p_i(s) D_i(p, s) - C, i = 1, 2.$$

where $p_i$ – product price of the firm $i$, $s_i$ – quality of the firm $i$'s product, $p = (p_1, p_2)$ – a vector of product prices of the competing firms, $s = (s_1, s_2)$ – a vector of product qualities, $D_i(p, s)$ – the demand function for the product of quality $s_i$ , $C$ – constant production costs.

The goal of each firm is its profit maximization, when at the first stage firms set the product quality and at the second stage they compete in product prices.

According to our proposition about the form of the density function, the explicit formulation of demand functions (19 and 20) differs depending on the value of the limits of integration. Therefore, demand functions view depends on the location of marginal consumers across the interval [0,1]. Thus, three co-locations of $\theta_1, \theta_2$ are possible:

1. $\theta_1, \theta_2 \in A$;
2. $\theta_1, \theta_2 \in B$;
3. $\theta_1 \in A, \theta_2 \in B$.

The proposition (Benassi et al., 2006) below allows us to cut the list of possible options to one case.

**Proposition 1.** *Consider any concave symmetric density $f(\theta)$ defined over $[0, 1]$, such that $f(0) = f(1) = 0$ and $f(1/2) \geq 1$. If $(\theta_1^*, \theta_2^*), \theta_2^* > \theta_1^*$ identifies the marginal consumers at the perfect Nash equilibrium in the two-stage game for vertical differentiated products, then $\theta_2^*$ is lower than the median of the distribution.*

*Proof.* Consider the derivative of the second firm profit with respect to its product price and equal it to zero according to the first order condition:

$$\frac{\partial \Pi_2}{\partial (p_2)} = 1 - F(\theta_2) - \frac{p_2}{s_2 - s_1} f(\theta_2) = 0.$$

Thus,

$$1 - F(\theta_2) = b(\theta_2).$$

where $b(\theta_2) = (a + \theta_2) f(\theta_2)$, $a = \dfrac{p_1}{s_2 - s_1} > 0$.

Notice some tendencies and properties of the equation above:

1. $1 - F(\theta_2)$ is a strictly decreasing from 1 to 0.
2. $b(\theta_2)$ is increasing till $f^{'}(\theta_2) \geq 0$ because $b^{'}(\theta_2) = f(\theta_2) + (\theta_2 + a)f^{'}(\theta_2)$.
   As $b(\theta_2)$ is a continuous function such as $b(1) = b(0) = 0$, then the maximum of the function $b(\theta_2)$ exists in some point $\widehat{\theta}_2$.
   When $f^{'}(\theta_2) \geq 0$ $\left(\text{which is true for sure for all } \theta_2 < \frac{1}{2}\right)$, then $b^{'}(\theta_2) > 0$.
   Thus, $\widehat{\theta}_2 > \frac{1}{2}$.
3. $b\left(\frac{1}{2}\right) = \left(a + \frac{1}{2}\right) f\left(\frac{1}{2}\right) > \frac{1}{2} = 1 - F\left(\frac{1}{2}\right)$
   This is equivalent to: $b\left(\frac{1}{2}\right) > 1 - F\left(\frac{1}{2}\right)$.
   As well $b(0) = af(0) = 0 < 1 - F(0) = 1$, which leads us to $b(0) < 1 - F(0)$.
   Thus, $\theta_2^* < \frac{1}{2}$.

First two observations allow us to conclude that there is unique maximum $\theta_2^* \in [0, \widehat{\theta}_2]$.

Now, we consider the interval $\theta_2 \in [\widehat{\theta}_2, 1]$ and show that there no extreme points there.

Let us introduce the function $\varphi(\theta_2) = 1 - F(\theta_2) - b(\theta_2)$.

– When $\theta_2 = \widehat{\theta}_2$, investigated function takes negative values $\varphi\left(\widehat{\theta}_2\right) < 0$.
– $\varphi^{'}(\theta_2) = -2f(\theta_2) - (a + \theta_2)f^{'}(\theta_2)$.
  And $\varphi\left(\widehat{\theta}_2\right) = -2f\left(\widehat{\theta}_2\right) - \left(a + \widehat{\theta}_2\right) f^{'}\left(\widehat{\theta}_2\right) = -f\left(\widehat{\theta}_2\right) - b^{'}\left(\widehat{\theta}_2\right) < 0$ (as $b^{'}\left(\widehat{\theta}_2\right) = 0$).
– As density function $f$ is decreasing in $\left[\widehat{\theta}_2, 1\right]$ and convex, then $f^{'}(\theta_2)$ is decreasing. Therefore, $\varphi^{'}(\theta_2)$ is increasing function.
– Taking into consideration that $f(1) = 0$ and $f^{'}(1) < 0$, we obtain that $\varphi^{'}(1) > 0$.
  As $\varphi^{'}(\theta_2)$ is increasing function, then $\varphi^{'} = 0$ in the unique point $\widetilde{\theta}_2$.
  $\varphi(1) = 0 \Longrightarrow \varphi(\theta)_2 < 0$ when $\widehat{\theta}_2 \leq \theta_2 \leq 1 \Longrightarrow 1 - F(\theta_2) < b(\theta_2)$ for any $\theta_2 \in \left[\widehat{\theta}_2, 1\right]$

Thus, we proved the proposition.

Therefore, to solve the problem only the first co-location of $\theta_1, \theta_2$ is possible. Thus, one case with explicit game formulation is considered.

Here, demand functions can be rewritten as follows:

$$D_1(p, s) = 2\left(\frac{p_2 - p_1}{s_2 - s_1}\right)^2 - 2\left(\frac{p_1}{s_1}\right)^2;$$

$$D_2(p,s) = 1 - 2\left(\frac{p_2 - p_1}{s_2 - s_1}\right)^2.$$

And profit functions of the investigated firms while producing goods of quality $s_i$ are:

$$\Pi_1(p,s) = p_1\left(2\left(\frac{p_2 - p_1}{s_2 - s_1}\right)^2 - 2\left(\frac{p_1}{s_1}\right)^2\right) - C; \tag{21}$$

$$\Pi_2(p,s) = p_2\left(1 - 2\left(\frac{p_2 - p_1}{s_2 - s_1}\right)^2\right) - C. \tag{22}$$

The sub-game perfect Nash equilibrium is obtained using the backward induction.

Suppose first, that firms have chosen their product quality and compete by its prices knowing the quality choice of their competitor.

The first order condition is presented below:

$$\begin{cases} \dfrac{\partial \Pi_1}{\partial p_1} = 2\left(\dfrac{p_2 - p_1}{s_2 - s_1}\right)^2 - 2\left(\dfrac{p_1}{s_1}\right)^2 - p_1\left(\dfrac{4(p_2 - p_1)}{(s_2 - s_1)^2} - \dfrac{4p_1}{s_1^2}\right) = 0 \\ \dfrac{\partial \Pi_2}{\partial p_2} = 1 - 2\left(\dfrac{p_2 - p_1}{s_2 - s_1}\right)^2 - \dfrac{4p_2(p_2 - p_1)}{(s_2 - s_1)^2} = 0 \end{cases} \tag{23}$$

Thus, the solution of the system is:

$$p_1^*(s) = \frac{s_1\sqrt{6}}{6}\frac{z - 3}{\sqrt{(3z - 1)(z - 3)}}; \tag{24}$$

$$p_2^*(s) = \frac{s_1\sqrt{6}}{6}\frac{z(z - 3)}{\sqrt{(3z - 1)(z - 3)}}. \tag{25}$$

where $z = 2 + \sqrt{1 + \dfrac{3\left(s_2 - s_1\right)^2}{s_1^2}}$

Checking the second order condition, we see that both second derivatives are negative in equilibrium:

$$\begin{cases} \dfrac{\partial \Pi_1^2}{\partial p_1^2}\left(p_1^*, p_2^*\right) < 0 \\ \dfrac{\partial \Pi_2^2}{\partial p_2^2}\left(p_1^*, p_2^*\right) < 0 \end{cases} \tag{26}$$

Therefore, we showed that the solutions (24 and 25) are prices in equilibrium.

Let us move to the second stage of the game - the choice of product quality.

By substituting equilibrium prices (24 and 25) into the profit functions (21 and 22) correspondingly, we get:

$$\Pi_1^*(s) = \frac{\sqrt{6}}{3} \cdot \frac{\left(3(s_2 - s_1)^2 + s_1(z-3)\right)^2}{\sqrt{(3z-1)(z-3)}^3} - C; \tag{27}$$

and

$$\Pi_2^*(s) = \frac{\sqrt{6}}{9} \cdot \frac{s_1(z-3)\left(3(s_2 - s_1)^2 + s_1(z-3)\right)}{\sqrt{(3z-1)(z-3)}^3} - C. \tag{28}$$

It can be easily checked that the second firm's profit is always increasing in its product quality, so we can confirm that the second firm chooses the maximum feasible product quality from the interval $[\underline{s}, \overline{s}]$, which means

$$s_2^* = \overline{s}.$$

Substituting this result to the first order condition of the first firm and solving the equation in $s_1$, we get that the first firm maximizes its profits by setting

$$s_1^* = 0,4954\overline{s}.$$

Therefore evaluated at the optimal quality the prices of the two firms products are:

$$p_1^* = 0,24806\overline{s};$$

and

$$p_2^* = 0,061592\overline{s}.$$

Generalizing the results of the investigation we should mention that optimal product quality of the firm which produce a higher-quality product is the highest possible product quality $\overline{s}$. The optimal product quality of another firm varies depending on the market conditions.

## 4. Results of Comparison

So, the paper presents the results of investigation of two game-theoretical models, which are the models of quality competition for vertically differentiated products: when consumers are distributed uniformly and non-uniformly over some industrial market. In our research we focused on the investigation of uncovered market case which is a more realistic one. Moreover, in this case consumer distribution influences not only the demand shares but also the size of market demand.

For each proposed model the unique sub-game perfect Nash equilibrium is received.

We consider two critical cases when the distribution is uniform and when it is triangular. The research shows that in the second case the market coverage is higher.

Besides, high quality goods production brings the higher profit in all cases for all values of distribution parameters.

Therefore, our investigation has proven an idea that a more concentrated inclination to quality leads to the extended market coverage and increases the profit of the firm which produces higher-quality goods.

## 5. Conclusion

To summarize the results of the investigation it should be mentioned that the solution of game-theoretical models is obtained for all reasonable values of initial parameters for the uncovered market case. On the basis of the outcome of the research it is possible to evaluate quantitatively the following parameters in case of industrial competition for vertically differentiated product:

– product quality estimation for each competing firms;
– product price in competition;
– demand faced by competing firms in equilibrium;
– firms profit in equilibrium.

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# Optimal Hierarchies in Firms: a Theoretical Model*

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**Abstract** A normative economic model of management hierarchy design in firms is proposed. The management hierarchy is sought to minimize the running costs. Along with direct maintenance expenses these costs include wastes from the loss of control. The results comprise the analytic expressions for the optimal hierarchy attributes: span of control, headcount, efforts distribution, wages differential, etc, as functions of exogenous parameters. They allow analyzing the impact of environment parameters on a firm's size, financial results, employees' wages and shape of hierarchy. The detailed analysis of this impact can help drawing up policy recommendations on rational bureaucracy formation in firms.

**Keywords:** organizational structure, optimal hierarchy, manager, effort.

## 1. Introduction

The notion of transaction costs (or "economic system exploiting costs") forms the basis of neoinstitutional economic theory and the modern theory of the firm. As O. E. Williamson (1975) notes, economizing on transaction costs is the main goal of any economic institution. The internal structure of modern firms usually takes the form of management hierarchy. Transaction costs are produced inside the hierarchy and greatly influenced by its shape and other attributes. At present the attributes of management hierarchy are universally recognized to exert key influence on the effectiveness of management (H. Mintzberg (1983)). Thus, the analysis of management hierarchies (organization structures) gives clues to deeper understanding of the nature of the firm.

Interest in normative models of management hierarchies increases in the time of context of the continuing processes of business globalization (mergers, absorptions, vertical and horizontal integration). The crucial problem of huge modern corporations is the rational organization of their bureaucracy. Severe competition for the global markets makes not only the financial results but the very existence of a corporation dependent on the efficiency of its management structure. Increasing pace of change in production and management technologies, financial turmoils require fast and adequate changes in the organization structure of a firm, and normative models of a hierarchy design must provide the aid in the solution of these sophisticated problems.

In this paper the transaction costs approach is combined with the original mathematical results in an optimal hierarchy design ( S. P. Mishin (2004); M. V. Goubko (2006)) to formulate and study the models of multi-layer management hierarchies.

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Along with "direct" maintenance expenses (salaries, bonuses, options, office rents, stationary, etc) due to the management staff, transaction costs in this model also include wastes from the so-called "loss of control" (O. E. Williamson (1967)). The questions addressed by the model are: how many managers the firm must hire, when headcount should be increased or decreased, how managers wages depend on their positions, whether corporate information systems implementation results in a flatter management hierarchy, when the growth of the firm is advantageous, etc.

## 2. Literature Review

Since the beginning of 20th century transaction costs have become the central point of a new approach to the theory of the firm. Market mechanisms were recognized to lead to costs allowing the rationality of alternative forms of institutions. The main topics of interest were "When do markets fail? What alternative modes of organization are available? What are the limits of these alternative modes?" Answering these questions was focused on the main alternative organization form – the management hierarchy – and demanded the advanced modeling of the internal structure of a firm.

One of the early formal models of intra-firm hierarchy was introduced by M. J. Beckmann (1960). He limits administrative costs to managerial wages. Imposing restrictions on the minimum span of control (the number of immediate subordinates of a manager) and the maximum wage differential between subsequent layers of hierarchy he proves that administrative costs rise approximately linearly with the firm growth. So, he concludes, these costs cannot limit the maximum size of a firm.

Later in his famous article O. E. Williamson (1967) introduces the important notion of loss of control. He argues that in real world the efficiency of a manager's control is limited by natural bounds of human attention and communication. It is claimed that only some fraction $\alpha < 1$ of a manager's orders and directions can be successfully implemented by his subordinates. Williamson supposes the output of any productive worker to be directly governed by the cumulative loss of control through the chain of command – the chain of managers "above" this worker. If hierarchy has $l$ layers of managers then the output $y$ of any worker is defined as $y = x \cdot \alpha^l$, where $x$ is some constant. Assuming constant span of control and wage differential Williamson shows that the loss of control makes the revenue of a firm concave in its size and results in a finite optimal firm size even with linear (in size) administrative costs.

Relying on this approach G. A. Calvo and S. Wellisz (1978) proposed the model of hierarchical monitoring. They internalized $\alpha$ interpreting it as an employee effort (e.g. time the manager engages in monitoring or worker spends in production). The only task of a manager is the control of his immediate subordinates' efforts that, in turn, depend on the quality of monitoring (it increases in the manager effort and decreases in his span of control). The employee effort is proved to be a decreasing function of the span of control of his immediate superior and to be unimodal on wage. The authors introduce two models. In the first, the loss of control is not cumulative, thus $y \sim \alpha_1$, where $y$ is the output of a productive worker and $\alpha_1$ is his effort. In the second, the loss of control is cumulative and the output $y$ of any worker is proportional to the product of efforts in the whole chain of command: $y \sim \alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_l$. G. Calvo and S. Welliz were the first to set the formal problem of optimal hierarchy design: to determine the number of productive workers, the number of layers in a hierarchy, the span of control and the wage for every layer to maximize the profit of a firm. The authors do not solve this problem explicitly but prove that firm's ability to grow crucially depends on the details of the monitoring mechanism in use.

Y. Qian in (Y. Qian (1994)) employs the hierarchy design technique developed by M. Keren and D. Levhari (1983) to analyze the model where managers in a hierarchy engage both in monitoring and in production activities. Among the other results Y. Qian shows that the optimal employee's wage and effort level rise from the bottom to the top of a hierarchy, and the optimal span of control is always greater than $e \approx 2.71$. He also proves the profit of a firm to be a concave increasing function of a firm size. Thus, Y. Qian agrees with M. Beckmann in that the loss of control in a management hierarchy cannot limit the growth of a firm.

Another model of hierarchical authority and control was introduced by S. Rosen (1982). He incorporates the labor market (the market of managerial skills) into the model of hierarchy design. The goal is to describe an equilibrium distribution of firms by their size along with explaining the market mechanisms for manager wages formation. The distinctive feature of his model is that every potential employee has a unique vector of skills that influences his effectiveness as a productive worker, a first-layer manager, a second-layer manager, etc. For the special case of two-layer firms with constant returns technology S. Rosen finds the equilibrium prices for the worker and manager skills. He shows that in equilibrium more able managers govern the firms of a greater size. The model of S. Rosen also justifies the power relation $C = S^{\gamma}$ between the size $S$ of a firm and the compensation $C$ of its top-manager. The presented power relation is supported well by the extensive empirical literature on top-managers wages (see, for instance D. H. Ciscel and D. M. Carroll (1980)).

Among the recent publications on optimal hierarchy design one can also mention K. J Meagher (2003) and A. Patacconi (2005). The other approaches to the optimal hierarchy problem (see the survey in M. V. Goubko (2006)) include "knowledge hierarchies" of L. Garicano and A.W. Beggs, "computer science approach" of R. Radner, T. Van Zandt, P. Bolton and M. Dewatripont, "teams theory approach" of J. Cremer, M. Aoki, J. Geanakoplos and P. Milgrom, "decision-making hierarchies" of R.K. Sah and J.E. Stiglitz, and contracts theory approach.

## 3. The Model

Define a productive technology of a firm. A manufacturing firm chooses what to produce from a set of final products (goods or services). A production technology for every product $p$ requires a set $N(p)$ of productive workers. Assume every product requires a distinct set of workers, so $N$ can be used as a synonym of a final product. Consider a single-product firm that can choose the only product at a moment.

The revenue function $R(N, z)$ depends on the product $N$ and its output volume $z$. No matter what product and volume the firm chooses, it bears two types of costs. The first are product-specific costs that do not depend on the internal structure of a firm (these could be raw material costs, marketing expenses, etc). The second are structure-specific costs that depend both on the product $N$ and on how the firm has organized the production (e.g. employees' wages). Since the point of this paper is the internal structure problem, suppose the product-specific costs are already accounted for in the revenue function $R(\cdot)$.

The simplest revenue function usually employed in the literature is a linear one: $R(N, z) = \pi(N) \cdot z$ (the firm buys raw materials and sells a final product at a constant price). In our model a bit more complicated revenue function is adopted: $R(N, z) = \pi(N) \cdot \ln(a(N) \cdot z)$ where $a(N)$ and $\pi(N)$ are some product-specific parameters. This function is concave in output and captures the narrowness of market for any given product. In general, the shape of a revenue function may be more complicated but, as is shown below, the logarithmic relation simplifies much the formal analysis.

Now describe a product $N$ manufacturing technology. In the literature (O.E. Williamson (1967); G. A. Calvo and S. Wellisz (1978); S. Rosen (1982); Y. Qian (1994)) every worker $w \in N$ is usually assumed to produce a uniform output $z_w$, so the total output $z$ is just a sum: $z = \sum_{w \in N} z_w$. This approach ignores the complementarity of employees' contributions. At the same time such complementarity is universally recognized (see P. R. Milgrom and J. Roberts (1992)) to be the main reason for the existence of firms per se. In contrast, we adopt here an extreme case of very strong complementarity – the Leontief technology $z = \min_{w \in N} z_w$. It supposes every worker to provide a single unit of local product for a single unit of final product to be produced (the units of measure for the local outputs are assumed to be chosen accordingly). "Local" outputs are non-substitutable.

Allow planning in a firm to be highly centralized, i.e. the principal (the owner of the firm) chooses the plan of production $x$ to be executed by the firm. However, the worker $w \in N$ affects his output $z_w$ by choosing the effort level $\xi_w \in [0, 1]$ (non-maximal effort $\xi_w < 1$ means some degree of shirking). Workers' effort levels are not directly observed by the principal, so, monitoring is required to build effective incentive schemes for workers (see the discussion in G. A. Calvo and S. Wellisz (1978); Y. Qian (1994)). This monitoring task is due to the managerial hierarchy built over the set of workers. This hierarchy is modeled by a directed tree, with productive workers being its leaves, managers being its intermediate nodes, and the top-manager being its root, while the edges showing subordination.

Informally, the problem set is that of the principal – to choose the product $N$, the plan $x$, and to organize the efficient execution of this plan, i.e. to find out how many managers to hire (including the top-manager), how to subordinate both workers to managers and managers to higher-layer managers in order to obtain better efforts (and thus, the output) at a lower cost.

Denote a set of managers in a hierarchy by $M$. Every manager has a set of immediate subordinates (they could be workers or other managers). Suppose a manager $m \in M$ has $k$ immediate subordinates. Then denote $(\xi_j(m))_{j=1,\ldots,k}$ the vector of manager's $m$ efforts ($\xi_j(m) \geq 0, j = 1, \ldots, k$), $j$-th component being the effort referred to monitoring and control of $j$-th immediate subordinate. Along with a monitoring function the manager's effort plays an immediate role in production. It acts upon the output of all the workers who are directly or indirectly (through the chain of managers) controlled by the manager. So if the worker $w \in N$ chooses the effort level $\xi_w$, his immediate superior $m_1$ chooses the effort level $\xi_1$ to control the worker $w$, manager's $m_1$ superior $m_2$ chooses the effort level $\xi_2$ to control $m_1$, and so on up to the top-manager who chooses the effort level $\xi_l$, then the output of worker $w$ is given by $z_w = x \cdot \xi_w \cdot \xi_1 \cdot \xi_2 \cdot \cdots \cdot \xi_l$. [1]

[1] This is the formula of "cumulative loss-of-control" technology discussed in O. E. Williamson (1967); G. A. Calvo and S. Wellisz (1978); S. Rosen (1982);

Now introduce the utility functions of employees. A productive worker $w \in N$ seeks to maximize the difference $u_w = \sigma_w - c(x, \xi_w)$ between his wage $\sigma_w$ and the cost function $c(x, \xi_w)$ that depends both on a plan (what the worker is expected to do) and the worker's effort level. Such cost function arises naturally as an individual rationality constraint in the presence of labor market – both the worker and the principal know well how high certain responsibilities (plan $x$) and effort levels are valued by market. Similarly, every manager $m \in M$ maximizes the difference $u_m = \sigma_m - K(m, H)$, where $K(m, H)$ is the cost of maintaining a manager $m$ in hierarchy $H$.

Costs $K(m, H)$ of a manager $m$ may depend both on his position in hierarchy $H$ and on the effort levels he exerts. Consider the manager $m$ governing (directly or indirectly) a group of workers $s \in N$. Suppose the manager $m$ has $k$ immediate subordinates that govern groups of workers $s_1, \ldots, s_k$ ($s = s_1 \bigcup \cdots \bigcup s_k$) and the manager $m$ has chosen the vector of efforts $\xi = (\xi_1, \ldots, \xi_k)$ to control them. The costs of the manager $m$ may depend both on the set $s$ (the larger is the group under control, the more complicated is the task of the manager) and the planned production volume $x$ (the control of the execution of a more ambitious plan requires more efforts and costs). The costs must also depend on the span of control $k$ (it can be very costly to directly control, for example, 1000 immediate subordinates). Also allow the costs of a manager to depend on how the group s is divided among his immediate subordinates. At the end, the costs must increase in manager's efforts. So one can write

$$K(m, H) = K(x, s_1, \ldots, s_k, \xi_1, \ldots, \xi_k).$$[2]

Take for simplicity a special form of a manager cost function, one of that allowing complete analytic calculation of optimal hierarchy attributes. For an arbitrary group of workers $s$ define its measure by $\mu_s = x|s|$ (it increases both in plan $x$ and in group's size $|s|$). It is implied below that the costs of the manager depend on the groups $s_1, \ldots, s_k$ measures rather that on the groups itself. Consider the constant elasticity of substitution cost function (see D. McFadden (1963)):

$$K(m, H) = K(\mu_1, \ldots, \mu_k, \xi_1, \ldots, \xi_k) = \left( \sum_{i=1}^{k} \mu_i^{\lambda} \cdot (-\ln \xi_i)^{-\delta} \right)^{\epsilon},$$

where $\lambda \in [0, 1]$, $\delta \in [0, 1]$, and $\epsilon \in [0, +\infty)$ are parameters (product-dependent in general).

This function satisfies the monotonicity conditions specified hereinabove. Also note the cost approaches infinity as any effort $\xi_i$ tends to the unity. This implies the impossibility of "total control". The parameter $\epsilon$ accounts for the cost function elasticity with respect to the workload $\sum_{i=1}^{k} \mu_i^{\alpha} \cdot (-\ln \xi_i)^{-\delta}$. One can think of $1/\epsilon$ as of

Y. Qian (1994). So the same argumentation may be used to justify it (imperfect communication in O. E. Williamson (1967), a specific monitoring mechanism in G. A. Calvo and S. Wellisz (1978), an intermediate "managerial" product in Y. Qian (1994); S. Rosen (1982)). In general, the monitoring effort of the manager may differ from his "productive" effort. Nevertheless, we assume them to be the same (see S. Rosen (1982) for detailed reasoning).

[2] As $s = s_1 \bigcup, \ldots, \bigcup s_k$, the whole group $s$ is accounted here. The function changes as the span of control $k$ changes, so $k$ is also accounted for in this notation.

the manager effectiveness measure. The parameter $\lambda$ describes the elasticity of workload with respect to the size of the group under control. In (S. P. Mishin (2004); M. V. Goubko (2006)) $\lambda$ is interpreted as a degree of standardization of management information in a firm – the less $\lambda$ is, the more standardized the manager's work is, thus the manager's workload increases slower in the size of a unit (problems in big units become "typical"). Lastly, $\delta$ accounts for the workload elasticity with respect to the managerial effort. Parameters $\lambda$, $\epsilon$, and $\delta$ may also be influenced by alternative economic factors.

The wages for all employees are set centrally by the principal on the basis of information elicited from monitoring, so the wage of an employee depends on his observed effort. In general, monitoring may be imperfect so the effectiveness of an incentive scheme $\sigma$ for an employee may depend on the degree of monitoring inaccuracy. Herein the case of perfect monitoring is considered, i.e. managers elicit true efforts of their immediate subordinates and pass this information to the principal with no distortion. Although not benevolent, managers do not distort the information (in non-cooperative framework), as their compensation does not depend on their reports, but solely depends on their own efforts reported by their immediate superiors. Top-manager is monitored directly by the principal at no cost.

Therefore, the principal faces a set of separate principal-agent incentive problems with perfect information. It is known (see A. Mas-Collel et al (1995); D. A. Novikov and S. N. Petrakov (1999)) that in such setting an optimal incentive scheme gives a zero payment for all but one efforts vector where the compensation is equal to employee's cost. Thus, the principal can gain any efforts from employees by just compensating for their costs. So the principal merely balances the output (revenue) and the total costs of the employees.

Now the optimal organization design problem can be stated formally: to choose the set of workers (product) $N$, the plan $x$, the hierarchy of managers $H$, and the effort levels for every manager and productive worker to maximize the profit

$$F = R(N, z) - \sum_{w \in N} c(x, \xi_w) - \sum_{m \in M} K(m, H).$$

## 4. The Results

For the stylized setting defined above one can completely solve the optimal hierarchy problem. The Leontief technology along with the monotonicity of costs with respect to efforts implies the equality of the local outputs $z_w$ ($w \in N$) in optimal hierarchy. The logarithmic revenue function then enables additive decomposition of the managers' contributions to the profit[3], so every manager's effort can be optimized separately[4]. Denote for short

$$\alpha := \frac{\lambda + \delta}{1 + \delta}, \quad \beta := \frac{\epsilon(1 + \delta)}{1 + \epsilon\delta}, \quad \tau := \frac{1}{1 + \epsilon\delta}, \quad n := |N|.$$

[3] I.e. the profit $F$ can be represented as a sum $\sum_{m \in M} f_m(\cdot) + \sum_{w \in N} f_w(\cdot)$, where manager's $m$ contribution $f_m$ depends on the plan $x$, manager's efforts vector, and his position in a hierarchy, and the worker's $w$ contribution $f_w$ depends on the plan $x$, and the effort $\xi_w$.

[4] Similar decomposition approach is used by J. Geanakoplos and P. Milgrom (1991) in their analysis of hierarchical planning with bilinear production costs.

The following result states the optimal effort levels of a manager that the principal must elicit.

**Lemma 1.** *If a manager $m$ in some hierarchy controls groups of workers with measures $\mu_1, \dots, \mu_k$, then his optimal efforts vector $(\xi_1, \dots, \xi_k)$ and the maximal contribution are given by*

$$\xi_i = \exp\left[-\left(\frac{xn(1-\tau)}{\pi(N)\tau}\right)^{\tau} \cdot \mu_i^{\alpha-1}\left(\sum_{i=1}^{k}\mu_i^{\alpha}\right)^{\beta-1}\right], i = 1, \dots, k, \tag{1}$$

$$f_m^{max} = -\frac{1}{\tau}\left(\frac{\pi(N)\tau}{nx(1-\tau)}\right)^{1-\tau}\left(\sum_{i=1}^{k}\mu_i^{\alpha}\right)^{\beta}. \tag{2}$$

See the appendix 1 for the proof.

It is easy to see from (2) that manager's contribution is negative as in the model the managers are just the source of costs. Thus one can speak about the *cost of manager's maintenance* $K^*(m, H) := -f_m^{max}$ that consists of both the manager's compensation and wastes from the loss of control he generates.

The cost of maintenance depends only on manager's position in a hierarchy. Namely, the measures $\mu_1, \dots \mu_k$ of groups controlled by $k$ immediate subordinates of a manager $m$ determine manager's $m$ costs. The costs obey *constant elasticity* with respect to the size of a unit under a manager's control. This means that if all $k$ measures $\mu_1, \dots \mu_k$ are multiplied by any positive number $A$, then the cost $K^*(m, H)$ is multiplied by $A^{\gamma}$ where $\gamma$ is some constant. From (2) one can easily find that $\gamma = \alpha\beta$.

The next step of the solution is to find the shape of optimal hierarchy. This requires choosing the management hierarchy to minimize the total maintenance costs of the managers it consists of. In general it is an extremely complex discrete optimization problem. But, fortunately, for the case of constant elasticity cost functions it has a closed form solution developed in (M. V. Goubko (2006)).

The optimal hierarchy is shown there to be *uniform*, i.e. every manager in a hierarchy has the same span of control and seeks to break the subordinated group of workers to pieces in the same proportion from the viewpoint of their measures[5].

The cost function (2) is studied in detail in (M. V. Goubko (2006)). The optimal hierarchy is shown to be symmetric (i.e. every manager seeks to divide the subordinate group of workers equally among his immediate subordinates)[6].

The optimal span of control $r$ is then determined as[7]

$$r = \left[\frac{\beta(1-\alpha)}{\beta-1}\right]^{\frac{1}{1-\alpha\beta}}, \tag{3}$$

[5] Note that the purely uniform hierarchy may not exist due to the finiteness of the set $N$. But, as proven in (M. V. Goubko (2006)), in any case, the optimal hierarchy is "roughly uniform" and the analytic formula for the uniform hierarchy costs is a good estimate for the costs of the optimal hierarchy in a big organization.

[6] The symmetry of the optimal hierarchy may seem obvious but surprisingly it holds only for a certain range of model parameters (fortunately, the most interesting one).

[7] This formula presents the estimate span of control. Real optimal span of control is one of the two nearest integers.

while the estimate of optimal hierarchy maintenance costs is given by the following expression (for the most common case when $\alpha\beta \neq 1$):

$$\sum_{m \in M} K^*(\cdot) = (nx)^{-(1-\tau)}[nx^{\alpha\beta} - (nx)^{\alpha\beta}] \cdot \pi(N)^{1-\tau}\tau^{-\tau}(1-\tau)^{-(1-\tau)}\frac{r^\beta}{r - r^{\alpha\beta}}. \quad (4)$$

See the Appendix 2 for the proof of these formulas.

Thus, given the product $N$, a good estimate for the management headcount is $|M| = (n-1)/(r-1)$. Note that it does not depend on the value of plan $x$ and is linear with respect to the number $n$ of productive workers. Thus, changes in plans and work intensity does not require change of of management hierarchy shape. With the extensional growth of production (the number of workers) the bureaucracy goes up proportionally.

Having found the optimal span of control and the managers effort levels[8] one can analytically write down the expression for the profit $F(N, x)$ of a firm as a function of the product $N$ and the plan $x$ (the formula is omitted for short). Thus, planning of $N$ and $x$ becomes a standard optimization problem.

Also one can obtain some comparative static results on how the span of control, managers headcount, salary and efforts distribution depend on the model parameters (the degree of standardization $1/\lambda$ and the managers' ability $1/\epsilon$).

Simple calculations give a surprising result: the optimal span of control increases with the decrease of standardization – the less standard problems do managers solve, the less managers the firm must have. The explanation of such span of control behavior is that manager cost function implies that the less standardization is (the more $\lambda$ is), the greater is the manager's aspiration to pass the problems to his immediate superior. Thus the workload of top-management inevitably rises while the significance of middle-layer managers falls. So it becomes less costly for the firm to spare of some middle-tier managers even suffering from the top-management overload. The relation between the span of control and the managers' ability is more predictable – the span of control rises with the ability (i.e. with the decrease of $\epsilon$).

The equilibrium manager's effort is given by (1). In the optimal hierarchy $\mu_i = \mu/r$, so one can write:

$$\xi_i = \exp\left[-B\mu^{\alpha\beta-1}r^{\beta(1-\alpha)}\right], \quad \text{where } B := \left(\frac{xn(1-\tau}{\pi(N)\tau})\right). \quad (5)$$

Thus, the monitoring effort increases in the measure $\mu$ of the unit under control if $\alpha\beta < 1$ and decreases otherwise. Therefore, the loss of control rises to the top of the hierarchy if $\alpha\beta = (\lambda+\delta)\epsilon/(1+\epsilon\delta) > 1$. This "pathological" behavior hurts much the profit of the firm and, as is shown below, the inequality $\alpha\beta < 1$ is the condition of the ability of unrestricted growth of the firm.

More precisely, given the linear price law $\pi(N) = \pi \cdot n$, if $\alpha\beta > 1$, then the profit is unimodal in $n$, so there exists the limit of the firm's growth, otherwise there may be no limit. Both cases are possible with reasonable values of parameters, so deeper parameters identification is needed to specify the real situation.

At the end investigate the dependence between the manager's wage and the size of the unit he controls. As the manager's wage $w_m$ compensates his costs

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[8] The calculation of optimal workers efforts is obvious.

$\left(\sum_{i=1}^{k} \mu_i^{\lambda}(-\ln \xi_i)^{-\delta}\right)^{\epsilon}$ in equilibrium, so, from (5):

$$w_m = B^{-\epsilon\delta} r^{\frac{\epsilon(1-\lambda)}{1+\epsilon\delta}} \mu^{\frac{(\lambda+\delta)\epsilon}{1+\epsilon\delta}}.$$

As, by definition, $\alpha\beta \equiv (\lambda+\delta)\epsilon/(1+\epsilon\delta)$, the wage obeys the power law in the size of the unit with the exponent $\alpha\beta$. So, if $\alpha\beta < 1$, the wage is concave in the size of the unit under control (given the plan $x$), otherwise the wage is convex. From the empiric literature on managerial wages the exponent of managerial wage is known to be in the range $[0.2, 0.4]$. In most real-world organizations the span of control varies from 4 to 10. These observations along with the formula for the optimal span of control help us to identify the range of potentially relevant parameters $\lambda$ and $\epsilon$. The area of interest is defined by the intervals $\lambda \in [0.05, 0.25]$, $\epsilon \in [1.15, 1.60]$ (given $\delta = 0.1$).

## 5. Perspectives

The prospective studies are devoted to the subsequent elimination of the model restrictions.

First, the assumption of a common plan $x$ can be relaxed. The prospect is that allowing for individual plans $x_w$ for every worker will not change the conclusions (although the formal proof may tangle). Then, every productive worker may be endowed with the individual technology-dependent cost function. The topical question is whether this complication results in the asymmetry of the optimal hierarchy. Also, different types of manager cost functions can be investigated along with giving them a fully-fledged economic explanation.

Imperfect and asymmetric information is known to be one of the main roots of the market failures the hierarchical control must resolve. So the most challenging line of the research is a generalization of the model towards accounting for imperfect information. An obvious way is the introduction of imperfect monitoring. In this case the wage of an employee may depend not only on the efforts exerted but also on the accuracy of monitoring that depends on the workload and the efforts of his immediate superior (as shown in simple models of imperfect monitoring by G. A. Calvo and S. Wellisz (1978); Y. Qian (1994). In general, this complication may require the development of advanced techniques for the optimal hierarchy search.

Every manager may be endowed with personal characteristics (a type). The principal then faces an adverse selection problem (see A. Mas-Collel et al (1995)) when assigning compensations. A standard incentive compatible scheme then results in information rents. These rents influence a manager's "effective cost" for the principal. The point is how the degree of information asymmetry influences the shape of the optimal hierarchy.

Yet another topical line of the inquiry is the study of incentives decentralization (the situation when the principal gives managers rights and resources to implement incentive schemes for their subordinates) and its impact on management effectiveness. While in the world of perfect information costless mechanisms of such decentralization are possible (see S. P. Mishin (2004); M. V. Goubko (2006)), this may not hold in the presence of asymmetric information. In the literature do exist models of incentive contracts decentralization in adverse selection and moral hazard environment but they restrict attention to the study of the simplest hierarchies (two agents with one principal), and the generalization of these results to the case of complex managerial hierarchies is still a question at issue.

## 6. Conclusion

The normative model of optimal hierarchy design in a firm is developed. The model accounts for revenue effects of the size of a firm, employees' costs and efforts, monitoring costs, etc. These features have not been combined before in the models of multi-layer hierarchies. The results of the analysis include the optimal monitoring efforts subject to a manager's position in a hierarchy, the optimal managerial headcount and the span of control, efficient employees' wages and the optimal profit of a firm.

These results allow analyzing the impact of environment parameters on a firm's size, its financial results, employees' wages and the shape of the optimal hierarchy. The detailed analysis of this impact will allow drawing up policy recommendations on rational bureaucracy formation in firms, big corporations and holdings. For the specific enterprise the model can answer the following important questions of organizational design:

1. How many managers should an organization employ and how many subordinate workers should these managers have?

2. How much does the maintenance of control system cost?

3. How will the growth of an organization increase the management expenses? Does this growth require radical restructuring the control system?

4. How should an organizational structure change in response to the new management technologies, production modernization and standardization, environment changes, etc.?

## Appendix

### 1. The proof of the Lemma 1.

Given the Leontief technology $z = \min_{w \in N} z_w$ the economy on nonproductive costs requires the optimal efforts to equalize the outputs of all workers, so every $z_w$ must be equal to $z$.

Identical transformation of profit formula then yields:

$$F = \frac{\pi(N)}{n} \sum_{w \in N} \left[ \ln a(N)x + \ln \xi_w + \sum_{j=1}^{l(w)} \ln \xi_j(w) \right] - \sum_{w \in N} c(x, \xi_w) -$$

$$- \sum_{m \in M} \left( \sum_{i=1}^{k(m)} \mu_i(m)^\lambda (-\ln \xi_i(m))^{-\delta} \right)^\epsilon .$$

Here $l(w)$ is the length of the worker's $w$ chain of command, $\xi_j(w)$ is the $j$-th managerial effort level in this chain of command, $k(m)$ is the manager's $m$ span of control, $\mu_i(m)$, $i = 1, \ldots, k(m)$, are the measures of manager's $m$ subordinate groups, and $\xi_i(m)$, $i = 1, \ldots, k(m)$, are his efforts levels.

Note that if the immediate subordinates of the manager $m$ in the hierarchy $H$ control the groups of workers $s_1(m), \ldots, s_k(m) \subseteq N$, then the effort level $\xi_i(m)$ of manager $m$ is accounted $|s_i(m)|$ times in the first sum of the above formula. So one can regroup the elements of managerial efforts and write

$$F = \pi(N)\ln(a(N)x) + \sum_{w\in N}\left[\frac{\pi(N)}{n}\ln\xi_w - c(x,\xi_w)\right] +$$

$$+\sum_{m\in M}\left[\frac{\pi(N)}{n}\sum_{i=1}^{k(m)}|s_i(m)|\ln\xi_i(m) - \left(\sum_{i=1}^{k(m)}\mu_i(m)^{\lambda}(-\ln\xi_i(m))^{-\delta}\right)^{\epsilon}\right].$$

Remember that by definition $\mu_i(m) := x|s_i(m)|$ so, finally one obtains the following expression for the profit:

$$F = \pi(N)\ln(a(N)x) + \sum_{w\in N}\left[\frac{\pi(N)}{n}\ln\xi_w - c(x,\xi_w)\right] +$$

$$+\sum_{m\in M}\left[\frac{\pi(N)}{nx}\sum_{i=1}^{k(m)}\mu_i(m)\ln\xi_i(m) - \left(\sum_{i=1}^{k(m)}\mu_i(m)^{\lambda}(-\ln\xi_i(m))^{-\delta}\right)^{\epsilon}\right]. \quad (6)$$

The problem is to maximize (6) by choosing the efforts levels given the outputs $z_w = x\cdot\xi_w\cdot\xi_1(w)\cdot,\ldots,\cdot\xi_{l(w)}(w)$ for all workers being equal. Below these constraints are omitted in the optimization. The unconstrained efforts levels for all managers are determined. Then the optimal hierarchy supported by these effort levels is found. Finally, we show this optimal hierarchy to obey symmetry, so the constraints of the local outputs equality are automatically satisfied in it.

The profit (6) is additive in the contributions of any worker or manager, so one can find the optimal efforts separately for any employee. Let $k$ immediate subordinates of some manager $m$ in the hierarchy $H$ to control the groups of workers with measures $\mu_1,\ldots,\mu_k$. Then to find the optimal effort levels $\xi_1,\ldots,\xi_k$ of manager $m$ one must maximize the contribution $f_m$ of this manager to the profit of the firm. From (6) the formula for the contribution is obtained:

$$f_m = \frac{\pi(N)}{nx}\sum_{i=1}^{k}\mu_i(m)\ln\xi_i - \left(\sum_{i=1}^{k}\mu_i^{\lambda}(-\ln\xi_i)^{-\delta}\right)^{\epsilon}. \quad (7)$$

From the first-order conditions the optimal efforts and the maximal contribution of a manager are calculated:

$$\xi_i = \exp\left[-\left(\frac{xn(1-\tau)}{\pi(N)\tau}\right)^{\tau}\cdot\mu_i^{\alpha-1}\left(\sum_{i=1}^{k}\mu_i^{\alpha}\right)^{\beta-1}\right], i=1,\ldots,k,$$

$$f_m^{max} = -\frac{1}{\tau}\left(\frac{\pi(N)\tau}{nx(1-\tau)}\right)^{1-\tau}\left(\sum_{i=1}^{k}\mu_i^{\alpha}\right)^{\beta}.$$ [9]

## 2. The proof of formulas (3), (4).

[9] Remember the definitions of $\alpha$, $\beta$, and $\tau$ introduced above.

It is proven by M. V. Goubko (2006) that the optimal hierarchy tends to be uniform. Uniform hierarchy is completely defined by its attributes: the span of control $r \in 2, 3, \ldots$, and the proportion $x = (x_1, ..., x_r)$.[10]

Also from M. V. Goubko (2006) the general formula to calculate the attributes of optimal uniform hierarchy given the cost function $K^*(\mu_1, \ldots, \mu_k)$ with the constant elasticity $\gamma$ is

$$(r, x) = Arg \min_{k=2,\ldots,n} \min_{y \in D_k} \frac{K^*(y_1, \ldots, y_k)}{|1 - \sum_{i=1}^{k} y_i^{\gamma}|}, \tag{8}$$

where $D_k$ is $k$-dimensional simplex. This formula holds for $\gamma \neq 1$.

For the cost function $K^*(\cdot) \sim \left(\sum_{i=1}^{k} \mu_i^{\alpha}\right)^{\beta}$ (M. V. Goubko (2006)) showed that the optimal hierarchy is symmetric for $\alpha \in [0, 1]$, $\beta \in [1, 6]$, i.e. $(x_1, \ldots, x_k) = (1/k, \ldots, 1/k)$.

Substitute this $x$ in (8), then

$$r = Arg \min_{k=2,\ldots,n} \frac{k^{(1-\alpha)\beta}}{|1 - k^{1-\alpha\beta}|}. \tag{9}$$

Allow $k$ be non-integer. Then, from the first-order conditions the expression (3) for the optimal span of control is obtained. As the function minimized in (9) is unimodal, the optimal span of control will be one of the nearest integers to (3).

The general formula for the cost of uniform hierarchy with the span of control $r$ and proportion $x$ is (see M. V. Goubko (2006)):

$$\left| (\sum_{w \in N} \mu_w)^{\gamma} - \sum_{w \in N} \mu_w^{\gamma} \right| \frac{K^*(x_1, \ldots, x_r)}{|1 - \sum_{i=1}^{r} x_i^{\gamma}|},$$

where $\mu_w$ is the measure of single worker $w \in N$.

Substituting $r$ from (3), $x_i = 1/r$, and mentioning $\mu_w = x$ for all $w$, yield exactly the expression (4).

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[10] The proportion $x$ is an element of $r$-dimensional simplex. Its components determine the measures of subordinate groups for every manager. If, for instance, some manager in an uniform hierarchy controls group of workers of measure $\mu$ and has $r$ immediate subordinates, then these subordinates control groups of workers with measures $x_1 \cdot \mu, \ldots, x_r \cdot \mu$.

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# How to Play Macroscopic Quantum Game

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**Abstract** Quantum games are usually considered as games with strategies defined not by the standard Kolmogorovian probabilistic measure but by the probability amplitude used in quantum physics. The reason for the use of the probability amplitude or "quantum probabilistic measure" is the nondistributive lattice occurring in physical situations with quantum microparticles. In our paper we give examples of getting nondistributive orthomodular lattices in some special macroscopic situations without use of quantum microparticles.

Mathematical structure of these examples is the same as that for the spin one half quantum microparticle with two non-commuting observables being measured. So we consider the so called Stern-Gerlach quantum games. In quantum physics it corresponds to the situation when two partners called Alice and Bob do experiments with two beams of particles independently measuring the spin projections of particles on two different directions In case of coincidences defined by the payoff matrix Bob pays Alice some sum of money. Alice and Bob can prepare particles in the beam in certain independent states defined by the probability amplitude so that probabilities of different outcomes are known. Nash equilibrium for such a game can be defined and it is called the quantum Nash equilibrium.

The same lattice occurs in the example of the firefly flying in a box observed through two windows one at the bottom another at the right hand side of the box with a line in the middle of each window. This means that two such boxes with fireflies inside them imitate two beams in the Stern-Gerlach quantum game. However there is a difference due to the fact that in microscopic case Alice and Bob freely choose the representation of the lattice in terms of non-commuting projectors in some Hilbert space. In our macroscopic imitation there is a problem of the choice of this representation(of the angles between projections). The problem is solved by us for some special forms of the payoff matrix. We prove the theorem that quantum Nash equilibrium occurs only for the special representation of the lattice defined by the payoff matrix. This makes possible imitation of the microscopic quantum game in macroscopic situations. Other macroscopic situations based on the so called opportunistic behavior leading to the same lattice are considered.

In this paper we continue the investigation (Grib et al., 2006) of macroscopic situations described by the same mathematical formalism as some simple (spin one half and spin one) quantum systems. In these situations stochasticity is described by some wave function as vector in finite dimensional Hilbert space with non-commuting operators in it as observables.

So one has the complementarity property for such systems. These situations can occur in economics (Soros, 2004, Choustova, 2005), biology (Khrennikov, 2006) etc so that chance in these sciences must not necessarily be described by the standard Kolmogorovian probability measure as it is usually supposed to be but by the more general quantum formalism. In (Grib et al., 2006) it was shown that new type of Nash equilibrium can arise in these cases. Differently from the microworld the Planck's constant does not play any role in our examples.

## 1. The Firefly in a Box

Here we consider some other than in our papers (Grib et al., 2006, Grib and Rodrigues, 1999, Grib et al., 2004) example named "the firefly in a box" (Svozil, 1998). This example will be used by us in order to show the connection between the Boolean lattice with probabilistic description of chance and the non Boolean nondistributive lattice with the quantum probability measure on it. The example is the simplification of the well known more complex Foulis "firefly in a box" case (Foulis, 1999, Cohen, 1989). We take it because in our paper (Grib et al., 2006) we found Nash equilibrium for a quantum game based on Hasse diagram for this simplified case. It occurred that in quantum case there are Nash equilibrium more profitable than the classical ones. These Nash equilibrant correspond to realizations of the nondistributive lattice in terms of certain noncommutative projectors in Hilbert space of the spin one half system. We shall not discuss other known examples (Wright urn (Wright, 1991)) leading to nondistributive lattices because we did not investigate Nash equilibrium for these examples. The rule for the quantum macroscopic game for our example can be formulated as follows. A firefly (surely a macroscopic agent) is roaming around

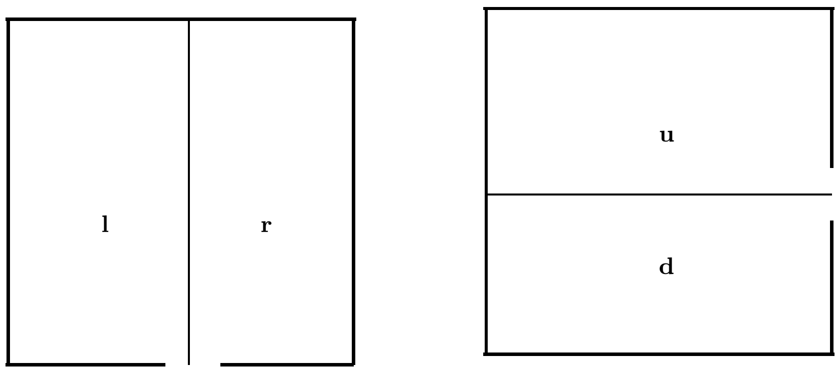


**Fig.1.** The firefly in a box with two windows.

a box and some observer (other macroscopic agent) can see it either through the window at the bottom of the box or through the window at the right side of it. Each window has a thin line perpendicular to it drawn at it's center so that an observer can see the firefly in one or another halves of the box.

An observer cannot look at the same moment at two windows at once so that one has two incompatible experiments. Let us call possible observable situations as "left", "right", "up" and "down". The outcomes of the experiments can be described by the nondistributive lattice employed by Birkhoff and von Neumann (Birkhoff and von Neumann, 1936) for the spin one half system with two complementary observables – (different spin projections) – being measured. This is an orthocomplemented lattice. All rigorous mathematical definitions can be found in (Birkhoff, 1993). However for better understanding of the paper we must give some necessary definitions and conjectures here.

The lattice L is a partially ordered set (S, $\leq$) with two operations $\vee, \wedge$ so that each pair $x, y \in L, x \neq y$ has a supremum $x \vee y$ and an infimum $x \wedge y$. There are elements $\oslash \in L, I \in L$ such that $x \vee \oslash = x$, $x \wedge \oslash$, $x \vee I = I$, $x \wedge I = x$.

The lattice is complemented if for $\forall x \in L$ exists at least one complement $x'$ such that $x \wedge x' = \oslash$, $x \vee x' = I$. The elements of the lattice are orthogonal $x \perp y$ if $x \leq y'$. The operations $\wedge, \vee$ can sometimes be understood as logical disjunction and conjunction. Then it is supposed that if x is true than $x \vee y$ is true, if $y$ is true then $x \vee y$ is true. However in quantum logic "if" is not equal to "always if" (iff). So in general it is not correct to think that from $x \vee y$ true follows that either of them is true.

Negation in quantum logic is realized through orthogonality. There is some discussion in literature on the problem of logical interpretation of lattices (see (Pykacz, 1996)) for the general case so that different views arise from different definitions. For Boolean sublattices of the non Boolean lattice it is possible to give usual logical interpretation of the lattice operations.

The success of quantum physics shows that the idea of Birkhoff and von Neumann of the nondistributive lattice with quantum probabilistic measure on it as justification of use of the probability amplitude is a correct one. So it seems reasonable to conclude that in other cases when the same lattices occur one must obtain the quantum mechanical mathematical formalism. One can draw the so called Hasse diagram for the lattice so that lines correspond to partial order,going up one can obtain intersection of lines at $\vee$, going down one can obtain intersection at $\wedge$. Here

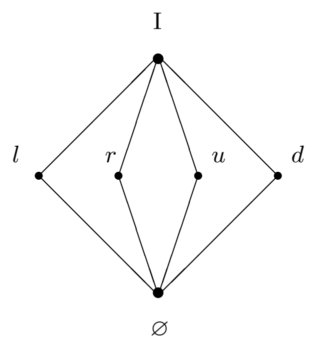


**Fig.2.** Hasse diagram of the nondistributive lattice of the firefly in a box example.

"l", "r", "u", "d" are elements (logical atoms) of the lattice describing different experimentally testable propositions for the firefly on Fig.1. Elements "l" and "r" as well as "u" and "d" are orthogonal. One also has for the two lattice operations $\wedge$ ("and"), $\vee$ ("or")

$$l \vee r = u \vee d = l \vee u = r \vee d = l \vee d = r \vee u = I$$

$$l \wedge r = l \wedge u = l \wedge d = r \wedge u = u \wedge d = r \wedge d = \oslash$$

which means that "l" or "r" is always true while "l" and "r" is always false etc. For example "r" and "u" is always false because "experimentally" there is no such observable element at the disposal of the observer due to the impossibility of simultaneous observation of the corresponding situations. The lattice is nondistributive because

$$l \wedge (r \vee d) = l \wedge I = l \neq (l \wedge r) \vee (l \wedge d) = \oslash \vee \oslash = \oslash.$$

If the firefly randomly moves inside the box the observer can describe the outcomes of his observations as some representation of the nondistributive lattice in terms of projectors ("yes - no" questions) in two dimensional real space. He (she) defines the "quantum" probability of the outcomes from some wave function $|\Psi\rangle$ taken as the two dimensional vector by

$$w_\alpha = \langle\Psi|\hat{P}_\alpha|\Psi\rangle, \alpha \in \{l, r, u, d\}, \tag{1}$$

here one has

$$w_l + w_r = w_u + w_d = 1. \tag{2}$$

So one has the wave function and non-commuting operators – the projectors $\hat{P}_\alpha$ for the "firefly in a box" example. To organize the game first consider the game in which quantum microparticles are used. Call it the Stern-Gerlach quantum game (Grib et al., 2006). Two partners Alice and Bob are sitting close to accelerators and prepare two beams of particles (protons) with spin one half. Then they do the Stern-Gerlach experiment measuring two different spin projections of their particles. There is some payoff matrix given by the table 1. The meaning of it is that in case Alice

**Table1.** payoff of Alice

| strategies | Bob | | | |
|---|---|---|---|---|
| Alice | **1** | **2** | **3** | **4** |
| **1** | 0 | 0 | $c_3$ | 0 |
| **2** | 0 | 0 | 0 | $c_4$ |
| **3** | $c_1$ | 0 | 0 | 0 |
| **4** | 0 | $c_2$ | 0 | 0 |

gets result **1** and Bob gets **3** Bob pays to Alice the sum of money $c_3$ prescribed by the payoff matrix etc. There are some frequencies $p_1$ of getting **1** by Alice in a series of her measurements and there are frequencies $q_3$ of getting **3** by Bob. These frequencies are defined by the probabilities of certain outcomes which can be calculated by the rules of quantum physics if one knows the wave functions of particles prepared by Alice and Bob in their beams. The average profit of Alice can be calculated as

$$\overline{H}_A = c_1 p_3 q_1 + c_3 p_1 q_3 + c_2 p_4 q_2 + c_4 p_2 q_4$$
$$p_1 + p_3 = 1, \quad p_2 + p_4 = 1, \quad q_1 + q_3 = 1, \quad q_2 + q_4 = 1 \tag{3}$$

The strategies of Alice and Bob in this game are described by the wave functions of particles in their beams. It is supposed that Alice and Bob have special experimental setups to produce their particles in certain states with some fixed wave function. However they are not free in their choice of measuring spin projections. It is supposed that both partners know what projections are to be measured, For macroscopic situations described by the same lattice as the Stern-Gerlach quantum game type one can be interested to find answers on the questions: what is the meaning of the "preparation" of the wave function and what is the meaning of measuring different non-commuting observables from the point of view of our macroscopic agents?

To find the answer one must take into account the fact that this lattice can be embedded into some Boolean lattice. The physical meaning of the embedment is simple: the observer cannot check some situations described by some more general Boolean lattice due to the character of his (her) experiments. These elements of the lattice are "hidden variables" for the observer and the non-distributivity of the lattice is the payment for his (her) "ignorance". A general theory of realizing of quantum logical lattices as "concrete logics" obtained from Boolean lattices was developed in (Kochen and Specker, 1967).

Surely due to Kochen-Specker's theorem (Kochen and Specker, 1967) not all quantum logical lattices can be embedded into Boolean lattices, that is why quantum theory of microparticles is basic and is not a hidden variable theory. The other objection is breaking of Bell's inequalities for relativistic systems if entangled states are considered (Bell, 1988).

However for our aim to find macroscopic situations with macroscopic agents with behavior described by the quantum rules (i.e. by the Born-Luders rule for calculation of probabilities) the class of such "quasi-classical" lattices (Grib and Rodrigues, 1999, Svozil, 1998) embedded in Boolean ones is wide enough. To construct the Boolean lattice divide the box on four parts.

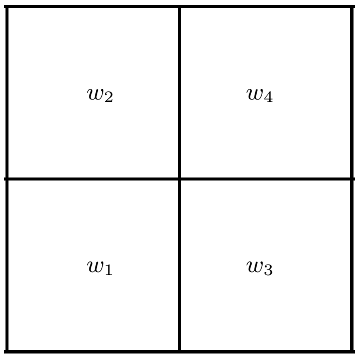


**Fig.3.** Probability of different firefly in a box positions from it's "own" point of view.

The firefly can occur in any of the four parts. Construct the Boolean lattice based on elements $1, 2, 3, 4$ as it's atoms. One obtains the nondistributive lattice of the Fig.2 from the distributive lattice by considering only composite elements "l", "r", "u", "d" of the second row of the boolean lattice while the atomic elements $1, 2, 3, 4$ as well as the third row composed from triples are unobservable.

So the main lesson is that one can expect obtaining of the quantum rules in situations when one has stochastic processes which are secondary to some basic non observable ones. This is typical for situations on the stock market, in some complex biological systems etc. The other important feature is complementarity as impossibility of checking all properties at the same moment of time. What is the connection of the probabilities on the Boolean lattice and the quantum probability on the nondistributive one?

Denoting $w_a$ : $w_1, w_2, w_3, w_4$ the probabilities for atoms of the Boolean lattice one obtains the equations

$$\begin{aligned} w_1 + w_2 &= w_l \\ w_3 + w_4 &= 1 - w_l \\ w_1 + w_3 &= w_d \\ w_2 + w_4 &= 1 - w_d \end{aligned} \tag{4}$$

From these equations it is easy to see that the "strange" quantum rule for one and the same object (the firefly) is not strange at all if it is considered not for the elementary events but for the complex ones!

$$w_l + w_r + w_u + w_d = 2 \tag{5}$$

It seems that any distribution $w_1, w_2, w_3, w_4$ leads to some fixed wave function and some fixed representation of the lattice in terms of projectors on some fixed directions on the plane. However one can see that distributions leading to appearance of two ones or zeros for complex events of the second floor are prohibited due to the definition of disjunction in the quantum logical lattice while it is possible in the Boolean case. This means that logical atoms of the first floor cannot be definitely determined from the point of the observer for the quantum logical case being totally "hidden" for him (her). One can ask the question: what prohibits the firefly to occur in the corner **1**?

Is it possible for the observer to get **1** for the outcome "left" and **1** for the outcome "down"? The answer is surely positive. But for the quantum system it is impossible to get eigenfunction for non-commuting spin operators. Is it a contradiction?

The answer is "no"! It is here where something like the wave packet collapse idea comes into the play. Quantum theory does not forbid to get in observations of complementary observables positive results. It only says that if one prepares the wavefunction as an "eigenfunction of the projector on the "left" part and performs many observations of complementary observables then one obtains some probability distribution for the complementary observable "up"-"down". This distribution is not 1, 0 because the firefly has some freedom and the only instruction for him given on the preparation stage is to be "somewhere" in the "left" part. The observer cannot always let him be in the corner1,so in many experiments with the same "left" instruction the results will be sometimes "up", sometimes "down" with frequencies obtained from the wave function and the representation of observables in the form of operators.

If the distributions $w_l, w_d$ are known the distribution $w_1, w_2, w_3, w_4$ obtained from it is not unique: it is defined up to some arbitrary $w_i$ where $w_i$ is some probability for fixed $i$.

This is just the manifestation of "indefiniteness" of the quantum situation in comparison to Boolean one as we just said before.

## 2. Stern-Gerlach Quantum Game

The macroscopic quantum game considered previously in (Grib et al., 2006) called the macroscopic Stern-Gerlach quantum game can be organized for the "firefly in a box" case as follows.

There are two partners Alice and Bob and two boxes with fireflies there. Alice can try to choose some classical probability distribution for Boolean elementary outcomes 1, 2, 3, 4 with limitations discussed before. This can be done by "training" of the firefly stimulating it to come more often to this or that part of the box. For example any observation by Alice of the part of the box is accompanied by the flash of light with some prolongation in time stimulating the firefly to react. Different times of observation result in different frequencies for the firefly to occur in some part of the box. Supposing that Alice is interested only in $l, r, u, d$ outcomes she cannot define exact distributions for Boolean elementary outcomes but only some class of it. However from the quantum point of view definition of frequencies for $l, r, u, d$ means definition of the wave function and the representation of the atoms of the non Boolean lattice in terms of non-commuting projectors, i. e. definition of the angle between different spin projections. This can be called "the preparation stage".

In special cases getting $1, 0$ for some of the complementary possibilities she can speak about some fixed wave function as an eigenfunction of some spin projection operator. However in general case it is not the case.

Same manipulations are made by Bob. However neither Alice nor Bob have knowledge of the training procedures of the partner.

Then the game begins. Alice and Bob with their trained fireflies look at the results of their observations accompanied by flashes of light. In cases defined by the rules of the game when for example Alice gets some fixed result "$\alpha$" while Bob gets "$\beta$" Bob must pay money to Alice etc.

There is some payoff matrix defining in what cases Bob pays Alice some money as it was defined in (Grib et al., 2006). The profit depends on the frequencies of the outcomes. The average profit is calculated by using the quantum rule for projectors $\hat{P}^a_\alpha$ for Alice, $\hat{P}^b_\alpha$ for Bob

$$\begin{gathered}\overline{H} = \langle \Psi_A | \langle \Psi_B | \hat{H} | \Psi_B \rangle \Psi_A \rangle; \\ \hat{H} = c_3 \hat{P}^a_u \otimes \hat{P}^b_d + c_1 \hat{P}^a_d \otimes \hat{P}^b_u + c_4 \hat{P}^a_l \otimes \hat{P}^b_r + c_2 \hat{P}^b_r \otimes \hat{P}^b_l .\end{gathered} \tag{6}$$

which in terms of the "Boolean philosophy" means calculation of

$$\overline{H} = c_3 w_u w^b_d + c_1 w_d w^b_u + c_4 w_l w^b_r + c_2 w_r w^b_l , \tag{7}$$

$$w_u + w_d = w_l + w_r = w^b_u + w^b_d = w^b_l + w^b_r = 1$$

where $w_\alpha, w^b_\alpha$ are probabilities used by Alice and Bob. One can recognize in (7) formula (3) with $w^b_\alpha$ playing the role of $q_\alpha, p_\alpha$.

Here one must make some remarks about some peculiar features of this calculation. The representation of the non Boolean lattice in terms of non-commuting projectors is not unique. It is defined up to some angles $\theta, \tau$. Existence of different representations of the non Boolean lattice parameterized by the angles is the manifestation of the freedom of the observer to choose measurement of any spin observable for the real quantum system. There is also another freedom for the observer manifested in the choice of the wave function.

Let us defined

$$\begin{aligned} w_u &= \cos^2 \alpha, \qquad & w_l &= \cos^2(\alpha - \theta) \\ w^b_u &= \cos^2 \beta, \qquad & w^b_r &= \cos^2(\beta - \tau) \end{aligned}$$

If after this definition of the angle Alice and Bob cannot change the angles then their freedom is now limited by choosing only some special distributions for the firefly in the complementary experiment. The "quantum logic" leads to arising of a special "quantum correlation" between complementary observations. This correlation according to (Grib et al., 2006) can be expressed by the constraint

$$\frac{(w_u + w_l - 1)^2}{cos^2\theta} + \frac{(w_l - w_u)^2}{sin^2\theta} = 1. \tag{8}$$

Different choice of the angles leads to the different constraint. Different Nash equilibrium can be found for different angles. It was shown in (Grib et al., 2006) for the Hasse diagram considered in this paper that some of these equilibrium are more profitable for partners others are less. So one can put the hypothesis that it is the more profitable equilibrium that can play the role of the principle of choice of the representation of the lattice.

## 3. Eigenequilibrium

There is some special case of the payoff matrix discovered in our paper (Grib et al., 2006) in which our macroscopic quantum game is totally defined. This case was called by us the case of "eigenequilibrium".

In quantum game on the quadrangle (Grib et al., 2006) with payoff matrix (see tabl.1) the average payoff of Alice can be written as

$$\langle H \rangle = \frac{1}{4}\left(g(x,y) + \operatorname{tr} C\right)$$

where

$$g(x,y) = -\langle x, Ay\rangle + \langle x, M_\theta^\dagger \omega\rangle - \langle M_\tau^\dagger \omega, y\rangle,$$

$A = M_\theta^\dagger C M_\tau$, and $x$, $y$ unit vectors on the plane: $|x| = 1,\ |y| = 1$. Here

$$M_\varphi = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \cos\varphi & \sin\varphi \end{bmatrix}, \quad n = c_1 + c_3,\ m = c_2 + c_4$$

$$\omega = \begin{bmatrix} c_3 - c_1 \\ c_4 - c_2 \end{bmatrix}, \qquad C = \begin{bmatrix} n & 0 \\ 0 & m \end{bmatrix}$$

An equilibrium $(x, y)$ is called *eigenequilibrium*, if it is an eigenvector of the matrix

$$\mathcal{A} = \begin{bmatrix} O & A \\ A^\dagger & O \end{bmatrix}$$

The following proposition proved in (Parfionov, 2008).

**Proposition 1.** *If the eigenequilibrium exists, then $\omega$ is a common eigenvector of the matrices $CM_\theta M_\theta^\dagger$ $CM_\tau M_\tau^\dagger$.*

A game is said to be *non-degenerate*, if

$$\varDelta = \begin{vmatrix} n & m \\ \omega_1^2 & \omega_2^2 \end{vmatrix} \neq 0$$

**Proposition 2.** *If the game is non-degenerate, then the necessary condition for the eigenequilibrium to exist is the coincidence of the angular parameters $\theta = \tau$. In this case their values are completely determined by the payoff coefficients of the game $\{c_j\}$:*

$$\cos 2\theta = \cos 2\tau = \frac{(m-n)\omega_1\omega_2}{\Delta} \tag{9}$$

Further finding *eigenequilibrium* of *non-degenerate* games, calculate $\theta$ using (9) and put $M = M_\theta$, $z = M^\dagger \omega$. In this case $A = A^\dagger = M^\dagger C M$ and the matrix $A$ non-negatively defined.

**Proposition 3 (Existence theorem).** *Let a vector $\omega$ be an eigenvector of the matrix $CMM^\dagger$ and $\langle Az, z\rangle \leqslant |z|^3$. Then the strategies $x = y = z/|z|$ form an eigenequilibrium.*

**Proposition 4 (Multiple Nash-equilibrium).** *Let a vector $\omega$ be an eigenvector of the matrix $CMM^\dagger$ and $\langle Az, z\rangle = |z|^3$. Then there are two eigenequilibrium $x = y = z/|z|$ $x = -z/|z|$, $y = z/|z|$.*

**Proposition 5 (Uniqueness theorem).** *Let there is a game with a non-degenerate equilibrium $\langle Az, z\rangle \neq |z|^3$. Then all possible equilibrium are exhausted by it.*

So, it occurs that optimal strategies of the players are defined not so by the representation of the ortholattice as by the ortholattice itself and by the payoff structure of the game.

For this case as in quantum Stern-Gerlach quantum game the angles are prescribed by the rule of the game and the only choices for Alice and Bob training their fireflies concern probability distributions satisfying the constraint with this angle.

The optimal choice corresponds to Nash equilibrium existing for this angle. For other choice of the angle Nash equilibrium does not exist and clever Alice and Bob will not use them at all.

## 4. Example of the Multiple Quantum Nash-Equilibrium

For

$$c_1 = 1; \quad c_2 = 2; \quad c_3 = 99; \quad c_4 = 98$$

the angles is equal $\theta = \tau = 45°$ and the optimal strategies of Alice and Bob are

$$p_1 = q_1 \approx 0,857; \quad p_2 = q_2 \approx 0,622; \qquad \langle H\rangle = 50$$

For

$$c_1 = 1;\ c_2 = 2;\ c_3 = 9;\ c_4 = 8$$

the angles is equal $\theta = \tau = 45°$ and there are *two* eigenequilibrium.

First equilibrium:

$$p_1 = q_1 = 0,9; \quad p_2 = q_2 = 0,8; \quad \langle H\rangle = 5$$

Second equilibrium:

$$p_1 = 0,1;\ q_1 = 0,9; \quad p_2 = 0,2;\ q_2 = 0,8; \quad \langle H\rangle = 5$$

## 5. Quantum Equilibrium Against the Classical One

If one considers the classical game with the same payoff matrix one can obtain (Grib et al., 2006) for the average profit:

$$h = (c_1^{-1} + c_3^{-1})^{-1} + (c_2^{-1} + c_4^{-1})^{-1}$$

It can be shown in some cases that the quantum equilibrium is more profitable than the classical one. Consider the following example: $c_1 = 1$, $c_2 = 9$, $c_3 = 10$, $c_4 = 2$ the optimal strategies of Alice and Bob are

$$p_1 = q_1 = \frac{130 + 9\sqrt{130}}{260} \approx 0.895, \quad p_2 = q_2 = \frac{130 - 7\sqrt{130}}{260} \approx 0.193$$

The optimal profit in the classical game is smaller than in the quantum one: $\langle h \rangle = 28/11$, $\langle H \rangle = 11/4$.

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# Solutions of Bimatrix Coalitional Games

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**Abstract** The PMS-vector is defined and computed in (Petrosjan and Mamkina, 2006) for coalitional games with perfect information. Generalization of the PMS-vector for the case of Nash equilibrium (NE) in mixed strategies is proposed in this paper.

**Keywords:** bimatrix games, coalitional partition, Nash equilibrium, Shapley value, PMS-vector, games with perfect information.

## 1. Introduction

The new approach to the solution of bimatrix coalitional games is proposed. Suppose $N$-person game $\Gamma$ with finite sets of strategies is given. The set of players $N$ is divided on two subsets (coalitions) $S$, $N\backslash S$ each acting as one player. The payoff of player $S$ $(N\backslash S)$ is equal to the sum of payoffs of players from $S$ $(N\backslash S)$. The Nash equilibrium (NE) in mixed strategies is calculated (in the case of multiple NE (Nash, 1951) the solution of the correspondly coalitional game will be not unique). Mathematical expectation of the payoffs coalition $S$ $(N\backslash S)$ in the NE in mixed strategies is allocated according to the Shapley value (Shapley, 1953). The resulting payoffs vector will be a generalization of the PMS-vector defined and computed in (Petrosjan and Mamkina, 2006) for coalitional games with perfect information. Then the payoff of coalitions $S$, $N\backslash S$ which appears with positive probability in the NE is allocated proportionally to the PMS-vector.

## 2. Statement of the Problem.

Suppose $N$-person game

$$\Gamma = \{N, X_1, \ldots, X_N, H_1, \ldots, H_N\}$$

is given. The set of players $N$ is divided into two separated coalitions $S$ and $N\backslash S$ each acting as one player. Rename the players of coalition $S$ from 1 to $s = |S|$, and the players of coalition $N\backslash S$ from $s+1$ to $N$. For each of the player the strategy sets are defined $X_i = \left\{x_i^{j_i}\right\}$, where $j_i = \overline{1, m_i}$, $i = \overline{1, N}$.

Denote by

$$x = \left(x_1^{j_1}, x_2^{j_2}, \ldots, x_N^{j_N}\right)$$

the $n$-tuple of strategies in the game. Let $H_i(x)$, $i \in N$, be the payoff of player $i$, $x \in \prod\limits_{i \in N} X_i$. The payoff of coalition $S$ $(N\backslash S)$ is equal to the sum of payoffs of players from $S$ $(N \backslash S)$:

$$H_{S\,(N\backslash S)}\,(x) = \sum_{i \in S(N\backslash S)} H_i\,(x).$$

Then we get the coalitional bimatrix game

$$\Gamma\left(\tilde{A},\, \tilde{B}\right) = \left\{N, X_{\mathrm{S}}, X_{N\backslash S}, H_{\mathrm{S}}, H_{N\backslash S}\right\},$$

where

$$\tilde{A} = \begin{pmatrix} H_S\left(x_1^1, \ldots, x_s^1, x_{s+1}^1, \ldots, x_N^1\right) & \ldots & H_S\left(x_1^1, \ldots, x_s^1, x_{s+1}^{m_{s+1}}, \ldots, x_N^{m_N}\right) \\ \ldots & \ldots & \ldots \\ H_S\left(x_1^{m_1}, \ldots, x_s^{m_s}, x_{s+1}^1, \ldots, x_N^1\right) & \ldots & H_S\left(x_1^{m_1}, \ldots, x_s^{m_s}, x_{s+1}^{m_{s+1}}, \ldots, x_N^{m_N}\right) \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} H_{N\backslash S}\left(x_1^1, \ldots, x_s^1, x_{s+1}^1, \ldots, x_N^1\right) & \ldots & H_{N\backslash S}\left(x_1^1, \ldots, x_s^1, x_{s+1}^{m_{s+1}}, \ldots, x_N^{m_N}\right) \\ \ldots & \ldots & \ldots \\ H_{N\backslash S}\left(x_1^{m_1}, ..., x_s^{m_s}, x_{s+1}^1, ..., x_N^1\right) & \ldots & H_{N\backslash S}\left(x_1^{m_1}, \ldots, x_s^{m_s}, x_{s+1}^{m_{s+1}}, ..., x_N^{m_N}\right) \end{pmatrix},$$

$$\dim \tilde{A} = \dim \tilde{B} = \prod_{i \in S} m_i \times \prod_{i \in N\backslash S} m_i\,.$$

It's required to find the optimal imputation rule for each coalition and in some sense optimal strategies for the coalitions.

## 3. Solution of the Problem

1. We shall consider the case when the matrixes $\tilde{A}$ and $\tilde{B}$ can be reduced to square matrixes $A$ and $B$: $\det A \neq 0$, $\det B \neq 0$. In other cases the iterational methods can be used.

Solve the bimatrix game $\Gamma\,(A,\, B)$, i. e. find a Nash equilibrium (NE) in the mixed strategies by formulas

$$x = v_2 u B^{-1};\ y = v_1 A^{-1} u,$$

where $v_1 = 1/\left(uA^{-1}u\right)$, $v_2 = 1/\left(uB^{-1}u\right)$, $u = (1, \ldots, 1)$, using the theorem about a completely mixed equilibrium (Petrosjan et al., 1998, p. 135).

In the case of multiple NE (Nash, 1951) the solution of the corresponding coalitional game will be not unique.

2. Calculate the NE value in mixed strategies:

$$E\,(\bar{x},\, \bar{y}) = \left[v\,(S)\,,\, v\,(N\backslash S)\right],$$

where

$$v\,(S) = \sum_{i \in X_S} \sum_{j \in X_{N\backslash S}} a_{ij}\xi_i\eta_j,\ \ v\,(N\backslash S) = \sum_{i \in X_S} \sum_{j \in X_{N\backslash S}} b_{ij}\xi_i\eta_j,$$

$$x = \{\xi_i\}_{i \in X_S}\,, y = \{\eta_j\}_{j \in X_{N\backslash S}}\,.$$

One can show that $v\,(S) \geq \sum_{i \in S} v_i$, where $v_i$ − maximal guaranteed payoff of $i$-th player, $i \in S$, under condition that the players from coalition $N\backslash S$ use mixed strategy from NE. This follows from the superadditivity of the characteristic function defined as maximal guaranteed payoff of the coalition $S$.

3. In (Petrosjan and Mamkina, 2006) the definition of PMS-vector in pure strategies for coalitional games with perfect information has been given. Find PMS-vector (imputation) in mixed strategies as the mathematical expectation over the 2-tuples of strategies generated by NE.

In the beginning define PMS-vector in mixed strategies. Let the game

$$\Gamma = \{N, X_1, \dots, X_N, H_1, \dots, H_N\}$$

in normal form with the coalitional partition

$$\Sigma = \{S_1, \dots, S_l\}\ ,\ \ l \leq n,\ \ S_i \cap S_j = \emptyset,\ \ \forall\ i \neq j,$$

be given.

Consider the game in normal form

$$\Gamma_\Sigma = \{N, X_1, \dots, X_l, H_1, \dots, H_l\}$$

between $l$ players, where the players are coalitions from partition $\Sigma$. Consider coalition $S_i$, consisting of $s_i$ players.

Notions:

– $m_j$ is the number of strategies of player $j$;

– $X_j = \{x_j^k\}_{k=\overline{1,m_j}}$ is the set of the strategies of player $j$;

– $X_{S_i} = \prod\limits_{j \in S_i} X_j$ is the set of the strategies of coalition $S_i$, i. e. Cartesian product of the sets of players' strategies, which are included into coalition $S_i$;

– vector $x_i \in X_{S_i}$ of dimension $s_i$ is the strategy of player $i$ in the game $\Gamma_\Sigma$;

– $H_{S_i} = \sum\limits_{j \in S_i} H_j$, i. e. payoff of player $S_i$ is a sum of payoffs of the players from coalition $S_i$.

– $l_{S_i} = |X_{S_i}| = \prod\limits_{j \in S_i} l_j$ is the number of pure strategies of coalition $S_i$;

– $l_\Sigma = \prod\limits_{i=\overline{1,l}} l_{S_i}$ is the number of $l$-tuples in pure strategies in the game $\Gamma_\Sigma$.

Let $\bar{\mu} = (\bar{\mu}_1, \dots, \bar{\mu}_l)$ be $l$-tuple NE in mixed strategies in the game $\Gamma_\Sigma$, where each mixed strategy of coalition $S_i$ is a vector

$$\bar{\mu}_i = \left(\bar{\mu}_i^1, \dots, \bar{\mu}_i^{l_{S_i}}\right)\ ,\ \ \bar{\mu}_i^j \geq 0\,,\ \ j = \overline{1, l_{S_i}},\ \ \sum_{j=1}^{l_{S_i}} \bar{\mu}_i^j = 1\,.$$

Denote a payoff of coalition $S_i$ in NE by

$$v\,(S_i) = \sum_{k=1}^{l_\Sigma} p_k H_k\,(S_i), i = \overline{1,l},$$

where

$$p_k = \prod_{i=\overline{1,l}} \bar{\mu}_i^{j_i},\ \ j_i = \overline{1, l_{S_i}},\ \ k = \overline{1, l_\Sigma},$$

– probability of the payoff's realization $H_k\,(S_i)$ of coalition $S_i$ , when players choose their pure strategies $x_{j_i}$ in $l$-tuple NE in mixed strategies $\bar{\mu}$, i. e.

$$H_k\,(S_i) = \sum_{j \in S_i} H_j\,(\bar{x}_1, \dots, \bar{x}_l).$$

The value $H_k(S_i)$ is random variable. There could be many $l$-tuple NE in the game, therefore, $v(S_1)$, ...., $v(S_l)$, are not uniquely defined.

Consider for each coalition $S_i \in \Sigma$, $i = \overline{1,l}$, a cooperative game $G_{S_i}$ supposing that the players which are not included into the coalition $S_i$, use NE strategies from $l$-tuple $\bar{\mu}$.

**Definition 1.** Let $w(S_i : K) = v(K)$ be characteristic function in the cooperative game $G_{S_i}$, where $K \subset S_i$. Divide payoff $w(S_i) = v(S_i)$ between the players of coalition $S_i$, according to Shapley value (Shapley, 1953) $Sh = (Sh_1, \dots, Sh_{S_i})$:

$$Sh_i = \sum_{\substack{S' \subset S \\ S' \ni i}} \frac{(s'-1)!\,(s-s')!}{s!} \left[w(S') - w(S' \setminus \{i\})\right] \quad \forall\, i = \overline{1,\, s}\,, \tag{1}$$

where $s = |S|$ ($s' = |S'|$) is the number of elements of set $S$ ($S'$) and $w(S')$ is a maximal guaranteed payoff of the subcoalition $S' \subset S$. Denote

$$Sh(S_k) = (Sh(S_k : 1), \dots, Sh(S_k : s_k)) \,,$$

where $s_k$ is the number of elements of set $S_k$. Moreover $w(S_i) = \sum\limits_{j=1}^{s_i} Sh(S_i : j)$. Then PMS-vector for the NE in mixed strategies in the game $\Gamma_\Sigma$ is defined as

$$\mathrm{PMS}(\Gamma_\Sigma) = (\mathrm{PMS}_1(\Gamma_\Sigma), \dots, \mathrm{PMS}_N(\Gamma_\Sigma)) \,,$$

where

$$\mathrm{PMS}_j(\Gamma_\Sigma) = Sh(S_i : j)\,, \; j \in S_i, \; i = \overline{1,l}.$$

4. Divide payoffs $H_k(S_i)$ of coalition $S_i$ for every $i = \overline{1,l}$ occurring with positive probability when $l$-NEtuple in the game $\Gamma_\Sigma$ is played, proportionally to the components of Shapley value:

$$\lambda_j(S_i) = \frac{Sh(S_i : j)}{\sum\limits_{j \in S_i} Sh(S_i : j)}\,, \quad j \in S_i, \quad i = \overline{1,\, l}\,.$$

One can show that is $Sh(S_i : j) = \sum\limits_{k=1}^{l_\Sigma} p_k H_{jk}(S_i)$, where $H_{jk}(S_i) = \lambda_j H_k(S_i)$, $j \in S_i$ $\forall i = \overline{1,l}$. Then in bimatrix coalitional game matrixes of players' payoffs $j = \overline{1,\, N}$, are defind as follows

$$A_j = \lambda_j A\,,\; j \in S; \; B_j = \lambda_j B, \;\; j \in N \backslash S\,.$$

## 4. Examples

*Example 1.* Let there be 3 players in the game. Each of them has two strategies (see table 1). The payoffs of each player are defined for all three-tuples.

1. Compose and solve the coalitional game, i. e. find NE in mixed strategies in the game:

$$\begin{array}{ccll}
 & & \eta = 3/7 & 1-\eta = 4/7 \\
 & & 1 & 2 \\
0 & (1,\,1) & [6,\,1] & [3,\,2] \\
0 & (2,\,2) & [4,\,3] & [4,\,2] \\
\xi = 1/3 & (1,\,2) & [4,\,5] & [6,\,3] \\
1-\xi = 2/3 & (2,\,1) & [8,\,1] & [3,\,2]\,.
\end{array}$$

It's clear, that first matrix row is dominated by the last one and the second is dominated by third. One can easily calculate NE and we have

$$y = \left(3/7 \; 4/7\right), \; x = \left(0 \; 0 \; 1/3 \; 2/3\right) .$$

**Table 1.**

| The strategies | | | The payoffs | | | The coalitions ((I, II), III) | The NE strategies | | | The NE payoffs | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| I | II | III | I | II | III | | I | II | III | I | II | III |
| 1 | 1 | 1 | 4 | 2 | 1 | ((1,1),1) | 1 | 1 | $\bar{y} = \begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \end{pmatrix}$ | $2\frac{2}{7}$ | 2 | $1\frac{4}{7}$ |
| 1 | 1 | 2 | 1 | 2 | 2 | ((1,1),2) | | | | | | |
| 1 | 2 | 1 | 3 | 1 | 5 | ((1,2),1) | 1 | 2 | $\bar{y} = \begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \end{pmatrix}$ | $4\frac{1}{7}$ | 1 | $3\frac{6}{7}$ |
| 1 | 2 | 2 | 5 | 1 | 3 | ((1,2),2) | | | | | | |
| 2 | 1 | 1 | 5 | 3 | 1 | ((2,1),1) | 2 | 1 | $\bar{y} = \begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \end{pmatrix}$ | $2\frac{5}{7}$ | $2\frac{3}{7}$ | $1\frac{4}{7}$ |
| 2 | 1 | 2 | 1 | 2 | 2 | ((2,1),2) | | | | | | |
| 2 | 2 | 1 | 0 | 4 | 3 | ((2,2),1) | 2 | 2 | $\bar{y} = \begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \end{pmatrix}$ | 0 | 4 | $2\frac{3}{7}$ |
| 2 | 2 | 2 | 0 | 4 | 2 | ((2,2),2) | | | | | | |

Then the probabilities of payoffs's realization of the coalitions $S$ and $N \backslash S$ in mixed strategies (in NE) are as follows:

$$\begin{array}{ccc} & \eta_1 & \eta_2 \\ \xi_1 & 0 & 0 \\ \xi_2 & 0 & 0 \\ \xi_3 & 1/7 & 4/21 \\ \xi_4 & 2/7 & 8/21 \end{array} .$$

The Nash value of the game in mixed strategies is calculated by formula:

$$E\left(x,\, y\right) = \frac{1}{7}\left[4,\, 5\right] + \frac{2}{7}\left[8,\, 1\right] + \frac{4}{21}\left[6,\, 3\right] + \frac{8}{21}\left[3,\, 2\right] = \left[\frac{36}{7},\, \frac{7}{3}\right] = \left[5\frac{1}{7},\, 2\frac{1}{3}\right] .$$

Rewrite table 1 in table 2.

**Table 2 comments.** In table 2 pure strategies of coalition $N \backslash S$ and its mixed strategy $y$ are given horizontally at the right side. Pure strategies of coalition $S$ and its mixed strategy $x$ are given vertically. Inside the table players' payoffs from the coalition $S$ and the total payoff of the coalition $S$ are given at the right side.

**Table 2.**

| | | | | | The strategies of $N\backslash S$, the payoffs of $S$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $y$ | **0,43** | | | **0,57** | | |
| | Math. Expectation | | $x$ | | 1 | | S | 2 | | S |
| Strategies of $S$ | 2,286 | 2,000 | **0,00** | 1, 1 | **4** | **2** | 6 | **1** | **2** | 3 |
| | 4,143 | 1,000 | **0,33** | 1, 2 | **3** | **1** | 4 | **5** | **1** | 6 |
| | 2,714 | 2,429 | **0,67** | 2, 1 | **5** | **3** | 8 | **1** | **2** | 3 |
| | 0,000 | 4,000 | **0,00** | 2, 2 | **0** | **4** | 4 | **0** | **4** | 4 |
| | **v1** | **v2** | | | **v1** | **v2** | | **v1** | **v2** | |
| | 2,286 | 2,000 | | min 1 | 3 | 2 | | 1 | 2 | |
| | 0,000 | 1,000 | | min 2 | 0 | 1 | | 0 | 1 | |
| | **2,286** | **2** | | max | **3** | **2** | | **1** | **2** | |

2. Divide the game's Nash value in mixed strategies according to Shapley's value (1):

$$Sh_1 = v\{\mathrm{I}\} + \tfrac{1}{2}\left[v\{\mathrm{I},\mathrm{II}\} - v\{\mathrm{II}\} - v\{\mathrm{I}\}\right],$$
$$Sh_2 = v\{\mathrm{II}\} + \tfrac{1}{2}\left[v\{\mathrm{I},\mathrm{II}\} - v\{\mathrm{II}\} - v\{\mathrm{I}\}\right].$$

Find the maximal guaranteed payoffs $v\{\mathrm{I}\}$ and $v\{\mathrm{II}\}$ of players I and II. For this purpose fix a NE strategy of a third player as

$$\bar{y} = \begin{pmatrix} 3/7 & 4/7 \end{pmatrix}.$$

**Table 2 comments (continuation).** Denote mathematical expectations of the players' payoffs from coalition $S$ when mixed NE strategies are used by coalition $N\backslash S$ by $E_{S(i,\,j)}(\bar{y}),\, i, j = \overline{1,2}$. In table 2 the mathematical expectations are located at the left, and values are obtained by using the following formulas:

$$\begin{aligned}
E_{S(1,\,1)}(\bar{y}) &= \left(\tfrac{3}{7}\cdot 4 + \tfrac{4}{7}\cdot 1;\ \tfrac{3}{7}\cdot 2 + \tfrac{4}{7}\cdot 2;\ \tfrac{3}{7}\cdot 1 + \tfrac{4}{7}\cdot 2\right) = \left(2\tfrac{2}{7};\ 2;\ 1\tfrac{4}{7}\right);\\
E_{S(1,\,2)}(\bar{y}) &= \left(\tfrac{3}{7}\cdot 3 + \tfrac{4}{7}\cdot 5;\ \tfrac{3}{7}\cdot 1 + \tfrac{4}{7}\cdot 1;\ \tfrac{3}{7}\cdot 5 + \tfrac{4}{7}\cdot 3\right) = \left(4\tfrac{1}{7};\ 1;\ 3\tfrac{6}{7}\right);\\
E_{S(2,\,1)}(\bar{y}) &= \left(\tfrac{3}{7}\cdot 5 + \tfrac{4}{7}\cdot 1;\ \tfrac{3}{7}\cdot 3 + \tfrac{4}{7}\cdot 2;\ \tfrac{3}{7}\cdot 1 + \tfrac{4}{7}\cdot 2\right) = \left(2\tfrac{5}{7};\ 2\tfrac{3}{7};\ 1\tfrac{4}{7}\right);\\
E_{S(2,2)}(\bar{y}) &= \left(\tfrac{3}{7}\cdot 0 + \tfrac{4}{7}\cdot 0;\ \tfrac{3}{7}\cdot 4 + \tfrac{4}{7}\cdot 4;\ \tfrac{3}{7}\cdot 3 + \tfrac{4}{7}\cdot 2\right) = \left(0;\ 4;\ 2\tfrac{3}{7}\right).
\end{aligned}$$

Third element here is mathematical expectation of payoffs of the player III (see table 1 too).

Then, look at the table 1 or table 2,

$$\left.\begin{aligned}
\min H_1(x_1 = 1,\, x_2,\, \bar{y}) &= \min\left\{2\tfrac{2}{7};\ 4\tfrac{1}{7}\right\} = 2\tfrac{2}{7};\\
\min H_1(x_1 = 2,\, x_2,\, \bar{y}) &= \min\left\{2\tfrac{5}{7};\ 0\right\} = 0;
\end{aligned}\right| \quad v_1 = \max\left\{2\tfrac{2}{7};\ 0\right\} = 2\tfrac{2}{7};$$
$$\left.\begin{aligned}
\min H_2(x_1,\, x_2 = 1,\, \bar{y}) &= \min\left\{2;\ 2\tfrac{3}{7}\right\} = 2;\\
\min H_2(x_1,\, x_2 = 2,\, \bar{y}) &= \min\{1;\ 4\} = 1;
\end{aligned}\right| \quad v_2 = \max\{2;\ 1\} = 2.$$

Thus, maxmin payoff for player 1 is $v\{\mathrm{I}\} = 2\frac{2}{7}$ and for player 2 is $v\{\mathrm{II}\} = 2$. Hence,

$$Sh_1(\bar{y}) = v_1 + \tfrac{1}{2}\left(5\tfrac{1}{7} - v_1 - v_2\right) = 2\tfrac{2}{7} + \tfrac{1}{2}\left(5\tfrac{1}{7} - 2\tfrac{2}{7} - 2\right) = 2\tfrac{5}{7};$$
$$Sh_2(\bar{y}) = 2 + \tfrac{3}{7} = 2\tfrac{3}{7}.$$

Thus, PMS-vector is equal:

$$\mathrm{PMS}_1 = 2\frac{5}{7};\ \ \mathrm{PMS}_2 = 2\frac{3}{7};\ \ \mathrm{PMS}_3 = 2\frac{1}{3}.$$

Now dividing the payoffs of coalition $S$ in pure strategies proportional to the Shapley vector we get:

$$\lambda_1 = \frac{PMS_1}{PMS_1 + PMS_2} = \frac{2\frac{5}{7}}{5\frac{1}{7}} = \frac{19}{36},\ \lambda_2 = \frac{PMS_2}{PMS_1 + PMS_2} = \frac{2\frac{3}{7}}{5\frac{1}{7}} = \frac{17}{36}.$$

Hence, the newly defind payoffs of players I and II from coalition $S$ are:

$$A_{\mathrm{I}} = \lambda_1 A = \tfrac{19}{36}\begin{pmatrix} 4 & 6 \\ 8 & 3 \end{pmatrix} = \begin{pmatrix} 2\frac{1}{9} & 3\frac{1}{6} \\ 4\frac{2}{9} & 1\frac{7}{12} \end{pmatrix},$$
$$A_{\mathrm{II}} = \lambda_2 A = \tfrac{17}{36}\begin{pmatrix} 4 & 6 \\ 8 & 3 \end{pmatrix} = \begin{pmatrix} 1\frac{8}{9} & 2\frac{5}{6} \\ 3\frac{7}{9} & 1\frac{5}{12} \end{pmatrix},$$

and the matrix of the game became equal to

$$\begin{array}{llcc} & & \eta = 3/7 & 1-\eta = 4/7 \\ \xi = 1/3 & +(1,\,2) & \left[\left(2\frac{1}{9},\, 1\frac{8}{9}\right),\, 5\right] & \left[\left(3\frac{1}{6},\, 2\frac{5}{6}\right),\, 3\right] \\ 1-\xi = 2/3 & +(2,\,1) & \left[\left(4\frac{2}{9},\, 3\frac{7}{9}\right),\, 1\right] & \left[\left(1\frac{7}{12},\, 1\frac{5}{12}\right),\, 2\right] \end{array}.$$

## 5. Conclusion

In this paper the algorithm of getting imputation proportional to the PMS-value in the NE in mixed strategies, and the example which show realization of the proposed approach are given.

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# Precautionary Policy Rules in an Integrated Climate-Economy Differential Game with Climate Model Uncertainty

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**Abstract** The paper introduces structural uncertainty in an integrated climate-economy differential game such that the probability distribution of climate sensitivity is unknown. This is generated by perturbing a continuous-time version of the climate model in Nordhaus (1992) and Nordhaus and Yang (1996). Instead of analyzing choices of regional representative consumers, we define social profit from regional production as the payoff to regional policymakers. There are two types of players: Firstly, regional policymakers $j = 1, 2, \ldots, N$, who are tied to a region by acting as a sovereign regional social planner who can only enforce regional emissions reduction policies. Secondly, investors $i = 1, 2, \ldots, n$, who are not tied to any region who allocate investments between firms (production processes generating emissions) located in all regions $j = 1, 2, \ldots, N$. We identify policymakers' optimal policy responses to firms' investment responses as well as firms' optimal investment responses to policymakers' policy responses in a global subgame perfect Nash equilibrium when policymakers do not cooperate and compare it to when policymakers coordinate national policy rules

**Keywords:** Differential games, structural uncertainty, feedback Nash equilibrium, closed-loop equilibrium, uncertainty aversion.

## 1. Introduction

Climate change policy is subject to fundamental uncertainties concerning the underlying scientific information available. Policy makers' decision to take or not take measures today are based on scientists' projections, generated by computer climate models and evaluated for different emissions scenarios. A common measure when comparing projections is the change in equilibrium mean atmospheric temperature that results from a doubling of $CO_2$. An increase in atmospheric $CO_2$ changes net radiation at the tropopause (radiative forcing) in the atmosphere, which influences the energy balance of the climate system, and hence, changes the mean atmospheric temperature $T_t$. When comparing computer climate models, a conclusive component is equilibrium climate sensitivity, which is defined as the ratio between a steady-state change in mean atmospheric temperature $\Delta T$ and a steady-state change in radiative forcing $\Delta R_t$.

$$\frac{\Delta T}{\Delta R_t} \tag{1}$$

Climate sensitivity depends on several underlying physical feedback processes which are hard to predict. One of the most uncertain is the cloud effect, other are water vapor, albedo and vegetation effect (see e.g. Harvey (2000) and Hansen, Lacis, Rind and Russell, Climate Sensitivity: Analysis of Feedback Mechanisms (1984a)). A recent analysis by Roe and Baker (2007) shows how climate sensitivity and its probability distribution becomes unpredictable due to uncertainties in underlying physical feedback factors when they translate into uncertainty in climate sensitivity. The apparent uncertainty in climate sensitivity as also evident in IPCC, Climate Change 2007: Working Group I Report The Physical Science Basis (2007c) Executive Summary which states 'The equilibrium climate sensitivity is a measure of the climate system response to sustained radiative forcing. It is not a projection but is defined as the global average surface warming following a doubling of carbon dioxide concentrations. It is *likely* to be in the range 2°C to 4.5°C with a best estimate of about 3°C, and is *very unlikely* to be less than 1.5°C. Values substantially higher than 4.5°C cannot be excluded, but agreement of models with observations is not as good for those values.'

Roe and Baker (2007) conclude that for high temperature levels above the IPCC interval of 2.0$^o$C - 4.5$^o$C the probability distribution changes very little to changes in the variance in the underlying physical processes. Hence, their conclusion is that scientific research that reduces uncertainty in the underlying physical processes has little effect in reducing uncertainty in climate sensitivity at high temperature outcomes. For values above 2.0$^o$C - 4.5$^o$C, the upper fat tail of the probability distribution of equilibrium climate sensitivity would remain fat despite progress in understanding the underlying physical processes. Roe and Baker (2007) therefore conclude *'We do not therefore expect the range presented in the next IPCC report to be different from that in the 2007 report'* and *'we are constrained by the inevitable: the more likely a large warming is for a given forcing (i.e. the greater the positive feedbacks) the greater the uncertainty will be in the magnitude of that warming.'*

What does this message tell policymakers? Clearly, optimal reductions in $CO_2$ emissions would differ whether the decision is based on a climate model predicting an equilibrium temperature of 1.5$^o$C or a model predicting 4.5$^o$C or even higher. Secondly, the fundamental uncertainty concerns not only future outcomes but also future probability distributions of climate sensitivity. True or inferred probability distributions are not available from current data. This deeper type of uncertainty refers to *model uncertainty* rather than *variable uncertainty.* When comparing the projections of computer climate models based on current data and knowledge, the probability distributions differ among models as they are based on components resulting from scientists' ad hoc assumptions and guesswork (Harvey, 2000).

So which climate model should we then base our decisions on when looking for optimal emissions reductions? Nordhaus (1992) used for example a model that predicts an equilibrium temperature of app. 3$^o$C located in the mid range of the IPCC interval. A more precautionary policymaker would perhaps prefer to use a model with 4.5$^o$C equilibrium to be more certain that she is not bad off even in the case a model corresponding to the upper climate sensitivity range becomes true. Our approach in this paper is simply *to leave the question unanswered* by defining structural uncertainty in the climate model such that policymakers face a set of climate models corresponding to a range of climate sensitivities. These conditions of uncertainty better mimics the scientific uncertainties that IPCC, Climate Change

2007: Working Group I Report The Physical Science Basis (2007c), Roe and Baker (2007) and Allen and Frame (2007) find in the computer climate models.

In the literature on the theory of decision-making it is common to distinguish between risk and uncertainty. The former refers to a process where actual outcome of variable is unknown but its probability distribution (objective or subjective) is known or can be estimated from samples and corresponds to *variable uncertainty.* The latter refers to when outcomes as well as probability distributions are unknown and corresponds to *model or structural uncertainty.* Another approach, taking into account unknown probability distributions or structural uncertainty, is Weitzman (2007) though his analysis is highly abstract by only looking at the effects of uncertainty (unknown probability distribution) with a CRRA utility function in a two-period analysis without specifying the source of uncertainty. The discussion on unknown probability distributions is not new though. Already Knight (1971) suggested that for many choices, the assumption of known probability distributions is too strong. Moreover, Keynes (1921), in his treatise on probability, put forward the question whether we should be indifferent between two scenarios that have equal probabilities, but one of them is based on greater knowledge. Savage (1954) argued that we should, while Ellsberg (1961) showed in an experiment that in reality humans tend not to do so. A person that is facing two uncertain lotteries with the same (subjective) probability to success, but with less information provided in the second lottery, tends to prefer the first lottery where more information is available. Having Ellsberg's paradox in mind, Gilboa and Schmeidler (1989) formulated a maximin decision criterion, by weakening Savage's Sure-Thing Principle, to explain the result from the Ellsberg experiment. In plain words, the decision-maker is suggested to maximize expected utility under the belief that the worst case scenario will happen (a maximin decision criterion). This preference is usually referred to as uncertainty aversion (as opposed to risk aversion which is a tendency to avoid uncertain outcomes) which is about avoiding bad since due to the pessimistic climate model is correct. The maximin decision criterion has been applied before in static models by e.g. Chichilnisky (2000) and Bretteville Froyn (2005) with the general result that it leads to an increase in abatement effort. Roseta-Palma and Xepapadeas (2004) apply it in a dynamic model of a water management problem following Hansen and Sargent (2001).

In this paper we start from Roe and Baker (2007) and Allen and Frame (2007), that current data from underlying physical processes are not sufficient to predict climate sensitivity. Firstly, we introduce unknown probability distributions of climate sensitivity in an integrated assessment model based on a continuous-time version of the climate model used in Nordhaus (1992) and Nordhaus and Yang (1996). Secondly, we introduce a type of precautionary preferences among policymakers based on maximin decision criteria by adding a preference parameter for robustness in policymakers' payoff functions. Nordhaus (1992) and Nordhaus and Yang (1996) as well as Weitzman (2007) use a representative consumer CRRA utility function. However, since we are interested in policymakers' and firms' responses to each other rather than the choices of regional representative consumers, we instead define the social profit from physical and natural capital within each region as the payoff to the policymaker. We introduce two types of players, investors $i = 1, 2, \ldots, n$, investing in firms (production processes generating emissions) located in region $j = 1, 2, \ldots, N$, and who are not physically tied to any specific region, and policymakers $j = 1, 2, \ldots, N$, who are tied to a region by acting as a sovereign regional social planner who can only enforce regional policies while taking the policies of foreign policymakers as well as all investors as given in her optimal choice of regional policy. We then identify an analytically tractable feedback Nash equilibrium in policy strategies for $N$ asymmetric policymakers subject to the climate model with uncertain climate sensitivity. Finding analytically tractable solutions to nonlinear feedback Nash equilibria with asymmetric players is usually difficult. The analytical solution found in this paper opens up for analyzes of different game formulations with asymmetric players in climate models of the type in Nordhaus (1992) and Nordhaus and Yang (1996).

The following sections are organized in the following way. In section 2., the range of climate models and climate change impacts are presented. Section 3. presents players and payoff functions, optimization problems and the optimal non-cooperative and cooperative policy rules, which is followed by a summary in 4..

## 2. The Climate Models

In this section we present the set of climate models which are generated by perturbing a continuous-time version of the climate benchmark model in Nordhaus (1992) and Nordhaus and Yang (1996) describing the relationship between atmospheric concentration rate $M_t$, radiative forcing $R_t$, atmospheric temperature $T_t$ and deep ocean temperature $\tilde{T}_t$. The net radiation balance is $R_t = F_t + S_t$ where $F_t$ is outgoing radiation and $S_t$ ingoing radiation. In equilibrium $F_t = -S_t$ since $R_t = 0$. Increasing anthropogenic emissions results in radiative forcing $\Delta R_{ft}$ which moves the system away from the initial equilibrium towards a new equilibrium as $\Delta R_{ft} = -\Delta R_t$ and $\Delta R_t = \Delta F_t + \Delta S_t$. Accordingly, the steady-state mean temperature must change by $\Delta T_t$ between the two equilibria.

$$\Delta T_t = \lambda_0 \Delta R_{ft} + \sum_s f_s \Delta T_t \tag{2}$$

where $\lambda_0$ is the reference climate sensitivity in absence of underlying feedback factors $f_s$ such as cloud effect, albedo effect and vegetation effect.[1] Denoting $f \equiv \sum_s f_s$, (2) can be rewritten as

$$\Delta T_t = \lambda_0 \frac{1}{1-f} \Delta R_{ft} \tag{3}$$

where $1/(1-f)$ is the change in equilibrium temperature that derives from feedback factors also called the gain of the climate system and equals $\Delta T / \Delta T_0$ i.e. the proportion by which the system response has changed due to feedback. In Nordhaus (1992) and Nordhaus and Yang (1996) the discrete-time radiative forcing dynamics $R_t$ is

$$R_t = \frac{\lambda_1 ln(M_t/M_0)}{ln(2)} + O_t \tag{4}$$

[1] $\lambda_0 = 1/(4\sigma T_e^3)$ is found as the balanced radiative forcing at a blackbody planet, by $dT/dF$ from Stefan-Boltzmann relationship $F = -\sigma T^4$.

where $\lambda_1$ is a parameter conclusive for equilibrium climate sensitivity and $O_t$ is radiative forcing from non-anthropogenic sources. For analytical tractability of Isaacs-Bellman-Flemming equations, we approximate (4) by the square-root approximation in (6). Compared to (4), this approximation overestimates radiative forcing by at most app 5% in the range from current rate up to a tripling in $CO_2$ rate. However, the advantage is that it makes an analytical solution possible which is necessary for identifying subgame consistent cooperative solutions with time-consistent payment streams in forthcoming analyzes of this game model.

The climate model in (6) - (8) shows the atmospheric concentration rate $M_t$ influence on global mean atmospheric and upper ocean temperature $T_t$ and deep ocean temperature $\tilde{T}_t$ relative to the preindustrial level via the change in radiative forcing $R_t$ (measured in $Wm^{-2}$) in (6). Emissions $E_{ijt}$ originates from production in $j = 1, 2, \ldots, N$ regions where $\eta_{ij} q_{ijt}$ is abatement, $q_{ijt}$ being abatement effort undertaken in firm $i$ in region $j$ and $\eta_{ij} > 0$ an firm-specific efficiency parameter. In equation (5), the sum of net emissions flows $E_{ijt} - \eta_{ij} q_{ijt}$ at time $t \in [0, \infty)$ from production processes $i = 1, 2, \ldots, n$ in regions $j = 1, 2, \ldots, N$ accumulates to the global atmospheric concentration $CO_2$ stock $M_t$. $\xi_j > 0$ is the marginal atmospheric retention ratio and $\Omega > 0$ the rate of assimilation.

$$dM = \left[ \sum_{m=1}^{n} \sum_{k=1}^{N} \xi_j (E_{mkt} - \eta_{mk} q_{mkt}) - \Omega M_t \right] dt \tag{5}$$

$$R_t = \frac{\lambda_1 \sqrt{M_t/M_0} + \nu}{\alpha\sqrt{2}} + O_t \tag{6}$$

$$dT = \frac{1}{R_1} \left( R_t - \lambda T - \frac{R_2}{\tau_2}(T_t - \tilde{T}_t) \right) dt \tag{7}$$

$$d\tilde{T} = \frac{1}{R_2} \left( \frac{R_2}{\tau_2}(T_t - \tilde{T}_t) \right) dt \tag{8}$$

A change in radiative forcing $R_t$ affects the energy balance of the climate system and hence the global mean atmospheric temperature $T_t$ through the relationship (7) via the deep ocean temperature $\tilde{T}_t$ in (8). The physical constants are $\lambda$ as a component in the underlying feedback processes, $R_i$ the thermal capacity of the atmosphere and the upper ocean ($i = 1$) and the deep ocean ($i = 2$), respectively. $1/\tau_2$ is the transfer rate from the atmosphere and upper ocean layer to the deep ocean layer. The parameters $\alpha$ and $\nu$ are set to fit (4).

### 2.1. Introducing Climate Model Uncertainty

Roe and Baker (2007) find that for high temperature levels above the IPCC interval of 2.0$^o$C - 4.5$^o$C, the probability distribution of equilibrium climate sensitivity changes very little to changes in the variance in underlying physical processes. Hence, their conclusion is that scientific research that reduces uncertainty in the underlying physical processes has little effect in reducing uncertainty in climate sensitivity. The fatter tail of the probability distribution of climate sensitivity for values above 4.5$^o$C would remain despite progress in understanding the underlying physical processes. Roe and Baker (2007) conclude *'we are constrained by the* *inevitable: the more likely a large warming is for a given forcing (i.e. the greater the positive feedbacks) the greater the uncertainty will be in the magnitude of that warming.'*[2] If the world warms by $4^o$C, the conditions in the underlying feedback processes $f$, such as cloud and water vapor effects, may have changed from current conditions making it impossible to determine when warming stops. Hence, from all climate and physical data we can observe today, we cannot distinguish between a climate sensitivity of $4^o$C or $6^o$C. Another consequence is that dramatic changes in the physical processes may not be needed for dramatic changes in the climate sensitivity (Visser *et al.*, 2000).

In this paper, we want to reconstruct the conditions of uncertainty to be closer to real conditions of uncertainty as described in Roe and Baker (2007). We also introduce precautionary preferences based on the Gilboa and Schmeidler (1989) idea, however this turns out to be a problem since their axioms are based on static decision making and not sufficient for dynamic models. Specifically, they do not state how the decision-maker's beliefs are affected by new information (which could increase or decrease scientific uncertainty as time proceeds). We then follow Hansen, Sargent, Turmuhambetova and Williams, Robustness and Uncertainty Aversion (2001b) and suggest that a rational decision-maker updates her beliefs to new information due to scientific progresses by a rule derived from backward induction.[3]

However, we start by introducing a new approach to scientific uncertainty in climate modeling by perturbing climate sensitivity in the model (6) - (8), defining the following process

$$B_t = \hat{B}_t + \int_0^t \lambda_s ds \quad \lambda_s \in [\lambda_{min}, \lambda_{max}] \tag{9}$$

where $d\hat{B}$ is the increment of the Wiener process $\hat{B}_t$ on the probability space $(\Xi, \Phi, G)$ with variance $\sigma^2 \geq 0$ where $\{\hat{B}_t : t \geq 0\}$. Moreover, $\{\lambda_t : t \geq 0\}$ is a progressively measurable drift distortion, implying that the probability distribution of $B_t$ itself is distorted and the probability measure $G$ is replaced by another unknown probability measure $Q$ on the space $(\Xi, \Phi, Q)$. Hence, (6) is replaced by

$$\frac{\sigma[\lambda_t dt + d\hat{B}]\sqrt{M_t/M_0} + \nu dt}{\alpha\sqrt{2}} + O_t dt \tag{10}$$

Since both mean and probability distribution of the drift term $\lambda_t$ are unknown, (9) yields different statistics of climate sensitivity in (10) where the interval $[\lambda_{min}, \lambda_{max}]$ indicates the maximum model specification error, e.g. corresponding to the range of climate sensitivities as the $2^o$ - $4.5^o$ range by IPCC, Climate Change 2007: Working Group I Report The Physical Science Basis (2007c) or even wider ranges that the policymaker is willing to accept. Hence, (9) together with (10) describe a set of

[2] Allen and Frame (2007) goes further on this result and conclude that scientific research trying to narrow the uncertainties in the upper end of the tail is perhaps useless, and hence, in the choice of research as well as policy targets one should take this into account.

[3] Gilboa and Schmeidler (1989) view uncertainty aversion as a minimization of the set of probability measures while Hansen, Sargent, Turmuhambetova and Williams, Robustness and Uncertainty Aversion (2001b) set a robust control problem and let its perturbations be interpreted as multiple priors in max-min expected utility theory.

climate models with the only restriction that $\lambda_s$ is bounded by the constraint in (9). Since the mean as well as probability distribution of $\lambda_s$ is unknown, the problem of the policymakers becomes two-folded: (1) what $\lambda_t$ should today's policy decision be based upon and (2) by which rules should $\lambda_t$ be updated if more scientific knowledge about climate sensitivity is gained as time proceeds?

### 2.2. Regional Climate Change Impacts

All regions $j = 1, 2, \ldots, N$ face the same change in global mean surface temperature $T_t - T_0$ while benefit and damage costs may differ significantly across regions $j = 1, 2, \ldots, N$. A broad overview of climate change impacts is given in IPCC, Climate Change 2001: Impacts, Adaptation and Vulnerability. Contribution of Working Group II to the Third Assessment Report of the Intergovernmental Panel on Climate Change (2001b), IPCC, Climate Change 2007: Working Group II Report Impacts, Adaptation and Vulnerability (2007d) as well as Tol, Estimates of the Damage Costs of Climate Change Part I. Benchmark Estimates (2002a) and Tol, Estimates of the Damage Costs of Climate Change Part II. Dynamic Estimates (2002b). The considered impacts are often on natural capitals such as agriculture, forestry, water resources, loss of dry- and wetland (due to sea-level rise) and increased consumption of energy resources (heating and cooling).[4] In this model we introduce damages on regional natural capitals $x_{jt}$ that derive from a global mean temperature deviation $T_t - T_0$. Another reason is that the solution to the partial differential equation system is technically simplified by separating benefits and costs to a set of polluting stocks and a set of damaged stocks.

Regional physical capital accumulation in (11) follows the structure of Merton (1975) and Yeung (1995) with a Cobb-Douglas investment function and depreciation rate $\delta_{ij} > 0$ where $I_{ijt}$ is the investment by investor $i$ in region $j$ in period $t$

$$dk_{ij} = \left[ I_{ijt}^{1/2} k_{ijt}^{1/2} - \delta_{ij} k_{ijt} \right] dt \tag{11}$$

$$dx_j = \left[ r_j \left( 1 - \frac{x_{jt}^{1/2}}{K_j} \right) x_{jt}^{1/2} - \frac{\Psi_j (T_t - T_0)}{x_{jt}^{1/2}} x_{jt} \right] dt \tag{12}$$

$$i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, N \tag{13}$$

The equations of motion of $x_{jt}$ in (12) are specified forms of the relationship used in Hennlock, A Differential Game on the Management of Natural Capital Subject to Emissions from Industry Production (2005a) and Hennlock, An International Marine Pollutant Sink in an Asymmetric Environmental Technology Game (2008c) and consist of a modified natural growth function with intrinsic growth $r_j > 0$ and regional carrying capacity $\bar{x}_{jt} = K_j^2$. The loss of $x_{jt}$ due to a deviation in global mean temperature rise $T_t - T_0$, where $T_0$ is the 1990 mean temperature level, is determined by a non-linear endogenous decay rate $\Psi_j(T_t - T_0) x_{jt}^{1/2}$, suggesting that the damage from a given mean temperature deviation accelerates as the natural capital stock $x_{jt}$ decreases where $\Psi_j > 0$ is a region-specific damage parameter. (5)

[4] Most research has been conducted on the effects of sea level rise e.g. Titus and Narayan (1991).

to (13) define the dynamic system with $2 + M(1 + n)$ state variables. The introduction of the unknown variable $\lambda_t$ in (7) implies that the dynamics of the system corresponds to the set $[\lambda_{min}, \lambda_{max}]$ of climate models. Hence, climate model uncertainty in (6) - (9) also makes regional impacts uncertain for a given net emissions scenario $\sum^N \sum^n \{E_{ijt} - \eta_{ij} q_{ijt}\}$.

## 3. Players and Payoffs

There are two types of players, investors $i = 1, 2, \ldots, n$, investing in firms (production processes) located in region $j = 1, 2, \ldots, N$, and who are not physically tied to any specific region, and policymakers $j = 1, 2, \ldots, N$, who are tied to a region by acting as a sovereign regional social planner who can only enforce regional policies while taking the policies of foreign policymakers as well as all investors as given in her optimal choice of regional policy. If all $n + N$ players assume that the other players do their best, there is a global Nash equilibrium in policies and investment rates. Identifying a Nash equilibrium as benchmark, several patterns of cooperation are then possible which are further discussed in section 3.4.

### 3.1. Investor $i \in [1, n]$

Every investor $i \in [1, n]$ solves a stochastic optimization problem and allocates total investment $\sum_{k=1}^N I_{ikt}$ in period $t$ between firms (production processes) located in all regions $j \in [1, N]$ for the production of a good $y$ that is sold on the world market at unit price. Profit-maximization by each investor $i$ follows Hennlock, A Robust Feedback Nash Equilibrium in a Climate Change Policy Game (2008d), and is obtained by allocating investment (and thereby production activity) between the regions $j \in [1, N]$. The expected payoff of investor $i$, where $\epsilon$ is the expectation operator, is

$$\max_{u_{ikt}} \epsilon \int_0^\infty \sum_{k=1}^N \left\{ p(1 - u_{ikt}) y_{ikt} - \frac{c_{ik}(q_{ikt})^2}{E_{ikt}} \right\} e^{-\rho_i t} dt \tag{14}$$

Investor $i$ seeks the optimal expected cash dividend from each firm $i \in [1, n]$ in each period $t$ located in region $j \in [1, N]$ by controlling the share $u_{ijt} \in [0, 1]$ of net profit that is reinvested in regional capital stocks $k_{ij}$. By investing $I_{ijt}$ investor $i$ contributes to total industry output $y_{jt}$ in region $j$, which is $y_{jt} = \sum_{m=1}^n \phi_{mj} k_{mjt}^{1/2}$. The amount reinvested $I_{ijt}$ in period $t$ is the remainder $I_{ijt} = u_{ijt} y_{ijt}$. Investor $i$'s discount rate is $\rho_i > 0$. The last term in (14) is firm $i$'s abatement cost, which is quadratic in regional abatement effort $q_{ijt}$ due to capacity constraints as more local abatement effort $q_{ijt}$ is employed. Abatement cost is decreasing in $E_{ijt}$, suggesting that it requires more expensive techniques as $E_{ijt}$ becomes smaller. $c_{ij} > 0$ is an abatement cost parameter for production $i$ in region $j$. The total emissions flow from region $j$ is $E_{jt} = \sum_{m=1}^n E_{mjt} = \sum_{m=1}^n \varphi_{mj} y_{mjt}$ where $\varphi_{mj} > 0$ is a firm-specific emissions parameter and the firm-specific level of $q_{ijt}^*$ set by policymaker $j$ is taken as given by investor $i$ when seeking optimal investment rates $u_{ikt}^*$. Every investor $i$ maximizes the payoff function (14) subject to the dynamic system formed by (5) and (7) to (13) and $q_{ijt}^* \geq \forall i \in [1, n]$ and $\forall j \in [1, N]$. The investors' stochastic optimal problems are solved in appendix A.1. The feedback investment rate strategies in terms of parameter values are:

$$u_{ijt}^{*}=\left(\frac{a_{ij}}{2p}\right)^{2}\frac{1}{\phi_{ij}k_{ijt}^{1/2}}\in[0,1]\quad\forall i\in[1,n]\quad\text{and}\quad\forall j\in[1,N]\tag{15}$$

Investor $i$'s feedback Nash investment rate $u_{ijt}^{*}$ is decreasing in $k_{ijt}$, implying that the share of net profit used for investment is large in the beginning when $k_{ijt}$ is low during business start-up. As $k_{ijt}$ grows, investor $i$ reduces the share of net profit reinvested in capital stock $k_{ij}$ located in region $j$. The optimal investment rate rules in (15) are further discussed in Hennlock, A Robust Feedback Nash Equilibrium in a Climate Change Policy Game (2008d).

### 3.2. Policymaker $j \in [1, N]$

Policymaker $j = 1, 2, \ldots, N$ receives a social profit from production (or employment) within region $j$ which is proportional to total production level $y_{jt} = \sum_{m=1}^{n} \phi_{mj}k_{mjt}^{1/2}$ in region $j$ but also a loss of regional natural capital $x_{jt}$ due to global climate change $T_t - T_0$. In the Nash equilibrium, each policymaker $j \in [1, N]$ seeks the optimal emissions reductions $\eta_{ij}q_{ijt}^{*}$ in each firm $i$ located in region $j$ given that the remaining $N-1$ policy makers individually seek the optimal $\eta_{ik}q_{ikt}^{*}$ $\forall k \neq j$ and that every investor $i \in [1, n]$ individually seek optimal investment rate $u_{ijt}^{*}$. The expected payoff of policymaker $j \in [1, N]$ is[5]

$$\epsilon\int_{0}^{\infty}\sum_{m=1}^{n}\left\{\omega_j y_{mjt}+\psi_j x_{jt}^{1/2}-\frac{c_{mj}q_{mjt}^{2}}{E_{mjt}}\right\}e^{-\rho_j t}dt+\theta_j R(Q)\tag{16}$$

The first term is social benefit of employment that is assumed to be proportional to total regional production $y_{mjt} = \phi_{mj}k_{mjt}^{1/2}$ where the parameter $\omega_j > 0$. The second term is the benefit from the regional natural capital $x_{jt}$ in region $j \in [1, N]$ where $\psi_j > 0$ is a parameter. The last term within the brackets is the abatement cost function. Policy maker $j$'s discount rate is $\rho_j > 0$. Following Hansen, Sargent, Turmuhambetova and Williams, Robustness and Uncertainty Aversion (2001b), policymaker $j$'s payoff function can be written as (16) in a multiplier robust problem where $1/\theta_j \geq 0$ denotes the policy maker's preference for robustness which together with $R(Q)$, the finite entropy, act as Lagrangian multiplier in (16). The process $\{\lambda_s\}$ in (6) is unknown and will change the future probability distribution of $B_t$ having probability measure $Q$ relative to the distribution of $\hat{B}_t$ having measure $G$. The Kullback-Leibler distance between $Q$ and $G$ is

$$R(Q)=\int_{0}^{\infty}\epsilon_Q\left(\frac{|\lambda_s|^2}{2}\right)e^{-\rho_j t}ds\tag{17}$$

As long as $R(Q) < \Theta_j$ in (16) is finite

$$Q\left\{\int_{0}^{t}|\lambda_s|^2 ds<\infty\right\}=1\tag{18}$$

[5] Technically, it is straightforward to also let investors be uncertainty averse in this solution.

which has the property that $Q$ is locally continuous with respect to $G$, implying that $G$ and $Q$ cannot be distinguished with finite data, and hence, modeling a situation with a decision-maker that cannot know the future probability distribution when using current data. Following Hansen and Sargent (2001), policymaker $j \in [1, N]$ optimal problem of finding optimal emissions reductions for each firm $\eta_{ij} q^*_{ijt}$, can be written as:

$$\max_{q_{jt}} \min_{\lambda_{jt}} \epsilon \int_0^\infty \sum_{m=1}^{n} \left\{ \omega_j y_{mjt} + \psi_j x_{jt}^{1/2} - \frac{c_{mj} q^2_{mjt}}{E_{mjt}} + \frac{\theta_j \lambda_t^2}{2} \right\} e^{-\rho_j t} dt \tag{19}$$

subject to

$$dk_{ij} = \left[ (I^*_{ijt})^{1/2} k_{ijt}^{1/2} - \delta_{ij} k_{ijt} \right] dt \tag{20}$$

$$dx_j = \left[ r_j \left( 1 - \frac{x_{jt}^{1/2}}{K_j} \right) x_{jt}^{1/2} - \frac{\Psi_j (T_t - T_0)}{x_{jt}^{1/2}} x_{jt} \right] dt \tag{21}$$

$$dM = \left[ \sum_{m=1}^{n} \sum_{k=1}^{N} \xi_j (E_{mkt} - \eta_{mk} q_{mkt}) - \Omega M_t \right] dt \tag{22}$$

$$\begin{aligned} dT = \frac{1}{R_1} \Bigg( & \frac{\sigma[\lambda_{jt} dt + d\hat{B}] \sqrt{M_t / M_0} + \nu dt}{\alpha \sqrt{2}} + O_t dt \\ & - \lambda T dt - \frac{R_2}{\tau_2} (T_t - \tilde{T}_t) dt \Bigg) \end{aligned} \tag{23}$$

$$d\tilde{T} = \frac{1}{R_2} \left( \frac{R_2}{\tau_2} (T_t - \tilde{T}_t) \right) dt \tag{24}$$

$$i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, N \tag{25}$$

**Definition 1. Feedback Nash Equilibrium** If there exist $N$ value functions $W_j(\mathbf{k}, \mathbf{x}, M, T, \tilde{T}, t)$ where

$$\mathbf{k} = (k_{11}, k_{12}, \ldots, k_{1N}, k_{21}, k_{22}, \ldots, k_{2N}, k_{n1}, k_{n2}, \ldots, k_{nN}) \tag{26}$$

and $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ that satisfy

$$\begin{aligned} & W_j(\boldsymbol{k}, \boldsymbol{x}, M, T, \tilde{T}, t) = \\ \epsilon \int_0^\infty & \sum_{m=1}^{n} \left\{ \omega_j y_{mjt} + \psi_j x_{jt}^{1/2} - \frac{c_{mj} (q^*_{mjt})^2}{E_{mjt}} + \frac{\theta_j (\lambda^*_{jt})^2}{2} \right\} e^{-\rho_j t} dt \\ & \geq \epsilon \int_0^\infty \sum_{m=1}^{n} \left\{ \omega_j y_{mjt} + \psi_j x_{jt}^{1/2} - \frac{c_{mj} q^2_{mjt}}{E_{mjt}} + \frac{\theta_j \lambda^2_{jt}}{2} \right\} e^{-\rho_j t} dt \end{aligned} \tag{27}$$

for strategies $q^*_{jt}(k_j, t) \subseteq R^1$ and $\lambda^*_{jt}(M, t) \subseteq R^1$ given that $\lambda^*_{jt}(M, t) \equiv \arg\min W_j(\mathbf{k}, \mathbf{x}, \mathbf{L}, M, T, \tilde{T}, t) \quad \forall j \in N$ and which satisfy the state equations,

$$dk_{ij} = \left[(I^*_{ijt})^{1/2} k^{1/2}_{ijt} - \delta_{ij} k_{ijt}\right] dt \tag{28}$$

$$dx_j = \left[r_j \left(1 - \frac{x^{1/2}_{jt}}{K_j}\right) x^{1/2}_{jt} - \frac{\Psi_j(T_t - T_0)}{x^{1/2}_{jt}} x_{jt}\right] dt \tag{29}$$

$$dM = \left[\sum_{m=1}^{n} \sum_{k=1}^{N} \xi_j (E_{mkt} - \eta_{mk} q^*_{mkt}) - \Omega M_t\right] dt \tag{30}$$

$$dT = \frac{1}{R_1}\Bigg(\frac{\sigma[\lambda_{jt} dt + d\hat{B}]\sqrt{M_t/M_0} + \nu dt}{\alpha\sqrt{2}} + O_t dt \\ - \lambda T dt - \frac{R_2}{\tau_2}(T_t - \tilde{T}_t) dt\Bigg) \tag{31}$$

$$d\tilde{T} = \frac{1}{R_2}\left(\frac{R_2}{\tau_2}(T_t - \tilde{T}_t)\right) dt \tag{32}$$

$$i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, N \tag{33}$$

The feedback Nash controls strategies

$$\Gamma^*_{ijt} = \{q^*_{ij}(k_{ij}), \lambda^*_j(k_j)\} \quad \forall i \in [1, n] \quad \forall j \in [1, N] \tag{34}$$

provide a robust feedback Nash equilibrium solution of the game defined in (19) to (25) given (49) to (55) (Basar and Olsder, 1999).

The value functions in definition 3 satisfy the partial differential equation system (35) - (36). Using (49) - (55) and (19) - (25) and definition 2 and 3 yield the Isaacs-

Bellman-Fleming equation (Fleming and Richel, 1975) of policy maker $j$:

$$
\begin{aligned}
-\frac{\partial W_j}{\partial t} = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (35)\\
\max_{q_{jt}} \min_{\lambda_{jt}} \sum_{m=1}^{n} \left\{ \omega_j y_{mjt} + \psi_j x_{jt}^{1/2} - \frac{c_{mj} q_{mjt}^2}{E_{mjt}} + \frac{\theta_j \lambda_{jt}^2}{2} \right\} e^{-\rho_j t} \\
+ \sum_{m=1}^{n} \sum_{k=1}^{N} \frac{\partial W_j}{\partial k_{mk}} \left[ (I_{mkt}^*)^{1/2} k_{mkt}^{1/2} - \delta_{mk} k_{mkt} \right] \\
+ \sum_{k=1}^{N} \frac{\partial W_j}{\partial x_k} \left[ r_k \left( 1 - \frac{\sqrt{x_{kt}}}{K_k} \right) \sqrt{x_{kt}} - \frac{\Psi_k (T_t - T_0)}{\sqrt{x_{kt}}} x_{kt} \right] \\
+ \frac{\partial W_j}{\partial M} \left[ \sum_{m=1}^{n} \sum_{k=1}^{N} \xi_j (E_{mkt} - \eta_{mk} q_{mkt}) - \Omega M_t \right] \\
+ \frac{\partial W_j}{\partial T} \left[ \frac{1}{R_1} \left( \frac{\sigma \lambda_{jt} \sqrt{M_t / M_0} + \nu}{\alpha \sqrt{2}} + O_t - \lambda T - \frac{R_2}{\tau_2} (T_t - \tilde{T}_t) \right) \right] \\
+ \frac{1}{2} \frac{\partial^2 W_j}{\partial T^2} \sigma^2 M_t \\
+ \frac{\partial V_i}{\partial \tilde{T}} \left[ \frac{1}{R_2} \left( \frac{R_2}{\tau_2} (T_t - \tilde{T}_t) \right) \right]
\end{aligned}
$$

$$
i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, N \tag{36}
$$

The robust feedback Nash controls strategies

$$
\Gamma_{ijt}^* = \{ q_{ij}^* (k_{ijt}), \lambda_j^* (M_t) \} \quad \forall i \in [1, n] \quad \forall j \in [1, N] \tag{37}
$$

are given by maximizing the partial differential equations (35) with respect to (37) for the $N$ policymakers and solving for the robust policy rules.

### 3.3. Feedback Nash Equilbrium Policy Rules

Since, policymakers' payoff functions are time autonomous, these policy rules (38) are time consistent and subgame perfect policy strategies or responses to the evolution of firms' net emissions $E_{ijt}$. Hence, the policy rules are credible and efficient threats in every subgame starting at $t < \infty$ (Dockner *et al.*, 2000), given the policymakers' preferences for robustness. By announcing the policy rule $q_{ijt}^*(E_{ijt})$, a policymaker regulates net emissions $E_{ijt} - \eta_{ij} q_{ijt}^*$ for each firm $i$ at each instant of time. As expected, the optimal abatement effort is $q_{ijt}^* \geq 0$ for all $k_{ijt} \geq 0$ since $\partial W_j / \partial M_t \leq 0 \ \forall t$.

$$
q_{ij}^* (E_{ijt}) = - \frac{\partial W_j}{\partial M_t} \frac{\xi_j \eta_{ij}}{2 c_{ij}} E_{ijt} e^{\rho_j t} \tag{38}
$$

The policy rules $q_{ijt}^*(E_{ijt})$ state that abatement effort $q_{ijt}^*$ in firm $j$ is proportional to firm $j$'s total emissions levels $E_{ijt}$ and the size of the proportion is determined by a ratio between policymakers' shadow costs, $\partial W / \partial M_t$, of $CO_2$ and the firm's abatement cost parameter $c_{ij}$. The partials derivatives in (38) are policymakers' expected

Nash shadow cost of $CO_2$, which are identified in appendix A.2. by differentiating the value functions (84) using (96) and (97). Substituting the Nash shadow costs into (38) yields the robust feedback Nash equilibrium strategies in terms of model parameter values.

$$q^*_{ijt}(E_{ijt}) = \frac{\sum_{j=1}^{N} \frac{b_{jk}}{2} \Psi_j \left(\frac{\sigma}{R_1 \alpha \sqrt{2}}\right)^2}{\rho_j + \frac{R_2}{R_1 \tau_2}(1 - \frac{1}{1+\rho_j \tau_2})} \frac{\xi_j \eta_{ij}}{2 c_{ij} M_0 \theta_j (\rho_j + \Omega)} \cdot E_{ijt} \geq 0 \tag{39}$$

The optimal policy rules in (39) are determined by four categories of factors (i) the set of physical parameters in the climate models defined by (5) and (7) to (9), (ii) the sum of climate impacts parameters $\Psi_{ij}$ on regional natural capitals $x_{ij}$, (iii) the sum of parameters $\phi_{ij}$ from the payoffs of natural capitals and, (iv) the policymaker's preferences of time $\rho_j$ and robustness $1/\theta_j$. Since a single policymaker only takes into account the value of own regional natural capital, $b_{jk} = 0$ for all $k \neq j$ in (39) resulting in that the levels of shadow cost trajectories of $CO_2$ are the share $b_{jk}/\sum_j^N b_{jk}$ of the total level of the globally optimal shadow cost trajectories. Decreased time preference $\rho_j$ or increased preference for robustness $1/\theta_j$ shifts expected shadow cost trajectory upwards, implying stricter policy rules $q^*_{ij}(E_{ijt})$, i.e. greater $q^*_{ij}$ for given $(E_{ijt})$ levels.

Using investor $i$'s optimal investment rule $u^*_{ijt}$ from (83), which determines the growth of capital $k_{ij}$, gives the policymaker $j$'s optimal policy response expressed in terms of firms' investment rate strategies $u^*_{ijt}$. The net effect is that the policymaker's best policy response is to reduce emissions reductions to regional firms which re-investment a larger share of profit in regional production.

$$q^*_{ijt} = \frac{\sum_{j=1}^{N} \frac{b_{jk}}{2} \Psi_j \left(\frac{\sigma}{R_1 \alpha \sqrt{2}}\right)^2}{\rho_j + \frac{R_2}{R_1 \tau_2}(1 - \frac{1}{1+\rho_j \tau_2})} \frac{\xi_j \eta_{ij}}{2 c_{ij} M_0 \theta_j (\rho_j + \Omega)} \frac{a_{ij}}{4} \frac{1}{u^*_{ijt}} \geq 0 \tag{40}$$

$$k = 1, 2, \ldots, n \quad j = 1, 2, \ldots, N \tag{41}$$

The rule $\lambda^*_j(M_t)$ in (42) tells how the policymaker $j$ given her preference for robustness $(1/\theta_j)$ updates climate sensitivity in (6). A policymaker with no preference for robustness $(1/\theta_j \to 0)$ will not update the climate sensitivity but continue using the benchmark model $\lambda_{min}$ as in ordinary stochastic optimal control regardless discovery of new climate data.

$$\lambda^*_{jt} = -\frac{\partial W_j}{\partial T} \frac{\sigma \sqrt{M_t/M_0} e^{\rho_j t}}{\theta_j R_1 \alpha \sqrt{2}} \quad \lambda_{jt} \in [\lambda_{min}, \lambda_{max}] \tag{42}$$

$$\lambda^*_{jt} = \frac{\sum_{j=1}^{N} \frac{b_{jk}}{2} \Psi_j}{\rho_j + \frac{R_2}{R_1 \tau_2}(1 - \frac{1}{1+\rho_j \tau_2})} \frac{\sigma \sqrt{M_t/M_0}}{\theta_j R_1 \alpha \sqrt{2}} \geq 0 \quad \lambda_{jt} \in [\lambda_{min}, \lambda_{max}] \tag{43}$$

$$k = 1, 2, \ldots, n \quad j = 1, 2, \ldots, N \tag{44}$$

The comparative statics of investors' optimal investment rates in Hennlock, A Robust Feedback Nash Equilibrium in a Climate Change Policy Game (2008d) also convey to this analysis. Regions with high abatement costs attracts investments in dirty capital as the policymaker chooses a lower reductions for given emission flows. High abatement cost regions become pollution havens with high cash-dividends. The introduction of precautionary preference of the policymaker in region $j$ will increase emissions reductions, further lowering firms' shadow price of capital, which induce a response by investors to increase cash-dividend at the cost of lower reinvestment in the region. Not surprisingly, a region hosting a non-precautionary policymaker (low $1/\theta_j$) will attract greater reinvestments in regional firms.

An uncertainty averse (high $1/\theta_j$) policymaker $j$ faces a greater expected Nash shadow cost of $CO_2$ compared to a policymaker $k \neq j$ with $(1/\theta_k < 1/\theta_j)$ which induces greater levels of $q^*_{ij}$ for given $(E_{ijt})$ levels. For example, a one percentage increase in $1/\theta_j$ results in one percentage shift upwards in expected shadow cost of $CO_2$ trajectory for given $CO_2$ levels which in turn alters the policy rule $q^*_{ijt}(E_{ijt})$ by one percentage greater abatement $q^*_{ijt}$ for given emissions levels $(E_{ijt})$. As $\theta_j \to 0$, policymaker $j$'s expected Nash shadow cost of $M_t$ increases toward infinity and the policy maker bases policy on the $\lambda_{max}$ climate model. A low $\theta_j$ also makes expected Nash shadow cost of $M$ highly sensitive (quadratic) to the variance $\sigma$ of underlying climate sensitivity factors and thermal capacity of atmosphere and upper ocean $R_1$ in the climate model, resulting in significant increases in feedback Nash emissions reduction strategies $q^*_{ijt}$ $\forall \in [1, n]$. M

### 3.4. Cooperative Policy Rules

The analysis of cooperative solutions in this game are left to forthcoming studies. However, there are several interesting cooperative structures possible in the game: (i) cooperation between policymakers across regional borders, (ii) cooperation between the policymaker and the regional investors within each region, (iii) cooperation between investors within each region, (iv) cooperation between investors across regions while policymakers choose policies individually, (v) bilateral coalitions where policymakers cooperate across regions in one coalition while investors cooperate across regions in another coalition, (vi) global cooperation between all policymakers and all investors on policies and allocation of investments.

Due lack of space we only identify coordinated regional policy rules in case (i) when policymakers' form a 'grand coalition' and investors play individual Nash investment rate strategies, assuming that the payment streams fulfill individual rationality and some time consistent burden-sharing (Yeung and Petrosjan, Subgame Consistent Cooperative Solutions in Stochastic Differential Games (2004a) and Yeung and Petrosjan, Cooperative Stochastic Differential Games (2006b)). The cooperative policy rules $q^o_{ijt}(E_{ijt})$ are obtained by maximizing the sum of policymakers' payoffs given the dynamics in (5), the climate model in (6) - (9) and that each investor $i$ individually solves for optimal investment rate trajectories $u^*_{ijt}$. The expected joint payoff to all policymakers $j \in [1, N]$ is

$$\max_{q_{jt}} \min_{\lambda_{jt}} \epsilon \int_0^\infty \sum_{j=1}^{N} \sigma_j \sum_{m=1}^{n} \left\{ \omega_j y_{mjt} + \psi_j x_{jt}^{1/2} - \frac{c_{mj} q_{mjt}^2}{E_{mjt}} + \frac{\theta h_t^2}{2} \right\} e^{-\rho t} dt \qquad (45)$$

The preferences of time $\rho$ and robustness $1/\theta$ are assumed to be weighted averages of policymakers' individual preferences i.e. $\rho = \sum_{j=1}^{N} \sigma_j \rho_j$ and $\theta = \sum_{j=1}^{N} \sigma_j \theta_j$ with $\sum_{j=1}^{N} \sigma_j = 1$, assuming that not only instantaneous payoffs but also preferences of time and robustness are reflected by policymakers' bargaining power in negotiations. From appendix A.3, the grand coalition's optimal policy rules are

$$q^o_{ijt} = \frac{\sum_{j=1}^{N} \frac{b_{cj}}{2} \Psi_j \left( \frac{\sigma}{R_1 \alpha \sqrt{2}} \right)^2}{\rho + \frac{R_2}{R_1 \tau_2}(1 - \frac{1}{1+\rho\tau_2})} \frac{\xi_j \eta_{ij}}{2 c_{ij} M_0 \theta(\rho + \Omega)} E_{ijt} e^{\rho t} \geq 0 \tag{46}$$

$$\lambda^o_t = \frac{\sum_{j=1}^{N} \frac{b_{cj}}{2} \Psi_j}{\rho + \frac{R_2}{R_1 \tau_2}(1 - \frac{1}{1+\rho\tau_2})} \frac{\sigma \sqrt{M_t/M_0}}{\theta R_1 \alpha \sqrt{2}} \geq 0 \tag{47}$$

$$j = 1, 2, \ldots, N \tag{48}$$

The optimal abatement levels and updating of climate sensitivity will have the same structural form as (38) and (42) but global optimal shadow cost of atmospheric concentration rate $\partial W/\partial M_t$ and global mean temperature $\partial W/\partial T_t$ will be different as they carry the total marginal cost that each firm incurs on all regional natural capital stocks via the climate model dynamics. By symmetry with (43), $\lambda(M_t)$ tells how the global social planner, given $\theta$, would update equilibrium climate sensitivity in (6). Substituting the optimal polcy rules in (42) and (43) and (46) and (47) in the dynamic system (5) and (13) and solving gives the dynamics that corresponds to the Nash equilbrium and the cooperative solution respectively and is a straightforward operation left for the reader.

## 4. Concluding Comments

In the article *'Why is Climate Sensitivity So Unpredictable?'* Roe and Baker (2007) analyze the effects of uncertainty in underlying physical feedback processes on mean outcome and shape of probability distribution of climate uncertainty. They explain why the shape of probability distribution of climate sensitivity gets a thick high-temperature tail and that it is not likely that progress in understanding underlying physical processes will narrow the tail. Having this in mind, we introduced a new approach of uncertainty to integrated assessment modeling by letting policymakers face a set of climate models with different equilibrium climate sensitivity, and without knowing which of them is correct or which probability distribution of equilibrium climate sensitivity is correct. The climate models were generated by perturbing a continuous-time version of the climate model in Nordhaus (1992) and Nordhaus and Yang (1996) making it statistically impossible for policymakers to infer correct future probability distributions about climate sensitivity by using current data. These conditions of uncertainty better describe the real conditions that policymakers today actually are facing as shown in Roe and Baker (2007), Allen and Frame (2007) and IPCC, Climate Change 2007: Working Group I Report The Physical Science Basis (2007c). Our policymakers update their policy rules in the light of new climate data according to optimal rules which will change the predictions of equilibrium climate sensitivity and alter the climate model continuously as the game evolves. This is accordance with IPCC, Climate Change 1995: Economic and Social Dimensions of Climate Change. Contribution of Working Group III to the Second Assessment of the Intergovernmental Panel on Climate Change (1995a) which stated *'The challenge is not to find the best policy today for the next 100 years, but to select a prudent strategy and to adjust it over time in the light of new information'.* On the other hand, policymakers using ordinary stochastic optimal control and the expected utility criterion, would stick to the original climate model used as base for policy decisions, despite discovery of new climate data that would reject the model.

In the gap of knowledge concerning climate sensitivity, we introduced precautionary preference among policymakers. Experiments like Ellsberg's, show that Savage's Sure-Thing Principle is badly supported in real decisions. People rather tend to choose the alternative where there is more certain information about outcome. Following the technique by Hansen and Sargent (2001), we introduced a preference for robustness $1/\theta$ among policymakers, where robustness means a preference to choose policy trajectories that are optimal in case a climate model with high equilibrium climate sensitivity should be found to be correct in the future. A policymaker with a preference for precaution faces a greater expected shadow cost of atmospheric $CO_2$ which results in stricter policy rules $q^*(E_t)$ in the game. The stronger the precautionary preference, the policymaker's expected shadow cost of $CO_2$ trajectory shifts upwards, the stricter are policy rules $q^*(E_t)$, i.e. greater abatement for given emissions. This lowers investors' expected shadow price of physical capital and hence, investors' responses are to increase current cash dividends at the cost of lower reinvestments in the region.

A low preference of time ($\rho$) and/or a high preference for precaution ($1/\theta$) among policymakers shifts expected shadow cost of atmospheric $CO_2$ upward moving the non-cooperative Nash trajectories of emissions reduction closer toward cooperative trajectories and they may even coincide at finite upper capacity bounds of abatement effort. Hence, a revalue of time preferences and precautionary among policymakers in general toward lower time preferences and higher precautionaries will not only result in stricter policy rules but also support efforts to reach cooperative outcomes.

Evident forthcoming studies would be simulations using empirical data in this game model with asymmetric profit and cost functions and varying precautionary preferences across policymakers. The major contribution of this paper though, should rather lie in the opportunities to analyze cooperative structures given climate model uncertainty. The structural uncertainty in this model was embedded in the integrated assessment differential game in Hennlock, Optimal Policy Rules in an Integrated Climate-Economy Differential Game (2007b), using a continuous-time version of the climate model used in Nordhaus (1992) and Nordhaus and Yang (1996) such that analytical closed-loop solutions can be defined analytically. The advantage of analytical solutions, compared to numerical simulations, which in general have poorer reliability, is not only that they allow for deeper understandings in for example sensitivity analyzes, but they also make analyzes of cooperative structures possible when it comes to the study of fulfilling conditions for individual rationality and subgame consistent payment streams, which are conclusive for the stability of long-term cooperative solutions.

## 5. Appendix

### 5.1. Appendix A.1 Investor $i \in [1, n]$ Optimization Problem

Every investor $i$ maximizes payoff function (14) subject to the dynamic system (5) to (13) and $q^*_{ijt} \geq \forall i \in [1, n]$ and $\forall j \in [1, N]$ in definition 1. Investor $i$'s stochastic optimal problem is:

$$\max_{u_{ikt}} \epsilon \int_0^\infty \sum_{k=1}^N \left\{ (1 - u_{ikt}) y_{ikt} - \frac{c_{ik}(q^*_{ikt})^2}{E_{ikt}} \right\} e^{-\rho_i t} dt \tag{49}$$

subject to

$$dk_{ij} = \left[ I_{ijt}^{1/2} k_{ijt}^{1/2} - \delta_{ij} k_{ijt} \right] dt \tag{50}$$

$$dx_j = \left[ r_j \left( 1 - \frac{x_{jt}^{1/2}}{K_j} \right) x_{jt}^{1/2} - \frac{\Psi_j (T_t - T_0)}{x_{jt}^{1/2}} x_{jt} \right] dt \tag{51}$$

$$dM = \left[ \sum_{m=1}^n \sum_{k=1}^N \xi_j (E_{mkt} - \eta_{mk} q_{mkt}) - \Omega M_t \right] dt \tag{52}$$

$$dT = \frac{1}{R_1} \left( \frac{\sigma[\lambda_{jt} dt + d\hat{B}] \sqrt{M_t / M_0} + \nu dt}{\alpha \sqrt{2}} + O_t dt - \lambda T dt - \frac{R_2}{\tau_2} (T_t - \tilde{T}_t) dt \right) \tag{53}$$

$$d\tilde{T} = \frac{1}{R_2} \left( \frac{R_2}{\tau_2} (T_t - \tilde{T}_t) \right) dt \tag{54}$$

$$i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, N \tag{55}$$

**Definition 2.** If there exist $n$ value functions $V_i(\mathbf{k}, \mathbf{x}, M, T, \tilde{T}, t)$ where

$$\mathbf{k} = (k_{11}, k_{12}, \ldots, k_{1N}, k_{21}, k_{22}, \ldots, k_{2N}, k_{n1}, k_{n2}, \ldots, k_{nN}) \tag{56}$$

and $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ that satisfy

$$\begin{aligned} &V_i(\boldsymbol{k}, \boldsymbol{x}, M, T, \tilde{T}, t) = \\ &\epsilon \int_0^\infty \sum_{k=1}^N \left\{ (1 - u^*_{ikt}) y_{ikt} - \frac{c_{ik}(q^*_{ikt})^2}{E_{ikt}} \right\} e^{-\rho_i t} dt \\ &\geq \epsilon \int_0^\infty \sum_{k=1}^N \left\{ (1 - u_{ikt}) y_{ikt} - \frac{c_{ik}(q^*_{ikt})^2}{E_{ikt}} \right\} e^{-\rho_i t} dt \end{aligned} \tag{57}$$

for strategies $u^*_{ijt}(k_j, t) \subseteq R^1 \quad \forall i \in [1, n]$ and $\forall j \in [1, N]$ which satisfy the state equations,

$$dk_{ij} = \left[ I_{ijt}^{1/2} k_{ijt}^{1/2} - \delta_{ij} k_{ijt} \right] dt \tag{58}$$

$$dx_j = \left[ r_j \left( 1 - \frac{x_{jt}^{1/2}}{K_j} \right) x_{jt}^{1/2} - \frac{\Psi_j(T_t - T_0)}{x_{jt}^{1/2}} x_{jt} \right] dt \tag{59}$$

$$dM = \left[ \sum_{m=1}^{n} \sum_{k=1}^{N} \xi_j (E_{mk} - \eta_{mk} q_{mkt}^*) - \Omega M_t \right] dt \tag{60}$$

$$dT = \frac{1}{R_1} \left( \frac{\sigma[\lambda_{jt} dt + d\hat{B}] \sqrt{M_t/M_0} + \nu dt}{\alpha\sqrt{2}} + O_t dt - \lambda T dt - \frac{R_2}{\tau_2}(T_t - \tilde{T}_t) dt \right) \tag{61}$$

$$d\tilde{T} = \frac{1}{R_2} \left( \frac{R_2}{\tau_2}(T_t - \tilde{T}_t) \right) dt \tag{62}$$

$$i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, N \tag{63}$$

The feedback Nash controls strategies

$$\Gamma_{ijt}^* = \{u_{ijt}^*(k_{ijt})\} \quad \forall i \in [1, n] \quad \forall j \in [1, N] \tag{64}$$

provide a feedback Nash equilibrium solution of the game defined in (49) to (55) given (19) to (25) (Basar and Olsder, 1999).

The value functions in definition 2 satisfy the partial differential equation system (65) - (66). Using (49) - (55) and definition 1 and 2 yield the dynamic programming problem (Fleming and Richel, 1975) of investor $i$

$$\begin{aligned} -\frac{\partial V_i}{\partial t} = \max_{u_{ikt}} \sum_{k=1}^{N} \left\{ (1 - u_{ikt}) y_{ikt} - \frac{c_{ik}(q_{ikt}^*)^2}{E_{ikt}} \right\} e^{-\rho_i t} \\ + \sum_{m=1}^{n} \sum_{k=1}^{N} \frac{\partial V_i}{\partial k_{mk}} \left[ I_{mkt}^{1/2} k_{mkt}^{1/2} - \delta_{mk} k_{mkt} \right] \\ + \sum_{k=1}^{N} \frac{\partial V_i}{\partial x_k} \left[ r_k \left( 1 - \frac{\sqrt{x_{kt}}}{K_k} \right) \sqrt{x_{kt}} - \frac{\Psi_k(T_t - T_0)}{\sqrt{x_{kt}}} x_{kt} \right] \\ + \frac{\partial V_i}{\partial M} \left[ \sum_{m=1}^{n} \sum_{k=1}^{N} \xi_j (E_{mkt} - \eta_{mk} q_{mkt}^*) - \Omega M_t \right] \\ + \frac{\partial V_i}{\partial T} \left[ \frac{1}{R_1} \left( \frac{\sigma \lambda_{jt} \sqrt{M_t/M_0} + \nu}{\alpha\sqrt{2}} + O_t - \lambda T - \frac{R_2}{\tau_2}(T_t - \tilde{T}_t) \right) \right] \\ + \frac{1}{2} \frac{\partial^2 V_i}{\partial T^2} \sigma^2 M_t \\ + \frac{\partial V_i}{\partial \tilde{T}} \left[ \frac{1}{R_2} \left( \frac{R_2}{\tau_2}(T_t - \tilde{T}_t) \right) \right] \end{aligned} \tag{65}$$

$$i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, N \tag{66}$$

The feedback Nash controls strategies

$$\Gamma^*_{ijt} = \{u^*_{ij}(k_{ijt})\} \quad \forall i \in [1,n] \quad \text{and} \quad \forall j \in [1,N] \tag{67}$$

are given by maximizing the partial differential equations (65) with respect to (67) for $n$ players and solving for the feedback Nash control variables.

$$u^*_{ijt} = \left(\frac{1}{2}\frac{\partial V_i}{\partial k_{ij}}\right)^2 \frac{k_{ijt}^{1/2}}{\phi_{ij}} e^{2\rho_i t} \quad \forall i \in [1,n] \quad \text{and} \quad \forall j \in [1,N] \tag{68}$$

The partials derivatives in (68) are investor $i$'s expected feedback Nash shadow price of $k_{ij}$. In order to identify shadow price paths, the $n$ value functions $V_i(k, x, M, T, \tilde{T}, t)$ that satisfy definition 2 and the partial differential equation system formed by (65) - (66) must be identified.

**Proposition 1.**

$$V_i(\boldsymbol{k}, \boldsymbol{x}, M, T, \tilde{T}, t) = \tag{69}$$
$$\left(\sum_{m=1}^{n}\sum_{k=1}^{N} a_{imk} k_{mk}^{\frac{1}{2}} + \sum_{k=1}^{N} b_{ik} x_k^{\frac{1}{2}} + d_i M + e_i T + f_i \tilde{T} + g_i\right) e^{-\rho_i t}$$

*The value functions $\forall i \in [1,n]$ satisfy definition 2 and the partial differential equation system formed by system (65) - (66).*

*Proof.* The values of the undetermined coefficients $(a_{iij}, b_{ij}, d_i, e_i, f_i, g_i)$ for all investors $i \in [1,n]$ and regions $j \in [1,N]$ are determined by substituting (83) into the partial differential equations (65) for all $i \in n$ forms the $n$ indirect Isaacs-Bellman-Fleming equations of investors $i = 1, 2, \ldots, n$. The coefficients of the indirect values functions in proposition 1 are then determined by the block recursive equation system

$$\begin{aligned} \rho_i a_{iij} = p\phi_{ij} - \frac{(d_j \xi_j \eta_{ij})^2 \varphi_{ij} \phi_{ij}}{4c_j} - \frac{a_{iij}}{2}\delta_{ij} \\ + d_i \xi_j \varphi_{ij} \phi_{ij} + d_i \xi_j \eta_{ij} \frac{d_j \xi_j \eta_{ij} \varphi_{ij} \phi_{ij}}{2c_{ij}} \quad d_i = 0 \end{aligned} \tag{70}$$

$$\begin{aligned} \rho_i a_{imj} = -\frac{a_{imj}}{2}\delta_{mj} + d_i \xi_j \varphi_{mj} \phi_{mj} \\ + d_i \xi_j \eta_{mj} \frac{d_j \xi_j \eta_{mj} \varphi_{mj} \phi_{mj}}{2c_{mj}} \quad \forall m \neq i \end{aligned} \tag{71}$$

$$\rho_i b_{ij} = -\frac{b_{ij}\rho_i}{2K_j} \tag{72}$$

$$\rho_i d_i = \frac{e_i}{M_0 \theta_j}\left(\frac{\sigma}{R_1 \alpha \sqrt{2}}\right)^2 - d_i \Omega \tag{73}$$

$$\rho_i e_i = -\sum_{j=1}^{N} \frac{b_{ij}\Psi_j}{2} - e_i \frac{\lambda}{R_1} - e_i \frac{R_2}{R_1 \tau_2} + f_i \frac{1}{\tau_2} \tag{74}$$

$$\rho_i f_i = e_i \frac{R_2}{R_1 \tau_2} - f_i \frac{1}{\tau_2} \tag{75}$$

$$a_{iij} = \frac{\phi_i \left( p + d_i \xi_k \varphi_{ij} + \frac{(\xi_k \eta_{ij})^2 \varphi_{ij} \phi_{ij}}{c_{ij}} \left( \frac{d_i d_j}{2} - \frac{(-d_j)^2}{4} \right) \right)}{\rho_i + \delta_{ij}/2} \tag{76}$$

$$a_{imj} = d_i \xi_k \varphi_{mj} \phi_{mj} \left( 1 + \frac{d_k \xi_k \eta_{mj}^2}{2c_{mj}} \right) \quad \forall m \neq i \tag{77}$$

$$b_{ij} = 0 \tag{78}$$

$$d_i = \frac{e_i}{M_0 \theta_j} \left( \frac{\sigma}{R_1 \alpha \sqrt{2}} \right)^2 \frac{1}{\rho_i + \Omega} \tag{79}$$

$$e_i = \frac{-\sum_{j=1}^{N} \frac{b_{ij}}{2} \Psi_j}{\rho_i + \frac{\lambda}{R_1} + \frac{R_2}{R_1 \tau_2} (1 - \frac{1}{1 + \rho_i \tau_2})} \tag{80}$$

$$f_i = e_i \frac{R_2}{R_1 \tau_2 (\rho_i + \frac{1}{\tau_2})} \tag{81}$$

$$i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, N \quad m \in [1, n] \tag{82}$$

Since $b_{ij} = 0$, then $a_{imj} = 0$, $d_i = 0$ and $g_i = 0$, which further simplify (76). The coefficients $g_i$ in proposition 1 are uniquely determined by the coefficients in (76) - (82) and (92) - (99).

Substituting the feedback Nash shadow prices and costs into (68) yields the feedback strategies in terms of parameter values:

$$u_{ijt}^* = \left( \frac{a_{ij}}{2p} \right)^2 \frac{1}{\phi_{ij} k_{ijt}^{1/2}} \in [0, 1] \quad \forall i \in [1, n] \quad \text{and} \quad \forall j \in [1, N] \tag{83}$$

### 5.2. Appendix A.2 Policymaker $j \in [1, N]$ Optimization Problem

Every policymaker $j$ maximizes payoff function (19) subject to the dynamic system (20) to (25) and $u_{ijt}^* \geq \forall i \in [1, n]$ and $\forall j \in [1, N]$.

**Proposition 2.**

$$W_j(\boldsymbol{k},\boldsymbol{x},M,T,\tilde{T},t)= \tag{84}$$
$$\Big(\sum_{m=1}^{n}\sum_{k=1}^{N}a_{jmk}k_{mk}^{\frac{1}{2}}+\sum_{k=1}^{N}b_{jk}x_k^{\frac{1}{2}}+d_jM+e_jT+f_j\tilde{T}+g_j\Big)e^{-\rho_j t}$$

*The value functions $\forall j\in[1,N]$ satisfy definition 1 and the partial differential equation system formed by system (35) - (36).*

*Proof.* Substituting (39) and (43) into the partial differential equations (35) for all $j\in[1,N]$ forms the $N$ indirect Isaacs-Bellman-Fleming equations of policy makers $j=1,2,\ldots,N$. The coefficients of the indirect values functions in proposition 2 are then

$$\begin{aligned}\rho_j a_{jij} = \omega_j\phi_{ij} - \frac{(d_j\xi_j\eta_{ij})^2\varphi_{ij}\phi_{jj}}{4c_{ij}} - \frac{a_{jij}\delta_{ij}}{2} + d_j\xi_j\varphi_{ij}\phi_{ij}\\ + d_j\xi_j\eta_{ij}\frac{d_j\xi_j\eta_{ij}\varphi_{ij}\phi_{ij}}{2c_{ij}}\end{aligned} \tag{85}$$

$$\rho_j a_{jik} = -\frac{a_{jik}\delta_{ik}}{2} + d_j\xi_k\varphi_{ik}\phi_{ik}\left(1+\frac{d_k\xi_k\eta_{ik}^2}{2c_{ik}}\right) \quad k\neq j \tag{86}$$

$$\rho_j b_{jj} = \psi_j - \frac{b_{jj}r_j}{2K_j} \tag{87}$$

$$\rho_j b_{jk} = -\frac{b_{jk}r_k}{2K_k} \quad k\neq j \tag{88}$$

$$\rho_j d_j = \frac{e_j}{M_0\theta_j}\left(\frac{\sigma}{R_1\alpha\sqrt{2}}\right)^2 - d_j\Omega \tag{89}$$

$$\rho_j e_j = -\sum_{j=1}^{N}\frac{b_{ij}\Psi_j}{2} - e_j\frac{\lambda}{R_1} - e_j\frac{R_2}{R_1\tau_2} + f_j\frac{1}{\tau_2} \tag{90}$$

$$\rho_j f_j = e_j\frac{R_2}{R_1\tau_2} - f_j\frac{1}{\tau_2} \tag{91}$$

$$a_{jij} = \frac{\phi_{ij}}{\rho_j+\delta_{ij}/2}\left(\omega_j + \frac{(d_j\xi_j\eta_{ij})^2\varphi_{ij}}{4c_{ij}} + d_j\xi_j\varphi_{ij}\right) \tag{92}$$

$$a_{jik} = \frac{d_j\xi_k\varphi_{ik}\phi_{ik}}{\rho_j+\delta_{ik}/2}\left(1+\frac{d_k\xi_k\eta_{ik}}{2c_{ik}}\right) \quad \forall k\neq j \tag{93}$$

$$b_{jj} = \frac{\psi_j}{\rho_j + \frac{r_j}{2K_j}} \tag{94}$$

$$b_{jk} = 0 \quad k \neq j \tag{95}$$

$$d_j = \frac{e_j}{M_0\theta_j}\left(\frac{\sigma}{R_1\alpha\sqrt{2}}\right)^2 \frac{1}{\rho_j + \Omega} \tag{96}$$

$$e_j = \frac{-\sum_{j=1}^{N} \frac{b_{ij}}{2}\Psi_j}{\rho_j + \frac{\lambda}{R_1} + \frac{R_2}{R_1\tau_2}(1 - \frac{1}{1+\rho_j\tau_2})} \tag{97}$$

$$f_j = e_j \frac{R_2}{R_1\tau_2(\rho_j + \frac{1}{\tau_2})} \tag{98}$$

$$i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, N \quad m \in [1, n] \tag{99}$$

The undetermined coefficients in appendices A.2 and A.3 are uniquely defined, and hence, this corresponding feedback Nash equilibrium is unique. The coefficients $g_j$ in proposition 2 are uniquely determined by the coefficients in (76) - (82) and (92) - (99).

### 5.3. Appendix A.3 Global Optimal Policy Responses

The global planner $c$ maximizes the sum of payoff functions (45) subject to the dynamic system (20) to (25) and $u^*_{ijt} \geq \forall i \in [1, n]$ and $\forall j \in [1, N]$. The value function $W_c(k, x, M, T, \tilde{T}, t)$ is:

**Proposition 3.**

$$W_c(\boldsymbol{k}, \boldsymbol{x}, M, T, \tilde{T}, t) = \tag{100}$$
$$\left(\sum_{m=1}^{n}\sum_{k=1}^{N} a_{cmk}k_{cmk}^{\frac{1}{2}} + \sum_{k=1}^{N} b_{ck}x_k^{\frac{1}{2}} + dM + eT + f\tilde{T} + g\right)e^{-\rho t}$$

*The value function satisfy the partial differential equation system formed by system (35) - (36) but replacing the payoff function in (45).*

*Proof.* Substituting (46) - (47) into the partial differential equations (35) and replacing payoff functions to (45) forms the indirect Isaacs-Bellman-Fleming equations of the global planner. The coefficients of the indirect value function in proposition 3 are then

$$\rho a_{cij} = \omega_j\phi_{ij} + \frac{(d_c\xi_j\eta_{ij})^2\varphi_{ij}\phi_{jj}}{2c_{ij}} - \frac{a_{jij}\delta_{ij}}{2} + d_c\xi_j\varphi_{ij}\phi_{ij} \tag{101}$$

$$\rho a_{cik} = -\frac{a_{cik}\delta_{ik}}{2} + d_c\xi_k\varphi_{ik}\phi_{ik}\left(1 + \frac{d_c\xi_k\eta_{ik}^2}{2c_{ik}}\right) \quad k \neq j \tag{102}$$

$$\rho b_{cj} = \psi_j - \frac{b_{jj} r_j}{2K_j} \tag{103}$$

$$\rho d_c = \frac{e_c}{M_0 \theta_j} \left( \frac{\sigma}{R_1 \alpha \sqrt{2}} \right)^2 - d_c \Omega \tag{104}$$

$$\rho e_c = - \sum_{j=1}^{N} \frac{b_{ij} \Psi_j}{2} - e_c \frac{R_2}{R_1 \tau_2} + f_c \frac{1}{\tau_2} \tag{105}$$

$$\rho f_c = e_c \frac{R_2}{R_1 \tau_2} - f_c \frac{1}{\tau_2} \tag{106}$$

$$a_{cij} = \frac{\phi_{ij}}{\rho + \delta_{ij}/2} \left( \omega_j + \frac{(d_c \xi_j \eta_{ij})^2 \varphi_{ij}}{2c_{ij}} + d_c \xi_j \varphi_{ij} \right) \tag{107}$$

$$a_{cik} = \frac{d_c \xi_k \varphi_{ik} \phi_{ik}}{\rho + \delta_{ik}/2} \left( 1 + \frac{d_c \xi_k \eta_{ik}}{2c_{ik}} \right) \quad \forall k \neq j \tag{108}$$

$$b_{cj} = \frac{\psi_j}{\rho + \frac{r_j}{2K_j}} \tag{109}$$

$$d_c = \frac{e_c}{M_0 \theta} \left( \frac{\sigma}{R_1 \alpha \sqrt{2}} \right)^2 \frac{1}{\rho + \Omega} \tag{110}$$

$$e_c = \frac{- \sum_{j=1}^{N} \frac{b_{ij}}{2} \Psi_j}{\rho + \frac{R_2}{R_1 \tau_2} (1 - \frac{1}{1 + \rho \tau_2})} \tag{111}$$

$$f_c = e_c \frac{R_2}{R_1 \tau_2 (\rho + \frac{1}{\tau_2})} \tag{112}$$

$$i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, N \quad m \in [1, n] \tag{113}$$

The undetermined coefficients in appendices A.1 and A.3 are uniquely defined, and hence, this corresponding feedback Nash equilibrium is unique. The coefficient $g_c$ in proposition 7 is uniquely determined by the coefficients in (76) - (82) and (92) - (99).

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# Random Priority Two-Person Full Information Best-Choice Game with Disorder*

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**Abstract** The following version of the two player full information best-choice game is considered. Two players observe the sequence of independent identical distributed random variables with the object of choosing the largest. The distribution law of observations changes at the random moment. A random assignment mechanism is used to give an observation to one of the players. Each of the players can choose at most one observation. The class of adequate strategies and the suitable gain function for the problem is constructed. A numerical examples are given.

**Keywords:** best-choice, disorder.

## 1. Introduction

The following version of the two-player full information best-choice problem is considered. A production system is working in the GOOD state and there is a constant probability $\alpha$ that it falls into the BAD state (and remains there) at a disorder moment. The transition matrix is as following:

$$\begin{array}{c|cc} & \text{G} & \text{B} \\ \hline \text{G} & \alpha & 1-\alpha \\ \text{B} & 0 & 1 \end{array}$$

In the GOOD state system produces independent identical distributed random variables (iid r. v.) uniform on [0,1]. In the BAD state system produces iid r. v. uniform on $[0, b]$ ($b \in (0, 1)$ is a parameter).

Two players (Player I and Player II) observe sequentially the output $X_t$ of the system at each time $t = 1, 2, ..., N$ and independently decide either CONTINUE (i.e. reject $X_t$ and observe $X_{t+1}$), or STOP (i.e. accept and receive $X_t$). Recall is not allowed (i.e. the observation once rejected cannot be recalled later). Each player knows parameters $\alpha$, $b$ and $p$, but the real state of the system is unknown. The aim of the players is to maximize the expected value of the accepted observation during the given finite period of time $N$.

When some player accepts an observation at time $k$, then the other one will investigate the sequence of future observations having an opportunity to accept one of them. A random assignment mechanism is used for giving an observation to one of the players. This mechanism is defined by the lottery described by a random variable $\xi$ with uniform distribution on [0,1] and a number $p$ (priority $p \in [0, 1]$ is

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a parameter). Player I gets an observation if $\xi \le p$; otherwise Player 2 gets the observation. It also means that the players cannot accept the same observation at the same moment. One can say that the players have random priority to accept an observation.

A class of one-threshold strategies is studied. Each player sets threshold – $q_1$ and $q_2$ respectively – and accepts the first observation that greater than $q$ and rejecting all observations that less than $q$. Gain functions for the problem are constructed.

The asymptotic behavior of the solution is also studied. We propose to find the limit (as $N \to \infty$) decision of the task and numerical results for different values of parameters $\alpha$, $b$ and $p$.

M. Sakaguchi (Sakaguchi, 2000) studies the imperfect full-information best-choice problem with disorder: a decision-maker should estimate an *a posteriori* information about the true state of the system. The similar problem for the disorder-free imperfect observations case has been considered by Z. Porosinski and K. Szajowski (Porosinski and Szajowski, 2000).

Paper (Mazalov and Ivashko, 2007) deals with other version of the full-information best-choice problem with disorder in which a decision-maker aims to maximize a probability of choosing the largest observation from the random sequence.

The concurrent version of this game has been considered in (Ivashko, 2008). "Concurrent" means that assignment mechanism is used only if both players wish to accept an observation. This problem has a following economical interpretation. There are two sellers, each has a peace of resource (like a house to sell, billboard for rent and so on). The resource of the second player is more prefereable: the priority $p$ shows how many potential customers wish to buy it. Every buyer makes a proposition to one of the player (accordingly players' priorities) to buy a resource (player observes a random variable). If player rejects the proposition then buyer makes the same proposition to the other player.

## 2. Gain Functions

Let $p \le \frac{1}{2}$. It means, that the second player has more chances to get an observation if both players want to accept it. Hence, the second player' acceptance threshold is greater than the first player' one.

Expected gain of the player in case of the system is in the GOOD (BAD) state, there are $n - k$ observations rejected and Player I and Player II has thresholds $q_1$ and $q_2$ respectively, is described by the function $H_k^G(q_1, q_2)$ ($H_k^B(q_1, q_2)$).

We use a method of dynamical programming to find a solution in this game. For each player expected gain is as following:

$$H_k^G(q_1, q_2) = \max(x, H_{k-1}(q_1, q_2))$$

Optimal threshold can be founded as the decision of the optimization problem for each player:

$$q^* = \arg\max_{q_i} H_N^G(q_1, q_2),\ i = 1, 2.$$

It means that if the player have rejected all the observations he gets a zero.

Hereinafter we use the following proposition.

**Proposition 1.** *Assume* $A_k = A_{k-1}B + C$.
*Then* $\forall A, B, C > 0$

$$A_k = A_1 B^{k-1} + C \sum_{i=0}^{k-2} B^i. \tag{1}$$

*If* $A_1 = C$, *then* $A_k = A_1 \frac{1-B^k}{1-B}$.

The expected gain of the first player is described by the following recurrent function:

$$H_k^G(q_1, q_2) = \alpha \left[ \int_0^{q_1} H_{k-1}^G(q_1, q_2)dx + \int_{q_1}^{q_2} (px + (1-p)H_{k-1}^G(q_1, q_2))dx + \right.$$
$$\left. \int_{q_2}^{1} (px + (1-p)H_{k-1}^G(q_1, 1))dx \right] + I(q_1 < b)\tfrac{1-\alpha}{b} \left[ \int_0^{q_1} H_{k-1}^B(q_1, q_2)dx + \right.$$
$$I(q_2 < b) \left( \int_{q_1}^{q_2} (px + (1-p)H_{k-1}^B(q_1, q_2)dx + \int_{q_2}^{b} (px + (1-p)H_{k-1}^B(q_1, b))dx) \right) +$$
$$\left. I(q_2 > b) \int_{q_1}^{b} (px + (1-p)H_{k-1}^B(q_1, q_2))dx \right],$$

where

$$I(a < b) = \begin{cases} 1, & \text{if } a < b \\ 0, & \text{otherwise} \end{cases}$$

If the second player has already accepted an observation, then

$$H_k^G(q_1, 1) = \alpha \left[ \int_0^{q_1} H_{k-1}^G(q_1, 1)dx + \int_{q_1}^{1} xdx \right] + I(q_1 < b)\tfrac{1-\alpha}{b} \left[ \int_0^{q_1} H_{k-1}^B(q_1, b)dx + \right.$$
$$\left. \int_{q_1}^{b} xdx \right] = \left[ \alpha \tfrac{1-q_1^2}{2} + I(q_1 < b) \tfrac{b^2 - q_1^2}{2} \tfrac{1-\alpha}{b} \right] (\alpha q_1)^{k-1} +$$
$$\left[ \alpha \tfrac{1-q_1^2}{2} + I(q_1 < b)(1-\alpha) \tfrac{b^2-q_1^2}{2} \tfrac{1-(\frac{q_1}{b})^k}{b-q_1} \right] \tfrac{1-(\alpha q_1)^{k-1}}{1-\alpha q_1}.$$

In case of BAD state of the system:

$$H_k^B(q_1, q_2) = I(q_1 < b)\tfrac{1}{b} \left[ \int_0^{q_1} H_{k-1}^B(q_1, q_2)dx + I(q_2 < b) \left( \int_{q_1}^{q_2} (px + \right. \right.$$
$$\left. (1-p)H_{k-1}^B(q_1, q_2))dx + \int_{q_2}^{b} (px + (1-p)H_{k-1}^B(q_1, b))dx \right) + I(q_2 > b) \int_{q_1}^{b} (px +$$

$$(1-p)H^B_{k-1}(q_1,q_2))dx\Big] = I(q_1<b)\left[p\frac{b^2-q_1^2}{2b}\left(\frac{q_1+(1-p)(min(q_2,b)-q_1)}{b}\right)^{k-1}+\right.$$

$$\left.\left(I(q_2<b)(1-p)(b-q_2)\frac{b^2-q_1^2}{2}\frac{1-(\frac{q_1}{b})^{k-1}}{b-q_1}+p\frac{b^2-q_1^2}{2}\right)\frac{1-(\frac{q_1+(1-p)(min(q_2,b)-q_1)}{b})^{k-1}}{b-(q_1+(1-p)(min(q_2,b)-q_1))}\right],$$

$$H^B_k(q_1,b) = I(q_1<b)\frac{1}{b}\left[\int\limits_0^{q_1} H^B_{k-1}(q_1,b)dx + \int\limits_{q_1}^{b} xdx\right] = I(q_1<b)\frac{b^2-q_1^2}{2}\frac{1-(\frac{q_1}{b})^k}{b-q_1}.$$

Condition for the recursion completion is following:

$$H^G_0(q_1,q_2) = H^G_0(q_1,1) = H^B_0(q_1,q_2) = H^B_0(q_1,b) = 0.$$

Hence,

$$H^G_k(q_1,q_2) = (\alpha p\frac{1-q_1^2}{2} + I(q_1<b)p\frac{1-\alpha}{b}\frac{b^2-q_1^2}{2})(\alpha p q_1+\alpha(1-p)q_2)^{k-1}+$$

$$\left[\alpha(1-p)(1-q_2)\left(\left[\alpha\frac{1-q_1^2}{2}+I(q_1<b)\frac{b^2-q_1^2}{2}\frac{1-\alpha}{b}\right](\alpha q_1)^{k-2}+\right.\right.$$

$$\left.\left[\alpha\frac{1-q_1^2}{2}+I(q_1<b)(1-\alpha)\frac{b^2-q_1^2}{2}\frac{1-(\frac{q_1}{b})^{k-1}}{b-q_1}\right]\frac{1-(\alpha q_1)^{k-2}}{1-\alpha q_1}\right)+\alpha p\frac{1-q_1^2}{2}+$$

$$I(q_1<b)(1-\alpha)\left(p\frac{b^2-q_1^2}{2b}\left(\frac{pq_1+(1-p)min(q_2,b)}{b}\right)^{k-1}+\right.$$

$$\left(I(q_2<b)(1-p)(b-q_2)\frac{b^2-q_1^2}{2}\frac{1-(\frac{q_1}{b})^{k-1}}{b-q_1}+p\frac{b^2-q_1^2}{2}\right)\cdot$$

$$\left.\left.\frac{1-(\frac{pq_1+(1-p)min(q_2,b)}{b})^{k-1}}{b-(pq_1+(1-p)min(q_2,b))}\right)\right]\frac{1-(\alpha p q_1+\alpha(1-p)q_2)^{k-1}}{1-\alpha(pq_1+(1-p)q_2)}.$$

The second player's expected gain is following:

$$H^G_k(q_1,q_2) = \alpha\left[\int\limits_0^{q_1} H^G_{k-1}(q_1,q_2)dx + \int\limits_{q_1}^{q_2}(pH^G_{k-1}(1,q_2)+(1-p)H^G_{k-1}(q_1,q_2))dx+\right.$$

$$\left.\int\limits_{q_2}^{1}(pH^G_{k-1}(1,q_2)+(1-p)x)dx\right] + I(q_2<b)\frac{1-\alpha}{b}\left[\int\limits_0^{q_1} H^B_{k-1}(q_1,q_2)dx+\right.$$

$$\left.\int\limits_{q_1}^{q_2}(pH^B_{k-1}(b,q_2)+(1-p)H^B_{k-1}(q_1,q_2))dx + \int\limits_{q_2}^{b}(pH^B_{k-1}(b,q_2)+(1-p)x)dx\right],$$

where if the first player has already accepted an observation:

$$H^G_k(1,q_2) = \alpha\left[\int\limits_0^{q_2} H^G_{k-1}(1,q_2)dx + \int\limits_{q_2}^{1} xdx\right] + I(q_2<b)(1-\alpha)H^B_k(b,q_2) =$$

$$\left(\alpha\frac{1-q_2^2}{2}+I(q_2<b)(1-\alpha)\frac{b^2-q_2^2}{2b}\right)(\alpha q_2)^{k-1}+\left(\alpha\frac{1-q_2^2}{2}+\right.$$

$$\left.(1-\alpha)I(q_2<b)\frac{b^2-q_2^2}{2}\frac{1-(q_2/b)^k}{b-q_2}\right)\frac{1-(\alpha q_2)^{k-1}}{1-\alpha q_2},$$

In case of the BAD state of the system:

$$H_k^B(q_1,q_2) = I(q_2<b)\tfrac{1}{b}\left[\int_0^{q_1} H_{k-1}^B(q_1,q_2)dx + \int_{q_1}^{q_2}(pH_{k-1}^B(b,q_2)+\right.$$

$$(1-p)H_{k-1}^B(q_1,q_2))dx+$$

$$\left.\int_{q_2}^{b}(pH_{k-1}^B(b,q_2)+(1-p)x)dx\right] = I(q_2<b)\left[(1-p)\tfrac{b^2-q_2^2}{2b}\left(\tfrac{pq_1+(1-p)q_2}{b}\right)^{k-1}+\right.$$

$$\left.\left(\tfrac{b^2-q_2^2}{2}\tfrac{1-(\frac{q_2}{b})^{k-1}}{b-q_2}p(b-q_1)+(1-p)\tfrac{b^2-q_2^2}{2}\right)\tfrac{1-(\frac{pq_1+(1-p)q_2}{b})^{k-1}}{b-pq_1-(1-p)q_2}\right].$$

$$H_k^B(b,q_2) = I(q_2<b)\tfrac{1}{b}\left[\int_0^{q_2} H_{k-1}^B(b,q_2)dx + \int_{q_2}^{b} xdx\right] = I(q_2<b)\tfrac{b^2-q_2^2}{2}\tfrac{1-(\frac{q_2}{b})^k}{b-q_2},$$

Hence,

$$H_k^G(q_1,q_2) = \left(\alpha(1-p)\tfrac{1-q_2^2}{2}+\right.$$

$$\left.I(q_2<b)(1-p)\tfrac{1-\alpha}{b}\tfrac{b^2-q_2^2}{2}\right)(\alpha pq_1+\alpha(1-p)q_2)^{k-1}+\left[\alpha p(1-q_1)\left(\left(\alpha\tfrac{1-q_2^2}{2}+\right.\right.\right.$$

$$\left.I(q_2<b)(1-\alpha)\tfrac{b^2-q_2^2}{2b}\right)(\alpha q_2)^{k-2}+\left(\alpha\tfrac{1-q_2^2}{2}+(1-\alpha)I(q_2<b)\tfrac{b^2-q_2^2}{2}\tfrac{1-(q_2/b)^{k-1}}{b-q_2}\right)$$

$$\left.\times\tfrac{1-(\alpha q_2)^{k-2}}{1-\alpha q_2}\right)+$$

$$\alpha(1-p)\tfrac{1-q_2^2}{2}+I(q_2<b)(1-\alpha)\left((1-p)\tfrac{b^2-q_2^2}{2b}\left(\tfrac{pq_1+(1-p)q_2}{b}\right)^{k-1}+\right.$$

$$\left.\left.\left(\tfrac{b^2-q_2^2}{2}\tfrac{1-(\frac{q_2}{b})^{k-1}}{b-q_2}p(b-q_1)+(1-p)\tfrac{b^2-q_2^2}{2}\right)\tfrac{1-(\frac{pq_1+(1-p)q_2}{b})^{k-1}}{b-pq_1-(1-p)q_2}\right)\right]\times$$

$$\tfrac{1-(\alpha pq_1+\alpha(1-p)q_2)^{k-1}}{1-\alpha(pq_1+(1-p)q_2)}.$$

Recursion stops at $k=0$:

$$H_0^B(b,q_2) = H_0^B(q_1,q_2) = H_0^G(1,q_2) = H_0^G(q_1,q_2) = 0.$$

Tables 1 and 2 show optimal acceptance thresholds and expected gain for both players.

**Table1.** Optimal thresholds for Player I (Player II) with $n = 20$, $p = 0.333$

| $\alpha \setminus b$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|---|
| 0.1 | 0.082 (0.082) | 0.246 (0.246) | 0.409 (0.411) | 0.573 (0.575) | 0.736 (0.739) |
| 0.3 | 0.082 (0.082) | 0.246 (0.247) | 0.410 (0.411) | 0.573 (0.576) | 0.737 (0.740) |
| 0.5 | 0.083 (0.083) | 0.247 (0.249) | 0.411 (0.414) | 0.575 (0.578) | 0.738 (0.742) |
| 0.7 | 0.084 (0.085) | 0.251 (0.254) | 0.416 (0.420) | 0.580 (0.584) | 0.742 (0.748) |
| 0.9 | 0.591 (0.594) | 0.591 (0.594) | 0.437 (0.442) | 0.602 (0.608) | 0.760 (0.767) |

**Table2.** Expected gain for Player I (Player II) with $n = 20$, $p = 0.333$

| $\alpha \setminus b$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|---|
| 0.1 | 0.104 (0.118) | 0.274 (0.285) | 0.444 (0.454) | 0.616 (0.625) | 0.790 (0.798) |
| 0.3 | 0.155 (0.186) | 0.307 (0.331) | 0.464 (0.481) | 0.626 (0.638) | 0.792 (0.802) |
| 0.5 | 0.229 (0.267) | 0.359 (0.390) | 0.496 (0.520) | 0.642 (0.658) | 0.797 (0.808) |
| 0.7 | 0.331 (0.365) | 0.437 (0.466) | 0.549 (0.574) | 0.671 (0.689) | 0.807 (0.819) |
| 0.9 | 0.536 (0.581) | 0.536 (0.581) | 0.641 (0.657) | 0.733 (0.748) | 0.833 (0.845) |

### 2.1. Asymptotic solution

Lets consider the asymptotic case of this game ($n \to \infty$).
The first player's gain function:

$$H_k^G(q_1, q_2) = \Big[\Big(\alpha\tfrac{1-q_1^2}{2} + I(q_1 < b)(1-\alpha)\tfrac{b^2-q_1^2}{2(b-q_1)}\Big)\tfrac{\alpha(1-p)(1-q_2)}{1-\alpha q_1} + \alpha p\tfrac{1-q_1^2}{2} +$$
$$\Big(I(q_2 < b)(1-p)(b-q_2)\tfrac{b^2-q_1^2}{2(b-q_1)} +$$
$$I(q_1 < b)p\tfrac{b^2-q_1^2}{2}\Big)\tfrac{(1-\alpha)}{b-(pq_1+(1-p)min(q_2,b))}\Big]\tfrac{1}{1-\alpha(pq_1+(1-p)q_2)}.$$

The second player's gain function:

$$H_k^G(q_1, q_2) = \Big[\tfrac{\alpha p(1-q_1)}{1-\alpha q_2}\Big(\alpha\tfrac{1-q_2^2}{2} + (1-\alpha)I(q_2 < b)\tfrac{b^2-q_2^2}{2(b-q_2)}\Big) + \alpha(1-p)\tfrac{1-q_2^2}{2} +$$
$$\tfrac{I(q_2<b)(1-\alpha)}{b-pq_1-(1-p)q_2}\Big(\tfrac{b^2-q_2^2}{2(b-q_2)}p(b-q_1) + (1-p)\tfrac{b^2-q_2^2}{2}\Big)\Big]\tfrac{1}{1-\alpha(pq_1+(1-p)q_2)}.$$

Tables 3 and 4 show numerical results of optimal thresholds and expected gain for the asymptotical case. With infinite number of observations players have nearly all equal thresholds, but the expected gain for the second player is greater than for the first player.

### 2.2. Absolute priority ($p = 0$)

Lets consider the case of abcolute priority of the second player ($p = 0$). It means that the second player set's the acceptance threshold as he is alone (independent of the first player's strategy). The first player should take into account the fact that

**Table3.** Optimal thresholds for Player I (Player II) with $n = \infty$, $p = 0.333$

| $\alpha \setminus b$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|---|
| 0.1 | 0.099 (0.099) | 0.299 (0.299) | 0.499 (0.499) | 0.699 (0.699) | 0.899 (0.899) |
| 0.3 | 0.099 (0.099) | 0.299 (0.299) | 0.499 (0.499) | 0.699 (0.699) | 0.899 (0.899) |
| 0.5 | 0.099 (0.099) | 0.299 (0.299) | 0.499 (0.499) | 0.699 (0.699) | 0.899 (0.899) |
| 0.7 | 0.099 (0.099) | 0.299 (0.299) | 0.499 (0.499) | 0.699 (0.699) | 0.899 (0.899) |
| 0.9 | 0.595 (0.590) | 0.620 (0.620) | 0.499 (0.499) | 0.699 (0.699) | 0.899 (0.899) |

**Table4.** Expected gain for Player I (Player II) with $n = \infty$, $p = 0.333$

| $\alpha \setminus b$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|---|
| 0.1 | 0.116 (0.128) | 0.309 (0.317) | 0.504 (0.509) | 0.701 (0.703) | 0.900 (0.900) |
| 0.3 | 0.165 (0.195) | 0.339 (0.360) | 0.519 (0.532) | 0.707 (0.712) | 0.900 (0.901) |
| 0.5 | 0.238 (0.275) | 0.387 (0.416) | 0.546 (0.564) | 0.716 (0.725) | 0.901 (0.903) |
| 0.7 | 0.339 (0.372) | 0.462 (0.489) | 0.593 (0.614) | 0.737 (0.749) | 0.904 (0.906) |
| 0.9 | 0.537 (0.581) | 0.533 (0.580) | 0.679 (0.692) | 0.789 (0.799) | 0.915 (0.919) |

the first observation with a big value will be assigned to the second player.

**Fixed horizon** $N$

The first player's gain function:

$$H_k^G(q_1,q_2) = \Big[\alpha(1-q_2)\Big(\Big[\alpha\tfrac{1-q_1^2}{2} + I(q_1<b)\tfrac{b^2-q_1^2}{2}\tfrac{1-\alpha}{b}\Big](\alpha q_1)^{k-2} +$$
$$\Big[\alpha\tfrac{1-q_1^2}{2} + I(q_1<b)(1-\alpha)\tfrac{b^2-q_1^2}{2}\tfrac{1-(\frac{q_1}{b})^{k-1}}{b-q_1}\Big]\tfrac{1-(\alpha q_1)^{k-2}}{1-\alpha q_1}\Big) +$$
$$(1-\alpha)I(q_2<b)\tfrac{b+q_1}{2}\tfrac{1-(\frac{q_1}{b})^{k-1}}{1}\tfrac{1-(\frac{q_2}{b})^{k-1}}{1}\Big]\tfrac{1-(\alpha q_2)^{k-1}}{1-\alpha q_2)}.$$

The second player's gain function:

$$H_k^G(q_1,q_2) = \Big(\alpha\tfrac{1-q_2^2}{2} + I(q_2<b)\tfrac{1-\alpha}{b}\tfrac{b^2-q_2^2}{2}\Big)(\alpha q_2)^{k-1} + \Big[\alpha\tfrac{1-q_2^2}{2} +$$
$$I(q_2<b)(1-\alpha)\Big(\tfrac{b^2-q_2^2}{2b}\big(\tfrac{q_2}{b}\big)^{k-1} + \tfrac{b^2-q_2^2}{2}\tfrac{1-(\frac{q_2}{b})^{k-1}}{b-q_2}\Big)\Big]\tfrac{1-(\alpha q_2)^{k-1}}{1-\alpha q_2}.$$

Gain function's recursion stops at $k = 0$:

$$H_0^B(b,q_2) = H_0^B(q_1,q_2) = H_0^G(1,q_2) = H_0^G(q_1,q_2) = 0.$$

**Asymptotic solution**

**Table5.** Optimal thresholds for Player I (Player II) with $n = 50$, $p = 0$

| $\alpha \setminus b$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|---|
| 0.1 | 0.091 (0.091) | 0.273 (0.273) | 0.455 (0.456) | 0.637 (0.638) | 0.819 (0.820) |
| 0.3 | 0.091 (0.091) | 0.273 (0.274) | 0.455 (0.456) | 0.637 (0.638) | 0.819 (0.820) |
| 0.5 | 0.091 (0.092) | 0.273 (0.275) | 0.455 (0.457) | 0.637 (0.639) | 0.819 (0.821) |
| 0.7 | 0.408 (0.408) | 0.275 (0.277) | 0.457 (0.461) | 0.638 (0.642) | 0.819 (0.823) |
| 0.9 | 0.627 (0.627) | 0.627 (0.627) | 0.465 (0.470) | 0.646 (0.654) | 0.825 (0.832) |

**Table6.** Expected gain for Player I (Player II) with $n = 50$, $p = 0$

| $\alpha \setminus b$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|---|
| 0.1 | 0.098 (0.136) | 0.283 (0.310) | 0.469 (0.488) | 0.655 (0.669) | 0.842 (0.853) |
| 0.3 | 0.129 (0.221) | 0.301 (0.368) | 0.478 (0.521) | 0.659 (0.684) | 0.843 (0.856) |
| 0.5 | 0.196 (0.309) | 0.344 (0.432) | 0.501 (0.563) | 0.667 (0.704) | 0.844 (0.861) |
| 0.7 | 0.237 (0.408) | 0.421 (0.506) | 0.548 (0.616) | 0.689 (0.734) | 0.849 (0.869) |
| 0.9 | 0.483 (0.627) | 0.483 (0.627) | 0.645 (0.688) | 0.748 (0.785) | 0.866 (0.889) |

In conclusion we'll consider the asymptotic case of the absolute priority varsion of the game.
The first player's gain function:

$$H_k^G(q_1, q_2) = \left[\left(\alpha\tfrac{1-q_1^2}{2} + I(q_1 < b)(1-\alpha)\tfrac{b^2-q_1^2}{2(b-q_1)}\right)\tfrac{\alpha(1-q_2)}{1-\alpha q_1} +\right.$$

$$\left. I(q_2 < b)(1-\alpha)\tfrac{b+q_1}{2}\right]\tfrac{1}{1-\alpha q_2}.$$

The second player's gain function:

$$H_k^G(q_1, q_2) = \left[\alpha\tfrac{1-q_2^2}{2} + \tfrac{I(q_2<b)(1-\alpha)}{b-q_2}\tfrac{b^2-q_2^2}{2}\right]\tfrac{1}{1-\alpha q_2}.$$

One can show that in case of absolute priority game with infinite number of observations second player has a following independent pure strategy:

$$q_2^* = \begin{cases} \frac{1-\sqrt{1-\alpha^2}}{\alpha} & \text{if } \frac{1-\sqrt{1-\alpha^2}}{\alpha} \geq b \\ b-0 & \text{otherwise} \end{cases}$$

Hence, the expected gain is

$$H^* = \max(\tfrac{1}{1-\alpha b}\left(\alpha\tfrac{1-b^2}{2} + b(1-\alpha)\right), I(b < \tfrac{1-\sqrt{1-\alpha^2}}{\alpha})\tfrac{1-\sqrt{1-\alpha^2}}{\alpha})$$

**Table7.** Optimal thresholds for Player I (Player II) with $n = \infty$, $p = 0$

| $\alpha \setminus b$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|---|
| 0.1 | 0.099 (0.099) | 0.299 (0.299) | 0.499 (0.499) | 0.699 (0.699) | 0.899 (0.899) |
| 0.3 | 0.099 (0.154) | 0.299 (0.299) | 0.499 (0.499) | 0.699 (0.699) | 0.899 (0.899) |
| 0.5 | 0.099 (0.268) | 0.299 (0.299) | 0.499 (0.499) | 0.699 (0.699) | 0.899 (0.899) |
| 0.7 | 0.408 (0.408) | 0.299 (0.408) | 0.499 (0.499) | 0.699 (0.699) | 0.899 (0.899) |
| 0.9 | 0.627 (0.627) | 0.627 (0.627) | 0.499 (0.627) | 0.699 (0.699) | 0.899 (0.899) |

**Table8.** Expected gain for Player I (Player II) with $n = \infty$, $p = 0$

| $\alpha \setminus b$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|---|
| 0.1 | 0.103 (0.141) | 0.301 (0.325) | 0.500 (0.513) | 0.700 (0.705) | 0.900 (0.901) |
| 0.3 | 0.059 (0.154) | 0.318 (0.381) | 0.507 (0.544) | 0.701 (0.717) | 0.900 (0.902) |
| 0.5 | 0.132 (0.268) | 0.359 (0.444) | 0.527 (0.583) | 0.708 (0.735) | 0.900 (0.905) |
| 0.7 | 0.236 (0.408) | 0.300 (0.408) | 0.572 (0.635) | 0.725 (0.762) | 0.901 (0.909) |
| 0.9 | 0.483 (0.627) | 0.483 (0.627) | 0.543 (0.627) | 0.780 (0.809) | 0.911 (0.924) |

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# Rank-Order Innovation Tournaments

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**Abstract** Research efforts and outcomes are generally private information of innovative firms : research inputs are unobservable and the value of innovations is difficult to evaluate . This is the reason why rank-order tournaments are more adequate incentive schemes rather than a conventional contracting. The typical model considers a risk-neutral sponsor (commonly governments or private corporations) and a number of risk-neutral or risk-averse contestants, such as research teams, startup companies. The contestants are competing to find the "best" innovation. The winner obtains the prize and the losers get nothing in a "winner-take-all" game. The prize is thus awarded by the sponsor on the basis of relative rank rather than on the absolute performance. An innovation tournament belongs to the class of dynamic $n$-player two-stage games of imperfect information : at the "entry stage" each firm decides whether to participate, at the "contest stage" each contestant decides whether to invest in each period without knowing the rivals' choices. The game is solved by backward induction. Provided the objective function is quasiconcave, the tournament subgame has a unique symmetric equilibrium in pure strategies. This contribution reviews the innovation tournament models for different probability distributions of shocks.

**Keywords:** research tournament, innovation race, two-stage game, incomplete information.

Research efforts and outcomes are generally private information of innovative firms : research inputs are unobservable and the value of innovations is difficult to evaluate. This is the reason why rank-order tournaments are more adequate incentive schemes rather than a conventional contracting (Taylor, 1995). The theory of rank-order tournaments was pioneered by Lazear and Rosen, 1981 and promoted as optimum labor contracts. The tournaments refer to incentive compensation schemes which pay according to an individual's ordinal rank rather than on output levels. The research competition may be analyzed in a labor tournament framework (Carmichael, 1983, Ma, 1988, Roy Chowdhury, 2005, Zhou, 2006). The typical model considers a risk-neutral sponsor (commonly governments or private firms) and a number of risk-neutral or risk-averse contestants (such as research teams, startup companies). The contestants are competing to find the "best" innovation. The winner obtains the prize and the losers get nothing in a "winner-take-all" game. The prize is thus awarded by the sponsor on the basis of relative rank rather than on the absolute performance (Green and Stokey, 1986). Following Taylor, 1995, research tournaments and innovation races differ fundamentally : in tournaments the terminal date is fixed and the quality of innovations will vary, whereas in innovation races the date of discovery is unknown and the quality standard is fixed. An innovation tournament belongs to the class of dynamic $n$-player two-stage games of imperfect information : at the "entry stage" each firm decides whether to participate, at the "contest stage" each contestant decides whether to invest in each period without knowing the rivals' choices. The game is solved by backward induction : the tournament is first solved for given prizes, then the sponsor's expected profit is computed and the optimal prize is deduced. Provided the objective function is quasiconcave, the tournament subgame has a unique symmetric equilibrium in pure strategies (Taylor, 1995,Wolfstetter, 1999). In most innovation tournaments (Fullerton and McAfee, 1999, Taylor, 1995), the value of the winner prize is exogenous. In new tournament models (Baye and Hoppe, 2003), research inputs not only the determine the probability of winning but also the value of the winner prize. This contribution reviews the innovation tournament models for different probability distributions of shocks.

## 1. Theory of Rank-Order Tournaments

Rank-order tournaments belong to a variety of alternative compensation schemes. A rank-order tournament is a compensation scheme in which a contest earning will depend on the rank order of contestants rather than on their absolute outputs. The winner is paid more than his marginal product. This compensation scheme motivates a greater effort among the contestants. We will present the elementary model and solve it for two and more contestants [1].

### 1.1. Contracting framework

A tournament is played between several agents (workers of a firms, firms of an industry, etc). These agents are identical and perform similar but independent tasks. The agents and the firm (the principal) are supposed to be risk-neutral. The agents compete for fixed prizes by the firm with their efforts or actions.

The model consists of two main equations : the utility function and the agents' production function. We will consider one particular agent by the subscript $i$ and his rivals (all other contestants) by the subscript -*i*. The preferences of each agent over his earnings $y_k$ and his effort $a_k,\ k \in \{i, -i\}$ are represented by a von Neumann-Morgenstern utility function. This function is assumed to be additively separable in earnings $y$ and action (or effort, or investment level) $a$. We then have [2]

$$U(y_k, a_k) = u(y_k) - c(a_k),\ \texttt{for}\ k \in \{i, -i\}.$$

The marginal utility of earnings is positive($u' > 0$) and weakly decreasing ($u'' \leq 0$) with risk-averse agents. The cost $c(a)$ or disutility from efforts is positive and increasing ($c' > 0, c'' > 0$). The observable agents' production function is given by

[1] The efficiency of this compensation scheme has been compared to the piece rates on the labor market notably by Drago and Heywood, 1989. Experimental comparison have been presented by Bull et al., 1987

[2] This is the form given by Nalebuff and Stiglitz, 1983. The alternative utility function by Lazear and Rosen, 1981 is of the form: $U_k = U(y_k, a_k) = U\bigg(y_k - c(a_k)\bigg)$, for $k \in \{i, -i\}$, where the random payment is defined by the two-parameter class function

$$y_i = \begin{cases} w_1 & , \texttt{if}\ x_i > x_{-i}, \\ w_2 & , \texttt{otherwise}. \end{cases}$$

the random function of the effort level [3]

$$x_k = x(a_k, \tilde{\varepsilon}_k) \texttt{ for } k \in \{i, -i\},$$

where $x_i$, $x_{-i}$ are the observable output of the agents and $\tilde{\varepsilon}_i$, $\tilde{\varepsilon}_{-i}$ the individual noise. The probability distribution of the random variable (RV) $\tilde{\varepsilon}$ is known of the firm and agents, is zero mean and uncorrelated with effort. A simplified additive specification is

$$x_k = a_k + \tilde{\varepsilon}_k \texttt{ for } k \in \{i, -i\}.$$

The game consists of two stages with imperfect information. Nature draws outputs noises from the same distribution. This drawing is not revealed but the distribution of the outputs is common knowledge. At stage 1, the firm commits to pay two prizes, one ($w_1$) is for the winner with high-output level and the other ($w_2$) is for the low-output agent. The prizes are fixed arbitrarily. At stage 2, each agent chooses his effort when ignoring the rival's choice. Output are observed and prizes are paid. Given the strategies of agents, the pair of prizes $(w_1, w_2)$ is found such as to maximize an agent's expected utility function, subject to the competitive zero-profit constraint by firms.

### 1.2. Two-contestant rank order tournaments under risk neutrality

This detailed presentation will be restricted to the case of risk-neutral agents as in (Wolfstetter, 1999). The expected profit of the firm is zero for a competitive market. The information is imperfect.

**Agent problem.** The two players are denoted by the subscripts $i$ and *-i*. The problem of the agent $i$ is to maximize his expected utility subject to the zero expected profit condition. We have

$$\max_{a_i} W_i = P(a_i, a_{-i})\ w_1 + \Big(1 - P(a_i, a_{-i})\Big)\ w_2 - c(a_i),$$
$$\text{s.t.}$$
$$w_1 + w_2 - p.(a_i + a_{-i}) = 0,$$

where $P(a_i, a_{-i}) \equiv \texttt{Prob}\{x_i > x_{-i}\}$ is the probability of winning the tournament and $p$ the price of good. If the agent $i$ wins, the realization of the $\tilde{\varepsilon}$ must satisfy the inequality

$$a_i + \varepsilon_i > a_{-i} + \varepsilon_{-i}.$$

For a given $\varepsilon_{-i}$, the probability of occurrence is given by [4]

$$1 - F(a_{-i} - a_i + \varepsilon_{-i}).$$

[3] This is a simplification of the form given by Nalebuff and Stiglitz, 1983. The initial equation introduces a multiplicative common random environment $\tilde{\eta}$ of the form $x_i = \tilde{\eta}.a_i + \tilde{\varepsilon}_i$ for the $i$th agent. The alternative utility function by Lazear and Rosen, 1981 is of the additive form: $x_i = a_i + \tilde{\varepsilon}_i + \tilde{\eta}$ with zero mean.

[4]

*Proof.* From the probability $\texttt{Prob}\{a_i + \varepsilon_i > a_{-i} + \varepsilon_{-i}\}$, we deduce easily $\texttt{Prob}\{\varepsilon_i > a_{-i} - a_i + \varepsilon_{-i}\} = 1 - \texttt{Prob}\{\varepsilon_i > a_{-i} - a_i + \varepsilon_{-i}\} = 1 - F(a_{-i} - a_i + \varepsilon_{-i})$ ∎

The total probability of winning is obtained by integration over all values of the $\varepsilon_i$, weighted by the density $f(\varepsilon_{-i})$. We have

$$P(a_i, a_{-i}) = \int \left\{ 1 - F\Big( (a_{-i} - a_i) + \varepsilon_{-i} \Big) \right\} f(\varepsilon_{-i}) d\varepsilon_{-i}.$$

The pair of equilibrium efforts levels $(a_i^*, a_{-i}^*)$ will satisfy

$$a_k^* = \texttt{argmax} \Big\{ w_2 + F(a_k - a_l^*)(w_1 - w_2) - c(a_k) \Big\}, \ \texttt{for}\ k, l \in \{i, -i\}, k \neq l$$

**Tournament subgame.** The first- and second order conditions (FOC and SOC) are

$$\begin{aligned} (w_1 - w_2) \frac{\partial P(a_i, a_{-i})}{\partial a_k} - c'(a_k) \ &= 0, \ \texttt{for}\ k \in \{i, -i\}, \\ (w_1 - w_2) \frac{\partial^2 P(a_i, a_{-i})}{\partial a_k^2} - c''(a_k) \ &= 0, \ \texttt{for}\ k \in \{i, -i\}. \end{aligned}$$

The Nash equilibrium consists of a pair of optimal efforts, where each agent chooses his investment (or effort) as the best response to the rival's effort. The reaction function are determined by

$$\begin{aligned} (w_1 - w_2) \frac{\partial P(a_i, a_{-i})}{\partial a_i} - c'(a_i) \ &= 0, \\ (w_1 - w_2) \frac{\partial^2 P(a_i, a_{-i})}{\partial a_{-i}^2} - c''(a_{-i}) \ &= 0. \end{aligned}$$

Provided the objective function is quasiconcave, the best replies are unique [5]and the tournament has a unique equilibrium in pure strategies. Due to the symmetry of the reaction functions, the Nash equilibrium is such that $a_i = a_{-i} = a^*$ with the same probability of success $P(a_i, a_{-i}) = P(a_{-i}, a_i) = \frac{1}{2}$. The winner of the competition is designed by Nature. At the symmetric choice the player's increasing chance of giving more effort is

$$\frac{\partial P(a_i, a_{-i})}{\partial a_i} = \frac{\partial}{\partial a_i} \int \bigg( 1 - F(a_{-i} - a_i + \varepsilon_{-i}) \bigg) f(\varepsilon_{-i}) d\varepsilon_{-i} = \int F'_{a_i} f(\varepsilon_{-i}) d\varepsilon_{-i}.$$

Hence, for a symmetric effort, we have

$$\begin{aligned} \frac{\partial P(a, a)}{\partial a} &= \int f(\varepsilon_{-i}) f(\varepsilon_{-i}) d\varepsilon_{-i}, \\ &= \mathbf{E}[f(\varepsilon)] = f(0). \end{aligned}$$

According to the FOC, the optimal effort is implicitly defined by [6]

$$(w_1 - w_2) f(0) = c'(a^*).$$

The optimal effort is then strict monotone increasing in the prize spread $w_1 - w_2$.The chosen effort does not depend on the total amount of prizes.

[5] The concavity of the distribution function $F$ does not assure the quasiconcavity of the objective function. But if the cumulative distribution function (cdf) $F$ is concave, the objective function is also concave and then quasiconcave.

[6] This condition assumes a local optimum. To reach a global optimum, one supplementary incentive condition is necessary. According to that condition, the agent will be insured

**Proposition 1.** *The principal (or firm) can induce any feasible effort as an equilibrium of the tournament subgame by choosing an appropriate prize spread.*

**Equilibrium prizes.** Using the solution of the tournament subgame, we can determine the equilibrium prizes. The firm chooses those prizes that maximize the expected value of profits subject to the agents participation condition. We have

$$\max_{w_1,w_2} \left\{2\big(p.a^*(w_1,w_2) - \frac{1}{2}(w_1+w_2)\big\},\right.$$
$$\text{s.t.}$$
$$w_2 + \frac{1}{2}(w_1 - w_2) - c\bigg(a^*(w_1,w_2)\bigg) \geq 0.$$

Substituting the constraint into the objective function, we have the simplified decision problem

$$\max_a \; 2\big(p.a - c(a) + w_2\big)$$

The equilibrium effort $a$ then solves $c'(a) = p$. The equilibrium outcome $(a^*, w_1^*, w_2^*)$ of the tournament game is obtained from the conditions

$$c'(a) = p,$$
$$(w_1 - w_2)f(0) = c'(a),$$
$$\frac{1}{2}(w_1 + w_2) = p.a^*.$$

The optimal values of the pair of prizes $(w_1^*, w_2^*)$ is given by

$$w_1^* = p.a^* + \frac{c'(a^*)}{2f(0)},$$
$$w_2^* = p.a^* - \frac{c'(a^*)}{2f(0)},$$

These optimal prizes consist of a common part $p.a^*$ equal to the expected output $\frac{w_1+w_2}{2}$ and a prime of $\pm\frac{c'(a^*)}{2f(0)} = \pm\frac{w_1+w_2}{2}$ to the winner and to the looser respectively. The agents are then not necessary paid to their marginal productivity. The prime is the incitation to participate to the tournament.

**Numerical examples.** Let us consider (see Wolfstetter, 1999) two different distributions of the RVs $\tilde{\varepsilon}$ : a normal distribution $\mathcal{N}(0,\sigma^2)$ and a uniform distribution $\mathcal{U}\big([-\frac{1}{2},\frac{1}{2}]\big)$. If $\tilde{\varepsilon} \rightsquigarrow \mathcal{N}(0,\sigma^2)$, then $\tilde{\theta} = \tilde{\varepsilon}_{-i} - \tilde{\varepsilon}_i \rightsquigarrow \mathcal{N}(0,2\sigma^2)$. The probability density

to receive an utility level, superior to the utility reservation. For a minimal effort $\underline{a}$ with probability $P(\underline{a}, a^*)$, this condition implies $w_i(a^*) \geq w_i(a_{-i}, a_j^*)$. Then, we have $\frac{1}{2}(w_1 - w_2) - c(a^*) \geq P(a_{-i}, a_j^*)(w_1 - w_2) - c(a_{-i})$ with $(w_1 - w_2)f(0) = c'(a^*)$. The constraint is

$$\frac{c'(a^*)}{f(0)}\left(\frac{1}{2} - P(a_{-i}, a_{-i}^*)\right) \geq c(a^*) - c(a_{-i}).$$

Then, this constraint is easier to satisfy with higher minimal effort. Moreover, since $f(0) \to \infty$ when the variance $\sigma^2 \to 0$, a global optimum is achieved for higher variance.

function (pdf) is [7] $f(\theta) = 1/\sqrt{4\sigma^2\pi}\exp\bigl(-\theta^2/4\sigma^2\bigr)$ and $f(0) = 1/\sqrt{4\sigma^2\pi}$. We then have the optimum

$$\left(a^*, w_1^*, w_2^*\right) = \left(p,\ \frac{1}{2}p(p+2\sigma\sqrt{\pi}),\ \frac{1}{2}p(p-2\sigma\sqrt{\pi})\right).$$

If $\tilde{\varepsilon} \rightsquigarrow \mathcal{U}\bigl([-\frac{1}{2}, \frac{1}{2}]\bigr)$, then $\tilde{\theta} \rightsquigarrow \mathcal{U}\left([-1,1]\right)$.The pdf is [8]

$$f(\theta) = \begin{cases} 1+\theta & \texttt{for } \theta \in (-1,0], \\ 1-\theta & \texttt{for } \theta \in [0,1), \\ 0 & \texttt{otherwise}. \end{cases}$$

and we have $f(0) = 1$. In this case, the optimum is

$$\left(a^*, w_1^*, w_2^*\right) = \left(p,\ \frac{1}{2}p(p+1),\ \frac{1}{2}p(p-1)\right).$$

### 1.3. Multiple contestants tournaments

**The Lazear and Rosen's tournament problem.** Let us consider the Lazear-Rosen specification for the utility function and the output equation with risk-averse agents Lazear and Rosen, 1981. We have

$$\begin{aligned} U(y_k, a_k) &= U\bigl(y_k - c(a_k)\bigr),\ \texttt{for } k \in \{i, -i\},\ \texttt{with } U', c', c'' > 0,\ U'' \leq 0, \\ x_k &= a_k + \tilde{\varepsilon}_k + \tilde{\eta},\ \texttt{for } k \in \{i, -i\}, \end{aligned}$$

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7 The sum of i.i.d and normally distributed RVs is again normally distributed, with variance equal to the sum of variances. More generally, suppose the RV $\tilde{X}$ has a continuous pdf $f(x)$ with support $S \subset \mathbb{R}$. Let a RV $\tilde{Y}$, whose pdf is $g(y)$ with support $T \subset \mathbb{R}$, be a one-by-one differentiable function $\varphi$ of $\tilde{X}$. The pdf of $\tilde{Y}$ is

$$g(y) = f\bigl(\varphi^{-1}(y)\bigr).\frac{1}{|\varphi'(y)|}$$

or equivalently $g(y) = f\bigl(\psi(y)\bigr).|\psi'(y)|$, where $\psi = \varphi^{-1}$. The proof is easily obtained from the cumulative distribution function (cdf) of $\tilde{Y}$ say $G(y) = \texttt{Prob}\{\tilde{Y} \leq y\}$, considering both monotonic increasing ($\varphi' > 0$) and decreasing ($\varphi' < 0$) cases.

8 Suppose that the RV $\tilde{Y}$ is a linear transformation of RV $\tilde{X}$ with $\tilde{Y} = A.\tilde{X}$ where A is a real-valued invertible matrix. The joint pdf of $\tilde{Y}$ is

$$g(y) = f\bigl(A^{-1}y\bigr).\frac{1}{|\texttt{det}A|}.$$

In this case define $\tilde{Y} = \tilde{X}_1 - \tilde{X}_2$ and $\tilde{Z} = \tilde{X}_2$ then $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The pdf of $\tilde{Y}$ is $g(y) = \int f(y+z, z)dz$. If $\tilde{X}_1$ and $\tilde{X}_2$ are i.i.d uniformly distributed on the support $\bigl(-\frac{1}{2}, \frac{1}{2}\bigr)$, $\tilde{Y}$ is triangular distributed on the support $(-1, 1)$. We have the pdf

$$g(y) = \begin{cases} 1+y & \texttt{for } y \in (-1,0], \\ 1-y & \texttt{for } y \in [0,1), \\ 0 & \texttt{otherwise}. \end{cases}$$

where the random environment is described a common RV $\tilde{\eta}$ (indicating the general difficulties of a set of tasks) and by the individualistic noise $\tilde{\varepsilon}$ (giving the advantage or disadvantage of doing one task in particular). Moreover, we assume $\mathbf{E}\big[\tilde{\varepsilon}_k\big] = \mathbf{E}\big[\tilde{\varepsilon}_k, \tilde{\varepsilon}_l\big] = 0$ and $\mathbf{E}\big[\tilde{\eta}\big] = 1$. The functions $F$ and $f$ are respectively the distribution function and the density of the RVs $\tilde{\varepsilon}_k$. The principal problem is to choose the contract parameters and the effort levels of contestants so as to maximize the expected utility function subject to the zero-expected-profit condition and the incentive compatibility condition. The expected utility is given by [9]

$$\begin{aligned}\mathbf{E}[U_i] &= \big(1 - \texttt{Prob}\{x_i < x_{-i}\}\big)U(w_1, a_i) + \texttt{Prob}\{x_i < x_{-i}\}U(w_2, a_i) \\ &= \big(1 - F(a_{-i} - a_i)\big)U\big(w_1 - c(a_i)\big) + F(a_{-i} - a_i)\Big)U\Big(w_2 - c(a_i)\Big)\end{aligned} \tag{1}$$

The zero-expected-profit condition is such that the total prize $w_1 + w_2$ equals the expected outputs. We have

$$p.a_k = \Big(1 - F(a_{-i} - a_i)\Big)w_1 + F(a_{-i} - a_i)w_2.$$

The incentive compatibility condition is obtained in maximizing the expected utility function $\mathbf{E}[U_i]$ w.r.t. $a_i$. We have

$$\begin{aligned}\frac{\partial}{\partial a_i}\mathbf{E}\big[U_i\big] = f(a_{-i} - a_i)U\Big(w_1 - c(a_i)\Big) - c'(a_i)\Big(1 - F(a_{-i} - a_i)\Big)U\Big(w_1 - c(a_i)\Big) - \\ f(a_{-i} - a_i)U\Big(w_2 - c(a_i)\Big) - c'(a_i)F\Big(a_{-i} - a_i\Big)U'\Big(w_2 - c(a_i)\Big)\end{aligned} \tag{2}$$

At the symmetric Nash solution, with $a_i = a_{-i} = a$ and $P(a, a) = \frac{1}{2}$, the Lazear-Rosen problem is

$$\max_{w_1, w_2, a_i} \mathbf{E}\big[U_i\big] = \frac{1}{2}\Big\{U\Big(w_1 - c(a_i)\Big) + U\Big(w_2 - c(a_i)\Big)\Big\},$$

$$\text{s.t.}$$

$$p.a_i = \frac{w_1 + w_2}{2},$$

$$c'(a_i) = 2f(0).\frac{U\Big(w_1 - c(a_i\Big) - U\Big(w_2 - c(a_i)\Big)}{U'\Big(w_1 - c(a_i)\Big) + U'\Big(w_2 - c(a_i)\Big)}.$$

Taking second-order Taylor series expansions of the utility function and of the incentive compatibility condition around $z_i \equiv \frac{w_1 + w_2}{2} - c(a_i)$, we have the quadratic approximation (McLaughlin, 1988)

$$\max_{w_1, w_2, a_i} \mathbf{E}\big[U_i\big] = U\Big(\frac{w_1 + w_2}{2} - c(a_i)\Big) + \frac{1}{2}U''(z_i)\big(\frac{w_1 - w_2}{2}\big)^2,$$

$$\text{s.t.}$$

$$p.a_i = \frac{w_1 + w_2}{2},$$

$$c'(a_i) = f(0).(w_1 - w_2).$$

[9] Indeed, we have $\texttt{Prob}\{x_i < x_{-i}\} = \texttt{Prob}\{\tilde{\epsilon}_i - \tilde{\epsilon}_{-i} < a_{-i} - a_i\} = F(a_{-i} - a_i)$.

Substituting the constraints into the objective function yields the unconstraint problem

$$\max_{\Delta w}\ \mathbf{E}\big[U_i\big] = U\Bigg(p.a^*\Big(f(0)\Delta w\Big) - c\Big(a^*(f(0)\Delta w)\Big)\Bigg) + \frac{1}{2}U''(z_i)\Big(\frac{\Delta w}{2}\Big)^2,$$

where $\Delta w = w_1 - w_2$. Ignoring terms of order 3, we derive the marginal condition

$$\frac{\partial \mathbf{E}[U_i]}{\partial \Delta w} = U'(z_i).\frac{p-c'}{c''}.f(0) + \frac{U''(z_i)\Delta w}{4} = 0.$$

The optimal prize spread is deduced as

$$\Delta w^* = \frac{f(0)p}{f(0)^2 + \frac{A_a c''}{4}},$$

where $A_a = -U''/U'$ denotes the absolute risk aversion. The optimal prize spread is increasing in product prize, decreasing in the risk aversion coefficient and curvature of the cost of effort. The optimal effort $a^*$ is deduced from the last incentive compatibility condition. We have to solve the implicit equation

$$c'(a^*) = \frac{p}{1 + A_a \frac{c''}{4f(0)^2}}.$$

The optimal effort is increasing in product price and decreasing in the risk aversion coefficient and the curvature of the cost of effort.

**The generalization to multiple contestants tournaments.** The generalization of this model is given by McLaughlin, 1988, for a single top prize $w_n$. The distribution function for each i.i.d RVs $\tilde{\varepsilon}_{-i}$ evaluated at $\varepsilon_i + a_i - a_{-i}$ is

$$F\big(\varepsilon_i + a_i - a_{-i}\big) = \texttt{Prob}\big\{x_i > x_{-i}\big\}.$$

The subscript -$i$ denotes the $n-1$ other contestants. The probability of ranking $n$th is [10]

$$\overline{F}_n(n) = \int F\big(\varepsilon_i + a_i - a_{-i}\big)^{n-1} f(\varepsilon_i)d\varepsilon_i.$$

At the symmetric Nash equilibrium, we have

$$\bar{f}_n(n) = (n-1)\int F(\varepsilon_i)^{n-2} f(\varepsilon_i)^2 d\varepsilon_i.$$

[10] The probability of ranking $k$th from the bottom, given the effort $a^*$ of other $n-1$ contestants is

$$\bar{F}_n(k) = \binom{n-1}{k-1}\int F\big(\varepsilon_i + a_i - a^*\big)^{k-1} \times \Big(1 - F\big(\varepsilon_i + a_i - a^*\big)\Big)^{n-k} f(\varepsilon_i)d\varepsilon_i.$$

The $n$-contestant tournament problem for a single prize under the Lazear-Rosen specification [11] is

$$\max_{w_1,w_n,a_i}\ \mathbf{E}\big[U_i\big] = \big(1-\frac{1}{n}\big).U\Big(w_1 - c(a_i)\Big) + \frac{1}{n}U\Big(w_n - c(a_i)\Big),$$

s.t.

$$p.a_i = \big(1-\frac{1}{n}\big)w_1 + \frac{1}{n}w_n,$$

$$c'(a_i) = \frac{\Big\{U\Big(w_n - c(a_i)\Big) - U\Big(w_1 - c(a_i)\Big)\Big\}.\bar{f}_n(n)}{\frac{1}{n}U'\Big(w_n - c(a_i)\Big) + \big(1-\frac{1}{n}\big)U'\Big(w_1 - c(a_i)\Big)}.$$

Using a Taylor quadratic approximation of the problem, we deduce the optimal prize spread and effort for the $n$-contestant tournament. We obtain

$$\Delta w^* = \frac{\bar{f}_n(n)p}{\bar{f}_n(n)^2 + A_a\frac{c''}{4}},$$

$$c'(a^*) = \frac{p}{1 + A_a\frac{c''}{4\bar{f}_n(n)^2}}.$$

Let us consider how the solutions are affected by the tournament size. The prize spread is increasing in the number of contestants with $\mathtt{limit}_{n\to\infty}\Delta w^* = \infty$. Since a marginal increase in effort has a negligible effect on the probability of winning for large $n$, the prize spread must be high to induce enough effort. Ma, 1988 considers the problem of implementation of incentive contracts when a principal hires many agents and is not able to control their actions.When the actions are mutually observable there is a unique perfect equilibrium. When the actions are not observable, there may have multiple equilibria.

## 2. Innovation Tournaments

### 2.1. Innovation games

**Incentive research competition.** Governments, private corporations and even individuals are commonly sponsors of research tournaments. Taylor, 1995 relates

[11] The multiple-prize $n$-contestant tournament (McLaughlin, 1988)would be

$$\max_{\{w_k\},a_i}\ \mathbf{E}\big[U_i\big] = \frac{1}{n}\sum_{k=1}^{n}U\Big(w_k - c(a_i)\Big),$$

s.t.

$$p.a_i = \frac{1}{n}\sum_{k=1}^{n}w_k,$$

$$c'(a_i) = \frac{\sum_{k=1}^{n}\Big\{U\big(w_k - c(a_i)\big)\bar{f}_n(k)}{\frac{1}{n}\sum_{k=1}^{n}U'\Big(w_k - c(a_i)\Big)}.$$

some recent famous contests [12] : Frigidaire Co and Whirlpool Corp. were selected among 14 contestants to compete for a research that would use 25 to 50 percent less electricity and no chlorofluorocarbons ... (The New York Times, 8 July 1992). Whirlpool won the contest. The Federal Communication Commission held a tournament for higher definition television, Dow and IBM sponsor annual tournaments in which the winners receive grants to develop their projects. Other examples are given by C.R. Taylor. A research tournament is an incomplete contract designed to overcome the difficulties of conventional contracts (such as bilateral contracts). Moreover, the prospect of winning a specific prize (production contract, cash,etc.) provides incentives to exert significative research efforts. Rogerson, 1982 estimates the size of the prize implicit in each production contract of 12 major aerospace major research contests held by the US Department of Defense between 1964 and 1977. Knoeber and Thurman, 1994 are testing empirically the tournaments' theory. Their results are consistent with the predictions: unchanged prize differentials (or spread) will not affect the performance, in mixed (heterogeneous) tournaments more able agents choose less risky strategies, and tournament sponsors will attempt to discourage agents with unequal ability [13].

**Innovation games : patent races and innovation tournaments.** There is two classes of innovation games : the innovation or patent race and the tournaments (Baye and Hoppe, 2003, Taylor, 1995). Both games differ regarding the features and modeling. Innovation races are used to model the competition to be first: the date of discovery is uncertain and the standard quality is fixed. A larger prize reduces the expected amount of time to innovate. On the contrary, innovation tournaments are used to model the competition to be best : the terminal date is fixed and the quality of innovation varies. A larger prize raises the expected quality of winning invention. patent race models were pioneered by Dasgupta and Stiglitz, 1980a, 1980b, Lee and Wilde, 1980, Loury, 1979, Reinganum, 1981,1982. A simple model is the memoryless patent race where uncertainty is formalized by an exponential distribution. In the Loury's study, the relationship between the market structure and the R&D spending relies on this probabilistic assumption. The assumption is that the firm's probability to discover depends on the current R&D expenses [14]. On the contrary, there no need to restrict the RVs to be exponentially distributed in the tournament game. Baye and Hoppe, 2003 demonstrate the strategic equivalence of tournaments and patent-race games. It is shown that innovation tournaments produce not only negative externalities on R&D due to the incentives to win, but also positive externalities due to the competition between contestants.

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[12] A famous historic example of tournament is the Liverpool and Manchester Railway Company in 1829. This company offered a prize of £500 for the best transportation engine (Fullerton and McAfee, 1999).

[13] Becker and Huselid, 1992 show how tournaments are incentive, using empirical tests on a panel data set from auto racing. The tournament spread exerts incentive effects on individual performances. Ehrenberg and Bognanno, 1990 focus on professional golf tournaments and also prove that level and structures of the prizes influence significantly the players' performances.

[14] Further refinements of the basic patent-race model (Tirole, 1990) introduce a choice between less or more risky technologies by the firms, learning effects such as accumulated experiences on the probability of discovering by unit of time.

**The dynamic game design and model building.** A dynamic game generally is played by $n$ contestants participating to a two-stage game of imperfect information. Let us describe the two-contestant tournament Taylor, 1995, Wolfstetter, 1999. Nature draws i.i.d. output distortions. The drawing is not revealed either to the contestants before there decisions or to the sponsor. The distribution of the distortion is common knowledge. At the first step, the sponsor sets two prizes $w_1 > w_2$ and commits to pay $w_1$ to the high-output R&D and $w_2$ to the low output. At the second stage, each contestant choose his investment in R&D without observing the opponents' choices. The prizes are then paid.The tournament game is solved by backward induction using prior beliefs. The tournament subgame is first solved for given prizes. Thereafter, the sponsor computes the expected profit and find the optimal prizes. In the tournament subgame, the two contestants choose their investment simultaneously. The contestant $i$'s probability of winning must be increasing in his R&D spending and decreasing in that of his rival. The equilibrium investments satisfy the reaction functions. A unique symmetric equilibrium in pure strategies is achieved, provided the quasiconcavity of the objective function. The equilibrium investment spending is strictly increasing in the prize spread. Using the solution of the tournament subgame, the sponsor chose prizes that maximize the expected profits subject to a contestant's participation condition. The equilibrium of the whole game is then deduced. Zheng and Vukina, 2006 construct an empirical model of a rank-order tournament game. The authors estimate the structural parameters of the symmetric Nash equilibrium and simulate the model. The observed cardinal tournament with risk-averse agents fits the data better.

## 3. A R&D Tournament Model with Spillovers

### 3.1. Presentation

In the R&D tournament model by Zhou, 2006, $n$ risk- neutral contestants are investing research inputs (or efforts) in order to produce the best research output. These contestants may be firms competing a specific award from some public or private sponsor. The contestant who produces the best R&D output obtains a reward of $w_1$. Each other contestants who loses the contest is supposed to receive a weaker prize of $w_2$. The prize spread is $\Delta w = w_1 - w_2$. The $i$'s contestant is identified by the subscript $i$ and the $n-1$ rivals by -$i$ with $-i \in \{1, 2, \ldots, i-1, i+1, \ldots, n\}$. The contestant $i$'s R&D output is given by

$$x_i = a_i + \beta \sum_{j \neq i} a_j + \tilde{\varepsilon}_i, \tag{3}$$

where $a_i$ denotes the R&D input, $\beta$ $(0 \leq \beta \leq 1)$ the spillovers from the R&D output [15], and $\tilde{\epsilon}_i$ an individual noise. The $i$'s R&D cost $c(a_i)$ is increasing and convex in efforts such that $c', c'' > 0$. Let denote $P_i$ the probability that contestant $i$'s wins the contest. His expected payoff is

$$P_i w_1 + (1 - P_i) w_2 - c(a_i).$$

[15] Griliches, 1992 reviews the basic model of R&D spillovers. The importance of spillovers is also shown empirically by Bernstein, 1988. The author estimates the effects of intra- and interindustry R&D investment spillovers on costs and on the production structure. The absorptive capacity and knowledge accumulation may also affect a firm's R&D output (Zhou, 2006).

### 3.2. Expected payoff of the contestants

Since the winner is supposed to have the highest $x_i$, the condition is according to equation (3)

$$a_i + \beta \sum_{j \neq i} a_j + \varepsilon_i > a_{-i} + \beta \sum_{k \neq i} a_k + \varepsilon_{-i}, \ \texttt{for all} \ -i \neq i.$$

Then, we have

$$(1-\beta)(a_i - a_{-i} + \varepsilon_i > \varepsilon_{-i}, \ \texttt{for all} \ -i \neq i.$$

We assume that there exists a symmetric equilibrium of R&D inputs[16]. Given the R&D inputs of the rivals all equal to the same amount $a$, the contestant $i$'s probability of winning is

$$F^{n-1}\Big((1-\beta)(a_i - a) + \varepsilon_i\Big), \ \texttt{for a given} \ \varepsilon_i.$$

Integrating over all realizations of $\tilde{\varepsilon}_i$, the contestant $i$'s expected probability of winning is

$$\int F^{n-1}\Big((1-\beta)(a_i - a) + \varepsilon_i\Big) f(\varepsilon_i) d\varepsilon_i.$$

The expected payoff is

$$W_i = w_1 \int F^{n-1}\Big((1-\beta)(a_i - a) + \varepsilon_i\Big) f(\varepsilon_i) d\varepsilon_i + w_2 \left\{1 - \int F^{n-1}\Big((1-\beta)(a_i - a) + \varepsilon_i\Big) f(\varepsilon_i) d\varepsilon_i\right\} - c(a_i). \quad (4)$$

In condensed form, we also have

$$W_i = w_2 + \Delta w \int F^{n-1}\Big((1-\beta)(a_i - a) + \varepsilon_i\Big) f(\varepsilon_i) d\varepsilon_i - c(a_i). \quad (5)$$

### 3.3. Reaction functions of the contestants

The contestant $i$ chooses the input R&D $a_i$ to maximize the expected payoff. The FOC of this problem is

$$(1-\beta)\Delta w \int \frac{\partial}{\partial a_i} F^{n-1}\Big((1-\beta)(a_i - a) + \varepsilon_i\Big) f(\varepsilon_i) d\varepsilon_i - c'(a_i) = 0.$$

We then have

$$(1-\beta)\Delta w \int (n-1) F^{n-2}\Big((1-\beta)(a_i - a) + \varepsilon_i\Big) \times f\Big((1-\beta)(a_i - a) + \varepsilon_i\Big) f(\varepsilon_i) d\varepsilon_i - c'(a_i) = 0. \quad (6)$$

[16] The existence of a symmetric equilibrium in tournaments is discussed by Nalebuff and Stiglitz, 1983. A symmetric equilibrium is more likely to exist when the variance of the RVs is large.

At the symmetric equilibrium with $a_i = a$, a contestant chooses his R&D spending so as to equalize the marginal benefit of increasing his R&D spending and the marginal cost of these research spending. We have the rule

$$(1-\beta)\Delta w \int (n-1)F^{n-2}(\varepsilon_i)f^2(\varepsilon_i)d\varepsilon_i = c'(a). \tag{7}$$

A larger spillover $\beta$ will then decrease the contestant's incentive to spend. Let us assume an exponential distribution of the type $f(\varepsilon) = \theta e^{-\theta\varepsilon}$. The condition $f'(\varepsilon) = -\alpha^2 e^{-\theta\varepsilon} < 0$ is then satisfied. Introducing this exponential pdf into 7, we have the integral

$$\int_{-\infty}^{\infty} F^{n-1}\Big((1-\beta)(a_i-a)+\varepsilon_i\Big) = e^{-\alpha(1-\beta)(a_i-a)} \times \int_{-\infty}^{\infty} \alpha e^{-\alpha\varepsilon_i n} d\varepsilon_i = \frac{1}{n} e^{-\alpha(1-\beta)(a_i-a)}.$$

We then obtain the following expected payoff according to 7

$$W_i = w_2 + \Delta w \times \frac{1}{n} e^{-\alpha(1-\beta)(a_i-a)}.$$

The reaction functions for two players are

$$\alpha(1-\beta)\frac{1}{n}\Delta w - c'(a) = 0.$$

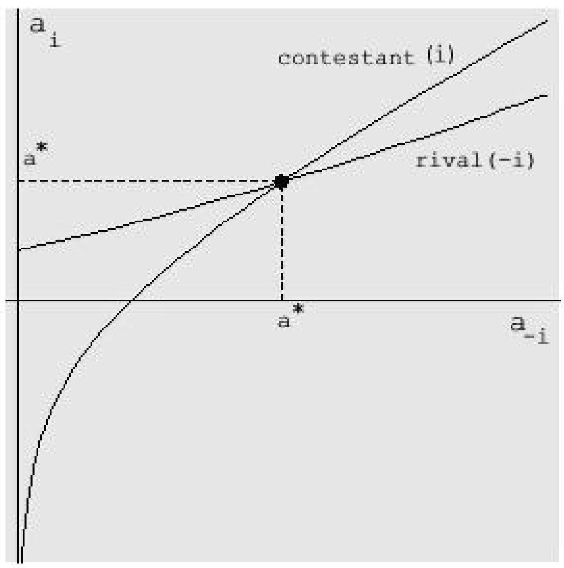


**Fig.1.** Reaction function of two contestants with exponential noise

### 3.4. Size of the tournament

**Proposition 2.** *The contestant's R&D spending varies as the pdf of the individual noise when the size of the tournament is increasing. The derivatives $\frac{da}{dn}$ and $f'(\varepsilon)$ have the same sign. Hence, if $f'((\varepsilon) < 0$ the contestant's R&D investment decreases when the number of contestants increases, if $f'(\varepsilon) = 0$ the R&D investment remains unchanged when the size of the tournament increases and if $f'(\varepsilon) > 0$ the R&D investment will increase with the size of the tournament.*

*Proof.* The integration by parts of the integral in equation(7) yields

$$(1-\beta)\varDelta w\left\{ f(+\infty) - 2\int F^{n-1}(\varepsilon_i)f'(\varepsilon_i)d\varepsilon_i \right\} = c'(a_i).$$

The derivation w.r.t the size of the tournament $n$ leads to

$$\begin{aligned}(1-\beta)\varDelta w\frac{d}{dn}\left\{ f(+\infty) - 2\int F^{n-1}(\varepsilon_i)f'(\varepsilon_i)d\varepsilon_i \right\} &= c''(a_i)\frac{da_i}{dn}, \\ (1-\beta)\varDelta w\int -\ln F(\varepsilon_i)F^{n-1}(\varepsilon_i)f'(\varepsilon_i)d\varepsilon_i &= c''(a_i)\frac{da_i}{dn}\end{aligned} \tag{8}$$

With fixed spread $\varDelta w$, we have

$$\mathtt{sgn}\left(\frac{da_i}{dn}\right) = \mathtt{sgn}f'(\varepsilon_i).$$

■

Zhou, 2006 illustrates this proposition with different distributions of the random variables. The exponential distribution $f(\varepsilon) = \theta\exp(-\theta\varepsilon),\ \theta > 0$ satisfies the condition $f'(\varepsilon) < 0$ and leads to the same result as Loury, 1979. For the uniform distribution of the RVs, we have $f'(\varepsilon) = 0$ with no change in the R&D spending when the number of opponents increases. R&D spending increase with a power function distribution like $f(\varepsilon) = \left(\frac{\varepsilon}{\gamma}\right)^{\theta},\ 0 < \varepsilon < \gamma,\ \theta > 0$ (Zhou, 2006).

## 4. Further Extensions

Further extensions of the basic tournament model have been considered with endogenous payoffs, introduction of multiplicative common shocks, multiple prizes, nonlinear output in effort and heterogeneous contestants.

### 4.1. Endogenous payoffs

Zhou, 2006 considers a tournament where the payoffs are endogenously determined by R&D spending. It is assumed that if the contestant $i$'s wins the contest with R&D output is $a_i + \beta\sum_{j\neq i} a_j + \varepsilon_i$, his reward will have the same amount. If the contestant looses, the reward will be zero. From the FOC for the contestant $i$'s expected payoff maximization, Zhou, 2006 shows two advantages : the payoff (conditioned of winning) is increased for a given probability of winning, and the expected probability of winning is also increased for a given reward. However, an increase of the prize to the winner may decrease each contestant's expected payoff and discourage the R&D spending.

### 4.2. Common shocks

The stochastic environment of the game already has individual-specific components $\tilde{\varepsilon}_j, \texttt{ for } j = 1, \ldots, n$. It may also have a common component $\tilde{\eta}$ with cdf $G(\eta)$. The random component will then represent the level of difficulty of a set of tasks, while the random component $\eta$ will be the individual's comparative advantage or disadvantage. These random disturbances are uncorrelated. The probability distributions $F(\varepsilon)$ and $G(\eta)$ are common knowledge. Both random disturbances affect the outputs. Two alternative forms may be considered for the contestant $i$'s output equation. The Lazear and Rosen, 1981 specification is additive

$$x_i = a_i + \tilde{\eta} + \tilde{\varepsilon}_i,\ a_i \geq 0,\ \mathbf{E}[\tilde{\varepsilon}_i] = \mathbf{E}[\tilde{\eta}] = 0.$$

The Nalebuff and Stiglitz, 1983 specification is multiplicative with

$$x_i = \tilde{\eta}.a_i + \tilde{\varepsilon}_i,\ \mathbf{E}[\tilde{\varepsilon}_i] = 0,\ \texttt{and } \mathbf{E}[\tilde{\eta}] = 1.$$

Green and Stokey, 1986 introduce common additive shocks and prove that tournaments dominate the individual-based contracts if the common shock is strong. Nalebuff and Stiglitz, 1983 use multiplicative shocks which affect the marginal product of effort and produce different results.

### 4.3. Multiple prizes

Nalebuff and Stiglitz, 1983 also consider tournaments with multiple prizes. The probability that the contestant $i$ obtains the $j$th position up from the bottom is

$$\texttt{Prob}\{x_i = j\} = \int \binom{n-1}{n-j} f(\varepsilon) F\Big(\eta(a_i - \bar{a}) + \varepsilon\Big)^{j-1} \times \Big\{1 - F\big(\eta(a_i - \bar{a}) + \varepsilon\big)\Big\}^{n-j} d\varepsilon. \quad (9)$$

At the symmetric equilibrium (where $a_i\bar{a}$), we compute

$$\frac{\partial}{\partial a_i}\texttt{Prob}\{x_i = j\} = \eta \int \binom{n-1}{n-j} f^2(\varepsilon) \Big(1 - F(\varepsilon)\Big)^{n-j-1} F(\varepsilon)^{j-2} \times \Big\{(j-1)\Big(1 - F(\varepsilon)\Big) - (n-j)F(\varepsilon)\Big\} d\varepsilon. \quad (10)$$

Thus by increasing the R&D inputs there an increased chance to have the position $j$.

### 4.4. Nonlinear outputs

When outputs are nonlinear in inputs,we may have the contestant $i$'s output equation

$$x_i = \varphi(\tilde{\eta}.a_i) + \tilde{\varepsilon}_i,$$

or

$$x_i = \varphi(\tilde{\eta}.a_i + \tilde{\varepsilon}_i).$$

This nonlinearities are studied by Nalebuff and Stiglitz, 1983. In the first situation, investing more at the symmetric solution increases the probability of winning by a factor proportional to $\varphi$.

### 4.5. Heterogeneous contestants

Relaxing the assumption of identical contestants also generalize the tournament game (Lazear and Rosen, 1981,McLaughlin, 1988). Each contestant knows his own abilities but ignores those of his opponents. Asymmetries in the knowledge of abilities produce inefficiencies. Lazear and Rosen, 1981 prove that if heterogeneous contestants do not self sort, the tournament is inefficient. Bhattacharya and Guasch, 1988 show how, by a proper design of contracts, the efficiency can be achieved with such tournaments with asymmetrically heterogeneity of agents. Jost and Kräkel, 2005 consider a sequential-move tournament with heterogeneous players. The agents' strategic behaviors differ from that one in simultaneous-move tournaments. Thus, the first playing agent may choose a preemptively high effort so that the follower will give up.

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# Nash and Stackelberg Solutions Numerical Construction in a Two-Person Nonantagonistic Linear Positional Differential Game⋆

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**Abstract** The paper suggests numerical methods for constructing Nash and Stackelberg solutions in a linear two-person positional differential game with terminal payoffs of players and polygonal constraints for players controls. Formalization of players' strategies in the game is based on formalization and the results of positional antagonistic differential games theory, developed by N. N. Krasovskii and his scientific school. The game is such, that it could be reduced to a game on the plane and the problem is transformed to solving non-standard optimal control problems. For the approximation of trajectories in these problems a set of computational geometry algorithms in plane is used, including convex hull construction, union and intersection of polygons and a Minkowski sum for polygons.

**Keywords:** nonantagonistic differential game, Nash solution, Stackelberg solution, algorithm.

## 1. Introduction

Problems of computation of solutions in antagonistic and nonantagonistic differential games are important (Başar and Olsder, 1999, Krasovskii and Subbotin, 1974, Krasovskii, 1985, Kleimenov, 1993). One can note algorithms proposed for antagonistic games in Isakova *et al.*, 1984, Vahrushev *et al.*, 1987, as well as in other studies of the same and other authors. Comparing to this, there are distinctly less studies concerning non-antagonistic games and they usually deal with linear quadratic games. The present paper describes algorithms for Nash equilibrium solutions and Stackelberg solutions in linear differential game with geometrical constraints for players' controls and terminal cost functionals of players.

The paper is organized as follows. Section 2 contains problem statement. Section 3 describes common method for Nash and Stackelberg solutions construction based on reduction of original problem to non-standard problems of (optimal) control. Section 4 solves Stackelberg problem approximating admissible trajectories via repetitive intersections of stable bridges and local attainability sets. Section 5 presents description of two algorithms. The first one builds a Nash solution with the

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help of auxiliary bimatrix game. The second one based on modification of previous algorithm and is aimed to construction of unimprovable (according to Pareto) Nash solutions. Brief description of the program implementation is given in Section 6. Results of numerical experiment for a material point motion in plane are presented in Section 7. Finally, Section 8 proposes possible study perspectives.

## 2. Formalization of Two-Person Nonantagonistic Positional Differential Game

Let dynamics of two-person nonantagonistic positional differential game (NPDG) be described by the equation

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u} + \mathbf{C}(t)\mathbf{v}, \quad t \in [t_0, \theta], \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is a phase vector, the controls $\mathbf{u} \in P \subset \mathbb{R}^k$ and $\mathbf{v} \in Q \subset \mathbb{R}^l$ are handled by Player 1 (P1) and Player 2 (P2) respectively, and $\theta$ is a fixed final time. Sets $P$ and $Q$ are convex polyhedra. Matrix functions $\mathbf{A}(t)$, $\mathbf{B}(t)$, and $\mathbf{C}(t)$ are continuous and have sizes $n \times n$, $n \times k$, and $n \times l$ respectively.

Player $i$ chooses his control in order to maximize the cost functional

$$I_i = \sigma_i(\mathbf{x}(\theta)), \quad i = 1, 2, \tag{2}$$

where $\sigma_i \colon \mathbb{R}^n \mapsto \mathbb{R}^1$ are given continuous and concave functions.

Suppose, that both players have complete information about the current position $(t, \mathbf{x}(t))$ of the game. The formalization of players' strategies and of motions generated by them in NPDG is similar to the formalization introduced for antagonistic positional differential games (APDGs) in (Krasovskii and Subbotin, 1974, Krasovskii, 1985) with the exception of technical details (Kleimenov, 1993). A pure strategy (or strategy for short) of P1 is identified with a pair $U \div \{\mathbf{u}(t, \mathbf{x}, \varepsilon), \beta_1(\varepsilon)\}$, where $\mathbf{u}(\cdot)$ is an arbitrary function depending on the position $(t, \mathbf{x})$ and on a positive precision parameter $\varepsilon$ and having values in $P$. The function $\beta_1 \colon (0, \infty) \mapsto (0, \infty)$ is a continuous monotone one and satisfies the condition $\beta_1(\varepsilon) \to 0$ if $\varepsilon \to 0$. The function $\beta_1(\cdot)$ has the following sense. For a fixed $\varepsilon$ the value $\beta_1(\varepsilon)$ is the upper bound for the step of a subdivision of the interval $[t_0, \theta]$ which P1 uses for forming step-by-step motion. A strategy $V \div \{\mathbf{v}(t, \mathbf{x}, \varepsilon), \beta_2(\varepsilon)\}$ of P2 is defined analogously.

Motions of two types: approximated (step-by-step) ones and ideal (limit) ones are considered as motions generated by a pair of strategies of players. Approximated motion $\mathbf{x}[\cdot, t_0, \mathbf{x}_0, U, \varepsilon_1, \Delta_1, V, \varepsilon_2, \Delta_2]$ is introduced for fixed values of players' precision parameters $\varepsilon_1$ and $\varepsilon_2$ and for fixed subdivisions $\Delta_1 = \{t_i^{(1)}\}$ and $\Delta_2 = \{t_j^{(2)}\}$ of the interval $[t_0, \theta]$ chosen by P1 and P2, respectively, under the conditions $\delta(\Delta_i) \leq \beta_i(\varepsilon_i)$, $i = 1, 2$. Here $\delta(\Delta_i) = \max_k (t_{k+1}^{(i)} - t_k^{(i)})$. A limit motion generated by the pair of strategies $(U, V)$ from the initial position $(t_0, \mathbf{x}_0)$ is a continuous function $\mathbf{x}[t] = \mathbf{x}[t, t_0, \mathbf{x}_0, U, V]$, for which there exists a sequence of approximated motions

$$\{\mathbf{x}[t, t_0^k, \mathbf{x}_0^k, U, \varepsilon_1^k, \Delta_1^k, V, \varepsilon_2^k, \Delta_2^k]\}$$

uniformly converging to $\mathbf{x}[t]$ on $[t_0, \theta]$ as

$$k \to \infty, \quad \varepsilon_1^k \to 0, \quad \varepsilon_2^k \to 0, \quad t_0^k \to t_0, \quad \mathbf{x}_0^k \to \mathbf{x}_0, \quad \delta(\Delta_i^k) \leq \beta_i(\varepsilon_i^k).$$

A pair of strategies $(U, V)$ generates a nonempty compact (in the metric of the space $\mathrm{C}[t_0, \theta]$) set $X(t_0, \mathbf{x}_0, U, V)$ consisting of limit motions $\mathbf{x}[\cdot, t_0, \mathbf{x}_0, U, V]$.

Now we introduce the following definition (Kleimenov, 1993).

**Definition 1.** A pair of strategies $(U^N, V^N)$ is called a Nash equilibrium solution ($NE$-solution) of the game, if for any motion $\mathbf{x}^*[\cdot] \in X(t_0, \mathbf{x}_0, U^N, V^N)$, any $\tau \in [t_0, \theta]$, and any strategies $U$ and $V$ the following inequalities hold

$$\max_{\mathbf{x}[\cdot]} \sigma_1(\mathbf{x}[\theta, \tau, \mathbf{x}^*[\tau], U, V^N]) \leq \min_{\mathbf{x}[\cdot]} \sigma_1(\mathbf{x}[\theta, \tau, \mathbf{x}^*[\tau], U^N, V^N]),$$
$$\max_{\mathbf{x}[\cdot]} \sigma_2(\mathbf{x}[\theta, \tau, \mathbf{x}^*[\tau], U^N, V]) \leq \min_{\mathbf{x}[\cdot]} \sigma_2(\mathbf{x}[\theta, \tau, \mathbf{x}^*[\tau], U^N, V^N]).$$

**Definition 2.** An $NE$-solution $(U^P, V^P)$, which is Pareto unimprovable with respect to the values $I_1, I_2$ (2) is called a $P$-solution.

Now let the following assumptions be fulfilled.

$1^0$. P1, called the leader, announced his strategy $U^* \div \{\mathbf{u}^*(t, \mathbf{x}, \varepsilon), \beta_1^*(\varepsilon)\}$ ahead of time to P2.

$2^0$. P2, called the follower, in view of P1's strategy $U^*$, chooses his rational strategy $V^*$ from the condition

$$\min \sigma_2(\mathbf{x}[\theta, t_0, \mathbf{x}_0, U^*, V]) \longrightarrow \max_V,$$

where the minimum of the function $\sigma_2$ is taken over the set $X(t_0, \mathbf{x}_0, U^*, V)$.

The problem of P1 is to find such a strategy $U^{S1}$, which ensures maximal value of his cost functional $\sigma_1(\mathbf{x}[\theta])$ (2) under the condition of the rationality of P2. (More detailed statement including the consideration of various variants of choice from the set of rational strategies for P2 can be found in (Kleimenov, 1993).)

**Definition 3.** A pair of strategies $(U^{S1}, V^{S1})$, where $V^{S1}$ is a rational strategy of P2 corresponding to announced strategy $U^{S1}$, is called an $S_1$-solution.

$S_2$-solution is defined analogously.

## 3. Auxiliary APDGs. Theorems on Structure of Solutions of NPDG

Now we consider auxiliary antagonistic positional differential games $\Gamma_1$ and $\Gamma_2$. Dynamics of both games is described by (1). In the game $\Gamma_i$ Player $i$ maximizes his payoff functional $\sigma_i(\mathbf{x}(\theta))$ (2) and Player $3-i$ opposes him.

It is known (see (Krasovskii, 1985)) that both games $\Gamma_1$ and $\Gamma_2$ have universal saddle points

$$\{\mathbf{u}^{(i)}(t, \mathbf{x}, \varepsilon), \quad \mathbf{v}^{(i)}(t, \mathbf{x}, \varepsilon)\}, \quad i = 1, 2 \tag{3}$$

and continuous value functions

$$\gamma_1(t, \mathbf{x}), \quad \gamma_2(t, \mathbf{x}). \tag{4}$$

The property of strategies (3) to be universal means that they are optimal not only for the fixed initial position $(t_0, \mathbf{x}_0) \in G$ but also for any position $(t_*, \mathbf{x}_*) \in G$ assumed to be initial one.

Now we formulate the following problems.

**Problem 1.** *Find measurable functions* $\mathbf{u}(t)$ *and* $\mathbf{v}(t)$, $t_0 \le t \le \theta$, *which generate a trajectory* $\mathbf{x}(t)$, $t_0 \le t \le \theta$, *satisfying the inequalities*

$$\gamma_i(t, \mathbf{x}(t)) \le \gamma_i(\theta, \mathbf{x}(\theta)),\ t_0 \le t \le \theta,\ i = 1, 2. \tag{5}$$

**Problem 2.**$i$ **(**$i = 1, 2$**).** *Find measurable functions* $\mathbf{u}(t)$ *and* $\mathbf{v}(t)$, $t_0 \le t \le \theta$, *which generate a trajectory* $\mathbf{x}(t)$, $t_0 \le t \le \theta$, *satisfying the inequality*

$$\gamma_{3-i}(t, \mathbf{x}(t)) \le \gamma_{3-i}(\theta, \mathbf{x}(\theta)),\ t_0 \le t \le \theta, \tag{6}$$

*and maximize the payoff functional* $\sigma_i(\mathbf{x}(\theta))$.

Let piecewise continuous functions $\mathbf{u}^*(t)$ and $\mathbf{v}^*(t)$, $t_0 \le t \le \theta$, generate a trajectory $\mathbf{x}^*(t)$, $t_0 \le t \le \theta$, of the system (1). Consider the strategies of P1 and P2

$$U^0 \div \{\mathbf{u}^0(t, \mathbf{x}, \varepsilon),\ \beta_1^0(\varepsilon)\}, \quad V^0 \div \{\mathbf{v}^0(t, \mathbf{x}, \varepsilon),\ \beta_2^0(\varepsilon)\},$$

where

$$\begin{gathered}
\mathbf{u}^0\left(t, \mathbf{x}, \varepsilon\right) = \begin{cases} \quad \mathbf{u}^*(t), & \text{if } \|\mathbf{x} - \mathbf{x}^*(t)\| < \varepsilon\varphi(t), \\ \mathbf{u}^{(2)}\left(t, \mathbf{x}, \varepsilon\right), & \text{if } \|\mathbf{x} - \mathbf{x}^*(t)\| \ge \varepsilon\varphi(t), \end{cases} \\
\mathbf{v}^0\left(t, \mathbf{x}, \varepsilon\right) = \begin{cases} \quad \mathbf{v}^*(t), & \text{if } \|\mathbf{x} - \mathbf{x}^*(t)\| < \varepsilon\varphi(t), \\ \mathbf{v}^{(1)}\left(t, \mathbf{x}, \varepsilon\right), & \text{if } \|\mathbf{x} - \mathbf{x}^*(t)\| \ge \varepsilon\varphi(t), \end{cases}
\end{gathered} \tag{7}$$

for all $t \in [t_0, \theta]$. Functions $\beta_i(\cdot)$ and positive increasing function $\varphi(\cdot)$ are chosen so that the following inequality

$$\left\| \mathbf{x}(t, t_0, \mathbf{x}_0, U^0, \varepsilon, \varDelta_1, V^0, \varepsilon, \varDelta_2) - \mathbf{x}^*(t)) \right\| < \varepsilon\varphi(t), \tag{8}$$

holds for $\varepsilon > 0$, $\delta(\varDelta_i) \le \beta_i(\varepsilon)$. Functions $u^{(2)}(\cdot)$ and $v^{(1)}(\cdot)$ are defined by (3). They can be interpreted as universal "penalty strategies" used when the partner refuses to follow the trajectory $\mathbf{x}^*(\cdot)$ at some moment of time $t \in [t_0, \theta]$. Penalty strategies were considered in (Kononenko, 1976, Tolwinski *et al.*, 1986).

The following results are valid (see (Kleimenov, 1993)).

**Theorem 1.** *Let the controls* $\mathbf{u}^*(\cdot)$ *and* $\mathbf{v}^*(\cdot)$ *be solution of Problem 1. Then the pair of strategies* $(U^0, V^0)$ *(7, 8) is an* $NE$*-solution. On the contrary, for any* $NE$*-solution there exists an equivalent solution of the same type having the form* $(U^0, V^0)$ *(7, 8) where* $\mathbf{u}^*(\cdot)$ *and* $\mathbf{v}^*(\cdot)$ *is a solution of Problem 1.*

**Theorem 2.** *Let Assumptions* $1^0$ *and* $2^0$ *be fulfilled. Let the controls* $\mathbf{u}^*(\cdot)$ *and* $\mathbf{v}^*(\cdot)$ *be solution of Problem 2.i. Then the pair of strategies* $(U^0, V^0)$ *(7, 8) is an* $S_i$*-solution. On the contrary, for any* $S_i$*-solution there exists an equivalent* $S_i$*-solution having the form* $(U^0, V^0)$ *(7, 8) where* $\mathbf{u}^*(\cdot)$ *and* $\mathbf{v}^*(\cdot)$ *is a solution of Problem 2.i.*

Thus, Theorems 1 and 2 establish correspondences between the sets of solutions of Problems 1 and 2.$i$, and the sets of $NE$- and $S_i$-solutions, respectively. These theorems determine a structure of solutions of the game. The existence theorems for $NE$- and $S_i$-solutions are corollaries of Theorems 1 and 2.

## 4. Stackelberg Solutions Building

The general idea for the algorithm is to search $\max_\alpha \sigma_i(\mathbf{x}_\alpha[\theta])$, where $\mathbf{x}_\alpha[\theta]$ are node points of a grid constructed for a *set of admissible trajectories final states* $D_i$. This $\mathbf{x}_\alpha[\theta]$ serves an endpoint for $S_i$-trajectory, which is then built back in time (controls $\mathbf{u}(t)$, $\mathbf{v}(t)$ may be found simultaneously). The $D_i$ set approximation is constructed by a procedure, that builds sequences of attainability sets (step-by-step in time), repeatedly throwing out the positions that do not satisfy (6). The procedure is described in brief below. More details (some designations differ) about it could be found in (Kleimenov and Osipov, 2003).

A special construction from the theory of antagonistic positional differential games called *stable bridge in pursuit-evasion game* is used. The aim of the follower in this game is to drive the phase vector to Lebesgue's set for the level function (for a chosen constant $c$) of his cost functional. Note, that any position at the bridge holds the inequality, but positions outside the bridge do not. Thus the bridge is used to find positions satisfying the inequality $\gamma_{3-i}(t,\mathbf{x}) \le c$.

A set $\widetilde{W}^c_{i,t}$ designates an approximation of a bridge (in the pursuit-evasion game) section in the time moment $t = \text{const}$. The following discrete scheme is used for building *admissible trajectories pipe* $G^c_i$ section approximation $\widetilde{G}^c_{i,t_k}$ (here $k$ runs through some time interval subdivision, $k = 0$ corresponds to $t_0$ moment and $k = N$ corresponds to $\theta$):

$$\widetilde{G}^c_{i,t_{k+1}} = \left[\widetilde{G}^c_{i,t_k} \oplus \delta_k(\mathbf{B}(t_k)P \oplus \mathbf{C}(t_k)Q)\right] \setminus \widetilde{W}^c_{i,t_{k+1}}, \tag{9}$$

where $\widetilde{G}^c_{i,t_0} = \{\, \mathbf{x}_0 \,\}$, $\delta_k = t_{k+1} - t_k$. Operation $A \oplus B = \{\, a + b \mid a \in A, b \in B \,\}$ denotes Minkowski sum of two sets $A$ and $B$.

We iterate through a sequence of $c$ values to make up $D_i$ of corresponding sequence of $D^c_i = \widetilde{G}^c_{i,\theta} \cap \widetilde{W}^c_{i,\theta}$ using (9) as follows:

1. We have some initial step value $\delta c > 0$ constrained by $\delta c_{\min} \le \delta c$.
2. Let $D_i$ be empty set, $c = c^{\max} = \max\limits_{\mathbf{x}\in\mathbb{R}^n} \sigma_{3-i}(\mathbf{x})$.
3. Build a pipe $\widetilde{G}^c_i$ and a set $D^c_i$ as in (9);
4. supplement $D_i := D_i \cup \{\, (\mathbf{x}, c) \mid \mathbf{x} \in D^c_i \,\}$.
5. If $\delta c \ge \delta c_{\min}$ then we choose next $c$ value:
   - if $\mathbf{x}_0 \in \widetilde{W}^c_{i,t_0}$ then a) return to the previous value $c := c + \delta c$, b) decrease step $\delta c$;
   - take next value $c := c - \delta c$;
   - repeat from item 3.
6. Quit.

One example of $S$-trajectories numerical computation results was presented in (Kleimenov *et al.*, 2006). Program, used for value function calculation, is based on results (Isakova *et al.*, 1984, Vahrushev *et al.*, 1987). An example of a bridge and a pipe is depicted on Fig.1. It illustrates all the basic constructions, that are described in the algorithm, for one iteration with fixed $c$ value.

## 5. Nash Solutions Building

The proposed algorithms use BM-procedure and modified BM-procedure, which are based on: the principle of non-decrease of player payoffs, the maximal shift in

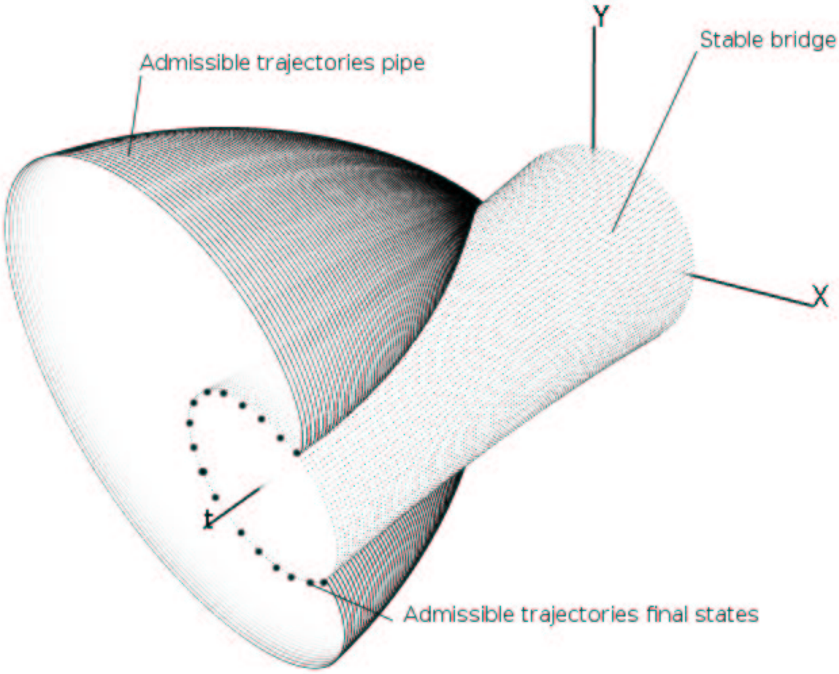


**Fig.1.** Stable bridge $\widetilde{W}_i^c$ and admissible trajectories pipe $\widetilde{G}_i^c$

the direction best for one and another player, and Nash equilibrium in auxiliary bimatrix games made up for each step in a subdivision of the time interval (see (Kleimenov, 1997)). The procedures implies that Player $i$ is interested in increasing the function $\gamma_i(t, \mathbf{x})$ along a trajectory.

The BM-procedure, described in subsection 5.1, allows to approximate a $NE$-solution, which is a $P$-solution and, in the same time, it mostly satisfies both players.

The modified BM-procedure, described in subsection 5.2, allows to approximate the rest of $P$-solutions as well.

### 5.1. BM-procedure

Suppose, that a position $(t_*, \mathbf{x}_*)$ is fixed. The system is moving from the position $(t_*, \mathbf{x}_*)$ to the position $(t^*, \mathbf{x}^*)$, where $t^* = t_* + h$ and $h$ is considered to be fixed provided $t_* + h \leq \theta$ holds. We are going to find $\mathbf{x}^*$ as well as $\mathbf{u}^*$ and $\mathbf{v}^*$ leading to that position (with $\mathbf{u}(t) \equiv \mathbf{u}^*$ and $\mathbf{v}(t) \equiv \mathbf{v}^*$, $t_* < t \leq t^*$).

A set

$$G(t^*; t_*, \mathbf{x}_*) = \{\, \mathbf{x}[t^*; t_*, \mathbf{x}_*, \mathbf{u}, \mathbf{v}] \mid \mathbf{u} \in P, \mathbf{v} \in Q \,\} \tag{10}$$

is an approximation of local attainability set of system (1) when moving from the position given to the next time moment.

Two sets $(i = 1, 2)$

$$W_i(t^*; t_*, \mathbf{x}_*) = \{\, \mathbf{x} \in \mathbb{R}^n \mid \gamma_i(t^*, \mathbf{x}) \geq \gamma_i(t_*, \mathbf{x}_*) \,\} \tag{11}$$

are approximated by known procedures of building maximal stable bridges in some pursuit-evasion games (see Krasovskii and Subbotin, 1974, Isakova *et al.*, 1984).

We analyze a set

$$H(t^*; t_*, \mathbf{x}_*) = G(t^*; t_*, \mathbf{x}_*) \cap W_1(t^*; t_*, \mathbf{x}_*) \cap W_2(t^*; t_*, \mathbf{x}_*) \tag{12}$$

in order to find $w^i(t^*; t_*, \mathbf{x}_*)$ , which is a maximum point of $\gamma_i(t^*, \mathbf{x})$ on it, $(i = 1, 2)$. Note, that these maximum points may be found either exactly on the set (and hence it should be simple to get corresponding controls $\mathbf{u}_{i0}$, $\mathbf{v}_{i0}$ leading to $w^i$) or in some neighborhood of $\mathbf{x}_*$. In the latter case one may try to maximize shift in the direction of a maximum point found. Below some details follow.

Consider vectors

$$\mathbf{s}^1(t^*; t_*, \mathbf{x}_*) = w^1(t^*; t_*, \mathbf{x}_*) - \mathbf{x}_*,$$
$$\mathbf{s}^2(t^*; t_*, \mathbf{x}_*) = w^2(t^*; t_*, \mathbf{x}_*) - \mathbf{x}_*.$$

Now we run through all the elements of polyhedra $P$ and $Q$ to find out controls $\mathbf{u}_{10}, \mathbf{u}_{20}, \mathbf{v}_{10}$, and $\mathbf{v}_{20}$:

$$\begin{gathered}
\max_{\mathbf{u}\in P, \mathbf{v}\in Q} \mathbf{s}^{1\top}[\mathbf{B}(t^*)\mathbf{u} + \mathbf{C}(t^*)\mathbf{v}] = \mathbf{s}^{1\top}[\mathbf{B}(t^*)\mathbf{u}_{10} + \mathbf{C}(t^*)\mathbf{v}_{10}],\\
\max_{\mathbf{u}\in P, \mathbf{v}\in Q} \mathbf{s}^{2\top}[\mathbf{B}(t^*)\mathbf{u} + \mathbf{C}(t^*)\mathbf{v}] = \mathbf{s}^{2\top}[\mathbf{B}(t^*)\mathbf{u}_{20} + \mathbf{C}(t^*)\mathbf{v}_{20}],\\
\text{holding the conditions}\\
\mathbf{x}[t^*; t_*, \mathbf{x}_*, \mathbf{u}_{10}, \mathbf{v}_{10}] \in H(t^*; t_*, \mathbf{x}_*),\\
\mathbf{x}[t^*; t_*, \mathbf{x}_*, \mathbf{u}_{20}, \mathbf{v}_{20}] \in H(t^*; t_*, \mathbf{x}_*).
\end{gathered} \tag{13}$$

Then, an auxiliary bimatrix $2 \times 2$ game $(A, B)$ is constructed. In this game P1 has two strategies: “to choose $\mathbf{u}_{10}$” and “to choose $\mathbf{u}_{20}$”. Similarly, P2 has two strategies: “to choose $\mathbf{v}_{10}$” and “to choose $\mathbf{v}_{20}$”. Therefore, payoff matrices of players are defined as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

$$\begin{aligned}
a_{ij} &= \gamma_1(t^*, \mathbf{x}[t^*; t_*, \mathbf{x}_*, \mathbf{u}_{i0}, \mathbf{v}_{j0}])),\\
b_{ij} &= \gamma_2(t^*, \mathbf{x}[t^*; t_*, \mathbf{x}_*, \mathbf{u}_{i0}, \mathbf{v}_{j0}])),\\
&\quad i, j = 1, 2.
\end{aligned}$$

Bimatrix game $(A, B)$ has at least one Nash equilibrium in pure strategies. It is possible to take a Nash equilibrium as both players' controls for semi-interval $(t_*, t^*]$ (or $[t_0, t^*]$ if $t_0 = t_*$). Such an algorithm of players' controls construction generates a trajectory, which is an $NE$-trajectory. When $(A, B)$ has two equilibria (1,1) and (2,2), then one is chosen after another in turn (they are being interleaved).

We take a solution of $(A, B)$ and try to fit both players' controls to maximize shift along the motion direction of this equilibrium, while staying inside $H$ (this may be done similar to (13)).

This way of players' controls generation is called *BM-procedure* here. To provide some Nash equilibrium in the game, controls generated by BM-procedure (giving BM-trajectory) must be paired with penalty strategies (see (7), (8)). Note, that each player watches for the trajectory and applies a penalty strategy when the other evades following BM-trajectory.

### 5.2. Modified BM-procedure

The BM-procedure introduced above can be modified by applying the additional inequality for every step $(t_*, x_*)$

$$\gamma_i(t^*, \mathbf{x}^*) \leq \gamma_i(t_*, \mathbf{x}_*) + \epsilon(t^*, t_*, \mathbf{x}_*), \tag{14}$$

where $\epsilon(\cdot,\cdot,\cdot)$ is some scalar function, which may be reformulated in terms of the set $H(t^*; t_*, \mathbf{x}_*)$. In trivial case it may be a constant, but even in simple examples constant value proved being impractical because $\gamma_i$ may grow with non-constant rate.

Having some $\epsilon$ value, we may force condition (14) by replacing the original $H$ set (12) with

$$H_i^\epsilon(t^*; t_*, \mathbf{x}_*) = H(t^*; t_*, \mathbf{x}_*) \backslash \{ \mathbf{x} \in \mathbb{R}^n \mid \gamma_i(t^*, \mathbf{x}) > \gamma_i(t_*, \mathbf{x}_*) + \epsilon(t^*, t_*, \mathbf{x}_*) \} \tag{15}$$

Modified BM-procedure with $i \in \{1, 2\}$ and fixed $\epsilon(\cdot,\cdot,\cdot)$ function is BM-procedure for which the set $H(t^*; t_*, \mathbf{x}_*)$ from (12) is substituted by $H_i^\epsilon(t^*; t_*, \mathbf{x}_*)$ from (15).

Then all other elements of BM-procedure may be applied to obtain a *modified BM-trajectory.* A set of endpoints of modified BM-trajectories is of a special interest as we want to approximate a part of the Nash trajectories endpoints set bound, which is Pareto unimprovable.

1. Let $K$ be an empty set.
2. Let some step $c > 0$ be fixed, $c := c_0$.
3. Try to build a modified BM-trajectory $T$, we may fail if computed $H_i^{\epsilon_c(\cdot)}$ approximation turns out to be empty at some step.
4. If we failed or endpoint of $T$ is in $K$ then quit.
5. Supplement $K := K \cup \{\text{endpoint of } T\}$.
6. Increase $c := c + c_0$.
7. Repeat from item 3.

The procedure is to be run twice: for $i = 1$ and for $i = 2$. This way both halves of the border approximation are built. It must be mentioned that the resulting approximation quality depends dramatically on choice of function $\epsilon(\cdot,\cdot,\cdot)$.

An $c^* \geq c_0$ may exist such that elements of $K$ corresponding to $c_0 \leq c \leq c^*$ approximate endpoints of $NE$-trajectories, which are not $P$-trajectories.

## 6. Program Implementation

A completely new approach to program implementation was successfully applied at first to $NE$-solution computation and after that — to $S_i$-solutions. Computational geometry algorithms library, developed by S. Osipov in `Fortran` somewhere in 80's, became outdated, so it was temporarily substituted with a C++ wrapper library, which builds upon polygon tessellation facilities from *OpenGL Utility Library* (*GLU*). GLU functions and structures for polygonal primitives are intensively used by algorithms, mentioned herein. Examples of the next section were obtained with GLU being used for polygons processing.Advantages of GLU include straightforward API in `C`, which is simple to use in almost any programming environment. Many implementations exist (usually bundled with operational system), both proprietary and open source.

Despite positive results achieved by the implementation, another library (which is an open source project) was tested, as a possible future base for our algorithms. It was *Computational Geometry Algorithms Library* (*CGAL*), which goal "is to provide easy access to efficient and reliable geometric algorithms in the form of a C++ library" (see `http://www.cgal.org/`). While GLU is a convenient external component, CGAL provides a complex framework to expand upon and is not bounded by hardware-supported double precision arithmetics.

In the case of $S_i$-solutions, OpenMP was adopted (discrete scheme (9) is run for different $c$ values in parallel). Tests on a machine with Intel Core 2 Duo processor demonstrated twofold run-time improvement for two-threaded computations against one-threaded.

## 7. An Example

The following vector equation

$$\begin{gathered} \ddot{\xi} = \mathbf{u} + \mathbf{v}, \quad \xi(t_0) = \xi_0, \quad \dot{\xi}(t_0) = \dot{\xi}_0 \\ \xi, \mathbf{u}, \mathbf{v} \in \mathbb{R}^2, \quad ||\mathbf{u}|| \le \mu, \quad ||\mathbf{v}|| \le \nu, \end{gathered} \tag{16}$$

describes the motion of a material point of unit mass on the plane $(\xi_1, \xi_2)$ under the action of a force $\mathbf{F} = \mathbf{u} + \mathbf{v}$. P1 (P2), who governs the control $\mathbf{u}$ ($\mathbf{v}$), tends to lead the material point as close as possible to the given target point $a^{(1)}$ ($a^{(2)}$) at the moment of time $\theta$. Then players' cost functionals are

$$\begin{gathered} \sigma_i(\xi(\theta)) = -||\xi(\theta) - a^{(i)}||, \\ \xi = (\xi_1, \xi_2), \quad a^{(i)} = (a_1^{(i)}, a_2^{(i)}), \quad i = 1, 2, \end{gathered} \tag{17}$$

where $\theta$ is final time.

By setting $y_1 = \xi_1$, $y_2 = \dot{\xi}_1$, $y_3 = \xi_2$, $y_4 = \dot{\xi}_2$ and making the following change of variables $x_1 = y_1 + (\theta - t)y_3$, $x_2 = y_2 + (\theta - t)y_4$, $x_3 = y_3$, $x_4 = y_4$ we get a system, which first and second equations are

$$\begin{aligned} \dot{x}_1 &= (\theta - t)(u_1 + v_1), \\ \dot{x}_2 &= (\theta - t)(u_2 + v_2). \end{aligned} \tag{18}$$

Further, (17) can be written

$$\sigma_i(\mathbf{x}(\theta)) = -||\mathbf{x}(\theta) - a^{(i)}||, \;\; \mathbf{x} = (x_1, x_2), \;\; i = 1, 2. \tag{19}$$

Since the cost functional (19) depends on variables $x_1$ and $x_2$ only and the right-hand side of (18) does not depend on other variables, one can conclude, that it is sufficient to consider only reduced system (18) with cost functionals (19).

Then initial conditions for (18) are given by formulae

$$x_i(t_0) = x_{0i} = \xi_{0i} + (\theta - t_0)\dot{\xi}_{0i}, \quad i = 1, 2.$$

It can easily be shown, that value functions in antagonistic differential games $\Gamma_1$ and $\Gamma_2$ are given by formulae (if $\mu \ge \nu$)

$$\begin{aligned} \gamma_1(t, \mathbf{x}) &= \min\{ -||\mathbf{x} - a^{(1)}|| - \frac{(\theta - t)^2}{2}(\mu - \nu), 0 \}, \\ \gamma_2(t, \mathbf{x}) &= \min\{ -||\mathbf{x} - a^{(2)}|| + \frac{(\theta - t)^2}{2}(\mu - \nu), 0 \} \end{aligned}$$

and universal optimal strategies (3) are given by

$$\mathbf{u}^{(i)}(t,\mathbf{x},\epsilon) = (-1)^i \mu \frac{\mathbf{x}-a^{(i)}}{||\mathbf{x}-a^{(i)}||},$$
$$\mathbf{v}^{(i)}(t,\mathbf{x},\epsilon) = -(-1)^i \nu \frac{\mathbf{x}-a^{(i)}}{||\mathbf{x}-a^{(i)}||}.$$

One can see modified BM-procedure results (with several $NE$-trajectories) on Fig.2 for the following setting: $\mu=\nu=1$, $t_0=0$, $\theta=2$, $\mathbf{x}_0=\xi_0=(0.6,0.9)$ (initial velocity is zero), $a^{(1)}=(-1,6)$, $a^{(2)}=(5,5)$, $\epsilon_c(t^*;t,\mathbf{x})=c(t^*-t)(\mu+\nu)(\theta-t)$. This variant, where $\mu=\nu$, was studied analytically in Kleimenov, 1993 (Section 1.13). Good coincidence of Fig.2 and Fig. 1.6, p. 53 of the book mentioned could be seen.

Another settings were used to demonstrate $S_i$- and BM-trajectories. Let the following conditions be given: $\mu=1.4$, $\nu=0.6$, $t_0=0$, $\xi_0=(0.5,0.5)$, $a^{(1)}=(4,5)$, and $a^{(2)}=(3.5,-2.5)$. Two variants of target points were considered:

(V1) $\theta=2.5$, $\dot{\xi}_0=(-0.25,-1)$, then $\mathbf{x}_0=(-0.125,-2)$, see Fig. 3 and Fig. 5;

(V2) $\theta=2.1$, $\dot{\xi}_0=(-1,-1)$, then $\mathbf{x}_0=(-1.6,-1.6)$, see Fig. 4.

Time step for $NE$-trajectory is 0.001, and time step for $S_i$-trajectories is 0.005.

The points $S^1$ and $S^2$ on Fig. 3 and Fig. 4 denote endpoints of $S_1$- and $S_2$-trajectories, respectively, while the point $N$ denotes $NE$-trajectory endpoint, generated by BM-procedure. On Fig. 3 symbol "$\times$" is used to show a point ($t=1.347$), where BM-procedure switches from nonantagonistic game to antagonistic one (V2 gives no antagonistic trajectory part).

Four modified BM-trajectories are given on Fig.3 and Fig.4 as well. The same $\epsilon_c(\cdot)$ function as for Fig.2 is used.

For V1, both players controls generating Nash trajectory are shown on Fig. 5. Note, that every 25th pair of vectors $\mathbf{u}$, $\mathbf{v}$ is presented there.

Further, on Fig. 3 and Fig. 4 border of relevant $D=(D_1\cap D_2)\backslash L$ set is painted. Here the symbol $L$ denotes a half-plane bounded by a line connecting points $a^{(1)}$ and $a^{(2)}$, not containing point $\xi_0$. Set $D$ contains all $NE$-trajectories endpoints, but, in general, there may be also points, which are not endpoints of any $NE$-trajectories. Sets $D_i$ were built with time step 0.005.

## 8. Conclusion

As it is seen today, there are at least two ways of development of the results presented. First, it seems to be possible to transparently generalize $NE$- and $S_i$-solution algorithms for non-linear systems with dynamics

$$\dot{\mathbf{x}}(t) = \mathbf{F}_1(t,\mathbf{x}(t),\mathbf{u}(t)) + \mathbf{F}_2(t,\mathbf{x}(t),\mathbf{v}(t)).$$

Second, software development may lead to a powerful and flexible framework simplifying solution computations in a class of differential games. The last program implementation uses techniques of generic programming, which is common for modern C++ software like CGAL. This supports flexibility of its structure and simplifies future modernizations. For example, the early experience allows to suggest, that facilities supplied by the library could give the algorithms literally new dimension: polygons could be changed to polyhedrons without deep change in generic algorithms constructing the solutions.

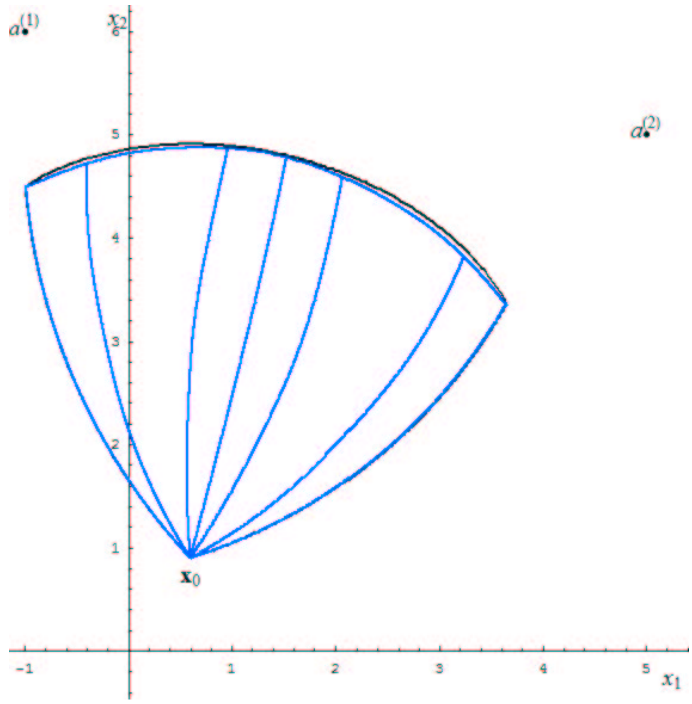


**Fig.2.** Modified BM-trajectories (the case $\mu = \nu$)

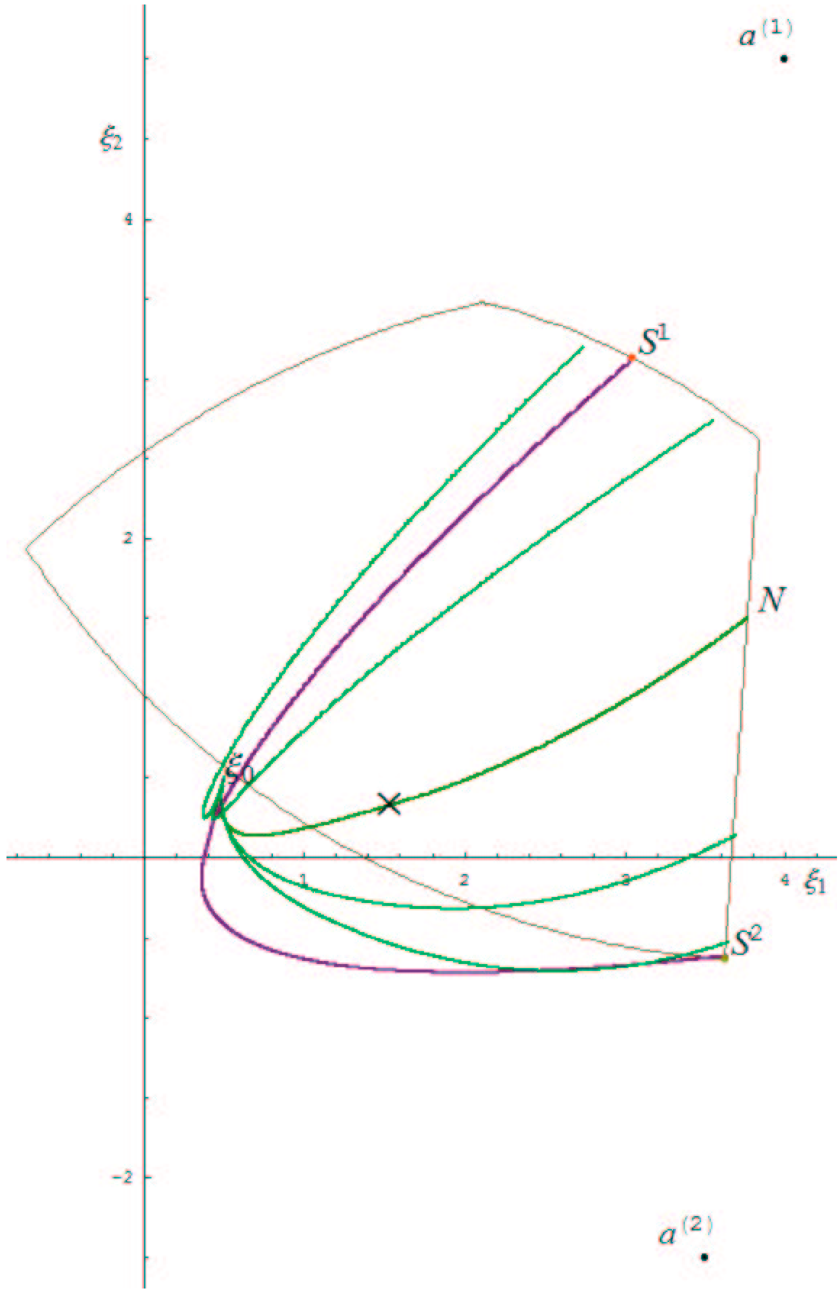


**Fig.3.** V1: $NE$-trajectories and $S_i$-trajectories

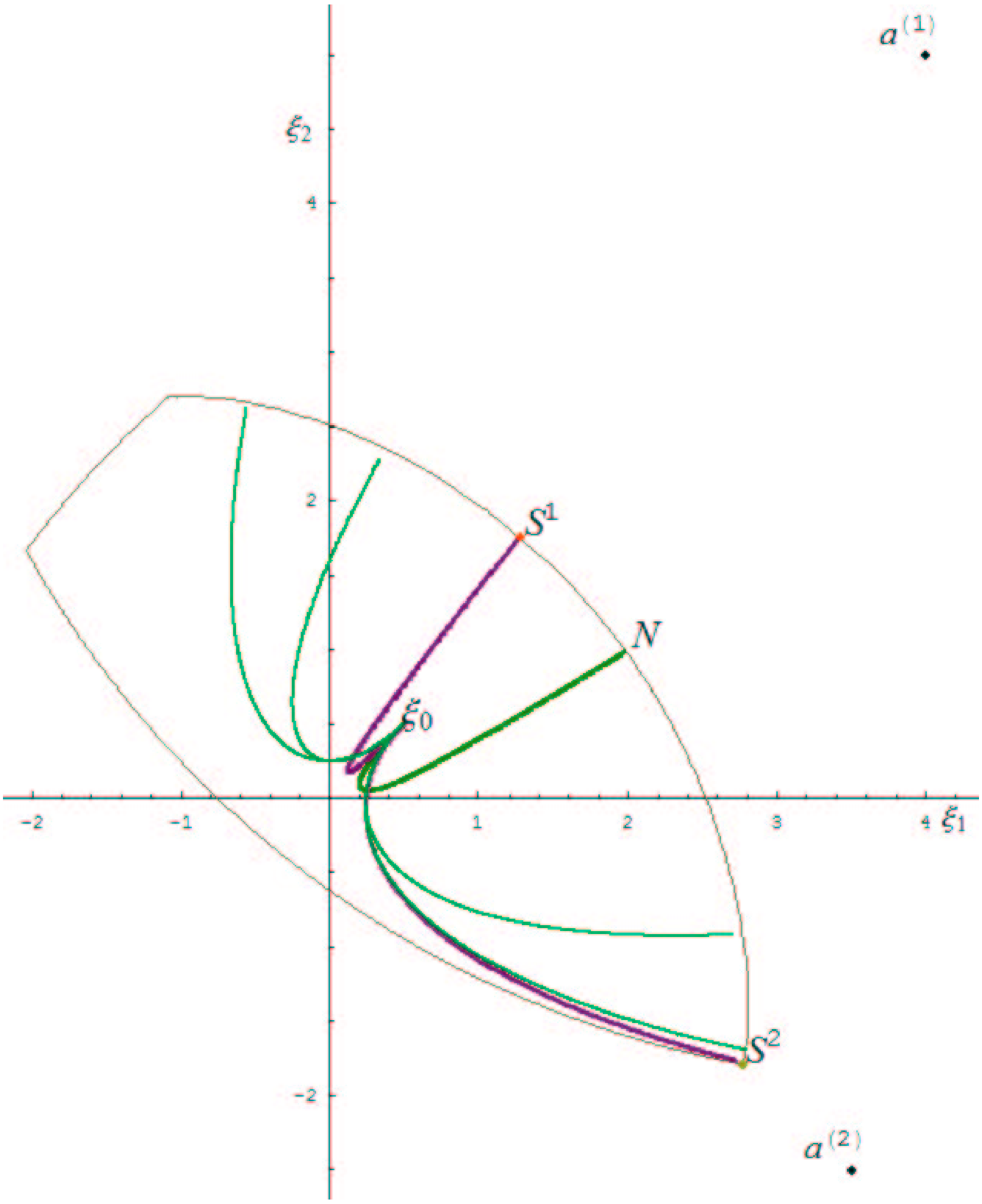


**Fig.4.** V2: $NE$-trajectories and $S_i$-trajectories

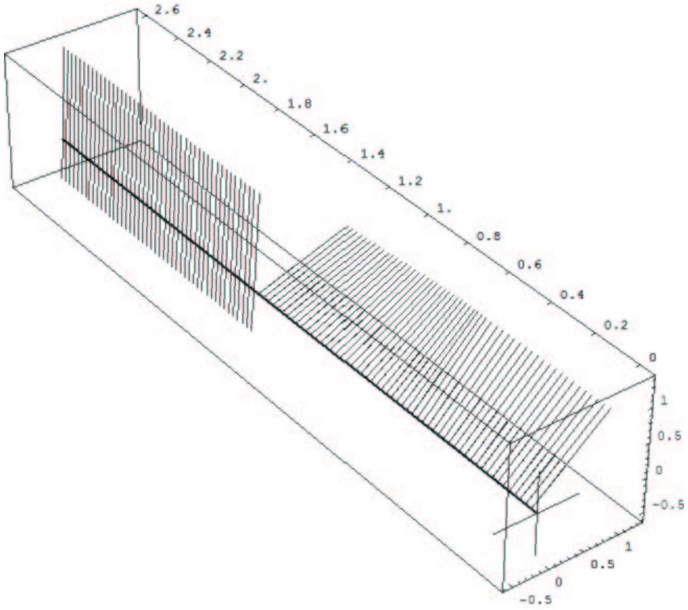


**Fig.5.** V1: Player controls generating $NE$-trajectory

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# D. W. K Yeung Condition for Dynamically Stable Joint Venture


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## 1. Model

Consider the case, when three firms form the joint venture to maximize joint profit (Yeung and Petrosyan, 2006). We have a planning period $[t_0, T]$. At the last moment firms technologies are terminated, and firms take the additional payoff. The firm $i$ profit is

$$\begin{array}{l}\int\limits_{t_0}^{T}\left[P_i\left[x_i(s)\right]^{1/2}-c_iu_i(s)\right]\exp\left[-r(s-t_0)\right]ds+\exp\left[-r(T-t_0)\right]q_i\left[x_i(T)\right]^{1/2} \\ i\in N=\{1,2,3\},\end{array} \tag{1}$$

where $P_i$, $c_i$ , and $q_i$ - positive constants, $r$ - the discount rate. Then firms will be called the players. The profit of player $i$, which is calculated by formula (1), will be called guaranteed payoff of player $i$ on the period $[t_0, T]$.

$x_i(s) \in R^+$ – technologys level of player $i$ at the moment $s$, this variable will be called the state of player $i$. $u_i(s) \in R^+$ – investment in technological advancement, this variable will be called the control of player $i$.

$P_i\left[x_i(s)\right]^{1/2}$ – the net operating revenue of player $i$ at technology level $x_i(s)$,

$c_iu_i(s)$ – the cost of investment,

$q_i\left[x_i(T)\right]^{1/2}$ – a salvage value of player $i$ technology at time $T$.

The evolution of technology level of firm $i$ follows the differential equation

$$\dot{x}_i(s)=\alpha_i[u_i(s)x_i(s)]^{1/2}-\delta x_i(s)x_i(t_0)=x_i^0,\quad i\in N=\{1,2,3\}, \tag{2}$$

where $\alpha_i[u_i(s)x_i(s)]^{1/2}$ – the addition to the technology, brought about by $u_i(s)$ amount a physical investment, $\delta$ the rate of obsolescence. When firms form the joint venture to maximize the joint profit, participating firms can gain more skills that would be difficult for them to obtain on their own. Therefore the evolution of firms technology is changed. We assume, that this evolution follows the system of differential equations

$$\begin{array}{l}\dot{x}_i(s)=\alpha_i[u_i(s)x_i(s)]^{1/2}+b_j^{[j,i]}[x_j(s)x_i(s)]^{1/2}+b_k^{[k,i]}[x_k(s)x_i(s)]^{1/2}-\delta x_i(s) \\ x_i(t_0)=x_i^0,\quad i,j,k\in N=\{1,2,3\}, i\neq j\neq k,\end{array} \tag{3}$$

where $b_j^{[j,i]}$ and $b_k^{[k,i]}$ – positive constants. In particular $b_j^{[j,i]}$, represent the technology transfer effect on firm $i$ by firm $j$. The profit of joint venture is sum of participating firms profits

$$\begin{array}{l}\int\limits_{t_0}^{T}\sum\limits_{i=1}^{3}[P_i[x_i(s)]^{1/2}-c_iu_i(s)]\exp[-r(s-t_0)]ds+ \\ +\sum\limits_{i=1}^{3}\exp[-r(T-t_0)]q_i[x_i(T)]^{1/2}.\end{array} \tag{4}$$

We introduce the definition of coalition in our game. The coalition $K$ is any consolidation of players from set $N = \{1, 2, 3\}$. When players form the coalition, each participant can gain more skills from his partners. As in joint venture $b_j^{[j,i]}$, is the technology transfer effect on firm $i$ by firm $j$.

The coalition payoff $K \subset N = \{1, 2, 3\}$ is the profit of coalition, which is determined by sum of participating firms profits. At the moment $t_0$ the coalition profit becomes:

$$\begin{aligned} &\int_{t_0}^{T} \sum_{i \in K} [P_i [x_i(s)]^{1/2} - c_i u_i(s)] \exp[-r(s - t_0)] ds + \\ &+ \sum_{i=1}^{3} \exp[-r(T - t_0)] q_i [x_i(T)]^{1/2}. \end{aligned} \tag{5}$$

The evolution of technology level of player $i$ in coalition $K$ follows the system of differential equations

$$\begin{aligned} &\dot{x}_i(s) = \alpha_i [u_i(s) x_i(s)]^{1/2} + \sum_{j \in K,\ j \neq i} b_j^{[j,i]} [x_j(s) x_i(s)]^{1/2} - \delta x_i(s) \\ &x_i(t_0) = x_i^0, \qquad i \in N = \{1, 2, 3\}. \end{aligned} \tag{6}$$

For notational convenience we express (6) as:

$$\dot{x}_K(s) = \{\dot{x}_i(s)\}_{i \in K} = f^K [s, x_K(s), u_K(s)], \qquad x_K(t_0) = x_K^0, \tag{7}$$

where $u_K$ is vector of controls $\{u_i\}_{i \in K}$ , $f^K [s, x_K, u_K]$ – is vector of values $f_i^K [s, x_K, u_i]$ for $i \in K$ . For maximization of coalition $K$ profit, we consider the optimal control problem $\omega \left[K, t_0, x_K^0\right]$ , which maximize (5) subject to (6).

The solution of this problem was considered by L.A. Petrosyan and D. W. K. Yeung (Yeung and Petrosyan, 2006) with method of dynamic programming. They introduced the twice continuously differentiable Bellman function $W^{(t_0)K}(t, x_K)$: $[t_0, T] \times \mathbb{R}^m \to \mathbb{R}$, which satisfy Bellman equation:

$$\begin{aligned} &-W_t^{(t_0)K}(t, x_K^t) = max \left\{ \sum_{i \in K} \left[ P_i \left[x_i(t)\right]^{1/2} - c_i u_i(t) \right] \exp\left[-r(t - t_0)\right] + , \right. \\ &\left. \sum_{i \in K} W_{x_i}^{(t_0)K}(t, x_K^t) f_i^K \left[t, x_K, u_K\right] \right\}, \\ &W^{(t_0)K}(T, x_K^T) = \sum_{i \in K} \exp\left[-r(T - t_0)\right] q_i \left[x_i(T)\right]^{1/2}, \\ &f^K \left[t, x_K^t, u_K^t\right] = \dot{x}_K \qquad K \subset N, \end{aligned} \tag{8}$$

where $P_i \left[x_i(t)\right]^{1/2} - c_i u_i(t)$ the payment received by player $i$ at moment $t$, which is discounted on this moment. Function $W^{(t_0)K}(t, x_K)$ determine max payoff of coalition $K$ on time interval $[t, T] \qquad t_0 \leq t \leq T$.

Consider the sub game $\omega\left[K, \tau, x_K^\tau\right]$ , which begin at moment $\tau \in [t, T]$ and state $x_K^\tau$ . We can show (Yeung and Petrosyan, 2006), that:

$$\begin{aligned} &\exp\left[-r(\tau - t_0)\right] W^{(\tau)K}(t, x_K^t) = W^{(t)K}(t, x_K^t), \qquad t_0 \leq \tau \leq t \leq T \\ &u_K^{(\tau)K*}(t, x_K^t) = u_K^{(t)K*}(t, x_K^t), \qquad t_0 \leq \tau \leq t \leq T. \end{aligned} \tag{9}$$

We suppose, that firms share their cooperative profits according to the Dynamic Shapley Value. For maximization of joint profit firms will use optimal controls on the period $[t_0, T]$ with accordant optimal trajectories. At moment $t_0$ and state $x_N^{t_0}$ firms agree, that firms $i$ share of profit be equal the accordant component of the Shapley Value:

$$\begin{aligned} &v^{(t_0)i}(t_0, x_N^0) = \sum_{K \subseteq N} \tfrac{(k-1)!(n-k)!}{n!} \left[W^{(t_0)K}(t_0, x_K^0) - W^{(t_0)K \setminus i}(t_0, x_{K \setminus i}^0)\right], \\ &i \in N. \end{aligned} \tag{10}$$

However, the Shapley Value has to be maintained throughout the venture horizon $[t_0, T]$. In particular, at moment $\tau \in [t_0, T]$ and state $x_N^\tau$ the following equality has to be maintained:

$$\begin{aligned} &v^{(\tau)i}(\tau, x_N^\tau) = \sum_{K \subseteq N} \tfrac{(k-1)!(n-k)!}{n!} \left[W^{(\tau)K}(\tau, x_K^\tau) - W^{(\tau)K \setminus i}(\tau, x_{K \setminus i}^\tau)\right], \\ &\tau \in [t_0, T]. \end{aligned} \tag{11}$$

Note, that $v^{(\tau)}(\tau, x_N^{\tau*}) = \left\{v^{(\tau)i}(\tau, x_N^{\tau*})\right\}_{i \in N}$ , as specified in (11), satisfies the basic principles of an imputation vector:

$$\begin{aligned} &\sum_{i=1}^{3} v^{(\tau)i}(\tau, x_N^{\tau*}) = W^{(\tau)N}(\tau, x_N^{\tau*}), \\ &v^{(\tau)i}(\tau, x_N^{\tau*}) > W^{(\tau)i}(\tau, x_N^{\tau*}), \qquad i \in N, \quad \tau \in [t_0, T]. \end{aligned} \tag{12}$$

The first equality in (12) means, that $v^{(\tau)}(\tau, x_N^{\tau*})$ satisfy the property of Pareto optimality throughout the game interval. The second equality shows that $v^{(\tau)}(\tau, x_N^{\tau*})$ guarantees the individual rationality throughout the game interval.

If condition (12) is maintained, the solution optimality principle - sharing profits according to the Shapley Value - is in effect at any moment throughout the game along the optimal state trajectory chosen at the outset. Hence, the time consistency is satisfied and no firms would have any initiative to depart the joint venture. Therefore, the dynamic imputation principle, leading in (12) dynamically stable or time consistent.

In our case the profits share of firm $i$ according to the Shapley Value becomes:

$$\begin{aligned} &v^{(\tau)i}(\tau, x_N^{\tau*}) = \\ &\tfrac{1}{3} W^{(\tau)i}(\tau, x_N^{\tau*}) + \tfrac{1}{6}\left(W^{(\tau)\{i,j\}}(\tau, x_N^{\tau*}) - W^{(\tau)j}(\tau, x_N^{\tau*})\right) + \\ &\tfrac{1}{6}\left(W^{(\tau)\{i,k\}}(\tau, x_N^{\tau*}) - W^{(\tau)k}(\tau, x_N^{\tau*})\right) + \\ &\tfrac{1}{3}\left(W^{(\tau)\{i,j,k\}}(\tau, x_N^{\tau*}) - W^{(\tau)\{j,k\}}(\tau, x_N^{\tau*})\right), \\ &i, j, k \in N = \{1, 2, 3\}, \quad \tau \in [t_0, T]. \end{aligned} \tag{13}$$

In the most cases the dynamic Shapley Value is dynamically unstable. In the course of time transitory changes appears, and conditions (11), (12) are not maintained. Therefore profits of players have to correct.

To realize the Dynamic Shapley Value, its necessary to compensate transitory changes at each moment $\tau \in [t_0, T]$. In other words, its necessary to distribute the gain joint profit. For realization the procedure of distribute components of the Shapley Value represents in following form:

$$\begin{aligned} v^{(t_0)i}(t_0, x_N^0) &= \sum_{K \subseteq N} \tfrac{(k-1)!(n-k)!}{n!} \left[ W^{(t_0)K}(t_0, x_K^0) - W^{(t_0)K\setminus i}(t_0, x_{K\setminus i}^0) \right] = \\ &= \int_{t_0}^{T} B_i(s) \exp\left[-r(s-t_0)\right] ds + \exp\left[-r(T-t_0)\right] q_i \left[x_i(T)\right]^{1/2}, \quad i \in N, \end{aligned} \tag{14}$$

where $B_i(s)$ - a payment, received by firm $i$ at moment $s$ after distribution the joint profit at moment $s$.

Moreover, for $i \in N$ and $t \in [t_0, T]$

$$v^{(t_0)i}(t, x_N^t) = \int_t^T B_i(s) \exp\left[-r(s-t_0)\right] ds + \exp\left[-r(T-t_0)\right] q_i \left[x_i(T)\right]^{1/2} \tag{15}$$

– the profit of player $i$ from cooperation on the period $[t, T]$ , where $x_N^t$ - the state of game at moment $t \in [t_0, T]$ .

The necessary condition, for $v^{(t_0)i}(t, x_N^{t*})$ to follow the equality (15) is that:

$$\begin{aligned} &v^{(t_0)i}(t, x_N^{t*}) = v^{(t)i}(t, x_N^{t*}) \exp\left[-r(t-t_0)\right], \\ &i \in N, \quad t \in [t_0, T]. \end{aligned} \tag{16}$$

Its necessary to find $B_i(s)$, when $v^{(t_0)i}(t, x_N^{t*})$ satisfy (14)-(16). Note that at each moment there is only distribution of joint profit, therefore the sum players profits are not changed, in other words

$$\sum_{i=1}^{3} B_i(s) = \sum_{i=1}^{3} P_i \left[x_i(s)\right]^{1/2} - c_i u_i(s), \qquad s \in [t_0, T]. \tag{17}$$

This dynamic analysis was considered in the book of L.A. Petrosyan and D.W.K Yeung ((Yeung and Petrosyan, 2006), chapter 6) In general case the payment, received by player $i$ at moment become:

$$\begin{aligned} B_i(\tau) = -\sum_{K \subseteq N} \tfrac{(k-1)!(n-k)!}{n!} &\left\{ \left[ W_t^{(\tau)K}(\tau, x_K^\tau) - W_t^{(\tau)K\setminus i}(\tau, x_{K\setminus i}^\tau) \right] \right. \\ &+ \left. \left( \left[ W_{x_N^{\tau*}}^{(\tau)K}(\tau, x_K^\tau) - W_{x_N^{\tau*}}^{(\tau)K\setminus i}(\tau, x_{K\setminus i}^\tau) \right] \right) \times f^N \left[\tau, x_N^{\tau*}, u^{(\tau)*N}(\tau, x_N^{\tau*})\right] \right\}, \end{aligned} \tag{18}$$

since the partial derivative $W^{(\tau)K}(\tau, x_K^\tau)$ with respect to $x_j$ , where $j \notin K$ will vanish, we can obtain (18):

$$
\begin{aligned}
&B_i(\tau) = -\sum_{K \subseteq N} \tfrac{(k-1)!(n-k)!}{n!} \Big\{ \Big[ W_t^{(\tau)K}(\tau, x_K^\tau) - W_t^{(\tau)K\setminus i}(\tau, x_{K\setminus i}^\tau) \Big] \\
&+ \sum_{j \in K} \Big[ W_{x_j^{\tau*}}^{(\tau)K}(\tau, x_K^\tau) \Big] f_j^N \Big[ \tau, x_N^{\tau*}, u_j^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] \\
&+ \sum_{h \in K\setminus i} \Big[ W_{x_h^{\tau*}}^{(\tau)K\setminus i}(\tau, x_K^\tau \setminus i) \Big] f_h^N \Big[ \tau, x_N^{\tau*}, u_h^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] \Big\} = \\
&- \sum_{K \subset N} \tfrac{(k-1)!(n-k)!}{n!} \Big\{ \Big[ W_t^{(\tau)K}(\tau, x_K^\tau) - W_t^{(\tau)K\setminus i}(\tau, x_{K\setminus i}^\tau) \Big] \\
&+ \Big[ W_{x_K^{\tau*}}^{(\tau)K}(\tau, x_K^\tau) \Big] f_K^N \Big[ \tau, x_N^{\tau*}, u_K^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] \\
&- \Big[ W_{x_{K\setminus i}^{\tau*}}^{(\tau)K\setminus i}(\tau, x_{K\setminus i}^\tau) \Big] f_{K\setminus i}^N \Big[ \tau, x_N^{\tau*}, u_{K\setminus i}^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] \Big\},
\end{aligned}
\tag{19}
$$

where $f_K^N \left[ \tau, x_N^{\tau*}, u_K^{(\tau)*N}(\tau, x_N^{\tau*}) \right]$ is vector, which include $f_i^N \left[ \tau, x_N^{\tau*}, u_i^{(\tau)*N}(\tau, x_N^{\tau*}) \right]$ for $i \in K$ . In our case the function $B_i(\tau)$ has form:

$$
\begin{aligned}
&B_i(\tau) = \\
&(-1) \Big( \tfrac{1}{3} \Big( W_t^{(\tau)i}(\tau, x_i^\tau) + \\
&W_{x_i}^{(\tau)i}(\tau, x_i^\tau) f_i^N \Big[ \tau, x_N^{\tau*}, u_i^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] \Big) + \\
&\tfrac{1}{6} \Big( W_t^{(\tau)\{i,j\}}(\tau, x_{\{i,j\}}^\tau) - W_t^{(\tau)j}(\tau, x_j^\tau) + \\
&W_{x_i}^{(\tau)\{i,j\}}(\tau, x_{\{i,j\}}^\tau) f_i^N \Big[ \tau, x_N^{\tau*}, u_i^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] + \\
&W_{x_j}^{(\tau)\{i,j\}}(\tau, x_{\{i,j\}}^\tau) f_j^N \Big[ \tau, x_N^{\tau*}, u_j^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] - \\
&W_{x_j}^{(\tau)j}(\tau, x_j^\tau) f_j^N \Big[ \tau, x_N^{\tau*}, u_j^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] \Big) + \\
&\tfrac{1}{6} \Big( W_t^{(\tau)\{i,k\}}(\tau, x_{\{i,k\}}^\tau) - W_t^{(\tau)k}(\tau, x_k^\tau) + \\
&W_{x_i}^{(\tau)\{i,k\}}(\tau, x_{\{i,k\}}^\tau) f_i^N \Big[ \tau, x_N^{\tau*}, u_i^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] + \\
&W_{x_k}^{(\tau)\{i,k\}}(\tau, x_{\{i,k\}}^\tau) f_k^N \Big[ \tau, x_N^{\tau*}, u_k^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] - \\
&W_{x_k}^{(\tau)k}(\tau, x_k^\tau) f_k^N \Big[ \tau, x_N^{\tau*}, u_k^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] \Big) + \\
&\tfrac{1}{3} \Big( W_t^{(\tau)\{i,j,k\}}(\tau, x_N^\tau) - W_t^{(\tau)\{j,k\}}(\tau, x_{\{j,k\}}^\tau) + \\
&W_{x_i}^{(\tau)\{i,j,k\}}(\tau, x_N^\tau) f_i^N \Big[ \tau, x_N^{\tau*}, u_i^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] + \\
&W_{x_j}^{(\tau)\{i,j,k\}}(\tau, x_N^\tau) f_j^N \Big[ \tau, x_N^{\tau*}, u_i^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] + \\
&W_{x_k}^{(\tau)\{i,j,k\}}(\tau, x_N^\tau) f_k^N \Big[ \tau, x_N^{\tau*}, u_i^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] - \\
&W_{x_j}^{(\tau)\{j,k\}}(\tau, x_N^\tau) f_j^N \Big[ \tau, x_N^{\tau*}, u_i^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] - \\
&W_{x_k}^{(\tau)\{j,k\}}(\tau, x_N^\tau) f_k^N \Big[ \tau, x_N^{\tau*}, u_i^{(\tau)*N}(\tau, x_N^{\tau*}) \Big] \Big) \Big).
\end{aligned}
\tag{20}
$$

Vector $B_i(\tau)_{i \in N}$, serves as a form equilibrating transitory compensation that guarantees the realization of the Shapley value imputation throughout the game horizon. Hence an instantaneous payment $B_i(\tau, x_N^{\tau*})$ to player $i \in N$ yields a dynamically stable solution to the joint venture.

Our problem to verify the D.W.K. Yeung condition for our model (Yeung, 2007).

The D.W.K. Yeung condition implies, that the maxim guarantied payoff of player $i$ on time interval $[t_0, T]$ has to be lower than sum of players payoff, received in cooperation on the time interval $[t_0, t]$ for each $t \in [t_0, T]$ plus maxim guarantied payoff of player $i$ in the time interval $[t, T]$ . D.W.K. Yeung condition guarantied the stability of cooperative agreement against unpredictable collapse of the coalition.

In our model D.W.K. Yeung condition has the following form:

Without use of profit redistribution $B_i(s)$:

$$\begin{aligned}
&\int_{t_0}^{T} \left[P_i \left[x_i^{t_0}(s)\right]^{1/2} - c_i u_i({}^{t_0}s)\right] \exp\left[-r(s-t_0)\right] ds + \\
&\exp\left[-r(T-t_0)\right] q_i \left[x_i^{t_0}(T)\right]^{1/2} \leq \\
&\int_{t_0}^{t} \left[P_i \left[x_i^{t_0\{i,j,k\}}(s)\right]^{1/2} - c_i u_i({}^{t_0\{i,j,k\}}s)\right] \exp\left[-r(s-t_0)\right] ds + \\
&\exp\left[-r(t-t_0)\right] \int_{t}^{T} \left[P_i \left[x_i^{t*}(s)\right]^{1/2} - c_i u_i({}^{t*}s)\right] \exp\left[-r(s-t)\right] ds + \\
&\exp\left[-r(T-t_0)\right] q_i \left[x_i^{t*}(T)\right]^{1/2}, \\
&i \in N = \{1,2,3\}.
\end{aligned} \tag{21}$$

With use of profit redistribution $B_i(s)$:

$$\begin{aligned}
&\int_{t_0}^{T} \left[P_i \left[x_i^{t_0}(s)\right]^{1/2} - c_i u_i({}^{t_0}s)\right] \exp\left[-r(s-t_0)\right] ds + \\
&\exp\left[-r(T-t_0)\right] q_i \left[x_i^{t_0}(T)\right]^{1/2} \leq \\
&\int_{t_0}^{t} B_i(s) \exp\left[-r(s-t_0)\right] ds + \\
&\exp\left[-r(t-t_0)\right] \int_{t}^{T} \left[P_i \left[x_i^{t*}(s)\right]^{1/2} - c_i u_i({}^{t*}s)\right] \exp\left[-r(s-t)\right] ds + \\
&\exp\left[-r(T-t_0)\right] q_i \left[x_i^{t*}(T)\right]^{1/2}, \\
&i \in N = \{1,2,3\}.
\end{aligned} \tag{22}$$

In our model players form a joint venture on the time period $[t_0, T]$, and then each player develops individually from moment $t$ and state $x_i^{t_0\{i,j,k\}}(t) = x_i^{t*}(t) \quad i \in N$ on. We have to confirm our assertions by quantitative results. Now we consider some examples of problem, represented above. All calculations were performed in program MAPLE.

## 2. Quantitative Results

### 2.1. Symmetric case

Firstly we consider the symmetric case of game, when all players have the same parameters. So $t_0$ =0 - initial moment of game. $T = 20$ - the terminal moment of game. $r = 0.2$ - the discount rate. $\delta = 0.05$ - the rate of obsolescence. $c1 = 0.5, c2 = 0.5, c3 = 0.5$ - constants, which determine costs of players investments in technological advancement. $q1 = 0.1, q2 = 0.1, q3 = 0.1$ - constants, which determine the salvage value of players technologies at last moment of game. $P1 = 0.1, P2 = 0.1, P3 = 0.1$ - constants, which determine net operating revenues of players. $\alpha1 = 0.3, \alpha2 = 0.3, \alpha3 = 0.3$ - constants, which determine technological addition of players. $b_1^{[2,1]} = b_1^{[3,1]} = b_2^{[1,2]} = b_2^{[3,2]} = b_3^{[1,3]} = b_3^{[2,3]} = 0.1$ - constants, which determine technology transfer effects between players.

We note by

$$Pr_i^K(t) = P_i\,[x_i(t)]^{1/2} - c_i u_i(t), \quad i \in K \subseteq N \tag{23}$$

the payment of player $i$ at moment $t$. The values of players state variables and players control are noted according to coalition $K$.

In the case of individual development graphics of players states dynamic and their profits, according to equations (2) and (23), becomes:

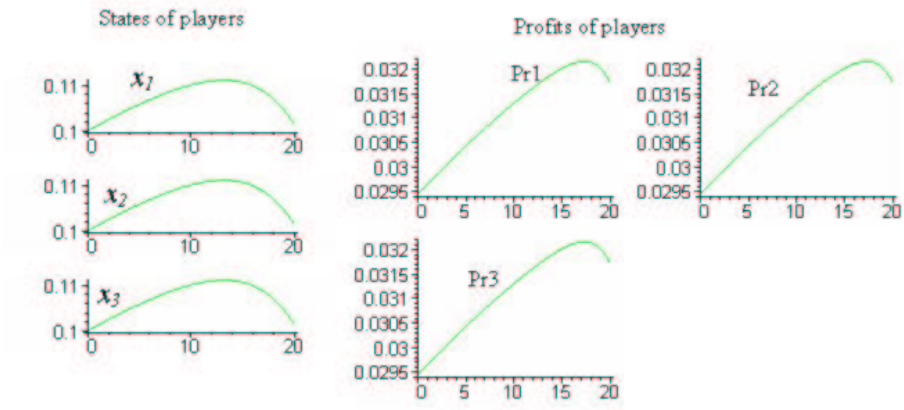


**Fig.1.**

As we see, all players develop in the same manner. In joint venture state variables of players, which develop according to system of equation (3), are represented below:

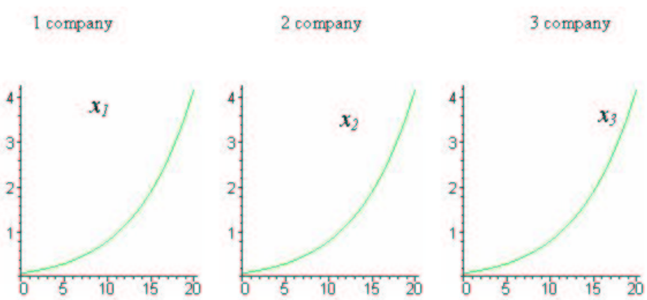


**Fig.2.**

Graphics of players profits are represented below. As we see all trajectories are the same for joint venture. The players profits are calculated by formula (23) for $K = N$ (Fig.3).

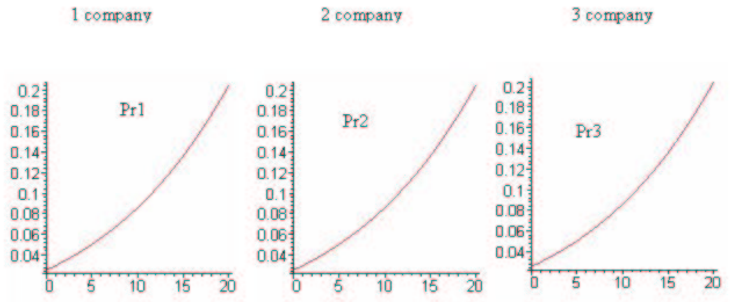


**Fig.3.**

In symmetric case profits share of players according to the Shapley value have to be equal, therefore, there are no profit redistribution. Graphics of functions Bi(t), calculated by (20), confirm that:

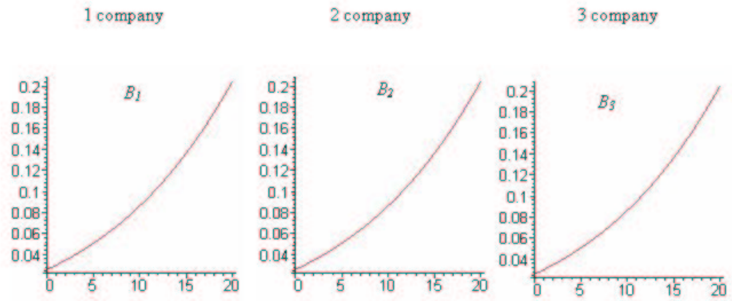


**Fig.4.**

We have to verify the correctness of our calculating and to make sure, that the noted profits of players do not change after redistribution.

Results of quantitative modeling are represented below. In this case there is no profit redistribution. There are some differences which are calculating errors. We have to verify, that Shapley Value is maintained throughout the game horizon, therefore we have to verify:

$$v^{(t_0)i}(t, x_N^t) = \int_t^T B_i(s) \exp\left[-r(s-t_0)\right] ds + \exp\left[-r(T-t_0)\right] q_i \left[x_i(T)\right]^{1/2}, \qquad (24)$$
$$i \in N, \qquad t \in [t_0, T].$$

The right part of equation (24) is depended only from $t$, we can denote it by $Ob_i(t)$. Components of the Shapley Value, calculated by formulas (10) and (11), we denote by $v_i(t)$. Values of Shapley Value components in some moment from $[t_0, T]$ are represented in table (1) with the players payoffs in joint venture after redistribution, right part of equation (24). As we see, there is dynamical stability of equation.

Now we can verify the D.W.K. Yeung condition for our model (see (21) and (22)). Choose some moment from game horizon, for example $t = 2,\ t = 5,\ t = 10$ and look, how the dynamics of players states and players payoffs changes.

Here we see graphics of firms states in joint venture with the breakup at moment 10, calculated according to (3) before breakup and according to (2) after breakup. Graphics represent with comparison of the same values in joint venture throughout the game horizon (Fig.5)

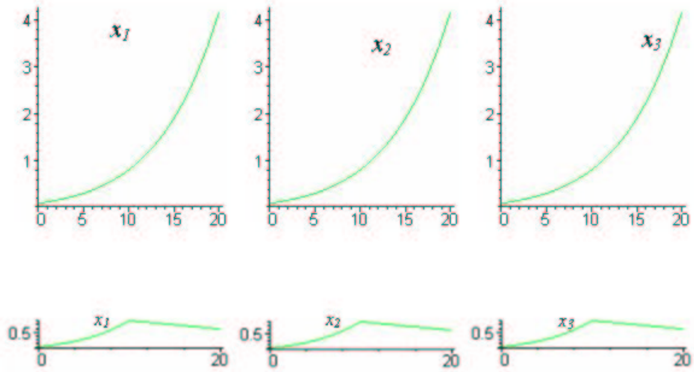


**Fig.5.**

As we see all players develop in the same manner. Graphics of players profits are equal too. There are graphics of players profits with profit redistribution in cooperation and without profit redistribution in cooperation. In the first case the profit of player $i$ is calculated by formula (20) before breakup, and by formula (20) for joint venture ($K = N$) in the second case. The profit of each player after breakup is calculated by formula (23) (Fig.6, Fig.7).

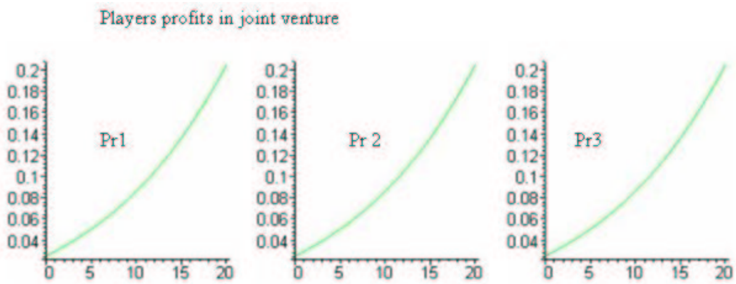


**Fig.6.**

Calculate the value of each players payoff in joint venture and compare it with guaranteed payoffs in the case of individual development. According to D.W.K.

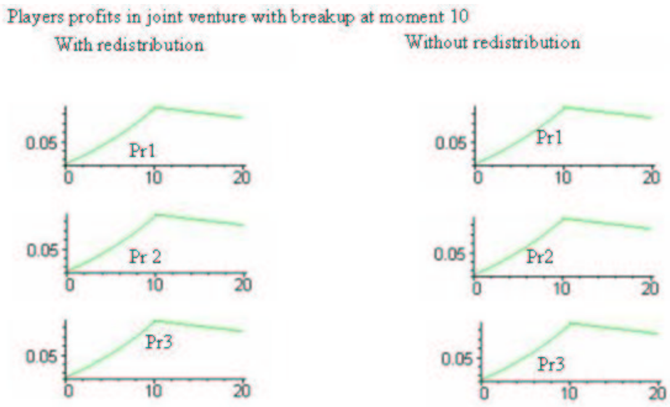


**Fig.7.**

Yeung condition the last value must be lower (or equal) at each moment $t \in [t_0, T]$. Calculate profits of all players for some moments of breaks. We have to calculate the payoffs both without profit redistribution, and with profit redistribution. Values of players payoffs are represented in table (2). As we see the D.W.K. Yeung condition is maintained.

### 2.2. Asymmetric case (P)

Let values $P1 = 0.1,\ P2 = 0.2,\ P3 = 0.05$. Look, how dynamics of players development and their payoffs are changed. Here are graphics of players state variables and their profits both for individual development and joint venture. In the first case all calculations are implemented according to equations (2), in the second case according to (3) (Fig.8, Fig.9).

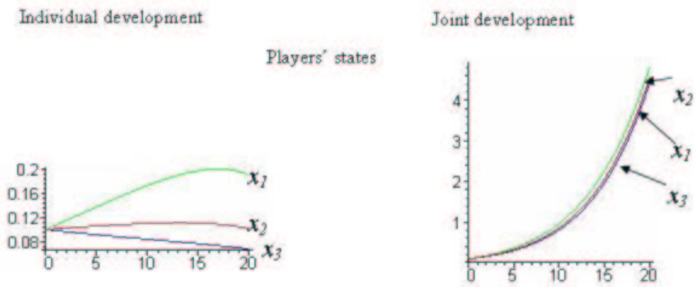


**Fig.8.**

Here are graphics of players profits in joint venture without profit redistribution and with profit redistribution. In the first case profit is calculated by formula (23) for joint venture ($K = N$ ), in the second case by formula (20) (Fig.10).

As we see there is some profit redistribution. We can illustrate it choosing a fixed time instant from game horizon.

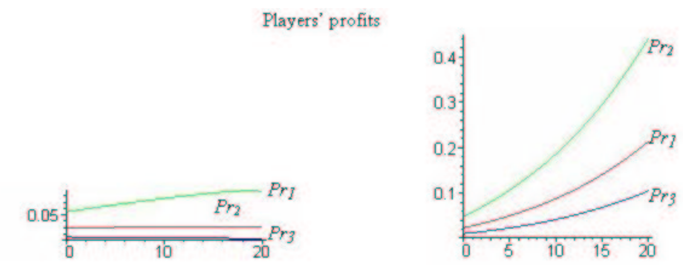


**Fig.9.**

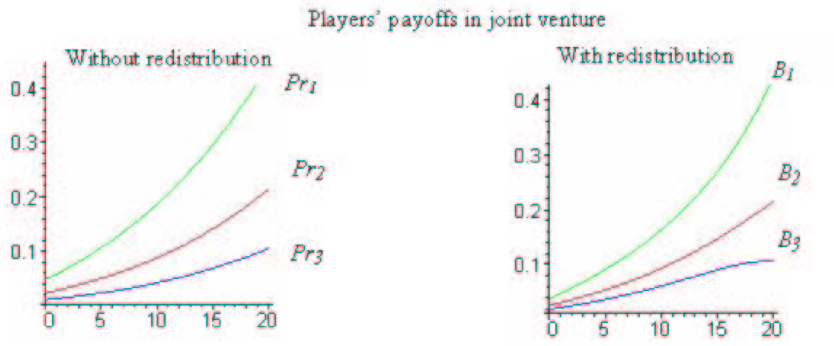


**Fig.10.**

Values of Shapley Value components in fixed time instants from $[t_0, T]$ are represented in table (3) with the players payoffs in joint venture after redistribution. The solution is dynamically stable.

Now we can verify the D.W.K. Yeung condition for our model (see (21) and (22)). Choose some moment from game horizon, for example $t = 2,\ t = 5,\ t = 10$ and look, how the dynamics of players states and players payoffs changes.

Here we see graphics of players state variables and firms profits in joint venture with the breakup at moment 10. players states are calculated according to (3) before breakup and according to (2) after breakup. There are graphics of players profit with profit redistribution in cooperation and without profit redistribution in cooperation. In the first case the profit of player $i$ is calculated by formula (20) before breakup, and by formula (23) for joint venture ($K = N$) in the second case. The profit of each player after breakup is calculated by formula (23). Graphics represent comparison of the values in joint venture throughout the game horizon (Fig.11, Fig.12, Fig.13).

Show some quantitative results. Calculate the value of each players payoff in this case and compare it with guaranteed players payoffs in the case of individual development. According to D.W.K. Yeung condition the last value must be lower (or equal) at each instant $t \in [t_0, T]$. Calculate profits of all players for some moments of coalition breakups. Values of players payoffs are represented in table (4).

As we see the D.W.K. Yeung Condition in this case is maintained. The players payoff in the case of individual development is lower than players payoff in coop-

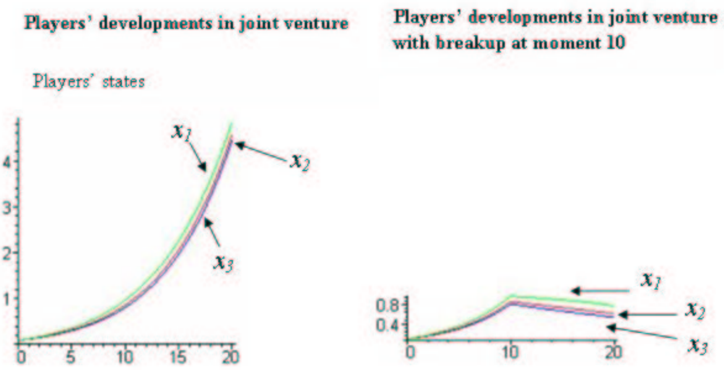


**Fig.11.**

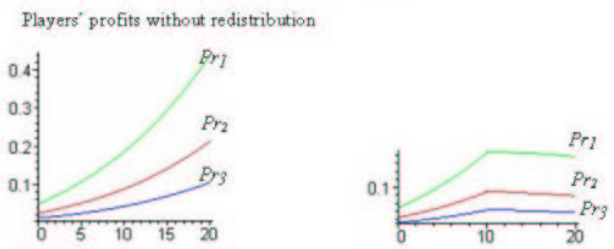


**Fig.12.**

erative development with breakup, and this condition is maintained both without profit redistribution and with profit redistribution.

### 2.3. Asymmetric case ($\alpha$)

Let values $\alpha_1 = 0.6,\ \alpha_2 = 0.3,\ \alpha_3 = 0.15$.

Here are graphics of players state variables and their profits both for individual development and joint venture. In the first case all calculations are implemented according to equations (2), in the second case according to (3) (Fig.14, Fig.15).

Here are graphics of players profits in joint venture without profit redistribution and with profit redistribution. In the first case profit is calculated by formula (23) for joint venture ($K = N$), in the second case by formula (20) (Fig.16).

As we see there is some profit redistribution. We can illustrate it by fixing some time instants from game horizon.

Values of Shapley Value components in time instant from $[t_0, T]$ are represented in table (5) with the players payoffs in joint venture after redistribution, right part of equation (24). Solution is dynamically stable.

Now we can verify the D.W.K. Yeung condition for our model (see (21) and (22)). Choose some moment from game horizon, for example $t = 2,\ t = 5,\ t = 10$ and look, how the dynamics of players states and players payoffs changes.

Here we see graphics of players state variables and firms profits in joint venture with the breakup at moment 10. Players state variables are calculated according to

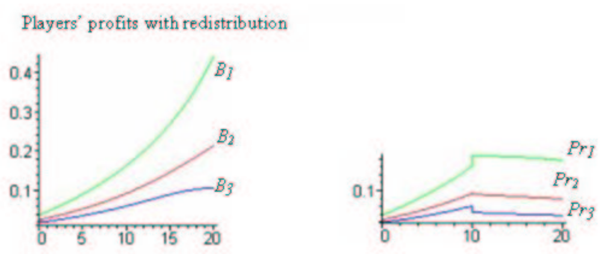


**Fig.13.**

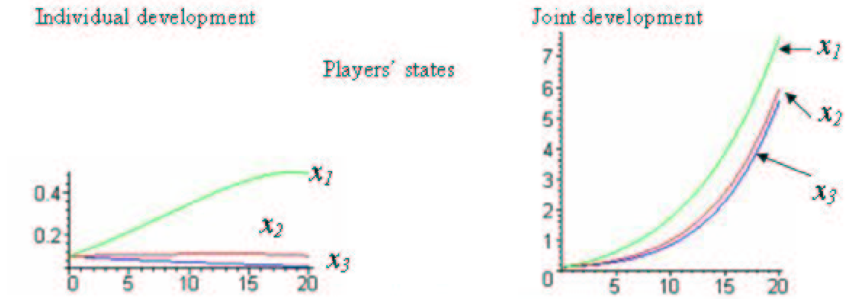


**Fig.14.**

(2) before breakup and according to (3) after breakup. There are graphics of players profits with profits redistribution in cooperation and without profit redistribution in cooperation. In the first case the profit of player i is calculated by formula (20) before breakup, and by formula (23) for joint venture ($K = N$) in the second case. The profit of each player after breakup is calculated by formula (23) (Fig.17, Fig.18, Fig.19).

Show some quantitative results. Calculate the value of each player payoff in this case and compare it with the maximal guaranteed players payoffs in the case of individual development. According to D.W.K. Yeung condition the last value must be lower (or equal) at each moment $t \in [t_0, T]$. Calculate profits of all players for some moments of breaks. We have to calculate the payoffs both without profit redistribution, and with profit redistribution. Values of players payoffs are represented in table 6.

As we see the D.W.K. Condition in this case is maintained. The players payoff in the case of individual development is lower than players payoff in cooperative development with breakup, and this condition is maintained both without profit redistribution and with profit redistribution.

### 2.4. Asymmetric case (q)

Let values $q1 = 0.2$, $q2 = 0.1$, $q3 = 0.05$. Here are graphics of players states and their profits both for individual development and joint venture. In the first case all calculations are implemented according to equations (2), in the second case according to (3) (Fig.20, Fig.21).

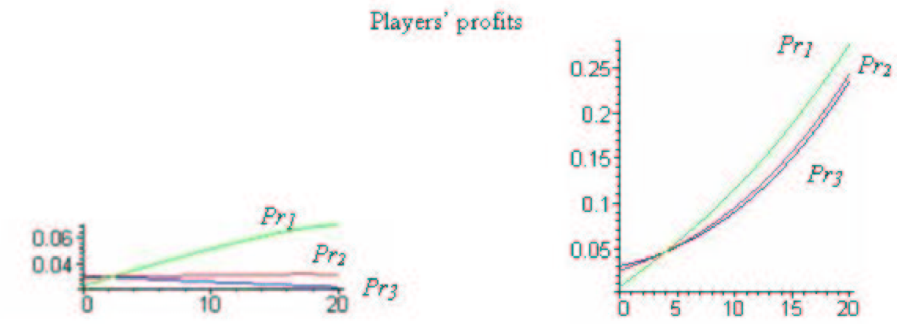
Players' profits
0.06
0.04
0
10
20
Pr1
Pr2
Pr3
0.25
0.2
0.15
0.1
0.05
0
5
10
15
20

**Fig.15.**

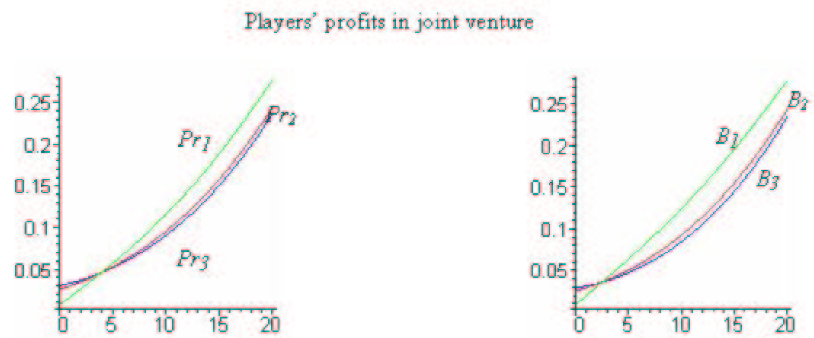
Players' profits in joint venture
0.25
0.2
0.15
0.1
0.05
0
5
10
15
20
Pr1
Pr2
Pr3
B1
B2
B3

**Fig.16.**

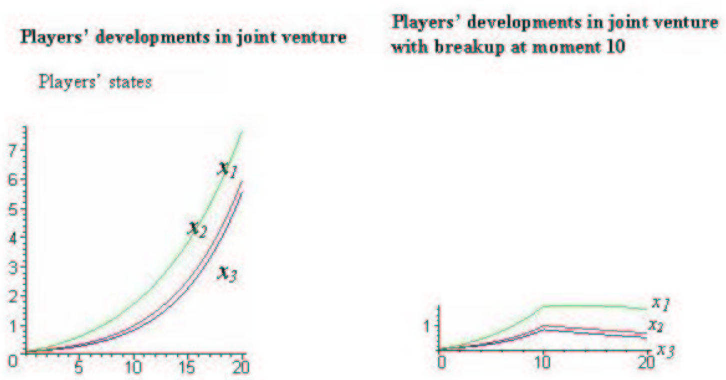


**Fig.17.**

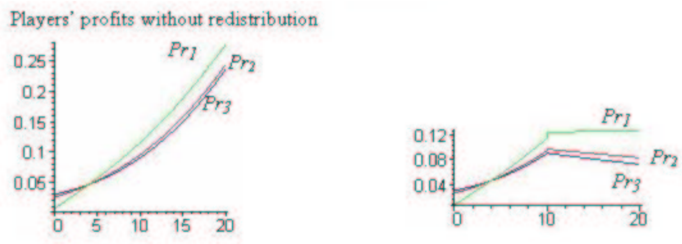


**Fig.18.**

Here are the graphics of players profits in joint venture without profit redistribution and with profit redistribution. In the first case profit is calculated by formula (23) for joint venture ($K = N$), in the second case by formula (20) (Fig.22).

As we see there is some profit redistribution. We can illustrate it in some fixed time instants from game horizon.

Values of Shapley Value components in time instant from $[t_0, T]$ are represented in table (7) with the players payoffs in joint venture after redistribution, right part of equation (24). Solution is dynamically stable.

Now we can verify the D.W.K. Yeung condition for our model (see (21) and (22)). Choose some moment from game horizon, for example $t = 2,\ t = 5,\ t = 10$ and look, how the dynamics of players states and players payoffs changes.

Here we see graphics of players state variables and firms profits in joint venture with the breakup at moment 10 (Fig.23, Fig.24, Fig.25).

Show some quantitative results. Calculate the value of each player payoff in this case and compare it with the maximal guaranteed players payoffs in the case of individual development. According to D.W.K. Yeung condition the last value must be lower (or equal) at each moment $t \in [t_0, T]$. Calculate profits of all players for some moments of breaks. We have to calculate the payoffs both without profit redistribution, and with profit redistribution. Values of players payoffs are represented in table 8.

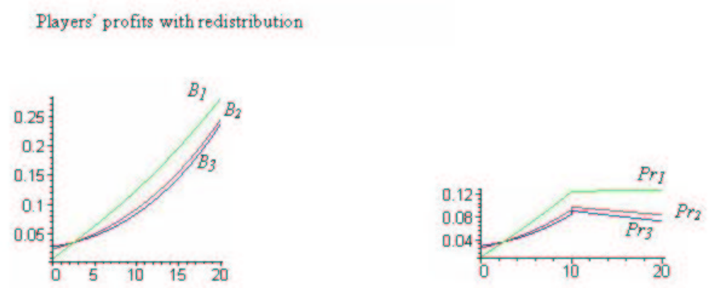


**Fig.19.**

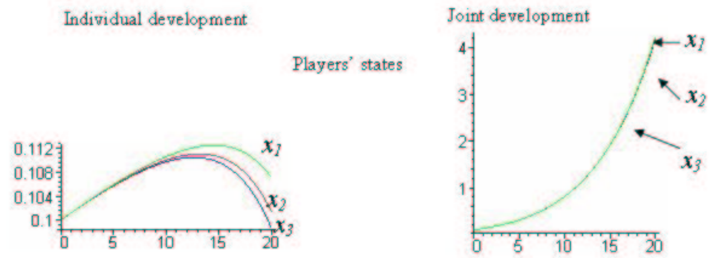


**Fig.20.**

As we see the D.W.K. Yeung condition in this case is maintained. The players payoff in the case of individual development is lower than players payoff in cooperative development with breakup, and this condition is maintained both without profit redistribution and with profit redistribution.

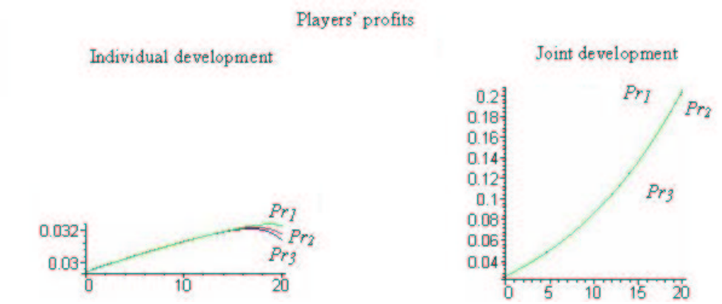


**Fig.21.**

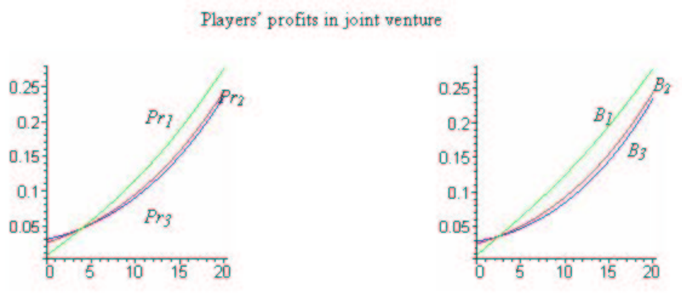


**Fig.22.**

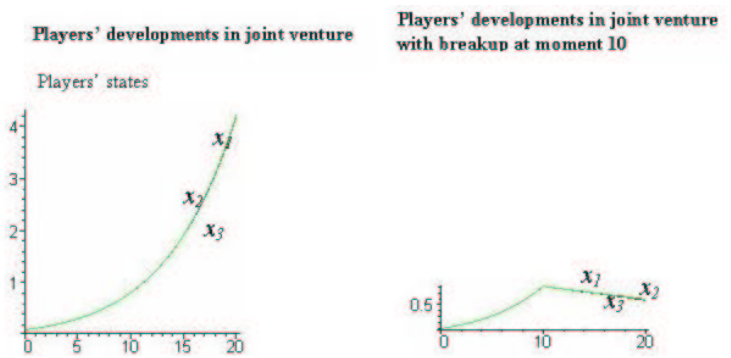


**Fig.23.**

## 3. Conclusion

As we see the D.W.K. Yeung condition in this case is maintained. The players payoff in the case of individual development is lower than players payoff in cooperative development with breakup, and this condition is maintained both without profit redistribution and with profit redistribution.

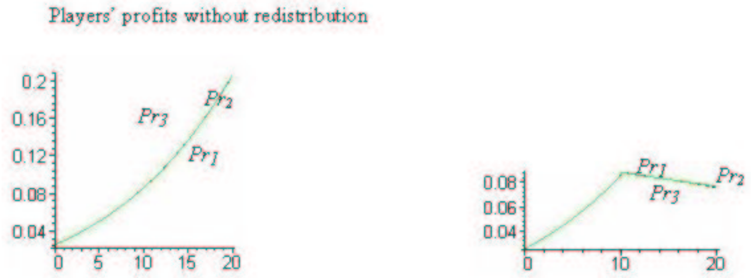


**Fig.24.**

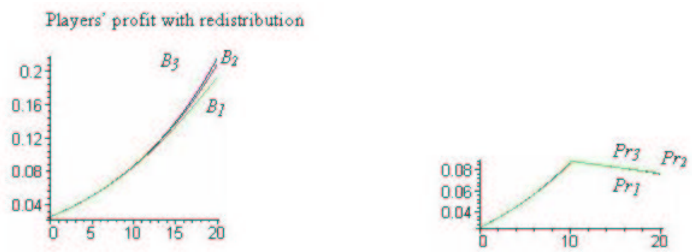


**Fig.25.**

**Table1.** components of the Shapley Value (symmetric case)

| $t$ | 0 | 7 | 13 | 17 | 20 |
|---|---|---|---|---|---|
| $v_1$ | 0,261 | 0,115 | 0,044 | 0,017 | 0,003 |
| $Ob_1$ | 0,261 | 0,115 | 0,044 | 0,017 | 0,003 |
| $v_2$ | 0,261 | 0,115 | 0,044 | 0,017 | 0,003 |
| $Ob_2$ | 0,261 | 0,115 | 0,044 | 0,017 | 0,003 |
| $v_3$ | 0,261 | 0,115 | 0,044 | 0,017 | 0,003 |
| $Ob_3$ | 0,261 | 0,115 | 0,044 | 0,017 | 0,003 |

**Table2.** Firms profits (symmetric case)

| Variants of firms behaviors | Firm1 | Firm2 | Firm3 |
|---|---|---|---|
| individually | 0,150 | 0,150 | 0,150 |
| Moment of breakup $t = 2$(with redistribution) | 0,173 | 0,173 | 0,173 |
| Moment of breakup $t = 2$(without redistribution) | 0,173 | 0,173 | 0,173 |
| Moment of breakup $t = 5$(with redistribution) | 0,203 | 0,203 | 0,203 |
| Moment of breakup $t = 5$(without redistribution) | 0,203 | 0,203 | 0,203 |
| Moment of breakup $t = 10$(with redistribution) | 0,236 | 0,236 | 0,236 |
| Moment of breakup $t = 10$(without redistribution) | 0,236 | 0,236 | 0,236 |

**Table3.** components of the Shapley Value (case P)

| $t$ | 0 | 7 | 13 | 17 | 20 |
|---|---|---|---|---|---|
| $v_1$ | 0,468 | 0,220 | 0,086 | 0,032 | 0,004 |
| $Ob_1$ | 0,469 | 0,220 | 0,086 | 0,032 | 0,004 |
| $v_2$ | 0,278 | 0,124 | 0,048 | 0,018 | 0,004 |
| $Ob_2$ | 0,278 | 0,124 | 0,048 | 0,018 | 0,004 |
| $v_3$ | 0,187 | 0,080 | 0,029 | 0,012 | 0,004 |
| $Ob_3$ | 0,187 | 0,080 | 0,029 | 0,012 | 0,004 |

**Table4.** Firms profits (case P)

| Variants of firms behaviors | Firm1 | Firm2 | Firm3 |
|---|---|---|---|
| individually | 0,318 | 0,149 | 0,072 |
| Moment of breakup t=2(with redistribution) | 0,349 | 0,177 | 0,097 |
| Moment of breakup t=2(without redistribution) | 0,368 | 0,173 | 0,083 |
| Moment of breakup t=5(with redistribution) | 0,390 | 0,211 | 0,128 |
| Moment of breakup t=5(without redistribution) | 0,429 | 0,203 | 0,096 |
| Moment of breakup t=10(with redistribution) | 0,435 | 0,250 | 0,162 |
| Moment of breakup t=10(without redistribution) | 0,496 | 0,237 | 0,113 |

**Table5.** components of the Shapley Value(case a)

| $t$ | 0 | 7 | 13 | 17 | 20 |
|---|---|---|---|---|---|
| $v_1$ | 0,278 | 0,124 | 0,048 | 0,018 | 0,004 |
| $Ob_1$ | 0,278 | 0,124 | 0,048 | 0,018 | 0,004 |
| $v_2$ | 0,468 | 0,220 | 0,086 | 0,032 | 0,004 |
| $Ob_2$ | 0,469 | 0,220 | 0,086 | 0,032 | 0,004 |
| $v_3$ | 0,187 | 0,080 | 0,030 | 0,012 | 0,004 |
| $Ob_3$ | 0,188 | 0,080 | 0,030 | 0,012 | 0,004 |

**Table6.** Firms profits (case a)

| Variants of firms behaviors | Firm1 | Firm2 | Firm3 |
|---|---|---|---|
| individually | 0,179 | 0,149 | 0,142 |
| Moment of breakup t=2(with redistribution) | 0,203 | 0,173 | 0,166 |
| Moment of breakup t=2(without redistribution) | 0,198 | 0,175 | 0,170 |
| Moment of breakup t=5(with redistribution) | 0,236 | 0,205 | 0,197 |
| Moment of breakup t=5(without redistribution) | 0,223 | 0,209 | 0,205 |
| Moment of breakup t=10(with redistribution) | 0,278 | 0,242 | 0,233 |
| Moment of breakup t=10(without redistribution) | 0,255 | 0,249 | 0,248 |

**Table7.** components of the Shapley Value(case q)

| $t$ | 0 | 7 | 13 | 17 | 20 |
|---|---|---|---|---|---|
| $v_1$ | 0,262 | 0,117 | 0,047 | 0,020 | 0,007 |
| $Ob_1$ | 0,262 | 0,117 | 0,047 | 0,020 | 0,007 |
| $v_2$ | 0,261 | 0,116 | 0,045 | 0,017 | 0,004 |
| $Ob_2$ | 0,261 | 0,116 | 0,045 | 0,017 | 0,004 |
| $v_3$ | 0,260 | 0,1152 | 0,044 | 0,016 | 0,002 |
| $Ob_3$ | 0,261 | 0,115 | 0,044 | 0,016 | 0,002 |

**Table8.** Firms profits (case q)

| Variants of firms behaviors | Firm1 | Firm2 | Firm3 |
|---|---|---|---|
| individually | 0,150 | 0,150 | 0,150 |
| Moment of breakup t=2(with redistribution) | 0,174 | 0,174 | 0,174 |
| Moment of breakup t=2(without redistribution) | 0,174 | 0,174 | 0,173 |
| Moment of breakup t=5(with redistribution) | 0,204 | 0,204 | 0,204 |
| Moment of breakup t=5(without redistribution) | 0,204 | 0,203 | 0,203 |
| Moment of breakup t=10(with redistribution) | 0,238 | 0,238 | 0,238 |
| Moment of breakup t=10(without redistribution) | 0,238 | 0,237 | 0,237 |

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# Brand and Generic Advertising Strategies in a Dynamic Monopoly with Two Brands *

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**Abstract** This paper considers a dynamic monopoly with two brands. It is an extension of the work "Brand and generic advertising strategies in a dynamic duopoly" (Bass, Krishnamurthy , Prasad, Sethi, 2005). A monopolist possessing two brands has to decide how to use both generic and brand advertising for each of his brands in order to maximize his profit. Thus, he needs a method or algorithm that he could use practically. This work offers such an algorithm. A multistage algorithm of a problem solution which is software-programmable have been developed. Symmetric case is analyzed as the example.

**Keywords:** dynamic monopoly, brand advertising strategy, generic advertising strategy, profit maximization, the Hamilton - Jacobi - Bellman equation, optimal control.

## 1. Introduction

For better understanding of the gist of the advertising types that are considered in this paper, we explain their meaning and goals that could be achieved with their help. Generic advertising is used by a group of producers , or cooperative, to promote products that are essentially homogenous. As a generic message promotes a type of commodity, all producers in the industry benefit from the generic campaign, including "free riders" who do not contribute funds to the advertising campaign. A successful generic advertising campaign will generally increase both the quantity sold of the commodity and the price paid by the consumer. Brand advertising promotes the particular characteristics of a given product brand, while generic advertising promotes consumption of the general commodity.

Therefore, to increase category demand the monopolist should use generic advertising and to increase the market share of one of two brands he should use brand advertising. So there is a question aroused: how to allocate advertising budget between two brands to get the maximal profit. This problem is very important to researchers. However, there are few works on this topic or ones associated with it. Moreover, these works deal either with static models that involve generic and brand advertising (Krishnamurthy 2000) or dynamic models which do not analyze the role of generic advertising explicitly (Sorger, 1989, Chintagunta and Vilcassim,

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1992, Erickson, 1992). Therefore, our work is up-to-date, because it studies a dynamic monopoly and considers the effects of the application of brand and generic advertising.

The rest of the paper is organized as follows: Section 2 describe the model. Section 3 presents the method that is used to find maximum of the profit. Section 4 deals with the algorithm for maximum profit finding. Section 5 considers symmetric case. Section 6 presents conclusions.

## 2. Problem Statement

Consider the following monopoly model. Suppose that there is one firm with two competing brands.The monopolist's goal is to maximize their profit using brand and generic advertising due to the fact that the advertising strategy is assumed to be the most important on the investigated market. The effect of advertising on demand is modelled separately depending on the type of advertising. We start with generic advertising. A firm's generic advertising for brand $i$=1,2 is denoted as $a_i(t)$.

Suppose that the increase in the category demand is shared unequally between two brands. Therefore, the effect of generic advertising on brand's sales can be presented as

$$\dot{S}_{i,g}(t) = \theta_i(k_1 a_1 + k_2 a_2), \quad i = 1, 2,$$

where $k_i$ - effectiveness of generic advertising of brand $i$, $\theta_i$ - allocation coefficient of brand $i$. We use $\theta_1 + \theta_2 = 1$.

According to (Sorger (1989)), the effect of brand advertising on brand's sales is

$$\dot{S}_{i,b}(t) = \rho_i u_i(t)\sqrt{S_{3-i}(t)} - \rho_{3-i} u_{3-i}(t)\sqrt{S_i(t)}, \quad i = 1, 2,$$

where $u_i(t)$ - is advertising of brand $i$ at time $t$, $\rho_i$ - effectiveness of advertising of brand $i$.

The total effect of generic and brand advertising on sales rate of brand $i$ is

$$\dot{S}_i(t) = \rho_i u_i(t)\sqrt{S_{3-i}(t)} - \rho_{3-i} u_{3-i}(t)\sqrt{S_i(t)} + \theta_i(k_1 a_1(t) + k_2 a_2(t)),$$

$$S_i(0) = S_{i0}, \quad i = 1, 2,$$

where $S_{i0}$ - is the initial sales of brand $i$.

Firm's profit is estimated as following

$$\max_{u_1(t),a_1(t),p_1(t),u_2(t),a_2(t),p_2(t)} V = \int_0^{\infty} e^{-rt} \left(p_1(t)S_1(t)(1 - b_1 p_1(t) + d_1 p_2(t)) + \right.$$

$$\left. + p_2(t)S_2(t)(1 - b_2 p_2(t) + d_2 p_1(t)) - \frac{c_1}{2}(a_1(t)^2 + u_1^2(t)) - \frac{c_2}{2}(a_2^2(t) + u_2^2(t))\right) dt,$$

where $r$ - discount rate, $S_i(t)$ - sales at time $t$, $p_i(t)$ - price charged at time $t$, $c_i(t)$ - advertising cost parameter, $\frac{c_i}{2}(a_i^2(t) + u_i^2(t))$ - the total advertising spending of brand $i$, $V(t)$ - profit function of the firm.

Rewriting the obtained equations separately for each brand we receive

$$\max_{u_1(t),a_1(t),p_1(t),u_2(t),a_2(t),p_2(t)} V = \int_0^{\infty} e^{-rt} \left(p_1(t)S_1(t)(1 - b_1p_1(t) + d_1p_2(t)) + \right.$$
$$\left. + p_2(t)S_2(t)(1 - b_2p_2(t) + d_2p_1(t)) - \frac{c_1}{2}(a_1(t)^2 + u_1^2(t)) - \frac{c_2}{2}(a_2^2(t) + u_2^2(t))\right) dt; \tag{1}$$

$$\begin{cases} \dot{S}_1(t) = \rho_1 u_1(t)\sqrt{S_2(t)} - \rho_2 u_2(t)\sqrt{S_1(t)} + \theta_1(k_1a_1(t) + k_2a_2(t)), S_1(0) = S_{10}; \\ \dot{S}_2(t) = \rho_2 u_2(t)\sqrt{S_1(t)} - \rho_1 u_1(t)\sqrt{S_2(t)} + \theta_2(k_1a_1(t) + k_2a_2(t)), S_2(0) = S_{20}. \end{cases} \tag{2}$$

## 3. Profit Maximization

This section presents the method of profit maximization for the monopolist. As it was mentioned before the monopolist uses generic and brand advertising.

The Hamilton - Jacobi - Bellman equation for system (1)-(2) is considered:

$$rV = \max_{u_1,a_1,p_1,u_2,a_2,p_2} \left\{ \begin{array}{c} p_1S_1(1 - b_1p_1 + d_1p_2) + p_2S_2(1 - b_2p_2 + d_2p_1) - \\ -\frac{c_1}{2}(a_1^2 + u_1^2) - \frac{c_2}{2}(a_2^2 + u_2^2) + \frac{\partial V}{\partial S_1}\Big(\rho_1 u_1\sqrt{S_2} - \\ -\rho_2 u_2\sqrt{S_1} + \theta_1(k_1a_1 + k_2a_2)\Big) + \frac{\partial V}{\partial S_2}\Big(\rho_2 u_2\sqrt{S_1} - \\ -\rho_1 u_1\sqrt{S_2} + \theta_2(k_1a_1 + k_2a_2)\Big). \end{array} \right. \tag{3}$$

From this, the first - order conditions for the optimal advertising decisions yield

$$\begin{aligned} u_1^* &= \frac{\rho_1}{c_1}\left(\frac{\partial V}{\partial S_1} - \frac{\partial V}{\partial S_2}\right)\sqrt{S_2}, \quad a_1^* = \frac{k_1}{c_1}\left(\theta_1\frac{\partial V}{\partial S_1} + \theta_2\frac{\partial V}{\partial S_2}\right); \\ u_2^* &= \frac{\rho_2}{c_2}\left(\frac{\partial V}{\partial S_2} - \frac{\partial V}{\partial S_1}\right)\sqrt{S_1}, \quad a_2^* = \frac{k_2}{c_2}\left(\theta_1\frac{\partial V}{\partial S_1} + \theta_2\frac{\partial V}{\partial S_2}\right). \end{aligned} \tag{4}$$

The first - order conditions for $p_1$ and $p_2$ yield

$$p_1 = \frac{S_2(t)(d_1S_1(t) + d_2S_2(t)) + 2b_2S_1(t)S_2(t)}{4b_1b_2S_1(t)S_2(t) - (S_1(t)d_1 + S_2(t)d_2)^2}$$

$$p_2 = \frac{S_1(t)(d_1S_1(t) + d_2S_2(t)) + 2b_1S_1(t)S_2(t)}{4b_1b_2S_1(t)S_2(t) - (S_1(t)d_1 + S_2(t)d_2)^2}$$

In the model the time interval $[0, T]$ is considered, where $T$ is assumed to be infinite, $T \to \infty$. Split this interval into n equal sections: $0 < t_1 < t_2 < ... < t_{n-1} < < t_n$, $t_n = T$. This sectioning is made such that profit function $V$ is linear and prices $p_1$, $p_2$ are constant for each small section.

Let us examine the method of profit maximization on the first time interval $[o, t_1]$.

Then, the optimal prices on the time interval $[0, t_1]$ are given by

$$
\begin{aligned}
p_1 &= \frac{S_{20}(d_1 S_{10} + d_2 S_{20}) + 2b_2 S_{10} S_{20}}{4b_1 b_2 S_{10} S_{20} - (S_{10} d_1 + S_{20} d_2)^2} \\
p_2 &= \frac{S_{10}(d_1 S_{10} + d_2 S_{20}) + 2b_1 S_{10} S_{20}}{4b_1 b_2 S_{10} S_{20} - (S_{10} d_1 + S_{20} d_2)^2}
\end{aligned} \tag{5}
$$

Make following substitution of variables:

$$
\begin{aligned}
m_1 &= p_1(1 - b_1 p_1 + d_1 p_2), \\
m_2 &= p_2(1 - b_2 p_2 + d_2 p_1).
\end{aligned} \tag{6}
$$

Plugging (4), (5) into (3), it is easy to obtain

$$
\begin{aligned}
rV = m_1 S_1 + m_2 S_2 + \frac{\rho_2^2}{2c_2}\left(\frac{\partial V}{\partial S_2} - \frac{\partial V}{\partial S_1}\right)^2 S_1 + \frac{\rho_1^2}{2c_1}\left(\frac{\partial V}{\partial S_1} - \frac{\partial V}{\partial S_2}\right)^2 S_2 + \\
+ \left(\frac{k_1^2}{2c_1} + \frac{k_2^2}{2c_2}\right)\left(\theta_1 \frac{\partial V}{\partial S_1} + \theta_2 \frac{\partial V}{\partial S_2}\right)^2.
\end{aligned} \tag{7}
$$

Suppose that the profit function $V$ is linear:

$$
V = \alpha_m + \beta_m S_1 + \gamma_m S_2 \tag{8}
$$

As we can see it satisfies the obtained equation (7).
Hence,

$$
\begin{aligned}
\frac{\partial V}{\partial S_1} &= \beta_m, \\
\frac{\partial V}{\partial S_2} &= \gamma_m,
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
r\beta_m &= m_1 + \frac{\rho_2^2}{2c_2}(\gamma_m - \beta_m)^2, \\
r\gamma_m &= m_2 + \frac{\rho_1^2}{2c_1}(\beta_m - \gamma_m)^2, \\
r\alpha_m &= \left(\frac{k_1^2}{2c_1} + \frac{k_2^2}{2c_2}\right)(\theta_1 \beta_m + \theta_2 \gamma_m)^2.
\end{aligned}
$$

Therefore,

$$
r\beta_m - r\gamma_m = m_1 - m_2 + \left(\frac{\rho_2^2}{2c_2} - \frac{\rho_1^2}{2c_1}\right)(\beta_m - \gamma_m)^2.
$$

Let $z = \beta_m - \gamma_m$, $a = \dfrac{\rho_2^2}{2c_2} - \dfrac{\rho_1^2}{2c_1}$, $b = m_1 - m_2$.

Using the change of variables mentioned above we get the quadratic equation

$$az^2 - rz + b = 0. \tag{10}$$

with the restriction

$$r^2 - 4ab \geq 0, a \neq 0. \tag{11}$$

Solving it, two $z$'s are obtained $z_{1,2} = \dfrac{r \pm \sqrt{r^2 - 4ab}}{2a}$.

So we can compute coefficients $\alpha_m$, $\beta_m$,$\gamma_m$ for each z. Therefore, all further calculations are made for both $z$'s

$$\begin{aligned} \gamma_m &= \frac{m_2}{r} + \frac{\rho_1^2}{2c_1 r} z^2; \\ \beta_m &= z + \gamma_m; \\ \alpha_m &= \frac{\left(\dfrac{k_1^2}{2c_1} + \dfrac{k_2^2}{2c_2}\right)(\theta_1\beta_m + \theta_2\gamma_m)^2}{r}. \end{aligned} \tag{12}$$

If $a = 0$ the root is

$$z_{1,2} = \frac{b}{r}.$$

That yields coefficients $\alpha_m$, $\beta_m$,$\gamma_m$ satisfy the following expressions:

$$\begin{aligned} \gamma_m &= \frac{m_2}{r} + \frac{\rho_1^2}{2c_1 r} \frac{(m_1 - m_2)^2}{r^2}; \\ \beta_m &= \frac{m_1 - m_2}{r} + \gamma_m; \\ \alpha_m &= \frac{\left(\dfrac{k_1^2}{2c_1} + \dfrac{k_2^2}{2c_2}\right)(\theta_1\beta_m + \theta_2\gamma_m)^2}{r}. \end{aligned} \tag{13}$$

From (4) and (9) it is seen that only one of three options is possible:

- if $z > 0$:

$u_1 = \dfrac{\rho_1}{c_1}(\beta_m - \gamma_m)\sqrt{S_{20}}, \quad u_2 = 0;$

- if $z < 0$:

$u_1 = 0, \quad u_2 = \dfrac{\rho_2}{c_2}(\gamma_m - \beta_m)\sqrt{S_{10}}$

- if $z = 0$:

$u_1 = 0, \quad u_2 = 0$

We analyze each option sequentially:

1) $u_1 = \dfrac{\rho_1}{c_1} (\beta_m - \gamma_m) \sqrt{S_{20}}, \quad u_2 = 0$

the solution of the system of differential equation system (2) is

$$\begin{cases} S_1(t) = e^{-\xi t} \left( \dfrac{\eta_2}{\xi} - S_{20} \right) + (\eta_1 + \eta_2)t - \dfrac{\eta_2}{\xi} + S_{10} + S_{20}, \\ S_2(t) = e^{-\xi t} \left( S_{20} - \dfrac{\eta_2}{\xi} \right) + \dfrac{\eta_2}{\xi}, \end{cases} \tag{14}$$

where

$$\xi = \frac{\rho_1^2}{c_1}(\beta_m - \gamma_m), \;\; \eta_1 = \theta_1(k_1a_1 + k_2a_2), \;\; \eta_2 = \theta_2(k_1a_1 + k_2a_2). \tag{15}$$

2) $u_1 = 0, \quad u_2 = \dfrac{\rho_2}{c_2} (\gamma_m - \beta_m) \sqrt{S_{10}}$ :

the solution of the system (2) changes to:

$$\begin{cases} S_1(t) = e^{-\xi t} \left( S_{10} - \dfrac{\eta_1}{\xi} \right) + \dfrac{\eta_1}{\xi}, \\ S_2(t) = e^{-\xi t} \left( \dfrac{\eta_1}{\xi} - S_{10} \right) + (\eta_1 + \eta_2)t - \dfrac{\eta_1}{\xi} + S_{10} + S_{20}. \end{cases} \tag{16}$$

where

$$\xi = \frac{\rho_2^2}{c_2}(\gamma_m - \beta_m), \quad \eta_1 = \theta_1(k_1a_1 + k_2a_2), \quad \eta_2 = \theta_2(k_1a_1 + k_2a_2). \tag{17}$$

3) $u_1 = 0, \quad u_2 = 0$

In this case

$$\begin{cases} S_1(t) = \theta_1(k_1a_1 + k_2a_2)t + S_{10}, \\ S_2(t) = \theta_2(k_1a_1 + k_2a_2)t + S_{20}. \end{cases} \tag{18}$$

Thus, substituting obtained coefficients $\alpha_m, \beta_m$ , $\gamma_m$ from (12) or (13) and $S_1(t)$, $S_2(t)$ either from (14), (15) or (16), (17) or (18) into (7), we respectively find $V_{1z_1}$ which corresponds to root $z_1$ and $V_{1z_2}$ which corresponds to root $z_2$. And then we find $V_1$ - the firm's profit on the first time interval [o, $t_1$] which is the maximum between $V_{1z_1}$ and $V_{1z_2}$. On the second time interval [$t_1$,$t_2$], the values of parameters $S_{10}$ and $S_{20}$ are changed to the values of parameters $S_1(t)$, $S_2(t)$ that were calculated on the previous time interval correspondingly. Finally, on time section [$t_{n-1}$,$T$], we get the firm's maximum profit $V_n$. The result of the method is $V$, which is calculated as the sum of all $V_i$, $i$=1,..$n$, obtained for each section [$t_{i-1}, t_i$].

## 4. The Algorithm for Maximum Profit Finding

The C++ program is written in the development environment MS Visual Studio 2005 to implement the algorithm for maximum profit finding.

In this section the algorithm for maximum profit finding is described. Accordingly to the algorithm the program finds the optimal solution. Analyze the algorithm on time interval $[o, t_1]$ as in previous section.

**Step 1:**

The monopolist inputs variables values $S_{10},\ S_{20},\ b_1,\ b_2,\ d_1,\ d_2,\ \rho_1,\ \rho_2,\ c_1,\ c_2,\ k_1,\ k_2,\ \theta_1,\ \theta_2,\ t,\ r, n$ where $n$ - is the number of intervals.
There are restrictions, which should be taken into account: $0 < d_1 < b_1, 0 < d_2 < b_2$.

**Step 2:**

According to (5) the prices $p_1$, $p_2$ are calculated.

**Step 3:**

At the third step we find $m_1$, $m_2$ from (6).

**Step 4:**

Solving the equation (10) in case when $a = 0$ (restriction (11)) and in case when $a \neq 0$ $z_{1,2}$ are received as follows:

$$z_{1,2} = \frac{r \pm \sqrt{r^2 - 4ab}}{2a} \qquad \text{or} \qquad z_{1,2} = \frac{b}{r}.$$

Then the calculations are made for each $z$, as it has been mentioned in section 3.

**Step 5:**

Using (12) or (13) the coefficients $\alpha_m,\ \beta_m,\ \gamma_m$ are estimated.

**Step 6:**

We choose the optimal advertising strategy:

• if $z > 0$, or $\beta_m > \gamma_m$, then

$$u_1 = \frac{\rho_1}{c_1}(\beta_m - \gamma_m)\sqrt{S_{20}}, \quad u_2 = 0;$$

$$a_1 = \frac{k_1}{c_1}(\theta_1\beta_m + \theta_2\gamma_m), \quad a_2 = \frac{k_2}{c_2}(\theta_1\beta_m + \theta_2\gamma_m).$$

• if $z < 0$

$$u_1 = 0, \quad u_2 = \frac{\rho_2}{c_2}(\gamma_m - \beta_m)\sqrt{S_{10}}$$

$$a_1 = \frac{k_1}{c_1}(\theta_1\beta_m + \theta_2\gamma_m), \quad a_2 = \frac{k_2}{c_2}(\theta_1\beta_m + \theta_2\gamma_m).$$

• if $z = 0$

$$u_1 = 0, \quad u_2 = 0$$

$$a_1 = \frac{k_1}{c_1}\left(\theta_1\beta_m + \theta_2\gamma_m\right), \quad a_2 = \frac{k_2}{c_2}\left(\theta_1\beta_m + \theta_2\gamma_m\right).$$

**Step 7:**

• for $z > 0$ according to (14) and (15)

$S_1(t),\ S_2(t)$ are received.

• for $z < 0$ the values of $S_1(t),\ S_2(t)$ are estimated in complience with (16) and (17)

• for $z = 0$ the values of $S_1(t),\ S_2(t)$ are estimated in accordance with (18).

**Step 8:**

Firm's profit $V_1$ is computed as:

$$V_1 = \alpha_m + \beta_m S_1 + \gamma_m S_2$$

**Step 9:**

So we received two values of firm's profit $V_1$ for each $z$. We choose the biggest one.

When algorithm is released for each time interval $[t_{i-1}, t_i]$,$i$=1,..$n$, the resulting firm's maximum profit $V$ is estimated as the sum of $V_i$, $i$=1,..$n$.

## 5. Symmetric Case

Let us look at a symmetric case. The data selected for this example have been chosen to illustrate the economic model and the work of the algorithm. From the results given below it is clear - the both of the brand have no brand advertisements, both have only generic advertisements. Both brands bring the same profit.

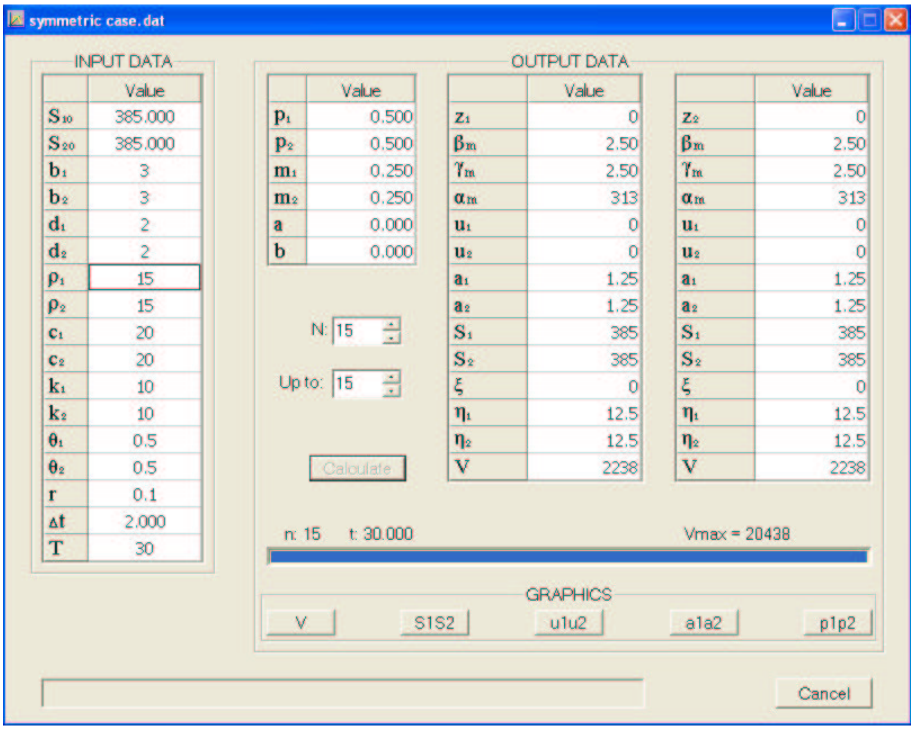


**Fig.1.** The example of program realization

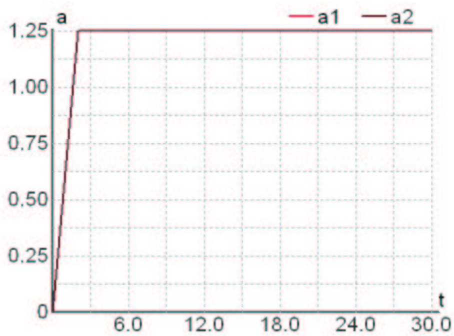


**Fig.2.** Generic advertising graph


**Fig.3.** The price graph

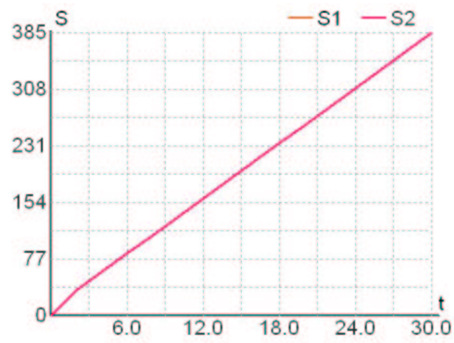


**Fig.4.** The firm sales graph

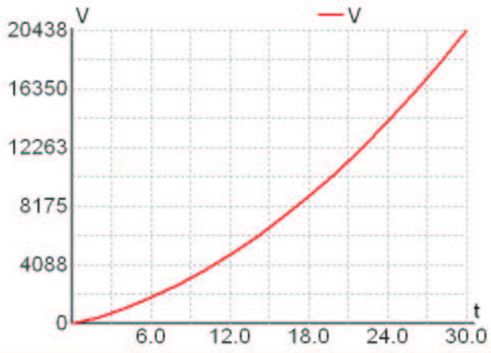


**Fig.5.** The firm profit graph

## 6. Conclusion

In a general case, it can be observed that in accordance to the calculations in the paper one of the two brands will not be advertised and its sales will be much less than those of the other one. So it is more profitable for a monopolist to refuse from one of the brands.

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# Three-Sided Matchings and Separable Preferences*

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**Abstract** In this paper we provide sufficient conditions for the existence stable matchings for three-sided systems.

## 1. Introduction

The two-sided matching model of Gale and Shapley (1962) can be interpreted as one where a non-empty finite set of firms need to employ a non-empty finite set of workers. Further, each firm can employ at most one worker and each worker can be employed by at most one firm. Each worker has preferences over the set of firms and each firm has preferences over the set of workers. An assignment of workers to firms is said to be stable if there does not exist a firm and a worker who prefer each other to the ones they are associated with in the assignment. Gale and Shapley (1962) proved that every two-sided matching problem admits at least one stable matching.

In this paper we extend the above model by including a non-empty finite set of techniques. A technique can be likened to a machine that is owned by a technologist who is neither a firm nor a worker, and which the firm and worker together use for production. Further each technologist owns exactly one technique. Each firm has preferences over the set of ordered pairs of workers and techniques, each worker has preferences over the set of ordered pairs of firms and techniques and each technologist has preferences over ordered pairs of firms and workers. Such models [see Alkan (1988)] are called three-sided systems. A matching in a three-sided system consists of disjoint triplets, each triplet comprising a firm, a worker and a technologist. A stable matching for a three-sided system is a matching which does not admit a triplet whose members are better off together than at their current designations. Alkan (1988) provided an example of a three-sided system that does not admit a stable matching. Danilov (2003) established the existence of a stable matching for lexicographic three-sided systems.

The preference of a firm is separable if its preference over workers is independent of the technique and its preference over techniques is independent of the worker. The preference of a worker is separable if its preference over firms is independent of the technique and its preference over techniques is independent of the firm. A three-sided system is said to be separable if preferences of all firms and workers are separable.

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Alternatively we assume that the preferences of the workers are lexicographic, with firms enjoying priority over techniques. If in addition the preferences of the firms are lexicographic, with workers enjoying priority over techniques, then the system is called lexicographic.

In this paper we show that if a three-sided system is lexicographic for workers and satisfies a property called Technical Specialization then there exists a stable matching. Technical Specialization says: given two distinct firm-worker pairs, the technique that is best for the firm in one pair is different from the technique that is best for the firm in the other. We also provide an example of a three-sided system with preferences of workers being both lexicographic as well as separable, that does not admit a stable matching. In this example the preferences of the firms are neither lexicographic nor separable.

Neither technical specialization nor the proof of theorem that establishes the existence of a stable matching when technical specialization is satisfied by a three-sided system, takes cognizance of the preferences of the technologists. In a way, the stable matching that is obtained may have resulted by 'coercing' the technologists. While this may make technical specialization an unpalatable assumption, it is worth remembering, that a stable matching for a three-sided system, does not require that every side of the system play an active role in determining its viability. Alternatively one may assume that the three-sided system is strongly separable i.e. separable for both firms and workers and lexicographic for workers. In such a scenario we need to assume that the preferences of firms and technologists over workers are in "agreement" (i.e. given a firm, a technologist ranks the workers in the same way that the firm does) to show that a three-sided system admits a stable matching. Agreement over workers in a strongly separable environment implies some kind of a hierarchy where the worker cares only about the firm and forms the bottom layer, whereas the technologists preferences over the workers "echoes" the preferences of the firm it is engaged with.

Following the tradition of Gale and Shapley (1962), we model our analysis in terms of a firm employing at most one worker. By present day reckoning, a firm employing at most one worker is usually a small road-side shop, rather than an industrial unit. Hence, it might appear as if our analysis has little if no relevance to the more common real world situations. However, it may well be a reasonable starting point for the cooperative theory of multi-sided systems. Roth and Sotomayor (1988) contain an elaborate discussion of matching models, where firms may employ more than one worker. It turns out in their analysis, that the cooperative theory for such firms is almost identical to the cooperative theory arising out of the Gale and Shapley (1962) framework. This occurs, since each firm can be replicated as often as the number of workers it can employ, with each replica having the same preferences over workers as the original firm. Further, the preferences of the workers between replicas of two different firms should be exactly the same as her preferences between the originals. On the other hand, the non-cooperative theory where each firm employs more than one worker is considerably different from the non-cooperative theory where firms may employ at most one. It is noteworthy that the cooperative theory for many-to-many two-sided matching models does not permit the same replication argument. This has been shown in Lahiri (2006).

The analysis reported in this paper, attempts at extending results pertaining to the existence of stable matchings in a labor market, by introducing technology as an essential determinant of the results that we obtain. Since our paper, is concerned with the cooperative theory of three-sided systems, the model that we use of a firm employing at most one worker, continues to provide valuable insights concerning the existence of stable matchings in labor markets.

## 2. The Model

Let $W$ be a non-empty finite set denoting the set of workers, $F$ a non-empty finite set denoting the set of firms and $T$ a non-empty finite set denoting the set of techniques. We assume for the sake of simplicity that the number of techniques ($|T|$) is equal to the number of firms ($|F|$) which in turn is equal to the number of workers ($|W|$).

Each $w \in W$ has preference over $F \times T$ defined by a linear order (i.e. anti-symmetric, reflexive, complete and transitive binary relation) $\geq_w$ whose asymmetric part is denoted $>_w$. Each $f \in F$ has preference over $W \times T$ defined by a linear order $\geq_f$ whose asymmetric part is denoted $>_f$. Each $t \in T$ has preference over $F \times W$ defined by a linear order $\geq_t$ whose asymmetric part is denoted $>_t$.

A three-sided system is given by the array $[\{\geq_f : f \in F\}, \{\geq_w : w \in W\}, \{\geq_t : t \in T\}]$.

A job-matching is a one-to-one function $m$ from $F$ to $W$. A technique matching is a one-to-one function $n$ from $F$ to $T$. Since $F$, $T$ and $W$ all have the same cardinality, every job-matching and every technique matching is of necessity a bijection. A pair $(m, n)$ where $m$ is a job-matching and n is a technique matching is called a matching (for the three-sided system).

A matching $(m, n)$ is said to be **stable** if there does not exist $f \in F$, $w \in W$ and $t \in T$ such that: $(w, t) >_f (m(f), n(f))$, $(f, t) >_w (m^{-1}(w), n(m^{-1}(w)))$ and $(f, w) >_t (m(n^{-1}(t), n^{-1}(t))$.

A three-sided system is said to be **separable for workers** if for all $w \in W$ there exists linear orders $P_w$ on $F$ and $Q_w$ on $T$ such that for all $(f, t), (f', t') \in F \times T$ : $(f, t) \geq_w (f', t)$ if and only if $fP_w f'$ and $(f, t) \geq_w (f, t')$ if and only if $tQ_w t'$.

A three-sided system is said to be **separable for firms** if for all $f \in F$ there exists linear orders $P_f$ on $W$ and $Q_f$ on $T$ such that for all $(w, t), (w', t') \in W \times T$ : $(w, t) \geq_f (w', t)$ if and only if $wP_f w'$ and $(w, t) \geq_f (w, t')$ if and only if $tQ_f t'$.

A three-sided system is said to be **separable** if it is separable for *both* firms and workers.

A three-sided system is said to be **lexicographic for workers** if for all $w \in W$ there exists linear orders $P_w$ on $F$ such that for all $f, f' \in F$ with $f \neq f'$ and $t, t' \in T$ : $fP_w f'$ implies $(f, t) >_w (f', t')$.

A three-sided system is said to be **lexicographic for firms** if for all $f \in F$ there exists linear orders $P_f$ on $W$ such that for all $w, w' \in W$ with $w \neq w'$ and $t, t' \in T$ : $wP_f w'$ implies $(w, t) >_f (w', t')$.

A three-sided system is said to be **lexicographic** if it is both lexicographic for workers as well as for firms.

A three-sided system is said to be **strongly separable** if it is separable and in addition lexicographic for workers.

Danilov (2003) proved that if a three-sided system is lexicographic, then it admits a stable matching. Boros, Gurvich, Jaslar and Krasner (2004) showed that if preferences are cyclic (i.e. workers care only about firms, firms care only about techniques and technologists care only about workers) and if there are exactly three agents on each side of a three-sided system, then a stable matching exists. Subsequently, Eriksson, Sjostrand and Strimling (2006) extended the same result to the case where there are four agents on each side of a three-sided system.

## 3. Existence of Stable Matchings

A three-sided system is said to satisfy **Technical Specialization** (TS) if there exists a function $\beta : F \times W \to T$ such that (a) for all $w, w_1 \in W$ and $f, f_1 \in F$ with $w \neq w_1$ and $f \neq f_1 : \beta(f, w) \neq \beta(f_1, w_1)$; (b) for all $w \in W, f \in F$ and $t \in T$ : $(w, \beta(f, w)) \geq_f (w, t)$.

**Theorem 1.** *Suppose a three-sided system that is lexicographic for workers satisfies TS. Then there exists a stable matching.*

*Proof.* Suppose preferences are lexicographic for workers and the system satisfies TS.

Hence for all $w \in W$ there exists linear orders $P_w$ on $F$ such that for all $f, f' \in F$ with $f \neq f'$ and $t, t' \in T : fP_w f'$ implies $(f, t) >_w (f', t')$.

For $f \in F$ let $P_f$ be the linear order on $W$ such that for all $w, w' \in W : wP_f w'$ if and only if $(w, \beta(f, w)) \geq_f (w', \beta(f, w'))$.

Consider the two-sided matching problem where the preference of a firm $f$ is given by $P_f$, and the preference of a worker $w$ is given by $P_w$.

As in Gale and Shapley (1962) we get a stable job-matching $m$, i.e. for all $w \in W$ and $f \in F$ : either $m(f)P_f w$ or $m^{-1}(w)P_w f$.

The technique-matching $n$ is defined as follows:
For all $f \in F : n(f) = \beta(f, m(f))$.

By TS, $n$ is well defined.

Suppose the matching $(m, n)$ is not stable. Thus, there exists $w \in W$, $f \in F$ and $t \in T$ such that: $(f, t) >_w (m^{-1}(w), n(m^{-1}(w)), (w, t) >_f (m(f), n(f))$ and $(f, w) >_t ((n^{-1}(t), m(n^{-1}(t))$.

Let $m^{-1}(w) = f_0$, and $n^{-1}(t) = f_1$.

Since the preferences of workers are lexicographic (with firms receiving priority over techniques), $(f, t) >_w (m^{-1}(w), n(m^{-1}(w))$ implies $fP_w f_0$.

However since $m$ is stable, $fP_w f_0$ implies $m(f)P_f w$.

Thus $(m(f), n(f)) = (m(f), \beta(f, m(f))) \geq_f (w, \beta(f, w))$.

Clearly $(w, \beta(f, w)) \geq_f (w, t)$.

Hence $(m(f), \beta(f, m(f))) \geq_f (w, t)$, contrary to our assumption.

Thus $(m, n)$ is stable. Q.E.D.

**Note:** The above proof is not valid if instead of assuming that preferences are lexicographic for workers, we assume that they are separable for them. The conflict arises since TS defines a best technique according to the preferences of the firms and not that of the workers.

The following example shows that if a three-sided system is merely lexicographic for workers then the existence of a stable matching is not guaranteed.

*Example 1.* Let $W = \{w_1, w_2\}$, $F = \{f_1, f_2\}$, $T = \{t_1, t_2\}$.

Assume that the system is lexicographic for workers with both $w_1$ and $w_2$ preferring $t_1$ to $t_2$ for any given firm $f$. Suppose that both $w_1$ and $w_2$ prefer $f_1$ to $f_2$.

Suppose $f_1$ prefers $(w_2, t_1)$ to $(w_1, t_1)$ to $(w_1, t_2)$ to $(w_2, t_2)$ and $f_2$ prefers $(w_1, t_1)$ to $(w_2, t_1)$ to $(w_2, t_2)$ to $(w_1, t_2)$.

Suppose that $t_1$ prefers $(f_2, w_1)$ to $(f_1, w_2)$ to $(f_2, w_2)$ to $(f_1, w_1)$ and $t_2$ prefers $(f_1, w_1)$ to $(f_1, w_2)$.

Let us consider the following four matchings:

1. $\{(f_1, w_1, t_1), (f_2, w_2, t_2)\}$;
2. $\{(f_1, w_1, t_2), (f_2, w_2, t_1)\}$;
3. $\{(f_2, w_1, t_1), (f_1, w_2, t_2)\}$;
4. $\{(f_2, w_1, t_2), (f_1, w_2, t_1)\}$.

Matching (1) is blocked by $(f_2, w_2, t_1)$ since $w_2$ prefers $(f_2, t_1)$ to $(f_2, t_2)$, $f_2$ prefers $(w_2, t_1)$ to $(w_2, t_2)$ and $t_1$ prefers $(f_2, w_2)$ to $(f_1, w_1)$.

Matching (2) is blocked by $(f_1, w_2, t_1)$ since $w_2$ prefers $(f_1, t_1)$ to $(f_2, t_1)$, $f_1$ prefers $(w_2, t_1)$ to $(w_1, t_2)$ and $t_1$ prefers $(f_1, w_2)$ to $(f_2, w_2)$.

Matching (3) is blocked by $(f_1, w_1, t_2)$ since $w_1$ prefers $(f_1, t_2)$ to $(f_2, t_2)$, $f_1$ prefers $(w_1, t_2)$ to $(w_2, t_2)$ and $t_2$ prefers $(f_1, w_1)$ to $(f_1, w_2)$.

Matching (4) is blocked by $(f_2, w_1, t_1)$ since $w_1$ prefers $(f_2, t_1)$ to $(f_2, t_2)$, $f_2$ prefers $(w_1, t_1)$ to $(w_1, t_2)$ and $t_1$ prefers $(f_2, w_1)$ to $(f_1, w_2)$.

Hence none of the four matchings are stable.

Further, $\beta(f_1, w_2) = \beta(f_2, w_1) = t_1$. This contradicts TS.

In Example 1 the preferences of workers are separable. Hence this example is an instance of separability for workers (alone) not being sufficient for the existence of a stable matching.

It is worth noting that TS is not necessary for the existence of a stable matching for a three-sided system, as the following example reveals.

*Example 2.* Let $W = \{w_1, w_2, w_3\}$, $F = \{f_1, f_2, f_3\}$ and $T = \{t_1, t_2, t_3\}$.

Suppose that for each $w \in W$ there exists a linear order $P_w$ on $F$ satisfying $f_1 P_w f_2 P_w f_3$ and for each $f \in F$ there exists a linear order $P_f$ on $W$ satisfying $w_1 P_f w_2 P_f w_3$. Suppose for each $w \in W$ there exists a linear order $Q_w$ on $T$ and for each $f \in F$ there exists a linear order $Q_f$ on $T$. Suppose $t_1 Q_a t_2 Q_a t_3$ for $a \in \{f_1, w_1, w_2\}$ and $t_3 Q_a t_2 Q_a t_1$ for $a \in \{w_3, f_2, f_3\}$. Further suppose that for all $w, w' \in W$, $f, f' \in F$ and $t, t' \in T$ with $w \neq w'$, $f \neq f'$ and $t \neq t'$: (a) $(w,t) >_f (w',t')$ if and only if $wP_f w'$; (b) $(w,t) >_f (w,t')$ if and only if $tQ_f t'$; (c) $(f,t) >_w (f',t')$ if and only if $fP_w f'$; (d) $(f,t) >_w (f,t')$ if and only if $tQ_w t'$.

In addition suppose that for all $t \in T$, $f' \in F$, $w' \in W$ and $i \in \{1,2,3\} : (f_i, w_i) \geq_t (f', w')$ if and only if $t = t_i$.

Towards a contradiction suppose that this system satisfies TS.

Then there exists a function $\beta : F \times W \to T$ such that (a) for all $w, w' \in W$ and $f, f' \in F$ with $w \neq w'$ and $f \neq f' : \beta(f,w) \neq \beta(f',w')$; (b) for all $w \in W$ and $f \in F : [(w, \beta(f,w)) \geq_f (w,t)$ for all $t \in T]$. Thus, $\beta(f_1, w_1) = t_1$ and $\beta(f_3, w_3) = t_3 = \beta(f_2, w_2)$. Thus this system does not satisfy TS.

However, the matching with the associated triplets being $(w_i, f_i, t_i)$ for $i = 1, 2, 3$ is indeed a stable matching.

A three-sided system $[\{\geq_f : f \in F\}, \{\geq_w : w \in W\}, \{\geq_t : t \in T\}]$ is said to satisfy **Agreement over Workers** if for $f \in F, t \in T and w' \in W : (w,t) >_f (w',t)$ implies $(f,w) >_t (f,w')$.

**Theorem 2.** *Suppose a three-sided system is strongly separable (i.e. separable for both firms and workers and lexicographic for workers) and satisfies Agreement over Workers. Then there exists a stable matching.*

*Proof.* Suppose that for all $w \in W$, there exists linear orders $P_w$ on $F$ and $Q_w$ on $T$ such that: (a) for all $f, f' \in F$ with $f \neq f'$ and $t, t' \in T : fP_w f'$ implies $(f,t) >_w (f',t')$; (b) for all $f \in F$ and $t, t' \in T$ with $t \neq t' : tQ_w t'$ implies $(f,t) >_w (f,t')$.

Suppose in addition that for all $f \in F$, there exists a linear order $P_f$ on $W$ and $Q_f$ on $T$ such that for all $w, w' \in F$ with $w \neq w'$ and $t, t' \in T$ with $t \neq t' : wP_f w'$ implies $(w,t) >_f (w',t)$ and $tP_f t'$ implies $(w,t) >_f (w,t')$.

Consider the two-sided matching model based on $F$ and $W$ where for each $f \in F$ and $w \in W$, preferences are given by $P_f$ and $P_w$ respectively. As in Gale and Shapley (1962), we get a job-matching $m$ that is stable, i.e. for all $w \in W$ and $f \in F :$ either $m(f)P_f w$ or $m^{-1}(w)P_w f$.

For $t \in T$, let $P_t$ be a linear order on $F$ such that for all $f, f' \in F$ with $f \neq f' : fP_t f'$ if and only if $(f, m(f)) >_t (f', m(f'))$.

Consider the two-sided matching model based on $F$ and $T$ where for each $f \in F$ and $t \in T$, preferences are given by $Q_f$ and $P_t$ respectively. As in Gale and Shapley (1962), we get a technique-matching $n$ such that for all $t \in T$ and $f \in F :$ either $n(f)Q_f t$ or $n^{-1}(t)P_t f$.

Towards a contradiction suppose that the matching $(m, n)$ is not stable.

Thus, there exists $w \in W$, $f \in F$ and $t \in T$ such that: $(f, t) >_w (m^{-1}(w), n(m^{-1}(w))), (w, t) >_f (m(f), n(f))$ and $(f, w) >_t (n^{-1}(t), m(n^{-1}(t)))$.

Since the preferences of workers are lexicographic with firms receiving priority over techniques, it must be either (a) $f = m^{-1}(w)$ and $tQ_w n(m^{-1}(w))$ or (b) $fP_w m_{-1}(w)$.

Suppose $f = m^{-1}(w)$.

Thus, $tQ_w n(m^{-1}(w))$.

Since preferences of firms are separable we must have $tQ_f n(f)$.

$tQ_f n(f)$ and the stability of the matching $n$ implies $(n^{-1}(t), m(n^{-1}(t))) >_t (f, m(f))$.

Thus, $(f, w) >_t (f, m(f))$.

This contradicts $w = m(f)$.

Hence suppose $fP_w m^{-1}(w)$. By the stability of the matching $m$, we must have $m(f)P_f w$.

Since preferences of firms are separable given $(w, t) >_f (m(f), n(f))$, the fact that we have $m(f)P_f w$ implies $tQ_f n(f)$.

$tQ_f n(f)$ and the stability of the matching $n$ implies $(n^{-1}(t), m(n^{-1}(t))) >_t (f, m(f))$.

Thus, $(f, w) >_t (f, m(f))$.

Since the three-sided system is assumed to satisfy agreement over workers and the preferences of firms are separable, $(f, w) >_t (f, m(f))$ implies $wP_f m(f)$, contradicting $m(f)P_f w$ as obtained earlier.

Thus $(m, n)$ is stable. Q.E.D.

In example 1 the preferences of the workers are lexicographic (with firms getting priority over technologists), but the preferences of the firms are not separable. Thus although the three-sided system satisfies agreement over workers, it does not admit a stable matching.

In the following example preferences are strongly separable but the system does not satisfy agreement over workers and does not admit a stable matching.

*Example 3.* Let $W = \{w_1, w_2\}$, $F = \{f_1, f_2\}$, $T = \{t_1, t_2\}$. Assume that the system is separable and lexicographic for workers (with firms receiving priority over technologists). Suppose that for any given firm both workers prefer $t_2$ to $t_1$ and that both workers prefer $f_1$ to $f_2$ .

Suppose the preferences of the firms are also lexicographic (although not in the sense that we have defined in this paper), with technologists receiving priority over workers. Suppose both firms prefer $t_2$ to $t_1$ and for any given technique prefer $w_2$ to $w_1$. Suppose $t_1$ prefers $(f_1, w_2)$ to $(f_1, w_1)$.

Suppose $t_2$ prefers $(f_2, w_2)$ to $(f_1, w_1)$ and $(f_1, w_1)$ to $(f_2, w_1)$ to $(f_1, w_2)$.

Let us consider the following four matchings:

1. $\{(f_1, w_1, t_1), (f_2, w_2, t_2)\}$;
2. $\{(f_1, w_1, t_2), (f_2, w_2, t_1)\}$;
3. $\{(f_2, w_1, t_1), (f_1, w_2, t_2)\}$;
4. $\{(f_2, w_1, t_2), (f_1, w_2, t_1)\}$.

Matching (1) is blocked by $(f_1, w_2, t_1)$ since $w_2$ prefers $(f_1, t_1)$ to $(f_2, t_2)$, $f_1$ prefers $(w_2, t_1)$ to $(w_1, t_1)$ and $t_1$ prefers $(f_1, w_2)$ to $(f_1, w_1)$.

Matching (2) is blocked by $(f_2, w_2, t_2)$ since $w_2$ prefers $(f_2, t_2)$ to $(f_2, t_1)$, $f_2$ prefers $(w_2, t_2)$ to $(w_2, t_1)$ and $t_2$ prefers $(f_2, w_2)$ to $(f_1, w_1)$.

Matching (3) is blocked by $(f_2, w_1, t_2)$ since $w_1$ prefers $(f_2, t_2)$ to $(f_2, t_1)$, $f_2$ prefers $(w_1, t_2)$ to $(w_2, t_1)$ and $t_2$ prefers $(f_2, w_1)$ to $(f_1, w_2)$.

Matching (4) is blocked by $(f_1, w_1, t_2)$ since $w_1$ prefers $(f_1, t_2)$ to $(f_2, t_2)$, $f_1$ prefers $(w_1, t_2)$ to $(w_2, t_1)$ and $t_2$ prefers $(f_1, w_1)$ to $(f_2, w_1)$.

Hence none of the four matchings are stable.

Note that given $t_2$, $f_1$ prefers $w_2$ to $w_1$, whereas given $f_1$, $t_2$ prefers $w_1$ to $w_2$.

Hence the system does not satisfy agreement over workers.

It is instructive to note that in example 3, although the preferences of the workers and firms are both lexicographic (though not in the sense in which it is defined here) with workers giving priority to firms over techniques, a stable matching does not exist, since firms accord priority to techniques over workers. The reciprocation of priority between firms and workers that was assumed by Danilov (2003) is absent in example 3.

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# A Game Model of Economic Behavior in an Institutional Environment *

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**Abstract** A game model proposed here helps to reveal a relation between institutions (i.e. norms and rules used in a society) and decisions of private agents. Players in the game are a government and numerous private agents. Activities of the private agents (the second player) are modeled as paths in an oriented graph with a finite set of nodes. The government (the first player) establishes and announces an institutional system - a set of actions (e.g. taxes, incentives etc.) on the arcs of the graph. A move of a private agent yields her and the government gains depending on the institutional system created by the government. The players try to maximize discounted sums of utilities given discount factors and horizons. The government has no information about a precise number and initial positions of the private agents. The basic question is: can the government establish a consistent institutional system corresponding to a Nash equilibrium? We show that in a specific case of myopic private agents a consistent institutional system does exist. A constructive proof is provided. A case of an almost myopic government is considered in detail. A possible application of the game model is a problem of effectiveness of the government control in science and R&D sector in Russia which became actual in connection with a reform started by the Russian government recently.

**Keywords:** iterated games, economic behavior, myopic agents, institutions, dynamic programming.

## 1. Introduction

Studying *institutions* (i.e. norms and rules used in a society) is an actual question in modern economics, sociology and management. The following problems related to institutions seem to be of great importance. How do institutions emerge? What is a relation between institutions and a behavior of economic agents? Why do institutions differ among countries (and among organizations)? Why is a transplantation of institutions from abroad often unsuccessful? To answer such questions game models are needed.

Generally there are two points of view on an emergence of institutions and correspondingly on a possibility of their changing. These views originate in pamphlets of philosophers of 17th and 18th centuries: Thomas Hobbes' "Leviathan" and Bernard Mandeville's "The Fable of Bees". The first of the views is that institutions are created by governments artificially and purposefully, and the second one is that institutions are a result of a game equilibrium and reflect interests of many participants. Our model unifies these points of view. Governments create institutions

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taking into account a further reaction of private agents. Here are opinions of two famous economists about the role of institutions. (Galbraith, 1983) defined *power* as "the ability of individuals or groups to win the submission of others to their purpose". Political and economic institutions often serve to force economic agents to act in a way suitable for a government. As (North, 1990) noted, "In the jargon of economists, institutions define and limit the set of choices of individuals". Our model uncovers mechanisms hinted in these quotations. The game model proposed here helps to reveal a relation between institutions and decisions of private agents.

A description of the model and a content of the further part of the paper are provided in the following Section.

## 2. The Model

Players in the game are a *government* and numerous *private agents*. Activities of the private agents (the second player) are modeled as paths in an oriented graph $\langle M, N\rangle$ with a finite set of nodes $M$ (states) and a set of arcs $N$ (actions). The government (the first player) establishes and announces an institutional system $P$. More precisely, for each node $i = 1, ..., n$ the government defines a set $p^i$ of actions on the arcs starting from the node $i$ to be used in response to actions of private agents. The institutions $p^i$ are taken from a set $A^i$ of feasible institutions. In reality the government's actions can be e.g. values of taxes and subsidies, other measures of incentives and punishment, etc. Thus the government commits to act in a definite way in response to actions of private agents. The whole institutional system $P$ consists of the institutions $p^i$:

$$P = (p^1, p^2, ..., p^n).$$

After the institutional system is announced the private agents choose their paths of actions. A move of a private agent from the node $i$ to the node $j$ yields her an instant utility gain $u(i, j, p^i)$. At the same time the government receives an instant utility gain $v(i, j, p^i)$. The gains depend on institutions created by the government. We will also use designation $u(i, j, P)$, $v(i, j, P)$. The preferences of different private agents are assumed to be identical, it means that the agents receive similar utilities if they choose the same actions. But the agents can be situated in different nodes, and naturally their choice depends not only on their preferences but on their initial states as well. Each private agent solves the following problem:

$$\max_{j_1,\ldots,j_n} \sum_{t=0}^{\widehat{T}} \widehat{\beta}^t u(i_t, i_{t+1}, P).$$

where $(i_t, i_{t+1}) \in N, 0 \le \widehat{\beta} \le 1$ is a discount factor of the private agent, $i_0$ is an initial state of the agent, and $P$ is an institutional system.

An agent is *myopic* if she possesses a zero discount factor $\widehat{\beta} = 0$ or a zero horizon $\widehat{T} = 0$. These two cases are close to each other though not identical. A difference between the two definitions of a myopic agent is discussed in Section 4. An optimal path for a myopic agent can be found stepwise.

The government maximizes a discounted sum of utilities related to the actions of the private agents given a discount factor $0 \le \beta \le 1$ . If the government deals with a single agent, the government's problem would be:

$$\max_P \sum_{t=0}^{T} \beta^t v(i_t, i_{t+1}, P)$$

A specific peculiarity of the model is that the government deals with many agents and possesses only a restricted information. The government has information on the agents' preferences but not enough information about their number and position. In particular the government has no information about how many agents are active now and in what nodes new active agents will appear in the future.

A question arises: is it possible for the government to establish a consistent institutional system corresponding to a Nash equilibrium under this partial information? Very often in economics and politics it is supposed that such an institutional system is possible. For example, governments in countries are often accused in not satisfying interests of one or other group of citizens. The U.S. is often criticized for so called double standards when different institutions are applied to different partners in international relations. We will see that in a special case of myopic private agents a consistent institutional system really exists in our model.

If the game with a single agent is considered as a game in an expanded form, its structure seems to be rather simple. The first player defines an institutional system $P$ i.e. her actions in all arcs of the graph. After thst the second player chooses a route in the graph. Despite the structure is simple, there may be three serious problems: (1) an enormous number of strategies; (2) a hard calculation of gains; (3) a presence of not a single but many "second players" with different initial nodes.

In Section 3 a constructive procedure solving the problem is proposed based on methods of dynamic programming and 'extremal' ('idempotent') algebra with operations $a \bigotimes b = a + \beta b, a \bigoplus b = \max\{a, b\}$. (See (Matveenko, 1990, Matveenko, 1998) for applications of extremal algebra to schemes of dynamic programming without or with discounting). An example of application of this constructive procedure to a version of iterated Prisoner's Dilemma is provided in Section 5.

Another possible application of our model (Section 6) is a problem of an effectiveness of the government control in science and R&D sector in Russia which became actual in connection with a reform started by the government recently. Here $M$ can be interpreted as a set of possible themes which can be explored by researchers, and $N$ is a set of possible directions of new research. The government would be more satisfied if the science deals with themes related to some definite practical issues such as nanotechnologies. The model shows that if the researchers are myopic (i.e. they are interested more in their current achievements and welfare than in long-term perspectives of their activities) then the government is really able to establish effective incentives under incomplete information. However, if the researchers are long-term-oriented, the government's problem is insolvable.

In Section 4 a detailed discussion of the notion of a myopic behavior of private agents and governments is provided.

## 3. The Basic Theorem

In this Section a constructive proof of the basic theorem is given.

**Theorem 1.** *Let us assume that, in each node $i$ and for each institution $p^i$, utilities $u(i, j, p^i), j = 1, 2, ..., n$ for alternative actions of a private agent are different: $u(i, j_1, p^i) \neq u(i, j_2, p^i)$ if $j_1 \neq j_2$. Then if the agents are myopic the government is able to create a consistent institutional system $P$.*

*Proof.* In each node $i$ a unique response of the private agent corresponds to each institution $p^i$. For a node $i$ denote $\overline{N}^i$ the set of arcs $(i, j)$ chosen by the agent under any institutions $P$. Then $\overline{N} = \overline{N}^1 \bigcup ... \overline{N}^n$ is the set of all arcs $(i, j)$ chosen by the agent under any institutional systems $P$. To each arc $s = (i, j) \in \overline{N}^i$ an institution $p^i(s)$corresponds under which the arc is chosen. If the agent chooses an arc $s$ under several different institutions $p^i$ then let $p^i(s)$ be an institution which provides the maximum utility $v(i, i, p^i)$ to the government.

As a result, a subgraph $\langle M, \overline{N} \rangle$ is constructed, to each arc $s = (i, j)$ of which an institution $p^i(s)$ corresponds. (The subgraph may consist of more than one connected components). Given discount factor $\beta \in (0, 1)$ a family of dynamic programming problems

$$\max_P \sum_{t=0}^{\infty} \beta^t u(i_t, i_{t+1}, P)$$

with different initial states $i_0 = i$ is defined. Denote these problems by $S_\beta(i)$ and their values by $V_\beta(i)$. The value function satisfies a recurrent relation of the dynamic programming (a Bellman equation)

$$V_\beta(i) = \max_{j=1,...,n} \{u(i, j) + \beta V_\beta(j)\}.$$

For each node $i$ an arc $(i, \overline{j}(i)$ exists for which the maximum is achieved:

$$V_\beta(i) = \{u(i, \overline{j}(i)) + \beta V_\beta(\overline{j}(i))\}.$$

Thus a policy function $\overline{j}(i)$ is defined (possibly not in a unique way) which shows what action of the private agent in the node $i$ is the most desirable for the government. Fixing the policy function $\overline{j}(i)$ the government establishes an institutional system

$$P = (p^1(1, \overline{j}(1)), ..., p^n(n, \overline{j}(n))).$$

If horizon $T$ is sufficiently large this institutional system provides the maximum discounted utility for the government independently on initial state $i_0$. $\square$

## 4. An Almost Myopic Government

It was already said that a myopic behavior can be defined in two ways: an agent possesses either a zero horizon or a zero discount factor. The first definition seems to be more natural from the point of view of the game theory. However this definition allows for a big degree of indeterminacy in the agent's choice. For example, for the following utilities matrix

$$U = \begin{pmatrix} 2 & 5 & 4 & 3 & 2 \\ 3 & 1 & 3 & 2 & 3 \\ 3 & 5 & 4 & 1 & 5 \\ 2 & 5 & 1 & 5 & 3 \\ 5 & 1 & 3 & 2 & 4 \end{pmatrix},$$

a choice of an agent with a zero horizon is indeterminate in each of the nodes $2, 3, 5$. Thus the agent with a zero horizon faces a risk of a disadvantageous choice. For example, if the agent makes her choice in the node $i = 4$ then the nodes $j = 2$ and $j = 4$ are equally attractive for her in the moment, however the long-run consequences of the choice are different, despite the agent (at the present time) may be not interested in studying these consequences.

The second definition of a myopia (a zero discount factor) leads to a concept of an *almost myopic agent* trying to diminish this risk. A zero discount factor can be treated as a limit of a sequence of diminishing positive discount factors. This approach provides a solution not only for the myopic agent but for an agent with a small discount factor as well.

We assumed above that the private agents are absolutely myopic, however the government can be almost myopic having a small discount factor. Below in this Section we show how the dynamic programming problem described in the previous Section can be solved for an almost myopic government. The argument $P$ or $p^i$ will be omitted for convenience.

Let us call *lexicographic maximal* (*l.m.*) such a path $\{i_0, i_1, ...\}$ that for any path $\{\widehat{i}_0 = i_0, \widehat{i}_1, ...\}$ the following takes place: if $v(\widehat{i}_t, \widehat{i}_{t+1} > v(i_t, i_{t+1})$ for an index $t$ then such an index $k < t$ exists for which $v(i_k, i_{k+1}) > v(\widehat{i}_k, \widehat{i}_{k+1})$. Notice that any l.m. path $\{i_t\}$ is stepwise optimal, i. e.

$$i_{t+1} \in Arg \max_{j \in M} v(i, j), t = 0, 1, ...$$

The inverse is true if all the sets

$$Arg \max_{j \in M} v(i, j), i \in M$$

are one-element.

**Theorem 2.** *There exists a number $\overline{\beta} \in (0, 1)$ such that for all discount factors $\beta \in (0, \overline{\beta})$ solutions of the problems $S_\beta(i), i \in M$ are l.m. paths and only they.*

*Proof.* For an initial node $i_0$ consider an l.m. path $\tau = \{i_0, i_1, ...\}$ and an arbitrary path $\widehat{\tau} = \{\widehat{i}_0 = i_0, \widehat{i}_1, ...\}$ which is not l.m. Then

$$v(\widehat{i}_s, \widehat{i}_{s+1}) \neq v(i_s, i_{s+1})$$

for some $s \geq 0$,

$$v(\widehat{i}_t, \widehat{i}_{t+1}) = v(i_t, i_{t+1})$$

for $0 \leq t < s$ if $s \neq 0$.

Hence

$$v(i_s, i_{s+1}) > v(\widehat{i}_s, \widehat{i}_{s+1}),$$

$$\sum_{t=0}^{\infty} \beta^t v(i_t, i_{t+1}) - \sum_{t=0}^{\infty} \beta_t v(\widehat{i}_t, \widehat{i}_{t+1})$$

$$= \beta^t \{v(i_s, i_{s+1}) - v(\widehat{i}_s, \widehat{i}_{s+1}) + \beta \Sigma_{t=0}^{\infty} \beta^t [v(i_{s+1+t}, i_{s+2+t}) - v(\widehat{i}_{s+1+t}, \widehat{i}_{s+2+t})]\}$$

$$> \beta^s [A + \frac{\beta}{1 - \beta} B]$$

where

$$A = \min_{i,j,k,l:v(i,j)>v(k,l)} (v(i,j) - v(k,l)) > 0,$$

$$B = \min_{i,j,k,l} [v(i,j) - v(k,l)] = \min_{i,j \in a(i)} v(i,j) - \max_{k,l \in a(k)} v(k,l) < 0.$$

If $\beta < A/(A-B)$ then $A + \beta B/(1-\beta) > 0$ and hence

$$\sum_{t=0}^{\infty} \beta^t v(i_t, i_{t+1}) > \sum_{t=0}^{\infty} \beta^t v(\widehat{i}_t, \widehat{i}_{t+1}),$$

thereby $\tau$ is a solution of the problem $S_\beta(i_0)$, and $\widehat{\tau}$ is not. □

*Example 1.* For the utility matrix $U$ pointed out above we have $A = 1, B = -4, A/(A-B) = 0.2$. The value $\beta = 0.2$ can be used.

Now let us describe an algorithm for constructing all l.m. paths. An example of its application is provided below. On the first iteration of the algorithm for each node $i \in M$ a number

$$a_i = \max_{i=1,...,n} v(i,j)$$

is calculated and a set of "successors" of the node $i$ is found:

$$A^{(1)}(i) = Arg \max_{j=1,...,n} v(i,j).$$

Let $h^{(1)}$ be a number of nonrecurring values among $a_i, i \in M$. The set of the nodes $M$ is decomposed into disjoint subsets (*classes*)

$$H_s^{(1)}, s = 1, ..., h^{(1)}$$

in such a way that nodes with equal values $a_i$ are located in the same class. The classes are numerated in ascending order of the values $a_i$. The order number $r^{(1)}(i)$ of the class containing the node $i$ is called a *rating* of the node $i$.

On the $k$-th iteration $k > 1$ the sets of successors and the ratings received on the previous iteration are modified in the following way. For $i = 1, ..., n$ the value

$$\overline{r}^{(k-1)} = \max_{j \in A^{(k-1)}(i)} r^{(k-1)}(j)$$

is calculated and it is set that

$$A^{(k)}(i) = Arg \max_{j \in A^{(k-1)}(j)} r^{(k-1)}(j).$$

If a class $H_s^{(k-1)}, s = 1, ..., h^{(k-1)}$ contains more than one element and the values $\overline{r}^{(k)}(i), i \in H_s^{(k-1)}$ differ, then this class is decomposed into new classes which are ordered in ascending order of the values $\overline{r}(i)$ . All the classes (both those conserved unchanged and newly created) are numerated anew, and after that to each node $i \in H_s^{(k)}$ a rating $r^{(k)}(i) = s$ is assigned.

The work of the algorithm is finished when all the sets $A^{(k)}(i), i = 1, ..., n$ become one-element or when all the sets $A^{(k)}(i), i = 1, ..., n$ and $H_s^{(k)}, s = 1, ..., h^{(k)}$ stabilize. Not more than $n$ iterations are needed for the stabilization.

Notice that on the $k$-th iteration the set of successors $A^{(k)}(i)$ is a set of nodes which immediately follow the node $i_0 = i$ on $k$-step l.m. paths. The ratings $r^{(k)}(i)$ provide a possibility to compare among themselves $k$-step l.m. paths initiating in different nodes.

*Example 2.* Let us construct l.m. paths for the case of the utility matrix $U$ given above in the beginning of the Section.

1-st iteration:
$A^{(1)}(1) = \{2\}, r^{(1)}(2) = 1,$
$A^{(1)}(2) = \{1, 3, 5\}, r^{(1)}(1) = r^{(1)}(3) = r^{(1)}(4) = r^{(1)}(5) = 2,$
$A^{(1)}(4) = \{2, 4\},$
$A^{(1)}(5) = \{1\}.$

2-nd iteration:
$\overline{r}^{(2)}(1) = 1, A^{(2)}(1) = \{2\}, r^{(2)}(2) = 1,$
$\overline{r}^{(2)}(2) = 2, A^{(2)}(2) = \{1, 3, 5\}, r^{(2)}(1) = 2,$
$\overline{r}^{(2)}(3) = 2, A^{(2)}(3) = \{5\}, r^{(2)}(3) = r^{(2)}(4) = r^{(2)}(5) = 3,$
$\overline{r}^{(2)}(4) = 2, A^{(2)}(4) = \{4\},$
$\overline{r}^{(2)}(5) = 2, A^{(2)}(5) = \{1\}.$

3-rd iteration:
$\overline{r}^{(3)}(1) = 1, A^{(3)}(1) = \{2\}, r^{(3)}(2) = 1,$
$\overline{r}^{(3)}(2) = 3, A^{(3)}(2) = \{3, 5\}, r^{(3)}(1) = 2,$
$\overline{r}^{(3)}(3) = 3, A^{(3)}(3) = \{5\}, r^{(3)}(5) = 3,$
$\overline{r}^{(3)}(4) = 3, A^{(3)}(4) = \{4\}, r^{(3)}(3) = r^{(3)}(4) = 4,$
$\overline{r}^{(3)}(5) = 2, A^{(3)}(5) = \{1\}.$ ,

4-th iteration:
$\overline{r}^{(4)}(1) = 1, A^{(4)}(1) = \{2\}, r^{(4)}(2) = 1,$
$\overline{r}^{(4)}(2) = 4, A^{(4)}(2) = \{3\}, r^{(4)}(1) = 2,$
$\overline{r}^{(4)}(3) = 3, A^{(4)}(3) = \{5\}, r^{(4)}(5) = 3,$
$\overline{r}^{(4)}(4) = 4, A^{(4)}(4) = \{4\}, r^{(4)}(3) = 4,$
$\overline{r}^{(4)}(5) = 2, A^{(4)}(5) = \{1\}, r^{(4)}(4) = 5.$

Thus l.m. paths go through the contours $\{1, 2, 3, 5, 1\}$ and $\{4\}$.

An influence of the discount factor $\beta$ on the asymptotics of optimal paths in models with a continuous set of states was studied by (Deneckere and Pelican, 1986) and (Boldrin and Montucchio, 1986) who demonstrated a possibility of a complex dynamics if the discount factor is between 0 and 1 but is far from the bounds of the segment. For models with a finite set of states such kind of dynamics was studied by (Matveenko, 1998).

## 5. An Example of the Game

An example of our model arises as a version of a repeated Prisoner's Dilemma where the first player establishes an institutional environment and then monitors the actions of the second player and, in fact, directs her behavior by use of the established institutions.

*Example 3.* The gains in the base game Prisoner's Dilemma are taken from (Mueller, 2003):

| Strategies | Not steal | Steal |
|---|---|---|
| Not steal (S) | (10,9) | (7,11) |
| Steal(T) | (12,6) | (8,8) |

The first player commits to execute 'governmental' functions and announces her further actions (i.e. an institutional system). Now it seems proper not to use terms 'not steal' and 'steal' in connection with the first player (the government) but to speak about her soft (S) and tough (T) actions. Notice that a resulting iterated game is not a game with simultaneous moves: the government answers the actions of the private agents but commits her reaction in advance.

The first player has 16 strategies (versions of an institutional system $P$). Three of them deserve a special attention. The first two institutional systems (Figures 1,2) are equally perfect, from the point of view of the government, if the agents are patient, what means here $2/3 < \widehat{\beta} < 1$ . The institutional system represented in Fig. 1 is a Tit-for-Tat or a grim strategy (it suits also for a standard simultaneous moves version of an iterated Prisoner's Dilemma).

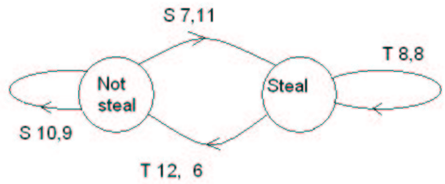


**Fig.1.** A Tit-for-Tat strategy of the first player and corresponding gains of the players. Under this institutional system the best way of behavior for a patient second player is not to steal, however an inpatient second player will steal.

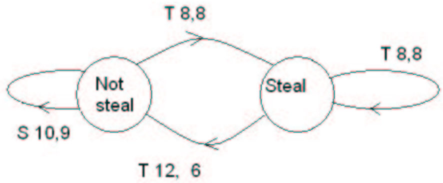


**Fig.2.** A 'tougher' strategy of the first player. If private agents are patient this institutional system provides the same gains to the players as the Tit-for Tat strategy. An impatient agent will continue to still if she steals initially.

It is easy to see that under the Tit-for-Tat strategy an impatient agent with a discount factor $0 < \widehat{\beta} < 2/3$ will continue to steal if she stole initially and will alternate stealing/not stealing if she did not steal initially. Under the 'tougher' strategy (Fig.2) an inpatient agent will not steal if she did not steal initially, however she will continue to steal if she did initially. In case of an impatient agent the best institutional system, from the point of view of the government, is shown in Fig. 3.

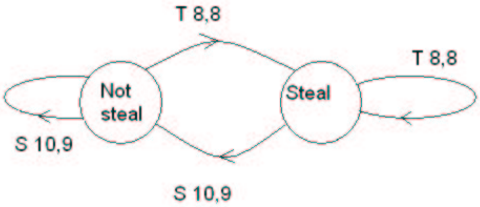


**Fig.3.** An institutional system which is the best, from the point of view of the government, in case of impatient private agents.

Now let us assume that the private agents are myopic and hence looking for their paths stepwise. The government can solve the game by use of the method described in Section 3. On the first step, a correspondence is constructed between arcs $s = (i, j) \in \overline{N}$ and institutions $p^i(s)$ under which the arcs are chosen by private agents. Notice that instead of 16 alternative full institutional systems $P$ we deal now only with 4 institutions $p^i$ in node $i =$ "*not steal*" and 4 institutions in node "*steal*".

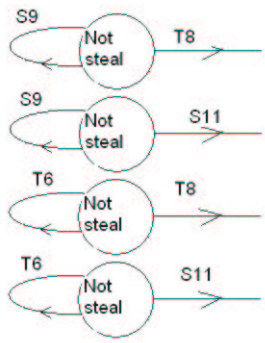


**Fig.4.** Four possible institutions in node "*not steal*" and gains of the second player.

In node "*not steal*" (see Fig. 4) the private agent will continue not to steal if the government plays S in response to non-stealing and T in response to stealing. Thus institution (S,T) corresponds to the arc (*not steal*, *not steal*). Under each of the three other institutions the private agent will steal. In this case the best play for the government is (T,T). Thus institution (T,T) corresponds to the arc (*not steal*, *steal*).

In node "*steal*", in a similar way, the following correspondence between arcs and institutions takes place. The arc (*steal*, *not steal*) corresponds to (S,T) played by the government in response to non-stealing and stealing correspondingly, and the arc (*steal*, *steal*) corresponds to (T,T).

On the second step, a dynamic programming problem is being solved with a matrix of gains of the first player corresponding to the institutions $p^i(s)$ found on the first step (see Fig. 5).

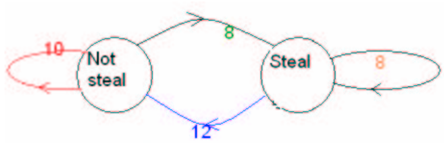


**Fig.5.** The dynamic programming problem of the first player.

It is easy to calculate the value function for our example:

$$V(Notsteal) = \frac{10}{1-\beta},$$

$$V(Steal) = \frac{12-2\beta}{1-\beta}.$$

In more complex cases a value function for such a problem can be often received as a result of a recurrent process of multiplication

$$x_{t+1} = A \bigotimes x_t, t = 0, 1, ...$$

where $x_0$ is an arbitrary $n$-dimensional initial vector with positive elements, $n$ is the number of nodes in the graph, $A$ is the matrix of gains of the first player, in our case

$$A = \begin{pmatrix} 10 & 8 \\ 12 & 8 \end{pmatrix},$$

and the multiplication of a matrix and a vector is defined in a natural way using elementary operations $a \bigotimes b = a + \beta b, a \bigoplus b = \max\{a, b\}$. A justification of the method and a discussion of conditions for its applicability are provided in (Matveenko, 1998).

The value function allows to find a policy function, i.e. to identify in all nodes desirable (for the first player) actions of the second player. For our example (see Figure 5 ) these actions are (*not steal*, *not steal*) and (*steal*, *not steal*).

On the third step the policy function is used to select corresponding institutions from the set of institutions found on the first step. As we could expect we come to the third institutional system described above in Fig. 3 which, as we remember, was suitable for impatient private agents.

## 6. Some Applications of the Model

Not only a Prisoner's Dilemma but some other kinds of iterated games (Snowdrift, etc.) can be studied by use of our model. Practical applications of such games are numerous: race of arms, corruption and anti-corruption, power safety, etc. Let us consider an example of a game of power safety.

*Example 4.* The second player - industrial countries - has two strategies: to increase or not to increase oil consumption. The first player - oil producing countries - can restrict or not restrict their oil production:

| Strategies | Increase consumption | Not increase consumptions |
|---|---|---|
| Restrict production | (5,-1) | (1,1) |
| Not restrict production | (3,3) | (2,2) |

OPEC usually uses a Tit-for-Tat strategy: it restricts production if buyers increase demand and does not restrict production if the demand is stable. Average gains of players without discounting are 2 and 2. Russian government proposed a non-restricting concept of a power safety promising an increase in oil production in response to demand. Average gains in this case are higher: 3 and 3.

An important for Russia example of our model is a game of the government and researchers (see also (Kraynov and Matveenko, 2007a, Kraynov and Matveenko, 2007b )). It is known that Science and R&D sector in Russia is financed mostly by the government and is very ineffective in a practical sense. Nobel laureate in physics Jorez Alferov formulated it in such way: "Our science is first class, not business class". In particular, in 2003 the state sector of science included 463 institutes of the Russian Academy of Sciences. Russian government started a reform of the Russian science. They picked out several priority directions, such as nanotechnologies and try to create an institutional system forcing researchers to act in a way leading to developing the priorities. The question is: can such a reform be successful or not? In our model the nodes of the graph can be interpreted as different subjects (themes) of research and arcs as different ways to use results for a new research. Our model answers that if the researchers are myopic in their preferences then in principle the government is able to create a consistent institutional system to manage the science. The assumption of a myopia of the researchers seems to be rather likely in the present time. However, if the discount factor (or the horizon) of the researchers increase, the system of management will stop working.

Another, more general, conclusion of the model is that if a government wants to create a consistent institutional system it will try to use all possibilities (such as media, school education) to create myopic citizens. In history we can find a lot of confirmations of this thesis.

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# Mutual Mate Choice Problem with Arrivals*

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**Abstract** In this paper mutual choice problem is considered. The individuals from two groups want to choose the partner from the other group. We present the multistage game, where free individuals from different groups randomly match with each other at each stage. The individuals choose each other by the quality. The distributions of the qualities at the first stage are uniform on [0, 1]. If they accept each other, they create a couple and leave the game. In this case each receives the partner's quality as a payoff. The remained players go to the next stage. At each stage the groups are reinforced by new individuals. At the last stage the individuals accept the partners lest remain alone. Each player aims to maximize her expected payoff. In this paper the optimal strategies are obtained. Also the statement in which the payoff of the player is equal to the arithmetic mean of the qualities of the couple is considered.

**Keywords:** mutual choice, dynamic game, equilibrium.

## 1. Introduction

In this paper mutual choice problem is considered. In the problem the individuals from two groups (females and males or employers and workers, etc.) want to form a long-term relationship with a member of the other group, i.e. to create a couple. We present the dynamic game with $n$ stages where free individuals from different groups randomly meet with each other in each stage. At the first stage each player meets the partner with the quality, which is the uniform distributed on [0, 1] random variable. If they accept each other they create a couple and leave the game. In this case each receives a payoff. Since at each stage the players leave the game then the distribution of the players by the quality changes. At the last stage the individuals who don't create the couple receive zero. Each player aims to maximize her expected payoff.

In this paper we present the different models of mutual choice problem. In the first model each player receives the partner's quality as a payoff and at each stage the groups are reinforced by new individuals. The new individuals arrive into the game due to individuals who create the couple at the previous stages produce the posterity.

In the second model the player who creates the couple receives the arithmetic mean of the qualities of the couple. Additionally if a player creates a couple then she pay $c$ for the contract.

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Unlike the best-choice problem where only one side makes the choice in given problem the two-sided choice is considered. Such problems have the application in biology and sociology, for the modeling of the market relations. In the literature such problems are named mating problem (see Alpern and Reyniers, 2005) or job search problem (see McNamara and Collins, 1990). The full information mutual choice problem is studied in (Alpern and Reyniers, 2005). The no-information mutual choice problem is considered in (Eriksson *et al.*, 2007).

In this paper for finding of the Nash equilibrium the statement of Alpern and Reyniers (see Alpern and Reyniers, 2005) is used. For presented problems the equilibrium strategies are obtained and numerical results are given. It is proved that for the large number of the stage in the problem with arrived individuals the optimal thresholds tend to the thresholds of the full information one-sided best-choice problem.

## 2. Mutual Choice Problem with Arrived Individuals

In this section we use the terms of the mating problem, i.e. females and males want choose the partner. Suppose that both groups have the same cardinal number. The initial distributions of qualities are both uniform on [0, 1]. The females' quality is denoted by $x$ and the males' is denoted by $y$ ($x, y \in [0, 1]$). The corresponding group denote $X$ and $Y$. Consider the multistage game in which at each stage the random meetings of the free individuals are modeled. In the game the players create the couple with a highly ranked individual of other group. If any partner don't accept other then the couple isn't created and both players remain into the game. If free individuals accept each other the couple is created and leaves the game. In this case each receives the partner's quality as a payoff. At each stage the groups are reinforced by new individuals. We find the equilibrium in this game in the form of threshold strategies.

At the beginning consider the game with two stages. The players' strategies are defined by one threshold $w : 0 \le w \le 1$. Suppose the players who create the couple at the first stage produce the posterity which together with remained individuals go to the second stage. Denote the number of new individuals as $\Delta$.

At the second stage there are $N_1 = w + (1 - w)w + \Delta$ individuals in the game. The individuals distribute by the quality uniformly, i.e. $N_1 = (1 + \Delta)w + (1 - w)(w + \Delta)$.

The the density of the distribution of the players by the quality follows

$$f(x) = \begin{cases} \frac{1+\Delta}{N_1}, \ 0 \le x < w; \\ \\ \frac{w+\Delta}{N_1}, \ w \le x \le 1. \end{cases}$$

The optimal $w$ can be find by the formula

$$w = \int_0^1 x f(x) dx.$$

Substituting the values of $f(x)$ we get the equation

$$w = \frac{1 + \Delta}{N_1} \int_0^w x dx + \frac{w + \Delta}{N_1} \int_w^1 x dx,$$

from this equation for the optimal threshold $w$ follows

$$\Delta = \frac{w(1-3w+w^2)}{2w-1}.$$

**Table1.** The optimal thresholds for the two stage game for the different $\Delta$.

| $\Delta$ | 0 | 0.1 | 0.5 | 1 | 10000 |
|---|---|---|---|---|---|
| $w$ | 0.382 | 0.404 | 0.442 | 0.461 | 0.5 |

Since the number of the individuals who create the couple at the first stage is equal to $(1-w)^2$ then let $\Delta = (1-w)^2\alpha$ where $\alpha$ is birth rate, $\alpha \geq 0$.

We obtain the following equation

$$(1-w)^2\alpha = \frac{w(1-3w+w^2)}{2w-1}.$$

In Table 2 the values of the threshold $w$ are presented for the different values $\alpha$.

**Table2.** The optimal thresholds for the two stage game for the different $\alpha$.

| $\alpha$ | 0 | 0.1 | 0.5 | 1 | 5 | 10 | 100 | 1000 |
|---|---|---|---|---|---|---|---|---|
| $w$ | 0.382 | 0.391 | 0.414 | 0.430 | 0.469 | 0.481 | 0.498 | 0.5 |

In the case of three stages for the players from the group $Y$ we denote the thresholds of the acceptance as $w_1$ and $w_2$ for the first and the second stages respectively. At the second stage there are $N_1 = w_1 + (1-w_1)w_1 + \Delta_1$ individuals in the game

Find the general number of the individuals at the third stage.

$$N_2 = w_2(1+\Delta_1) + \tfrac{(w_1-w_2)(1+\Delta_1)w_2(1+\Delta_1)}{N_1} + \tfrac{(1-w_1)(w_1+\Delta_1)w_2(1+\Delta_1)}{N_1} + \Delta_2,$$

where $\Delta_2$ is the number of new individuals at the third stage.

Note that

$$N_2 = w_2(1+\Delta_1)\left[2 - \tfrac{w_2(1+\Delta_1)}{N_1}\right] + \Delta_2.$$

The density of the distribution of the players by the quality at the second stage was found above. Denote its as $f_1(x)$,

$$f_1(x) = \begin{cases} \frac{1+\Delta_1}{N_1}, & 0 \leq x < w_1; \\ \frac{w_1+\Delta_1}{N_1}, & w_1 \leq x \leq 1. \end{cases}$$

Then the density of the distribution of the players by the quality at the third stage equal to

$$f_2(x) = \begin{cases} \frac{1+\Delta_1+\Delta_2}{N_2}, & 0 \le x < w_2; \\ \frac{\frac{w_2(1+\Delta_1)^2}{N_1}+\Delta_2}{N_2}, & w_2 \le x < w_1; \\ \frac{\frac{w_2(w_1+\Delta_1)(1+\Delta_1)}{N_1}+\Delta_2}{N_2}, & w_1 \le x \le 1. \end{cases}$$

We obtain the system of the equation which define the optimal thresholds $w_1$ and $w_2$,

$$\begin{cases} w_2 = \int\limits_0^1 y f_2(y)dy, \\ w_1 = \int\limits_0^{w_2} w_2 f_1(y)dy + \int\limits_{w_2}^1 y f_1(y)dy \end{cases}$$

or

$$\begin{cases} w_2 = \int\limits_0^{w_2} y\frac{1+\Delta_1+\Delta_2}{N_2}dy + \int\limits_{w_2}^{w_1} y\frac{\frac{w_2(1+\Delta_1)^2}{N_1)}+\Delta_2}{N_2}dy + \int\limits_{w_1}^{1} y\frac{\frac{w_2(w_1+\Delta_1)(1+\Delta_1)}{N_1}+\Delta_2}{N_2}dy, \\ w_1 = w_2\int\limits_0^{w_2} \frac{1+\Delta_1}{N_1}dy + \int\limits_{w_2}^{w_1} y\frac{1+\Delta_1}{N_1}dy + \int\limits_{w_1}^{1} y\frac{w_1+\Delta_1}{N_1}dy. \end{cases}$$

Let $\Delta_1 = \Delta_2 = \Delta$. In Table 3 the values of the threshold $w_1$ and $w_2$ are presented for the different values $\Delta$.

**Table3.** The optimal thresholds for the three stage game for different $\Delta$.

| $\Delta$ | 0 | 0.1 | 0.5 | 1 | 10000 |
|---|---|---|---|---|---|
| $w_1$ | 0.482 | 0.515 | 0.558 | 0.575 | 0.606 |
| $w_2$ | 0.322 | 0.366 | 0.417 | 0.434 | 0.461 |

Let the number of new individuals at the second stage is $\Delta_1 = (1-w_1)^2\alpha$ and at the third stage is

$$\Delta_2 = \left[(w_1-w_2)(1+\Delta_1)\left(1-\frac{w_2(1+\Delta_1)}{N_1}\right)+(1-w_1)(w_1+\Delta_1)\left(1-\frac{w_2(1+\Delta_1)}{N_1}\right)+(1-w_1)^2\right]\alpha.$$

In the expression for the $\Delta_2$ the number of the individuals who create the couples at the second and third stages stands in square brackets.

In Table 4 the numerical results for the optimal thresholds $w_1$ and $w_2$ are given for different values $\alpha$.

Since at the each stage new individuals arrive into the game then the optimal thresholds must be greater then the thresholds in the game without the reinforcing of the groups ($\Delta_1 = \Delta_2 = 0$).

Then consider the game with $n+1$ stages. The player from the group $Y$ uses the threshold strategy $w_1, ..., w_n$, where $0 < w_n \le w_{n-1} \le ... \le w_1 \le w_0 = 1$. At

**Table4.** The optimal thresholds for the three stage game for different $\alpha$.

| $\alpha$ | 0 | 0.1 | 0.5 | 1 | 5 | 10 | 100 | 1000 |
|---|---|---|---|---|---|---|---|---|
| $w_1$ | 0.482 | 0.498 | 0.529 | 0.548 | 0.590 | 0.603 | 0.622 | 0.625 |
| $w_2$ | 0.322 | 0.346 | 0.391 | 0.416 | 0.467 | 0.480 | 0.498 | 0.5 |

that at each stage the groups increase in $\Delta_i$ individuals , $i = 1, .., n$. To getting of the optimal thresholds we find the density of the distribution of the players by the qualities at each stage. For that find the general number of the individuals at each stage.

Initially the distribution is uniform. After first stage the players with the quality greater than $w_1$ can create the couples and leave the game. Since at the second stage new individuals arrive into the group then there are $N_1 = w_1 + (1 - w_1)w_1 + \Delta_1$ players in the game. After the second stage the players with the quality greater than $w_2$ can create the couples and also leave the game. At the third stage there are $N_2 = w_2(1+\Delta_1) + \frac{(w_1-w_2)(1+\Delta_1)w_2(1+\Delta_1)}{N_1} + \frac{(1-w_1)(w_1+\Delta_1)w_2(1+\Delta_1)}{N_1} + \Delta_2$ individuals.

At the stage $i+1$ $(i = 1, ..., n)$ the general number of the individuals in each group is equal to

$$N_i = w_i(1+\sum_{j=1}^{i-1}\Delta_j) + (w_{i-1}-w_i)(1+\sum_{j=1}^{i-1}\Delta_j)\frac{w_i(1+\sum_{j=1}^{i-1}\Delta_j)}{N_{i-1}}$$
$$+\sum_{k=2}^{i-1}(w_{k-1}-w_k)\left[\left[...[1+\Delta_1]\frac{w_2(1+\Delta_1)}{N_1} + ... + \Delta_{i-1}\right]\frac{w_i(1+\sum_{j=1}^{i-1}\Delta_j)}{N_{i-1}}\right.$$
$$\left.+(1-w_1)\left[\left[...[w_1+\Delta_1]\frac{w_2(1+\Delta_1)}{N_1} + ... + \Delta_{i-1}\right]\frac{w_i(1+\sum_{j=1}^{i-1}\Delta_j)}{N_{i-1}} + \Delta_i.\right.\right.$$

Otherwise its can be written in the following form

$$N_i = w_i(1+\sum_{j=1}^{i-1}\Delta_j)\left[2 - \frac{w_i(1+\sum_{j=1}^{i-1}\Delta_j)}{N_{i-1}}\right] + \Delta_i. \tag{1}$$

At that $\Delta_i$ can be found by the formulas $\Delta_i = \alpha\sum_{j=1}^{i-1}\bar{N}_j$, where $\bar{N}_i$ is the number of the individuals who create the couples at the $i$-th stage,

$$\bar{N}_i = N_{i-1} - \left[w_i(1+\sum_{j=1}^{i-1}\Delta_j) + (w_{i-1}-w_i)(1+\sum_{j=1}^{i-1}\Delta_j)\frac{w_i(1+\sum_{j=1}^{i-1}\Delta_j)}{N_{i-1}}\right.$$
$$+\sum_{k=2}^{i-1}(w_{k-1}-w_k)\left[\left[...[1+\Delta_1]\frac{w_2(1+\Delta_1)}{N_1} + ... + \Delta_{i-1}\right]\frac{w_i(1+\sum_{j=1}^{i-1}\Delta_j)}{N_{i-1}}\right.$$
$$\left.\left.+(1-w_1)\left[\left[...[w_1+\Delta_1]\frac{w_2(1+\Delta_1)}{N_1} + ... + \Delta_{i-1}\right]\frac{w_i(1+\sum_{j=1}^{i-1}\Delta_j)}{N_{i-1}} - \Delta_i\right].\right.\right.$$

Then the density of the distribution of the players by the quality at the stage $i+1$ $(i=1,...,n)$ is following

$$f_i(x)=\begin{cases} \dfrac{1+\sum\limits_{j=1}^{i}\Delta_j}{N_i}, & 0\le x<w_i;\\[2ex] \dfrac{(1+\sum\limits_{j=1}^{i-1}\Delta_j)\frac{w_i(1+\sum_{j=1}^{i-1}\Delta_j)}{N_{i-1}}+\Delta_i}{N_i}, & w_i\le x<w_{i-1};\\[2ex] \dfrac{\left[\left[...[1+\Delta_1]\frac{w_2(1+\Delta_1)}{N_1}+...+\Delta_{i-1}\right]\frac{w_i(1+\sum_{j=1}^{i-1}\Delta_j)}{N_{i-1}}+\Delta_i\right.}{N_i}, & w_k\le x<w_{k-1};k=2,...,i-1;\\[2ex] \dfrac{\left[\left[...[w_1+\Delta_1]\frac{w_2(1+\Delta_1)}{N_1}+...+\Delta_{i-1}\right]\frac{w_i(1+\sum_{j=1}^{i-1}\Delta_j)}{N_{i-1}}+\Delta_i\right.}{N_i}, & w_1\le x\le 1. \end{cases} \tag{2}$$

Now get the equations of the optimality by backward induction. The player's optimal expected payoff after $i$-th stage denote as $v_i(x), i=1,...,n$ if she deals with the partner with quality $x$. Then the equation of the optimality follows

$$v_n(x)=\max\{x,\int_0^1 yf_n(y)dy\},$$

$$v_i(x)=\max\{x,Ev_{i+1}(x_{i+1})\}, i=1,...,n-1.$$

Substituting expression (2) for the densities $f_i(x)$ we get the following confirmation

**Theorem 1.** *Nash equilibrium in the $(n+1)$-stage mutual choice game with arrived individuals is defined by thresholds $w_i$, $i=1,...,n$, which satisfy the recursion*

$$\begin{cases} w_n=\int\limits_0^1 xf_n(y)dy;\\ w_i\ =\int\limits_0^{w_{i+1}} w_{i+1}f_i(y)dy+\int\limits_{w_{i+1}}^1 yf_i(y)dy, i=1,2,...,n-1, \end{cases} \tag{3}$$

*where $f_i(x)$ is defined by expressions (2)*

Find the limiting expression for the optimal thresholds as the large $\alpha$.

**Lemma 1.** *For all $i=1,...,n$ $\lim\limits_{\alpha\to\infty} f_i(x)=1$.*

*Proof.* Proof is conducts by induction.

As $i=1$ we obtain

$$f_1(x)=\begin{cases} \frac{1+\Delta_1}{w_1+(1-w_1)w_1+\Delta_1}, 0\le x<w_1;\\[1ex] \frac{w_1+\Delta_1}{w_1+(1-w_1)w_1+\Delta_1}, w_1\le x\le 1. \end{cases}$$

Since $\Delta_1 = (1-w_1)^2\alpha$, then as $\alpha \to \infty$ $f_1(x) \to 1$ at the each interval.
Suppose that the proposition is true for $i = 1, ..., n-1$.
Prove its for $i+1$.
Consider $f_i(x)$ on interval $0 \leq x < w_i$.
Taking into account (1) we get

$$\frac{1+\sum\limits_{j=1}^{i}\Delta_j}{N_i} = \frac{1+\sum\limits_{j=1}^{i-1}\Delta_j + \Delta_i}{w_i(1+\sum\limits_{j=1}^{i-1}\Delta_j)\left[2-\frac{w_i(1+\sum\limits_{j=1}^{i-1}\Delta_j)}{N_{i-1}}\right]+\Delta_i}.$$

Analyse $N_i$ and $\Delta_i$: $N_i$ has power $i$ under variable $\alpha$, $\Delta_i$ also has power $i$. Therefore, taking into account $\lim\limits_{\alpha\to\infty}\frac{1+\sum\limits_{j=1}^{i-1}\Delta_j}{N_{i-1}} = 1$ we get that the limit of this expression as $\alpha \to \infty$ is equal to 1.

Analogically given proposition is proved for $f_i(x)$ on the intervals $w_k \leq x < w_{k-1}, k = 2, ..., i-1$, and $w_1 \leq x \leq 1$. Since $\lim\limits_{\alpha\to\infty}\frac{1+\sum\limits_{j=1}^{i-1}\Delta_j}{N_{i-1}} = 1$, we obtain the approval of the lemma.

By Lemma 1 $f_i(x) \to 1$ as $\alpha \to \infty$. Then (3) takes the following form.

**Corollary 1.** *As $\alpha \to \infty$ the optimal thresholds $w_i$ $(i = 1, ..., n)$ satisfy the following recurrent expressions*

$$w_i = \frac{1+w_{i+1}^2}{2}, i = 1, ..., n-1; w_n = 1/2. \tag{4}$$

Sequence (4) is named Moser sequence. It appears in the one-sided full information best-choice problem (see Moser, 1956). Thus, the solution of the mutual choice problem with arrived individuals for the large birth rate $\alpha$ agrees with the solution in the one-sided full information best-choice problem.

**3. Mutual Choice Problem with Cost of the Contract**

In the second model the player who creates the couple in the first stage receives as a payoff the arithmetic mean of the qualities of the couple. At the beginning consider the game with two stages. The players' strategies are defined by one threshold $u : 0 \leq u \leq 1$. Hence the rule of acceptance is to create the couple at the first stage if $\frac{x+y}{2} \geq u$, where $u$ is the expected payoff at the second stage, denote $2u = w$. Furthermore at the each stage player who create the couple pays additional cost of the contract $c$. We obtain the optimal value $w$ and the optimal strategies for players depending on the cost $c$.

At the beginning consider the case $0 \leq w \leq 1$.

The general number of individuals in each group at the second stage is equal to

$$N_1(w) = \int\limits_0^w (w-x)dx = w^2/2.$$

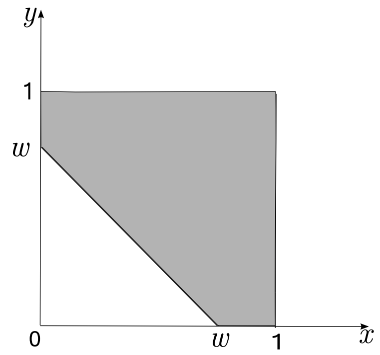


**Fig.1.**

The density of the distribution of the qualities at the second stage is following

$$f_1(x,w)=\begin{cases}\frac{w-x}{N_1(w)},\ 0\le x\le w;\\ \\ 0, \qquad w<x\le 1.\end{cases}$$

The optimal $w$ for the player is obtained from the equation

$$w/2-c=\frac{1}{N_1(w)}\int\limits_0^w x(w-x)dx,$$

$$w=6c.$$

The case $w\ge 1$ is presented on the Figire 1.

The general number of individuals in each group at the second stage is equal to

$$N_1(w)=w-1+\int\limits_{w-1}^{1}(w-x)dx=2w-w^2/2-1.$$

The density of the distribution of the qualities at the second stage is following

$$f_1(x,w)=\begin{cases}\frac{1}{N_1(w)},\ 0\le x\le w-1;\\ \\ \frac{w-x}{N_1(w)},\ w-1<x\le 1.\end{cases}$$

The optimal $w$ for the player is obtained from the equation

$$w/2-c=\frac{1}{N_1(w)}\int\limits_0^{w-1} xdx+\frac{1}{N_1(w)}\int\limits_{w-1}^{1} x(w-x)dx;$$

$$w/2-c=\frac{1-3w^2+w^3}{6-12w+3w^2}.$$

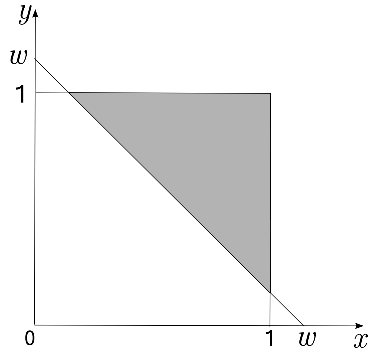


**Fig.2.**

Consider the case $n = 3$. Denote $w_1, w_2$ — the thresholds of the acceptance at each stage, $w_1 \le 1$. The general number of individuals in each group at the second stage is equal to $N_1(w_1)$ and the density of the distribution of the qualities at the second stage is equal to $f_1(x, w_1)$.

The general number of individuals in each group at the third stage is equal to

$$N_2(w_1, w_2) = \int\limits_0^{w_2} \frac{(w_2 - x)(w_1 - x)}{N_1(w_1)} dx.$$

The density of the distribution of the qualities at the third stage is following

$$f_2(x, w_1, w_2) = \begin{cases} \frac{(w_2 - x)(w_1 - x)}{N_1(w_1) N_2(w_1, w_2)}, & 0 \le x \le w_2; \\ 0, & w_2 < x \le 1. \end{cases}$$

The optimal $w_1, w_2$ for the player are obtained from the equations

$$\begin{cases} \frac{w_2}{2} - c = \int\limits_0^1 x f_2(x, w_1, w_2) dx, \\ \frac{w_1}{2} - c = \frac{1}{2} \left[ \int\limits_0^{w_2} w_2 f_1^{X+Y}(z, w_1) dz + \int\limits_{w_2}^1 z f_1^{X+Y}(z, w_1) dz \right], \end{cases}$$

$f_1(z, w_1)^{X+Y}$ is the density of the distribution of the sum of variables $X$ and $Y$.

We obtain the following equation

$$\begin{cases} \frac{w_2}{2} - c = \int\limits_0^{w_2} x \frac{(w_2 - x)(w_1 - x)}{N_1(w_1) N_2(w_1, w_2)} dx, \\ \frac{w_1}{2} - c = \frac{1}{2} \left[ \int\limits_0^{w_2} w_2 \left( \int\limits_0^z \frac{(w_1 - t)(w_1 - z + t)}{(N_1(w_1))^2} dt \right) dz + \int\limits_{w_2}^{w_1} z \left( \int\limits_0^z \frac{(w_1 - t)(w_1 - z + t)}{(N_1(w_1))^2} dt \right) dz \right. \\ \left. \qquad + \int\limits_{w_1}^{2w_1} z \left( \int\limits_{z - w_1}^{w_1} \frac{(w_1 - t)(w_1 - z + t)}{(N_1(w_1))^2} dt \right) dz \right]. \end{cases}$$

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# Ideal Money and Asymptotically Ideal Money

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## Revolutionary or Evolutionary Changes or Reforms of Systems of Money

Our topic is focused on an ideal, specifically on "Ideal Money", and it is not hard to see that there are naturally different routes by which a system of money might become either improved or might become, in some senses, more degraded and less worthy of praise. Change can come at a stroke, like when Alexander cut the Gordian knot, or it can come in a gradual fashion, through many smaller steps, and this latter can be classed as the pathway of "evolutionary change".

It is easy to illustrate cases of "revolutionary" reform or change in systems of money. A good example came in 1717 when Isaac Newton, supported by George II, fixed the value of the local UK currency to a precise amount of gold that defined the value of the currency (the "pound") in such a way that it was immediately recognizable throughout the Continent (of Europe) as of a fixed value in relation to generally accepted standards (of the time). (And this was the origin of the "gold standard".)

Another example of revolutionary change was when Argentina attempted to establish an internationally respectable system of money by means of a "currency board". (This attempt failed conspicuously, but the failure was rather similar to a bankruptcy event involving an ordinary commercial bank which simply turned out to have insufficient "capital".)

When the use of paper and printing was developed in China that made possible a "revolutionary" change, namely the introduction of paper money.

Jacques Rueff, F. A. von Hayek, and R. Mundell are notable scholars and economists who have particularly contributed to the theories of how a system or systems of money might be improved in an effectively revolutionary fashion. For example there has been a quite dramatic improvement in the (internationally perceived apparent) quality of the money used in the countries of Italy and Greece simply because they have moved through the revolutionary transition of renouncing the use of the lira or the drachma and have accepted the use of the newly established "euro" unit.

## Evolutionary Changes and Relevant Teachings

On the other hand (from the case of "revolutionary" changes) there is often the possibility that a system of money may gradually improve in quality, either through somewhat accidental circumstances (like a very favorable trade balance) or through the learning of good teachings of applicable varieties.

A series of American economists have been notable through their contributions which have enhanced the understanding of how systems of money actually function and particularly of how the dollar (US) and its value have been interacting with the relevant factors of influence. There has always been some "populist" thinking in the USA which can encourage ideas about money that are not well based in any scientific sense. And the teachings of some of the notable economists have sometimes given a more scientific perspective on the areas where the "populist" viewpoints have been influential.

M. Friedman acquired fame through teaching the linkage between the supply of money and, effectively, its value. In retrospect it seems as if elementary, but Friedman was as if a teacher who re-taught to American economists the classical concept of the "law of supply and demand", this in connection with money.

We can also note at this point that the understanding of the effects of the uncontrolled behavior of all the various "users" of a domestic money is the inclusive category of description into which the notable contributions of a series of American economists can be recognized. F. Kydland, R. Lucas, E. Phelps, and E. Prescott are notable American economists who have contributed to the better understanding of issues arising in the area of theories of "macroeconomics". Without arguing for a direct constitutional reform of the status quo of the dollar in the USA they have contributed much enlightenment in relation to the interactions between intelligent categories of the "users" of currencies (or in particular the dollar) and "the central authorities" (of central bank, treasury, state institutions, executive and legislative government).

The evolving recognition of the fact that the "users" of a currency become like players in a game and have optional strategies by means of which they will be able to seek to optimize according to their own particular economic interests leads to the recognition that the tasks of central planners and managers, of a state, are not as simple as if they had only to herd flocks of sheep.

Thus the "users", like the managers, can be viewed as players in interactive games. In particular, with this perspective, it is natural to think of the users as having "expectations" in relation to the future value of the domestic currency, compared either with real assets, foreign currencies, or indices of costs. These expectations may or may not be "well-founded" or "rational" but they will inevitably guide or influence the choices made by the "users".

### General Considerations and History

The special commodity or medium that we call money has a long and interesting history. And since we are so dependent on our use of it and so much controlled and motivated by the wish to have more of it or not to lose what we have we may become irrational in thinking about it and fail to be able to reason about it as if about a technology, such as radio, to be used more or less efficiently.

We present the argument that various interests and groups, notably including "Keynesian" economists, have sold to the public a "quasi-doctrine" which teaches, in effect, that "less is more" or that (in other words) "bad money is better than good money". Here we can remember the classic ancient economics saying called "Gresham's law" which was "The bad money drives out the good". The saying of Gresham's is mostly of interest here because it illustrates the "old" or "classical" concept of "bad money" and this can be contrasted with more recent attitudes which have been very much influenced by the Keynesians and by the results of their influence on government policies since the 30's.

**Digression on the Philosophy of Money**

It seems to be relevant to the politics of state decisions that affect the character of currency systems promoted by states that there are typical popular attitudes in relation to money. Although money itself is merely an artifact of practical usefulness in human societies and/or civilizations, there are some traditional or popular views associating money with sin or immorality or unethical or unjust behavior. And such views can have the effect that an ideal of good money does not seem such a good cause as an ideal of a good public water supply. There is also, for example, the Islamic concept which has the effect of classing as "usury" any lending of money at interest. (Here we can wonder about what sort of inflation rates might have been typical for any major varieties of money, such as Byzantine money, at the times actually contemporaneous with the Prophet Mohammed.)

In general, money has been associated in popular views with moral or ethical faults, like greed, avarice, selfishness, and lack of charity. But on the other hand, the existence of money often makes it easy to make valuable donations of philanthropic sorts and the parties receiving such contributions tend to find it most helpful when the donations are received as money!

But the New Testament story about "money changers" being driven from the Temple illustrates clearly the idea of putting the clearly mundane and possibly "unclean" utility of money at some distance from where that money would presumably continue to be received when used as a vehicle for donations.

Economics has been called "the dismal science" and it is certainly an area of studies where "the mundane" is appropriately studied.

And philosophically viewed, money exists only because humanity does not live under "Garden of Eden" conditions and there are specializations of labor functions. So we are always exchanging, mediated by money transfers, the differing fruits of our varied forms of labor.

**Welfare Economics**

A related topic, which we can't fully consider in a few paragraphs, is that of the efforts to be made by the national state and society in general for dealing with "social equity" and concerns for the general "economic welfare". Here the key viewpoint is methodological, as we see it. HOW should society and the state authorities seek to improve economic welfare generally and what should be done at times of abnormal economic difficulties or "depression"?

We can't go into it all, but we feel that actions which are clearly understandable as designed for the purpose of achieving a "social welfare" result are best. And in particular, programs of unemployment compensation seem to be comparatively well structured so that they can operate in proportion to the need. And public works projects allow the wealthy to pay through taxes to provide jobs for workers and these can produce valuable works if the projects are well planned.

**Money, Utility, and Game Theory**

In the sort of game theory that is studied and applied by economists the concept of "utility" is very fundamental and essential. Von Neumann and Morgenstern give a notably good and thorough treatment of utility in their book (on game theory and economic behavior). The concept of utility (mathematical) does indeed predate the book of Von Neumann and Morgenstern. And for example, as a concept, mathematical utility can be traced back to a paper published in 1886 in Pisa by G. B. Antonelli.

When one studies what are called "cooperative games", which in economic terms include mergers and acquisitions or cartel formation, it is found to be appropriate and is standard to form two basic classifications:

(1): Games with transferable utility.
(and)
(2): Games without transferable utility
(or "NTU" games).

In the world of practical realities it is money which typically causes the existence of a game of type (1) rather than of type (2); money is the "lubrication" which enables the efficient "transfer of utility". And generally if games can be transformed from type (2) to type (1) there is a gain, on average, to all the players in terms of whatever might be expected to be the outcome.

But this function of money in generally facilitating the transfer of utility would seem to be as well performed by the currency of Zimbabwe as by that of Switzerland. Or the question can be asked "How do 'good money' and 'bad money' differ, if at all, for the valuable function of facilitating utility transfer?". But if we consider contracts having a relatively long time axis then the difference can be seen clearly.

Consider a society where the money in use is subject to a rapid and unpredictable rate of inflation so that money worth 100 now might be worth from 50 to 10 by a year from now. Who would want to lend money for the term of a year?

In this context we can see how the "quality" of a money standard can strongly influence areas of the economy involving financing with longer-term credits.

And also, if we view money as of importance in connection with transfers of utility, we can see that money itself is a sort of "utility", using the word in another sense, comparable to supplies of water, electric energy or telecommunications. And then, if we think about it, we can consider the quality of money as comparable to the quality of some "public utility" like the supply of electric energy or of water.

**"Keynesians"**

The thinking of J. M. Keynes was actually multidimensional and consequently there are quite different varieties of persons at the present time who follow, in one way or another, some of the thinking of Keynes. And of course SOME of his thinking was scientifically accurate and thus not disputable. For example, an early book written by Keynes was the mathematical text "A Treatise on Probability".

The label "Keynesian" is convenient, but to be safe we should have a defined meaning for this as a party that can be criticized and contrasted with other parties.

So let us define "Keynesian" to be descriptive of a "school of thought" that originated at the time of the devaluations of the pound and the dollar in the early 30's of the 20th century. Then, more specifically, a "Keynesian" would favor the existence of a "manipulative" state establishment of central bank and treasury which would continuously seek to achieve "economic welfare" objectives with comparatively little regard for the long term reputation of the national currency and the associated effects of that on the reputation of financial enterprises domestic to the state.

And indeed a very famous saying of Keynes was " . . . in the long run we will all be dead . . . ".

**A Critique of the Science of the Keynesians**

It is difficult to make a criticism here because so much of the scientific research work, particularly of American economists, in the years since, say, "the thirties", has been in the area of the study of the topic called "macroeconomics" and most or almost all of this work has a "Keynesian" orientation.

I think there is a good analogy to mathematical theories like, for example, "class field theory". In mathematics a set of axioms can be taken as a foundation and then an area for theoretical study is brought into being. For example, if one set of axioms is specified and accepted we have the theory of rings while if another set of axioms is the foundation we have the theory of Moufang loops.

So, from a critical point of view, the theory of macroeconomics of the Keynesians is like the theory of plane geometry without the axiom of Euclid that was classically called the "parallel postulate". (It is an interesting fact in the history of science that there was a time, before the nineteenth century, when mathematicians were speculating that this axiom or postulate was not necessary, that it should be derivable from the others.)

So I feel that the macroeconomics of the Keynesians is comparable to a scientific study of a mathematical area which is carried out with an insufficient set of axioms. And the result is analogous to the situation in plane geometry, the plane does not need to be really flat and the area within a circle can expand hyperbolically as a function of the radius rather than merely with the square of the radius. (This picture suggests the pattern of inflation that can result in a country, over extended time periods, when there is continually a certain amount of gradual inflation.)

The missing axiom is simply an accepted axiom that the money being put into circulation by the central authorities should be so handled as to maintain, over long terms of time, a stable value.

Instead of this one can observe, in the context of the popularity of "Keynesian" orientations, that it is considered extremely undesirable that there should ever occur a period of deflation (where wages and prices might be forced to decrease) but that continual inflation is an acceptable consequence (of whatever actually causes it under the effective circumstances of the actual "management" of a national money system).

Looking backwards, in the period of time between 1717 and 1931 the Bank of England actually had to operate with the axiom accepted that we are viewing as comparable to Euclid's "parallel postulate". The theory of what can be done, in central banking, with a money value axiom being in effect is not an empty theory but this is an area which seems hardly to have been studied at all since the advent of "the Keynesians" in "the thirties".

Another aspect of "Keynesianism", in relation to scientific themes, is that it seems to me to be very much like a school of medical theory and to be oriented towards "therapeutic" procedures. But often a school of medical practice can be criticized from one or another point of view. For example, "What are the long term consequences of the continued application of the procedures of therapy?"

### The Machiavellian Perspective

A serious study of the phenomena of paper money or coinage as issued by state authorities would not be complete without consideration of a Machiavellian analysis of the "con games" that arise whenever the quality level of a money may seem different to different types of appraisers. And Machiavelli is very notable as an early "nonmathematical" game theorist (!!).

The issuer of a state-sponsored "legal tender" is comparable to the person of "Il Principe" in the writings of Machiavelli. And the Prince (in the Machiavellian sense) naturally has a circle of advisors and counselors (some of whom may be qualified to be called macro-economists or economists).

The advisors to the Prince will typically find it easier and more strategically wise not to criticize the fundamental structure of the Prince's provision, for his Principality, of a specific medium usable to facilitate exchanges of utility. And financial institutions, in the Principality, may have become adapted over time periods like at least a generation or maybe of several generations to the specific characteristics of the money system, perhaps the "legal tender", that is provided in the Principality.

If the (effective) position of being the Prince is rotating or like a political office with a "term limit" then it can easily happen that one Prince will want to spend heavily, on his own most favored projects, before the next Prince will come to power with his own quite different agenda and perceived system of preferences for state expenditures and taxation. And a current Prince may not infrequently be able to spend additional money without immediately raising taxes, thinking to leave that burden to his successor and to his successor's (legislative) Government. (And also such a Prince can naturally think that if his successor finds that the Treasury is relatively bare of resources and that tax income is limited that that successor will be discouraged from heavily spending on his own pet areas (which might be viewed as undesirable from the viewpoint of the current Prince).)

Thus, viewed in this fashion, systems of economic foundations (for labor, business, and exchanges) that have actually many areas of deficiency compared with the ideal possibilities (which can be imagined by consideration of foundations of a more ideal quality); these systems can yet persist over long time periods in a manner similar to that of the persistence of political and governmental systems that are ultimately judged to have been of an inferior or unfavorable sort.

From 1917 to 1989 (dating to the fall of the "Berlin wall") an economic system existed in Europe that, arguably, failed to efficiently motivate human entrepreneurial labor through a system of materially valuable rewards to the (entrepreneurial) workers.(In the future Socialism may, possibly, find a good solution for the problem of providing motivation for innovative works of practical value.)

And we can't really logically assume that human civilization has found the ultimate ideal of forms of social government in the times of the twentieth century. (One can imagine a future form of government where a highly advanced automaton (or array of computers) would function like the office of a City Manager with the human input to the government passing through the analogue of a City Council.)

### Ideal Money

Our proposal is that a preferable version of a general system for the transferring of utility, thus a "medium of exchange", would be structured so as to provide a medium with a natural (and reliable!) stability of value. And this stability of value would be particularly of benefit in connection with contracts or exchanges involving long time periods for the complete performance of the contract or exchange.

Classically, when gold or silver was used as the basis of a standard for exchanges, that objective was consequently achieved (even though neither of these two "precious metals" would be, in fact, perfectly stable in value by comparison with the other). The existence of a standard provided comparative certainty contrasting with the gamblers' situation that results when a lender must lend money without much of an assurance that in 30 years the value of it will not have been greatly eroded by inflation. Thus, faced with such value uncertainties, mortgage lenders must learn to lend, if they are lending their own money, at sufficiently high interest rates so as to have a fair chance of winning their gamble against inflation!

We published a paper entitled "Ideal Money" in the Southern Economic Journal (in 2002) and it was essentially the text of a keynote lecture that we gave on that topic at the meeting of the Southern Economic Association in Tampa, Florida. Of course, necessarily, on a topic with such a universal relevance to human affairs, it is difficult, really, to say something new. But there can be novelty in the details and in terms of the context and the times.

Our key proposal was/is that an index that can be called an ICPI or "Industrial Consumption Price Index" could be employed as a basis for the standardization of the value of money. This proposal is for an index based on the international prices of specific goods. For example like the prices for silver or copper as recorded daily at London.

The commodities or utilities or services for which their international prices could be used in an ICPI index should be wisely chosen so as to avoid those that might have comparatively rapidly changing prices. Exactly how an index should be constituted cannot be specified at this point but it can be noted that the problem of constituting a suitable index is quite analogous to that of constituting index measures for the prices of "Industrials" or "Transports" or "Utilities" like Dow Jones has long had for the stocks traded on the New York Stock Exchange. But of course one doesn't expect the value measure of a "basket" of commodities to rise as much, over long times, as the value of the Dow Jones Industrials index has risen in the past.

We also observed that a method of calculation could be employed that would use "moving averages" to achieve that the money value being defined would vary as smoothly and gradually as practicable with the passing of time.

But now we want to mention another possibility that arises because of the present day circumstances that are relevant to the international interactions of the various national currencies. It could be very difficult, and a slow process, to set up such a practical and useful system of conventions as the international metric system of measures (of length, volume, and weight). So it should not be expected that reform and progress, in the area of systems of money, will be very easily achieved.

Nowadays we see some new areas of competition between different major currencies of the world since the euro has come into existence and the psychological climate in which the "central bankers" are operating is recently changed by the theme that is next described.

**The Confessional of Targeting**

It was the observation of a new "line" that has become popular with those responsible for "central banking" functions relating to national currencies that gave us the idea for the theme of "asymptotically ideal" money.

The idea seems paradoxical, but by speaking of "inflation targeting" these responsible officials are effectively CONFESSING that, notwithstanding how they formerly were speaking about the difficulties and problems of their functions, that it is indeed after all possible to control inflation by controlling the supply of money (as if by limiting the amount of individual "prints" that could be made of a work of art being produced as "prints").

This popularity of the line of "inflation targeting" seems to have started in New Zealand, which is the place, among the USA, Canada, Australia, and New Zealand, which had the most depreciated dollar. And we can note also that New Zealand was hardly a place where any crisis of poverty really forced them to not maintain the value of their dollar but rather just a place where "Keynesian" thinking was probably very influential.

If now we think of a world of a number of major currencies and with all of these provided by central authorities that operate under some sort of a ritual of "inflation targeting" then, as things evolve, what SHOULD the targets be?

It is only really respectable that there should not be an arbitrary or capricious pattern of inflation, but how should a proper and desirable form of money value stability be defined?

Rapid inflation is easily measured, on a national level, by a domestically defined "cost of living" index. So if the cost of living, as measured by another agency than central banking authorities, were not rising (when expressed in terms of the domestic money) then one could feel assured that there was not inflation.

However this requirement is actually a little too strong (for a properly good money worthy to be called of "ideal" type)! It is actually quite natural for the calculated "cost of living" to be rising, even when measured, say, in terms of gold, whenever there is so much technological progress that the people in an area, without working harder, are lifted to a higher standard of living by the rapid progress, as if each human would become the beneficiary of the assistance of 3 robot helpers to do the work of his livelihood.

So in the last years of the era of the gold standard the "cost of living" measures were gradually rising, in "advanced countries", but it was not appropriate to view that as indicating inflation since the money was not losing value in relation to alternative options for "treasure hoarding", (such as gold!).

To be quite respectable, in a Gresham-advised sense, money needs only to be AS GOOD as other material commodities that might be hoarded. It does not really need to be so good (as time passes) that the cost of living statistic should remain constant.

But "inflation targeting", unless all major currencies would (somehow!) be able to be adopting and really employing the same target rate, would still provide the opportunity for "connoisseurs of quality" to rank the currencies in hierarchies of gradations of quality (like bond rating agencies rank the debt of commercial enterprises or like other rating agencies comparatively appraise various insurance companies). Those really having lower planned inflation rates would naturally be seen as superior in quality. (We should note that the INTERNATIONAL perspective relating to a currency is not how it relates to domestically measured costs in its home country but how it compares, on the international markets, with other currencies and commodities.)

What inflation targeting does is to open up the possibility that somehow the various major currencies may evolve to develop stability of value. And in this sense there could be "asymptotically ideal money" in that an evolving trend could lead to the value stability that would constitute a major improvement in quality.

## Currencies in Competition

It is observable that certain types of financial enterprises, such as large internationally operating insurance companies, tend to migrate to national homes where the national currency is of at least comparatively higher quality (such as, e. g., Switzerland).

In the near future there may be a smaller number of major currencies used in the world and these may stand in competitive relations among themselves. There is now the "euro" and the inflationary tradition of the Italian lira seems to be past history now. And there COULD be introduced, for example, a similar international currency for the Islamic world, or for South Asia, or for South America, or here or there.

And if "inflation targeting" were used as a "line" by the managers handling all of these various internationally prominent currencies then there would arise interesting possibilities for comparisons between these major currencies. Each of the currencies managed thusly would have its officially recognized status in terms of inflation as measured by the domestic index of costs of the state of the managers. But also, and this is what is more significant from an internationally oriented viewpoint, the various currencies would have rates of exchange so that they could be realistically compared in terms of their actual values.

And so the various currencies managed with "inflation targeting" would be comparable by users or observers who would be able to form opinions about the quality of the currencies. And what I want to suggest is that "the public" or the users, those for whom a medium of exchange functions as a basic utility, may develop opinions that are critical of currencies of lower "value quality". That is, the public may learn to demand better quality of that which CAN be managed to be of better quality or which can be managed to be of the lower quality observed in so many of the various national currencies in the 20th century.

So we can imagine the evolutionary possibility of "asymptotically ideal money". Starting with the idea of value stabilization in relation to a domestic price index associated with the territory of one state, beyond that there is the natural and logical concept of internationally based value comparisons. The currencies being compared, like now the euro, the dollar, the yen, the pound, the Swiss franc, the Swedish krona, etc. can be viewed with critical eyes by their users and by those who may have the option of whether or not or how to use one of them. This can lead to pressure for good quality and consequently for a lessened rate of inflationary depreciation in value.

Illustrating these optional choices that the public, the users of a money, may have, the people of Sweden recently had the opportunity of voting in a referendum on whether or not Sweden should join the eurocurrency bloc and replace the krona by the euro and thus use the same currency as Finland. The people voted against that, for various reasons. But it cannot be irrelevant whether or not the future quality of a currency is really assured or whether instead that it depends on the shifting sands of political decisions or the possibly arbitrary actions of a bureaucracy of officials.

The voters in the U.K. are expecting to have the opportunity to vote in a referendum relating to the adoption, for the U.K., of the euro (which is already adopted in Ireland). Here they have a dramatic conflict, since the pound was the original currency of "the gold standard", with its value pegged to gold in 1717 by Isaac Newton (who was then Master of the Mint).

In recent years the pound has had a comparatively good rating with regard to inflation, inferior to the rating of the Swiss franc but superior to most currencies of the world. So the British have the alternatives of accepting adoption of the euro when first voting, or after a delay, or never.

We can legitimately wonder how the speediness of its adoption or delays in its adoption might affect the policies operating to control the actual exchange value of the euro. The constitutional structure of the authority behind the euro is of the "paper money" character in that nothing is really guaranteed as far as the value of the euro is concerned. But this is typical of all currencies used in the world nowadays.

Of course when a currency, for a time, does have a specification of its value beyond that simply depending on supply and demand for a fiat money, like the money of Argentina had a peg to the U.S. dollar a few years ago, then international observers can wisely distrust the reliability of such a stabilization of its value. Such forms of value definition are not necessarily unsound, particularly when a small economy, like that of Panama, links its currency to that of a larger area like that of the USA. But it is obvious that this sort of thing puts a (paradoxical) burden on the foundation of the currency that is used as a reference basis.

For example, if all sorts of non-European countries decided to define the values of their currencies as on a par with the euro, without actually joining into any system of cooperative regulations associated with that, then the effect of that would seem likely to destabilize the stability of the euro if it would otherwise be highly stable and of good value quality.

### Insurance Companies, Commercial Banks, and State Banks

It can be difficult, psychologically, for good patriots to appreciate the comparison, but state banks, or whatever issues the money used in a state or in a group of states, are logically comparable to good or bad commercial banks or to good or bad insurance companies.

And it is observable that internationally operating commercial banks or insurance companies can be favored by being based where the conventional money is of relatively higher quality. The same principle also applies to the business of "investment banking" which is a differentiable specialty function of commercial banks or other financial companies.

### Savings, Savings Institutions, and Savings Rates

Another area where money quality is very relevant is in relation to the "savings rate". How will individual decision makers behave with regard to options for thrifty or more "spendthrift" behavior? It is arguable that the larger classes, in the sense of economically differentiated population strata, should be able to employ thrift options that are not extremely complex in character. And if the quality of the money is really good then simply to save in terms of the ordinary medium of exchange is at least a practical first step. So thus the existence of good money may naturally promote a higher "savings rate".

The history of "savings banks" and "credit unions" seems to illustrate social and economic developments that occurred during the time of stable money values of the gold standard era. Thus forms of financial institutions came into existence in the climate of "good money" which would not have evolved were the money of an obviously unstable value.

The process of capital investment by means of which enterprises prepare to have the competence for making successful products in the future naturally relates to the processes recognizable as involving savings decisions by individuals and households. And it can become as if paradoxical, when the official "savings rate" is found to be low, how the investment processes are occurring. The truth may be that the mass of the citizens of a state with an apparently advanced economy can become actually comparable to the people of an area being developed by colonialists and thus not be a leading force in relation to the advancement of the national economy.

### Relations to Law and Contracts

A concept that we thought of later than at the time of developing our first ideas about Ideal Money is that of the importance of the comparative quality of the money used in an economic society to the possible precision, as an indicator of quality, of the contracts for performances of future contractual obligations.

We have noted, as a matter of general theory, that money provides the practical means for the "transfer of utility" and that the distinction between "games with transferable utility" and "games with non-transferable utility" (or TU and NTU games) thus is naturally linked with the matter of whether or not there is available the means of money transfers to facilitate a good cooperative game solution (which could be that of a relatively simple game of bargaining).

But when there is the dimension of time also, incorporated into a contract for exchanges (such as for example a mortgage contract, or an annuity contract with an insurance company, or a contract for services to be performed over an extended period of time) then the quality of the money unit in terms of which the contract is written makes a big difference in the level of certainty of the contract terms.

Uncertainty perturbing the issue of the effective meaning of a contract is comparable to and analogous to a climate of lawlessness that would make contracts, in general, unreliable.

It is reasonable to expect that were the quality of a national currency very high (in terms of the stability of the value of the currency unit) then that interest rates on mortgages or on the national debt would become comparatively low (as "rational expectations" would interact with investment options for mortgage and investment bankers).

If there were only the alternatives of two varieties of money of which one of them would depreciate in value, compared with the other of them, AT A CONSTANT RATE, then it would be reasonable to expect that REAL interest rates, say for mortgages, could be the same whichever money were used. But the pattern to be expected when there is money that decreases in value compared with "real value" measures is that this continuous devaluation IS NOT AT A CONSTANT RATE (and the phenomenon of "surprise inflation", which has been much discussed by economists, is to be expected). (So of course, expected, it won't be entirely a "surprise", but yet, when "Keynesian" policies are strongly in effect, one must rationally expect inflation but also a degree of difficulty for precisely and quantitatively predicting that inflation!)

**Ideal Global Money, the Concept**

It is not a new concept in economic theory, that there could be material benefits whenever a number of separate currencies would be replaced by a single money. This caught the eye of John Stuart Mill, in particular.

But with a benefit there may also come the reverse of benefits and it is logical for human advisors of human societies to be wary when the world, in 2008, seems not yet ready for a globally effective federal government comparable to those operating on national levels in Berne or Washington.

My opinion is that if it were too easy to set up a form of "global money" that the version achieved might have characteristics of inferiority which would make it, comparatively, more like a relatively inferior national currency than like any of the more praiseworthy national or imperial currencies known to historical records.

But there is a good prospect for avoiding the establishment of another, possibly deceptive, currency of inferior quality. Here I think of the possibility that a good sort of international currency might EVOLVE before the time when an official establishment might occur.

We can observe that, in the world, that there continue to exist varieties of areas where religion, language, laws, and customs are quite variable. And one can suspect that indeed it is somewhat in the nature of Man and his cultures that this variableness is typical.

So my personal view is that a practical global money might most favorably evolve through the development first of a few regional currencies of truly good quality. And then the "integration" or "coordination" of those into a global currency would become just a technical problem. (Here I am thinking of a politically neutral form of a technological utility rather than of a money which might, for example, be used to exert pressures in a conflict situation comparable to "the cold war".)

A process analogous to this occurred when a number of European countries passed first through EFTA, then through the EU, then into an "exchange rates stabilization" and then into the structure of "the euro" (which is based in Frankfurt).

Another topic to consider is that if a few large-scale currencies are interacting then that the agencies (or states) responsible for each of these currencies might benefit by holding certain amounts of currency reserves in terms of the other major currencies. And also the authority managing one currency might calculate an appropriate index to define the standard value for that currency by using the observed values of the other major world currencies. (We mentioned before the use of less volatile values in connection with "smoothing out" the calculations of a normative ICPI index that would define a standard value for a currency.)

At the present time it can observed that the currency of China has a value stabilized in relation to an index based on the values of other currencies. So in the future it could be that the currency sponsored by Japan or by the Southeast Asia

Economic Cooperation Area might become stabilized in value with the aid of a index including the value of the currency of China.

**Evolution of Customs and Opinions**

In a large state like one of the "great democracies" it is reasonable to say that the people should be able, in principle, to decide on the form of a money (like a "public utility") that they should be served by, even though most of the actual volume of the use of the money would be out of the hands of the great majority of the people. But most typically the people would expect to be served by their elected representatives and not to make most of the relevant decisions in a direct fashion.

If it becomes a matter of strong and definite preferences that the money used should have definite characteristics of quality then, in principle, the people can demand that. For example formerly there was the drachma and now there is, in Greece, the euro instead of that. And the people seem to be pleased with the change.

So the quality of the medium or media of exchange that is/are used can be improved, if the improvement is really desired. Here we speak of quality in the sense of Gresham or like a bond rating agency.

But the famous classical "Gresham's Law" also reveals the intrinsic difficulty. Thus "good money" will not naturally supplant and replace "bad money" by a simple Darwinian superiority of competitive species. Rather than that, it must be that the good things are established by the voluntary choice of human agencies. And these responsible agencies, being naturally of the domain of politically derived authorities, would need to make appropriate efforts to achieve such a goal and to pay the costs that are entailed before their societies can benefit. And the benefits would come from the improvement in the quality of this public utility (money) which serves to facilitate the game-theoretic function of "the transfer of utility".

An example of an efficiently working global reform (at least in relation to electronic manufactures) is the metric system, with its central Bureau located near Paris. There is an example of a system of yardsticks where inflation is currently NOT in fashion.

# Studying Cooperative Games Using the Method of Agencies

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My work in this area really goes back, in inspiration, to 1996, when I thought of the basic idea. And since then I have been involved in concretely developing and testing this method (or approach) through computations relating to mathematical models of games of at first two and then three players. At the time, in 1996, I was actually in West Virginia at a summer science camp event that is called the National Youth Science Camp. I gave a general lecture about the topic of the relation between game theory and "theoretical biology" and I focused on how the biologists had considered biological contexts analogous to the "Prisoners' Dilemma" game that had been much studied by game theorists.

Then after I had made my presentation, and while I was still at the camp, the idea came to me that the theme studied with game theory by the theoretical biologists (the theme of the natural evolution of cooperation) could be the foundation for a non-arbitrary theoretical approach to the puzzle of the proper "evaluation" of cooperative games (so that we would gain an idea of how much the situation of being a player in a specific position of a specific game should be worth to that player).

This thinking evolved into the "method of agencies". We start with a game of three persons that is for simplicity assumed to be fully describable by a "characteristic function" which specifies the payoff utility amount that is immediately accessible to a single player or to any set of the players (as a coalition of players or as an individual player).

Then we consider, as a game played repeatedly, a transform of the original game which has the rule that coalitions can only be formed by means of elective actions of "acceptance" by which one player (or one agent) accepts the agency function of another player (or agent). The accepting party assigns all of his valuable power in the game to the party being accepted. And, as a matter of language, any player thus "accepted" would become an "agent" and would be in a position analogous to that of a lawyer who had been granted a "power of attorney".

The concept of agencies allows that a coalition of two of the players can be effectively formed if either of the two elects the other player to become the authorized agent representing the interests of both of the two players. In the case of a game only of two players that elected agent would already represent "the grand coalition" (or all the cooperative possibilities for the game). And in the case of a game of three players we naturally introduce the possibility for a second stage of cooperative coalescence so that an elected agent-player can either elect to accept representation by the remaining ("solo") player or, alternatively, such an agent-player may be accepted/elected by the remaining ("solo") player and thus become the representative of all three of the players.

We model the game situation as that of a repeated game (as if the game is infinitely or indefinitely repeated, with no "discounting") and we seek to find an equilibrium in that context. So this equilibrium concept is quite parallel to the concept of equilibrium under evolutionary pressures (or "natural selection") in Nature. Then we seek to find for the players, which are parties that have quite limited actual opportunities for bargaining and negotiative actions (because they have nothing like the full range of human verbal communications possibilities that they could use in the process of optimizing), a type of equilibrium such that each of the parties to the game is not capable of making any refinement of his/her strategic behavior pattern that would improve his/her payoff expectations prospect.

The mathematical work, to find the equilibria (and to find how they vary in three-person games as the sub-coalitions (of two players) vary in strength), becomes a matter of computations to find numerical approximate solutions of equations with as many as 42 variables, and this work is made feasible by modern computer resources and by software like Mathematica.

In principle, this work is equivalent to carrying out an exhaustive experiment on the behavior of specialized robot players that are designed to effectively bargain or negotiate for obtaining favorable outcomes in terms of "the division of the spoils" as regards the payoffs realized from the cooperative game.

I have recently been moving on into the consideration of variant modelings for the same type of three person games (which are "CF" games described entirely by the listing of a "characteristic function" giving the payoffs realizable by the separate action of the members of any sub-coalition and that available to the "grand coalition" of all players). These would be models involving "attorney-agents" that enter into the game like new players and represent specific coalitions. The attorney-agents would be robotic in their motivation (rather than like human lawyers) and that simplifies the analysis. So we hope thus to obtain illuminating data valuable for making comparisons of different model variants.

And one of the paradoxes connected with the possibility of modeling the coalition formation actions with robotic attorney-agents is that that could lead to an actual reduction of the number of strategic parameters that we would need to determine to calculate an equilibrium. (When we first considered the possibility of attorney-agents it SEEMED as if using them would necessarily lead to greatly more complex models!)

**Remarks on Complexity and Computation**

I spoke on the work of formulating and solving the equations for the equilibria in the work on these game models when I was invited to be a keynote speaker at the International Mathematica Symposium of 2003 in London. As mentioned above, for a game without a symmetry of the players, it became a matter of finding, numerically, the solution of 42 equations in 42 variables. (But if the defining payoffs data of the characteristic function made two of the players symmetrically situated then, to find similarly symmetric equilibria became a matter of solving 21 equations in 21 variables, and this is where most of our project calculating work was done.)

A game of merely three players would seem, at first consideration, to be a problem not very much more complicated, for theory, than a game of only two players. But this really seems to be true only for strictly non-cooperative games. (The recreational, but risky, game of "poker" is a good illustration of a naturally conventionally non-cooperative game.)

For a long time I have thought of the challenges of cooperative games of three or more players as being possibly parallel to the challenges historically faced by mathematicians seeking to find formulae for the roots of algebraic polynomial equations of degrees higher than two. The first breakthroughs on finding formulae for the roots of those equations did not come until late Renaissance times in Italy (while the analogous formula for quadratic equations was known to the Chaldeans). And then, as a consequence of those developments, mathematicians went on to the introduction of complex numbers (and this made it possible to always have understandable intermediate quantities when working with the discovered formulae for the roots (which could themselves all be real) of the equations of cubic or quartic degree.

It does seem that COOPERATIVE games of three or more players will naturally bring in much more complexity of structure and interpretation than games of two players or than non-cooperative games of three or more players, provided that we can actually develop a sort of theory for them that seems to be more than arbitrarily conventional.

But there is no general consensus yet among game theorists regarding what sort of theory should be developed or found.

## Demands and Acceptance Probabilities in the Case of Two Players

We first worked out the function of players' "demands" controlling their "probabilities of acceptance" (in a repeated game context) for the case of games of a simple bargaining type of two players. We present an explanation of this to prepare for and facilitate explaining the modeling structure for three (or more) players.

Originally, in our first trials of the new ideas, we studied a model bargaining problem where the set of accessible possibilities was enclosed by a parametrically described algebraic curve (forming the Pareto boundary). This was arranged so as to have a natural bargaining solution point at (u1=1/2, u2=1/2) (referring to the players' utility functions). The total bargaining problem was asymmetric, but modulo the theory of localized determination of the solution point, it was such that (u1,u2) = (1/2,1/2) should be the compromise bargain.

(We were surprised, however, when we found that if we used (as described below) different "epsilon numbers" (see below) for the players that that difference would unbalance the model's selection of a bargaining solution (!). Later, thinking about it, we realized that the use of appropriately matching epsilon numbers for the players could be naturally justified. In a game problem with transferable utility (like with our studies for 3 players) this amounts to using THE SAME epsilon number for all players.)

For Player 1 we let d1 stand for his demand (number) and e1 for his epsilon-number. The "epsilons" make the "reactive" behavior of a player depend smoothly on the numbers that the players choose as parameters of strategy so that we can obtain the system of equations to be solved for the equilibria by differentiating a player's expected payoff function with respect to each of the strategy parameters that that player controls. Then his "acceptance rate" a1 is defined in terms of these numbers plus also the data number u1b2 which is "the amount of utility that Player 1 is given by the Player 2 when Player 2 has become the agent for both of the players (and has selected a point on the Pareto boundary)" (and this data is observable by

Player 1 simply through the known history of Player 2's behavior in the repeated game).

We need a specific rule of relationship between d1 and a1 and this is (was) specified by the relations:

$$A1 = \text{Exp}[\ (u1b2 - d1)/e1\ ], \text{ and } a1 = A1/(1{+}A1).$$

(Which formulae have the effect that A1 is positive and that a1 is like a positive probability, between 0 and 1.) In a completely dual fashion, for Player 2:

$$A2 = \text{Exp}[\ (u2b1 - d2)/e2\ ], \text{ and } a2 = A2/(1{+}A2)\ .$$

As we remarked already, we discovered from the calculations that we needed to use e1 = e2 if we wanted desirable results! (But it seems that this can be justified as "impartial" if we consider another means for introducing probabilistic uncertainty affecting the consequences of demands; in particular, if the uncertainty resulted from "fuzziness" about the knowledge of the precise location of the Pareto boundary then that version of ignorance would affect the players in an impartial fashion.)

In the case of two players the players would simultaneously vote, with each player voting either to accept the other as the general agent or voting, in effect, for himself/herself instead. Then our first idea was to apply an "election rule" declaring that an election was void if both of the players voted for accepting (the other player) and only effective if only one player made a voting choice of acceptance. So then we specified that the election should be repeated with a certain probability (say probability (1-e4)) whenever both players had voted acceptance votes (and if that retrying process ultimately failed then the players finally were given the null reward {0,0} (in utilities) for failure to cooperate!).

This complicated the "payoff formula" somewhat but the VECTOR of payoffs, {PP1,PP2}, was ultimately calculable as functionally dependent on a1, a2, u1b2, and u2b1. (In this listing the utility amounts were regarded as resulting from STRATEGY CHOICES by the players where P1 would actually choose {u1b1,u2b1} as a point chosen BY P1 (!) on the Pareto boundary curve. So, from the curve, u1b1 could be interpreted as a function of u2b1, with P1 interpreted as simply choosing u2b1 strategically.)

The vector {PP1,PP2} becomes a function of a1, a2, u1b2, u2b1, u1b1, and u2b2 and this reduces to the 4 quantities first listed because of u1b1 and u2b2 being determined by the Pareto curve.

And a1 is a function of d1 and u1b2 with a2 similarly controlled by d2 and u2b1.

So, ultimately, we arrive at 4 equations in four variables for the condition of equilibrium. These derive from the partial derivatives of the payoff function for a player taken with respect to the parameters describing his strategic options.

Thus there are the partial derivatives of PP1 with respect to d1 and with respect to u2b1 and these are both to vanish. Then there are two dual equations derived from PP2.

(Later we learned that differentiating PP1 with respect to a1 directly, rather than viewing a1 as a function of d1, would give a simplified (but equivalent) version of the d1-associated equation.)

**Details of the Modeling for Three Players**

When there are three players instead of two we need to arrange to have two successive stages for "acceptance votes" where any one player could vote to accept the (unconstrained!) agency function of any other player. This principle continues to apply to players who had already themselves become agents. So two steps of coalescence of this sort result necessarily in the achievement of the "grand coalition" in the form that all three of the players are represented by one of them who, as the agent acting for the other two, can access all the resources of the grand coalition (which are simply v(1,2,3) since we simplify by considering a "CF game" that is DEFINED by the characteristic function given for it).

For the specific modeling we simplify further by having v(1,2,3)=1 and v(1)= v(2)=v(3)=0 and we call (for convenience with Mathematica, etc.) the values of the two-player coalitions by the names b3=v(1,2), b2=v(1,3), and b1=v(2,3). These three numbers, b1, b2, and b3 define the games of the family we studied. We finally obtained graphs illustrating how the calculated payoffs (to the players, based on our model) would vary, as b3 (or b1 and b2) would vary, compared with similar graphs for the Shapley value and the nucleolus (which are calculable for any CF game).

At the first stage of elections (in which every player is both a candidate to become an agent and also a voter capable of electing some other player to become his authorized representative agent) there are six possible votes of acceptance and we described the probabilities for each of these by the parameter symbols a1f2, a1f3, a2f1, a2f3, a3f1, and a3f2. (a3f2, for example, is the probability of the action of P3 (Player 3) to vote for P2 (which is to vote to accept P2 as his elected agent).

These probabilities, like all of the voting probabilities in the model, need to be related to demands, as we will explain.

After the first stage of elections is complete and one agency has been elected (Note that this requires some precision of the election rules that we need to specify.) then there remains one "solo player" and one coalition of two players of which one of the two (like a strong committee chairman) has been elected to be the empowered agent acting for both of them.

Then for the second stage of elections there are 12 numbers that describe the probabilities of "acceptance votes" (but only two of these numbers are truly relevant in connection with each of the six possible ways in which an agency had been elected as the first stage of agency elections). These numbers are a12f3 and a3f12, a13f2 and a2f13, a21f3 and a3f21, a23f1 and a1f23, a31f2 and a2f31, and a32f1 and a1f32.

Thus a12f3 is the probability of a vote by P1 representing the coalition, led by P1, of P1 and P2, voting for his (and his coalition's) acceptance of P3 as the final agent (and thus for P3 as effective leader, finally, of the grand coalition). Alternatively a3f12 is the probability of a vote by P3, as a solo player, to accept the leadership of the (1,2) coalition (as led by P1) to become also effective as his enabled agency and thus to access the resources of the grand coalition.

With the election process we need rules that specify simple outcomes (eliminating tie vote complications, etc.) so what we used was that if in any election there was more than one vote of acceptance that then a random event would select just one of those (two or three) acceptances to become the effective vote. This convention suggested the naturalness of allowing an election to be repeated when none of the voting players had voted for an acceptance.

The convention of repeating failed elections seemed to be a very favorable idea. (It also seems to favor some of our projected refinements or extensions of the modeling, as we explain later.) So, as variable parameters affecting the model structure, we introduced "epsilons" called e4 and e5 where the probability of repeating a failed election AT THE FIRST STAGE OF AGENCY ELECTIONS would be (1-e4) (this is expected to be a "high probability") and the similar probability applying in the event of election failures at the second stage would be (1-e5).

(We will say more below about the benefits of having elections that are typically repeated when no party votes.)

Besides the set (presented above) of 18 numbers describing the probabilities for votes of acceptance there is a set of 24 numbers that describe how the players choose (this is a strategy choice!) to allocate utility among themselves and these numbers are linked with the 12 differentiable possibilities for how some individual player happened to be elected to become the final agent.

These numbers are {u2b1r23,u3b1r23}, {u2b1r32,u3b1r32},
{u2b12r3,u3b12r3}, {u2b13r2,u3b13r2}, {u1b2r13,u3b2r13},
{u1b2r31,u3b2r31}, {u1b21r3,u3b21r3}, {u1b23r1,u3b23r1},
{u1b3r12,u2b3r12}, {u1b3r21,u2b3r21}, {u1b31r2,u2b31r2},
and {u1b32r1,u2b32r1}.

The notational pattern is that, e.g., u1b2r31 represents "the quota of utility allocated to Player 1 by Player 2 in situations where Player 2 was elected as final agent by the coalition of Players 3 and 1 when this coalition was led by Player 3". The "hidden allocations" are like u2b2r31 and these must be non-negative. u2b2r31 would be the amount that Player 2 would allocate to himself in this situation. Of course u2b2r31 = 1 - u1b2r31 - u3b2r31 because the resources, v(1,2,3), of the grand coalition are simply +1.

Another generic case is like u3b13r2 where the final agent (Player 1 here) was previously the agent in control of a coalition (coalition (1,3) here) and he allocates u3b13r2 to Player 3 and u2b13r2 to Player 2.

So these uxbxxxx numbers must all lie between zero and +1.

### Relations of Demands and Acceptance Probabilities

For most of the cases, in the modeling of the games of three players, the relations between the acceptance probabilities and the controlling "demands" (which demands are parameters that are strategy choices of the players) are natural extensions of the comparable relations for two player games (and this is simply because MOST of these numbers actually relate to "second stage" elections where the field is reduced to just two voters and two candidates!).

Thus we specify that a12f3 is to be controlled by a "demand" d12f3 which is made by Player 1, who is the competent voter in the situation (which is that (1,2) is led by Player 1 and that P3 is "solo"). The formulae controlling the relation mathematically are

$$\text{a12f3} = \text{A12f3}/(1 + \text{A12f3}) \text{ (with)}$$

$$\text{A12f3} = \text{Exp}[\ (\text{u1b3r12 - d12f3})/\text{e3}\ ]\ .$$

This is actually EXACTLY LIKE the relation used for a bargaining game of two players. (u1b3r12 corresponds to u1b2 there.) But here the perspective is thus:

"Player 1 is leading (1,2) and considering whether or not to accept P3 as the final agent, so in relation to this he looks at the utility payoff, u1b3r12, that he WOULD BE ALLOCATED by Player 3 in the event of that (effective) acceptance and he compares that number with his demand d12f3 and this comparison (modulated by e3) controls Player 1's probability of voting to accept Player 3 in the situation". (Note incidentally that Player 1 here appears as acting entirely in his selfish interest and disregarding any concerns of the (represented) Player 2 (!).)

Similarly we specify, for a2f13 as typical, that

$$\text{a2f13} = \text{A2f13}/(1 + \text{A2f13}) \text{ (with)}$$

$$\text{A2f13} = \text{Exp[ (u2b13r2 - d2f13)/e3 ]} .$$

So for the 12 acceptance probabilities relating to the possibilities for votes at the second stage of elections there are linked 12 demand numbers, as described above.

But for the first stage of elections the version of modeling (perhaps not optimal) that we happened to use had three demands that controlled the six a-numbers a1f2, etc. that applied to that stage of the process of elections. This was because we only allowed that a player should choose a single demand number with d1, d2, and d3 being these choices. Then each player's choice of his "demand" controlled BOTH of his probabilities for voting for acceptance (of one or another of the two other players).

Thus a relationship between d1 and the pair of a1f2 and a1f3 was created so that Player 1's (strategy) choice of d1 modulated his BEHAVIOR (as described by a1f2 and a1f3. In this relationship we used calculated utility expectation measures that we called q12 and q13. Here, to illustrate, q12 is "the expectation of the average receipt of utility, by P1, conditional on the assumption that P1 has achieved acceptance of P2 (at the first stage of elections) so that the coalition (2,1), led by Player 2 is formed to (enter into) play at the second stage of elections". This quantity q12 happens to be calculable entirely from e3, e5, and the quantities that describe the behavior of players P2 and P3.

The governing formulae relating d1 to a1f2 and a1f3 are then these:

$$\text{a1f2} = \text{A1f2}/(1 + \text{A1f2} + \text{A1f3}) \text{ and a1f3} = \text{A1f3}/(1 + \text{A1f2} + \text{A1f3}) \;;$$

with

$$\text{A1f2} = \text{Exp[ (q12 - d1)/e3 ] and A1f3} = \text{Exp[ (q13 - d1)/e3 ]} .$$

The structure is that A1f2 is a non-negative number which is large or small depending on how the rewards to be expected by P1 when P1 would manage to accept the agency offered by P2 compare with d1 while A1f3 similarly depends on the prospects if P1 becomes an acceptor of P3. Then the formulae (on the first of the lines of equations just above) give the definitions or constructions of a1f2 and a1f3 such that these can be the probabilities of exclusive events (since either P1 can vote for accepting P2 or P1 can vote for P3 (similarly) or P1 can simply decline to make any vote for an acceptance.

(The expressions derivable for q12 and q13 are not very long and they are dual under symmetry of P2 and P3, so for illustration,

$$\begin{gathered}\text{q12} = \text{((1 - a21f3)*(1 - a3f21)*b3*e5 + 2*a21f3*u1b3r21 + a3f21*((2 -}\\ \text{a21f3)*u1b21r3 - a21f3*u1b3r21))/ (2*(1 - (1 - a21f3)*(1 - a3f21)*(1 - e5)))}\end{gathered}$$

and q13 is dual to this.)

We can also remark that a technical point of detail enters into the actual calculation of the formula above: If it happens (which has probability e5 at each trial) that a second stage election failed after Player 1 had accepted Player 3 then in that case our rule was that the players P1 and P2 were to be each given a payoff of b3/2 while P3 is to be given zero (and this instance of the playing of the repeated game is then complete). (Thus technically our example game is an NTU game but we can still use Shapley value and the nucleolus because of generalizations of those.) (We adopted this convention of splitting b3 = v(1,2) equally between P1 and P2 just for simplicity when beginning our study. And in retrospect it seems good for the games with only b3 > 0 (and b1 = b2 =0). But it seems that it can distort evaluations in other cases (!).)

**The Equations for the Equilibrium Solutions**

From the quantities above, not including the demand numbers, the patterns of the actual (steady) behavior of the three players are fully described. These numbers, 18 a-numbers and 24 u-numbers, or 42 parameters in all, therefore describe the directly observed behavior of the players.

So we can compute the (moderately lengthy) terms of a vector payoff function, say {PP1,PP2,PP3} describing the payoff consequences to the players of their behavior with the terms being rational fractional expressions in these 42 quantities plus also a dependence on e4 and e5 (because of the effects of the chances of repeating failed elections).

It is, however, the d-numbers and the u-numbers (but not the a-numbers) that are officially the strategic choices of the players. For the proper set of equilibrium equations we need to work with these. This involves, in principle, the substitution for all of the a-numbers of the expressions derivable for them as functions of the d-numbers and the u-numbers. So suppose that the completion of these substitutions would give us an expanded vector payoff function {PP1du, PP2du, PP3du} in which all appearances of a-numbers have been replaced by the expressions that describe their reactive varying (as the players vary their behaviors in reaction to the observed actions of the other players). Then the set of 24 + 15 = 39 equilibrium equations for the strategic u-numbers and d-numbers are derivable by taking, for each of the strategic parameters, the partial derivative, with respect to it, of the payoff function of the player who is the controller of that strategy parameter.

Actually, however, we found, starting with the study of two-player cases, that we could take a modified route of derivation and arrive at simplified yet equivalent equations. But, skipping all the details, the PI worked with the project assistant Alexander Kontorovich in a program where the two workers independently derived the equations so that the results could provide good confirmations. Methods were also developed, working within Mathematica, that could exploit the symmetries of the game (with b1, b2 and b3 as symmetric symbols) and these gave both cross-checking benefits and also made it possible to derive multiple variant equations from one good calculation. In particular, the 24 equations associated with u-numbers, because of the 3-factorial (3!) symmetry, became 4 groups of six and only one of each group really needed direct calculation.

In the end, with the chosen simplifications, we transformed into equations NOT INCLUDING any of the d-numbers but including all the a-numbers. This needed the additional inclusion of three equations of the sort of an equation linking a1f2 and a1f3, which both depend on d1, which is being eliminated from the simplified equations. Thus there are 42 equations, involving as variables the a-numbers and the u-numbers, for a general game.

When the game has symmetries the equation set can be much reduced. If b1 = b2 = b3 then all of the two player coalitions have the same strength and then we can look for solutions involving the same behavior for all players. Then the equations reduced to merely 7 in number (and this was a good basis for finding the first solutions!). If merely b1 = b2 then the coalitions (1,3) and (2,3) have the same strength and we can look for solutions with P1 and P2 in symmetrically patterned behavior. This leads to a reduction to 21 equations, and we did most of our work on calculations with these 21 equations since that level of symmetry led to interesting differences among the various value concepts that could be compared.

### Prospective Model Improvements or Refinements

Because the project research has already exhibited results that compare very interestingly with the analogous results (in terms of predicting game payoffs) that are derivable from the Shapley value, from the nucleolus, or from models of the "random proposers" type, we can wonder if variations in detail of our modeling would affect these comparisons in one direction or another. Good ideas to enable or facilitate the study of games with 4 or more players are naturally also of interest.

One quite simple idea is that if by election a coalition of two players has been formed, with one of them elected as the agent authorized to act for both, that then it is not obviously in the interests of those two players to give more information unnecessarily to the third player, so the identity of the agent-leader who was elected in the formation of that coalition of two players may as well or better be kept secret. Then we find that very nice reductions of the quantity of the strategic data necessary for the players are a consequence.

For example, the a-numbers of the types illustrated by a1f23 and a1f32 would need to become coincident, with the same applying to the related strategically chosen d-numbers d1f23 and d1f32, simply because P1 WOULD NOT KNOW which of P2 and P3 had been elected to be (as it were) the chairman of the committee formed by the two of them.

Furthermore, u2b1r23 and u2b1r32 would likewise need to be the same number.

Thus with "secret coalitions", where the fact that the first stage of agency elections has succeeded would be known to all but where the remaining solo player would not be advised of WHO had elected WHOM in the formation of the coalition of the others; this change would reduce the total number of strategic parameters needed from 39 to 30. (And this with possibly no loss of good representation of the interactions of the players' interdependent interests.)

### Shifting the Agency Function to Attorney Agents

A variant scheme of modeling based on the introduction of "attorney agents", because of its possibly giving payoff outcomes that are closer to those derived from Shapley value calculations, is particularly of interest. (We found much earlier, for two-person bargaining games, that it was successful to use a variant of the modeling in which a single "attorney agent" would perform the agency functions that were originally being performed, alternatively, by one or the other of the original two players. In this variant the two players would need to simultaneously both vote to elect the attorney.)

When we first conceived of the possibility of approaching cooperative games via a repeated game modeling supporting the analogy with natural evolution, at that time the idea of attorneys as agents occurred as soon as did the idea of the players themselves becoming agents for each other. But our initial viewpoint was that the attorneys should be expected to lead to much more of complications in detail!

But now, after further consideration and stimulated by the hope of possibly finding an increased level of influence, on the model's predictions of payoffs, of 2-player coalitions of only small value, we have thought more about the possibilities and have learned that a shift to attorney agents, for the same type of 3-player games, would yield a reduction from 39 strategy dimensions to only 24 dimensions.

Each 2-player coalition would be represented by a dedicated attorney agent who would function somewhat like an "in-house" attorney working on corporate law (on a salary rather than on commissions!). We would model him/her/it as functioning like a robot that is motivated SIMPLY TO MAXIMIZE HIS/HER/ITS FREQUENCY OF BEING EMPLOYED. (Of course this means that the attorney becomes a player in the expanded model and that the attorney's payoff is determined by his/her/its frequency of employment.) And the attorney has strategies also and at equilibrium will be setting these strategic choices so as to maximize his/her/its expected payoff.

To become authorized to act the attorney agent for a set of two or three players would need to have received the simultaneous votes of all of the set of players of the represented coalition. But for this the players would not need, necessarily, to be voting for the coalition (and the agent) with a high probability since we could arrange that with probability (1-e4) that a failed election would be repeated (with here e4 being a small "epsilon" just like in the model already studied computationally). We say more about the elections below after touching on "reluctant acceptance" as a phenomenon.

It can be remarked now that the election results are naturally unambiguous if all members of a possible coalition must SIMULTANEOUSLY vote to authorize the election of the attorney representing them as coalesced members of that coalition. Of course, if a 5-player game were similarly modeled, with competing attorney agents to be alternatively elected, then of course it COULD happen that two non-overlapping coalitions of 2-player type would simultaneously be elected.

## Reluctant Acceptance Behavior

It was only as a consequence of actually working on the details of the research project that we discovered the apparent desirability of allowing the players to find the sort of an equilibrium in which they would only rarely, comparatively, vote to accept the agency function of another player.

Of course it is obvious enough that the acceptance action is quasi-altruistic, since the agent accepted is not at all constrained to consider properly the interests of a player accepting him/her EXCEPT through the structure of the repeated game context AND through the reactive behavior of the players built into the model. Thus a player will be ”DEMANDING” to be treated by a chosen standard (of benefits) in connection with any particular type of acceptance vote.

The players MUST be sometimes accepting, in a global sense, or they would never be gaining any of the benefits specified for the coalitions by the characteristic function.

So we found that simply providing a rule for the probable repetition of failed acceptance elections caused the calculable equilibria to shift, in line with the highness of the probability of election repetitions, so that the same sort of efficiency of reaching close to the Pareto boundary would be attained with lower probabilities of acceptance, in the voting, whenever the probability of repeating a failed election would be improved.

Thus the players could become as if wise negotiators waiting patiently for the other sides to make concessions!

There was another advantage found with arranging for ”asymptotically perfectly reluctant accepting” and this was that this idea seemed to remove what otherwise would appear as an arbitrary rule for the elections; the convention that if more than one voter voted, then a single voting action, chosen at random, would be certified and made effective.

**Pro-Cooperative Games and Evaluations of Games**

The forthcoming book of E. Maskin, which expands on his Presidential Address to the Econometric Society, has a theme that connects with our idea of ”Pro-Cooperative Games”. This is the theme of ”externalities” as realistic considerations that are not included in the formal description of a game (say as a ”CF game” in particular) and which COULD act, for example, to (effectively) prevent the formation of the grand coalition.

We began to see that in our games studied by our modeling method (with agencies) that if the strengths of all of the 2-player coalitions were quite large (and comparable to $v(1,2,3) = +1$) then that it could become reasonable, in a repeated game context, for there to be various stable equilibria. Thus any two of the players could be seen as being able to ”learn” that they are natural allies and then, through an alliance, gain the lion’s share of all the possible benefits from the game.

The concept of a ”pro-cooperative game” would be that of games where such an alliance of two players would not be able to thus benefit them. Then these games would be more properly suitable for being assigned a ”value” (by whatever means of evaluation would ultimately be selected).

**Comparisons with Random Proposers Modeling**

Several references in the Bibliography can be viewed as works that relate to the ”Nash program” (which was the suggestion that the study of cooperative games should, somehow, be reduced to that of non-cooperative games).

And in particular there are references to studies of games of more than two players that are based on a method of modeling with ”random proposers”. These studies have themselves cited the influence of earlier studies, by Rubinstein et al, of ”alternating offers” models that have been successful in studies of two party bargaining.

With the ”random proposers” models the mathematical calculations necessary to find equilibria are not so difficult as to limit the results available to numerical approximations (as they were found to be with our modeling based on elected agencies) and some really nice results have been obtained in the sense that relatively simple relations to the nucleolus and the Shapley value have appeared in the outcomes.

But what is the truth (if there is any truth!!)?

We suspect, actually, (and this can be viewed as a private opinion) that the "random proposers" modeling simplifies the bargaining and negotiating context by removing an element of the "free enterprise" type. So while nice results are deduced, in terms of mathematical simplicity, they may be only approximate results, in some sense.

With the modeling in terms of "agencies" it was as if all players are always "PROPOSERS" but only occasionally does a player become an ACCEPTOR (of the proposal of another player). So the element of "free enterprise" possibly enters as the players, on their own initiatives, select and decide upon which proposals to accept.

Researchers studying the "random proposals" models have observed that when a player becomes a "proposer" that this seems to give a differential advantage to him. Of course using random assignments, among the players, evens out the advantages, but that is not the same as a "free enterprise" process in its basic nature.

We feel that a truly ideal concept, in relation to the study of games, is to achieve the capacity to give valuable appraisals of a game situation to the players or prospective players of the game.

An "arbitration scheme" (in the words of Luce and Raiffa) could be developed on the basis of a theory that seemed helpful toward game appraisals. But we wish to remark that, in principle, there is some risk of building an "arbitration formula" into a modeling procedure used in studying games. So the arbitration formula cannot be validly DERIVED if it is already inserted as an assumption in the modeling!

## Relevant Existing Literature

The recent work of Abreu and Pearce notably involves, like our modeling scheme, the study of the repeated games context. In this context they are able to deal with both the cooperative and the competitive aspects of a situation of bargaining that is not of the simplest variety.

Harsanyi was an early pioneer explorer in the search for theoretical understanding of cooperative games including, particularly, games of the NTU category.

Rubinstein's 1982 paper in Econometrica influenced various later papers connecting bargaining and offers and acceptances.

Others, for example (Gul, 1989), (Osborne and Rubinstein, 1990), (Montero-Garcia, 1998), (Seidmann and Winter, 1998), (Ferreira, 1999), and (Ray and Vohra, 1999), in a general sense, look at coalitions from a "dynamical" viewpoint, understanding that the participants in a game-like situation must act appropriately for coalitions to actually form.

The papers of Baron and Ferejohn, of Okada, and of Gomes have made use of the "random proposers modeling" and are good sources in relation to that idea (which has antecedents going back to Rubinstein and Stahl).

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# Competition of Large-scale Projects:Game-theoretical Approach

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**Abstract** The process of competition of large-scale projects is studied in a setting motivated by real-life problems of optimization of gas and oil transportation networks and optimization of the corresponding investment. The employed mathematical model is a noncooperative game of several players with choice of time moments and payoff functions that contain improper integrals. It is assumed that the investigated processes are described by exponential functions. This assumption is reasonable because of the economic sense of the problem. Also, this assumption simplifies the mathematical model and the implementation of the corresponding algorithms. The use of exponential functions makes it possible to create effective codes for computer modelling of these problems. The article contains detailed consideration of the involved mathematical assumptions, description of the algorithms for finding points of Nash equilibrium and the best responses of investors to actions of other investors, and description of the developed software. An illustrative numerical example is given also.

**Keywords:** N-player non-cooperative game, Nash equilibrium solutions, competition of large-scale projects.

## 1. Introduction

Mathematical and computer models of the process of competition of investors to large-scale projects, such as construction of gas pipelines, was studied in many publications, see e.g. (Klaassen et al., 2002–Brykalov et al., 2005). In article (Klaassen et al., 2004), a description was obtained of Nash equilibrium points in a game of two investors who finance competing projects of gas pipeline construction. It was shown in (Brykalov et al., 2004) that if instead of a complete description, we are satisfied with an algorithm that enumerates all the points of Nash equilibrium, then we can consider a game of several investors and significantly relax the imposed mathematical requirements.

The games considered in the above cited articles provide mathematical models for the following situation. Several gas pipelines are being built by competing investors and are aimed at one and the same regional market of natural gas. When new gas pipelines come into operation, the amount of gas supplied to the market is increased, which obviously can lower the price of gas. The investors that put their pipelines into operation earlier can enjoy a high price of gas for some time. The investor who comes to the market first enjoys some period of monopoly. On the other hand, completing the construction of a pipeline later can be desirable for a number of reasons. In particular, it can reduce the price of construction. This naturally creates a kind of game between the investors. Various aspects of this game were studied in the above cited articles.

In particular, the above described research included mathematical and computer modelling of the Turkish gas market. These considerations are based on the assumption that the price of gas is set by the market itself. Some heuristic algorithms were proposed. A strict mathematical model was developed and published later in (Klaassen et al.,2004). This model has initiated further development of computer realizable algorithms and mathematical generalizations in many ways.

In the further research, an attempt was made to apply the developed technique to the Chinese market of natural gas. However, many of the assumptions used for modelling the Turkish market appeared to be invalid for the specific Chinese market. From the point of view of economics, the main difference is that in the case of China the prices are fixed not purely by a market mechanism. A mathematical model that takes into account these circumstances was given in (Nikonov, 2004). Methodically, the article (Nikonov, 2004) is a continuation of research (Klaassen et al.,2004), where the basic model is described. However, the assumptions of the model of Chinese market are essentially different and sometimes the opposite.

Below the results and algorithms of (Brykalov et al.,2004) are specified for a typical case when the process of construction and exploitation of the gas pipelines is described with the help of exponential functions (Brykalov et al.,2005). This allows us to simplify many of considerations and imposed conditions. On the other hand, this supposition is not too restrictive as exponential functions frequently arise in connection with problems of this type and in research on economics in general. It is convenient to work with exponential functions as each of them is described by two parameters only. Algorithms in (Brykalov et al.,2004) require finding intersection points of the corresponding graphs. In the case of exponential functions, this is reduced to an elementary equation. There is no need to employ numerical methods. Because of that, algorithms from (Brykalov et al.,2004) in the case of exponential functions can be effectively realized in the form of computer codes, see (Brykalov et al.,2005). Below we analyze in detail the mathematical requirements that arise in these problems in the case of exponential functions, present algorithms for finding best responses of participants and points of Nash equilibrium, and describe the corresponding software.

## 2. Basic Assumptions and Problem Statement

We study a mathematical model of the investment process in the form of a game of several players. There are $n$ players, where $n \geq 2$. These players can be treated as investors or managers supervising the construction of several gas pipelines. These projects compete with each other as they are aimed at the same regional gas market. Assume that the construction of the gas pipelines starts at one and the same time moment $t = 0$. The player number $i$ chooses the commercialization moment $t_i$ of the corresponding project. At this time moment, the construction of the gas pipeline number $i$ is finished and its commercial exploitation starts. So, the gas supplied by this pipeline is available at any time moment $t \geq t_i$. Thus, the actions of a player are treated as the choice of the commercialization moment (Klaassen et al., 2004). This is a laconic and convenient description, which is informative enough with respect to the stages of both construction and exploitation of the pipeline.

Let $C_i(t_i) = \gamma_i e^{-q_i t_i}$ be the total investment needed for finishing the construction of the pipeline $i$ at the time moment $t_i$. Here $\gamma_i$, $q_i$ are positive parameters. Let us also consider the cost reduction rates

$$a_i(t_i) = -C_i'(t_i) = \alpha_i e^{-q_i t_i},$$

where $\alpha_i = q_i \gamma_i$.

We understand the expression $\{1, ..., i-1, i+1, ..., n\}$ for $i = 1$ as the set $\{2, ..., n\}$, and for $i = n$ as the set $\{1, ..., n-1\}$.

Let for any number $i = 1, ..., n$ and set $H \subset \{1, ..., i-1, i+1, ..., n\}$, positive numbers $\beta_{iH}$, $p_{iH}$ be given. For any time moment $t > 0$, the value $b_{iH}(t) = \beta_{iH} e^{-p_{iH} t}$ is assumed to be the benefit rate player $i$ receives by means of sales of gas at the time moment $t$ under the condition that at this time all pipelines $j \in H$ and only they supply gas to the market together with pipeline $i$. Until player $i$ has made the choice of the commercialization moment $t_i$, the benefit rate $\beta_{iH} e^{-p_{iH} t}$ can be considered to be 'virtual' because it is not known yet whether $t$ will be larger than $t_i$, that is, whether the corresponding pipeline will be in operation at the time moment $t$. The parameters $\beta_{iH}$, $p_{iH}$ depend on the price in the regional market and also the cost of extraction of gas and its transportation along a pipeline. In its turn, the price depends on the amount of gas available on the market, and so, on what pipelines are already in operation. This explains the dependence of parameters $\beta_{iH}$, $p_{iH}$ on the set $H$. When construction of new pipelines is completed, the amount of available gas increases on the market. Increase of supply results in decrease of the price. Thus, let us assume the following: If $G \subset H \subset \{1, ..., i-1, i+1, ..., n\}$ and $G \neq H$ then the following inequalities hold

$$\beta_{iG} > \beta_{iH}, \qquad p_{iG} \leq p_{iH}. \tag{1}$$

We also assume that for any number $i = 1, ..., n$ one has

$$p_{i\{1,...,i-1,i+1,...,n\}} < q_i. \tag{2}$$

Note that inequalities (1),(2) imply: $p_{iG} < q_i$ for any set $G \subset \{1, ..., i-1, i+1, ..., n\}$. In the expressions $\beta_{iH}$, $p_{iH}$, the set $H$ can, in particular, be empty: $H = \emptyset$. In this case, only player $i$ can be present in the market, which corresponds to monopoly. We assume that the parameter $\alpha_i$ in the expression for the cost reduction rate and the value $\beta_{i\emptyset}$, which corresponds to the monopoly case, satisfy the inequality

$$\alpha_i > \beta_{i\emptyset} \tag{3}$$

that should hold for all numbers $i = 1, ..., n$. In connection with inequality (3), see Assumption 4 in (Nikonov, 2004), and also Assumption 2.2 and Remark 2.1 in (Klaassen et al., 2004).

For any $i = 1, ..., n$, denote by $A_i$ the set of all numbers of the form

$$t = \frac{\ln\alpha_i - \ln\beta_{iH}}{q_i - p_{iH}}, \tag{4}$$

where $H$ takes as values all the subsets $H \subset \{1, ..., i-1, i+1, ..., n\}$. The number of these subsets is $2^{n-1}$. So, $A_i$ is a finite set with no more than $2^{n-1}$ elements. (Values of some of the expressions (4) can coincide, which decreases the number of elements.) Note that inequalities (1),(3) imply that the numerator of fraction (4) is positive, and inequalities (1),(2) imply that the denominator is positive as well. Thus, the elements of the set $A_i$ are positive. Note also that equality (4) is obtained when the variable $t$ is found from the equation

$$\alpha_i e^{-q_i t} = \beta_{iH} e^{-p_{iH} t}. \tag{5}$$

From geometrical point of view, $A_i$ is the set of abscissas of the points of intersection of the graphs of functions in (5) for all $H$.

As $\alpha_i = q_i \gamma_i$, the expression (4) can be written in the form

$$\frac{\ln q_i + \ln \gamma_i - \ln \beta_{iH}}{q_i - p_{iH}}.$$

For any given numbers $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$, denote by

$$G_i(t) = G_i(t|t_1, ..., t_{i-1}, t_{i+1}, ..., t_n) = \{j \neq i \,:\, t_j \leq t\}$$

the set of all rivals of player $i$ who are present in the market at the time moment $t$. For any $t \geq t_i$, the actual benefit rate $b_i(t)$ of player $i$ at time moment $t$ is defined by

$$b_i(t) = b_i(t|t_1, ..., t_{i-1}, t_{i+1}, ..., t_n) = b_{iG_i(t)}(t) = \beta_{iG_i(t)} e^{-p_{iG_i(t)} t}.$$

The total benefit for player $i$ is

$$B_i(t_1, ..., t_n) = \int_{t_i}^{\infty} b_i(t|t_1, ..., t_{i-1}, t_{i+1}, ..., t_n) dt = \int_{t_i}^{\infty} \beta_{iG_i(t)} e^{-p_{iG_i(t)} t} dt.$$

It should be noted that here the numbers $\beta_{iG_i(t)}, p_{iG_i(t)}$ can change with the growth of $t$. For values of $t$ that exceed all the numbers $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$, the function to be integrated is an exponent of the form

$$\beta_{i\{1,...,i-1,i+1,...,n\}} e^{-p_{i\{1,...,i-1,i+1,...,n\}} t},$$

where the parameters $\beta_{i\{1,...,i-1,i+1,...,n\}}$, $p_{i\{1,...,i-1,i+1,...,n\}}$ no longer depend on $t$. As the number $p_{i\{1,...,i-1,i+1,...,n\}}$ is positive, we see that the improper integral is finite. The total profit $P_i(t_1, ..., t_n)$ of player $i$ is the total benefit of this player minus the total investment in the construction of the corresponding pipeline:

$$P_i(t_1, ..., t_n) = -C_i(t_i) + B_i(t_1, ..., t_n) = -\gamma_i e^{-q_i t_i} + \int_{t_i}^{\infty} \beta_{iG_i(t)} e^{-p_{iG_i(t)} t} dt.$$

Thus, we have an $n$-person game of timing. Strategies $t_i$ of players $i$ in this game are positive numbers. Any collection of strategies $(t_1, ..., t_n)$ of all players determines the payoff $P_i(t_1, ..., t_n)$ to each player. Here the strategy $t_i$ is the commercialization moment, and the payoff $P_i(t_1, ..., t_n)$ is the profit of investor $i$.

## 3. Finding Best Responses and Points of Nash Equilibrium

Let us recall two widely used definitions of game theory and apply them to the considered case. A strategy $t_i$ of player $i$ is called a best response of this player to strategies $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$ of other players $1, ..., i-1, i+1, ..., n$ if

$$P_i(t_1, ..., t_{i-1}, t_i, t_{i+1}, ..., t_n) = \max_{s>0} P_i(t_1, ..., t_{i-1}, s, t_{i+1}, ..., t_n).$$

The best response exists if the maximum in the right-hand side is attained at some point. This point might happen to be not unique. So, there might exist several best responses of a player to a fixed collection of strategies of other players. A collection of strategies $t_1, ..., t_n$ of players $1, ..., n$ is called a Nash equilibrium if for every $i = 1, ..., n$, the strategy $t_i$ is a best response of player $i$ to the strategies $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$ of other players $1, ..., i-1, i+1, ..., n$. A Nash equilibrium corresponds to the case when neither of the players is interested in changing the strategy provided all the other players are not changing their strategies.

**Theorem.** *For an arbitrary collection of strategies* $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$ *of players* $1, ..., i-1, i+1, ..., n$*, there exists at least one best response* $t_i$ *of player* $i$ *to these strategies, and each best response* $t_i$ *belongs to the set* $A_i$*.*

*Proof* of the existence of the best response is reduced to a direct application of Proposition 2 in (Brykalov et al., 2004), taking into account properties of exponential functions. From Proposition 1 in (Brykalov et al., 2004), it directly follows that the best response belongs to the set $A_i$. In the considered case, the set $D_i$ introduced in (Brykalov et al., 2004) happens to be empty due to the continuity of the functions employed.

**Corollary.** *If a collection of strategies* $t_1, ..., t_n$ *is a Nash equilibrium, then* $t_i \in A_i$ *for every* $i = 1, ..., n$*.*

*Proof* of Corollary consists in application of the second part of Theorem together with the definition of Nash equilibrium.

It was mentioned above that in the considered case the sets $A_i$ are finite. Because of that, the above statements provide the basis for algorithms for direct finding of best responses of a player to strategies of other players and for checking if a given collection of strategies of all players forms a Nash equilibrium. Let us describe these algorithms. We assume that all the above imposed conditions are satisfied, the natural number $n \geq 2$ is fixed.

### 3.1. Best Response Algorithm

The input data of the algorithm:

(i) an integer $i$, such that $1 \leq i \leq n$;

(ii) positive numbers $\gamma_i$, $q_i$;

(iii) positive numbers $\beta_{iH}$, $p_{iH}$ for all subsets $H \subset \{1, ..., i-1, i+1, ..., n\}$;

(iv) strategies $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$ of players $1, ..., i-1, i+1, ..., n$.

The output of the algorithm is a nonempty finite set $S$, which consists of positive numbers and contains no more than $2^n$ elements. Here $S$ is the set of all best responses of player $i$ to the strategies $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$ of the rest players $1, ..., i-1, i+1, ..., n$.

Sequence of actions of the algorithm:

Step 1. For each subset $H \subset \{1,...,i-1,i+1,...,n\}$ find the number $\frac{\ln q_i + \ln \gamma_i - \ln \beta_{iH}}{q_i - p_{iH}}$ and form the set $A_i$ of all these numbers.

Step 2. For all $s \in A_i$ calculate the values

$$v(s) = -\gamma_i e^{-q_i s} + \int\limits_s^\infty \beta_{iG_i(t)} e^{-p_{iG_i(t)} t} dt,$$

where $G_i(t) = \{j \neq i \, : \, t_j \leq t\}$.

Step 3. Find the set $S$ of all points $s \in A_i$ at which the maximum of function $v(s)$ on the finite set $A_i$ is attained.

Indeed, we see from Theorem that the output of this algorithm is the set of all best responses, and that this set is nonempty.

Now we can use Corollary and the definition of Nash equilibrium as the basis for constructing an algorithm for checking this property.

### 3.2. Nash Equilibrium Verification Algorithm

The input data of the algorithm:

(i) positive numbers $\gamma_i$, $q_i$, $\beta_{iH}$, $p_{iH}$ for all integers $i = 1,...,n$ and all subsets $H \subset \{1,...,i-1,i+1,...,n\}$;

(ii) a collection of strategies $t_1,...,t_n$ of all players.

The output of the algorithm is YES if the collection of strategies $t_1,...,t_n$ is a Nash equilibrium and NO otherwise.

Sequence of actions of the algorithm:

Step 1. Put $i := 1$.

Step 2. For player $i$ and strategies $t_1,...,t_{i-1},t_{i+1},...,t_n$ of other players $1,...,i-1,i+1,...,n$, with the help of the Best Response Algorithm find a nonempty finite set $S$ of best responses of player $i$ to these strategies.

Step 3. If $t_i \notin S$, finish the work of algorithm with the output NO.

Step 4. If $t_i \in S$ and $i < n$, put $i := i+1$ and go to Step 2.

Step 5. If $t_i \in S$ and $i = n$, finish the work of algorithm with the output YES.

**Remark 1.** It can be seen from Corollary that all the Nash equilibrium points belong to the set $N = A_1 \times ... \times A_n$. As the set $A_i$ for every number $i$ contains no more than $2^{n-1}$ elements, we have that the set $N$ is finite and contains no more than $2^{((n-1)^2)}$ elements. Application of Nash Equilibrium Verification Algorithm to all collections of strategies $(t_1,...,t_n) \in N$ allows one to find all the points of Nash equilibrium in the considered game.

**Remark 2.** From the point of view of mathematics, the imposed conditions can be somewhat relaxed by discarding inequality (2) and allowing the parameters $\gamma_i$, $q_i$ to equal zero. In this case, a fraction of the form (4) can happen to be undefined (can contain zero in denominator or under the sign of logarithm) or can happen to be negative. Only positive numbers should be included into the set $A_i$, while indefinite and negative fractions should be ignored. Here the set $A_i$ can be empty. Here the assertion of Theorem about the existence of the best response becomes invalid, however, if the best response exists, then it still belongs to the set $A_i$. Note that Corollary also remains valid. Both algorithms need only insignificant changes.

**Remark 3.** The advantages of the considered model are its simplicity, small number of parameters, and the possibility to work with this model in explicit form without employing any difficult numerical methods. Some drawback can be seen in the finiteness of values $C_i(0) = \gamma_i$. According to the economical sense of the problem, these values should be infinite, because no amount of investment, however large, can allow to complete the construction in a very small time. In order to avoid this drawback, one could take the functions $C_i(t_i)$ in the form employed in Section 5 of (Klaassen et al., 2004), however that would complicate the model, and one would have to numerically find the corresponding points of intersection of the graphs. The above-mentioned shows that the values $\gamma_i$ ideally should be chosen large enough, so that these values $C_i(0) = \gamma_i$ do not interfere in the process of finding the points of Nash equilibrium. In case when the algorithm gives collections of strategies with the presence of zero components, one should change the parameters of the model (or even use a different model).

## 4. Computer Realization of the Algorithm and a Numerical Example

The above described algorithm for finding points of Nash equilibrium was realized in the form of a Delphi 7 computer code. The code provides a convenient interface including graphical illustrations. We give some results produced by the code for the case of three players ($n = 3$) for illustrative purposes.

The code allows to choose interactively the number of players (investors). After this choice is made, the code asks to fill in tables with characteristics of the participants. An example of the tables with parameters of players in the case $n = 3$ is shown in Fig. 1. The first box of each table contains parameters of the function that characterizes the cost of construction ($A = \alpha_i, I = q_i$). The other boxes describe the functions $b_i(t)$ in cases when only one player number $i$ acts on the market, two players in the corresponding combinations are present, and at last, all the three players take part. Here $B$ with indices equals $\beta_{iH}$ for the corresponding $H$, and similarly $q = p_{iH}$.

Note that in the case of three players, for each $i = 1, 2, 3$ one has $2^{(3-1)} = 4$ variants of the set $H$. For each of these variants, the parameters are fixed that describe the benefit function of the player.

As an illustration, the code shows the graphs of all the employed functions $a_i(t), b_i(t) = \beta_{iH}e^{-p_{iH}t}$. On these graphs, the intersection points are marked, which are needed to construct the sets $A_i$, and the corresponding values of abscissa are given. In Fig. 2, the graphs are shown of the functions that characterize players with the values of parameters given in Fig. 1.

Here, according to the algorithm, for each player we find the set $A_i$ whose elements are the points that can be expected to give the maximal values of the benefit function of the player. These are the abscissas of the intersection points of the corresponding graphs Fig. 3. Their values can be found as the solutions of equation (5).

**Table5.** The optimal thresholds for the two stage game for the different $c$.

| $c$ | 0 | 0.1 | 0.167 | 0.2 | 0.3 | 0.5 |
|---|---|---|---|---|---|---|
| $w$ | 0 | 0.6 | 1 | 1.16 | 1.5 | 2 |

**Table6.** The optimal thresholds for the three stage game for the different $c$.

| $c$ | 0.1 | 0.1196 |
|---|---|---|
| $w_1$ | 0.8366 | 1 |
| $w_2$ | 0.484 | 0.579 |

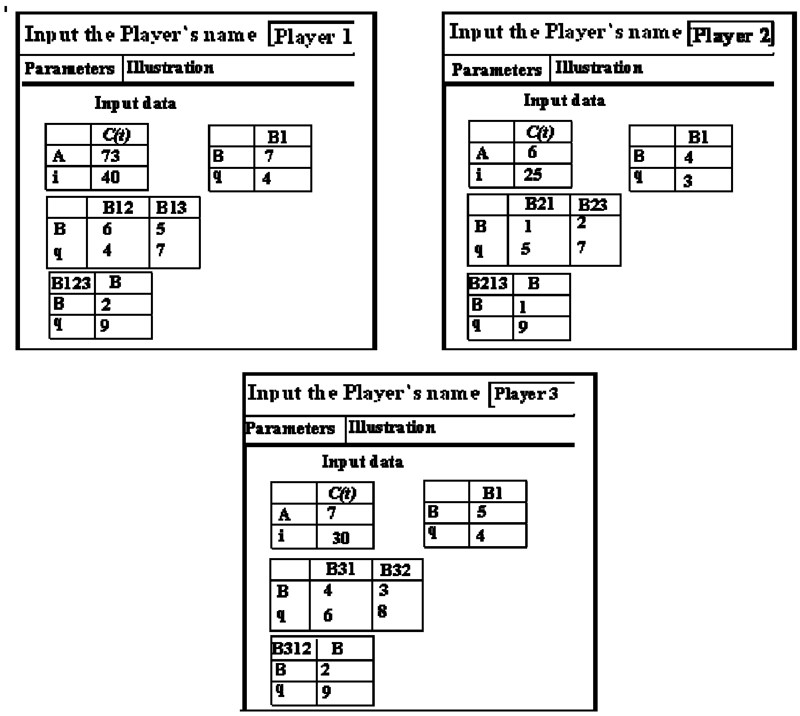
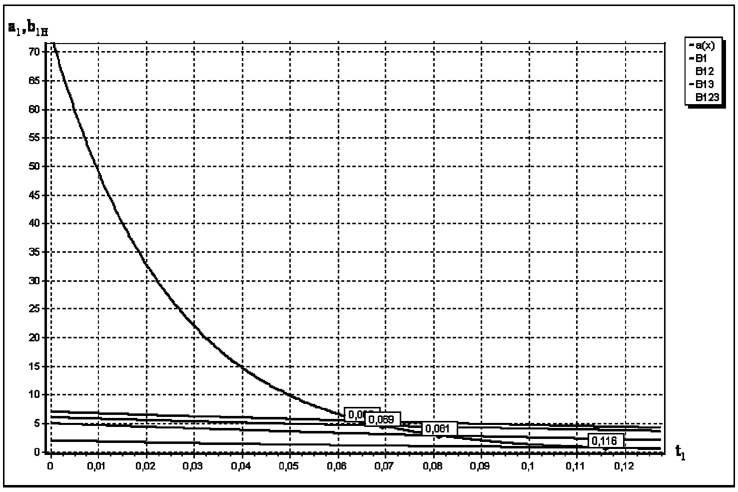
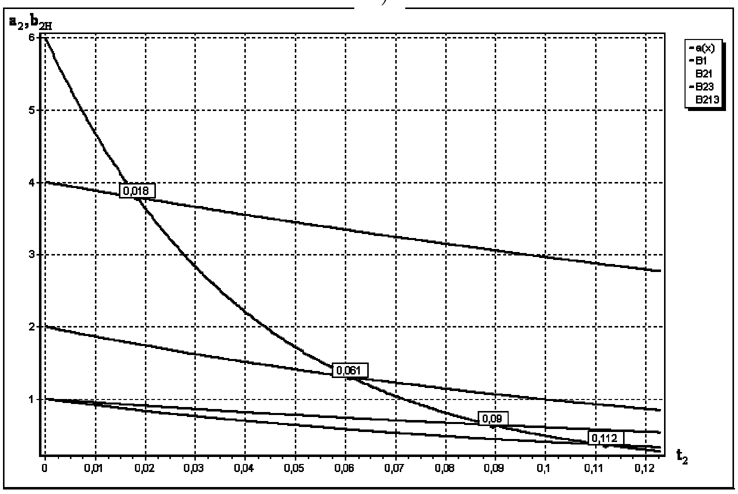


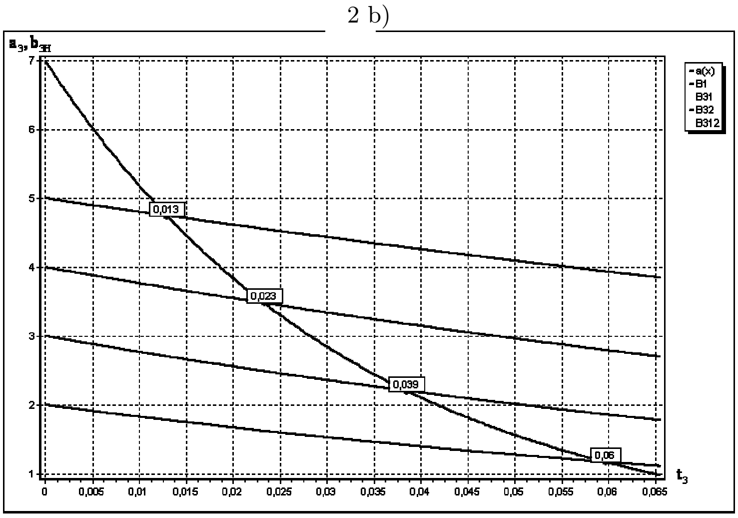

**Fig.1.** Input parameters of the players




2 a)




2 b)




2 c)

**Fig.2.** Graphs of functions that characterize the players: a) for player 1, b) for player 2, c) for player 3

| | | | | |
|---|---|---|---|---|
| №1 | 0,063 | 0,081 | 0,116 | 0,065 |
| №2 | 0,089 | 0,061 | 0,111 | 0,018 |
| №3 | 0,023 | 0,038 | 0,059 | 0,012 |

**Fig.3.** Sets $A_i,\ i = 1, 2, 3$

The code forms for the players the sets $A_1, A_2, A_3$ and the points of Nash equilibrium:

$$(0.081,\ 0.111,\ 0.012);$$
$$(0.116,\ 0.061,\ 0.012);$$
$$(0.116,\ 0.018,\ 0.038).$$

Let us demonstrate the work of the Nash equilibrium verification algorithm. Consider, e. g., the point (0.116, 0.061, 0.012). Taking into account the inequality $t_2 > t_3$, we see that the benefit function of the first player has the form

$$P_1(t_1|t_2,t_3) =$$

$$= \begin{cases} \int\limits_{t_1}^{\infty} b_{1H_{123}}(t)dt - C_1(t_1), \text{for } t_1 > t_2 > t_3 \\ \int\limits_{t_1}^{t_2} b_{1H_{13}}(t)dt + \int\limits_{t_2}^{\infty} b_{1H_{123}}(t)dt - C_1(t_1), \text{for } t_2 > t_1 > t_3 \\ \int\limits_{t_1}^{t_3} b_{1H_1}(t)dt + \int\limits_{t_3}^{t_2} b_{1H_{13}}(t)dt + \int\limits_{t_2}^{\infty} b_{1H_{123}}(t)dt, \text{for } t_2 > t_3 > t_1. \end{cases}$$

As $t_1 > t_3$, we can construct the benefit function of the second player in the form:

$$P_2(t_1|t_1,t_3) =$$

$$= \begin{cases} \int\limits_{t_2}^{\infty} b_{2H_{213}}(t)dt - C_2(t_2), \text{for } t_2 > t_1 > t_3 \\ \int\limits_{t_2}^{t_1} b_{2H_{23}}(t)dt + \int\limits_{t_1}^{\infty} b_{2H_{213}}(t)dt - C_2(t_2), \text{for } t_1 > t_2 > t_3 \\ \int\limits_{t_2}^{t_3} b_{2H_2}(t)dt + \int\limits_{t_3}^{t_1} b_{2H_{23}}(t)dt + \int\limits_{t_1}^{\infty} b_{2H_{213}}(t) - C_2(t_2)dt, \text{for } t_1 > t_3 > t_2. \end{cases}$$

Similarly, we construct the benefit function of the third player, taking into account that $t_1 > t_2$.

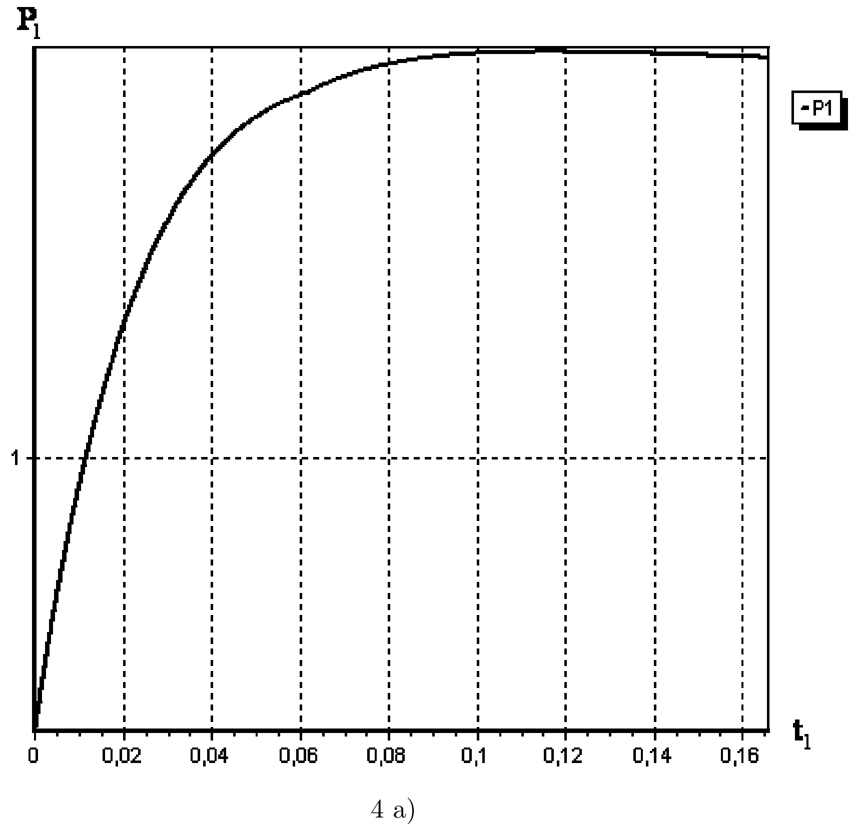
Substituting the considered values of the coefficients, we obtain the functions $a_1(t_1) = 73e^{-40t_1}$, $b_{1H_1}(t_1) = 7e^{-4t_1}$, $b_{1H_{13}}(t_1) = 5e^{-7t_1}$, $b_{1H_{123}}(t_1) = 2e^{-9t_1}$, $a_2(t_2) = 6e^{-25t_2}$, $b_{2H_2}(t_2) = 4e^{-3t_2}$, $b_{2H_{23}}(t_2) = 2e^{-7t_2}$, $b_{2H_{213}}(t_2) = e^{-9t_2}$. Since $C_1(t_1) = \frac{73}{40}e^{-40t_1} + \frac{73}{40}$, one has that

$$P_1(t_1|t_2,t_3) =$$

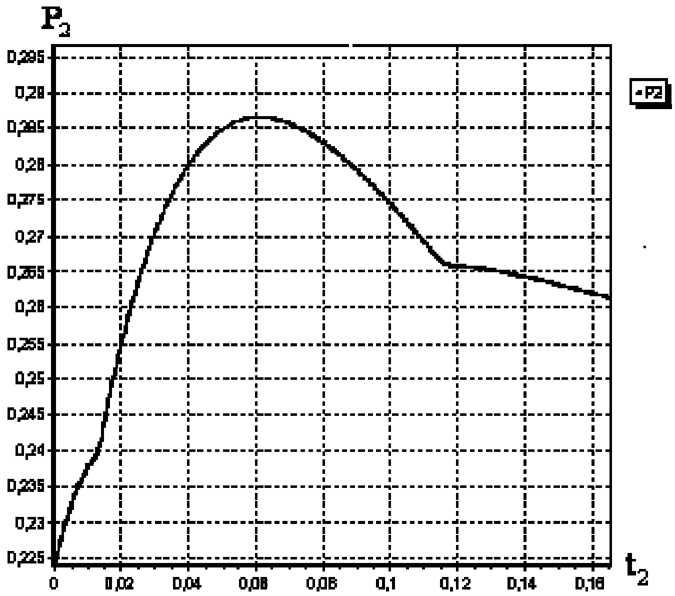
$$= \begin{cases} \frac{2}{9}e^{-9t_1} + \frac{73}{40}e^{-40t_1} - \frac{73}{40}, \text{for } t_1 > t_2 > t_3 \\ -\frac{5}{7}(e^{-7t_2} - e^{-7t_1}) + \frac{2}{9}e^{-9t_2} + \frac{73}{40}e^{-40t_1} - \frac{73}{40}, \text{for } t_2 > t_1 > t_3 \\ -\frac{7}{4}(e^{-4t_3} - e^{-4t_1}) - \frac{5}{7}(e^{-7t_2} - e^{-7t_1}) + \frac{2}{9}e^{-9t_2} + \frac{73}{40}e^{-40t_1} - \frac{73}{40}, \text{for } t_2 > t_3 > t_1. \end{cases}$$

It is easy to see that the number $t_1$ =0.116 is the best response of the first player to the strategies $t_2$ =0.061, $t_3$ =0.012 of the other players, i. e., the maximum point of the function $P_1(t_1|0.061; 0,012)$ for $t_2 > t_3$. As the considered number is obtained as a solution of the equation $a_1(t) = b_{1H_{123}}(t)$, this number is a stationary point of the function and belongs to the set $A_1$. It can be checked directly that this number is indeed the point of maximum. Similarly, the number $t_2$ =0.061 is the best response of the second player to the strategies $t_3 = 0.012$, $t_1 = 0.116$ , and the number $t_3 = 0.012$ is the best response of the third player to the strategies of the other players.

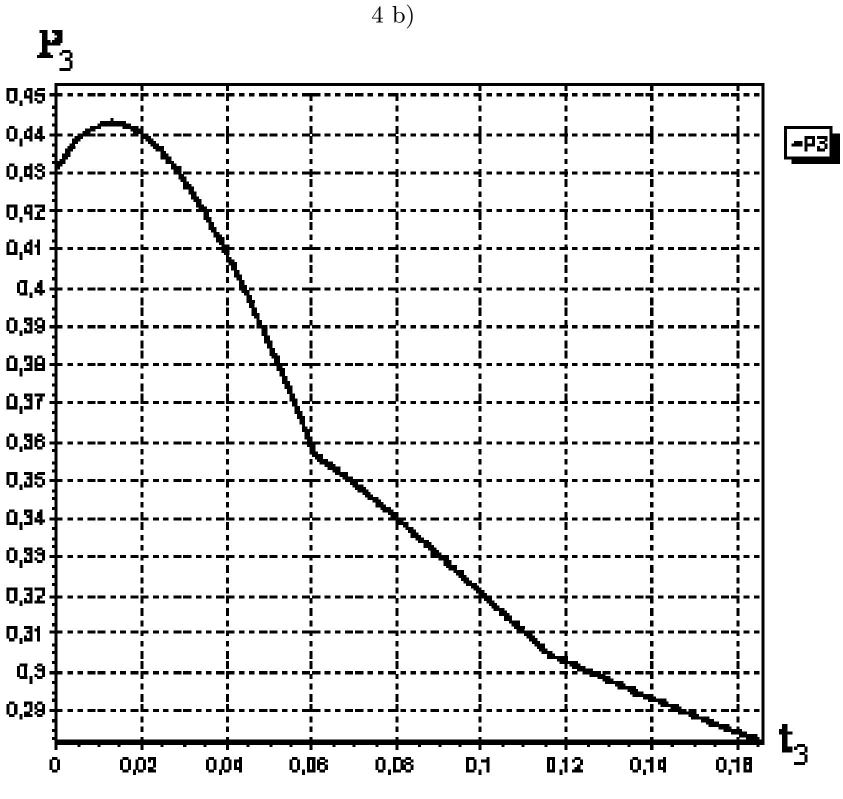
For illustrative purposes, the code gives the graphs of benefit functions of the players and the values of the benefit functions at the points of Nash equilibrium. The graphs for the considered example are presented in Fig. 4. The break points of the curves correspond to the discontinuity points of the benefit function, i. e. the moments when new investors enter the market.




4 a)




4 b)




4 c)

**Fig.4.** Benefit functions of the players at an equilibrium point: a) the function $P_1(t_1|0.061, 0.012)$, b) the function $P_2(t_2|0.116, 0.012)$, c) the function $P_3(t_3|0.116, 0.061)$

## 5. Conclusion

In the paper the process of competition of large-scale projects is studied in a setting motivated by real-life problems of optimization of gas and oil transportation networks and optimization of the corresponding investment. A mathematical model of the above situation is constructed as a non-cooperative game of several players in which the moments of time when the participants enter the market are to be chosen.

It is assumed that the payoff functions that contain improper integrals in investigated processes are described by exponential functions. This reasonable from the economic point of view assumption simplifies the mathematical model and allows us to formulate the conditions for best responses of participants in terms of explicit formulas. For the case under consideration the effective codes for computer modelling are created in the paper.

The article contains detailed consideration of the involved mathematical assumptions, description of the algorithms for finding points of Nash equilibrium and the best responses of investors to actions of other investors, and description of the developed software. An illustrative numerical example is also given .

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# A Generalized Model of Hierarchically Controlled Dynamical System

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**Abstract** The idea of the paper is to combine in the same model several concepts from game theory, graph theory, and controlled dynamical systems theory, namely: 1) a directed graph without contours and loops; 2) a game of n players in normal form; 3) a cooperative game; 4) a dynamical system controlled by several subjects. The combination permits to describe complex dynamical systems with hierarchical structure (particularly organizational and environmental systems) more completely and to take into consideration different interactive and interdependent aspects of the systems. Some examples are considered such as environmental control, corruption modeling, and linear multistage games with hierarchical matrices.

**Keywords:** hierarchical game theory, directed graphs, controlled dynamical systems.

## 1. Introduction

Management deals with organizational systems. The following characteristics (at least) are immanent for those systems.

1. Hierarchy. The hierarchical structure is generated by the relation of subordination between employees and departments in the organization.
2. Conflicts. The conflicts are conditioned by presence of many employees and departments with different interests using limited resources and aiming distinct goals.
3. Coalitions. The organizational coalitions are both official departments and other unions and different groups having common interests.
4. Dynamics. The state of organization changes in time due to managerial efforts and external impact.

These characteristics are studied by different disciplines and described by different mathematical models, as a rule separately. The idea of the paper is to combine in the same model several concepts from game theory, graph theory, and controlled dynamical systems theory, namely: 1) a directed graph without contours and loops; 2) a game of n players in normal form; 3) a cooperative game; 4) a dynamical system controlled by several subjects.

The combination permits to describe complex dynamical systems with hierarchical structure (particularly organizational and environmental systems) more completely and to take into consideration different interactive and interdependent aspects of the systems. Some examples are considered such as environmental control, corruption modeling, and linear multistage games with hierarchical matrices.

## 2. The Model

A generalized model of hierarchically controlled dynamical system may be represented as follows:

$$H = \left\langle N, A, S, C, \{U_i\}_{i \in N}, \{J_i\}_{i \in N} \right\rangle \tag{1}$$

where $N = \{1, \ldots, n\}$ is a set of players; $A$ is a binary relation of hierarchy on $N$; $S$ is a set of coalition structures on $N$, $s \in S\colon s = \{K_1, \ldots, K_m\}$, $K_1 \cup \ldots \cup K_m = N$, $K_i \cap K_j = \emptyset$, $K_1, \ldots, K_m \subset N$; $C$ is a controlled dynamical system, $C\colon x^t = x^{t-1} + f\left(x^t, U_1^t, \ldots, U_n^t\right), x_0 = x^0$, $t = 1, \ldots, T$; $U_i$ is a set of strategies of the $i$-th player; $J_i\colon U_1 \times U_2 \times \ldots \times U_n \to \mathbb{R}$ is a payoff function of the $i$-th player.

The following properties are supposed to be fulfilled:

**P1–hierarchy:** the binary relation $A$ is a strict order relation;
**P2–stratification:** each coalition structure $s \in S$ is ordered;
**P3–economic rationality:** each player $i \in N$ tends to maximize $J_i$.

Let's define a Neumann-Morgenstern characteristic function:

$$v(K) = \max_{u_K \in U_K} \min_{u_{N \setminus K} \in U_{N \setminus K}} \sum_{i \in K} J_i\left(u_K, u_{N \setminus K}\right), \quad K \subset N \tag{2}$$

where $u_K$ is a set of strategies of the players from $K$, $u_{N \setminus K}$ is a set of strategies of the players from $N \setminus K$.

The model (1) together with the characteristic function (2) allows to combine four known concepts:

1. a directed graph without contours and loops $D = (N, A)$ with additional set of ordered structures of the vertices $S$ which characterizes a hierarchical structure of the system;
2. a game of $n$ players in normal form $G = \left\langle N, \{U_i\}_{i \in N}, \{J_i\}_{i \in N} \right\rangle$ which represents the system as a set of independent rational individuals having conflict and searching for compromise;
3. a cooperative game $\Gamma_v = \langle N, v \rangle$ which allows to describe coalitions, united actions and rational imputations;
4. a dynamical system controlled by several subjects inside and beyond the organization $x^t = x^{t-1} + f\left(x^t, U_1^t, \ldots, U_n^t\right), x_0 = x^0$.

## 3. An Environmental Interpretation

The following interpretation of the model (1) for a problem of the sustainable development of a simple ecological-economic system is possible: $N = \{L, F\}$, where $L$ (Leader) is an environmental protection agency, $F$ (Follower) is an enterprise; $A = \{(L, F)\}$ defines the administrative and economic dependence of $F$ from $L$; $S = \{S_1, S_2\}, S_1 = \{\{L\}, \{F\}\}$ – isolated behavior, $S_2 = \{\{L, F\}\}$ – cooperative behavior of $L$ and $F$; $C$ is defined by (5):

$$J_L = \sum_{t=1}^{T} \left[ g_L^t \left( p^t, q^t, u^t, x^t \right) - M\,\rho \left( x^t, X_L^t \right) \right] \rightarrow \max_{p^t \in P^t, q^t \in Q^t} \tag{3}$$

$$J_F = \sum_{t=1}^{T} g_F^t \left( p^t, u^t, x^t \right) \rightarrow \max_{u^t \in U(q^t)} \tag{4}$$

$$x^t = x^{t-1} + f \left( x^{t-1}, u^t \right), \quad x_0 = x^0, \quad t = 1, \ldots, T \tag{5}$$

where $Q$ is a set of Leader's administrative strategies; $P$ is a set of Leader's economic strategies; $U$ is a set of Follower's strategies (environmental impacts); $J_L$ is Leader's payoff function considering the sustainability requirement $x^t \in X_L^t$; $M\,\rho\,(x^t, X_L^t)$ is a penalty function; $J_F$ is Follower's payoff function; $x^t$ is a state vector of the ecological-economic system in the moment $t$; $x_0$ is a known initial state. So, $U_L = \{Q, P\}$; $U_F = U$; $J_L$, $J_F$ are defined above as (3), (4).

Methods of management (compulsion, impulsion, conviction) which permit Leader to provide sustainability $x^t \in X_L^t, t = 1, \ldots, T$, are formalized as Stackelberg equilibriums of a special form for a hierarchical game of $L$ and $F$. The equilibriums formalize administrative, economic, and psychological methods in management respectively (Ougolnitsky, 2002; 2004).

## 4. Corruption Modeling

The phenomenon of corruption is intensively studied in the past two decades (Klitgaard, 1991; Bac, 1996; Bardhan, 1996; Rose-Ackerman, 1997). The principal-agent-client model may serve as a basic one for the corruption investigation (Levin and Zirik, 1998).

So, corruption is closely connected with hierarchy. In particular, corruption arises in the hierarchical ecological-economic systems. That's why we introduce corruption in the model (3)–(5) in its derivative form with implicit description of dynamics (Ougolnitsky, 2002; 2004). We use a simplified version of the basic model with two hierarchically ordered players: Leader and Follower. This approach is also considered in (Denin, 2008).

The derivative model of a simple ecological-economic system with hierarchical structure considering corruption is represented as:

$$J_v = \sum_{t=1}^{T} \left[ g_v^t \left( p^t, q^t, u^t, \beta^t \right) - M\,\rho \left( u^t, U_v^t \right) \right] \rightarrow \max \tag{6}$$

$$p^t \in P^t, \quad q^t \in Q^t$$

$$J_u = \sum_{t=1}^{T} \left[ g_u^t \left( p^t, u^t, \beta^t \right) - M\,\rho \left( u^t, U_u^t \right) \right] \rightarrow \max \tag{7}$$

$$u^t \in U^t \left( q^t \right), \quad \beta^t \in B^t$$

The explications for the model (6)–(7) are given in Table 1.

Let's classify the corruption phenomena in dependence with types of privileges given to Follower by Leader for bribe. Let's distinguish three types of corruption:

**Table1.** Model notations

| Notation | Mathematical definition | Environmental interpretation |
|---|---|---|
| $u^t$ | $\left(u_1^t, \ldots, u_n^t\right) \in \mathbb{R}_+^n$ | The set of Follower's strategies in the year $t$: resource extraction, production of goods, pollution |
| $\beta^t$ | $\left(\beta_1^t, \ldots, \beta_n^t\right) \in \mathbb{R}_+^n$ | The set of Follower's additional strategies which may be interpreted as a bribe to Leader |
| $p^t$ | $\left(p_1^t, \ldots, p_n^t\right) \in \mathbb{R}_+^n$ | The set of Leader's strategies influencing Follower's control function: taxes, penalties, privileges |
| $q^t$ | $\left(q_1^t, \ldots, q_n^t\right) \in \mathbb{R}_+^n$ | The set of Leader's strategies influencing Follower's set of admissible strategies: quotas, limits, restrictions |
| $T$ | $0 < T \leq \infty$ | The period of consideration |
| $P^t, Q^t$ | $P^t \subset \mathbb{R}_+^m, Q^t \subset \mathbb{R}_+^n$<br>$t = 1, \ldots, T$ | The sets of Leader's strategies which may vary in time |
| $U^t\left(q^t\right)$ | $U^t\left(q^t\right) \subset \mathbb{R}_+^n$<br>$t = 1, \ldots, T$ | The set of Follower's strategies depending on Leader's strategies in the year $t$ |
| $B^t$ | $B^t \subset \mathbb{R}_+^n$<br>$t = 1, \ldots, T$ | The set of admissible bribes from Follower to Leader |
| $g_v^t$ | $g_v^t \colon \mathbb{R}_+^m \times \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^k \to \mathbb{R}$ | Leader's payoff function in the year $t$ |
| $g_u^t$ | $g_u^t \colon \mathbb{R}_+^n \times \mathbb{R}_+^m \times \mathbb{R}_+^k \to \mathbb{R}$ | Follower's payoff function in the year $t$ |
| $\rho$ | $\rho\left(x^t, X\right) = \begin{cases} 0, & x^t \in X \\ > 0, & x^t \notin X \end{cases}$ | A conditional function checking whether the state vector belongs to a given set (for example, a set of sustainable development) |
| $M$ | $M \to \infty$ | Penalty constant |
| $U_v^t, U_u^t$ | $U_v^t, U_u^t \subset \mathbb{R}_+^n$<br>$t = 1, \ldots, T$ | The sets of Follower's strategies satisfying the sustainable development conditions for Leader and Follower respectively |
| $J_v, J_u$ | $J_v \in \mathbb{R}, J_u \in \mathbb{R}$ | Leader and Follower's payoffs in the whole period $T$ respectively |

*p-corruption.* In this case Leader gives to Follower tax privileges for a bribe:

$$p^t = p^t\left(\beta_p^t\right) \tag{8}$$

In the simplest case the dependence may be a linear function:

$$p^t = p_0 - \gamma\,\beta_p^t, \quad \gamma > 0$$

*q-corruption.* In this case Leader gives to Follower quota privileges for a bribe:

$$q^t = q^t\left(\beta_q^t\right) \tag{9}$$

In the simplest variant the dependence may be a linear function as in the previous case:

$$q^t = q_0 - \delta\,\beta_q^t, \quad \delta > 0$$

*a-corruption.* In this case Leader extends for a bribe a set of strategies satisfying the sustainable development conditions; in other words, Leader softens the conditions:

$$U_v^t = U_v^t\left(\beta_q^t\right) \tag{10}$$

In fact, the a-corrupted Leader neglects the sustainable development conditions for a bribe. So this type of corruption is the most dangerous for environment and may results in critical implications.

Let's consider the following problem as a model example:

$$\begin{aligned}
J_v &= \sum_{t=1}^{T}\left[c^t\,p^t\,u^t - g_2\left(p^t,q^t\right) - M\,\rho\left(u^t,U_v^t\right)\right] \to \max \\
g_2\left(p^t,q^t\right) &= \frac{p^t+q^t}{\left(1-p^t\right)\left(1-q^t\right)} \\
& 0 \le p^t \le 1, \quad 0 \le q^t \le 1 \\
J_u &= \sum_{t=1}^{T} c^t\,\left(1-p^t\right)\,u^t \to \max \\
& 0 \le u^t \le 1-q^t \\
U_v^t &= \left[0,a^t\right] \quad 0 \le a^t \le 1 \quad t = 1,\ldots,T
\end{aligned}$$

The introduction of corruption factor in the basic model (Ougolnitsky, 2002; 2004) consists in that Follower gives a part of his payoff to Leader as a bribe. In exchange Leader provides to Follower some privileges in taxes, quotas, or sustainable development conditions.

Let's consider the different types of corruption as defined above.

*p-corruption*

$$J_v = \sum_{t=1}^{T} \left[ c^t \, p_\beta^t \, \left(1 - \beta_p^t\right) \, u^t + c^t \, \beta_p^t \, u^t - g_2 \left(p^t, q^t\right) - M \, \rho \left(u^t, U_v^t\right) \right] \to \max \quad (11)$$

$$J_u = \sum_{t=1}^{T} \left[ c^t \, \left(1 - p_\beta^t\right) \, \left(1 - \beta_p^t\right) \, u^t \right] \to \max \quad (12)$$

$$p_\beta^t = p^t - \gamma \, \beta_p^t + d^t, \quad \gamma > 0 \quad (13)$$

$$0 \le p_\beta^t \le 1, \quad 0 \le q^t \le 1 \quad (14)$$

$$0 \le \beta_p^t \le 1 \quad (15)$$

$$0 \le u^t \le 1 - q^t \quad (16)$$

$$U_v^t = \left[0, a^t\right] \quad 0 \le a^t \le 1 \quad t = 1, \ldots, T \quad (17)$$

Thus, a bribe factor $\beta_p^t$ permits Follower to get some tax privileges. The parameter $d^t$ characterizes so-called "toughness" of corruption. Let's consider that the value $d^t = 0$ corresponds to a "tough" corruption, and the value $d^t = \gamma$ to a "soft" one.

*q-corruption*

$$J_v = \sum_{t=1}^{T} \left[ c^t \, p^t \, \left(1 - \beta_q^t\right) \, u^t + c^t \, \beta_q^t \, u^t - g_2 \left(p^t, q^t\right) - M \, \rho \left(u^t, U_v^t\right) \right] \to \max \quad (18)$$

$$J_u = \sum_{t=1}^{T} \left[ c^t \, \left(1 - p^t\right) \, \left(1 - \beta_q^t\right) \, u^t \right] \to \max \quad (19)$$

$$0 \le p^t \le 1, \quad 0 \le q_\beta^t \le 1 \quad (20)$$

$$q_\beta^t = q^t - \delta \, \beta_q^t + d^t, \quad \delta > 0 \quad (21)$$

$$0 \le \beta_q^t \le 1 \quad (22)$$

$$0 \le u^t \le 1 - q_\beta^t \quad (23)$$

$$U_v^t = \left[0, a^t\right] \quad 0 \le a^t \le 1 \quad t = 1, \ldots, T \quad (24)$$

In this case Follower can get a bigger resource quota giving a bribe in exchange. As in the previous case, the value $d^t = 0$ corresponds to a "tough" corruption, and the value $d^t = \delta$ to a "soft" one.

*a-corruption*

$$J_v = \sum_{t=1}^{T} \left[c^t\, p^t\, \left(1-\beta^t_\alpha\right)\, u^t + c^t\, \beta^t_\alpha\, u^t - g_2\left(p^t, q^t\right) - \right.$$
$$\left. -M\,\rho\left(u^t, U^t_v\left(\alpha^t\right)\right)\right] \to \max \tag{25}$$

$$J_u = \sum_{t=1}^{T} \left[c^t\, \left(1-p^t\right)\, \left(1-\beta^t_\alpha\right)\, u^t\right] \to \max \tag{26}$$

$$0 \le p^t \le 1, \quad 0 \le q^t \le 1 \tag{27}$$

$$\alpha^t = \lambda\,\beta^t_\alpha + d^t, \quad \lambda > 0 \tag{28}$$

$$0 \le \beta^t_\alpha \le 1 \tag{29}$$

$$0 \le u^t \le 1 - q^t \tag{30}$$

$$U^t_v = \left[0, a^t + \alpha^t\right] \quad 0 \le a^t + \alpha^t \le 1 \quad t = 1, \ldots, T \tag{31}$$

In this case Follower can use more resources due to an expansion of the domain of strategies satisfying to the relaxed (for a bribe in exchange) sustainable development conditions. As in two previous cases let's consider that the value $d^t = 0$ corresponds to a "tough" corruption, and the value $d^t = a^t$ to a "soft" one.

Let's investigate the model for different types of corruption and different management methods (compulsion, impulsion, conviction) according to (Ougolnitsky, 2002; 2004) and compare the results with results for the model without corruption factor (a "pure" model). Here we give the results only for the case of compulsion. Index $t$ is omitted because an optimal solution is chosen for the whole period of consideration.

*q-corruption.* According to the definition let's consider that the value $q_\beta = q - \delta\,\beta_q$ corresponds to a "tough" corruption, and the value $q_\beta = q + \delta\,(1-\beta_q)$ — to a "soft" one.

1. *"Soft" corruption*

   The Follower problem is:

$$J_u = c\,(1-p)\,(1-\beta_p)\,u \to \max_{q,\beta_p}$$
$$\text{where } u \in [0, 1-q_\beta]\,, \quad q_\beta = q - \delta\,\beta_q$$

   It is evident that a quantity of resource used by Follower is equal exactly to the quota defined by Leader. Therefore $u^* = 1 - q_\beta = 1 - q + \delta\,\beta_q$. So the Follower's problem may be represented as follows:

$$J_u = c\,(1-p)\,(1-\beta_p)\,(1 - q + \delta\,\beta_q) \to \max_{q,\beta_p}$$

   From the equality $(J_u)'_{\beta_q} = 0$ we get $\beta^*_q = \frac{q+\delta-1}{2\,\delta}$, that is a point of local maximum for the function $J_u\,(\beta_q)$. From the condition $\beta_q \ge 0$ this is true if $\delta \ge 1 - q$ else $\beta^*_q = 0$. Taking into consideration $u \le 1$ we get an inequality $\delta \le 1 + q$. Otherwise Follower chooses the strategy which provides him the absence of Leader's quota, i. e. if $\delta > 1 + q$ then Follower's optimal strategy is equal to $\beta^*_q = \frac{q}{\delta}$.

So we get

$$\beta_q^* = \begin{cases} 0, & \delta \in [0, 1-q] \\ \frac{q+\delta-1}{2\,q}, & \delta \in [1-q, 1+q] \\ \frac{q}{\delta}, & \delta \in [1+q, +\infty) \end{cases} \tag{32}$$

$$J_u^* = \begin{cases} c\,(1-p)\,(1-q)\,, & \delta \in [0, 1-q] \\ c\,(1-p)\,\frac{(1+\delta-q)^2}{4\,\delta}, & \delta \in [1-q, 1+q] \\ c\,(1-p)\,\left(1-\frac{q}{\delta}\right), & \delta \in [1+q, +\infty) \end{cases} \tag{33}$$

Now consider Leader's payoff function. It depends on value of the parameter $\delta$:

$$J_v = \begin{cases} c\,p\,(1-q)\,, & \delta \in [0, 1-q] \\ c\,\left[p\,\frac{(1+\delta-q)^2}{4\,\delta} + \frac{(\delta+q-1)\,(\delta+1-q)}{4\,\delta}\right], & \delta \in [1-q, 1+q] \\ c\,\left[p\,\left(1-\frac{q}{\delta}\right)+\frac{q}{\delta}\right], & \delta \in [1+q, +\infty) \end{cases} \tag{34}$$

Let's consider the three cases separately.

(a) $\delta \in [0, 1-q]$
From (34) it is evident that $q^* = 1-a$. Then $J_v = c\,p\,a$, otherwise, the equilibrium is the same as in the "pure" model (without corruption).

(b) $\delta \in [1-q, 1+q]$
The sustainable development condition means that $u \in [0, a]$. Taking into consideration Follower's optimal reaction we get $\delta \le 2\,a - 1 + q$. From an identical inequality $2\,a-1+q \le 1+q$ follows that it is rational to search for an optimal solution on $\delta$ in the segment $[1-q, 2\,a-1+q]$. For the correct problem formulation it is necessary that $1-q \le 2\,a-1+q$, or $1-q \le a$.

(c) $\delta \in [1+q, +\infty)$
In this case we get from (34) that $q^* = 1$. Hence we will search for a Leader's optimal strategy on $q$ from the following conditions:

$$\delta \in [1-q, 2\,a-1+q]\,, \quad q \in [1-a, 1]$$

Then Leader's optimal strategy is equal to: $q_\beta^* = 1 - 2\,a\,\beta_q$. Besides, $J_v^* = c\,(1+p)\,\frac{a}{2}$; $J_u^* = c\,(1-p)\,\frac{a}{2}$.
The investigation results may be represented as follows:

$$(p^*, q^*, u^*, \beta^*) = \left(p, 1-a, a, \frac{1}{2}\right)$$

$$J_v^* = c\,(1+p)\,\frac{a}{2}$$

$$J_u^* = c\,(1-p)\,\frac{a}{2}$$

By this means "soft" $q$-corruption permits Leader to increase his payoff essentially. As for Follower's payoff, it diminishes twice in comparison with the "pure" model (Ougolnitsky, 2004).

2. *"Tough" corruption*
As in the case of "soft" corruption Follower's optimal strategy is equal to $u^* = 1 - q_\beta$. Then Follower's problem may be represented as

$$J_u = c\,(1-p)\,(1-\beta_q)\,(1-q-\delta\,(1-\beta_q)) \to \max_{q,\beta_q}$$

Equating the first derivative on $\beta_q$ of the function $J_u$ to zero we get a point of local maximum $\beta_q^* = \frac{2\,\delta+q-1}{2\,\delta}$. From (22) this is true if $\delta \geq \frac{1-q}{2}$. Otherwise the local maximum of the function $J_u$ is reached in the point $\beta_q^* = 0$. Hence

$$\beta_q^* = \begin{cases} 0, & \delta \in \left[0, \frac{1-q}{2}\right] \\ \frac{2\,\delta+q-1}{2\,\delta} & \delta \in \left[\frac{1-q}{2}, +\infty\right) \end{cases}$$

$$J_u^* = \begin{cases} c\,(1-p)\,(1-q-\delta)\,, & \delta \in \left[0, \frac{1-q}{2}\right] \\ c\,(1-p)\,\frac{(1-q)^2}{4\,\delta}, & \delta \in \left[\frac{1-q}{2}, +\infty\right) \end{cases}$$

So the "tough" corruption is also not profitable for Follower in any conditions. To find Leader's optimal strategy it is necessary to consider two cases:

(a) $\delta \in \left[0, \frac{1-q}{2}\right]$, then Leader's payoff function is equal to: $J_v = c\,p\,(1-q-\delta)$. It is evident that Leader's optimal strategies satisfy the condition $\delta + q = 1 - a$, hen the payoff function has the form $J_v^* = c\,p\,a$.

(b) $\delta \in \left[\frac{1-q}{2}, +\infty\right)$, then Leader's payoff function is equal to: $J_v = c\left[p\,\frac{(1-q)^2}{4\,\delta} + \frac{(2\,\delta+q-1)\,(1-q)}{4\,\delta}\right]$. From the condition $u \leq a$ we may deduce for this case the condition $q \in [1-2\,a, 1]$. Then the maximum is reached when $\delta$ is infinitely big or $q = 1-2\,a$: $\lim\limits_{\delta\to+\infty} J_v = c\,a$.

So,

$$q_\beta^* = 1 - 2\,a + N\,(1-\beta_q)\,,$$

when $N$ is infinitely big;

$$J_v^* \cong c\,a$$

The investigation results may be represented as follows:

$$(p^*, q^*, u^*, \beta^*) = \left(p, 1-a, a, 1-\frac{a}{N}\right)$$

($N$ is infinitely big)

$$J_v^* \cong c\,a$$
$$J_u^* = 0$$

So the "tough" corruption permits Leader to control the whole resource and Follower's payoff is close to zero.

*a-corruption* The set of sustainable development strategies has a form

$$U_v = [0, a + \alpha\,(\beta_\alpha)]\,, \text{where } \alpha\,(\beta_\alpha) = \lambda\,\beta_\alpha - d$$

1. *"Soft" corruption*
In this case $\alpha\,(\beta_\alpha) = \lambda\,\beta_\alpha$. Then Follower's problem may be represented as

$$J_u\,(p,q) = c\,(1-p)\,(1-\beta_\alpha)\,u \to \max_{u,\beta_\alpha}$$

It is evident that $u^* = 1 - q$. Taking into consideration that Leader's optimal strategy is equal to $q^* = 1 - (a + \alpha\,(\beta_\alpha))$ we get

$$J_u\,(p,q) = c\,(1-p)\,(1-\beta_\alpha)\,(a + \lambda\,\beta_\alpha) \to \max_{u,\beta_\alpha}$$

The solution of the equation $(J_u)'_{\beta_\alpha}$ gives for the function $J_u(\beta_\alpha)$ the point of local maximum $\beta^*_\alpha = \frac{\lambda - a}{2\lambda}$. But from (29) this statement is true only if $\lambda \geq a$. Otherwise Follower must choose a strategy without bribe, i.e. $\beta^*_\alpha = 0$. The condition $a + \alpha \leq 1$ gives that if $\lambda \geq 2 - a$ then Follower must use the whole possible resource, or $\beta^*_\alpha = \frac{1-a}{\lambda}$. Hence

$$\beta^*_\alpha = \begin{cases} 0, & \lambda \in [0, a] \\ \frac{\lambda - a}{2\lambda} & \lambda \in [a, 2-a] \\ \frac{1-a}{\lambda} & \lambda \in [2-a, +\infty) \end{cases}$$

$$J^*_u = \begin{cases} c\,(1-p)\,a, & \lambda \in [0, a] \\ c\,(1-p)\,\frac{(\lambda+a)^2}{4\lambda}, & \lambda \in [a, 2-a] \\ c\,(1-p)\left(1 - \frac{1-a}{\lambda}\right), & \lambda \in [2-a, +\infty) \end{cases}$$

Therefore Follower's choice depends on the "wideness" of the set of sustainable development strategies. If the set is wide enough then Follower does not give any bribe to Leader and looses nothing in comparison with the "pure" case. If the set is not so wide then Follower can increase his payoff in comparison with the "pure" case using a bribe. It is evident that Follower's payoff increases if a value of parameter $\lambda$ increases.

Hence

$$(p^*, q^*, u^*, \beta^*) = \left(p, 0, 1, \frac{1-a}{2-a}\right)$$

$$J^*_v = c\,\frac{p+1-a}{2-a}$$

$$J^*_u = c\,(1-p)\,\frac{1}{2-a}$$

Respectively, Leader's payoff considering Follower's optimal strategy is

$$J_v = \begin{cases} c\,p\,a, & \lambda \in [0, a] \\ c\left[p\,\frac{(\lambda+a)^2}{4\lambda} + \frac{(\lambda-a)\,(\lambda+a)}{4\lambda}\right], & \lambda \in [a, 2-a] \\ c\left[p\,\left(1 - \frac{1-a}{\lambda}\right) + \frac{1-a}{\lambda}\right], & \lambda \in [2-a, +\infty) \end{cases}$$

The maximum of Leader's payoff function on $\lambda$ is reached in the point $\lambda^* = 2-a$. In this case $J^*_v = c\,\frac{1}{2-a}\,(p+1-a)$.

2. *"Tough" corruption*

Analogously to the previous cases we get the following results for Follower's payoff function:

$$\beta^*_\alpha = \begin{cases} \frac{1}{2}, & \lambda \in [0, 2] \\ \frac{1}{\lambda} & \lambda \in [2, +\infty) \end{cases}$$

$$J^*_u = \begin{cases} c\,(1-p)\,\frac{\lambda}{4}, & \lambda \in [0, 2] \\ c\,(1-p)\left(1 - \frac{1}{\lambda}\right), & \lambda \in [2, +\infty) \end{cases}$$

In distinction to the "soft" corruption in this case to give a bribe is always profitable for Follower.

The maximum of Leader's payoff function is reached when $\lambda^* = 2$, from what follows:

$$(p^*, q^*, u^*, \beta^*) = \left(p, 0, 1, \frac{1}{2}\right)$$
$$J_v^* = c\,\frac{1+p}{2}$$
$$J_u^* = c\,\frac{1-p}{2}$$

Let's summarize the results of comparative analysis for different types of corruption. It is evident that corruption is profitable for Leader in all cases. Besides, the "tough" corruption is always more profitable for Leader and less profitable for Follower than the "soft" one.

In the case of compulsion the "tough" $a$-corruption is more profitable for Leader than the "tough" $q$-corruption if $p > 2\,a - 1$. This condition is identically true if $a < \frac{1}{2}$. So, if the conditions of sustainable development are tough ($a < \frac{1}{2}$) then $a$-corruption is economically profitable for Leader because it allows to avoid the conditions. In the same time $a$-corruption may be profitable for Follower too that makes the $a$-corruption even more dangerous. For example, in the case of compulsion the "soft" $a$-corruption is always more profitable for Follower than its absence in the "pure" case. If $a < \frac{1}{2}$ then even the "tough" $a$-corruption gives Follower a bigger payoff than in the "pure" case.

In the case of impulsion the "tough" $a$-corruption is more profitable for Leader than the "tough" $p$-corruption if $a < 1 - \frac{\varepsilon}{2}$ (i. e. practically always).

## 5. Linear Multistage Games with Hierarchical Structure

Organizational systems have a hierarchical structure. The most adequate mathematical formalism for structural description is a directed graph (digraph) in which the vertices represent structural elements and the arcs represent directed relations between elements. It is also possible to take into consideration quantitative characteristics of the elements and relations by introducing weights (values) of the vertices and arcs respectively.

The hierarchy means that some structural elements have a priority in comparison with others. That's why for description of the hierarchical structures it is necessary to use digraphs of a special type. It is the specific digraphs and their incidence matrices that are used to describe the hierarchical structures in the proposed approach.

Let's call connected digraphs without contours and loops the strictly hierarchical ones ($SH$-digraphs). Their incidence matrices we also should call the strictly hierarchical ones.

It is known that a vertex without input arcs exists in every digraph without contours. We should say that all vertices without input arcs form the first layer in the digraph $SH$.

It is possible to use a digraph representing an organizational structure as a base for a game theoretic model formulation in a way that follows.

Let $SH = (Y, Z)$ be a strictly hierarchical digraph with $n$ vertices. Then we should call a strictly hierarchical non-cooperative two-person game generated by $SH$ a linear multistage game in which payoff functions are the following:

$$J_1 = \left(r^{T+1}, x^{T+1}\right) + \sum_{t=1}^{T} \left(r^t, x^t\right) + \sum_{t=0}^{T} \left(s^t, u^t\right) \tag{35}$$

$$J_2 = \left(l^{T+1}, x^{T+1}\right) + \sum_{t=1}^{T} \left(l^t, x^t\right) + \sum_{t=0}^{T} \left(m^t, v^t\right) \tag{36}$$

The equations of controlled process are

$$x^{t+1} = A_t' \, x^t + B_t' \, u^t + C_t' \, v^t, \quad t = 0, 1, \ldots, T \tag{37}$$

The initial conditions are

$$x^0 = x_0$$

The first player tends to maximize the payoff function (35) by choosing strategies $u^t$ with restrictions

$$D_t' \, u^t \le d^t \tag{38}$$

The second player tends to maximize the payoff function (36) by choosing strategies vt with restrictions

$$G_t' \, v^t \le g^t, \quad t = 0, 1, \ldots, T \tag{39}$$

where $x^t, u^t, v^t, r^t, s^t, l^t, m^t, d^t, g^t, x^0$ are vectors from $\mathbb{R}^n$.

A set $(x_i^t, u_i^t, v_i^t, r_i^t, s_i^t, l_i^t, m_i^t, f_i^t, d_i^t, g_i^t, x_{0i})$ is a value of the vertex $y_i \in Y$.

For vertices from the first layer $y_i \in L_1$

$$d_i^t = g_i^t = 0 \tag{40}$$

For vertices from the last layer $y_i \in L_m$

$$u_i^t = v_i^t = 0 \tag{41}$$

$A_t = \left\|a_{ij}^t\right\|$, $B_t = \left\|b_{ij}^t\right\|$, $C_t = \left\|c_{ij}^t\right\|$,

$D_t = \left\|d_{ij}^t\right\|$, $G_t = \left\|g_{ij}^t\right\|$ are strictly hierarchical matrices.

A set $\left(a_{ij}^t, b_{ij}^t, c_{ij}^t, d_{ij}^t, g_{ij}^t\right)$ is a value of the arc $z_{ij} \in Z$.

Let's introduce for each player a Hamilton function

$$H_1\left(p^t, x^t, u^t, v^t\right) = \left(p^t, A_t' \, x^t + B_t' \, u^t + C_t' \, v^t + \left(r^t, x^t\right) + \left(s^t, u^t\right)\right) \tag{42}$$

$$H_2\left(q^t, x^t, u^t, v^t\right) = \left(q^t, A_t' \, x^t + B_t' \, u^t + C_t' \, v^t + \left(l^t, x^t\right) + \left(m^t, v^t\right)\right) \tag{43}$$

where $p^t, q^t$ are conjugate variables according to (37) and defined by formulas

$$p^{t-1} = r^t + A \, p^t, \quad p^T = 0 \tag{44}$$

$$q^{t-1} = l^t + A \, q^t, \quad q^T = 0 \tag{45}$$

The necessary conditions of optimality of strategies $u^*$, $v^*$ in the game (35)–(41) are given by the following conditions (Gavrilov, 1969):

$$\max_{u^t \in U_t} H_1\left(p^t, x^*, u^t, v^*\right) = H_1\left(p^t, x^*, u^*, v^*\right) \tag{46}$$

$$\max_{v^t \in V_t} H_2\left(q^t, x^*, u^*, v^t\right) = H_2\left(q^t, x^*, u^*, v^*\right) \tag{47}$$

where $x^*$ is a trajectory of the system (37) generated by the optimal strategies $u^*, v^*$; $U_t, V_t$ are sets of admissible strategies given by conditions (38), (39) respectively.

To find maximums by (46), (47) it is sufficient to consider the Hamilton functions (42), (43)

$$H_u\left(p^t, u^t\right) = \left(s^t, u^t\right) + \left(p^t, B_t' u^t\right) \tag{48}$$

$$H_v\left(q^t, v^t\right) = \left(m^t, v^t\right) + \left(q^t, C_t' v^t\right) \tag{49}$$

Recurrent formulas (44), (45) permit to calculate $p^t, q^t$:

$$\begin{aligned} p^t &= r^{t+1} + A_t r^{t+2} + \ldots + A_t^{T-t-1} r^T \\ q^t &= l^{t+1} + A_t l^{t+2} + \ldots + A_t^{T-t-1} l^T \\ t &= 0, 1, \ldots, T-1 \end{aligned}$$

So the Hamilton functions (48), (49) are linear functions of the variables $u^t, v^t$ respectively:

$$H_u\left(u^t\right) = \left(K_u^t, u^t\right), \quad H_v\left(v^t\right) = \left(K_v^t, v^t\right),$$

where $K_u^t$ is a vector value which does not contain $u^t$ and depends on the parameters $A^t$, $B^t$, $s^t$, $r^t$;
$K_v^t$ is a vector value which does not contain $v^t$ and depends on the parameters $A^t$, $C^t$, $m^t$, $l^t$.

The conditions of nonnegativity of the strategies arise naturally in applications:

$$u^t \geq 0, \quad v^t \geq 0$$

To satisfy the necessary optimality conditions (46), (47) it is sufficient to solve the pair of linear programming problems

$$\begin{aligned} &\left(K_u^t, u^t\right) \to \max, \quad D_t' u^t \leq d^t, \quad u^t \geq 0, \quad t = 0, 1, \ldots, T \\ &\left(K_v^t, v^t\right) \to \max, \quad G_t' v^t \leq g^t, \quad v^t \geq 0, \quad t = 0, 1, \ldots, T \end{aligned}$$

These linear programming problems have the same type as the initial game (35)–(41). That game is explicitly generated by an organizational structure given by the strictly hierarchical digraph $SH$.

## 6. Conclusion

We think it is worthwhile to combine several mathematical concepts in the same model for describing organizational dynamics and management processes. These concepts are: a directed graph without contours and loops, a game of $n$ players in normal form, a cooperative game, a dynamical system controlled by several subjects. A generalized model (1) of hierarchically controlled dynamical system based on the idea is proposed in this paper and partly illustrated by some examples.

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# Proportionality in Bargaining Games: *status quo*-Proportional Solution and Consistency


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**Abstract** The *status quo*-proportional solution for bargaining games is studied. The consistency of the solution is proved, and it is used to give an axiomatic characterization of this solution. The logarithmic transformation of player's utilities is considered and is used to establish the relation between some bargaining solutions: the Nash bargaining solution, the egalitarian, the utilitarian and the *status quo*-proportional solutions.



**Keywords:** bargaining games, bargaining solutions, Nash solution, *status quo*-proportional solution, consistency.


## 1. Introduction

In this paper we study the consistency property of the *status quo*-proportional solution defined by the author in (Pechersky, 2007), where an axiomatic approach was developed to define uniquely the *proportional excess* function on the space of positively generated NTU games, which generalizes to NTU games the proportional TU excess $v(S)/x(S)$. The properties of proportional excess and corresponding nucleolus and prenucleolus were studied. In particular, it was shown that for non-leveled bargaining games with positive *status quo* point the nucleolus (and prenucleolus) defines the Pareto optimal point of a feasible set proportional to the *status quo* point. Therefore this solution was called the *status quo*-proportional solution (to distinguish it from the proportional solution due to Kalai (Kalai, 1977)).

It was shown that for the class $\mathcal{G}$ of comprehensive (**0**-comprehensive), non-leveled bargaining games $(q, Q)$ with positive *status quo* points $q$ the *status quo*-proportional solution is the unique solution satisfying Pareto optimality, scale covariance, anonymity and strong monotonicity axioms.

It is not difficult to show that the strong monotonicity axiom can be replaced by independence of irrelevant alternatives axiom. So the system of axioms characterizing the *status quo*-proportional solution becomes an analogues of that for Nash solution (the difference is in the classes of bargaining games: $\mathcal{G}$ with positive *status quo* points for the *status quo*-proportional solution, and bargaining games with convex feasible sets and zero *status quo* points for the Nash solution).

Most of the axiomatic theory of bargaining has been written under the assumption of a fixed number of agents. However, the model has been enriched by allowing the number of agents to vary (Thomson, 1994). Axioms specifying how solutions could, or should, respond to such changes have been formulated and new characterizations of the main solutions as well as of new solutions generalizing them have been developed. In particular, Lensberg (Lensberg, 1988) used a stability (consistency) axiom to give a characterization of the Nash solution without independence of irrelevant alternatives.

Our aim in proposed paper is to consider the consistency property for the *status quo*-proportional solution and give corresponding axiomatic characterization. Then we consider the logarithmic transformation of player's utilities to establish the relation between some bargaining solutions, in particular, the Nash bargaining solution, the egalitarian, the utilitarian and the *status quo*-proportional solutions.

The paper is organized as follows. Section 2 provides definitions and notations. In Section 3 we study the consistency property for the *status quo*-proportional solution and compare the solution and the Nash bargaining solution. In Section 4 we apply the logarithmic transformation of player's utilities and consider some relations between the bargaining solutions.

## 2. Definitions and Notations

Let $\mathcal{N}$ be the set of natural numbers and let $\mathcal{P}$ be the family of nonempty, finite subsets of $\mathcal{N}$. The members of $\mathcal{P}$ will be denoted $N, N', \ldots$. $\mathcal{N}$ may be thought of as the set of all potential players, and $\mathcal{P}$ is the family of all subsets of those players that may conceivably become involved in some bargaining problem.

Let $N \in \mathcal{N}$ be a non-empty finite set of players.

*Bargaining game.* A bargaining game with the players set $N \in \mathcal{P}$ is a pair $(q, Q)$, where $q \in \mathbb{R}^N$ is the *status quo* point, and $Q \subset \mathbb{R}^N$. When interpreting this pair one can think as follows: if the players act separately the only possible outcome for the players is $q$ giving utility $q_i$ to player $i \in N$. If all players cooperate they can potentially agree on an arbitrary outcome $x \in Q$.

We suppose that every bargaining game $(q, Q)$ satisfies the following properties:

(a) $Q \subset \mathbb{R}^N_+$ is compact and comprehensive, i. e. $x \in Q$, $y \in \mathbb{R}^N_+$, and $x \geq y$ imply $y \in Q$;

(b) $Q$ is non-leveled, i. e.

$$x, y \in \partial Q, \quad x \geq y \;\Rightarrow\; x = y;$$

(c) $q \geq \mathbf{0}$ and there is $x \in Q$ such that $x > q$.

Let $x, y \in \mathbb{R}^N$. We will write $x \geq y$, if $x_i \geq y_i$ for all $i \in N$; $x > y$ , if $x_i > y_i$ for all $i \in N$. Denote

$$\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \,:\, x \geq \mathbf{0}\},$$

$$\mathbb{R}^N_{++} = \{x \in \mathbb{R}^N \,:\, x > \mathbf{0}\},$$

where $\mathbf{0}$=(0,0,...,0). Further we denote by $e_N$ the vector $e_N = (1, 1, \ldots, 1) \in \mathbb{R}^N$.

Let $A \subset \mathbb{R}^N_+$. If $x \in \mathbb{R}^N$, then $x + A = \{x + a \,:\, a \in A\}$ and $\lambda A = \{\lambda a \,:\, a \in A\}$. The boundary of $A$ is denoted by $\partial A$. For $x, y \in \mathbb{R}^N$ we denote by $[x, y]$ (or $[y, x]$) the segment of the form $[x, y] = \{z \in \mathbb{R}^N : \; z = \lambda x + (1 - \lambda) y, \, \lambda \in [0, 1]\}$.

In what follows we keep mostly the notations used by Lensberg (Lensberg, 1988). Let $N \in \mathcal{P}$. Denote by $\Omega^N$ the family of all bargaining games with the players set $N$ satisfying the properties (a)–(c).

A *bargaining solution* is a map

$$F : \bigcup_{N \in \mathcal{P}} \Omega^N \to \bigcup_{N \in \mathcal{P}} \mathbb{R}^N_+,$$

which associates with each element $N \in \mathcal{P}$ and each $(q,Q) \in \Omega^N$ a unique point of $Q$, $F(q,Q)$ being called the solution of bargaining game $(q,Q)$.

Now we address to the axioms, which will play an important role in the paper. We formulate them in form suitable for the case of variable set of players.

**Pareto optimality (PO).** For every $N \in \mathcal{P}$, for every $(q,Q) \in \Omega^N$ it follows from $F(q,Q) = x$ that there is no $y \in Q$ such that $y \geq x$, $y \neq x$.

For every $N, N' \in \mathcal{P}$ such that $|N| = |N'|$ let us denote by $\Gamma^{N,N'}$ the family of one-to-one correspondences $\gamma : N \to N'$. It will be convenient to consider sometimes $\gamma \in \Gamma^{N,N'}$ as a function from $\mathrm{I\!R}^N$ in $\mathrm{I\!R}^{N'}$, defined by $y = \gamma(x)$, if $y_{\gamma(i)} = x_i$ for every $i \in N$. For every $(q,Q) \in \Omega^N$ we define also $\gamma(q,Q) \equiv (\gamma(q), \gamma(Q))$.

**Anonymity (AN).** For every $N, N' \in \mathcal{P}$ such that $|N| = |N'|$, for every $\gamma \in \Gamma^{N,N'}$ and every $(q,Q) \in \Omega^N$

$$F(\gamma(q,Q)) = \gamma(F(q,Q)).$$

For every $N \in \mathcal{P}$ denote by $\Lambda^N$ the family of such mappings $\mathrm{I\!R}^N$ in $\mathrm{I\!R}^{N'}$ that for any $\lambda \in \Lambda^N$ there is such $a \in \mathrm{I\!R}^N_{++}$ that for every $i \in N$ and every $x \in \mathrm{I\!R}^N$

$$\lambda_i(x) = a_i x_i.$$

It will be convenient to use the following notation for $\lambda \in \Lambda^N$ and $(q,Q) \in \Omega^N$:

$$\lambda(q,Q) \equiv (\lambda(q), \lambda(Q)).$$

**Scale covariance (SC).** For every $N \in \mathcal{P}$, every $(q,Q) \in \Omega^N$ and every $\lambda \in \Lambda^N$

$$F(\lambda(q,Q)) = \lambda(F(q,Q)).$$

**Remark 1.** Lensberg uses term Scale Invariance for the axiom of this type, but we prefer Scale Covariance.

**Independence of Irrelevant Alternatives (IIA).** For all $N \in \mathcal{P}$, for all $(q,Q), (q,Q') \in \Omega^N$, if $Q' \subset Q$ and $F(q,Q) \in (q,Q')$, then $F(q,Q') = F(q,Q)$.

For every $N, N' \in \mathcal{P}$ such that $N \subset N'$ and $x \in \mathrm{I\!R}^N$ let us denote by $H^x_N$ the hyperplane in $\mathrm{I\!R}^N$ of the form

$$H^x_N = \{y \in \mathrm{I\!R}^N \, : \, y_{N' \setminus N} = x_{N' \setminus N}\}.$$

We denote also for every $N, N' \in \mathcal{P}$ such that $N \subset N'$ and $x \in \mathrm{I\!R}^{N'}$ the restriction of $x$ on $\mathrm{I\!R}^N$ by $x_N$.

For $Q \subset \mathrm{I\!R}^{N'}$ and $x \in Q$ denote by $t^x_N(Q)$ the projection of the set $H^x_N \cap Q$ on $\mathrm{I\!R}^N$.

Now we can formulate two consistency axioms which are akin to Lensberg's axioms (Lensberg, 1988), but use non-zero *status quo* points.

**Bilateral stability (B.STAB).** For every $N, N' \in \mathcal{P}$ such that $N \subset N'$ and $|N| = 2$, for every $(q,Q) \in \Omega^N$ and every $(r,R) \in \Omega^{N'}$ if $q = r_N$ and $Q = t^x_N(R)$, where $x = F(r,R)$, then $F(q,Q) = x_N$.

The following axiom is a strengthened form of the previous one.

**Multilateral stability (M.STAB).** For every $N, N' \in \mathcal{P}$ such that $N \subset N'$, for every $(q, Q) \in \Omega^N$ and every $(r, R) \in \Omega^{N'}$ if $q = r_N$ and $Q = t_N^x(R)$, where $x = F(r, R)$, then $F(q, Q) = x_N$.

We denote for every $N \in \mathcal{P}$ by $\Omega_0^N$ the family of all bargaining games in $\Omega^N$ satisfying properties (a) and (c) with $q = \mathbf{0}$ and convex $Q$. In this case every bargaining game $(q, Q)$ can be simply denoted by $Q$. By $\Omega_+^N$ we denote the family of all bargaining games in $\Omega^N$ satisfying properties (a)–(c) with $q > \mathbf{0}$.

**Remark 2.** We do not give here the original axioms due to Lensberg, because they can be easily derived from just mentioned simply by replacing non-zero *status quo* points $q$ by $\mathbf{0}$ and taking into account that $\lambda_i(\mathbf{0}) = 0$.

## 3. The Nash Solution and the *status quo*-Proportional Solution

### 3.1. Solutions and IIA

As we have mentioned above one of our goal is to show that the *status quo*-proportional solution is closely related to the Nash bargaining solution in the sense that they both are defined by the same system of axioms but for different classes of bargaining games.

It is well known that the Nash bargaining solution (Nash, 1950) is characterized by the following theorem.

**Theorem 1.** *A solution $F$ satisfies PO, AN, SC and IIA if and only if for all $N \in \mathcal{P}$ and all $Q \in \Omega_0^N$, $F(Q) = NS(Q)$, where NS denotes the Nash bargaining solution.*

Let us recall the definition of the *status quo*-proportional solution (*sq*-proportional solution in what follows).

Let $N \in \mathcal{P}$. Then the *sq*-proportional solution PS is defined for every $(q, Q) \in \Omega_+^N$ by

$$PS(q, Q) = \mu(q, Q)q,$$

where $\mu(q, Q) = \max\{t \in \mathbb{R}_+ : tq \in Q\}$.

To characterize the *sq*-proportional solution we need one more axiom.

**Strong monotonicity (SM).** For every $N \in \mathcal{P}$ and all $(q, Q), (q, Q') \in \Omega^N$, if $Q \subset Q'$, then $F(q, Q') > F(q, Q)$.

In (Pechersky, 2007) the following theorem has been proved (we give it with appropriate modification to the case of variable set of players).

**Theorem 2.** *The* sq-*proportional solution is the unique solution on $\Omega_+ = \bigcup_{N \in \mathcal{P}} \Omega_+^N$ satisfying PO, SC, AN and SM.*

It is easy to note that the proof holds if we replace the SM by IIA. So we can state the following theorem.

**Theorem 3.** *The* sq-*proportional solution is the unique solution on $\Omega_+$ satisfying PO, SC, AN and IIA.*

Thus the Nash solution and the *sq*-proportional solution are both characterized by PO, SC, AN and IIA. But the first is defined on $\Omega_0 = \bigcup_N \Omega_0^N$, and the latter on $\Omega_+$.

### 3.2. Solutions and Consistency

The main result of the paper by Lensberg (Lensberg, 1988) is the following theorem.

**Theorem 4.** *A solution on $\Omega_0$ satisfies PO, AN, SC and M.STAB if and only if it is the Nash solution.*

Now turn to the *sq*-proportional solution.

**Proposition 1.** *The* sq-*proportional solution on $\Omega_+$ satisfies PO, SC, AN and M.STAB.*

*Proof.* PO, SC and AN are proved in Proposition 7 in (Pechersky, 2007). So we have to check only M.STAB.

Since *sq*-proportional solution satisfies SC, we can consider for all $N \in \mathcal{P}$ the bargaining games of the form $(e_N, Q)$. Then $PS(e_N, Q) = \mu e_N$ for corresponding number $\mu$.

Let now $N, N' \in \mathcal{P}$ be such that $N \subset N'$ and $(e_{N'}, R) \in \Omega_+^{N'}$. Then define $(q, Q)$ as follows: $q = e_N$ and $Q = t_N^{\mu e_{N'}}(R)$, where $\mu e_{N'} = PS(e_{N'}, R)$. Clearly, $\mu e_N \in t_N^{\mu e_{N'}}(R)$ is Pareto optimal, and so $\mu e_N = PS(e_N, Q)$. □

**Corollary 1.** *The* sq-*proportional solution satisfies B.STAB.*

**Proposition 2.** *If a solution $F$ on $\Omega_+$ satisfies PO, AN, SC and B.STAB, then for all $N \in \mathcal{P}$ with $|N| = 2$ and all $(q, Q) \in \Omega_+^N, F(q, Q) = PS(q, Q)$.*

*Proof.* Let $N \in \mathcal{P}$ with $|N| = 2$, and let $(q, Q) \in \Omega_+^N$. Without loss of generality we can assume (by AN), that $N = \{1, 2\}$, and (by SC) that $q = e_N$.

We will define a bargaining game $(r, R) \in \Omega_+^{N'}$ with $|N'| = 3$ such that $r = e_{N'}, PS(r, R) = \mu e_{N'}$ for some $\mu$ and $t_N^{\mu e_{N'}}(R) = Q$.

By AN we can assume that $N' = \{1, 2, 3\}$. Let us consider the set $Q \subset \mathbb{R}_+^N$. By property (c) there is $x \in \partial Q$ such that $x = \mu e_N$ with $\mu > 1$. Define $\varepsilon = \mu - 1$. Then $x = (1 + \varepsilon, 1 + \varepsilon)$, $\varepsilon > 0$. By non-levelness condition the set $\partial Q$ can be represented in the following manner:

$$\partial Q = \{(x_1, x_2) \in \mathbb{R}_+^N : \ x_2 = f(x_1)\}$$

for some strictly decreasing function $f$.

Let $f(0) = \alpha$ and $f^{-1}(0) = \beta$. Without loss of generality we can assume that $\alpha \leq \beta$. Note also that by non-levelness $\alpha > 1 + \varepsilon$. Let now $\gamma : \{1, 2\} \to \{2, 3\}$ and $\gamma' : \{2, 3\} \to \{3, 1\}$. Denote $Q^{23} = \gamma(Q)$ and $Q^{31} = \gamma'(Q^{23})$. Let $R_1 \subset \mathbb{R}_+^{N'}$ be defined by $R_1 = Q_1 \cup Q_2 \cup Q_3$, where

$$Q_1 = Q^{23} \times \{\mu e_{\{1\}}\}, Q_2 = Q^{31} \times \{\mu e_{\{2\}}\}, Q_3 = Q \times \{\mu e_{\{3\}}\}.$$

By construction $\mu e_{N'} \in R_1$ (cf. Fig. 1).

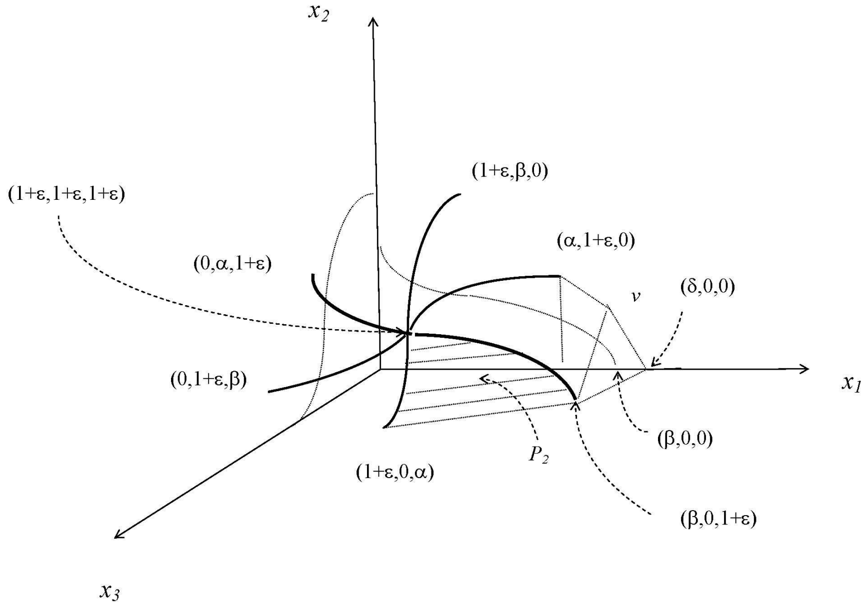


Fig. 1

Our aim is to define the set R as a "hull" of the set $R_1$. To do this consider first an arbitrary $a \in [0, 1+\varepsilon]$. Then this $a$ defines (by non-levelness) exactly two points $y(a) \in \partial Q_1$ and $z(a) \in \partial Q_3$ (we consider $Q_1, Q_2, Q_3$ as two-dimensional sets) such that $y_2(a) = z_2(a) = a$ (note that $y(a) = z(a)$ for $a = 1+\varepsilon$). Let now

$$P_2 = \bigcup_{a \in [0,1+\varepsilon]} P(a),$$

where $P(a)$ is the segment $[z(a), y(a)]$.

Similarly define the sets $P_1$ and $P_3$.

The following construction will be divided into the three steps.

(1) Consider for every $b \in [1+\varepsilon, \alpha]$ (recall that by non-levelness $\alpha > 1+\varepsilon$) two points $z(b) \in \partial Q_3$ and $w(b) \in \partial Q_2$ (they are defined uniquely) with $z_1(a) = w_1(b) = b$. Define $E(b) = [z(b), w(b)]$.

(2) Since $\alpha > 1+\varepsilon$ (by non-levelness) then $f(\alpha) < 1+\varepsilon$. Let $v = (\beta, f(\alpha), 0)$. Consider the segment $S_3 = [(\alpha, 1+\varepsilon, 0), v]$. Then every $b \in [\alpha, \beta]$ defines two points $z(b) \in \partial Q_3$ and $w(b) \in S_3$ with $z_1(b) = w_1(b)$. Define $E(b) = [z(b), w(b)]$. If $\alpha = \beta$, then we omit this step and take $v = (\alpha, 1+\varepsilon, 0)$.

(3) Now take the point $h = (\delta, 0, 0)$ with $\delta > \beta$, and consider two segments $S_3' = [v, h]$ and $S_2 = [(\beta, 0, 1+\varepsilon), h]$. Then every $b \in [\beta, \delta]$ defines two points $z(b) \in S_2$ and $w(b) \in S_3'$ with $z_1(b) = w_1(b)$ (for $b = \delta$ they coincide. Define $E(b) = [z(b), w(b)]$.

Now define $E_1 = \bigcup_{b \in [1+\varepsilon, \delta]} E(b)$

Similarly define the sets $E_2$ and $E_3$.

Finally, let us define $R$ to be the comprehensive hull of the set $P_1 \cup P_2 \cup P_3 \cup E_1 \cup E_2 \cup E_3$.

Let us check that $R$ satisfies the properties for the bargaining game $(e_{N'}, R)$ to be in $\Omega_+^{N'}$.

The comprehensiveness holds by construction. Now check the non-levelness. Consider first the set $P_2$, and take an arbitrary $x_2 = a < 1+\varepsilon$. Clearly, the segment $P(a) = [y(a), z(a)] \subset P_2$, and it is parallel to the plane $x_2 = 0$. Further, $y_1(a) = 1+\varepsilon$ and $z_3(a) = 1+\varepsilon$ for all $a$. Every point in $P(a)$ can be represented uniquely as $u(a) = \lambda y(a) + (1-\lambda)z(a)$ for an appropriate $\lambda \in [0,1]$. So any two different points $u^1(a), u^2(a) \in P(a)$ differ both in the first and in the third coordinates. So there are no segments in $P_2$ parallel to axis $x_1$ or $x_3$.

Let now suppose that there is a segment $I$ in $P_2$ parallel to axis $x_2$. This means that for every point $v \in I$ we have $v_1 = c_1$ and $v_3 = c_3$ for some numbers $c_1, c_3 > 1+\varepsilon$. Consider two different points $v^1, v^2 \in I$. Then

$$v^1 = \lambda_1 y(a_1) + (1-\lambda_1)z(a_1), v^2 = \lambda_2 y(a_2) + (1-\lambda_2)z(a_2)$$

for some appropriate numbers $\lambda_1, \lambda_2 \in [0,1]$ and $a_1, a_2 > 1+\varepsilon$.

Suppose $a_1 < a_2$. Since $f$ is strictly decreasing, we have $y_3(a_1) > y_3(a_2), y_1(a_1) = y_1(a_2) = 1+\varepsilon, z_1(a_1) > z_1(a_2)$ and $z_3(a_1) = z_3(a_2) = 1+\varepsilon$. Then

$$c_1 = \lambda_1(1+\varepsilon) + (1-\lambda_1)z_1(a_1) = \lambda_2(1+\varepsilon) + (1-\lambda_2)z_1(a_2),$$

$$c_3 = \lambda_1 y_3(a_1) + (1-\lambda_1)(1+\varepsilon) = \lambda_2 y_3(a_2) + (1-\lambda_2)(1+\varepsilon).$$

From the first equation it follows that $\lambda_2 < \lambda_1$, and from the second that $\lambda_2 > \lambda_1$, a contradiction. Of course the same assertion holds for $P_1$ and $P_3$.

The proof that $E_1$ (and also $E_2$ and $E_3$) does not contain segments, which are parallel to the axes, is akin to just given. But we should take into account not only the monotonicity of $f$, but also the construction of the segments $S_2, S_3$ and $S_3'$ (see steps (2) and (3) above).

Thus the set $R$ is such that the bargaining game $(e_{N'}, R) \in \Omega_+^{N'}$ (recall that the point $x = (1+\varepsilon, 1+\varepsilon, 1+\varepsilon)$ belongs to $R$).

Since $R$ is invariant under rotations of axes, $F(e_{N'}, R) = x$.

Now let us check that $t_N^{\mu e_{N'}}(R) = Q$. It is sufficient to show that every $v \in \partial(Q_3)$ is Pareto optimal in $R$. To prove this let us take an arbitrary point $v \in \partial Q_3$. If $x_1 > 1+\varepsilon$ then $y_1 \le 1+\varepsilon$ for every $y \in (E_2 \cup R_1 \cup E_3)$. For every $y \in (E_1 \cup R_3)$ we have $y_3 < x_3$ or $y_3 = x_3$, but in the last case $y_1 < x_1$, or if $y_1 > x_1$, then $y_2 < x_2$ (since $f$ is strictly decreasing.

Finally consider an arbitrary $y \in R_2, y \ne x$. Then $y \in [y(a), z(a)]$ for some $a \in [0, 1+\varepsilon]$. When $y = z(a)$ then $z_1(a) < x_1$, or $z_1(a) > x_1$ but $z_2(a) < x_2$.

Let now $y \ne z(a)$. If $a > x_2$ then $y_2 < x_2$. If $a \le x_2$, then $z_1(a) \le x_1, y_1(a) = 1+\varepsilon$. Then

$$y_1 = \lambda y_1(a) + (1-\lambda)z_1(a) = \lambda(1+\varepsilon) + (1-\lambda)z_1(a) < x_1$$

for any $\lambda \in (0,1]$ (since $x_1 > 1+\varepsilon$).

The case $x_1 < 1+\varepsilon$ is akin to just discussed.

As we have mentioned above $F(e_{N'}, R) = \mu e_{N'}$. Then by B.STAB $F(e_N, Q) = F(e_N, t_N^{\mu e_{N'}}(R)) = \mu e_N$.

□

The following theorem summarize the above discussion.

**Theorem 5.** *If a solution $F$ on $\Omega_+$ satisfies PO, AN, SC and B.STAB, if and only if it is the* status quo*-proportional solution.*

*Proof.* Let us consider an arbitrary $N' \in \mathcal{P}$, and let $(r, R) \in \Omega_+^{N'}$. By AN we can assume that $q = e_{N'}$. Let $x = F(e_{N'}, R)$. Then, by B.STAB, for every $N \subset N'$, $|N| = 2$ we have $F(e_N, Q) = x_N$, where $Q = t_N^x(R)$. By Proposition 2 $F(e_N, Q) = PS(e_N, Q) = \mu e_N$ for an appropriate $\mu$. Hence $x_i = x_j = \mu$ for $i, j \in N$ and every $N \subset N'$. Therefore $x_i = x_j = \mu$ for all $i, j \in N'$, and so $x = \mu e_{N'}$.

□

**Remark 3.** It is clear that B.STAB can be replaced by its stronger version M.STAB.

Thus the Nash solution and the *sq*-proportional solution can be characterized by the same systems of axioms. The difference is that the first one is defined on $\Omega_0$, and the second one on $\Omega_+$.

## 4. Some Relations Between Bargaining Solutions

In the section we consider some relations between four bargaining solutions: the Nash solution, the *sq*-proportional solution, the egalitarian solution, and the utilitarian solution.

First we will consider what happens with solutions when we apply the logarithmic transformation of utilities.

Let now $N = \{1, 2, \ldots, n\}$ be fixed, and $x \in \mathbb{R}_{++}^N$. Denote

$$LN(x) = (\ln(x_1), \ln(x_2), \ldots, \ln(x_n)).$$

Then we can define the bargaining game $LN(q, Q) = (LN(q), LN(Q))$, where

$$LN(Q) = \{(y_1, y_2, \ldots, y_n) : \exists x \in Q : y = LN(x)\}.$$

Here we should note that to preserve positivity we need to restrict our attention to the bargaining games $(q, Q)$ with $q \geq e_N$ and $q$-comprehensive sets $Q$. So if we denote $Q_q^+ = comp(Q \cap (q + \mathbb{R}_+^N))$, define $LN(q, Q) = (r, R)$, where $r = LN(q)$ and $R = LN(Q_q^+) \cap \mathbb{R}_+^N$, then we avoid the problem.

However, this problem is not of key importance in our discussion, because Scale Covariance of the Nash solution and the *sq*-proportional solution allows to consider the bargaining games with just mentioned properties. Besides we are interested mostly in the structure of relations between the solutions.

Let us consider the Nash solution. It is not difficult to check that for every $Q \in \Omega_0^N$

$$LN(NS(Q)) = US(LN(Q)),$$

where $US$ is the utilitarian solution.

Indeed, let $x = NS(Q)$, then

$$\prod_{i\in N} x_i \geq \prod_{i\in N} y_i \quad \forall y \in Q.$$

Then

$$\ln \prod_{i\in N} x_i \geq \ln \prod_{i\in N} y_i \;\Rightarrow \sum_{i\in N} \ln x_i \geq \sum_{i\in N} \ln y_i.$$

Then if we denote $z = LN(x)$, then

$$\sum_{i\in N} z_i \geq \sum_{i\in N} w_i \quad \forall w \in LN(Q).$$

Hence $z = US(LN(Q))$.

**Remark 4.** The Nash solution is defined for the bargaining games with convex sets $Q$ and is (strongly) Pareto optimal for every bargaining game. Hence, by concavity of ln, all (strongly) Pareto optimal points transform into the extremal points in $LN(Q)$. So $z$ will be the unique point maximizing the sum of player's utilities.

Recall also that the the Utilitarian solution does not depend on the *status quo* point and is characterized by Pareto optimality, Symmetry, Positive Homogeneity and Additivity axioms (cf., for example, Pechersky, 2006).

Let now consider the *sq*-proportional solution. Then it is clear that

$$LN(PS(q, Q)) = ES(LN(q, Q)),$$

where $ES$ is the egalitarian solution defined for a bargaining game $(r, R)$ by

$$ES(r, R) = r + \delta e_N,$$

where $\delta = \max\{t > 0 : \; r + te_N \in R\}$.

Indeed, let $x = PS(q, Q)$. This means that $x = \mu q$, where $\mu = \max\{t > 0 : tq \in Q\}$. Then $\ln x_i = \ln \mu + \ln q_i$ for every $i$. Hence $z = LN(x) = LN(q) + \delta e_N$, where $\delta = \ln \mu$.

The egalitarian solution for bargaining games with $q = \mathbf{0}$ and convex, comprehensive sets $Q$ is characterized by (weak) PO, SM, Symmetry and Positive Homogeneity axioms (cf., Kalai, 1977).

**Remark 5.** The definition of the Nash solution and the egalitarian solution for $q \neq \mathbf{0}$ requires the Translation Invariance axiom.

Our discussion we conclude with the scheme, which demonstrates the relations between discussed solutions (see. Fig. 2).

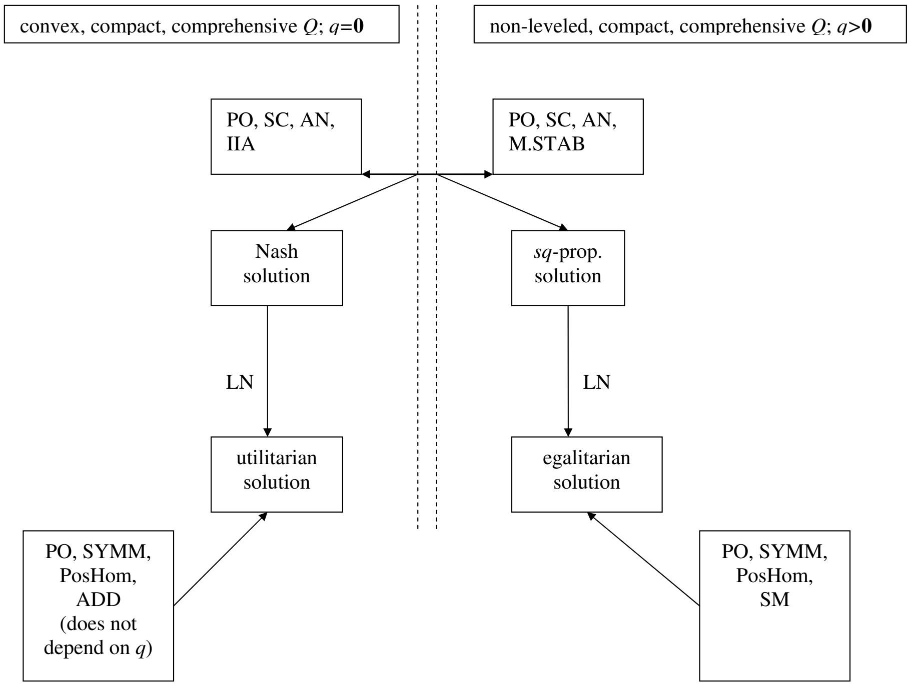


Fig. 2

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# Conditions for Sustainable Cooperation

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**Abstract** There are three important aspects which must be taken into account when the problem of stability of long-range cooperative agreements is investigated: time-consistency of the cooperative agreements, strategic stability and irrational behavior proofness. The mathematical results based on imputation distribution procedures (IDP) are developed to deal with the above mentioned aspects of cooperation. We proved that for a special class of differential games time-consistent cooperative agreement can be strategically supported by Nash equilibrium. We also consider an example where all three conditions are satisfied.

**Keywords:**differential game, cooperative solution, time-consistency of the cooperative agreements, payoff distribution procedures (PDP), imputation distribution procedures (IDP), strategic stability, irrational behavior proofness

## 1. Introduction

Cooperation is a basic form of human behavior. And for many practical reasons it is important that cooperation remains stable on a time interval under consideration. There are three important aspects which must be taken into account when the problem of stability of long-range cooperative agreements is investigated.

1. *Time-consistency (dynamic stability) of the cooperative agreements.* Time-consistency involves the property that, as the cooperation develops cooperating partners are guided by the same optimality principle at each instant of time and hence do not possess incentives to deviate from the previously adopted cooperative behavior.
2. *Strategic stability.* The agreement is to be developed in such a manner that at least individual deviations from the cooperation by each partner will not give any advantage to the deviator. This means that the outcome of cooperative agreement must be attained in some Nash equilibrium, which will guarantee the strategic support of the cooperation.
3. *Irrational behavior proofness.* This aspect must be also taken in account since not always one can be sure that the partners will behave rational on a long time interval for which the cooperative agreement is valid. The partners involved in the cooperation must be sure that even in the worst case scenario they will not loose compared with non cooperative behavior.

The mathematical tool based on payoff distribution procedure (PDP) or imputation distribution procedure (IDP) is developed to deal with the above mentioned aspects of cooperation.

## 2. Continuous Time Case

Consider $n$-person differential game $\Gamma(x_0, T - t_0)$ with prescribed duration and independent motions on the time interval $[t_0, T]$. Motion equations have the form:

$$\dot{x}_i = f_i(x_i, u_i), \quad u_i \in U_i \subset R^\ell, x_i = (x_{i1,\dots,x_{im}}) \in R^m, f_i = (f_{i1}, \dots, f_{im}) \in R^m$$
$$x_i(t_0) = x_i^0, \quad i = 1, \dots, n. \tag{1}$$

It is assumed that the system of differential equations (1) satisfies all conditions necessary for the existence, prolongability and uniqueness of the solution for any $n$-tuple of measurable controls $u_1(t), \dots, u_n(t)$.

The payoff of player $i$ is defined as:

$$H_i(x_0, T - t_0; u_1(\cdot), \dots, u_n(\cdot)) = \int_{t_0}^{T} h_i(x_0; x(\tau))d\tau,$$

where $h_i(x_0; x)$ is a continuous function and $x(\tau) = \{x_1(\tau), \dots, x_n(\tau)\}$ is the solution of (1) when open-loop controls $u_1(t), \dots, u_n(t)$ are used and $x(t_0) = \{x_1(t_0), \dots, x_n(t_0)\} = \{x_1^0, \dots, x_n^0\} = x_0$.

Suppose that there exist an $n$-tuple of open-loop controls $\bar{u}(t) = (\bar{u}_1(t), \dots, \bar{u}_n(t))$ and the trajectory $\bar{x}(t)$, $t \in [t_0, T]$, such that

$$\max_{u_1(t),\dots,u_n(t)} \sum_{i=1}^{n} H_i(x_0, T - t_0; u_1(t), \dots, u_n(t)) =$$
$$= \sum_{i=1}^{n} H_i(x_0, T - t_0; \bar{u}_1(t), \dots, \bar{u}_n(t)) = \sum_{i=1}^{n} \int_{t_0}^{T} h_i(x_0; \bar{x}(\tau))d\tau. \tag{2}$$

The trajectory $\bar{x}(t) = (\bar{x}_1(t), \dots, \bar{x}_n(t))$ satisfying (2) we shall call "optimal cooperative trajectory".

Let $N = \{1, \dots, n\}$ be the set of players. Define in $\Gamma(x_0, T - t_0)$ characteristic function in a classical way:

$$V(x_0, T - t_0; N) = \sum_{i=1}^{n} \int_{t_0}^{T} h_i(x_0; \bar{x}(\tau))d\tau,$$
$$V(x_0, T - t_0; \emptyset) = 0,$$
$$V(x_0, T - t_0; S) = Val\, \Gamma_{S, N \setminus S}(x_0, T - t_0), \tag{3}$$

where $Val\, \Gamma_{S,N\setminus S}(x_0, T - t_0)$ is a value of zero-sum game played between coalition $S$ acting as first player and coalition $N \setminus S$ acting as player 2, with payoff of player $S$ equal to:

$$\sum_{i \in S} H_i(x_0, T - t_0; u_1(\cdot), \dots, u_n(\cdot)).$$

Define $L(x_0, T - t_0)$ as imputation set in the game $\Gamma(x_0, T - t_0)$ (see Neumann and Morgenstern (1947)):

$$L(x_0, T - t_0) = \{\alpha = (\alpha_1, \dots, \alpha_n) :$$
$$\alpha_i \geq V(x_0, T - t_0; \{i\}), \quad \sum_{i \in N} \alpha_i = V(x_0, T - t_0; N)\}. \tag{4}$$

**Regularized game $\Gamma_\alpha(x_0, T-t_0)$.** For every $\alpha \in L(x_0, T-t_0)$ define the noncooperative game $\Gamma_\alpha(x_0, T-t_0)$, which differs from the game $\Gamma(x_0, T-t_0)$ only by payoffs defined along optimal cooperative trajectory $\bar{x}(\tau)$, $\tau \in [t_0, T]$.

Let $\alpha \in L(x_0, T-t_0)$. Define the imputation distribution procedure (IDP) (see Petrosjan (1993)) as function $\beta(\tau) = (\beta_1(\tau), \dots, \beta_n(\tau))$, $\tau \in [t_0, T]$ such that

$$\alpha_i = \int_{t_0}^{T} \beta_i(\tau) d\tau. \tag{5}$$

Denote by $H_i^\alpha(x_0, T-t_0; u_1(\cdot), \dots, u_n(\cdot))$ the payoff function in the game $\Gamma_\alpha(x_0, T-t_0)$ and by $x(\tau)$ the corresponding trajectory, then

$$H_i^\alpha(x_0, T-t_0; u_1(\cdot), \dots, u_n(\cdot)) = H_i(x_0, T-t_0; u_1(\cdot), \dots, u_n(\cdot))$$

if there does not exist such $t \in (t_0, T]$ that $x(\tau) = \bar{x}(\tau)$ for $\tau \in (t_0, t]$. Let $t = \sup\{t' : x(\tau) = \bar{x}(\tau), \tau \in [t_0, t']\}$ and $t > t_0$, then

$$\begin{aligned} H_i^\alpha&(x_0, T-t_0; u_1(\cdot), \dots, u_n(\cdot)) = \\ &= \int_{t_0}^{t} \beta_i(\tau) d\tau + H_i(\bar{x}(t), T-t; u_1(\cdot), \dots, u_n(\cdot)) = \\ &= \int_{t_0}^{t} \beta_i(\tau) d\tau + \int_{t}^{T} h_i(\bar{x}(t); x(\tau)) d\tau. \end{aligned}$$

In a special case, when $x(\tau) = \bar{x}(\tau)$, $\tau \in [t_0, T]$ (if $x(\tau)$ is an optimal cooperative trajectory in the sense of (2)), we have

$$H_i^\alpha(x_0, T-t_0; \bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot)) = \int_{t_0}^{T} \beta_i(\tau) d\tau = \alpha_i.$$

By the definition of payoff function in the game $\Gamma_\alpha(x_0, T-t_0)$we get that the payoffs along the optimal trajectory are equal to the components of the imputation $\alpha = (\alpha_1, \dots, \alpha_n)$.

Consider the current subgames (see Neumann and Morgenstern (1947)) — $\Gamma(\bar{x}(t), T-t)$ along $\bar{x}(t)$ and current imputation sets $L(\bar{x}(t), T-t)$. Let $\alpha(t) \in L(\bar{x}(t), T-t)$. Suppose that $\alpha(t)$ can be selected as differentiable function of $t$, $t \in [t_0, T]$.

**Definition 2.1.** The game $\Gamma_\alpha(x_0, T-t_0)$ is called *regularization* of the game $\Gamma(x_0, T-t_0)$ ($\alpha$-regularization) if the IDP $\beta$ is defined in such a way that

$$\alpha_i(t) = \int_{t}^{T} \beta_i(\tau) d\tau$$

or

$$\beta_i(t) = -\alpha_i'(t). \tag{6}$$

From (6) we get

$$\alpha_i = \int_{t_0}^{t} \beta_i(\tau) d\tau + \alpha_i(t), \tag{7}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in L(x_0, T-t_0)$, and $\alpha(t) = (\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t)) \in L(\bar{x}(t), T-t)$. Suppose now that

$M(x_0, T - t_0) \subset L(x_0, T - t_0)$ is some optimality principle in the cooperative version of the game $\Gamma(x_0, T - t_0)$, and $M(\bar{x}(t), T - t) \subset L(\bar{x}(t), T - t)$ is the same optimality principle defined in the subgames $\Gamma(\bar{x}(t), T - t)$ with initial conditions on the optimal trajectory. $M$ can be $c$-core, $HM$-solution, Shapley Value, Nucleous e.t.c. If $\alpha \in M(x_0, T - t_0)$, and $\alpha(t) \in M(\bar{x}(t), T - t)$ the condition (7) gives us the time consistency of the chosen imputation $\alpha$, or the chosen optimality principle. Then we have the *time consistency (dynamic stability) of the chosen cooperative agreement.*

Consider now the problem of *strategic stability* of cooperative agreements. Based on imputation distribution procedure $\beta$, satisfying (5) we can prove the following basic theorem.

**Theorem 2.1.** *In the regularization of the game $\Gamma_\alpha(x_0, T - t_0)$ for every $\varepsilon > 0$ there exist an $\varepsilon$-Nash equilibrium (Nash (1951)) with payoffs $\alpha = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$.*

*Proof.* The proof is based on actual constraction of the $\varepsilon$-Nash equilibrium in piecewise open-loop (POL) strategies with memory.

Remind the definition of POL strategies with memory in differential game. Denote by $\hat{x}(t)$ any admissible trajectory of the system (1) on the time interval $[t_0, t]$, $t \in [t_0, T]$.

The strategy $u_i(\cdot)$ of player $i$ is called POL if it consists from the pair $(a, \sigma)$, where $\sigma$ is a partition of time interval $[t_0, T]$, $t_0 < t_1 < \ldots < t_l = T$ $(t_{k+1} - t_k = \delta > 0)$, and a mapping $a$ which corresponds to each point $(\hat{x}(t_k), t_k)$, $t_k \in \sigma$ an open-loop control $u_i(t)$, $t \in [t_k, t_{k+1})$.

Consider a family of associated with $\Gamma(x, T - t)$, but not with $\Gamma_\alpha(x, T - t)$ zero-sum games $\Gamma_{\{i\}, N\setminus\{i\}}(x, T - t)$ from the initial position $x$ and duration $T - t$ between the coalition $S$ consisting from a single player $i$ and the coalition $N \setminus \{i\}$ with player's $i$ payoff equal to

$$H_i(x, T - t; u_1(\cdot) \ldots, u_n(\cdot)).$$

The payoff of player $N \setminus \{i\}$ in $\Gamma_{\{i\}, N\setminus\{i\}}(x, T - t)$ equals to $(-H_i)$. Let $\hat{u}(x, t; \cdot)$ be the $\varepsilon$-optimal POL strategy of player $N \setminus \{i\}$ in $\Gamma_{\{i\}, N\setminus\{i\}}(x, T - t)$. Note, that $\hat{u}(x, t; \cdot) = \{\hat{u}_j(x, t; \cdot)\}$, $j \in N \setminus \{i\}$.

Let $\hat{x}(t) = \{\hat{x}_1(t), \ldots, \hat{x}_n(t)\}$ be the segment of an admissible trajectory of (1) defined on the time interval $[t_0, \underline{t}]$, $t \in [t_0, T]$. For each $\underline{i} \in \{1, \ldots, n\}$ define $\bar{t}(i) = \sup\{t_i : \hat{x}_i(t_i) = \bar{x}_i(t_i)\}$ and $\bar{\bar{t}}(j) = \min_i \bar{t}(i) = \bar{t}(j)$. $\bar{\bar{t}}(j)$ lies in one of the intervals $[t_k, t_{k+1})$, $k = 0, 1, \ldots, l-1$. Thus, $\bar{t}(i) - t_0$ is the length of the time interval starting from $t_0$ on which $x_i(t)$ coincides with $\bar{x}_i(t)$ — the $i$-th component of the cooperative trajectory $\bar{x}(t)$. And $\bar{\bar{t}}(j) - t_0$ is the length of the time interval starting from $t_0$ on which $x(t)$ coincides with cooperative trajectory $\bar{x}(t)$.

Define the following strategies of player $i \in N$.

$$u_i^*(\cdot) = \begin{cases} \bar{u}_i(t) & \text{for } (\hat{x}(t_k), t_k) \text{ on the optimal cooperative} \\ & \text{trajectory } \bar{x}(t) \ \ (\hat{x}(\tau) = \bar{x}(\tau), \tau \in [t_0, t_k]); \\ \hat{u}_i(\hat{x}(t_{k+1}), t_{k+1}; \cdot) & i\text{-th component of the } \varepsilon/2\text{-optimal POL} \\ & \text{strategy of player } N \setminus \{j\} \text{ in the game} \\ & \Gamma_{\{j\}, N\setminus\{j\}}(x(t_{k+1}), T - t_{k+1}), \text{ if } t_k \le \bar{\bar{t}}(j) < t_{k+1}; \\ \text{arbitrary} & \text{for all other positions.} \end{cases}$$

Show that $u^*(\cdot) = (u_1^*(\cdot), \dots, u_n^*(\cdot))$ is $\varepsilon$-Nash equilibrium in $\Gamma_\alpha(x_0, T - t_0)$. The following equality holds

$$H_i(x_0, T - t_0; u^*(\cdot)) = H_i(x_0, T - t_0; u_1^*(\cdot), \dots, u_n^*(\cdot)) = \int_{t_0}^{T} \beta_i(t)dt = \alpha_i. \quad (8)$$

Consider the $n$-tuple $(u^*(\cdot)||u_i(\cdot))$ where player $i$ changes his strategy $u_i^*(\cdot)$ on $u_i(\cdot)$.

We have to show that

$$H_i(x_0, T - t_0; u^*(\cdot)) \geq H_i(x_0, T - t_0; u^*(\cdot)||u_i(\cdot)) - \varepsilon. \quad (9)$$

for all $i \in N$ and all POL $u_i(\cdot)$ of player $i$.

It is easy to see that when the $n$-tuple $u^*(\cdot)$ is played the game develops along the optimal trajectory $\bar{x}(t)$. If in $(u^*(\cdot)||u_i(\cdot))$ the trajectory $\bar{x}(t)$ is also realized then (8) will be equality and thus true. Suppose now that in $(u^*(\cdot)||u_i(\cdot))$ the trajectory $x(t)$ different form $\bar{x}(t)$ is realized. Then let

$$\bar{t} = \inf\{t : \bar{x}(t) \neq x(t)\}.$$

and $\bar{t} \in [t_{k-1}, t_k)$. Since the motion of players are independent we get $\bar{x}_m(t_k) = x_m(t_k)$ for $m \in N \setminus \{i\}$ and $\bar{x}_i(t_k) \neq x_i(t_k)$ (but $\bar{x}_j(t_{k-1}) = x_j(t_{k-1})$ for $j \in N$). Then from the definition of $u^*(\cdot)$ it follows that the players $m \in N \setminus \{i\}$ will use their strategies $\hat{u}_m(\hat{x}(t_k), t_k; \cdot)$ which are $\varepsilon/2$-optimal in a zero-sum game $\Gamma_{\{i\}, N\setminus\{i\}}(x(t_k), T - t_k)$ against the player $i$ which deviates from the optimal trajectory on a time interval $[t_{k-1}, t_k)$.

If the players from the set $N \setminus \{i\}$ will use their strategies $\hat{u}_m(\hat{x}(t_k), t_k; \cdot)$, player $i$ starting from position $x(t_k), t_k$ will get not more than

$$V(x(t_k), T - t_k; \{i\}) + \frac{\varepsilon}{2},$$

where $V(x(t_k), T - t_k; \{i\})$ is the value of the game $\Gamma_{\{i\}, N\setminus\{i\}}(x(t_k), T - t_k)$. Then the total payoff of player $i$ in $\Gamma_\alpha(x_0, T - t_0)$ when the $n$-tuple of strategies $(u^*(\cdot)||u_i(\cdot))$ is played cannot exceed the amount

$$\int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + V(x(t_k), t_k; \{i\}) + \frac{\varepsilon}{2} + \int_{t_{k-1}}^{t_k} h_i(x_i(\tau))d\tau. \quad (10)$$

But the payoff of player $i$ when the $n$-tuple $u^*(\cdot)$ is played is equal to

$$\alpha_i = \int_{t_0}^{T} \beta_i(\tau)d\tau = \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + \int_{t_{k-1}}^{T} \beta_i(\tau)d\tau = \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + \alpha_i(t_{k-1}). \quad (11)$$

By the definition of IDP (see (5), (6)), $\alpha_i(t_{k-1}) \in L(\bar{x}(t_{k-1}), T - t_{k-1})$,

$$\int_{t_{k-1}}^{T} \beta_i(\tau)d\tau = \alpha_i(t_{k-1}) \geq V(\bar{x}(t_{k-1}), T - t_{k-1}; \{i\}). \quad (12)$$

From the continuity of the function $V$ and continuity of the trajectory $x(t)$ by appropriate choice of $\delta > 0$ $(t_{k+1} - t_k = \delta)$ the following inequalities can be guaranteed:

$$|V(\bar{x}(t_{k-1}), T - t_{k-1}; \{i\}) - V(x(t_k), T - t_k; \{i\})| < \frac{\varepsilon}{4},$$

$$\int_{t_{k-1}}^{T} \beta_i(\tau)d\tau = \alpha_i(t_{k-1}) \geq V(x(t_k), T - t_k; \{i\}) - \frac{\varepsilon}{4}.$$

Compare $\alpha_i(t_{k-1})$ and $V(x(t_k), t_k; \{i\}) + \varepsilon/2 + \int_{t_{k-1}}^{t_k} h_i(x_i(\tau))d\tau$. By choosing $\delta = t_{k+1} - t_k$ sufficiently small one can achieve that the integral $\int_{t_{k-1}}^{t_k} h_i(x_i(\tau))d\tau$ will be also small (less than $\varepsilon/4$).

Adding to both sides of (12) the amount $\int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau$ and using the previous inequality we get

$$\begin{aligned}
\alpha_i = \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + \alpha_i(t_{k-1}) &\geq \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + V(\bar{x}(t_{k-1}), T - t_{k-1}; \{i\}) \geq \\
&\geq \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + V(x(t_k), T - t_k; \{i\}) - \frac{\varepsilon}{4} \\
&\geq \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + V(x(t_k), T - t_k; \{i\}) - \frac{\varepsilon}{4} + \int_{t_{k-1}}^{t_k} h_i(\tau)d\tau - \frac{\varepsilon}{4} \\
&\geq \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + V(x(t_k), T - t_k; \{i\}) + \int_{t_{k-1}}^{t_k} h_i(\tau)d\tau - \frac{\varepsilon}{2} \\
&\geq \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + V(x(t_k), T - t_k; \{i\}) + \int_{t_{k-1}}^{t_k} h_i(\tau)d\tau + \\
&\qquad + \frac{\varepsilon}{2} - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}.
\end{aligned} \tag{13}$$

Here first four addends in the right part of the inequality constitute the upper bound of player $i$ payoff when $(u^*(\cdot)||u_i^*(\cdot))$ is played. But $\alpha_i$ is the payoff of player $i$ when $u^*(\cdot)$ is played, and we get

$$\begin{aligned}
H_i(x_0, T - t_0; u^*(\cdot)) = \alpha_i &\geq \\
&\geq \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + V(x(t_k), T - t_k; \{i\}) + \int_{t_{k-1}}^{t_k} h_i(\tau)d\tau + \frac{\varepsilon}{2} - \varepsilon \geq \\
&\geq H_i(x_0, T - t_0; u^*(\cdot)||u_i(\cdot)) - \varepsilon
\end{aligned} \tag{14}$$

and we get (9). The theorem is proved. □

This means that the cooperative solution (any imputation) can be strategically supported in a regularized game $\Gamma_\alpha(x_0, T - t_0)$ (realized in a specially constructed Nash equilibrium) by the Nash equilibrium $u^*(\cdot)$ defined in the Theorem 2.1.

Conditions for the *irrational behavior proofness* of the cooperative solutions. Suppose now that in some intermediate instant of time the irrational behavior of some player (or players) will force the other players to leave the cooperative agreement, then the *irrational behavior proofness* condition (see D.W.K. Yeung (2007)) requires that the following inequality must be satisfied

$$V(x_0, T - t_0; \{i\}) \leq \int_{t_0}^{t} \beta_i(\tau)d\tau + V(\bar{x}(t), T - t; \{i\}), \quad i \in N. \tag{15}$$

If the IDP $\beta(t)$ can be chosen in such a way, that both time-consistency and *irrational behavior proofness* conditions are satisfied (the strategic stability as we have shown follows from time-consistency) the cooperative agreement about the choice of the imputation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is stable.

From (15) we have the following condition for IDP $\beta(\tau) = (\beta_1(\tau), \beta_2(\tau), \ldots, \beta_n(\tau))$:

$$\beta_i(\tau) \geq -\frac{d}{d\tau} V(\bar{x}(\tau), T-\tau; \{i\}), \quad i = 1, \ldots, n. \tag{16}$$

In (16) $V(\bar{x}(\tau), T-\tau; \{i\})$ is the value of the zero-sum game played with coalition $N \backslash \{i\}$ as one player and player $\{i\}$ with the coalitional payoff equal to $[-H_i(\bar{x}(\tau), T-\tau; u_1, \ldots, u_n)]$. Suppose that $y(t), t \in [\tau, T]$ is the trajectory of this zero-sum game, when the saddle point strategies are played. We suppose that for each initial conditions $\bar{x}(\tau), T-\tau, \tau \in [t_0, T]$ such saddle point exist (if not we can consider $\varepsilon$-saddle point in piecewise open loop srategies which for every given $\varepsilon \geq 0$ exist always, but the following formulas in this case are to be considered with $\varepsilon$-accuracy).

Then we can write

$$V(\bar{x}(\tau), T-\tau; \{i\}) = \int_\tau^T h_i(\bar{x}(\tau); y(t))dt,$$

where $y(\tau) = \bar{x}(\tau)$. From (16) we have

$$\beta_i(\tau) \geq -\frac{d}{d\tau} \int_\tau^T h_i(\bar{x}(\tau); y(t))d =$$
$$= -[-h_i(\bar{x}(\tau); y(\tau)) + \int_\tau^T \sum_{l=1}^n \sum_{k=1}^m \frac{\partial h_i(\bar{x}(\tau), y(t))}{\partial x_{lk}} f_{lk}(\bar{x}(\tau), \bar{u}(\tau))dt] =$$
$$= h_i(\bar{x}(\tau); \bar{x}(\tau)) - \int_\tau^T \sum_{l=1}^n \sum_{k=1}^m \frac{\partial h_i(\bar{x}(\tau), y(t))}{\partial x_{lk}} f_{lk}(\bar{x}(\tau), \bar{u}(\tau))dt$$

or

$$\beta_i(\tau) \geq h_i(\bar{x}(\tau); \bar{x}(\tau)) - \int_\tau^T \sum_{l=1}^n \sum_{k=1}^m \frac{\partial h_i(\bar{x}(\tau), y(t))}{\partial x_{lk}} f_{lk}(\bar{x}(\tau), \bar{u}(\tau))dt.$$

## 3. Discrete Time Case

In what follows as basic model we shall consider the game in extensive form with perfect information.

**Definition 3.1.** A game tree is a finite oriented treelike graph $K$ with the root $x_0$.

We shall use the following notations. Let $x$ be some vertex (position). We denote by $K(x)$ a subtree $K$ with the root in $x$. We denote by $Z(x)$ immediate successors of $x$. The vertices $y$, directly following after $x$, are called alternatives in $x$ ($y \in Z(x)$). The player who makes a decision in $x$ (who selects the next alternative position in $x$), will be denoted by $i(x)$. The choice of player $i(x)$ in position $x$ will be denoted by $\bar{x} \in Z(x)$.

Let $N = \{1, \ldots, n\}$ — be the set of all players in the game.

**Definition 3.2.** A game in extensive form with perfect information (see Kuhn (1953)) $G(x_0)$ is a graph tree $K(x_0)$, with the following additional properties:

– The set of vertices (positions) is split up into $n+1$ subsets $P_1, P_2, \dots, P_{n+1}$, which form a partition of the set of all vertices of the graph tree $K$. The vertices (positions) $x \in P_i$ are called players $i$ personal positions, $i = 1, \dots, n$; vertices (positions) $x \in P_{n+1}$ are called terminal positions.
– In each vertex (position) $x$ the system of real numbers $h(x) = (h_1(x), \dots, h_n(x))$is defined. $h_i(x)$ is interpreted as stage payoff of player $i$ in the vertex (position) $x$.

**Definition 3.3.** A strategy of player $i$ is a mapping $U_i(\cdot)$, which associate to each position $x \in P_i$ a unique alternative $y \in Z(x)$.

As in the previous case denote by $H_i(x; u_1(\cdot), \dots, u_n(\cdot))$ the payoff function of player $i \in N$ in the subgame $G(x)$ starting from the position $x$.

$$H_i(x; u_1(\cdot), \dots, u_n(\cdot)) = \sum_{i=1}^{l} h_i(x_i')$$

where $x = (x_1', x_2', \dots, x_l')$ is the path realized in the subgame $G(x)$, when the $n$-tuple of strategies $(u_1(\cdot), \dots, u_n(\cdot))$ is played, $x_1' = x$.

Denote by $\bar{u}(\cdot) = (\bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot))$ the $n$-tuple of strategies and the trajectory (path) $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_m)$, $\bar{x}_m \in P_{n+1}$ such that

$$\max_{u_1(\cdot),\dots,u_n(\cdot)} \sum_{i=1}^{n} H_i(x_0; u_1(\cdot), \dots, u_n(\cdot)) =$$
$$= \sum_{i=1}^{n} H_i(x_0; \bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot)) = \sum_{i=1}^{n} (\sum_{k=0}^{m} h_i(\bar{x}_k)). \tag{17}$$

The path $\bar{x} = (\bar{x}_0, \dots, \bar{x}_m)$ satisfying (17) we shall call "optimal cooperative trajectory".

Define in $G(x_0)$ characteristic function in a classical way

$$\begin{aligned} &V(x_0; N) = \sum_{i=1}^{n} (\sum_{k=0}^{m} h_i(\bar{x}_k)), \\ &V(x_0; \emptyset) = 0, \\ &V(x_0; S) = Val\ \Gamma_{S, N\setminus S}(x_0), \end{aligned}$$

where $Val\ \Gamma_{S,N\setminus S}(x_0)$ is a value of zero-sum game played between coalition $S$ acting as first player and coalition $N \setminus S$ acting as player 2, with payoff of player $S$ equal to

$$\sum_{i\in S} H_i(x_0; u_1(\cdot), \dots, u_n(\cdot)).$$

Define $L(x_0)$ as imputation set in the game $G(x_0)$.

$$L(x_0) = \left\{ \alpha = (\alpha_1, \dots, \alpha_n) : \alpha_i \geq V(x_0; \{i\}), \quad \sum_{i\in N} \alpha_i = V(x_0; N) \right\}.$$

**Regularized game $G_\alpha(x_0)$.** For every $\alpha \in L(x_0)$ define the noncooperative game $G_\alpha(x_0)$, which differs from the game $G(x_0)$ only by payoffs defined in the vertexes(positions) along optimal cooperative path $\bar{x} = (\bar{x}_0, \dots, \bar{x}_m)$. Let $\alpha \in L(x_0)$.

Define the imputation distribution procedure (IDP) as function $\beta_k = (\beta_1(k), \ldots, \beta_n(k))$, $k = 0, 1, \ldots, m$ such that

$$\alpha_i = \sum_{k=0}^{m} \beta_i(k). \tag{18}$$

Suppose in the situation $(u_1(\cdot), \ldots, u_n(\cdot))$ the path $(x_0, \ldots, x_{l'})$ is realized. Define by $H_i^{\alpha}(x_0; u_1(\cdot), \ldots, u_n(\cdot))$ the payoff function in the game $G_\alpha(x_0)$

$$H_i^{\alpha}(x_0; u_1(\cdot), \ldots, u_n(\cdot)) = \sum_{k=0}^{r} \beta_i(k) + \sum_{k=r+1}^{l'} h_i(x_k),$$

where $r$ is defined as $max\{k : x_k = \bar{x_k}\} = r$.

By the definition of the payoff function in the game $G_\alpha(x_0)$ we get that the payoffs along the optimal cooperative trajectory are equal to the components of the imputation $\alpha = (\alpha_1, \ldots, \alpha_n)$ $(H_i^{\alpha}(x_0; \bar{u}_1(\cdot), \ldots, \bar{u}_n(\cdot)) = \alpha_i)$.

Consider current subgames $G(\bar{x}_k)$ along the optimal path $\bar{x}$ and current imputation sets $L(\bar{x}_k)$. Let $\alpha^k \in L(\bar{x}_k)$.

**Definition 3.4.** The game $G_\alpha(x_0)$ is called regularization of the game $G(x_0)$ ($\alpha$-regularization) if the IDP $\beta$ is defined in such a way that

$$\alpha_i^k = \sum_{j=k}^{m} \beta_i(j)$$

or $\beta_i(k) = \alpha_i^k - \alpha_i^{k+1}$, $i \in N$, $k = 0, 1, \ldots, m-1$, $\beta_i(m) = \alpha_i^m$, $\alpha_i^0 = \alpha_i$.

**Theorem 3.1.** *In the regularization of the game $G_\alpha(x_0)$ there exist a Nash equilibrium with payoffs $\alpha = (\alpha_1, \ldots, \alpha_n)$.*

*Proof.* Along the cooperative path we have

$$\alpha_i^k \geq V(\bar{x}_k; \{i\}), \quad i \in N, k = 0, 1, \ldots, m.$$

since $\alpha^k = (\alpha_1^k, \ldots, \alpha_n^k) \in L(\bar{x}_k)$ is an imputation in $G(\bar{x}_k)$ (note that here $V(\bar{x}_k; \{i\})$ is computed in the subgame $G(\bar{x}_k)$ but not $G_\alpha(\bar{x}_k)$). In the same time

$$\alpha_i^k = \sum_{j=k}^{m} \beta_i(j)$$

and we get

$$\sum_{j=k}^{m} \beta_i(j) \geq V(\bar{x}_k; \{i\}), \quad i \in N, k = 0, 1, \ldots, m. \tag{19}$$

But $\sum_{j=k}^{m} \beta_i(j)$ is the payoff of player $i$ in the subgame $G_\alpha(\bar{x}_k)$ along the cooperative path, and from (19) using the arguments similar to those in the proof of Theorem 2.1 one can construct the Nash equilibrium with payoffs $\alpha = (\alpha_1, \ldots, \alpha_n)$ and resulting cooperative path $\bar{x} = (\bar{x}_0, \ldots, \bar{x}_m)$.

The *irrational behavior proofness* condition in this case will be

$$\sum_{j=0}^{l} \beta_i(j) + V_i(\bar{x}_{l+1}; \{i\}) \geq V_i(x_0; \{i\}), \quad 0 \leq l \leq m, i \in N. \tag{20}$$

## 4. Example

In this example as an imputation we shall consider Shapley value (Shapley (1953)). Using the proposed regularization of the game we shall see that there exist a Nash equilibrium with payoffs equal to the components of the Shapley value.

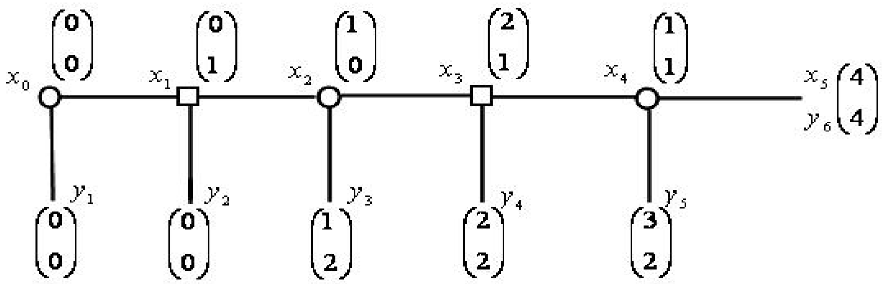


**Fig.1.** Game $G(x_0)$

In the game $G(x_0)$, $N = \{1,2\}$, $P_1 = \{x_0, x_2, x_4\}$, $P_2 = \{x_1, x_3\}$, $P_3 = \{y_1, y_2, y_3, y_4, y_5, y_6\}$, $h(x_0) = (1,0)$, $h(x_1) = (0,1)$, $h(x_2) = (1,0)$, $h(x_3) = (2,1)$, $h(x_4) = (1,1)$, $h(x_5) = (4,4)$, $h(y_1) = (0,0)$, $h(y_2) = (0,0)$, $h(y_3) = (1,2)$, $h(y_4) = (2,2)$, $h(y_5) = (3,2)$, $h(y_6) = h(x_5) = (4,4)$. The cooperative path is $\bar{x} = \{\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5\}$.

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|---|
| $V(x;\{1\})$ | 0 | 0 | 2 | 4 | 5 | 4 |
| $V(x;\{2\})$ | 0 | 3 | 5 | 4 | 5 | 4 |
| $V(x;\{1,2\})$ | 15 | 15 | 14 | 13 | 10 | 8 |
| $Sh(x;\{1\})$ | $7,5$ | 6 | 5,5 | 6,5 | 6 | 4 |
| $Sh(x;\{2\})$ | 7,5 | 9 | 8,5 | 6,5 | 4 | 4 |
| $\beta_1(x) = \beta_1(j)$ | 1,5 | 0,5 | -1 | 0,5 | 2 | 4 |
| $\beta_2(x) = \beta_2(j)$ | -1,5 | 0,5 | 2 | 2,5 | 0 | 4 |

It can be easily seen that the inequality (19)

$$\sum_{j=k}^{m} \beta_i(j) \geq V(\bar{x}_k; \{i\})$$

for $i \in N$ holds in this case and the *irrational behavior proofness* condition (20) is also satisfied:

$$\sum_{j=0}^{l} \beta_i(j) + V_i(\bar{x}_{l+1}; \{i\}) \geq V(x_0; \{i\}), \quad i = 1, 2, \quad 1 \leq l \leq 4.$$

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# Analysing Plural Normative Interpretations in Social Interactions ⋆

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**Abstract** In this study, we deal with the legitimate plurality of interpretations of a given normative system. Distinct individuals or organisations often favour divergent interpretations of the same enforced norms or principles. This plurality, we argue, does not make it impossible that norms or principles be considered effective. Indeed, their susceptibility to various interpretations might be viewed as a constituent of their ability to impose structure onto social interaction. In this article, we suggest that interpretative plurality can be thought of in terms of alternative systems of individual and coalitional power. Power, in turn, is modelled through the game-theoretic notion of an effectivity function. This can prove useful, it is argued, to describe the contrast between formal authority and real power in organisations, and to improve our understanding of how interpretative controversies might account for real-authority migration.

**Keywords:** Authority, Effectivity function, Game Form, Interpretation, Pluralism, Power.

## 1. Introduction

Recent research suggests that pluralism in the interpretation of norms (including very general norms such as "principles") plays a crucial role in social dynamics and the dynamic play of social mechanisms. In short, the fact that possible interpretations are plural is not just to be considered a static problem for the "meaning" of norms at a given time. It also proves essential to the understanding of the way norms get applied in actual societies.

This will be exemplified in a selection of topics, of which the 1st part of our talk will consist: we select a few topics from recent analyses which document the need for a more unified approach. There are various relevant mechanisms (which are fairly well documented by now in extant studies) through which the plurality of legitimate (or plausible) interpretations has an impact on the dynamics of the implementation of norms or principles in the real world.

It is fairly commonplace to observe that the way norms or principles will impact social life is not always obvious from the meaning of the norms or principles themselves. Or, alternatively, we might say that their ultimate meaning can only be assessed from real-life behaviour and interpretative tasks (this has been privileged in the economic approach to rules developed by Reynaud (2003). To be sure, these features of reality are constrained and influenced by operating principles and norms.

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But they are not fully determined by them, because most norms, policies principles are partially indeterminate, which might account for specific properties in social life (Calvert and Johnson, 1999, Matland, 1995, Moor, 2005).

Dialogue plays a role in such cases. When there is a plurality of interpretations, this plurality calls for argument if social agents are to interact with a tolerable degree of coordination. In institutional settings, for example, the accepted meaning of norms or principles is gradually determined in successive steps and this process has a lot to do with coordination tasks as well as cognitive functionings. It is quite obvious to several researchers that we lack a unified methodology and toolbox for dealing with this emergent problematic. In this article explore a possible avenue for such a unification, using game-theoretic modelling. We first propose a general framework for modelling. Then we bring in specific topics, for which such a modelling is relevant. Finally, we sketch some integrative steps for political theory and organisation management.

## 2. Framework and Motivation from Problems in Normative Analysis

### 2.1. General framework

*The social interaction*

For a finite set $X$, we denote $\mathcal{X} = \{G \subset X, G \neq \emptyset\}$, $\mathcal{X}^* = \mathcal{X} \bigcup \{\emptyset\}$ with $|X|$ referring to the number of elements in $X$. If $Y \subset X$, we denote $Y^c = \{y \notin Y, y \in X\}$.

For two finite sets $S$ and $X$, the symbol $X_S$ is shorthand for the Cartesian product : $\prod\limits_{i \in S} X_i$. And : $X_{S^c} = \prod\limits_{i \notin S} X_i$.

Let $N = \{1, ..., n\}$ be the set of players. A coalition is a set $S \in \mathcal{N}$. The set of possible strategies for $i \in N$ is a finite set, denoted by $\Sigma_i$. A joint strategy (or action, with a common "intention" or not) of the members of coalition $S$ is an element of $\Sigma_S$. We denote the set of the joint strategies of the members of $N$ by: $\Sigma = \prod\limits_{i \in N} \Sigma_i$ .

*Norms and interpretations*

A *normative system* $\mathcal{F}$ is a subset of $\mathcal{P}(\Sigma)$. An *interpretation* is a partition of the set of strategies in two disjoint sets : the set of rule-abiding actions and the set of the other actions. Among the elements of $\mathcal{F}$ it will often be possible to delineate a family of rule-abiding actions which constitute a "dominant interpretation" of the valid rules at a given time. To sum up, a rule-abiding action is an element $\theta$ of $\Sigma$ ; an interpretation is a set of actions which are considered acceptable with respect to the rules (or rule-abiding for short) such as $\{ \theta 1, \ldots \theta \mathrm{p} \}$.

A normative system such as $\mathcal{F}$ is a set of interpretations ; formally speaking, this is expressed by equating it with a subset of the set of parts of $\Sigma$. This means that the plurality of interpretations is viewed as an internal characteristic of the very notion of a valid normative system.

We hold that an interpretation can be decomposed into a Cartesian product of normatively correct sets of individual actions. Then a *set of interpretations* can be described as a family:

$$\mathcal{F} = \left( \prod_{\mathrm{i} \in N} \theta_{\mathrm{i},}^{\lambda} \right)_{(\lambda \in \Lambda)} {}^{1}.$$

[1] We shall write alternatively : $\lambda \in \Lambda$ or $\theta^\lambda \in \Lambda$ .

where $\Lambda$ is an indexation set.

It may be noted that a normative system, here, is equated with a set of interpretations in the sense we just mentioned.

A family of *consequence functions* is a family :

$$\nu = (\nu^{\lambda})_{\lambda \in \Lambda}$$

where $\nu^{\lambda}$ is a surjective application :

$$\nu^{\lambda} : \theta^{\lambda} \to A_{\lambda},$$

such that:

$$\bigcup_{\lambda \in \Lambda} \nu\left(\theta^{\lambda}\right) = A.$$

A normative system $\mathcal{F}$ may be a code or law, subject to interpretation ; it expresses a rule for behaviour which is accepted as a social norm.

### 2.2. Motivation from problems in normative analysis

**Penalties which are dependent on interpretations.** The plurality of interpretations paves the way for behaviour patterns which have the following properties. They are not clearly illegal; they are indeed legal given some preferred interpretation; and they might constitute obstacles to the proper exercise of power (with the intended results) in conformity with some interpretation of the valid norms. This appears to hold when penalties are inflicted on the basis of disputable interpretations of the enforced norms or principles. Such *adverse behaviour* on the part of other agents can rely on legal devices (such as nullifying steps with respect to decisions of other agents, administrative penalties or judicial penalties) as well as other devices (such as reputation or credibility losses, humilitating political retreats which are made compulsory by the behaviour of others, lost inter-institutional support on other issues which (result from the threat potential of agents towards one another).

Such penalties can be assessed on the basis of the consequences of actions. Moreover, they result from deliberate initiatives (on the part of the other agents) to inflict penalties with a view to given interpretations of norms. There is little doubt that penalties of both categories are important when it comes to describe and explain the process of implementing the rules. At one extreme, if there is no penalty whatever the behaviour, we simply say that the rules are not being put into operation. They are simply not enforced. But this is just an extreme case.

From the viewpoint of an agent in a law-abiding political world, we might say that the results of the actions he takes may or may not encapsulate penalties, depending on the arguments the other agents receive as convincing, insofar as the interpretation of enforced norms and principles is at stake. Thus, the understanding of persuasion and argument, and more generally the ways some interpretations can gain influence, is important for the analysis of the implementation of rules. We hold that it stands to benefit from game-theoretic analyses of power.

Insofar as regulatory efforts in institutional settings involve conditional penalties, they should take into account the way intra-organisational deliberation involves interpretative choices about the regulatory environment. In addition, regulatory initiatives should take inter-organisational interpretative exchanges into account, when it comes to benchmark the expectations about the impact of regulatory changes. These are difficult questions and it is fair to say that present-day political, economic or organisational theory is hardly capable of dealing with them in an instructive way. But they are essential questions too, and our choice of concepts and analytical framework is meant to give us some understanding of them.

**Interpretation and the migration of real power.** In connection with the notion of "formal power", we pay attention to a group of interpretations which are considered most relevant (among all the credible or plausible ones) because they find support in educated of influent legal opinion. For example, such interpretations are put forward in the institutional communication of highly influential institutions; or else, they are proposed as the correct interpretations by a group of influential lawyers. Among the key features of the social processes through which norms are put into operation, we find the evolution of dominant or received interpretations, which are used by all agents as benchmark interpretations (even if agents have incentives to promote alternative views in a strategic fashion).

Indeed, we might be interested in making explicit the interpretations which are compatible with observed behaviour (under the hypothesis that social agents behave in a lawlike manner). This is what matters if we are concerned with the benchmarked expectations of social agents about the consequences of their actions. This is our own proposition for giving a precise meaning to the notion of real power, which has been variously approached and contrasted with formal authority by Aghion and Tirole (1997), Backhaus (2001), Thorlakson (2006).

To illustrate, let us consider a principle such as:
(P) "respect for unbiased competition in economic matters" (for example, in the European Union).

This very general principle can surely be said to impact actual outcomes – that is to say, what happens to people and firms, their revenues, work and production, etc. But this ultimate layer is not reached immediately. There is an implementation process which involves, first, the selection of some interpretations (ideally a single one) from the set of possible interpretations of the principle.

Typically, some choices will be made concerning anti-trust regulation, control of merger processes and the control of State operations (e.g. financing operations to the benefit of specific firms, regions or economic domains). Such choices are documented, for example, in court rulings, European directives, specific political decisions in member States, detailed administrative guidelines, etc. These choices can be very detailed in some cases and thus they build up what legitimately counts as an "implementation" of the underlying general principle. By this we mean that the principle is not just formulated in the abstract. It is a proper object for argument.

It might well be the case (as it turns up more often than not) that these interpretative choices leave room for subsequent interpretation to a great extent. To simplify matters, let us suppose that a ECJ ruling or a European directive brings about a set of clear, unambiguous restrictions on interpretation possibilities[2]. Then usually there is no single compulsory interpretation of, say, principle (P). A variety of interpretations are still in potential use. All we can say is that some *a priori* possible interpretations, which might have been entertained by a person who is competent in the language, must be ruled out. For example, if it is made clear that mergers in a given sector are forbidden, then it should be clear that any merger is

[2] To be sure, this is not always the case. It is possible that some documents, which are aimed at clearing matters up, bring about confusion to some degree because the meaning is not clear. But for the sake of argument, we leave such matters aside at this point.

forbidden, given that it occurs unambiguously in the relevant sector – and sometimes it can safely be said that we are indeed in the relevant activity sector.

This kind of indeterminacy can impact the allocation of power among institutional actors. Even if some formal rules for the allocation of power are effective (valid), some re-allocations of real authority might be supported by particular judgments or interpretations which are related to the enforced norms or principles. From the emerging literature on authority migration in federal systems[3], it is already quite clear that this is a very important mechanism to account for gains in real capacities to influence outcomes, which cannot be fully predicted from the formal rules of power allocation.

We should also pay attention to the following scenario: interpretative difficulties arise because dramatically new situations emerge. Such a scenario is certainly not uncommon in politics, for example, given that there is such an important role for unexpected novelty in the field (Calvert and Johnson, 1999). New situations keep emerging, and the enforcement of rules can hardly be deduced from the rules themselves in a strict sense.

In the space left open by a set of constraints, observation of actual behaviour will give indications as to which normative interpretations can plausibly be considered "dominant" in a given domain. In the eyes of a given actor (for example, an institutional actor I), it is possible that some subset of interpretations of a set of principles (for example, principle (P) and some constitutional power-allocating principles) is the preferred set. Interpretations in this subset encapsulate a given view of the principle(s). It might well be the case, however, that the observed association of chosen actions and resulting outcomes is flatly inconsistent with this world's view. Then we can safely infer that I's preferred interpretations are not dominant. By contrast, the remaining, possible interpretations (those which are consistent with the observations) belong to a "dominant" group of interpretations in some sense: they enable one to describe the nature of the available strategies in the real world. They enable one to describe what happens when people act in such and such ways.

Among the mechanisms which appear to play a role in the correlated evolution of real power and dominant interpretations, we can think of differential *expertise* capabilities. Let us quote Thomson and Hosli's statement about the interview they conducted among EU officials:

"Some of the practitioners indicated that the Parliament's position is weakened by the lack of technical policy expertise among MEPs compared with the Council whose Member State representatives are supported by large national bureaucracies."[4]

## 3. Towards a Formal Treatment of Normative Pluralism in Rule-Based Processes

### 3.1. Effectivity as a model of power

The game-theoretical notion of effectivity has been used to model power (Abdou and Keiding, 1991, Peleg, 1998) and this will be the case here too. For each valid interpretation $\lambda \in \Omega$, we consider an effectivity function which describes the distribution of power, given universal abiding by the rules in the framework of this particular interpretation.

[3] See : Thorlakson (2006).
[4] Thomson and Hosli (2006).

For a given normative system, given the plurality of interpretations, the associated *system of power* is a family of effectivity functions $\left(E^{\lambda}\right)_{\lambda \in \Omega}$ (with $\Omega \subset \Lambda$).
On the one hand, in connection with the notion of "formal power", we pay attention to a group of interpretations which are considered specially relevant (among all the credible or plausible ones) because they find support in educated of influent legal opinion. For example, such interpretations are put forward in the institutional communication of highly influential institutions; or else, they are proposed as the correct interpretations by a group of influential lawyers.
Broadly speaking, a group $S$ has the power to prevent the social state from being in set non-$B$ if (and only if) it is *effective* for the realisation of states in $B$ (it can make sure that no social state outside $B$ will prevail).
For each group $S$, we consider a family of sets of alternatives with respect to which S is effective. An "effectivity function" is a description of such sets for each subset of players. This expresses the distribution of power among agents. Formally, an effectivity function is a mapping:

$$E : \mathcal{N} \rightarrow \mathcal{P}\left(\mathcal{A}\right)$$

such that :

$$\forall B \in \mathcal{A} \quad \text{and} \quad \forall S \in \mathcal{N}, \quad B \in E(N) \quad \text{and} \quad A \in E(S).$$

Thus "real power" and ''formal power" turn out to be alternative systems of powers, brought about by the respectively concerned interpretations. That is to say, they can be equated, respectively, with distinct families of effectivity functions from $\mathcal{N}$ to $\mathcal{P}\left(\mathcal{A}\right)$.

### 3.2. Alpha - and beta effectivity

The following notions can be useful for applied analysis, when it comes to describe structures of institutional power. We will rely on them in 3.3 for the purpose of defining a norm-based society.

Coalition $S$ is $\alpha$ – *effective* to block a set of alternatives $B^c \in \mathcal{A}_\lambda$ by a action that is permissible under interpretation $\lambda$ (let us call it $\sigma_S \in \theta_S^\lambda$) such that $\nu\left(\sigma_S, \sigma_{S^c}\right) \in B$ for all $\sigma_{S^c} \in \theta_{S^c}^\lambda$.

An $\alpha$ – *legal power system is a family of effectivity functions* $(E_\alpha^\lambda)$ such that for each considered interpretation $\lambda$, the corresponding effectivity function associates to each coalition, the sets of alternatives over which the coalition is $\alpha$ – *effective.*

Coalition $S$ is $\beta$ – *effective* to block a set of alternatives $B^c \in \mathcal{A}_\lambda$ by an action that is correct under interpretation $\lambda$ if, for all $\sigma_{S^c} \in \theta_{S^c}^\lambda$ , there exists $\sigma_S \in \theta_S^\lambda$ such that $\nu\left(\sigma_S, \sigma_{S^c}\right) \in B$.

A $\beta$- *legal power system is a family of effectivity functions* $(E_\beta^\lambda)$ such that for each considered interpretation $\lambda$, the corresponding effectivity function associates to each coalition, the sets of alternatives over which the coalition is $\beta$ – *effective.*

### 3.3. The notion of a norm-based society

We now introduce the notion of a norm-based society, as
triplet $(G, \mathcal{F}, \kappa)$ where $G = \{N, \Sigma_{1,\ldots,}\Sigma_n, A\}$ is a game form and $\kappa \in \{\alpha, \beta\}$.
The $\alpha$ and $\beta$ effectivity specifications refer to alternative conceptions of garantees enjoyed by individuals and groups in social interaction. We have:

$$\forall B \in \mathcal{A}, \quad \text{on a:} \quad B \in E_\alpha^\lambda(S) \rightarrow B \in E_\beta^\lambda(S).$$

$S$ can be $\alpha$ – effective for $B$ without being $\beta$ – effective for $B$.
Such descriptions enable us to depict the possibilities, for coalition $S$, to warrant to itself the specific ruling-out of some states of affairs, according to distinct schemes :
- either in the sense of enjoying such guarantees through one's actions no matter what the others do (description "$\alpha$") ;
- or in the sense of enjoying such guarantees through one's actions conditional on the strategies of the others, that is to say, through adequately "complementing" the choices of the other people (description "$\beta$") ;

A discrepancy appears to exist between real and formal power when an effectivity function that expresses real power (that is to say, it belongs to the "real" power system $E$) turns out to be $\mathcal{F}$-incompatible for the $\kappa$specification of effectivity. Formally, then, we have:

$$E_{\kappa}^{\lambda} \neq E$$

where $\lambda$ is an interpretation that belongs to the formal-power system.

In such a pattern, it can be said that $E$ expresses real authority in the norm-based society, in a way that cannot be reconciled with a given "dominant" (formal) interpretation of the existing norms. These notions allow a more unified description of the way in which the enforced rules can be associated with a plurality of interpretations, the existence of dominant interpretations. A norm-based society is a society which can be said to be faithful to the rules according to some interpretations at least. Thus we rely on a description of the structure of power which depicts access and control of outcomes through actions in a unified way, stressing the following features: (1) it is able to incorporate penalties of various kinds as components of outcomes, and (2) it is sensitive to interpretations.

## 4. Conclusion

The plurality of plausible interpretations both constrains and helps explain the social mechanisms through which principles or norms come into existence as coordination and cooperative devices in real-world social life. These mechanisms are argumentative in nature: they take place in language and reason-giving. Sometimes, interpretation is suggested by acts only, but usually it is backed by explanation or justification. We presume our conceptual suggestions might benefit from a formal approach, because the problems we deal with are intrinsically formal in nature, after all. They involve general forms of social interaction, generic mappings of strategies onto social outcomes, rather than particular social or historical circumstances.

It is fairly obvious that implementing norms or principles in real-world communities involves decisions concerning the allocation of *power*, understood as real authority: the ability to gain access to outcomes, or to influence outcomes, through one's action or control. Such decisions are sensitive to interpretative choices. Depending on how we interpret norms or principles, what counts as their "implementation" - or their effectiveness in the real world, to put it differently - will involve contrasting power (or control) structures.

The implementation of norms will result in the selection of a structure (or family of structures) of social interaction. This is the general idea which we capture by means of the effectivity-function concept. In the perspective we took here, this concept should provide guidance for an improved understanding of shifting authority in organisations or institutions, especially as it relates to interpretative issues.

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# Uncertainty Aversion and Equilibrium*

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**Abstract** If rationality is not mutual knowledge in a game then standard expected utility theory requires a rational player to have a specific belief about the behaviour of a non-rational opponent. This paper argues that this problem does not arise under Choquet expected utility theory, which does not require a player's beliefs to be additive. Non-additive beliefs allow the formalization of the idea that a player who faces a non-rational opponent faces genuine uncertainty. Optimal strategies can then be derived from assumptions about the rational player's attitude towards uncertainty. This paper investigates the consequences of such a view of strategic interaction. We formulate equilibrium concepts, called Choquet-Nash equilibrium in normal forms and perfect Choquet equilibrium in extensive forms, that solves the infinite regress that arises in this situation and study existence and properties of these equilibria in normal and extensive form games.

**Keywords:** extensive form games, equilibrium, non-additive beliefs, uncertainty aversion.

## 1. Introduction

Game theory is about rational decision-making under strategic interaction. Players are assumed to be rational in a decision-theoretic sense: they act as if they possess a utility function over outcomes and beliefs given by a probability distribution over states, and maximise subjective expected utility (Savage, 1954). Beliefs in turn are derived from assumptions based on mutual knowledge of rationality.

In normal form games this leads to the game-theoretic solution concepts of dominance, rationalizability, equilibrium and correlated equilibrium (Tan and Werlang, 1988; Aumann and Brandenburger, 1995). In extensive form games this leads to equilibrium refinements (see, e.g., Selten, 1965; van Damme, 1992).

Yet this view leads to difficulties in extensive games (see, e.g., Selten, 1975, 1978; Binmore, 1987; Reny, 1993). Equilibrium refinements assume that players continue to regard their opponents as rational even after they have observed them deviating from rational play. More fundamentally, the natural explanation of experimental evidence is that players are not necessarily rational. But then rationality

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cannot even be mutual knowledge at the beginning of the game. Game theory suffers from the defect that its solution concepts are based on rationality alone.

What is a rational strategy if rationality is not mutual knowledge? This paper attempts to address this question by applying a weaker definition of decision-theoretic rationality to games. Schmeidler (1989) has shown that Savage's sure-thing Principle can be weakened so that players can still be modelled as maximising expected utility subject to their beliefs, but now beliefs do no longer have to be additive. Whereas Savage's approach reduces uncertainty to risk, Schmeidler's Choquet expected utility theory (henceforth CEU) gives rise to a qualitative difference between risk and uncertainty.

This difference is important in games if we distinguish between rational and non-rational players as in Kreps et al. (1982). A rational player is one who chooses his strategy as to maximise utility given his beliefs. A rational player who faces a rational opponent can anticipate her strategy if he knows her utility function and can anticipate her beliefs. Consequently, a rational player who faces a rational opponent faces risk, in the sense that his beliefs are given by objective probabilities determined by best-reply considerations. Thus his beliefs are necessarily additive.

On the other hand, a rational player who faces a non-rational opponent faces true uncertainty, if all he knows is that a non-rational player does not necessarily choose a utility-maximising strategy. Under CEU, a rational player's beliefs reflect his attitude towards uncertainty. If the player is not uncertainty-neutral, his beliefs will not be additive.

Schmeidler (1986, 1989) has also shown that maximisation of expected utility under uncertainty aversion is equivalent to minimax behaviour under uncertainty neutrality. We argue, therefore, that the rational strategy of an uncertainty-averse player when facing a non-rational opponent is maximin play, if there is neither a theory nor an empirical regularity that allows further restriction of the rational player's beliefs.

Overall, a player who is rational in the sense of CEU will therefore take both possibilities into account: that the opponent may be rational and that she may not be. He will therefore evaluate a strategy with the utility that results in case the opponent is rational and with the strategy's minimax utility in case the opponent is non-rational. The overall expected utility of the strategy is thus the weighted sum, where the weights come from a common prior about the opponent's rationality.

Clearly, a rational player will now choose the strategy that maximises expected utility. He will also anticipate that a rational opponent will do the same, and will anticipate that the rational opponent will anticipate that, ad infinitum. This gives, as usual, rise to an infinite hierarchy of beliefs.

This paper presents an equilibrium concept, called Choquet-Nash equilibrium, that cuts through this infinite regress that arises in games if rationality is not mutual knowledge. We formulate the equilibrium concept and discuss existence and properties in normal form games. An equilibrium for extensive form games also has to specify how players update their beliefs. Using the Dempster-Shafer rule, that generalizes Bayes' rule to non-additive probabilities, we define an equilibrium concept, called perfect Choquet equilibrium, for extensive form games and study its properties.

We show that in normal form games Choquet-Nash equilibria always exists, for every prior probability of mutual rationality. If the game is zero-sum, Choquet-

Nash equilibria coincide with Nash equilibria. Thus the solution concept avoids the indeterminacy associated with rationalizability. If the game is nonzero-sum, every Nash equilibrium is a Choquet-Nash equilibrium, but not vice versa: A Choquet-Nash equilibrium may be non-rationalizable. Thus in normal form games Choquet-Nash equilibria are a proper generalization of Nash equilibria, thus allowing to model the dependence of the equilibrium outcome on the degree to which rationality is mutual knowledge.

In extensive form games, probability-zero histories only arise if rationality is mutual knowledge at the beginning of the game. For extensive games we show that, as a consequence, perfect Choquet equilibrium can differ fundamentally from subgame-perfect equilibrium and its refinements.

This paper joins a growing literature that applies CEU to games. The first of these were Dow and Werlang (1994) and Klibanoff (1993). Dow and Werlang (1994) consider normal form games in which players are CEU maximisers. Klibanoff (1993) similarly considers normal form games in which players follow maximin-expected utility theory (Gilboa and Schmeidler, 1989), which is closely related to CEU. In Hendon et al. (1995) players have belief functions, which amounts to a special case of CEU. Extensions and refinements have been proposed by Eichberger and Kelsey (1994), Lo (1995*a*) and Marinacci (1994). These authors consider normal form games and do not distinguish between rational and non-rational players. The paper closest to ours is Mukerji (1994), who considers normal form games but distinguishes between rational and non-rational players.[1] Extensive games have been studied by Lo (1995*b*) and Eichberger and Kelsey (1995). Lo (1995*b*) extends Klibanoff's approach to extensive games, Eichberger and Kelsey (1995) are the first to use the Dempster-Shafer rule in extensive games. They do not distinguish between rational and non-rational players.

This paper is organized as follows. Section 2 outlines the decision-theoretic approach to game theory, based on expected utility theory in subsection 2.1 and based on CEU in subsection 2.2. In section 3 the Choquet-Nash equilibrium is motivated and defined for normal form games. Section 4 contains a series of examples to illustrate the properties of Choquet-Nash equilibria. Section 5 defines the Dempster-Shafer updating rule in subsection 5.1 and the perfect Choquet equilibrium for extensive form games in subsection 5.2. Section 6 contains examples of extensive games to illustrate their properties. Section 7 concludes.

## 2. Decision-Theoretic Rationality and Nash Equilibrium

### 2.1. Expected Utility Theory

A game in normal form is defined by specifying the set of players $N$, for each player a set of strategies $S_i$ and each player's von Neumann-Morgenstern utility function $u_i$. Thus players are assumed to be rational: when faced with uncertainty they maximise subjective expected utility. This concept of rationality has been axiomatized by Savage (1954). The appendix contains details of his construction.

In a game rational beliefs must not only satisfy Savage's axioms, but must in addition be consistent with what players know about the structure of the game and

[1] Another difference is that Mukerji (1994) formulates the equilibrium in terms of beliefs, whereas we formulate it directly in mixed strategies, which is not equivalent under non-additive beliefs.

about each other's rationality. In this way it is possible to derive game-theoretic solution concepts in Savage's framework from additional assumptions.

In particular, a rational player not only has a belief about the opponents' actions, but can also anticipate that rational opponents will hold such beliefs about himself. Consequently, a rational player will also form a belief about these opponents' beliefs. But again, he can anticipate that rational opponents who know that the player is rational do this as well, and this gives rise to an infinite regress. A Nash equilibrium solves this infinite regress, i.e. is consistent with such a hierarchy of beliefs.[2]

If rationality is not mutual knowledge the question thus arises how a rational player should act if if he *knew* that the opponent is not rational. In that case Savage's axioms imply that the rational player has a belief given by a unique probability measure over the opponent's actions.[3] If neither a theory of bounded rationality nor a stable empirical regularity of non-rational behaviour is available, there seems to be no way to derive this belief. On the other hand, if there is no restriction on this belief the folk theorem (see, e.g., Fudenberg and Maskin, 1986) applies and all feasible and individually rational payoffs can be achieved in some equilibrium.

The idea of this paper is that a weaker rationality concept allows further assumptions about the *rational* player from which rational actions can be derived. On this basis a solution concept can be defined that is consistent with the infinite regress that arises if rationality is not mutual knowledge.

Weakening the underlying rationality concept is also of independent interest. Savage's theory of rationality is normative (Savage, 1967), the question what a rational strategy is if rationality is not mutual knowledge is a descriptive one. A less restrictive concept of rationality will also be descriptively more adequate.

The next subsection describes a weaker version of decision-theoretic rationality.

### 2.2. Choquet Expected Utility Theory

Several axioms in Savage's framework are empirically very restrictive, in particular the assumptions that the preference relation over acts is complete, transitive and satisfies the Sure-Thing Principle.

Choquet expected utility theory weakens the sure-thing principle.[4] The descriptive validity of the sure-thing principle is questioned by the Allais paradox, the Ellsberg paradox and similar findings, and it places a high demand on a player's rationality. Under Choquet expected utility, the sure-thing principle is not assumed

[2] We do not intend to give a formal or precise argument here, see e.g. Tan and Werlang (1988) or Aumann and Brandenburger (1995) for details.

[3] The celebrated Kreps et al. (1982) approach to bounded rationality follows this line. They show that it is possible to reconcile experimental evidence with game theoretic solutions by assuming a specific belief about the behaviour of non-rational players. This paper tries to extend their approach by relaxing the requirement of having a specific belief.

[4] Choquet expected utility has been developed by Schmeidler (1989) in the Anscombe and Aumann (1963) framework, in which acts ("horse lotteries") lead to additive probability measures over events ("roulette lotteries"). Gilboa (1987) has extended this approach to the Savage framework. Other important contributions are, e.g., Wakker (1989) and Sarin and Wakker (1992, 1994).

to hold for all acts, but only for acts that are comonotonic. Two acts[5] $f, f' \in \mathcal{F}$ are comonotonic iff $\neg\exists\omega, \omega' \in \Omega : f(\omega) \succ f(\omega')$ and $f'(\omega) \prec f'(\omega')$, i.e. both acts give rise to the same preference over states. In the following table, $f$, $g$ and $h$ are pairwise comonotonic, $f$ (or $g$ or $h$) and $h'$ are not.

**Table1.** Comonotonic and non-comonotonic acts.

| | $\omega_1$ | $\omega_2$ |
|---|---|---|
| $f$ | 10 | 6 |
| $g$ | 16 | 0 |
| $h$ | 10 | 0 |
| $h'$ | 0 | 4 |

Now consider objective mixtures between the above acts:

**Table2.** Mixtures between comonotonic and non-comonotonic acts.

| | $\omega_1$ | $\omega_2$ |
|---|---|---|
| $\frac{1}{2}f + \frac{1}{2}h$ | 10 | 3 |
| $\frac{1}{2}g + \frac{1}{2}h$ | 13 | 0 |
| $\frac{1}{2}f + \frac{1}{2}h'$ | 5 | 5 |
| $\frac{1}{2}g + \frac{1}{2}h'$ | 8 | 2 |

Restricting the sure-thing principle to comonotonic acts means that if the player is indifferent between $f$ and $g$ then he must also be indifferent between $\frac{1}{2}f + \frac{1}{2}h$ and $\frac{1}{2}g + \frac{1}{2}h$, but he may, e.g., strictly prefer $\frac{1}{2}f + \frac{1}{2}h'$ to $\frac{1}{2}g + \frac{1}{2}h'$. The reason is that mixtures of non-comonotonic acts can be interpreted as "hedging", i.e. distributing utility across states. Uncertainty aversion means that the player prefers objective mixing (the hedging strategy) to his subjective weighting of pure acts.[6]

Schmeidler (1989) has shown that behaviour that is rational in this weaker sense can still be described by expected-utility maximisation. Players do still act as if they possess a von Neumann-Morgenstern utility function and beliefs, and take expected values.

These beliefs, however, are no longer given by a probability measure over events, but a capacity, i.e. non-additive measure over events. Formally: a capacity $v$ maps $\Sigma$ into $[0,1]$ such that (i) $v(\emptyset) = 0$, (ii) $v(\Omega) = 1$ and (iii) $E \subseteq E' \Longrightarrow v(E) \leq v(E')$.

Property (iii) weakens the finite-additivity requirement for finitely-additive measures: $E \cap E' = \emptyset \Longrightarrow v(E \cup E') = v(E) + v(E')$. Thus non-additive beliefs (which still may, but in general needn't be additive) have a qualitative flavour.

[5] Acts $f \in \mathcal{F}$ map states $\omega \in \Omega$ into consequences $z \in Z$. For a formal statement of the sure-thing Principle see the appendix.

[6] This forces these weights to add to less than 1. For further discussion of comonotonicity see, e.g., Chew and Wakker (1996).

The expectation of a real-valued random variable $X$ with respect to a non-additive measure $v$ is defined in the sense of Choquet (1953). Formally:[7]

$$\int X dv := \int_0^\infty v(X \geq t)dt + \int_{-\infty}^0 [v(X \geq t) - 1]dt.$$

Schmeidler's theorem can now be stated:

**Theorem 1 (Schmeidler, 1989).** *If, in the Anscombe and Aumann (1963) version of the Savage framework, the sure-thing principle is restricted to comonotonic acts then there exist a utility function* $u : Z \rightarrow \mathbb{R}$*, bounded and cardinal,*[8] *and a capacity* $v : \Sigma \rightarrow [0,1]$*, unique and non-atomic, such that* $f \succeq f' \iff \int u(f)dv \geq \int u(f')dv$.

It seems that we did not gain much so far: in order to derive a rational player's action when facing a non-rational opponent we must now specify the capacity $v$ instead of the measure $p$. However, Choquet expected utility allows, in contrast to Savage's subjective expected utility, the introduction of an additional assumption about rational preferences over acts that characterizes the player's attitude towards uncertainty.

Formally, uncertainty aversion can be characterized in terms of the capacity[9] $v$. Note that the domain $\Sigma$ of $v$ is a lattice with respect to set inclusion. $v$ displays uncertainty aversion iff it is supermodular, i.e. $v(E)+v(E') \leq v(E \cap E')+v(E \cup E')$.

Maximisation of Choquet expected utility under uncertainty aversion can now be related to restrictions on *sets* of *additive* beliefs. Formally, the capacity $v$ is a set function just like a cooperative game. The core of $v$ can be defined analogously: $C(v) := \{p : \Sigma \rightarrow [0,1] |$ *$p$ a finitely additive measure such that* $p(E) \geq v(E),\ \forall E \in \Sigma\}$. The core is nonempty if $v$ is supermodular (Shapley, 1971).

Schmeidler (1986, 1989) has shown that maximisation of Choquet expected utility under uncertainty aversion is equivalent to maximinimization of subjective expected utility over the core.

**Theorem 2 (Schmeidler, 1986, 1989).**

$$\max_{f \in A} \int u(f)dv = \max_{f \in A} \min_{p \in C(v)} \int_\Omega u(f)dp.$$

Let us summarize the argument so far: Starting from Savage's subjective expected utility theory, the restriction of the sure-thing principle to comonotonic acts leads to the maximisation of Choquet expected utility, in which beliefs may, but needn't be additive. Under uncertainty aversion, maximisation of Choquet expected utility is equivalent to maximinimise over a set of additive beliefs given by the core of the non-additive belief.

As a result, we are led to the question which restrictions can or should be placed on the set of additive beliefs, over which the player maximinimizes in order to determine the rational action vis-á-vis a non-rational opponent. We suggest to make such a restriction part of the definition of an equilibrium.

[7] As usual we write $v(X \geq t)$ for $v(\{\omega \in \Omega | X(\omega) \geq t\})$. The integrals on the right hand side are extended Riemann integrals. If $v$ is additive this is the usual expectation.

[8] i.e. unique up to affine transformations

[9] Schmeidler (1989) expresses uncertainty aversion in the Anscombe-Aumann framework directly in terms of acts and proves that these definitions are equivalent.

## 3. Normal-Form Games

We consider finite[10] 2-player games in normal form in which rationality is not mutual knowledge.

**Assumption 1:** *(Choquet Rationality)*
*Rational players are assumed to be rational in the sense of Choquet expected utility.*

To avoid complications and to be able to concentrate on the issues that arise from lack of mutual knowledge of rationality alone, we make the usual assumption that players have a common prior $\epsilon$.

**Assumption 2:** *(Common Prior)*
*Rational players believe that the probability that the opponent is rational is* $1-\epsilon$. $\epsilon$ *is common knowledge among the rational players, i.e. it is a common prior.*[11]

Under the objective prior, a rational player will evaluate a strategy with the expected payoff, given that he faces either a rational or a non-rational opponent. The strategy of the rational opponent will be determined endogenously. A rational player must therefore evaluate the strategy in case he faces a non-rational opponent. For Choquet-rational players this will depend on their attitude towards uncertainty.

**Assumption 3:** *(Uncertainty Aversion)*
*Rational players are uncertainty averse.*[12]

There is both an empirical and a pragmatic justification for imposing this assumption. First, uncertainty aversion seems to be an empirical phenomenon. It is usually proposed as an explanation for the Ellsberg paradox. Dow and Werlang (1992) have proposed uncertainty aversion as an explanation of the phenomenon that there is a whole range of asset prices (as opposed to a unique price) for which an agent neither buys an asset nor sells the asset short. In general, just as risk aversion is usually assumed in decision making under risk, uncertainty aversion seems a reasonable assumption in decision making under uncertainty.[13] Second, there are pragmatic justifications for uncertainty aversion. Under uncertainty aversion the core of a non-additive belief is nonempty. Moreover, there is a long tradition of previous literature dealing with risk aversion, uncertainty aversion and the minimax criterion.[14] More fundamentally, however, the alternative assumptions are uncertainty-appeal, which seems implausible, or uncertainty neutrality, in which case Choquet expected utility reduces to expected utility theory, which implies that a player has a *unique* belief about how non-rational opponents play.

Assumption 3 implies minimax-behaviour over a set of additive beliefs. It remains to determine the size of this set.

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[10] The concept can easily be generalized to a finite number of players.

[11] In particular, it is treated as an objective probability.

[12] Note that we are in no way suggesting that it is rational to be uncertainty averse, just as Assumption 2 does not suggest that it is rational to have a common prior. Uncertainty aversion is an empirical assumption about rational players.

[13] Note, however, that the notions of risk aversion and uncertainty aversion are logically independent. Formally, risk aversion corresponds to a property of the utility function, uncertainty aversion corresponds to a property of beliefs. So it is theoretically possible, for example, to be both risk-loving and uncertainty-averse.

[14] "A minimax solution seems, in general, to be a reasonable solution of the decision problem when an a priori distribution in $\Omega$ does not exist or is unknown to the experimenter." (Wald, 1950)

Any restriction should come from a theory of bounded rationality. However, no such theory seems available. It could also come from a stable empirical regularity that could, for instance, be observed in experiments. But such regularities are only available for specific games. Moreover, these regularities will not result from non-rational play alone.

Thus we are led to argue that no restriction at all can be placed on this set. As a consequence, it is rational for an uncertainty-averse player who maximises Choquet expected utility to play his maximin-strategy against a non-rational opponent.

**Assumption 4:** *(Unrestricted Non-Rationality)*
*Rational players treat non-rational players as unpredictable, i.e. no a priori restriction is imposed on non-rational players.*

A rational player will now anticipate that a rational opponent will maximise his utility given his beliefs, that the rational opponent has the same prior probability about being matched with a rational player and also reacts with minimax play to non-rational opponents. But, surely, the rational opponent will anticipate this as well, and the player should anticipate this as well, ad infinitum. This gives, as usual, rise to an infinite hierarchy of beliefs.

In a Choquet-Nash equilibrium this infinite regress does not lead to an inconsistency. Each rational player maximises utility given his beliefs. These beliefs take the common prior into account and that a rational opponent does the same. Under Assumptions 1 – 4 a Choquet-Nash equilibrium is a strategy profile such that a rational, uncertainty-averse player has no incentive to deviate.

**Definition 1.** Let $G$ be a 2-player normal form game with finite pure strategy sets $S_1, S_2$ and utility functions $u_1, u_2$. Denote mixed strategies by $\sigma_i \in \Sigma_i$, $i = 1, 2$. A strategy profile $(\sigma_1^*, \sigma_2^*) \in \Sigma_1 \times \Sigma_2$ is a *Choquet-Nash equilibrium* iff

$$\sigma_1^* \in \arg\max_{\sigma_1 \in \Sigma_1} [(1-\epsilon)u_1(\sigma_1, \sigma_2^*) + \epsilon \min_{\sigma_2 \in \Sigma_2} u_1(\sigma_1, \sigma_2)]$$

and

$$\sigma_2^* \in \arg\max_{\sigma_2 \in \Sigma_2} [(1-\epsilon)u_2(\sigma_1^*, \sigma_2) + \epsilon \min_{\sigma_1 \in \Sigma_1} u_2(\sigma_1, \sigma_2)].$$

Note that if $\epsilon = 0$, i.e. if rationality is mutual knowledge, this definition reduces to the Nash equilibrium condition.

**Theorem 3.** *For every $\epsilon \in [0, 1]$ a Choquet-Nash equilibrium exists.*

*Proof.* The proof is a standard argument. We can define a maximin-correspondence[15] $\hat{\sigma_j}(\sigma_i) := \arg\min_{\sigma_j} u_i(\sigma_i, \sigma_j)$ Since $S_j$ is finite, this correspondence is well defined on a compact, convex domain and upper-hemicontinuous. Thus we can define best-reply correspondences $\sigma_i^* = \sigma_i^*(\sigma_i, \sigma_j, \epsilon) = \arg\max_{\sigma_i}[(1-\epsilon)u_i(\sigma_i, \sigma_j) + \epsilon u_i(\sigma_i, \hat{\sigma_j}(\sigma_i))]$ which are again, for arbitrary $\epsilon$, well-defined on a convex, compact domain and upper hemi-continuous. Kakutani's fixed point theorem then gives the existence of a fixed point, which is a Choquet-Nash equilibrium. □

Note the important difference between subjective expected utility and Choquet expected utility in justifying the minimax-strategy against non-rational opponents. Under subjective expected utility the maximin-strategy is rational only if the

[15] Recall that a minimax-strategy maximises a player's security level and that a maximin-strategy holds the opponent down to his security level.

rational player believes that the non-rational opponent minimaxes him. This belief seems difficult to justify. Under Choquet expected utility the maximin-strategy is rational because the rational player cannot exclude the possibility that the non-rational opponent plays, perhaps by chance, a minimax-strategy and because he reacts aversely towards the uncertainty created by the lack of possibility to forecast a non-rational opponent's play.

## 4. Choquet-Nash Equilibria in Normal Form Games

In this section we go through a series of examples to illustrate the properties of Choquet-Nash equilibria (henceforth CNE) in normal form games. We will only consider two-player games.

### 4.1. Strict Dominance

Consider the following version of the prisoner's dilemma:

| | $L$ | $R$ |
|---|---|---|
| $T$ | 2,2 | 0,3 |
| $B$ | 3,0 | 1,1 |

Both players have a strictly dominant strategy. Consequently, it does not matter to them whether the opponent is rational or not, a strictly dominant strategy is the only rational one. (B,R) is also the unique CNE, because a strictly dominant strategy is of course also a minimax strategy.

### 4.2. Iterated Strict Dominance

Consider the following game, (similar to Fudenberg and Tirole (1991, p.6)):

| | $L$ | $R$ |
|---|---|---|
| $T$ | 1,1 | $-99$,0 |
| $B$ | 0,1 | 0,0 |

In this game playing L is a strictly dominant strategy for player 2 (who chooses columns). Consequently, iterated strict dominance yields T as the unique rational strategy for player 1, if rationality is mutual knowledge. In particular, (T,L) is the unique equilibrium and the unique rationalizable strategy profile of the game.

Still, (T,L) is not a plausible profile. For T to make sense for player 1 he must be convinced that player 2 is rational. However, player 2 does not gain that much from playing L instead of R. In addition, player 1 does not gain much from playing T instead of B if player 2 plays L, but he looses very much when he does so if player 2 plays R.

The CNE in this game depends on $\epsilon$. In every CNE, player 2 will play L because this is his strictly dominant strategy. However, if $\epsilon$ is not sufficiently low, player 1 will play B. Note that this shows that non-rationalizable strategies may be CNE-strategies.

### 4.3. Weak Dominance

| | $L$ | $R$ |
|---|---|---|
| $T$ | 2,2 | 0,2 |
| $B$ | 2,0 | 1,1 |

This game has two Nash equilibria. (T,L) is the payoff-dominant Nash equilibrium, but it involves weakly dominated strategies. (B,R) is a Nash equilibrium in weakly dominant strategies but it is payoff-dominated.

The literature on cautious rationalizability maintains that it is not rational to play weakly dominated strategies. This is formalized through the concept of caution, according to which a player's belief should assign some positive probability to every possible strategy of the opponent. This solution concept thus excludes the Nash equilibrium (T,L).

In contrast, (T,L) is a CNE if $\epsilon = 0$. Thus it is not a priori nonrational to play the payoff dominant equilibrium strategies in weakly dominated strategies if rationality is mutual knowledge. However, (B,R) is the only CNE if $\epsilon > 0$. In this sense CNE replaces the ad hoc concept of caution.

### 4.4. Payoff Dominance

| | $L$ | $R$ |
|---|---|---|
| $T$ | 2,2 | 0,0 |
| $B$ | 0,0 | 1,1 |

This game has three Nash equilibria, (T,L), (B,R) and a Nash equilibrium in mixed strategies. As can be checked easily, the mixed Nash equilibrium strategies are also the players' minimax strategies. It follows that the mixed Nash equilibrium is also a CNE independently of the degree $\epsilon$ of mutual knowledge of rationality, thus providing a rationale for mixed equilibria.[16] Further, for every $\epsilon$ for which (B,R) is a CNE (T,L) is a CNE as well. The opposite direction is not true, i.e. (T,L) is a CNE for a lower degree of mutual knowledge of rationality than (B,R). In this sense CNE provides some rationale for payoff dominance.

### 4.5. Risk Dominance

| | $L$ | $R$ |
|---|---|---|
| $T$ | 9,9 | 0,7 |
| $B$ | 7,0 | 8,8 |

In this game the payoff dominant equilibrium (T,L) is risk dominated by the equilibrium (B,R). Here, for every $\epsilon$ for which (T,L) is a CNE, (B,R) is a CNE as well. The opposite direction is not true, i.e. (B,R) is a CNE for a lower degree of mutual knowledge of rationality than (T,L). In this sense CNE provides a rationale for risk dominance.

The last two examples show CNE allows a formal argument whether payoff or risk dominance should have precedence.

### 4.6. Battle of the Sexes

Consider the following Battle-of-the-Sexes game:

| | $L$ | $R$ |
|---|---|---|
| $T$ | 3,1 | 0,0 |
| $B$ | 0,0 | 1,3 |

[16] However, this only holds for mixed equilibria in minimax strategies, not for all mixed equilibria.

This game has 3 Nash equilibria, (T,L), (B,R) and $p^* = \frac{3}{4}$, $q^* = \frac{1}{4}$. (T,L) and (B,R) are also Choquet-Nash equilibria for $\epsilon < \frac{1}{4}$. The mixed strategy Nash equilibrium, however, is only a Choquet equilibrium for $\epsilon = 0$. Thus, Choquet-Nash equilibria represent a formal approach to distinguish between the plausibility of mixed equilibria according to the specific game in question.

### 4.7. Zero-Sum Games

Consider the following zero-sum game ("Matching Pennies"):

| | $L$ | $R$ |
|---|---|---|
| $T$ | 1,−1 | −1,1 |
| $B$ | −1,1 | 1,−1 |

This game has a unique Nash equilibrium in mixed strategies $p^* = Prob(T) = \frac{1}{2}$, $q^* = Prob(L) = \frac{1}{2}$. Note that this is also the unique CNE. This is easy to see: Since it is already rational to play the minimax-strategy against rational opponents and since minimaxing is also rational against non-rational opponents it is overall rational. Note that this holds for all zero-sum games.

Remember, however, that *all* strategy profiles are rationalizable. Note that this shows that Choquet-Nash equilibria differ from rationalizability.

### 4.8. Duopoly

Consider the Cournot duopoly, i.e. the quantity-setting game between two identical profit-maximizing firms. It is clear that the minimax-strategies will in general depend on the specification of both the demand and the cost functions. As a consequence, the CNE will also depend on these specifications.

In contrast, consider the Bertrand duopoly, i.e. the price-setting game. Here the competitive prices are both equilibrium prices and minimax prices. As a result, the CNE coincides with the competitive equilibrium, independent of the specification of the demand and cost functions.

This finding reinforces the Bertrand paradox that these two models of duopolistic competition do not lead to the same outcome, given that the specification of the strategies is the modeller's choice. It lends support to the – counterintuitive – hypothesis that the Bertrand model is the more robust model of duopolistic competition.

## 5. Extensive-Form Games

In normal forms Choquet-Nash equilibrium generalizes Nash equilibrium, taking potential lack of mutual knowledge of rationality into account. Rationality may indeed be mutual knowledge however, in which case the analysis does not add anything new.

This changes dramatically in extensive form games. On the one hand, mutual knowledge of rationality may well be an assumption to start with, but it cannot be maintained after a player deviated from whatever action is defined as rational. Lack of mutual knowledge of rationality thus arises endogenously in extensive forms[17].

[17] Selten (1975) argues explicitly that we must define the rational solution as a limiting form of bounded rationality. But of course trembles do not capture the essence of bounded rationality, nor can the assumption be justified that trembles are independent.

On the other hand, a rational strategy must certainly not give a rational player an incentive to pretend not to be rational. As a consequence, in extensive forms Choquet-Nash equilibria will not be just a generalization of equilibrium analysis, but differ substantially from equilibrium refinements.

In order to define the analogue of Choquet-Nash equilibrium for extensive forms we must first specify how rational players update their beliefs about the opponents after observing their actions. Since beliefs are not necessarily additive, Bayes' rule does not apply.

### 5.1. Updating Non-Additive Beliefs

Capacities can be updated according to the Dempster-Shafer rule (Dempster, 1968; Shafer, 1976) which generalizes Bayes' rule.

**Definition 2.** Let $v$ be a capacity $v : \Sigma \to [0,1]$. Let $A, B \in \Sigma$. The posterior capacity according to the *Dempster-Shafer rule* is given by:

$$v(A|B) := \frac{v(A \cup \overline{B}) - v(\overline{B})}{1 - v(\overline{B})}.$$

There are no probability-zero-events unless rationality was mutual knowledge at the beginning of the game, i.e. $\epsilon_0 = 0$. In all other cases the Dempster-Shafer rule yields a well-defined posterior capacity.[18]. But even if $\epsilon_0 = 0$ a probability-zero-event arises only after a non-rational action, because, by definition, rational players should play rationally, and all players were assumed to be rational. Consequently, even in this case it can be argued that a player's non-rationality is revealed.

Applying the Dempster-Shafer rule to a prior $\epsilon$ that the opponent is non-rational and the rational opponent chooses a certain action $A$, with probability $\sigma$ gives the posterior $\epsilon'$ in the following way:[19]

$$1 - \epsilon' = v(R|A) = \frac{(1-\epsilon)\sigma}{1 - (1-\epsilon)(1-\sigma)}.$$

### 5.2. Perfect Choquet Equilibrium

Consider now a finite two-player game in extensive form. In order to extend our equilibrium concept we add the Dempster-Shafer updating rule as a perfection requirement.

**Assumption 5:** *(Dempster-Shafer Rule)*
*Rational players use the Dempster-Shafer rule to update their belief about the opponents' rationality whenever this is possible, i.e. as long as an observed play did not have capacity zero.*

Under Assumptions 1 – 5 a perfect Choquet equilibrium is again a strategy profile from which a rational, uncertainty-averse player will not deviate.

**Definition 3.** Let $\Gamma$ be a 2-player game in extensive form[20] with perfect recall. Let the finite pure strategy set of player $i$ at information set $h_i$ be $S_{i,h_i}$ and his utility functions $u_i$. Denote (local) mixed strategies at information set $h_i$ by $\sigma_{i,h_i} \in \Sigma_{i,h_i}$.

[18] See Gilboa and Schmeidler (1993) for more on updating ambiguous beliefs.
[19] For details see appendix 2.
[20] For a formal definition see Selten (1975) or Kreps and Wilson (1982). The restriction to 2 players is for simplicity only.

Player $i$'s behaviour strategy in $\Gamma$ is $\sigma_i = (\sigma_{i,h_i})_{h_i \in H_i}$, where $H_i$ denotes the set of all information sets of player $i$. A strategy profile $(\sigma_1^*, \sigma_2^*) \in \Sigma_1 \times \Sigma_2$ is a *perfect Choquet equilibrium* iff

$$\sigma_{i,h_i}^* \in \arg \max_{\sigma_{i,h_i} \in \Sigma_{i,h}} [(1 - \epsilon_{h_i}) u_i(\sigma_{i,h_i}, \sigma_{i,h_i'}^*, \sigma_j^*) + \epsilon_{h_i}) \min_{\sigma_j \in \Sigma_j} u_i(\sigma_{i,h_i}, \sigma_{i,h_i'}^*, \sigma_j)],$$

for both players $i$ and $j$ and for all information sets $h_i$ and $h_i'$, $h_i' \neq h_i$. Here $\nu_{h_i}$ is player $i$'s posterior belief that the opponent is rational, given that the information set $h_i$ has been reached. This belief comes from a common prior, updated by the Dempster-Shafer rule:

$$1 - \epsilon_{h_i} = \frac{(1 - \epsilon_{\tilde{h_i}})\sigma^*(h_i)}{1 - (1 - \epsilon_{\tilde{h_i}}) \cdot [1 - \sigma^*(h_i)]}.$$

Here, $\tilde{h_i}$ is player $i$'s information set that immediately precedes $h_i$ and $\sigma^*(h)$ is the probability that the information set $h_i$ is reached by the equilibrium strategies. The common prior assumption means that by default $\epsilon_\emptyset = \epsilon$.

The intended interpretation is this: In equilibrium, a rational player must not have an incentive to deviate, and this must hold at every information set. So, at each information set, an equilibrium strategy must maximise his expected utility given his beliefs. A rational player will have two kinds of beliefs. The first belief specifies at which node in the information set the player thinks he is if his opponent is rational, i.e. the belief specifies what a rational opponent does. In equilibrium, this belief must be consistent with the player's own play at different information sets $\sigma_{i,h_i'}$, and the rational opponent's equilibrium strategies $\sigma_j^*$. Since the updating rule pins down rational beliefs at every information set, and since in equilibrium beliefs are correct, this first belief needn't be made explicit. The second belief $\nu_{h_i}$ specifies whether the opponent is regarded as rational or not. This belief results from the belief whether the opponent is rational at the preceding information set of that player and the opponent's rational strategies, combined by the Dempster-Shafer rule. Players start with a common prior and are uncertainty averse, so that they evaluate a strategy with its security level when facing a non-rational opponent.

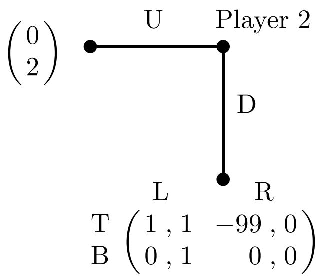
Note that in a perfect Choquet equilibrium the equilibrium path is supported by a different solution concept, i.e. minimax play, off the equilibrium path. Consequently, the solution concept does not suffer from the logical deficiency of subgame perfection, where the equilibrium path is supported by equilibrium reasoning off the equilibrium path.

## 6. Perfect Choquet Equilibria in Extensive Form Games

In this section we discuss examples of extensive games to clarify the concept of a perfect Choquet equilibrium (henceforth PCE).

### 6.1. Iterated Strict Dominance with Outside Option

Consider the following game:

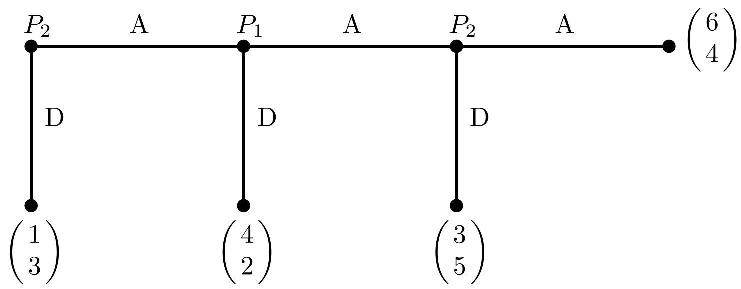



This game is similar to the iterated-strict-dominance normal form game discussed in section 5, except that player 2 now has a strictly dominant outside option. Clearly, player 1 can anticipate that his choice between T and B will only matter after a non-rational choice by player 2, in which case T is very risky.

Since U is a strictly dominant strategy for player 2, it is the only PCE-strategy for player 2. However, it follows that player 1, after observing D, can conclude that his opponent is non-rational.[21] If player 1 believes, however, that the opponent is non-rational he will play B, because player 1 is uncertainty-averse. It follows that the only PCE in this game, independent of the degree of mutual knowledge of rationality $\epsilon$, is (U,B,L).

### 6.2. The Centipede Game

Consider the following version of the Centipede game:

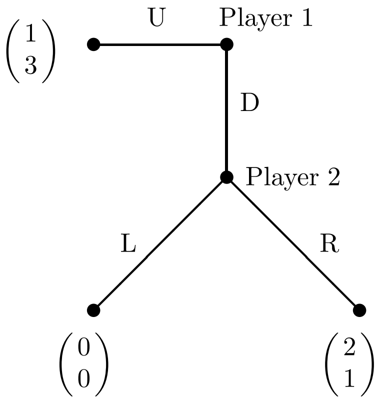



Clearly, the only PCE-strategy for player 2 at his last information node is D, because this is strictly dominant and his beliefs about player 1 do not matter. Consider now player 1. If he plays D, he gets 4, if he plays A he will either meet a rational or a non-rational opponent. The rational opponent will play D. Player 1 does not know what a non-rational player 2 will do, but he cannot exclude that a nonrational player 2 will play D and player 1 is uncertainty averse. Consequently, the only PCE strategy for player 1 is D.

[21] If $\epsilon = 0$, D is a capacity zero event. As argued earlier, this event also reveals the player's non-rationality. This can be formalized by the Dempster-Shafer rule and a limit argument for $\epsilon \longrightarrow 0$.

### 6.3. The Chain-Store Stage Game:

The following game is the stage game of a Chain-Store game. Here we are interested in the one-shot version only.




This game has two Nash equilibria, (U,L) and (D,R). Only (D,R) is a subgame-perfect equilibrium.

In every PCE, player 2 will play R, because R maximises his utility and it does not matter whether player 1 is rational or not. Player 2 is in a decision situation, not in a game situation. For player 1 the PCE will depend on his belief, i.e. the prior probability $\epsilon$, whether the opponent is rational or not. If player 1 plays U with probability $p$, then his expected payoff will be $1 \cdot p$ from U and $(1-p) \cdot [(1-\epsilon)2 + \epsilon 0]$ from D, because with probability $1-\epsilon$ player 2 is rational and play his equilibrium strategy R, with probability $\epsilon$ player 2 will not be rational, and because player 1 is uncertainty averse he cannot exclude that the resulting payoff is 0. The overall payoff from $p$ is thus $2 - p - 2\epsilon(1-p)$, so if $\epsilon > \frac{1}{2}$ the PCE strategy for player 1 is U, if $\epsilon < \frac{1}{2}$ the PCE strategy for player 1 is D.

## 7. Conclusion

Game Theory assumes that rationality is mutual knowledge. Yet this assumption is empirically questionable, and it cannot be maintained in extensive games after a deviation from rational play. This raises the normative question what a rational strategy is if rationality may, but need not be mutual knowledge.

We drew a distinction between rational and non-rational players and modelled rational players as rational in the sense of Choquet expected utility theory. This means that rational players maximise expected utility given their beliefs, but that these beliefs need not be additive. As a consequence rational play depends on players' attitude towards uncertainty.

The assumptions that players are uncertainty averse and that there is no theory of non-rationality that restricts beliefs about non-rational play gives rise to the concept of Choquet-Nash equilibrium that solves the infinite regress of beliefs in normal form games. A Choquet-Nash equilibrium exists for every common prior probability of mutual rationality. This solution concept generalizes Nash equilibrium, but differs both from rationalizability and from the Kreps et al. (1982)

approach, and sheds light on equilibrium and dominance arguments in analyzing normal forms.

The assumption that players use the Dempster-Shafer rule to update their non-additive beliefs in extensive form games and that players conclude from probability zero events that they face a non-rational opponent gives rise to the concept of perfect Choquet equilibrium in extensive forms. This solution concepts avoids the logical difficulty associated with Nash equilibrium refinements in the sense that no equilibrium arguments are used off the equilibrium path. Perfect Choquet equilibria differ from equilibrium refinements.

This work can be extended in several ways. First, the foundations of this solution concept and its mathematical structure need further study. In particular, the solution concept can be refined if more restrictions are imposed on beliefs about non-rational play. Secondly, the solution concepts can be more systematically applied to classes of normal and extensive form games. This also raises the issue of economic applications, for instance in mechanism design. Finally, the solution concept can be taken as a reference point and compared with experimental evidence.

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## Appendix

### 1. First Appendix

This appendix summarizes the concept of rationality as axiomatized by Savage (1954).

The basic elements of this theory are an infinite set $\Omega$ of states, $\omega \in \Omega$, an $\sigma$-algebra $\mathcal{A}$ of events, $E \in \mathcal{A} \subseteq 2^{\Omega}$, and a set $Z$ of outcomes, $z \in Z$. An act $f$, $f \in \mathcal{F}$, is a measurable function that associates an outcome with each state, formally: $f : \Omega \to Z, f^{-1}(z) \in \mathcal{A}, \forall z \in Z$. The player has a preference ordering $\succeq$ over acts, i.e. $\succeq \subseteq \mathcal{F} \times \mathcal{F}$.

Savage's Theorem says that if this preference relation satisfies the following seven postulates then players act as if they possess a utility function over consequences and beliefs given by a probability measure over states, and players maximise subjective expected utility.

The first postulate (P1) says that the preference relation is complete and transitive, i.e. a complete preorder. Formally: $f \succeq f' \;\vee\; f \preceq f',\;\; \forall f, f' \in \mathcal{F}$ and $f \succeq f' \;\wedge\; f' \succeq f'' \Rightarrow f \succeq f'',\;\; \forall f, f', f'' \in \mathcal{F}$.

From the preference relation over acts three more preference relations can be derived. First, we can identify outcomes with those acts that lead with certainty to that outcome, and thus derive a preference relation over outcomes, also denoted by $\succeq$. Formally:[22] $\succeq \subseteq Z \times Z$, with $z \succeq z'$ iff (if and only if) $f =_{\Omega} z, f' =_{\Omega} z'$ and $f \succeq f'$. It is clear that (P1) implies that this is also a complete preorder.

[22] We write $f =_E z$ for $f(\omega) = z,\ \forall \omega \in E$, for $f \in \mathcal{F}$ and $z \in Z$.

Second, we can define a qualitative probability relation $\geq$ over events: Consider two acts $f$ and $f'$ and two outcomes $z$ and $z'$ with $z \succeq z'$, such that $f$ leads to outcome $z$ if event $E$ obtains and to $z'$ otherwise and $f'$ leads to the outcome $z$ if $E'$ obtains and to $z'$ otherwise. If the player prefers $f$ to $f'$ it must be because he believes that $E$ is more probable than $E'$. Formally:[23] $\geq \subseteq \mathcal{A} \times \mathcal{A}$, with $E \geq E'$ iff $f = (z_E, z'_{\overline{E}}), f' = (z_{E'}, z'_{\overline{E'}})$ and $f \succeq f', z \succeq z'$.

The second postulate (P2) demands that this qualitative probability relation is well-defined, i.e. that the relation between events does not depend on the particular acts chosen to define it. Whenever two acts differ on some event, then the preference between every other two acts that differ only on that event and agree with the first two acts there, must be the same. This is the sure-thing principle. Formally: if $f =_{\overline{E}} g, f' =_{\overline{E}} g', f' =_E f$ and $g' =_E g$ then $f \succeq g \iff f' \succeq g'$, $\forall E \in \mathcal{A}$ and $\forall f, f', g, g' \in \mathcal{F}$.

The third preference relation that can be derived from preferences between acts is a relation over acts conditional on events, denoted by $\succeq_E$. An act is preferred to another conditionally on event $E$ iff every other two acts, that agree with the first two acts on that event and with each other on its complement, are preferred in the same way. Formally: $f \succeq_E g$ iff $\forall f', g' \in \mathcal{F}$ : if $f' =_{\overline{E}} g'$ then $f' =_E f, g' =_E g \iff f' \succeq g'$.

The third postulate (P3) says that conditional preference is consistent with the relation over outcomes. Formally: if $E$ is non-null[24] and if $f =_E z, f' =_E z'$ then $f \succeq_E f' \iff x \succeq y$, $\forall f, f' \in \mathcal{F}$, $\forall z, z' \in Z$ and $\forall E \in \mathcal{A}$.

The fourth postulate (P4) demands consistency between the qualitative probability relation and the relation between outcomes. Formally: if $x \succ y, x' \succ y'$ then $(x_E, y_{\overline{E}}) \succeq (x_{E'}, y_{\overline{E'}}) \iff (x'_E, y'_{\overline{E}}) \succeq (x'_{E'}, y'_{\overline{E'}})$, $\forall E, E' \in \mathcal{A}$ and $\forall x, x', y, y' \in Z$.

The fifth postulate (P5) demands that the relation between outcomes is non-trivial, i.e. that not all outcomes are equivalent. Formally: $\exists z, z' \in Z : z \succ z'$.

The sixth postulate (P6) demands, roughly, that no state is an atom, i.e. that there is always a partition of the state space such that it is possible for each cell of that partition to substitute an arbitrary outcome for the act on that cell without altering the preference between any two acts. Formally: whenever $f \succ g$ then there is a finite partition $(\Omega_i)_{i=1,\ldots,n}$ of $\Omega$ such that $\forall \Omega_i$ : if $f' =_{\Omega_i} x, f' =_{\overline{\Omega_i}} f$ then $f' \succ g$ and if $g' =_{\Omega_i} x, g' =_{\overline{\Omega_i}} g$ then $f \succ g'$, $\forall f, g \in \mathcal{F}$ and $\forall x \in Z$.

The seventh and final postulate (P7) allows the extension of simple, i.e. finitely-valued, acts to general acts. Formally: if $\forall \omega \in E : f \succeq f'(\omega)$ then $f \succeq_E f'$, $\forall f, f' \in \mathcal{F}$, $\forall E \in \mathcal{A}$, and analogously for $f \preceq f'(\omega)$.

Savage's Theorem can now be stated formally:

**Theorem 4 (Savage, 1954).** *If $\succeq \subseteq \mathcal{F} \times \mathcal{F}$ satisfies (P1) – (P7) then there exist a utility function $u : Z \to \mathbb{R}$, bounded and cardinal,*[25] *and a probability measure $p : \mathcal{A} \to [0,1]$, unique, non-atomic and finitely additive, such that $f \succeq f' \iff \int u(f)dp \geq \int u(f')dp$.*

## 2. Second Appendix

[23] $\overline{E}$ is the complement of $E$ and we write $(z_E, z'_{\overline{E}})$ for $f(\omega) = \begin{cases} z \text{ , if } & \omega \in E \\ z' \text{ , if } & \omega \in \overline{E} \end{cases}$.

[24] An event $E$ is null iff $f \succeq_E f', \forall f, f'$, i.e. all acts are indifferent conditional on $E$.

[25] i.e. unique up to affine transformations

Applying the Dempster-Shafer rule to a prior $\epsilon$ that the opponent is non-rational and the rational opponent chooses a certain action $A$, with probability $\sigma$ gives the posterior $\epsilon'$ in the following way.

Formally, all given data are additive capacities:

$$\begin{aligned} v(R = \text{ player is rational}) &= 1-\epsilon, \\ v(\overline{R} = \text{ player is non-rational}) &= \epsilon, \\ v(A = \text{ rational action}|R) &= \sigma, \\ v(\overline{A} = \text{ non-rational action}|R) &= 1-\sigma, \end{aligned}$$

If non-rational play is unrestricted, the associated capacity of a non-rational player is given by

$$\begin{aligned} v(A|\overline{R}) &= 0, \\ v(\overline{A}|\overline{R}) &= 0. \end{aligned}$$

According to the Dempster-Shafer rule the posterior is given by

$$v(R|A) = \frac{v(R \cup \overline{A}) - v(\overline{A})}{1 - v(\overline{A})}$$

and since the opponent is either rational or not we must have

$$v(R|A) + v(\overline{R}|A) = 1.$$

Since

$$v(\overline{A}|\overline{R}) = 0 = \frac{v(\overline{A} \cup R) - v(R)}{1 - v(R)} = \frac{v(\overline{A} \cup R) - (1-\epsilon)}{\epsilon}$$

we have for $\epsilon \neq 0$:

$$v(\overline{A} \cup R) = 1 - \epsilon.$$

Similarly:

$$v(\overline{A}|R) = 1 - \sigma = \frac{v(\overline{A} \cup \overline{R}) - v(\overline{R})}{1 - v(\overline{R})} = \frac{v(\overline{A} \cup \overline{R}) - (\epsilon)}{1-\epsilon},$$

so that for $\epsilon \neq 1$:

$$v(\overline{A} \cup \overline{R}) = (1-\epsilon) \cdot (1-\sigma) + \epsilon = 1 - \sigma(1-\epsilon).$$

Now

$$\begin{aligned} & v(R|A) + v(\overline{R}|A) = 1 \\ \Longleftrightarrow\ & \frac{v(R \cup \overline{A}) - v(\overline{A})}{1 - v(\overline{A})} + \frac{v(\overline{R} \cup \overline{A}) - v(\overline{A})}{1 - v(\overline{A})} = 1 \\ \Longleftrightarrow\ & v(R \cup \overline{A}) + v(\overline{R} \cup \overline{A}) = 1 + v(\overline{A}), \end{aligned}$$

so that substituting gives

$$v(\overline{A}) = (1-\epsilon)(1-\sigma).$$

Consequently, the Dempster-Shafer rule becomes

$$v(R|A) = \frac{(1-\epsilon) - (1-\epsilon)(1-\sigma)}{1-(1-\epsilon)(1-\sigma)} = \frac{(1-\epsilon)\sigma}{1-(1-\epsilon)(1-\sigma)}.$$

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# Compliance Pervasion and the Evolution of Norms: the Game of Deterrence Approach

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## 1. Introduction

It is a well known fact that multiplication of interactions within a given group of individuals generates norms and values, which in turn shape the group's culture, and hence the behaviors prevalent within that group. Now observation of daily life shows that contradictions might occur between individual behaviors and norms prevailing within the group. Depending on their nature, these contradictions may either foster the group' dynamic and thus pave the way toward progress, or on the opposite play a negative role in terms of consistency and hence social efficiency. Such will be the case in particular when non compliance with a given legal or regulatory corpus becomes pervasive. Two conflicting sets of norms have then been developed, social practice on the one hand, and the legal and regulatory corpus, the latter being now unfitted to the group or the society[1].

Various states of development of the group can be propitious to the emergence of such contradiction. In particular, the pace of globalization and the accelerated rhythm of scientific and technological progress is known to be a source of constant change in individual behaviors as well as in the regulatory and legal corpuses supposed to regulate human societies. The growing complexity of the norms system resulting from these changes may produce inconsistencies and a poor relevance of the regulatory and legal corpus. This may in turn significantly decrease the expected individuals' degree of compliance . This problematic is epitomized quite well by the fast developing issues pertaining to Intellectual Property Rights.

At the methodological level, cultural dimensions of groups and societies have been the subject of many studies. In particular, interest has been sharply growing for the impact of cultural factors on business management (Trompenaar & Hampden-Turner 1997, Schein 1999, Hofstede.& Hofstede 2004 ): how norms and values carried by social practices within a given culture should be confronted by management in setting its own norms and rules?

On its side, Game Theory has shown interest in cultural issues (Casson 1991, Bednar & Page, 2000), and modeled through an evolutionary approach the pervasion of behaviors within a given population. The existing literature contains also many

[1] This lack of fitness may have two kinds of sources: either the values carried by the legal and regulatory corpus are not adequate, or enforcement is inefficient.

papers or books dealing with economic models of Law (Blair & al 2005) and application of game theoretic tools to legal issues (Baird & al 1994).
Leaning partially on these works as well as on an elementary and quite well spread out case of Intellectual Property Rights, the present paper will propose a methodology for analyzing the conditions on the set of norms that should prevail at both levels of social practice and of legal or regulatory corpuses to make compliance pervasive.
More precisely, we shall start from a Basic Copyrights Game confronting the author of a work of mind and an end user of this work, and analyze conditions under which the end user will be compliant with the legal corpus at two different levels.
The first one can be called static / individual, since it considers a single end user facing a single author in a one shot game. Different cases will be considered based on the players perceptions of the various possible states of the world in terms of their acceptability. To take these perceptions into account as well as the standard utilities associated with the possible states of the world, we shall alternatively resort to two types of matrix games: standard Games of Nash and Games of Deterrence, which are qualitative games aiming at analyzing acceptability threshold effects.
The second level of analysis will be called dynamic /collective. It will assess the evolution of behaviors within the group through using the results already obtained, in analyzing the strategies properties of a Replicator Dynamic based evolutionary game of deterrence (Ellison & Rudnianski 2006). In particular, depending on the specific structure of the matrix supporting the evolutionary game, and hence on the specific type of graph of deterrence associated with that game, it will be possible to assess pervasion of compliance with norms.
By so doing, the analysis will thus enable to point out how the impact of a specific set of norms on the individual will orientate the evolution of the group.

## 2. The Static / Individual Level

### 2.1. The Basic Copyright Game

As already mentioned in the introduction, the Basic Copyright Game describes the interaction between:

- The author of a work of mind
- an end user of the work

The end user can be:

- compliant (C)
- non compliant (NC)

When facing a violation of his / her intellectual property rights, the author can be:

- defensive (D): take the violator to court
- passive (P): do not react

The interaction can then be represented by the following matrix:
where:

- $a - c$ represents the loss of value for the author caused by infringement with no response

| | | **Author** | |
|---|---|---|---|
| | | D | P |
| **End user** | C | (x,a) | (x,a) |
| | NC | (y,b) | (z,c) |

**Fig.1.** the Basic Copyrights Game

- $b-c$ represents the loss for the author due to litigation when facing infringement of his / her intellectual property rights
- $z-x$ represents the value of infringement for the end user
- $x-y$ is the penalty for the end user having infringed the author's intellectual property rights

Now, looking at the end user's outcomes, and considering that when facing evidence of no compliance, the court will issue a judgment against the infringer , it is then quite straightforward that $z > x > y$
If we then look at the author's outcomes, we can similarly consider that:

- $a > b$ : the author can never benefit from infringement, and even if he / she obtains compensatory damages and the total reimbursement of litigation cost, time and efforts devoted by the author to manage the issue will never be totally compensated
- $a > c$

Now what remains to compare is $b$ and $c$.
Discarding the possibility that $b = c$ as improbable, we have then to consider two possible cases.
The first case, $b > c$, corresponds to situations where the author's loss due to infringement is higher than the cost of winning in front of a court.
In such situations the game displays a single (pure) equilibrium: (C,D).
In other words, anticipating that he / she will be taken to court in case of infringement, a rational end user will choose to be compliant.
From the game theoretic point of view the problem is over.
The second case, $b < c$, corresponds to situations where damages or benefits resulting from an IPR violation of IPR are small compared to litigation costs. This happens for instance when an end user illegally downloads a song or a limited number of music albums on Internet. This case is much more spread out than the first one, and is related to the issue of norms that we wish to address here.
Therefore in the sequel we shall always assume that: $a > c > b$
Then the game displays two Nash equilibria: (C, D) and (NC, P).
These two equilibria are diametrically opposed in that sense that each one allocates an opposite strategy to each player.
To discriminate between the two equilibria, several approaches can be envisaged. One is the classical Selten's sub-game perfectness. We shall come back to it later. Another one which we shall consider here is based on the concept of threshold that we shall now introduce.

## 2.2. Acceptability thresholds

In daily life, decision makers usually do not resort to optimization tools to make their choices. There are a number of reasons for that: they do not master the tools,

they lack information, time etc. Nevertheless, "rational" decision makers need to base their choices on some bounded rationality principles which they can use as discriminating tools. A quite spread out principle is acceptability: a decision maker will select a decision if he / she considers the outcome of that decision acceptable to him/her.
This amounts to considering that for each player there are only two possible outcomes:

- Acceptable outcome noted 1
- Unacceptable outcomes noted 0

A rational player will of course look for an acceptable outcome.
In the Basic Copyrights Game, for each player, the four possible states of the world are associated with three ordered outcomes. It follows that there are four possibilities to position the acceptability threshold of each player. A particular acceptability threshold will establish a correspondence (consistent with the outcomes ordering) between the set of possible outcomes (or the set of states of the world associated with it) and {0,1}.
Thus, the four possible acceptability thresholds for each player are:

- Threshold 1: all outcomes are unacceptable
- Threshold 2: all outcomes except for the best one are unacceptable
- Threshold 3: all outcomes except for the worst one are acceptable
- Threshold 4: all outcomes are acceptable.

It follows that the Basic Copyrights Game can be associated with 16 possible binary games, that we shall call games of deterrence for reasons that will appear later. A game characterized by threshold i for the end user and threshold j for the author will be denoted by (i,j)
All 16 game matrices are given in the appendix.
Switching from the Basic Copyright Game to anyone of these binary games changes the player's perspective and hence may constrain his/her choice.

### 2.3. Strategies playability and deterrence

Let us consider for instance the binary game such that

- for the end user, only being taken to court is unacceptable
- for the author, supposedly rich and powerful, one all states of the world are acceptable

This means that the binary game is (3,4). The corresponding game matrix is:

Strategies D and P are equivalent and always guarantee the author an acceptable outcome (i.e. whatever the end user's choice). They will therefore be termed *safe*.
The same goes with strategy C of the end user.
Now it is just common sense that in such a binary game, there is no reason for

**Author**

| **End user** | | D | P |
|---|---|---|---|
| | C | (1,1) | (1,1) |
| | NC | (0,1) | (1,1) |

**Fig.2.** Game (3,4)

which a player should not play a safe strategy. Therefore the latter will be considered *playable*.
By contrast resorting to strategy NC may lead to an unacceptable outcome for the end user if the author plays D. NC will therefore be termed *dangerous*. Moreover the end user cannot exclude that the author might play D, while on the other hand the former has a playable strategy. Therefore strategy NC should be considered *not playable*.
Now one should not consider that only safe strategies are playable while all dangerous strategies are not playable. Le us for instance consider game (4,2) which corresponds to a situation in which:

- all states of the world are acceptable for the end user
- the author considers that non compliance is always unacceptable

The game matrix is:

**Author**

| **End user** | | D | P |
|---|---|---|---|
| | C | (1,1) | (1,1) |
| | NC | (1,0) | (1,0) |

**Fig.3.** Game (4,2)

The two strategies of the end user are safe, while the two strategies of the author are dangerous. Indeed the author cannot exclude that the end user will decide to be non compliant, in which case, whatever the author's choice, the latter outcome will be unacceptable. Nevertheless, in a game a player *must* play. Therefore, given the fact that his / her two strategies are equivalent, the author can play any one of them, which in turn are playable.
Now it is obvious that playability in this second example is of a different nature than playability in the first one: in the first example a strategy was playable because it was a "good" strategy, while in the second example the author's strategies are playable, only because the author has no better choice.
We shall therefore distinguish between those two types of playability.
The first one is will be termed *positive playability* while the second one will be termed *playability by default*. So a strategy will be termed *playable* if it is either positively playable or playable by default.
Now the reasoning developed in the framework of the two previous examples shows that the playability properties of a player's given strategy might depend on the playability of the other player's strategies.
More precisely, given a strategy $\phi$ of the end user and a strategy $\psi$ of the author, $\psi$ is said to be *deterrent vis--vis* $\phi$ iff:

1. $\psi$ is playable (positively or by default)
2. the outcome of implementing ($\phi$,$\psi$) is unacceptable for the end user
3. the end user has another strategy $\phi'$which is positively playable .

Thus in the first example here above, the author's defensive strategy D is deterrent vis--vis the non compliance strategy NC of the end user.

### 2.4. Games of deterrence solutions and equilibria

The two concepts of playability have been introduced here above in a quite intuitive manner. To avoid any ambiguity, the above definitions can be generalized and stated more formally. Let $S_E$ (card $S_E$ = n), and $S_A$ (card $S_A$ = p) be the strategic sets of the end user and the author respectively.
We can then consider finite bi-matrix games ($S_E$, $S_A$, $M_E$, $M_A$, S) in normal form where possible outcomes are taken from the set {0,1}.
For any strategic pair $(\phi, \psi) \in S_E \times S_A$

- $M_E(\phi, \psi) = 1$ defines an acceptable outcome for the end user
- $M_E(\phi, \psi) = 0$ defines an unacceptable outcome for the end user.

Similar definitions apply by analogy to the author with matrix $M_A$.
A strategy $\phi$ of the end user is said to be *safe* iff $\forall \psi \in S_A, M_E(\phi, \psi) = 1$
A non-safe strategy is said to be dangerous.
Let $J(\phi)$ be an *index called index of positive playability*, such that :
If $\phi$ is *safe* then $J(\phi) = 1$
If not

$$J(\phi) = \prod_{\psi \in S_A} [1 - (1 - M_E(\phi, \psi)J(\phi))](1 - j_E)(1 - j_R)$$

with

$$j_E = \prod_{\psi \in S_E} (1 - J(\psi)); j_R = \prod_{\phi \in S_A} (1 - J(\phi))$$

If $J(\phi) = 1$, strategy $\phi \in S_E$ is said to be *positively playable.*
If there are no positively playable strategies in $S_E$, that is if $j_E = 1$, all strategies $\phi \in S_E$ are said to be *playable by default.*
Similar definitions apply by analogy to strategies $\psi$ of $S_A$ .
A strategy $s \in S_E \cup S_A$ is *playable* iff it is either *positively playable or playable by default.*
The system **S** of all $J(\phi)$,$\phi \in S_E$,and$J(\psi)$,$\psi \in S_A$, is called the *playability system of the game.*
A solution of **S** is a consistent set

$$\{J(\phi_1), J(\phi_2)...J(\phi_n), J(\psi_1), J(\psi_2)...J(\psi_n)\}$$

In the general case, there is no uniqueness of the solution.
Consider for instance game $(3, 3)$ where the court case is the only state of the world unacceptable for both players. One can easily establish that the game displays two solutions:

- $\{J(C) = 1, J(NC) = 1, J(D) = 0, J(P) = 1\}$

- $\{J(C) = 1, J(NC) = 0, J(D) = 1, J(P) = 1\}$

A strategic pair $(\phi, \psi) \in S_E \times S_A$ is said to be an *equilibrium* of this game if both strategies are playable for some solution of the playability system.
In game (3,3) each solution has two equilibria:

- (C,P) and (NC,P) for the first solution
- (C,P) and (C,D) for the second solution

We see that these two solutions differ by the two strategic pairs which are Nash equilibria in the standard game.
As far as the safe strategies are concerned we see that these strategies remain safe and hence positively playable, whatever the solution under consideration.

### 2.5. Graphs of deterrence

Given a game of deterrence $(S_E, S_A, M_E, M_A, S)$, we shall call graph of deterrence, a bipartite graph G on $S_E \times S_A$ such that given $(\phi, \psi) \in S_E \times S_A$, there is an arc of origin $\phi$ (resp.$\psi$),and extremity $\psi$ (resp.$\phi$), iff $M_A(\phi, \psi) = 0$, (resp.$M_E(\phi, \psi) = 0$)
Solving the playability system **S** amounts to determining playabilities of the graph vertices. Since a graph can be broken down into paths and circuits, we shall call:

- E-path (resp. A-path) a path the root of which is an element of $S_E$ (resp. of $S_A$);
- Γ-graph, a graph, that includes neither an E-path nor an A-path.

It has been shown (Rudnianski,1991) that :

1. if G is an E-path (resp. A-path), the only positively playable strategy for the end user (resp the author) is the root, while all strategies of the author (resp. of the end user) are playable by default;
2. if G is a primary circuit, all strategies of both players are playable by default;

Moreover, it has been shown (ibid) that through appropriate cuts, it is always possible to break down the graph of deterrence into connected parts, each one being an E-path, an A-path, or a Γ-graph. Hence, depending on the presence of these elementary components in the graph, one can distinguish between 7 types of games : type E, type A, type Γ, type E-A, type E-Γ, type A-Γ, type E-A-Γ.
It has also been shown [ibid] that :

- if G is an E-path, the only positively playable strategy for the end user is the root, while all strategies of the author are playable by default;
- if G is a primary circuit, all strategies of both players are playable by default;
- if G is a Γ-*graph*,a solution of S satisfies :
  i. for any strategy $s_0$,

$$J(s_0) = \neg[j_E \vee j_A] \wedge \exists s \in N(s_0) : J(s) \wedge \forall s' \in N'(s_0) : [\neg J(s')]$$

  Where $N(s_0)$ (resp. $N'(s_0)$) is the set of the first strategies met when following G backward from $s_0$ and belonging to the same strategic set as $s_0$ (resp. to the other);

ii. on a path, the vertices positive playability is determined by the parity of their distance to the origin of the path;
iii. each player has at least one non positively playable strategy.

This typology leads in turn to the Classification Theorem (ibid) :

(1) Given a game of deterrence, its playability system's solution set is not empty.
(2) The game type defines the properties of the solution set.

It follows from (1) that every game of deterrence has an equilibrium, but the above shows that this equilibrium may not be unique.
The graph of deterrence associated with game (3,4) is:

$$C \quad P \quad D \rightarrow NC$$

So (3,4) is an E-A type game.
Likewise the graph of deterrence associated with game (4,2) is:

$$C \quad D \leftarrow NC \rightarrow P$$

So (4,2) is an E-type game.
Last in the case of game (3,3), the graph of deterrence is :

$$C \quad P \quad D \leftrightarrow NC$$

Hence, (3,3) is an E-A-Γ type game.

### 2.6. The effect of acceptability thresholds

It stems from the above discussions that acceptability thresholds impact the game type and hence the solutions and equilibria associated with that game. As already stated, with the Basic Copyrights Game there are 4 possible acceptability thresholds per player, and hence 16 possible games of deterrence. One can then systematize the analysis conducted above on three of these 16 games, and associate with each one its solution set. The results of this analysis can be summarized on the following 4 x 4 table in which:

- pp means positively playable
- pd means playable by default
- np means not playable

Now the conclusions given by the table in each of the 16 cases still need to be translated into strategic choices.
One can reasonably assume that:

- If nothing differentiates the two strategies of a player in a given game of deterrence, the player will then proceed to a random choice.
- In case of a game with several solutions, a strategy playable in all solutions should be preferred to one which is not playable in some of the solutions.
- if for any solution of the game the positive playability index of a player's strategy is superior or equal to the positive playability index of another strategy of the same player, then the latter will select the first rather than the second.

**Author**

| End User | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | • All strategies pd | • All strategies pd | • C & NC pd<br>• P safe, D np | • C & NC pd<br>• P & D safe |
| 2 | • All strategies pd | • All strategies pd | • C np & NC pp or C & NC pd<br>• P safe & D np | • C & NC pd<br>• P & D safe |
| 3 | • C safe, NC np<br>• P & D pd | • C safe & NC np<br>• P & D pp or P & D pd | • C & P safe<br>• NC pp & D np or NC np & D pp | • C safe & NC np<br>• P & D safe |
| 4 | • C & NC safe<br>• P & D pd | • C & NC safe<br>• P & D pd | • C & NC safe<br>• P safe & D np | • All strategies safe |

**Fig.4.** Acceptability thresholds and game solutions

**Author**

| **End User** | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | • Random choice for both players | • Random choice for both players | • End user : random choice<br>• Author : P | • Random choice for both players |
| 2 | • Random choice for both players | • Random choice for both players | • End user : NC<br>• Author : P | • Random choice for both players |
| 3 | • End user : C<br>• Author : random choice | • End user : C<br>• Author : random choice | • End user : C<br>• Author : P | • End user: C<br>• Author : random choice |
| 4 | • Random choice for both players | • Random choice for both players | • End user : random choice<br>• Author : P | • Random choice for both players |

**Fig.5.** Acceptability thresholds and strategic choices

On the basis of these assumptions the players' strategic choices are given by the table of Fig.5:

It stems straightforwardly from the above table that compliance is granted only for those end users whose acceptability threshold is 3, according to which the only unacceptable state of the world for the end user is the one corresponding to a court case.
It is noticeable that this conclusion is valid whatever the author's acceptability threshold, and in particular even in those cases where a court case is unacceptable for the author himself / herself.
In other words, an end user whose acceptability threshold is 3 will not look at the acceptability thresholds of the author. This means in turn that from a legal point of view, how the law treats the author doesn't matter at all. What counts is how the law treats the non compliant end user.
This observation might look weird, with regard to the Basic Copyrights Game model, since in the latter - as under many legal corpuses, one could assume that there is no disconnection in a court case within the fate of the infringer (end user) and the one of the infringement's victim (the author). In other words, what the second gets should depend on what the first one has to pay. So it seems at first glance that the outcomes of the two parties are intertwined.
But in reality, there are two reasons at least for which the situations of the two parties can be assessed separately.
The first one, and the most common, simply stems from the fact that outcomes and utilities are two different categories. Just as two end users having two different acceptability thresholds, will have different perceptions of at least one state of the world, there is no rationale for pretending that an end user and an author will always have identical assessments of a given state of the world.
The second reason lies in the mechanism of penalties and damages, which should be considered as two different dimensions of legal provisions associated with an Intellectual Property Rights court case. One can imagine a legal corpus that will not strictly compensate the author for his / her loss due to infringement, while inflicting the infringer a penalty much higher than the loss supported by the author, with the objective to strengthen deterrence of non compliance. Whether this objective can be reached or not is an issue that will be addressed in the sequel.
Identical conclusions can be drawn from the above table if one switches now from the end user to the author. Indeed, for authors which acceptability threshold is 3 (according to which the only unacceptable state of the world is the court case), being passive will be the only rational choice. The table shows that this choice doesn't depend on the end user's particular acceptability threshold. In other words, a frightful author will always remain passive. This will occur in particular if the end user's acceptability threshold is 2 (all states of the world, except infringement with no court case, are unacceptable). In such case, the end user will be non compliant and the author will not react.

### 2.7. Acceptability thresholds and game type

We have seen that each matrix game of deterrence can be associated in a one to one mapping with a graph of deterrence, which in turn defines the game type. Let us now analyze the relation between game types and compliance.

To make compliance compulsory, it is necessary that simultaneously strategy C be playable and strategy NC be non playable. This implies that C must be positively playable. Now as compliance should occur whatever the author's acceptability threshold, C must be safe. Hence the game type may only be one among the following four: E, E-A, E-Γ , E-A-Γ.
If NC is a root, then NC is safe and constitutes a playable alternative to C, in which case compliance is not compulsory.
If the game-type is E, NC belongs to an E-path while all the author's strategies are playable by default.
If NC is a root, then NC is safe and constitutes a playable alternative to C, in which case compliance is not compulsory.
Let us then assume that NC is not a root. This means that NC is a successor of an author's strategy.
Given that the interaction is modeled by a 2 x 2 game, one can easily see that there are only four possible graphs of deterrence for the above condition to occur. These are:

1. $C \rightarrow P \rightarrow NC \rightarrow D$
2. $C \rightarrow D \rightarrow NC \rightarrow P$
3. $C \rightarrow P \rightarrow NC$
   $\downarrow$
   $D$
4. $C \rightarrow D \rightarrow NC$
   $\downarrow$
   $P$

Each of the games associated with the above graphs of deterrence displays contradictions with the characteristics of the Basic Copyrights Game. Thus:

1. In the first game, the strategic pair (C,P) implies an unacceptable outcome for the author, while for the latter non compliance with no response is acceptable
2. In the second game the strategic pair (C,P) implies an unacceptable outcome for the author, on the opposite of (NC,P). Likewise going to court is acceptable for the author in case of non compliance while being passive is not
3. In the third game, non compliance is always acceptable for the author, while compliance is always unacceptable. For the end user court case is acceptable, while infringement with no response from the author is unacceptable.
4. In the fourth game non compliance is always acceptable for the author, while compliance is always unacceptable

It follows that the case of an E-type game in which NC is not playable can be discarded.
If we then look at the three remaining candidate game types E-Γ, E-A-Γ, and E-A,we can easily show that these types do not contradict the characteristics of the Basic Copyrights Game.
On the whole, hence compulsory compliance can occur only in the framework of E-Γ, E-A-Γ, and E-A type games.[2]
Now, one can generalize the mapping already realized in section 2.5 above between

[2] This doesn't mean that as soon as the game type is one of the three above compliance is compulsory. Consider for instance E-A-Γ. The only Basic Copyrights game of this

three of the 16 possible games of deterrence deriving from the Basic Copyrights Game and their graphs of deterrence, and hence their game type.
This will enable to assess how acceptability thresholds affect game types, and then in turn how these game types can be related to compliance or non compliance.
One can easily show that this mapping is the one given in the table hereunder:

| | | **Author** | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| **End User** | 1 | • Γ-type | • Γ-type | • A-Γ-type | • A-type |
| | 2 | • Γ-type | • Γ-type | • A-Γ-type | • A-type |
| | 3 | • E-Γ-type | • E-Γ-type | • E-A-Γ-type | • E-A-type |
| | 4 | • E-type | • E-type | • E-A-type | • E-A-type |

**Fig.6.** Acceptability thresholds and game type

So we see here again that acceptability threshold 3 for the end user implies three possible game types E-Γ , E-A-Γ , and E-A which occurrence depends on the acceptability thresholds of the author:

- for thresholds 1 and 2 of the latter the game type is E-Γ
- for threshold 3 the game type is E-A-Γ
- for threshold 4 the game type is E-A.

Observation of the table shows that:

- All 7 possible types of games are represented.
- Game types E-Γ and E-A-Γ occur only when the end user's threshold is 3.
- Game type E-A may occur when the end user's threshold is 3 or 4.

Now one may ask to what extent the conclusions here above depend on the specific assumption made on the Basic Copyrights Game, namely that $b < c$.
So let us momentarily assume that the opposite is true, i.e. that $b > c$, and look again for conditions under which compliance of the end user is compulsory.
The reasoning made here above can be developed again.
First, for compliance to take place whatever the author's acceptability threshold,

type is game (3,3) which displays two solutions: one for which NC is not playable and one for which NC is positively playable. In the latter case, compliance of course is not compulsory, while it is in the first one.

it is necessary that C be safe, which implies game types E, E-Γ , E-A-Γ , and E-A. Secondly, examination of the four possible games associated with game type E leads to the same conclusion: each of these games contradicts some characteristics of the Basic Copyrights Game and hence all should be discarded.
So what is left to examine is the distribution of game types according to the author's acceptability threshold.
It is straightforward that for author's thresholds 1 and 4, for which all outcomes are equivalent, nothing is changed. One can also easily see that nothing is changed for threshold 2, since in that case only the best outcome a is acceptable. The only change will occur for author's threshold 3. In this case the game type is E-A just like in the case where $b < c$. But the difference with the latter case is that now the game displays a single solution for which NC is not playable.

### 2.8. From acceptability thresholds to postures

If compliance is compulsory for those end users whose acceptability threshold is 3, nevertheless the problem of ensuring compliance is not yet solved. Indeed acceptability threshold is conveyed by the player's perception of the various states of the world. This perception can a priori be considered as "built in" the player, and therefore not susceptible to be changed, at least in the interaction framework envisaged till now.
More precisely a player's perception includes several dimensions: the norms and values of the society including those which are carried by the legal and regulatory corpuses, possibly those of the group to which he / she belongs (in the present case end users or authors), his/her personal values, previous experience and personal situation.
It follows that a change in acceptability threshold at the individual level may happen only if some factors associated with one of the dimensions mentioned here above are modified.
This could be the case for instance if for the end user the work of mind's value is small, or on the opposite big, or if a change occurs in the legal or regulatory corpus, making penalties for infringement smaller or bigger.
At the individual level we may nevertheless modify the interaction framework in order to reintroduce the author as a tool which may help to change the end user's perception. More precisely, we shall successively consider two possibilities:

- Mere communication
- Strategic postures

By mere communication we mean that the author's acceptability threshold is static, and that he / she will try to communicate this threshold to the end user, in the hope that the latter's perception will change if necessary, and eventually be defined by acceptability threshold 3.
To understand how such mechanism could possibly work, we first have to come back to the concept of acceptability. Till now it has been considered that the acceptability of each state of the world was assessed independently, i.e. according to criteria which did not take other states of the world into account.
Now in real life, this may often not be the case. Facing two unpleasant states of the world resulting each one from a different decision, an actor could either consider both as unacceptable and then select his/her decision randomly (as it has been the

case for the author in game (4,2)), or reintroduce a hierarchy between those two states of the world, by considering that the least unpleasant state becomes relatively acceptable while the most unpleasant remains unacceptable.

The same thing can be said if instead of considering two unpleasant states of the world, the actor would on the opposite consider two appealing ones. He/she could either take his/her decision randomly as it was the case for the author in game (3,4) or order the two sates of the world, and consider that the most appealing is acceptable while the least appealing is unacceptable.

At first glance, this approach seems to contradict the process followed till now, and which lead through a dichotomization of the set of states of the world, from the cardinal utilities of the Basic Copyrights Game to the corresponding cluster of games of deterrence. But in reality there is no contradiction for two related reasons. First, utilities ordering can be achieved through a repeated dichotomization process, by which at each stage one takes away the most (un)pleasant element(s) from the set of states of the world. This amounts to enrich the existing ordering. The process is repeated till no distinction can be made between the remaining states of the world. Secondly in the present case where we consider acceptability thresholds, the above developments have shown that the particular positioning of a player's acceptability threshold impacts his /her outcome in the Basic Copyrights Game, which is after all what he/she will get after interaction with the other player has taken place.

In that respect, for the end user, selecting acceptability threshold 3 will result in an outcome of x, which is the second best, while selecting acceptability threshold 1 or 4 for instance generates a risk of getting y which is the worst outcome, as soon as the author's acceptability threshold is not 3.

So if we suppose momentarily that the end user is risk averse, and if the acceptability threshold of the author is not 3, it is in the interest of the latter to let the end user know about it.

Now nothing guarantees that such communication is efficient, for instance in the case where the end user is not risk averse. It might then be necessary to go one step further and consider that the players acceptability thresholds are not simply perceptions, but strategic tools which they can use in order to defend their interests. For this reason acceptability thresholds will now become *postures*. Such postures are of current use in many areas of real life, where they can support threats or negotiations, in the field of international relation as well as in the fields of business and social affairs.

Consider for instance an end user who looks for optimizing his/her expected value in the Basic Copyrights Game. What posture should he/she adopt?

To answer that question it is necessary to associate with each pair of acceptability thresholds the pair of expected utilities that both players would get in the Basic Copyrights Game. This will define a postures meta-game which matrix is:

| | | **Author** | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| **End User** | 1 | • $(x/2 + y/4 + z/4, a/2 + b/4 + c/4)$ | • $(x/2 + y/4 + z/4, a/2 + b/4 + c/4)$ | • $(x/2 + z/2, a/2 + c/2)$ | • $(x/2 + y/4 + z/4, a/2 + b/4 + c/4)$ |
| | 2 | • $(x/2 + y/4 + z/4, a/2 + b/4 + c/4)$ | • $(x/2 + y/4 + z/4, a/2 + b/4 + c/4)$ | • $(z, c)$ | • $(x/2 + y/4 + z/4, a/2 + b/4 + c/4)$ |
| | 3 | • $(x, a)$ | • $(x, a)$ | • $(x, a)$ | • $(x, a)$ |
| | 4 | • $(x/2 + y/4 + z/4, a/2 + b/4 + c/4)$ | • $(x/2 + y/4 + z/4, a/2 + b/4 + c/4)$ | • $(x/2 + z/2, a/2 + c/2)$ | • $(x/2 + y/4 + z/4, a/2 + b/4 + c/4)$ |

**Fig.7.** The postures meta-game

Four cases must a priori be distinguished depending on the relative utilities :

- For the end user the relative utilities of:
  - the expected utility of compliance: x
  - the expected utility of non compliance: y/2 + z/2
- For the author the relative utilities of:
  - the expected utility of a random choice by the two players: a/2+b/4+c/4
  - the utility associated with an infringement followed by no judicial procedure: c

Let us first consider the end user. Given the assumption already made in the Basic Copyrights Game, one can reasonably infer that the net value $z - x$ generated by infringement is smaller than the loss x-y generated by the penalties and litigation costs that the end user would have to pay if a court case is to take place. In other words: $2x > y + z$.
It then stems from this condition and the general assumptions made on the players utilities ($z > x > y$ and $a > c > b$) that the strategic pairs (3,1), (3,2) and (3,4) are Nash equilibria.
Let us then switch to the author.
If the author's utility associated with an infringement followed by no reaction is greater than his / her expected utility associated with a random choice by the two players (in other words if $c > a/2 + b/4 + c/4$), then the strategic pair (2,3) is also a Nash equilibrium, and hence the postures meta-game displays four Nash equilibria: (2,3), (3,1), (3,2) and (3,4).

If $c < a/2 + b/4 + c/4$, then the postures meta-game displays only three Nash equilibria: (3,1), (3,2) and (3,4)
Now, coming back to Figure 5, these results can be interpreted as follows.
If the author's utility associated with an infringement followed by no reaction is greater than his / her expected utility associated with a random choice by the two players, then the end user might adopt as a Nash strategy either posture 2 or posture 3.
In the first case Figure 5 shows that the Nash equilibrium of the postures meta-game corresponds to the strategic pair such that the end user infringes and the author remains passive.
On the opposite, if the author's utility associated with an infringement followed by no reaction is smaller than his / her expected utility associated with a random choice by the two players, the end user will necessarily adopt posture 3 as a Nash strategy. The Nash equilibrium in the postures meta-game then corresponds to the strategic pair such that the end user remains compliant while the author plays randomly.
Last, one can see on the matrix that postures 1 and 4 of the end users are equivalent and - weakly - dominated by posture 2.
Likewise postures 1, 2 and 4 of the author are equivalent.

## 2.9. The Basic Copyrights Game and the Chain Store Paradox

Selten's Chain Store Paradox deals with a chain store that faces successively the threat of entry of q potential competitors in n different cities. In each city, the potential competitor can either enter the market (strategy E) or not enter (strategy NE), while the chain store can accept (strategy A) the entry or refuse it (strategy R).
At the level of a single city, if the potential competitor doesn't enter whether the chain store accepts or refuses the result is the same: the competitor keeps the whole market in the city under consideration.
If the potential competitor enters the market and the chain store accepts this entry, both parties share the existing city market. If the chain store refuses the entry, then both parties engage a costly price war.
The game can be represented by the following matrix:

| | | **Chain Store** | |
|---|---|---|---|
| | | R | A |
| **Competitor** | NE | (0,2) | (0,2) |
| | E | (-1,-1) | (1,1) |

**Fig.8.** The Chain Store Paradox

The matrix game displays two Nash equilibria: (NE, R) and (E,A)
The issue dealt with by the model is twofold:

1) Is is possible to find a refinement of the Nash equilibrium that enables a preference order between the two equilibria?
2) By refusing the entry of competitors in the early stages of the game, can the chain store build a reputation effect that will deter subsequent potential competitors to enter?

Through introducing the concept of sub-game perfectness and using backward induction, Selten shows that there is only one sub-game perfect equilibrium (E,A), and no reputation effect can deter potential competitors to enter the market.
Now, the analysis developed above shows that the Chain Store Paradox model is identical to the Basic Copyrights Game.
The chain store and the competitor in the former model correspond respectively to the author and the end user in the latter. Likewise the chain store strategies A and R correspond to the author's strategies P and D respectively, while the competitor's strategies E and NE correspond to the end user's strategies C and NC respectively.
Moreover the matrix structure is exactly the same in the two games, and the Nash equilibria in the Chain Store Paradox correspond exactly to the Nash equilibria in the Basic Copyrights Game.
Nevertheless the conclusions obtained about the Basic Copyrights Game through the process developed above, do not coincide entirely with the conclusions obtained by Selten.
Sometimes deterrence doesn't work, but sometimes it does. All depends on the players' acceptability thresholds (which are not taken into account in the Chain Store Paradox which considers only utilities). Applying the results of the Basic Copyrights Game analysis leads to the conclusion in the Chain Store Paradox that if the competitor's acceptability threshold is 3, he / she will not enter the market. This conclusion just meets common sense since it means that if price war is the only state which is unacceptable for the competitor, the latter, by deciding not to enter can guarantee himself / herself an acceptable outcome.
On the whole, one can consider that introducing acceptability thresholds through players' perceptions or postures enables to find solutions that may eliminate the paradox.

## 3. The Dynamic / Collective Case

### 3.1. From perceptions and postures to evolution of norms and values

The Basic Copyrights Game can be considered as a metaphor of interactions between two groups of individuals within a given population: the group of end users on the one hand, and the group of authors on the other.
Although it is obvious that inside a group two individuals may differ to the point that they will have different perceptions or adopt different postures, one may nevertheless consider that sometimes individuals' behaviors are driven by the norms and values of the society to which these individuals belong.
These norms and values, which define what is considered acceptable and what is not, are a product of the society's culture, but they are also strongly and directly influenced by the legal and regulatory corpuses that proceed - partially at least - from that culture. In that respect and with regard to the issue of compliance pervasion, one can consider that the two players of the Basic Copyrights Game play asymmetric roles, in the sense that authors may represent also the interests of the whole society. The legal and regulatory corpus through encouraging or refraining the authors to go to court in case of infringement will therefore shape the contours of what the society considers as a fair balance between pervasion of innovation on the one hand and pervasion of compliance on the other (see Blair and al).

If we now look at the individual level, the norms and values (and more particularly the acceptability thresholds which are considered here) that command behaviors have different sources, which act as many layers of the individual's culture: norms and values of the whole society, or of the particular group to which the individual belongs (end users or authors), experience of the individual ...
As the aim of the present paper is to introduce a methodology, in the sequel we shall not dwell into details, and just consider for the sake of simplicity that the perception or the posture of a given party (end user or author) is a feature of the group to which this party belongs.
On this basis, we shall successively analyze how compliance pervasion evolves for various pairs of acceptability thresholds, and then analyze in a particular case how norms and values conveyed by these acceptability thresholds can themselves evolve.

### 3.2. Norms and evolution of behaviors

To model the evolution of compliance in a simple way we shall resort to the classical Replicator Dynamic.
One can easily establish that the system of equations of the latter associated with the Basic Copyrights Game is:

$$\theta_C{}' = \theta_C(1-\theta_C)[x - z + \theta_D(z - y)]$$

$$\theta_D{}' = \theta_D(1-\theta_D)(1-\theta_C)(b - c)$$

Where:

- $\theta_C$ represents the proportion of the end users who are compliant
- $\theta_D$ represents the proportion of the authors who are defensive

One can notice that:

- The proportion of end users who are compliant remains constant if one of the following conditions is satisfied:
  - The end users group is totally homogeneous (i.e. all end users are either compliant or non compliant)
  - The proportion of defensive authors equals the ratio between the value of infringement for the end user and the sum of that value and of the penalty if there is a court case

Furthermore compliance increases as soon as the end users group is not homogeneous and the above ratio is positive.

- The proportion of authors who are defensive remains constant if one of the following conditions is satisfied:
  - The authors group is totally homogeneous
  - All end users are compliant
  - The authors do not either gain or lose anything in litigation (it has already been mentioned that such case is quite improbable)

Furthermore, under the conditions stated in section 2.1 ($c > b$), if the author's group is not totally homogeneous, the proportion of defensive authors decreases.
If we then take acceptability thresholds into consideration by switching to the game of deterrence representation, the following table gives the corresponding system of equations for each pair of norms:

**Author**

| **End User** | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | • $\theta'_C/\theta_C = 0$<br>• $\theta'_D/\theta_D = 0$ | • $\theta'_C/\theta_C = 0$<br>• $\theta'_D/\theta_D = 0$ | • $\theta'_C/\theta_C = 0$<br>• $\theta'_D/\theta_D = -(1-\theta_D)(1-\theta_C)$ | • $\theta'_C/\theta_C = 0$<br>• $\theta'_D/\theta_D = 0$ |
| 2 | • $\theta'_C/\theta_C = -(1-\theta_D)(1-\theta_C)$<br>• $\theta'_D/\theta_D = 0$ | • $\theta'_C/\theta_C = -(1-\theta_D)(1-\theta_C)$<br>• $\theta'_D/\theta_D = 0$ | • $\theta'_C/\theta_C = -(1-\theta_D)(1-\theta_C)$<br>• $\theta'_D/\theta_D = -(1-\theta_D)(1-\theta_C)$ | • $\theta'_C/\theta_C = -(1-\theta_D)(1-\theta_C)$<br>• $\theta'_D/\theta_D = 0$ |
| 3 | • $\theta'_C/\theta_C = \theta_D(1-\theta_C)$<br>• $\theta'_D/\theta_D = 0$ | • $\theta'_C/\theta_C = \theta_D(1-\theta_C)$<br>• $\theta'_D/\theta_D = 0$ | • $\theta'_C/\theta_C = \theta_D(1-\theta_C)$<br>• $\theta'_D/\theta_D = -(1-\theta_D)(1-\theta_C)$ | • $\theta'_C/\theta_C = \theta_D(1-\theta_C)$<br>• $\theta'_D/\theta_D = 0$ |
| 4 | • $\theta'_C/\theta_C = 0$<br>• $\theta'_D/\theta_D = 0$ | • $\theta'_C/\theta_C = 0$<br>• $\theta'_D/\theta_D = 0$ | • $\theta'_C/\theta_C = 0$<br>• $\theta'_D/\theta_D = -(1-\theta_D)(1-\theta_C)$ | • $\theta'_C/\theta_C = 0$<br>• $\theta'_D/\theta_D = 0$ |

**Fig.9.** Norms and the Replicator Dynamic

The above table shows that the 16 pairs of norms and the corresponding games can be gathered in six categories associated each one with a specific dynamic of the parties' behaviors / strategies:

- Category 1 includes games (1,1), (1,2), (1,4), (4,1), (4,2), (4,4).
  Whatever the game in this category, the initial profile of each one of the two groups (end users and authors) remains the same: there is no evolution. As far as games (1,1), (1,4), (4,1) and (4,4) are concerned, the interpretation is quite obvious. In each one of these games, all four possible states of the world are considered equivalent by the end users' group as well as by the authors' group. So there is no rationale to change behaviors and hence no evolution.
  If we then look at games (1,2) and (4,2), while end users still consider all possible states of the world as equivalent, this is no more the case for the authors, who consider that there are two acceptable states of the world, namely those associated with end users' compliance. So, one could expect some evolution in the authors' behavior. That this evolution doesn't occur can be explained by the fact that in both games by the two strategies of the authors are equivalent, and hence there is no rationale for an author to change.

- Category 2 includes games (1,3) and (4,3).
  For the reasons just mentioned, in both games the end users have no reason to modify their behaviors whatever change may occur in the authors' behaviors. Now as far as the latter are concerned, the above table shows that two main cases should be distinguished:
  - If the authors' group is totally homogeneous, or if all end users are compliant, then the corresponding equation shows there is no evolution in the authors' behavior.
    - This conclusion can be understood quite straightforwardly in the case where all end users are compliant: indeed as the users' group profile remains the same whatever the authors' group profile, and both authors' strategies lead to the same outcome in this case, there is no rationale for the authors to change.
    - when the authors' group is totally homogeneous, the conclusion stems in particular from the features of the Replicator Dynamic according to which species' reproduction requires that the species exists [3] .
  - If the authors' group is not totally homogeneous and the end users' group is not totally compliant, the above equations show that the proportion of defensive authors decreases with time, which can be understood since the passive strategy is -weakly- dominant in both games.

- Category 3 includes games (2,1), (2,2), (2,4).
  In those three games the authors' profile remains constant which stems from

[3] This particular feature may be considered a weakness of the Replicator Dynamic, since it cannot take into account evolutions during which new species appear. Nevertheless, as it suffices that one unit of a different species be present in the group scrutinized, one can reasonably assume there is a high probability that such condition is satisfied when the group under consideration is large, for instance at the national level which is the one at which one should envisage the impact of legal corpuses.

the fact that in these three games authors consider equally unacceptable (games (2,1) and (2,2)) or acceptable (game (2,4)) the consequences of being passive or defensive when facing an infringement of their intellectual property rights.
As far as the end users are concerned, conclusions are similar to those obtained for the authors in category 2. In particular, if the end user's group is not homogeneous and not all authors are defensive, the proportion of compliant end users decreases

- Category 4 includes games (3,1), (3,2), (3,4).
  The profile of the authors' group remains constant for the same reasons than in category 3.
  Regarding the end users, again two main cases should be considered:
  - In the first one, either the end users' group is totally homogeneous, or all authors are passive, in which case the end users' group profile remains constant.
    If the end users' group is totally homogeneous, the fact that no change occurs stems from the features of the Replicator Dynamic, while if all authors are passive the conclusion stems from the fact that either end user's strategy leads to the same outcome
  - If the end users' group is not totally homogeneous, and not all authors are passive, the proportion of compliant end users increases, which can be explained by the fact that for the three games under consideration the compliance strategy is -weakly- dominant.

- Category 5 is comprised of a single game: (2,3).
  If the end users' group is totally homogeneous or if all authors are defensive, the end users' group profile remains constant, which in the latter case can be explained by the fact that both end users strategies lead to the same outcome.
  If not, the proportion of compliant end users decreases, which can be explained by the fact that compliance is a -weakly - dominated strategy.
  Likewise if the authors' group is totally homogeneous or if all end users are compliant, the authors' group profile remains constant, which in the latter case stems from the fact that both authors' strategies lead to the same outcome.
  If not, the proportion of defensive authors decreases, which can be explained by the fact that being defensive is a - weakly - dominant strategy.
  So in this game the evolution leads to a situation where authors will be passive while end users will be non compliant.

- Category 6 is comprised of a single game: (3,3).
  For the end users, the situation is the same than in category 4, while for the authors it is the same than in category 5.
  This implies an evolution toward compliance for the end users and toward being defensive for the authors.

On the whole, discarding for the sake of simplicity the case of initially homogeneous groups, the following table summarizes how norms impact the evolution of behaviors:

**Author**

| **End User** | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | • Both profiles constant | • Both profiles constant | • Users profile constant<br>• Defense decreases | • Both profiles constant |
| 2 | • Compliance decreases<br>• Authors profile constant | • Compliance decreases<br>• Authors profile constant | • Compliance decreases<br>• Defense decreases | • Compliance decreases<br>• Authors profile constant |
| 3 | • Compliance increases<br>• Authors profile constant | • Compliance increases<br>• Authors profile constant | • Compliance increases<br>• Defense decreases | • Compliance increases<br>• Authors profile constant |
| 4 | • Both profiles constant | • Both profiles constant | • Users profile constant<br>• Defense decreases | • Both profiles constant |

**Fig.10.** The impact of norms on behaviors evolution

We see that evolution toward compliance is fostered when end users adopt the set of norms 3, while on the opposite it is unfavoured when the latter adopt the set of norms 2. Sets of norms 1 and 4 are neutral with respect to that evolution.
If we then turn toward the authors, we see that sets of norms 1, 2 and 4 are neutral with respect to evolution toward a defensive attitude, while set of norms 3 is favorable to an evolution toward a passive attitude.
It follows that if non compliance and defensiveness are considered as representing a form of aggressive behaviour, the game is asymmetric: on the one hand non compliance may increase, be constant or decrease, while on the other defensiveness can only be constant or decrease. The reason for such an asymmetry stems from the features of the Basic Copyrights Game: if the end user may benefit or lose from being non compliant with respect to an initial situation where he / she was compliant, the author can never benefit from being defensive with respect to the same initial situation.
Moreover, the above table shows that compliance evolution doesn't depend on the set of norms of the authors group, and likewise that evolution of defensiveness doesn't depend on the norms prevailing within the users group.
These conclusions are fully consistent with the ones found in the analysis of the individual / static case, of which they can be considered an extension.

### 3.3. The norms meta-game

The concept of posture introduced in the analysis of the individual / static case was used to model the consequences of the end user's and author's acceptability thresholds. The same process can be developed with respect to norms. But now the outcomes associated with a pair of norms are not the outcomes of the Basic Copyrights Game associated with the particular game of deterrence which these norms define, but the outcomes of the Basic Copyrights Game that would result from the evolution of behaviors triggered by the set of norms under consideration.
To define such a norms' meta-game, we need to complete the determination of those outcomes. For a real life problem the extra information required would be the initial profile of each one of the two groups. Indeed, in the case where the profile of one or of the two groups remains constant, this profile will be the basis on which the corresponding Basic Copyrights Game outcomes will be assessed. These profiles should derive from statistical studies.
Now as in the framework of this paper no such statistical study is available, we shall simply discard the cases where the groups are totally homogeneous, and therefore enable to focus on "proper" evolution.
On this basis, we can associate with each pair of norms a strategic pair, with which we shall in turn associate with a pair of outcomes. Thus under the above assumption, the table of strategic pairs is the following:

Now, just like in the individual / static case one can think of the two groups (end users and authors) as two abstract players interacting in a game in which the strategic sets are the norms sets. If at first glance such identification might look a little weird, it can nevertheless be justified on two grounds.
The first one is the evolutionary framework. We shall come back to that in the next section. The second one pertains more directly to the design of a legal corpus. The law maker should be interested in simulating the impact of laws on the society, for

**Author**

| **End User** | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | • Random choice for both players | • Random choice for both players | • End user: random choice<br>• Author: P | • Random choice for both players |
| 2 | • End user: NC<br>• Author: random choice | • End user: NC<br>• Author: random choice | • End user: NC<br>• Author: P | • End user: NC<br>• Author: random choice |
| 3 | • End user: C<br>• Author: random choice | • End user: C<br>• Author: random choice | • End user: C<br>• Author: P | • End user: C<br>• Author: random choice |
| 4 | • Random choice for both players | • Random choice for both players | • End user: random choice<br>• Author: P | • Random choice for both players |

**Fig.11.** Norms and long term behaviors

which these laws are set up, and in particular to know the optimal pair of norm sets for a given legal corpus.
The norms' meta-game can thus be represented by the matrix of Fig.12:

| | | **Author** | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| **End User** | 1 | $(x/2+y/4+z/4, a/2+b/4+c/4)$ | $(x/2+y/4+z/4, a/2+b/4+c/4)$ | $(x/2+z/2, a/2+c/2)$ | $(x/2+y/4+z/4, a/2+b/4+c/4)$ |
| | 2 | $(y/2+z/2, b/2+c/2)$ | $(y/2+z/2, b/2+c/2)$ | $(z, c)$ | $(y/2+z/2, b/2+c/2)$ |
| | 3 | $(x, a)$ | $(x, a)$ | $(x, a)$ | $(x, a)$ |
| | 4 | $(x/2+y/4+z/4, a/2+b/4+c/4)$ | $(x/2+y/4+z/4, a/2+b/4+c/4)$ | $(x/2+z/2, a/2+c/2)$ | $(x/2+y/4+z/4, a/2+b/4+c/4)$ |

**Fig.12.** The norms meta-game

We see that except for the row corresponding to the set of norms 2 for the end user, this matrix is the same than the matrix of the postures meta-game.
The two games display the same properties as far as Nash equilibria are concerned, except that in the norms' meta-game (2,3) is a Nash equilibrium whatever the relative values for the authors of litigation cost and loss caused y infringement (provided of course that $c > b$).
This means that with the assumptions of the Basic Copyrights Game, in the long run these values do not impact the choice of norms. This is quite a significant information for the law maker when designing the system of penalties and compensatory damages that should apply to copyrights issues.
Another social benefit of the norms' meta-game is that it shows that at *equilibrium*, only two end users' sets of norms matter: the sets 2 and 3. On the opposite, any of the four norms sets pertaining to authors can be an element of an equilibrium, but norms 1, 2 and 4 are equivalent strategies for the norms' meta-game.

### 3.4. The dynamics of norms

In the previous sections it was assumed that while the parties behaviors could evolve, the corresponding evolutions took place within the framework of fixed norms and values for the end users as well as for the authors. Even if the norms and values could be the object of a choice by the concerned parties as in the norms' meta-game, once the choice has been made, the norms were there to stay.
If there is no doubt that this framework is relevant for relating behaviors evolutions, with norms of the end users and the authors, one may nevertheless question the realism of a norms "fixity principle".
After all, if a set of norms is less efficient than another, in that sense that by adopting the latter a given group would do better than by adopting the former, within an evolutionary context it might be difficult to justify why the group under consideration would stick to the first one.
On the opposite, one can reasonably assume that changes will affect the group's set of norms, in a way that members of the group will possibly improve their efficiency.
To develop an extensive study of norms' dynamic is out of scope.
We may nevertheless focus on a quite significant case, based on the results of the norms' meta-game.
Let us thus consider the following sets of norms:

- For the end users group, the sets 2 and 3 which are the only ones for which a Nash equilibrium may occur in the norms' meta-game.
- For the authors group, as sets 1, 2 and 4 are equivalent in the norms' meta-game, we can select one of them, say set 2 which is an element of equilibrium (3,2), and add set 3 which is an element of equilibrium (3,2).

There is another justification for deleting norms' sets 1 and 4 of both groups: with respect to the Basic Copyrights Game, neither is discriminatory, i.e. they do not offer the group who has adopted one of these norms' sets a way to order the possible states of the world, and hence to select a particular strategy.
We can then consider the following reduced 2 x 2 game: Keeping in mind the general

| | | **Author** 2 | **Author** 3 |
|---|---|---|---|
| **End user** | 2 | $(y/2+z/2, b/2+c/2)$ | $(z, c)$ |
| | 3 | $(x, a)$ | $(x, a)$ |

**Fig.13.** The norms reduced meta-game

assumptions made above:

- $z > x > y$

- $a > c > b$
- $2x > y + z$

the Nash equilibria of this reduced game, i.e. (2,3) and (3,2) are Nash equilibria in the non reduced meta-game, and they furthermore represent all outcomes pairs that may occur in the latter.
So what is to be considered now is the case of protracted interactions between the groups, such that in each group individuals have at each stage of the dynamic of interaction, the possibility to adopt one of the two sets of norms considered here above.
To conduct the analysis we shall once again resort to the Replicator Dynamic model [4] , which will now be based on the game matrix here above.
To avoid confusion with the analysis conducted above, let us denote by:

- $\phi_2$ the proportion of the end users who adopt the norms set 2
- $\psi_3$ the proportion of the authors who adopt the norms set 3

One can show that the equations of the Replicator Dynamic write:

- $\phi_2' = (1/2)\phi_2(1 - \phi_2)[(z - x) - (x - y) + \psi_3[(z - x) + (x - y)]]$
- $\psi_3' = (1/2)\phi_2\psi_3(1 - \psi_3)(c - b)$

Let us first consider the evolution of $\phi_2$.
As for the dynamic of behaviors, if the group of end users is totally homogeneous (all end users choose $\phi_2$ or all choose $\phi_3$), it will remain so.
If not, the first equation here above shows that the evolution of the set of norms adopted by the end users depends on two main factors:

- The respective values of unpunished infringement on the one hand, and of the penalties and litigation cost for such infringement on the other
- The proportion of authors adopting norms' set 3.

If the value of infringement for the end user is bigger than or equal to penalties and litigation cost, the above equation shows that the proportion of end users adopting norms' set 2 will continuously increase, which after all meets common sense. It means in particular that in this case, the proportion of authors adopting norms' set 3 doesn't matter.
In other words if in the legal corpus the penalties and / or litigation costs are smaller than the value of unpunished infringement, the group of end users will consider that compliance is as unacceptable as court case, and will not take into consideration the distribution of norms inside the authors' group.
In the opposite case, i.e. when the value of infringement for the end user is smaller than the penalties and litigation cost, the evolution of norms within the end users' group depends on the proportion of authors adopting norms' set 3.
If $\psi_3 < [(x - y) - (z - x)]/[(x - y) + (z - x)]$, the proportion of end users adopting norms' set 2 decreases.
If $\psi_3 > [(x - y) - (z - x)]/[(x - y) + (z - x)]$, the proportion of end users adopting norms' set 2 increases.

[4] As already noticed, this model may present drawbacks, but displays the advantage to be simple enough for not generating intricacies that would blur the presentation of the methodology that the present paper wishes to propose.

Let us now switch to the authors' group.
Once again the authors' group is totally homogeneous, it will remain so for ever.
If not, the second equation of the Replicator Dynamic shows that if $\phi_2 > 0$, $\psi_3$ will always increase. This means in turn that if at initial time, neither group is totally homogeneous, the evolution of norms for the two groups will be the following:

- The end users will eventually adopt norms' set 2
- The authors will eventually adopt norms' set 3.

Of course, whether the direction of evolution of the end users norms will always remain the same or will switch from decrease to increase depends on the initial value of $\psi_3$. If the initial value of the latter is small enough, $\phi_2$ will first decrease till it reaches a minimum for $\psi_3 = [(x - y) - (z - x)]/[(x - y) + (z - x)]$, and then increase.
It follows that the interaction will eventually lead the end users to be non compliant and the authors to remain passive in case of infringement to their copyrights.
This means that out of the two possible Nash equilibria in the reduced meta-game, evolution will favor (NC,P).
This pessimistic conclusion (as far as compliance is concerned) stems from the assumptions of the Basic Copyrights Game according to which $c > b$ [5] .
More precisely, as long as the authors are better off by not reacting to an infringement of their intellectual property rights, the severity of the penalties inflicted to non compliant end users is useless in the long run. To prevent the occurrence of such a situation there are theoretically two possibilities.
The first one consists in a significant decrease of litigation costs. But such a possibility doesn't seem very realistic given lawyers' usual fees, and the fact that the judicial system usually wants to prevent proliferation of court cases that would follow such a decrease. Moreover even if these obstacles were overcome, the decrease of litigation costs would also benefit end users, and thus accelerate evolution toward non compliance.
The second possibility is a closer coupling between outcomes of the two parties resulting from a court case, through ensuring that an increased penalty for the non compliant end user benefits the author till the point that b becomes greater than c. In this case the initial proportion of end users adopting norms' set 2 may initially increase if the proportion of authors adopting norms' set 3 is high enough, then decrease as well as $\psi_3$, in which case the equilibrium eventually selected will be ($\psi_3$, $\phi_2$). As a consequence 50% of the authors will be defensive, while all end users will be compliant.

## 4. Conclusions

Leaning on the example of the Basic Copyrights Game, the present paper has proposed a methodology for analyzing compliance issues at two levels.
The static / individual level, has considered that the player's view of the states of the world does not entirely coincide with (cardinal or ordinal) utilities, but focuses on how acceptable the players consider these states. This is done through

[5] It can be noticed that if we go back to the bridge established between the Basic Copyrights Game and Selten's Chain Paradox, the above conclusions meet the ones obtained by Selten, according to which the competitor will enter and the monopolist will accept that entry

associating the original quantitative game with a set of games of deterrence, each one corresponding to a specific pair of acceptability thresholds.
These thresholds can be considered either as fixed norms of the players or possible postures which the latter can use to build a reputation promoting their interests. A parallel between the Basic Copyrights Game and Selten's Chain Store Paradox, has shown that at the static/ individual level, deterrence may occur.
The second level, called dynamic/collective, has analyzed how the behaviours of two interacting populations (end users and authors) evolve with norms.
Associating with each pair of norms the behaviours resulting from evolution, has enabled to build a game in which the two populations' strategies are the norms these populations adopt. An evolutionary process based on behaviours efficiencies has been generated which leads to the selection of particular sets of norms.
It has been shown that in the case under consideration the level of penalties inflicted to non compliant end users cannot by itself generate compliance pervasion. This pervasion also requires that the author victim of an infringement should get a share of these penalties important enough, to keep credible the threat that he / she would react in case of an infringement of his / her rights.
More generally, starting from a simple case, the model has proposed elements of a methodology paving the way to a comprehensive game theoretic tool for analyzing the interaction between evolution of norms and pervasion of behaviours in a variety of social, economic, legal or political contexts.

*Many thanks to Pr Hélène Bestougeff from Unversité de Marne la Vallée and ORT France for reviewing the paper.*

## Appendix

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | D P<br>C (0,0) (0,0)<br>NC (0,0) (0,0) | D P<br>C (0,1) (0,1)<br>NC (0,0) (0,0) | D P<br>C (0,1) (0,1)<br>NC (0,0) (0,1) | D P<br>C (0,1) (0,1)<br>NC (0,1) (0,1) |
| 2 | D P<br>C (0,0) (0,0)<br>NC (0,0) (1,0) | D P<br>C (0,1) (0,1)<br>NC (0,0) (1,0) | D P<br>C (0,1) (0,1)<br>NC (0,0) (1,1) | D P<br>C (0,1) (0,1)<br>NC (0,1) (1,1) |
| 3 | D P<br>C (1,0) (1,0)<br>NC (0,0) (1,0) | D P<br>C (1,1) (1,1)<br>NC (0,0) (1,0) | D P<br>C (1,1) (1,1)<br>NC (0,0) (1,1) | D P<br>C (1,1) (1,1)<br>NC (0,1) (1,1) |
| 4 | D P<br>C (1,0) (1,0)<br>NC (1,0) (1,0) | D P<br>C (1,1) (1,1)<br>NC (1,0) (1,0) | D P<br>C (1,1) (1,1)<br>NC (1,0) (1,1) | D P<br>C (1,1) (1,1)<br>NC (1,1) (1,1) |

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# A Game Theoretic Approach for Selecting Optimal Strategies of Fertiliser Application

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**Abstract** The paper describes the application of game theoretic approach in resource management with specific application to development of optimal strategies of phosphorus applications for soil fertilisation. This approach allows resource managers to consider not only competitive strategies, which were treated as the Nash equilibrium game solutions but the strategies which imply cooperation between farmers. These strategies were modelled as the cooperative Pareto optima of the game. The objective function of the game has been developed in order to reflect both economic advantages of phosphorus applications and the environmental losses associated with these applications expressed as dollar values. The paper presents algorithms for finding competitive and cooperative solutions of the game for the particular case when no time scheduling is included in the game parametrization. The results obtained in the paper showed that the cooperative solutions lead to much lesser environmental impacts than that in the case of non-cooperative strategies.

**Keywords:** water quality, game theory, phosphorus, fertilisers.

## 1. Introduction

Phosphorus pollution is a major factor affecting the waterways in Western Victoria and in many other catchments in Australia. One of the major strategies in reducing phosphorus contamination is the adoption by farmers of less harmful and more efficient fertilisation strategies. The present paper outlines the research work game theoretic modelling for resource management and provides some quantitative assessment of optimal strategies in phosphorus application in the Hopkins catchment region. Its main objective is the development of theoretical and computational model to predict optimal amounts of phosphorus applied for land fertilisations in the Hopkins River catchment in order to assess and minimise its impacts on the health of rivers in this area. The methodology used to achieve this aim relies on the tools of Game Theory developed for solving the non-zero sum multiple player games.

This paper formulates a general game-theoretic model, based on information regarding farmers' practices and environmental factors, on application of fertiliser which will assist in reducing agricultural pollution, especially those associated with phosphorus contamination, in the Hopkins catchment. The two main classes of game solutions, non-cooperative and cooperative (Owen, 2001), are addressed in this work. The non-cooperative approach will result in a set of solutions which are Nash equilibrium and the cooperative approach has been implemented by the cooperative Pareto optima. Among the key research questions addressed in this paper will be the benefit of applying game theory for improving the health of waterways in the Hopkins catchment area.

Game theory may be broadly described as a mathematical theory which was developed to model how rational human beings or organizations make decisions in a competitive environment or conflict situation. It allows researchers to find optimal strategy of behaviour for players involved in the game. In this work the game theoretic approach has been employed for modelling the strategies of phosphorus application by farmers of the Hopkins Basin. In the context of the present work players are farmers (or households) applying fertilisers on their paddocks.

The main advantage of a game-theoretic approach in resource management is that it allows one to consider and compare competitive as well as cooperative actions of agents sharing limited resources. Non-cooperative games are games where each player or groups of players are antagonistic to each other. The main objective of non-cooperative games is to find optimal strategies for which players can use against each other to optimize one or more utility functions. In cooperative games, players are able to form coalitions and utilities are transferable (shared) between members of these coalitions. The main objective here is to understand how cooperation could lead to better distribution of utilities to all players, in comparison to players engaging in pure competition between themselves. However, the cooperation can be modelled in the non-cooperative games via cooperative Pareto optima. This approach was utilised in the present work.

The paper formulates the model composition of the game when players are 30 households in the Hopkins catchment. The information on their land use structures were taken form the survey specially implemented within the current project. This survey also provided the information on the phosphorus application policies characterizing each of these households. The objective function for each of these players has been define as a sum of their crops revenues, total cost of phosphorus used and environmental penalties associated with the current level of pollution, which was calculated using the wide range of data on economic impacts associated to the water quality deterioration.

This paper treats the problem of phosphorus application completely as economic question. It formulates the conceptual model which allows users to estimate the optimal values of phosphorus applied for fertilisation from the point of view of utility maximization, leaving the hydrological problem of quantification of the nutrient concentration in the waterways out of consideration.

## 2. Game Theory and Resource Modeling

The application of game theory to natural resource management problems is fairly recent and is at its developmental stage. Most applications appear to be on the cooperative aspect and is applied at different levels, from global to regional and local. The objective is usually to reduce environmental hazards through a detailed analysis of prevailing conditions and availability of resources which allow for a reasonable set of strategies. Lund and Palmer (1997) put up a good case for resolving conflict arising from management of water resources using tools from game theory. Game theoretic approaches in water resource management was further developed by Ratner and Yaron (1990). This paper presents an analysis of the economic potential in regional cooperation of water usage in irrigation under conditions characterized by a general trend of increasing salinity. Income maximizing solutions for the region were obtained and related income distribution schemes derived using cooperative game theory algorithms and shadow cost pricing. An illuminating use of cooperative game in modeling greenhouse gas emissions was also presented by Filar and Gaertner (1996). Their methodology considered different regions of the world, classified according to their production of greenhouse gases, as players in a cooperative game and arrived at a fair allocation of emission reductions. Ray (2000) discussed the significant role that cooperative games and correlated strategies could play in the proper management of our environment. Basaran and Bölen (2005) conducted a case study in northern Turkey using game theory to obtain a better understanding of the decision making process and its consequences on a drainage basin. A similar study was also undertaken, with cooperation at the country level, by Dinar (2004). In Hermans (2004), different experiences from various countries were used to demonstrate limitations and successes of a game-theoretic approach for water resource management.

The focus of this study is on phosphorus pollution of waterways and the problem has been considered mainly from a non-cooperative perspective but will touch upon the cooperative aspect which are treated as collusive cooperative Pareto optima in a non-cooperative game. The type of pollution that is of concern in this paper is an example of non-point source (NPS) pollution which has been discussed in Segerson (1993) and Xepapadeas (1999). The chief characteristic of NPS pollution is the inability of regulators to observe emissions by individual dischargers, leading to games where there is an asymmetric pattern of information. All that the regulators can observe are the ambient concentration of the pollutants without being able to detect the sources of these emissions with full certainty. It is to be noted that the aggregate pollution will also affect the polluters themselves, directly or indirectly, so it is important that some measure of cooperation is achieved between the dischargers of the pollutant. NPS was also considered in Cocharada et al. (2002), where a regulator is unable to use conventional policies to regulate pollution.

## 3. Case Study Region and Rationale

Providing increased protection to Australian rivers is one of the nation's top priorities and it is justified by the increasing industrial and demographic impacts to the environment, as well as the severe droughts which visited this country on a fairly regular basis. The Hopkins catchment from the Hopkins basin, situated in Western Victoria (Figure 1), was selected as the case study area because water quality issues are very topical for this region. The pollution of water by nutrients, especially phosphorus, is a major environmental problem in this catchment. Therefore, improvement of water quality in the streams of the Hopkins catchment is one of the priorities of the Hopkins regional strategy indicated in the regional development plan of the local Catchment Management Authorities (Glenelg-Hopkins CMA, 2003).

The Hopkins catchment is predominantly an agricultural area with the sheep (wool and prime lamb) industry dominating, and some dairy and cereal crop production. The climate is moderately dry with average rainfall of about 700 mm from records kept over the last 120 years. In extremely dry years, annual rainfall could go below 350 mm and in very wet years, annual rainfall has reached 1000 mm (Glenelg-Hopkins CMA, 2003). The distribution of average precipitations over the catchment area is presented in Figure 2.

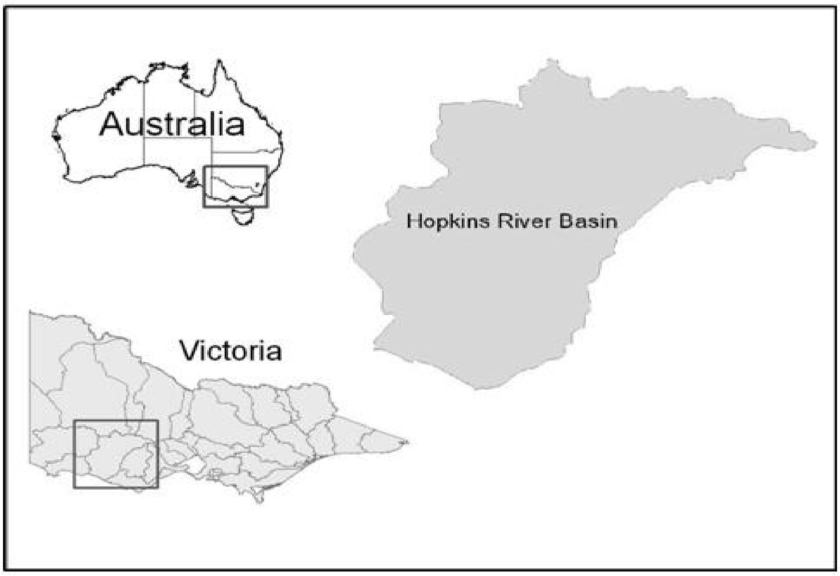


**Fig.1.** Location of the Hopkins catchment

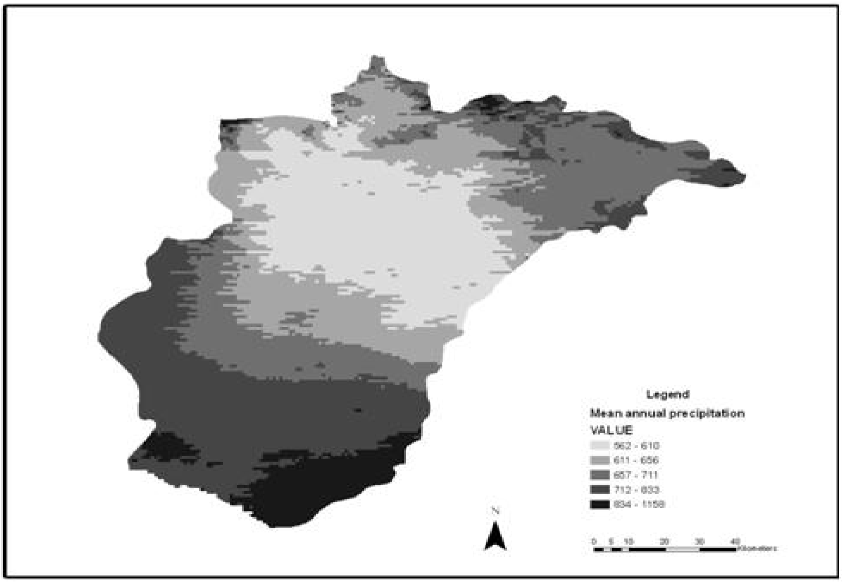


**Fig.2.** Average annual rainfall in the Hopkins catchment

Phosphorus pollution is a major factor causing severe detriment to the waterways of Western Victoria and this is mainly attributable to fertiliser usage. The major source of water pollution in the Hopkins catchment area is associated with intensive agricultural activities and usage of fertilisers by local farmers. The nutrient load, primarily phosphorus, could be better regulated and controlled by applying more efficient policies of fertilisation. The only way to achieve this would be in careful scheduling of fertiliser applications and monitoring quantities of fertilisers introduced, thus ensuring that demands of farmers' crops are met and pollution limits for the region adhered to.

There is not a great deal of data on phosphorus pollution in the region. However, regular phosphorus measurements are taken from eleven stations in the Hopkins river basin, three of which are located on the Hopkins River (Figure 3). This information is very useful in estimating the phosphorus load in surface water, thereby providing an indirect measure of phosphorus usage through farming activities. Figure 4 is an example of phosphorus measurements taken from a location (Cudgee) in the region. (Source: Victorian Resources Data Warehouse: http://www.vicwaterdata.net/vicwaterdata/.)


**Fig.3.** Phosphorus measurement locations in the Hopkins River Basin

## 4. Justification of Selected Approach

One of the most common approaches for resource management modeling in environmental economics is to apply various optimization techniques, especially linear programming, to appropriate objective (revenue) functions. These optimization techniques assume that the major driving force of all economic agents, farmers in our case, is revenue maximization. Environmental parameters can also be incorporated

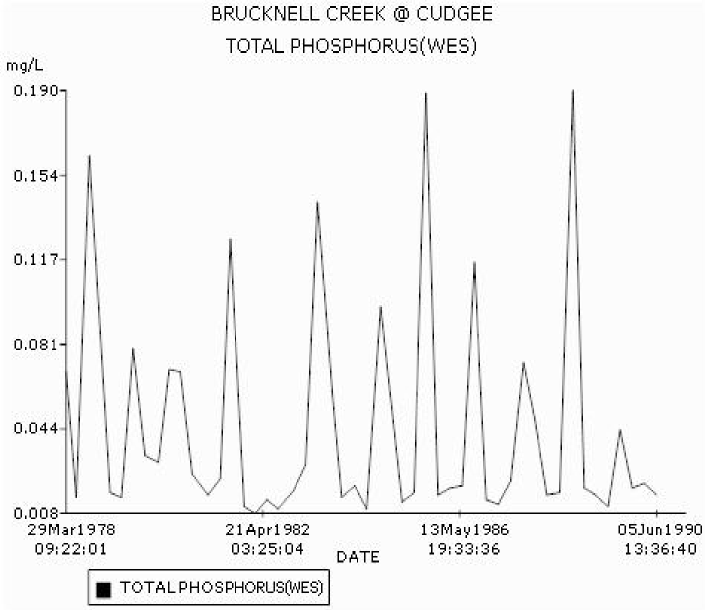


**Fig.4.** Phosphorus concentration in the Hopkins River measured at Cudgee (see location in Figure 3)

in these objective functions after being expressed in some monetary equivalent. The relevant references can be found for instance in Scoccimarro et al. (1999). As the objective of each agent is to increase its own revenue, often at the expense of others, this approach meant that all farmers (or groups of farmers) are posited in a competitive framework. The theory of non-cooperative games are especially suited to model such framework, where group of agents are intent on maximizing their own revenue. However, sometimes cooperation, rather than pure competition, plays a more significant role in resource management because most community members share the same environmental concerns on issues such as water quality, soil salinity, biodiversity, etc.

In the next section, we propose a game-theoretic approach to model the source of phosphorus pollution in the Hopkins Catchment. The players' optimal strategies in case of competition are given by the set of Nash equilibria. A Nash equilibrium is a combination of strategies, or profiles, which, if adopted by all players, will render it infeasible for anyone to gain by unilaterally deviating from adopting them, i.e. they are *stable* (Pindyck and Rubinfield, 2001). This type of equilibrium does not necessarily provide the best outcome to all players, but they are stable with respect to the behaviors of others. Collusion between agents, i.e. cooperation, can lead to *Pareto optima*. A Pareto optimum is a combination of strategies where there are no other combinations which are preferred by *all players* and strictly preferred by *at least one player* (Owen, 2001).

## 5. Game Theory Formulation to Phosphorus Pollution of Waterways

### 5.1. The Players

In this section a static, non-cooperative game model of phosphorus pollution in the Hopkins catchment region is formally presented, which will be referred to as the *Hopkins project*. As a preliminary remark, a *strategic form non-cooperative game* $\Gamma$ is defined as the system represented by

$$\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$$

where $N = \{1, 2, \ldots, n\}$ represents the set of players, $S_i$ is the set of *pure strategies* available to Player $i$ and $u_i(s)$ is a function defined on the Cartesian product set $S = \prod_{i\in N} S_i$ which represents the payoff or *utility* to Player $i$ when a combination of strategies, or *profile*, $s \in S$ is selected by the players. If chance is involved in a game, i.e. a lottery is played, then the payoff is an *expected value*, as commonly defined in probability theory.

In the Hopkins project, each farmer household is represented as a player. The number of players, $n$, should not be too large. Otherwise, the analysis would be intractable. We will denote Player $i$ by $P_i$.

### 5.2. The Strategies

The strategy set $S_i$, $i = 1, 2, \ldots, n$, available to each player, consists of tuples

$$s_i = (\alpha_i, t_i) = (\alpha_i^1, \alpha_i^2, \ldots, \alpha_i^R, t_i^1, t_i^2, \ldots, t_i^R)$$

where $R$ is the number of crops fertilised using phosphorus and

$$\begin{aligned} \alpha_i^r &= \text{the amount of phosphorus used by } P_i \text{ for crop } r \text{ per unit area} \\ t_i^r &= \text{the scheduling of the application of phosphorus by } P_i \text{ for crop } r, \\ &\quad \forall\ r = 1, 2, \ldots, R. \end{aligned}$$

That is, each member of $S_i$ consists of the amount and time of application of phosphorus to crop $r$ planted by the farmers, $r = 1, 2, \ldots, R$. It is allowed for the $\alpha_i^r$ to vary continuously within the interval $A_r = [A_{r1}, A_{r2}]$ , i.e., irrespective of the player, there is a minimum quantity $A_{r1}$ and a maximum quantity $A_{r2}$ of phosphorus that can be applied to crop $r$. Similarly, the time of application, $t_i^r$, also takes values in an interval $T_r = (t_{r1}, t_{r2}]$ where $t_{r1}$ is the minimum time and $t_{r2}$ the maximum time of application. Note that it is sometimes more realistic for $T_r$ to be a finite set, e.g. $T_r = \{1, 2, \ldots, 12\}$ when phosphorus is applied once - a -month; however, an interval for mathematical convenience and tractability has been chosen.

Thus, $s_i \in S_i = \prod_{r=1}^R A_r \times \prod_{r=1}^R T_r$ for any $i$ and $s = (s_1, s_2, \ldots, s_n) \in S = \prod_{i\in N} S_i$ represents a profile adopted by all players. In the sequel, we also use the notation $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ to represent a profile adopted by all players *except* $P_i$.

### 5.3. The Pay-off Function

This will be measured by a *profit function* which has as its components the price obtained for the farm produce and the negative impact of environmental degradation. Before displaying the payoff function the following terms are defined:

For each strategy $(\alpha_i, t_i) \in S_i$ executed by $P_i$, let

$$
\begin{aligned}
\gamma &= \text{Cobb-Douglas constant}\\
q_i^r(\alpha_i^r, t_i^r) &= \text{proportion of phosphorus that is released into farmland devoted to crop } r;\\
1 - q_i^r(\alpha_i^r, t_i^r) &= \text{proportion of phosphorus that flow into the effluent river systems as a consequence thereof;}\\
E(t_i^r) &= \text{(negative) environmental impact manifested as cost per unit application of phosphorus;}\\
A_i^r &= \text{total quantity of land devoted to crop } r \text{ by } P_i;\\
W_i^r(t_i^r) &= \text{amount of water available at time } t_i^r;\\
Q^r(t_i^r) &= \text{quantity of crop } r \text{ produced per unit area per unit phosphorus}^\gamma \text{ per unit of water}^{1-\gamma};\\
p_r &= \text{price (revenue) obtained per unit of crop } r \text{ sold;}\\
\alpha_i^0 &= \text{base quantity of phosphorus in soil of user } i \text{ per unit area;}\\
F &= \text{price per unit of phosphorus fertiliser ;}\\
\beta_{ij} &= \text{environmental influence indirectly induced on } P_i \text{ by } P_j\\
\text{and} \quad L &= \text{toxicity threshold level, i.e. the amount of phosphorus in the effluent river systems above which there will be a negative environmental impact.}
\end{aligned}
$$

The payoff function accrued by Player $i$ if all players adopted the profile

$$s = ((\alpha_1, t_1), (\alpha_2, t_2), \ldots (\alpha_n, t_n))$$

can be expressed as

$$
\begin{aligned}
u_i(s) = \sum_{r=1}^{R} \Bigg[ & p_r Q^r(t_i^r) A_i^r \left[\alpha_i^r q_i^r(\alpha_i^r, t_i^r) + \alpha_i^0\right]^\gamma W_i^r(t_i^r)^{1-\gamma} - F A_i^r \alpha_i^r \\
& - \sum_{j=1}^{N} \beta_{ij} E(t_j^r) A_j^r (\alpha_j^r (1 - q_j^r(\alpha_j^r, t_j^r)) - L) I(\alpha_j^r (1 - q_j^r(\alpha_j^r, t_j^r)) > L) \Bigg] \qquad (1)
\end{aligned}
$$

where $0 \le \beta_{ij} \le 1$ are constants and $I(A)$ refers to the indicator of the set $A$, i.e. $I(A) = 1$ if event $A$ has occurred, and equals to 0 otherwise.

The rationale for (1) is as follows: not all phosphorus that were used are released into farmland, a proportion of this flowed into the effluent river systems, producing a negative environmental impact if the total amount released exceeded a toxicity threshold level. Fixed proportions $\beta_{ij}$ $i \neq j$, of this environmental influence are indirectly induced by other players on $P_i's$ domain and have a negative impact on $P_i's$ payoff. The term in $\beta_{ii}$ represents Player $i$'s own impact with constants

$$\beta_{ii} = 1 - \sum_{j=1, j \neq i}^{N} \beta_{ji}, \ \ \forall\, i \in \{1, 2, \ldots, N\}$$

Note that the functions $E(.)$ and $Q^r(.)$ depend only on the time of application $t_i^r$ and not on $\alpha_i^r$. This assumption is not unreasonable since these quantities, expressed as amounts per unit application, should be independent of the total amount of phosphorus being applied.

### 5.4. Optimal Strategies

**Nash Equilibrium.** The Hopkins project as outlined above is an example of a non-zero sum, $n$-person game. A strategy profile

$$s^* = (s_1^*, s_2^*, \dots, s_n^*) \in S$$

is a Nash equilibrium if it satisfies the following property:

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \quad \forall s_i \in S_i \text{ and } \forall i \in N. \tag{2}$$

Thus, a Nash equilibrium is stable for all players since any unilateral deviation from equilibrium by a player, given that all other players adhere to their best strategies, will result in a possible decrease of its revenue.

If the various functions forming $u_i(.), i = 1, 2, \dots, n$, render them *concave* with respect to the variables $\alpha_i^r$ and $t_i^r$, and, in addition, the optimum occurs in the interior of $S$, then the Nash equilibrium of the game is the solution of the following systems of equations involving first-order partial derivatives:

$$\begin{aligned} \left.\frac{\partial u_i}{\partial \alpha_i^r}\right|_{s^*} &= 0 \\ \left.\frac{\partial u_i}{\partial t_i^r}\right|_{s^*} &= 0 \\ r &= 1, 2, \dots, R; i = 1, 2, \dots, n. \end{aligned} \tag{3}$$

**Pareto Optima.** The derivation of a Pareto optimum falls into the class of multi-objective decision problems where a vector-valued function, $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_n(\mathbf{x}))$, $\mathbf{f} : A \to R^n, A \subset R^k$, is optimized according to some vector optimization criterion.

A point $\mathbf{x}^*$ is a *Pareto optimum* if there exist no other feasible $\mathbf{x}$ such that $f_j(\mathbf{x}) \geq f_j(\mathbf{x}^*) \quad \forall j$ with at least one $j$ such that $f_j(\mathbf{x}) > f_j(\mathbf{x}^*)$.

In the context of our model therefore, a strategy profile $s^*$ defines a Pareto optimum if there exists no profile $s$ such that

$$u_i(s) \geq u_i(s^*) \qquad \forall i \in N$$

with at least one $i \in N$ such that

$$u_i(s) > u_i(s^*).$$

A Pareto optimum is obtainable only if players enter into a binding cooperative agreement. One method of obtaining a Pareto optimum is to solve the *weighting problem* which is defined as follows:

For some set of weights $\mathbf{w} = (w_1, w_2, \dots, w_n) \in R^n$ such that $w_j \geq 0, \forall j$ and $\sum_{j=1}^n w_j = 1$, find $x^*$ such that

$$\mathbf{w}.\mathbf{f}(\mathbf{x}^*) = \max_{\mathbf{x} \in A} \mathbf{w}.\mathbf{f}(\mathbf{x}) \tag{4}$$

where the symbol $\mathbf{x}.\mathbf{y}$ refers to the scalar product between vectors $\mathbf{x}$ and $\mathbf{y}$. The following sufficient conditions for the existence of the Pareto optimum point $x^*$ are given in Chankong and Haimes (1983) (cf. Theorem 4.6):

$x^*$ is a Pareto optimum point of a multi-objective decision problem if one of the following two conditions hold:

(i) there exists a set of weights $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in R^n$ such that $w_j > 0, \forall j$ and $\sum_{j=1}^{n} w_j = 1$, such that $x^*$ solves the *weighting problem*;
(ii) $x^*$ is the unique solution of the *weighting problem.*

Thus, in our model, using the first condition above, Pareto optimum solutions can be obtained by maximizing the following weighted sum of the players' payoff functions:

$$Z(s) = \sum_{i=1}^{n} w_i\, u_i(s) \tag{5}$$

provided a set of weights $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ can be found satisfying $w_i > 0, \forall i$ and $\sum_{i=1}^{n} w_i = 1$. Therefore, sufficient first order conditions for the Pareto optimum solutions are given by

$$\begin{aligned} \left.\frac{\partial Z}{\partial \alpha_i^r}\right|_{s^*} &= 0 \\ \left.\frac{\partial Z}{\partial t_i^r}\right|_{s^*} &= 0 \\ r = 1, 2, \ldots, R; i &= 1, 2, \ldots, n. \end{aligned} \tag{6}$$

This project is particularly concerned with cooperative Pareto optima mentioned above.

## 6. Application of the Model

### 6.1. Formulation of the Simplified Game

For this project, a special survey of farmers in the Hopkins River catchment region was conducted. This allows us to learn about attitude of landowners to phosphorus application and to collect information about timing and quantity of phosphorus application in this region. Detailed description of this survey is beyond the scope of the present paper and is described in a separate work (Schlapp and Schreider, 2007).

Results from this survey can be used in applications of the model. In a first run of the model the time component of the strategy was not considered and thus the strategy set is $s_i = (\alpha_i^1, \alpha_i^2, \ldots, \alpha_i^R)$. So the variables to be determined are the $\alpha_i^r$, phosphorus application by player $i$ on crop $r$ over the season. As a consequence,

$$\begin{aligned} E(t_i^r) &= E \\ Q^r(t_i^r) &= Q^r \\ W_i^r(t_i^r) &= W_i^r \end{aligned}$$

become exogenous parameters. Furthermore for simplicity sake the $\alpha_i^r$ dependence from the absorption constant $q_i^r(\alpha_i^r, t_i^r)$ is removed. That is $q_i^r(\alpha_i^r, t_i^r) = q_i^r$ is an exogenous parameter. Then (1) becomes,

$$u_i(\alpha_i^r) = \sum_{r=1}^{R} \Bigg[ p_r Q^r A_i^r [\alpha_i^r q_i^r + \alpha_i^0]^\gamma (W_i^r)^{1-\gamma} - F A_i^r \alpha_i^r \\ - \sum_{j=1}^{N} \beta_{ij} E A_j^r (\alpha_j^r (1-q_j^r) - L) I(\alpha_j^r (1-q_j^r) > L) \Bigg] \quad (7)$$

and (5) becomes

$$Z(s) = \sum_{i=1}^{N} \sum_{r=1}^{R} \Bigg[ w_i p_r Q^r A_i^r [\alpha_i^r q_i^r + \alpha_i^0]^\gamma (W_i^r)^{1-\gamma} - w_i F A_i^r \alpha_i^r \\ - \sum_{j=1}^{N} w_j \beta_{ji} E A_i^r (\alpha_i^r (1-q_i^r) - L) I(\alpha_i^r (1-q_i^r) > L) \Bigg]. \quad (8)$$

In order to find the Nash equilibrium the following must be solved,

$$\frac{\partial u_i}{\partial \alpha_i^r} = 0 \qquad r = 1, 2, \ldots, R; i = 1, 2, \ldots, N.$$

That is, solve $R \times N$ equations for the $R \times N$ variables $\alpha_i^r$. The solutions yielded are

$$\alpha_i^r = \frac{W_i^r \left[ \frac{F + \beta_{ii} E (1-q_i^r) I(\alpha_i^r (1-q_i^r) > L)}{\gamma p_r Q^r q_i^r} \right]^{1/\gamma - 1} - \alpha_i^0}{q_i^r}. \quad (9)$$

Second order conditions were verified confirming individual utilities $u_i(\alpha_i^r)$, for each $i = 1, \ldots, N$, are maximised with respect to $\alpha_i^r$, $r = 1, \ldots, R$.

In order to find a Pareto optimum the following must be solved,

$$\frac{\partial Z}{\partial \alpha_i^r} = 0 \qquad r = 1, 2, \ldots, R; i = 1, 2, \ldots, N.$$

That is, solve $R \times N$ equations for the $R \times N$ variables $\alpha_i^r$. The solutions yielded are

$$\alpha_i^r = \frac{W_i^r \left[ \frac{w_i F + \sum_{j=1}^{N} w_j \beta_{ji} E (1-q_i^r) I(\alpha_i^r (1-q_i^r) > L)}{w_i \gamma p_r Q^r q_i^r} \right]^{1/\gamma - 1} - \alpha_i^0}{q_i^r}. \quad (10)$$

Second order conditions were verified confirming the combined utility $Z$ is maximised with respect to $\alpha_i^r$ for $i = 1, \ldots, N$ and $r = 1, \ldots, R$.

### 6.2. Estimation of Parameters

Blue-green algae blooms due to increased phosphorus in the water is selected as the most significant environmental impact in the region. The environmental impact constant $E$ is the cost per unit of phosphorus above threshold $L$ in the effluent rivers systems. Read et al. (1999) estimated that the expected annual impacts of toxic blue-green algae blooms would be $\$1,820,807$ per year for the combined Glenelg and Hopkins Catchments. Of this, $\$963,927$ is directly attributable to farm dams (domestic and stock water supplies) throughout the basins highlighting just how directly phosphorus use can negatively impact farmers. Phosphorus loads in the combined Catchments are estimated to be 265,194 kg per year. Thus a justifiable estimate for $E$ is,

$$E = \frac{\$1,820,807\,\text{p.a.}}{265,194\,\text{kg p.a.}} = \$6.87/\text{kg}.$$

With regards to the threshold level $L$, Schlapp and Schreider (2007) show phosphorus levels in the waterways of the region to be above 0.118 mg/L for increasing periods of the year. Holmes (2002) displays EPA guidelines for phosphorus levels to be 0.030 mg/L. Read et al. (1999) estimate pasture runoff contributes $15-20\%$ of total phosphorus to surface waters of Glenelg-Hopkins region. Thus current levels of phosphorus in waterways are unacceptably high already even with the effects of farming removed (Nash, 2007). For this reason we set the safe threshold level,

$$L = 0$$

Within the Glenelg-Hopkins region we consider three main types of farming enterprise; wool, prime lamb and regular cropping (wheat, oats and canola). Thus we define,

$$\begin{aligned} r = 1 &\Leftrightarrow \text{wool} \\ r = 2 &\Leftrightarrow \text{prime lamb} \\ r = 3 &\Leftrightarrow \text{regular cropping} \end{aligned}$$

Wool, prime lamb and regular cropping prices are obtained from average prices listed in Department of Primary Industries (2006). For regular cropping the average was taken from wheat and oats prices.

$$\begin{aligned} p_1 &= \$7.78/\text{kg} \\ p_2 &= \$3.03/\text{kg CW (carcass weight)} \\ p_3 &= \$0.16/\text{kg} \end{aligned}$$

The water constants $W_i^r$ will depend on the location of the player $i$ and the type of crop $r$. Within the region of the survey, annual precipitation can be separated into four separate categories as can be seen in (Schlapp and Schreider, 2007). These ranges are,

$$[562-600), [600-650), [650-700), [700-750)\ \text{mm p.a.}$$

For crops $r = 1, 2$ the water constants for each player $i$ will be the yearly rainfall based on location. However due to regular cropping only taking place usually from mid-April through to mid-December (Saul et al., 1999) the water constants will be just be the average rainfall over this period for $r = 3$.

In determining the absorption constant $q_i^r$ readers are referred to Murray et al. (2004) in which numerous studies regarding phosphorus runoff are discussed. Dougherty et al. (2004) also shows results for phosphorus runoff under different rainfall simulations and times since fertiliser application. Both papers indicate that phosphorus runoff is proportional to the amount of fertiliser applied. The proportion of runoff depends more on the quantity and intensity of rainfall since application of fertiliser. In a study by Olness et al. (1980), 2.9% of applied phosphorus was lost in rotationally grazed pasture. Thus taking 2.9% as runoff rate a reasonable estimate for absorption constants,

$$q_i^1 = q_i^2 = 97.1\% \qquad i = 1, 2, \ldots, N.$$

Zhang et al. (2003), China, show $0.4 - 1.2\%$ runoff of applied phosphorus in paddy soil under wheat. Withers et al. (2001), UK, show 0.7% runoff of applied phosphorus in cultivated seed beds. Thus taking 1.0% as runoff rate a reasonable estimate for absorption constant $q_i^3$,

$$q_i^3 = 99.0\% \qquad i = 1, 2, \ldots, N.$$

The constant $\alpha_i^0$ represents the base phosphorus levels in soil for user $i$. Since the clay type soils of the region naturally have very low phosphorus levels and an inability to retain phosphorus well, an initial estimate is,

$$\alpha_i^0 = 0$$

The price of phosphorus, $F$, is calculated from results in Department of Primary Industries (2006). An average of \$42/ha was spent on fertilisers with an average of 10kg/ha phosphorus applied. Thus a reasonable estimate for the price of fertiliser is,

$$F = \$4.20/\text{kg}$$

Department of Primary Industries (2006) Wool production results show an average of 6.2kg/ha/100mm of rainfall. This is effectively $\frac{620}{100^{1-\gamma}}$kg/ha/100mm$^{1-\gamma}$. Furthermore, an average of 10 kg of phosphorus was used to produce this quantity, so a reasonable estimate for $Q^1$ is,

$$Q^1 = \frac{620}{10^\gamma \times 100^{1-\gamma}} \text{kg Wool/ha/100mm}^{1-\gamma}/\text{kg P}^\gamma$$

In a similar fashion $Q^2$ and $Q^3$ are obtained as

$$Q^2 = \frac{1790}{10^\gamma \times 100^{1-\gamma}} \text{kg CW Prime Lamb/ha/100mm}^{1-\gamma}/\text{kg P}^\gamma$$
$$Q^3 = \frac{13.45 \times 100^{1-\gamma}}{10^\gamma} \text{kg Crop/ha/100mm}^{1-\gamma}/\text{kg P}^\gamma$$

where 13.45 is the average of Wheat and Oats figures.

With regards to the constants $\beta_{ij}$, these are of the form

$$\beta_{ij} = \frac{k}{x_{ij}^2}$$

where $x_{ij}$ is distance from Player $i$ to Player $j$, $i \neq j$. The constant $k = 0.25$ and the smallest unit of distance is 1.

An infinite number of Pareto optima exist though the particular cooperative Pareto optimum sought here is that with weights

$$w_i = \frac{1}{N} \qquad i = 1, 2, \ldots, N.$$

Intuitively this is the Pareto optimum that maximises the sum of utilities.

The areas $A_i^r$ are provided from the survey.

After calibration of the model the value of the Cobb-Douglas production constant $\gamma$ is

$$\gamma = 0.115.$$

The calibration process of the parameter $\gamma$ was based on the assumption that all farmers in the region act reasonably well in sense of fertiliser applications and follow the recommendations of the Department of Primary Industries, Victoria, which assign the threshold of phosphorus application within the interval of between 13 to 18 kg/ha.

### 6.3. Results and Sensitivity Analysis

The Nash equilibrium and a Pareto optimum solutions of the game are given in Table 1. It is worth noting that many of the Pareto optimum solutions for players are similar whilst Nash equilibrium solutions are not. This is due to the independence in the Pareto optimum solutions of the $\beta$ constants, given an equal weights $w_i$. That is, most parameters required for the solution, that is $W_i^r$, $q_i^r$, and so on, are the same for groups of players. With regards to the zero valued solutions this simply indicates the particular player has zero area devoted to the particular crop production. However, the major interest is to compare these solutions with the real values of phosphorus application which is implemented in next Section.

A sensitivity analysis of the average difference in non-zero Nash equilibrium and Pareto optimum solutions was undertaken. Values of parameters $E$, $p_r$, $\gamma$, $1-q_i^r$, were altered and resultant changes in average differences in Nash equilibrium and Pareto optimum solutions measured. Figure 5 displays this analysis. A 25% increase and decrease in the environmental constant $E$ resulted in a 23.1% increase and a 23.9% decrease respectively in the average difference. A 25% increase and decrease in the crop price vector $p_r$ resulted in a 29.8% increase and a 27.8% decrease respectively in the average difference. Finally a 25% increase and decrease in the Cobb-Douglas constant $\gamma$ resulted in a 46.0% increase and a 33.4% decrease respectively in the average difference in solutions. As it is shown in Figure 5 the response in percentage change in the difference in Nash equilibrium and Pareto optimum solutions is very much linear with respect to percentage change in the parameters tested in the vicinities analyzed.

Figure 6 shows the results of sensitivity analysis implemented in respect of price of phosphorus. The analysis of relationships of solutions of the game, expressed as a average application of phosphorus for all 3 crops considered in this project, showed that the double increase of the price of phosphorus (which really happened since the year the survey was implemented) leads to the almost double reduction of phosphorus application. The fact that solutions are a power function of price $F$ is reflected in solutions (9) and (10).

## 7. Discussion and Conclusions

The important outcome of the paper is that it formulates the general concept of a game theoretic approach in developing the optimal strategies for water quality management associated with phosphorus pollution in an agricultural region in eastern Victoria, Australia. This methodology can be easily extended to other regions with similar structure of industries and similar nature of environmental problematic. However, a bridge has to be established between the model that was developed

**Table1.** Nash equilibrium and Pareto optimum solutions for phosphorus application $\alpha$ (kg/ha) with $\gamma = 0.115$. Zero values indicates that given cropping enterprise is not practised on the property.

| Player | Nash Wool | Nash P Lamb | Nash Crop | Pareto Wool | Pareto P Lamb | Pareto Crop |
|---|---|---|---|---|---|---|
| 1 | 12.15 | 13.88 | 0.00 | 11.79 | 13.46 | 0.00 |
| 2 | 11.05 | 12.62 | 14.44 | 10.91 | 12.46 | 14.38 |
| 3 | 10.17 | 0.00 | 13.36 | 10.13 | 0.00 | 13.34 |
| 4 | 10.95 | 12.50 | 14.40 | 10.91 | 12.46 | 14.38 |
| 5 | 0.00 | 12.48 | 14.39 | 0.00 | 12.46 | 14.38 |
| 6 | 10.95 | 0.00 | 14.40 | 10.91 | 0.00 | 14.38 |
| 7 | 10.18 | 11.62 | 13.37 | 10.13 | 11.56 | 13.34 |
| 8 | 0.00 | 0.00 | 15.53 | 0.00 | 0.00 | 15.51 |
| 9 | 10.17 | 0.00 | 13.36 | 10.13 | 0.00 | 13.34 |
| 10 | 10.93 | 12.48 | 14.39 | 10.91 | 12.46 | 14.38 |
| 11 | 10.95 | 12.50 | 14.40 | 10.91 | 12.46 | 14.38 |
| 12 | 0.00 | 11.60 | 13.36 | 0.00 | 11.56 | 13.34 |
| 13 | 10.93 | 12.48 | 14.39 | 10.91 | 12.46 | 14.38 |
| 14 | 10.98 | 12.53 | 14.41 | 10.91 | 12.46 | 14.38 |
| 15 | 10.17 | 0.00 | 13.36 | 10.13 | 0.00 | 13.34 |
| 16 | 12.84 | 14.65 | 0.00 | 12.66 | 14.45 | 0.00 |
| 17 | 12.81 | 0.00 | 16.73 | 12.66 | 0.00 | 16.66 |
| 18 | 0.00 | 14.88 | 0.00 | 0.00 | 14.45 | 0.00 |
| 19 | 13.16 | 15.02 | 0.00 | 12.66 | 14.45 | 0.00 |
| 20 | 13.05 | 14.90 | 0.00 | 12.66 | 14.45 | 0.00 |
| 21 | 12.34 | 14.09 | 0.00 | 11.79 | 13.46 | 0.00 |
| 22 | 11.98 | 0.00 | 0.00 | 11.79 | 0.00 | 0.00 |
| 23 | 0.00 | 13.85 | 15.67 | 0.00 | 13.46 | 15.51 |
| 24 | 12.22 | 13.95 | 15.71 | 11.79 | 13.46 | 15.51 |
| 25 | 12.21 | 0.00 | 0.00 | 11.79 | 0.00 | 0.00 |
| 26 | 12.41 | 14.16 | 15.79 | 11.79 | 13.46 | 15.51 |
| 27 | 12.30 | 14.05 | 0.00 | 11.79 | 13.46 | 0.00 |
| 28 | 12.12 | 13.84 | 15.66 | 11.79 | 13.46 | 15.51 |
| 29 | 0.00 | 14.08 | 15.76 | 0.00 | 13.46 | 15.51 |
| 30 | 12.10 | 0.00 | 0.00 | 11.79 | 0.00 | 0.00 |

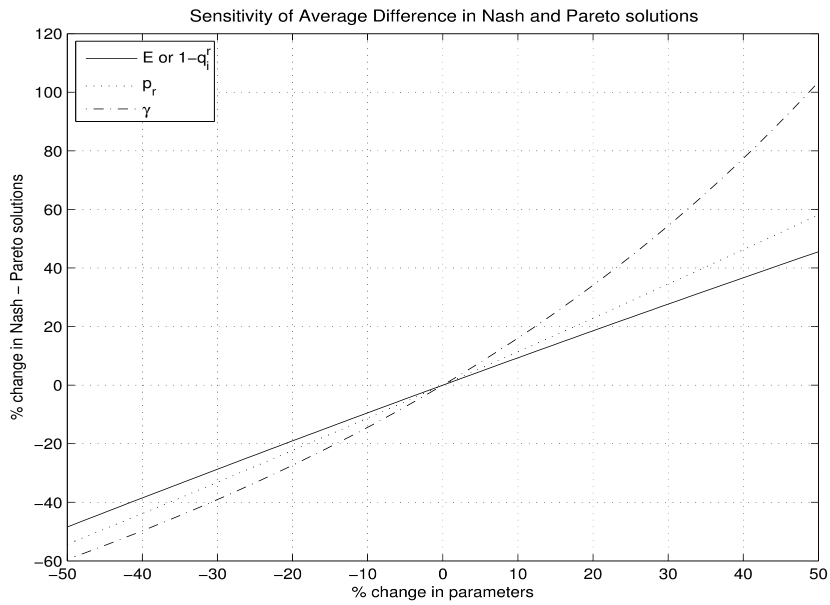


**Fig.5.** Sensitivity Analysis of the average difference in Nash equilibrium and Pareto optima solutions

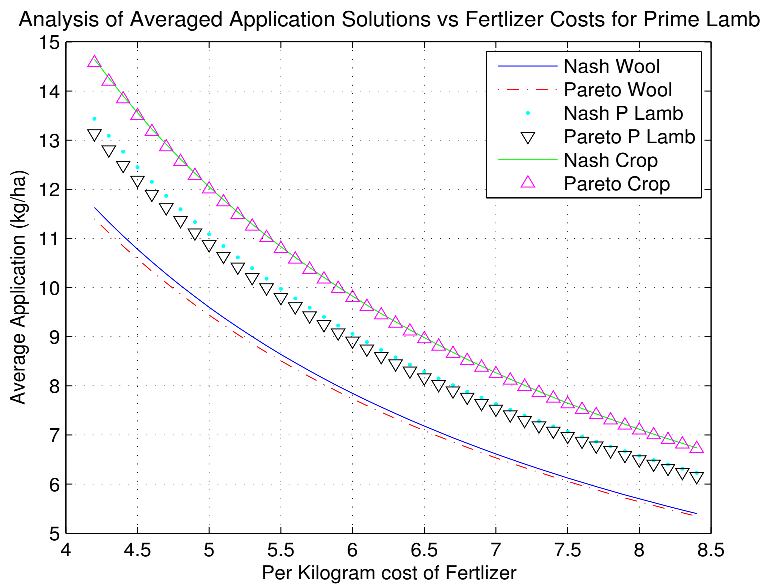


**Fig.6.** Sensitivity Analysis of Nash equilibrium and Pareto optima solutions subject to fertiliser costs

and the decision support tool that can be used by resource managers in the region considered. The key requirement to the model developed in this work is that all parameters included in the developed model should be measurable, thereby allowing appropriate data to be collected. Unfortunately, this requirement was not satisfied for all parameters included in the model and some of their values were estimated by trial and error or taken form the external sources.

As a first pass approach the solutions of game were obtained for a particular case where no time scheduling of phosphorus application was included in the game parametrization, because these dates were not available from the project's household survey. Table 1 displays both Nash equilibrium and cooperative Pareto optimum solutions for the model in terms of amount of phosphorus applied per hectare obtained for runoff values reported in Olness et al. (1980), quoted in Murray et al. (2004), Zhang et al. (2003) and Withers et al. (2001). As it follows from this Table, the amount of phosphorus applied is consistently larger for the case when farmers follow the competitive strategy for all types of land use practicing in the Hopkins catchment covered by survey.

More interesting analysis is possible when the amount of excess phosphorus is computed for each household considered in the present work. Table 2 reports the results of comparative analysis of actual values of phosphorus applied by farmer for the period of survey and the game solution for obtained for the Nash equilibrium and Pareto optimum. Positive difference between those values indicates farmers over fertilised their paddocks, whereas negative values indicate under fertilisation. The values in Table 2 are given in kilograms per hectare. After summation of these values multiplied to the areas under particular crop for the particular farm the total phosphorus excess can be calculated. Excess phosphorus is defined here as a difference between current practised application of phosphorus and that given by the game theoretic solution. These values are 2,649, -1,478 and 48,073 kg of excess phosphorus for competitive Nash equilibrium solution for, wool, prime lamb and crop areas, respectively. For cooperative Pareto solutions these values are 4,530, -101 and 53,163 excess kilos for the same industries. It means that total over fertilisation is about 49 tons of phosphorus if the competitive paradigm is considered, whereas the total excess phosphorus reaches the value of 53 tons if the cooperation is introduced. That means the advantage of cooperation for just 30 farms in the Hopkins Basin is more than 4 tons of excess phosphorus, equivalent to approximately 50 tons of Single Super (a widely used fertiliser), per 30 properties in the Hopkins basin. This value would be dramatically increased if all 1340 households of the Hopkins catchments are included in the analysis. One important finding of the survey conducted during the project implementation is that the model described in the present report cannot be realized to its fullest extent as yet because not all parameters included in the model are easily measurable. The most important obstacle from the survey, it appears that no detailed nor precise schedule is available and this is mainly due to variations in expected climate, enterprise mix and budgets. The responses to this question, when they are available, are usually within a margin of error of plus or minus a month.

Table 3 displays values of utilities (payoff functions (7)) for players given they adopt either the Nash or cooperative Pareto strategy. From this Table the benefits of cooperation can be seen because almost all (29 out of 30) household demonstrate the increase of utility when switching from competitive to cooperative strategy. An

**Table2.** Differences in survey responses to applied phosphorus and Nash equilibrium and Pareto optimum solutions for $\alpha$ (kg/ha) with $\gamma = 0.115$

| Player | Nash Wool | Nash P Lamb | Nash Crop | Pareto Wool | Pareto P Lamb | Pareto Crop |
|---|---|---|---|---|---|---|
| 1 | 4.90 | 3.17 | 0.00 | 5.26 | 3.59 | 0.00 |
| 2 | 2.85 | 1.28 | -0.54 | 2.99 | 1.44 | -0.48 |
| 3 | -1.07 | 0.00 | 8.54 | -1.03 | 0.00 | 8.56 |
| 4 | -0.45 | -2.00 | 7.50 | -0.41 | -1.96 | 7.52 |
| 5 | 0.00 | 2.82 | 3.71 | 0.00 | 2.84 | 3.72 |
| 6 | 2.05 | 0.00 | 16.88 | 2.09 | 0.00 | 16.90 |
| 7 | 1.20 | -0.25 | 14.01 | 1.25 | -0.19 | 14.03 |
| 8 | 0.00 | 0.00 | -2.39 | 0.00 | 0.00 | -2.37 |
| 9 | 4.83 | 0.00 | 6.64 | 4.87 | 0.00 | 6.66 |
| 10 | -5.77 | -7.32 | -2.46 | -5.75 | -7.30 | -2.45 |
| 11 | 6.96 | 5.41 | 3.51 | 7.00 | 5.45 | 3.53 |
| 12 | 0.00 | -11.60 | 6.09 | 0.00 | -11.56 | 6.11 |
| 13 | 3.83 | 2.28 | 15.88 | 3.85 | 2.30 | 15.89 |
| 14 | 2.27 | 0.72 | 4.65 | 2.34 | 0.79 | 4.68 |
| 15 | -5.62 | 0.00 | -2.46 | -5.58 | 0.00 | -2.44 |
| 16 | 2.27 | 0.46 | 0.00 | 2.45 | 0.66 | 0.00 |
| 17 | -5.05 | 0.00 | -16.73 | -4.90 | 0.00 | -16.66 |
| 18 | 0.00 | 4.08 | 0.00 | 0.00 | 4.51 | 0.00 |
| 19 | 0.80 | -1.06 | 0.00 | 1.30 | -0.49 | 0.00 |
| 20 | 5.15 | 3.30 | 0.00 | 5.54 | 3.75 | 0.00 |
| 21 | -1.64 | -3.39 | 0.00 | -1.09 | -2.76 | 0.00 |
| 22 | 1.67 | 0.00 | 0.00 | 1.86 | 0.00 | 0.00 |
| 23 | 0.00 | 4.35 | 5.33 | 0.00 | 4.74 | 5.49 |
| 24 | 1.43 | -0.30 | 11.67 | 1.86 | 0.19 | 11.87 |
| 25 | -2.21 | 0.00 | 0.00 | -1.79 | 0.00 | 0.00 |
| 26 | -3.31 | -5.06 | 6.01 | -2.69 | -4.36 | 6.29 |
| 27 | 3.92 | 2.17 | 0.00 | 4.43 | 2.76 | 0.00 |
| 28 | -4.37 | -6.09 | 4.35 | -4.04 | -5.71 | 4.50 |
| 29 | 0.00 | -7.08 | -8.76 | 0.00 | -6.46 | -8.51 |
| 30 | -0.25 | 0.00 | 0.00 | 0.06 | 0.00 | 0.00 |
| Excess P (kg) | 2,649 | -1,478 | 48,073 | 4,530 | -101 | 48,734 |
| Sum Excess P (kg) | | | 49,244 | | | 53,163 |

interesting point is that the largest differences in players utilities are seen when constants $1 - \beta_{ii}$ are closer to 1, which means these farmers are located within a close neighbourhood of others. This indicates these players have more influence on other farmers and vice versa and thus cooperation is more pronounced. This increase of utilities is obtained because of increase of the environmental component of this objective function (1). The monetary component, or total revenues of the households are always larger in the case of competitive Nash solutions. Table 4 displays these results.

**Table3.** Values of Utilities for the Nash equilibrium and Pareto optimum and their differences

| Player | Nash Utility | Pareto Utility | Pareto - Nash | $1 - \beta_{ii}$ |
|---|---|---|---|---|
| 1 | 124,371.03 | 124,390.84 | 19.81 | 0.59 |
| 2 | 1,093,291.93 | 1,093,297.31 | 5.38 | 0.24 |
| 3 | 468,736.78 | 468,737.54 | 0.76 | 0.09 |
| 4 | 328,778.42 | 328,778.95 | 0.52 | 0.06 |
| 5 | 433,704.61 | 433,704.93 | 0.32 | 0.04 |
| 6 | 574,938.73 | 574,939.30 | 0.58 | 0.07 |
| 7 | 163,844.10 | 163,845.14 | 1.04 | 0.10 |
| 8 | 203,678.15 | 203,680.98 | 2.83 | 0.08 |
| 9 | 566,875.56 | 566,876.72 | 1.16 | 0.09 |
| 10 | 195,429.90 | 195,430.19 | 0.29 | 0.03 |
| 11 | 477,443.84 | 477,444.24 | 0.40 | 0.06 |
| 12 | 284,312.54 | 284,313.28 | 0.73 | 0.06 |
| 13 | 503,511.92 | 503,512.42 | 0.50 | 0.03 |
| 14 | 438,110.90 | 438,111.48 | 0.58 | 0.11 |
| 15 | 38,942.50 | 38,944.43 | 1.93 | 0.08 |
| 16 | 44,930.07 | 44,938.41 | 8.34 | 0.27 |
| 17 | 583,944.43 | 583,945.61 | 1.19 | 0.22 |
| 18 | 117,222.15 | 117,234.11 | 11.96 | 0.56 |
| 19 | 98,469.89 | 98,482.21 | 12.32 | 0.74 |
| 20 | 113,006.16 | 113,017.73 | 11.57 | 0.59 |
| 21 | 402,675.99 | 402,669.07 | -6.92 | 0.88 |
| 22 | 129,404.81 | 129,416.61 | 11.80 | 0.32 |
| 23 | 984,308.61 | 984,313.50 | 4.89 | 0.56 |
| 24 | 372,779.12 | 372,786.54 | 7.42 | 0.70 |
| 25 | 169,330.60 | 169,351.55 | 20.95 | 0.69 |
| 26 | 241,000.98 | 241,008.75 | 7.77 | 0.98 |
| 27 | 163,342.99 | 163,382.33 | 39.34 | 0.82 |
| 28 | 164,261.76 | 164,279.85 | 18.09 | 0.54 |
| 29 | 502,192.14 | 502,210.09 | 17.94 | 0.87 |
| 30 | 82,303.63 | 82,326.40 | 22.77 | 0.51 |

Some important parameters of the model were assigned to their numerical values based on the analysis of the literature review of research works implemented in the region (say, the value of the environmental impact constant $E$), and outside the region (absorption constants $q$). Some of the parameters were unmeasurable and were treated as free parameters of the model (for instance, the value of $k$ and $\gamma$, characterizing the spatial proximity of the players and the Cobb-Douglas constant, respectively). We treated these parameters as values to be calibrated, where calibration process was based on minimizing the difference between modelled phos-

**Table4.** Values of Production and Environmental Costs for the Nash equilibrium and Pareto optimum

| Player | Nash Production | Pareto Production | Nash Environment | Pareto Environment |
|---|---|---|---|---|
| 1 | 125,541.48 | 125,525.43 | 1,170.45 | 1,134.60 |
| 2 | 1,096,136.64 | 1,096,111.42 | 2,844.71 | 2,814.11 |
| 3 | 470,463.99 | 470,457.72 | 1,727.21 | 1,720.17 |
| 4 | 330,437.28 | 330,432.42 | 1,658.85 | 1,653.48 |
| 5 | 435,370.28 | 435,367.60 | 1,665.66 | 1,662.66 |
| 6 | 577,271.65 | 577,264.80 | 2,332.92 | 2,325.50 |
| 7 | 164,504.09 | 164,502.05 | 659.99 | 656.91 |
| 8 | 204,191.05 | 204,190.48 | 512.90 | 509.49 |
| 9 | 568,686.93 | 568,681.32 | 1,811.37 | 1,804.60 |
| 10 | 196,094.92 | 196,094.22 | 665.02 | 664.04 |
| 11 | 479,241.54 | 479,237.15 | 1,797.70 | 1,792.91 |
| 12 | 285,232.05 | 285,230.22 | 919.50 | 916.94 |
| 13 | 505,174.54 | 505,172.64 | 1,662.63 | 1,660.22 |
| 14 | 440,043.84 | 440,034.34 | 1,932.94 | 1,922.86 |
| 15 | 39,176.78 | 39,176.45 | 234.28 | 232.02 |
| 16 | 45,555.15 | 45,551.94 | 625.08 | 613.53 |
| 17 | 586,762.37 | 586,728.22 | 2,817.95 | 2,782.60 |
| 18 | 118,133.82 | 118,119.28 | 911.67 | 885.17 |
| 19 | 99,376.59 | 99,362.29 | 906.70 | 880.07 |
| 20 | 113,894.40 | 113,879.85 | 888.24 | 862.12 |
| 21 | 404,180.98 | 404,119.49 | 1,504.99 | 1,450.42 |
| 22 | 130,452.63 | 130,441.86 | 1,047.83 | 1,025.26 |
| 23 | 986,512.16 | 986,464.17 | 2,203.55 | 2,150.67 |
| 24 | 374,517.01 | 374,479.50 | 1,737.89 | 1,692.97 |
| 25 | 170,601.50 | 170,577.82 | 1,270.91 | 1,226.27 |
| 26 | 242,113.30 | 242,086.93 | 1,112.32 | 1,078.18 |
| 27 | 165,067.95 | 165,043.25 | 1,724.95 | 1,660.92 |
| 28 | 165,309.97 | 165,294.34 | 1,048.20 | 1,014.49 |
| 29 | 503,763.91 | 503,722.71 | 1,571.77 | 1,512.62 |
| 30 | 83,452.31 | 83,442.66 | 1,148.68 | 1,116.26 |

phorus application amounts and those recommended by the resource management authorities. In order to address the questions about how the changes in these model parameters affect the modelling results the sensitivity analysis was implemented. This analysis can answer the question how sensitive the results are to the different variations in the model parameters. This analysis shows that changes in solutions are usually lesser than changes in the model parameter values. The only exception is the parameter $\gamma$ whose increase can cause more significant changes in solutions (it doesn't happen in the case of decrease of this parameter). These results of sensitivity analysis are shown in Figure 5.

The major results obtained in this work are quite important. The game theoretic model provided some optimal thresholds for phosphorus application in the region, lesser than the currently practising quantities. Orientation to this optimal thresholds in fertilisation can lead to the significant improvement of water quality in the region and decrease the upstream impacts to the lowland and coastal areas in the catchment considered. The cooperation between farmers, which leads to development of one agreed strategy of phosphorus application, can result in even lesser environmental impact to the region compared with the non-cooperative strategy when all farmers maximise their utilities individually. This cooperation can be implemented through the system of farmers community groups that are quite influential in the region and whom supported our project from the very initial stages of its implementation.

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# A Practicable Cost-Allocation Method for Cooperative Settings*

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**Abstract** Cooperative game-theory offers solutions to cost-allocation in cooperative settings. Because of their complexity, these solutions are hard to apply in practice, for instance in market-like situations where each participant offers a commodity and asks for others' commodities, especially if the productions of such commodities are interdependent. We present an approach which provides strategically stable solutions in the sense of the core of a cooperative game and which does not depend on a characteristic function. Therefore, it may serve as a strategic guideline for the design of cost-sharing mechanisms that benefit larger and/or decentralized cooperations with structures of internal markets.

**Keywords:** cost-allocation, cooperative markets, subsidy-free prices

## 1. Introduction

In this paper we analyze traditional *cost-allocation games*[1] and present a practicable cost-allocation method that remedies the insufficiencies of traditional solutions. This presentation bases on the considerations and results elaborated in (Selders & Ehrhart, 2008).

A main question that arises in a cooperation, where the efforts of many lead to one joint result, is how aggregate costs should be distributed (allocated) among the different players in a way to provide a stable and voluntary cooperation. One can identify two shortcomings of traditional cost-allocation games that address this issue. First of all, famous solution concepts like the core or the Shapley-Value are based on the formulation of a game in coalitional form and require the use of the characteristic function, which assigns to each player and each coalition of players their stand-alone outcome. This requirement imposes severe restrictions on the application to real questions in practice, because the problem becomes very complex if the number of players is large.[2] Therefore, it is not astonishing that game-theoretic solution concepts have hardly ever been applied in practice.[3]

This leads us to our second point in question. Most of the examined games are not very complex, because they possess only one cost-generating source, for example

* The authors thank Hervé Moulin for his valuable suggestions.

[1] See (Young, 1985, Young, 1994). Cost-allocation games belong to cooperative game theory. A game is called cooperative if the players can negotiate binding agreements on their strategies which lead to a joint result of the game.

[2] The characteristic function leads to a calculation complexity of $2^n$, where $n$ is the number of players in the game.

[3] See (Schotter & Schwödiauer, 1980) and (Biddle & Steinberg, 1985).

communication costs or the cost of an airstrip.[4] This presents another limitation for real life applications, particularly if they are characterized by a complex interaction structure.

Consider, for example, the situation presented in Fig. 1. Here, a group of

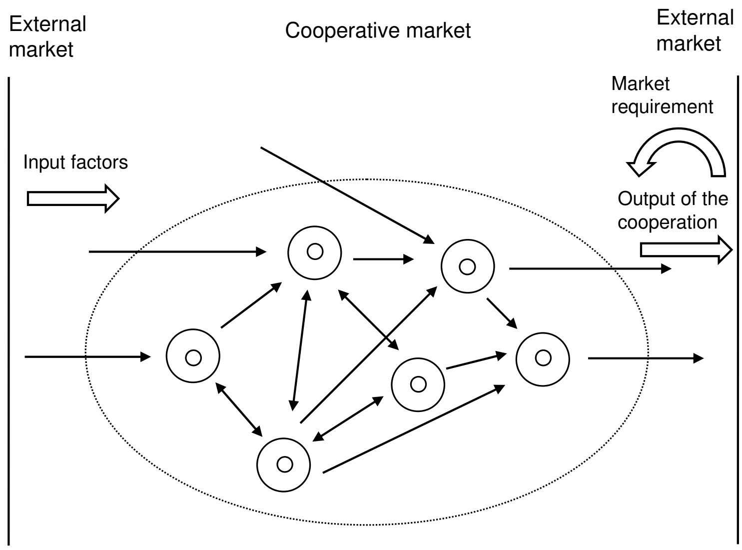


**Fig.1.** A general form of a cooperative market.

players forms a cooperation to fulfill an exogenous requirement. One possible interpretation of the cooperation is an internal market that is formed within a corporation. Another possibility is that several companies jointly invest in a plant that refines their byproducts, which then are sold to the market. Independent of the specific application, this market-like *network model* exhibits a high amount of interactions between the players. If we assume that every player delivers multiple goods or services to his peers, complexity increases further.

Suppose that within a society of $n$ players, player $i$ generates the revenue $r_{ij}$ with services and goods delivered to $j$. His total revenue $R_i$ is

$$R_i = \sum_{j \neq i} r_{ij}.$$

Assuming for the moment that there are no external input factors and denoting fixed costs by $FC_i$, $i$'s total costs $TC_i$ are

$$TC_i = \sum_{j \neq i} r_{ji} + FC_i.$$

[4] Prominent examples are the Aircraft Landing Fees (Littlechild & Thompson, 1977) and the Internal Telephone Billing Rates (Billera *et al.*, 1978).

With these two simple relations, in Table 1. we construct a (Leontief) transaction matrix to formulate the network illustrated in Fig. 1..[5]

| | Production | | | | | Revenue |
|---|---|---|---|---|---|---|
| Consumption | $r_{11}$ | $\cdots$ | $r_{1i}$ | $\cdots$ | $r_{1n}$ | $R_1$ |
| | $\vdots$ | $\ddots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ |
| | $r_{i1}$ | $\cdots$ | $r_{ii}$ | $\cdots$ | $r_{in}$ | $R_i$ |
| | $\vdots$ | $\ddots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ |
| | $r_{n1}$ | $\cdots$ | $r_{ni}$ | $\cdots$ | $r_{nn}$ | $R_n$ |
| Fixed costs | $FC_1$ | $\cdots$ | $FC_i$ | $\cdots$ | $FC_n$ | |
| Total costs | $TC_1$ | $\cdots$ | $TC_i$ | $\cdots$ | $TC_n$ | |

Table 1: Transaction matrix

This kind of economy requires a very flexible modeling to provide a practicable solution, which is not the case if it were to be modeled with a characteristic function.[6] To generate the characteristic function, all coalitions have to be addressed, resulting in the calculation of $R_i$ and $TC_i$ for all $i$ in $2^{n-1}$ different matrixes.[7]

Hence, the unmanageable application of the coalitional form, which hinders the use of game-theoretic methods in practice, raises the question of finding an approach that can be verified with respect to the desired game-theoretic criteria but that does not depend on the characteristic function,[8] In Sect. 2. we propose the property of stability as the appropriate criterion, formulated by the concept of the core of a coalitional game, which ensures that each player and every coalition has an incentive to participate in the game. With the subsequent analysis in Sects. 3. and 4., we present an alternative approach of a practicable cost-allocation method that provides a stable solution and that is based on the general type of economy illustrated above.

In Sect. 3., we transform the conditions of the core into the context of sharing priced commodities instead of sharing costs.[9] Since our cooperative market (in the broader sense) possesses all the characteristics of labor division, we define the players to be exclusive holders of their technologies – even if they are so, only because of an agreement with the other players. Using a similar perspective as in contestable market theory, we use a price system of subsidy-free prices, as developed by (Faulhaber, 1975) and (Sharkey & Telser, 1978), to define stability and to

[5] This model was developed by Wassily (Leontief, 1966). See (Livingstone, 1969) for an application to accounting problems.

[6] The demand functions and the cost functions of the players can of course be non-linear. For details about the cost functions see Sect. 3.

[7] The matrix for $\emptyset$ is trivial. The matrix given in Table 1. corresponds to the matrix for the grand coalition $N$.

[8] We acknowledge the fact that there are cost-allocation methods in practice that deal with this kind of situations, but they do not offer any of the desired game-theoretic characteristics, especially not stability in the sense of the core. See (Schichtel, 1981) and (Wißler, 1997).

[9] We do not elaborate how our proposed solution, which is a price mechanism, relates to (sometimes highly controversial) topics in specific applications, for example the one of internal transfer pricing in companies. We focus solely on a cooperative solution, which implicitly sets the assumption that the user of our method favors a cooperative approach over a non-cooperative one.

allocate costs for one single producer.[10] For the case of one single multi-output monopoly, the connection to coalitional cost sharing games was demonstrated by (Moulin, 1988), using insights provided by (Telser, 1978), (Sharkey & Telser, 1978) and (Scarf, 1986).

The main purpose of this paper is presented in Sect. 4. Here, we generalize the existing theory to be applicable to the network model. The main challenge lies in the accurate modeling of the relationship between the various players that can be consumers and producers at the same time, who produce and trade an arbitrary number of (disjunct) commodities. To allow for a realistic model, we consider that all technologies are interdependent, that is, the prices set by one player affect all cost-functions – and therefore all prices – of all other players. For our generalized model, we prove that the property of subsidy-free prices is equivalent to the core solution in the corresponding game in coalitional form and that the existence of a system of positive subsidy-free prices is guaranteed. In this vein, a core solution can be computed in a practicable way. Instead of only being relevant for the regulation of a multi-output monopoly, subsidy-free prices appear to be a valuable tool to allocate costs in situations where traditional solution concepts fail to be practicable.

## 2. The Coalitional Cost-Sharing Game

Let us first consider the simple form of a cost-sharing game in a society $N \in \{1, \ldots, n\}$ with $n$ parties (players) that each have a need for a certain commodity. The question arises whether each player $i \in N$ should produce his requirements[11] by himself or, as a member of a coalition $S \subseteq N$, should produce the aggregate requirements of the coalition and share the production costs with the other coalition members.

This question is addressed by a *cooperative game in coalitional form* $(N, c)$ with *transferable utilities*. Here, the *characteristic function* $c(\cdot)$ assigns to each *coalition* $S$ the least costs that $S$ has to bear for fulfilling its members' needs, if all other players together make a stand against $S$.[12] For the cooperation of two or more players to be worthwhile, the characteristic function has to be *subadditive*.[13] In this case, $c(N)$ are the least total costs for fulfilling the needs of all players. The characteristic function is *submodular* (the game is called *concave*), if the advantages of a player joining a coalition is bigger for a larger coalition.[14]

Assuming a subadditive characteristic function, we now examine how to allocate total costs among the players. The *cost-allocation* is a vector $\mathbf{x} = (x_1 \ldots, x_n)$ whose component $x_i$ describes the costs that player $i \in N$ has to bear. A cost-allocation has to fulfill certain criteria in order to guarantee that all players agree to form the grand coalition and share one single source of supply. First, a cost-

[10] See (Mirman *et al.*, 1985) for an excellent survey and discussion.

[11] Since $i$ initially requests quantities of desired products and/or services, we speak of requirements rather than demands. These may also be given externally, e.g. by the external market.

[12] Note that the definition of a coalition also includes the empty coalition $\emptyset$, each player $i \in N$, and the *grand coalition* $N$.

[13] A characteristic function $c(\cdot)$ is called *subadditive*, if $c(S \cup T) \leq c(S) + c(T)$ holds for all $S, T \subseteq N$ with $S \cap T = \emptyset$.

[14] A characteristic function $c(\cdot)$ is called *submodular*, if $c(S \cup T) + c(S \cap T) \leq c(S) + c(T)$ holds for all $S, T \subseteq N$.

allocation has to be *efficient*, meaning that the total costs have to be at their lowest level and that these costs have to be shared exactly,

$$\sum_{i \in N} x_i = c(N)\,. \tag{1}$$

Second, no player and no coalition should be worse off by participating in the game than by standing alone. This is only fulfilled if the total costs that the members of a coalition have to bear are not greater than the coalition's stand-alone costs. Hence, we say that an allocation $\mathbf{x}$ fulfills the *stand-alone principle*, if for all $S \subseteq N$

$$\sum_{i \in S} x_i \leq c(S) \tag{2}$$

holds. Both principles together define the well-known concept of the *core* $\mathcal{C}(\cdot)$:

$$\mathcal{C}(N,c) := \left\{ \mathbf{x} \in \mathbf{R}^n : \sum_{i \in N} x_i = c(N) \text{ and } \sum_{i \in S} x_i \leq c(S) \text{ for all } S \neq \emptyset \right\}.$$

The core contains all allocations that offer no incentive to any player and any coalition not to participate in the game. Hence, the core is also known as the set of *(strategically) stable allocations*, which do not invite players and coalitions to "block" the allocation by obtaining a better result on their own. Therefore, stable allocations are seen as a necessary condition for a voluntary cooperation of the players.[15] We consider this type of strategic stability to be of utmost importance for cost-allocation and suggest that the outcomes of an appropriate method for cost-allocations should at least fulfill the properties of the core. A detailed argumentation can be found in (Selders & Ehrhart, 2008). We focus on allocation methods that propose stable (core) allocations, which offer players incentives to take part in the game and bear their proposed cost share. Then, voluntary cooperation is secured even if players pursue individual objectives.

However, we have merely resolved half of the given task: by only defining the core and considering its strategic advantages as appropriate for cost-allocation, we do not address the issue of guaranteeing a practicable way of putting our theoretic claims into practice. Therefore, we shall reformulate our requirements into an applicable method of allocating joint costs which does not depend on the unmanageable characteristic function.

## 3. Cost-Sharing Through Subsidy-Free Prices

We now introduce a practicable method to allocate costs, which satisfies the necessary conditions for strategic stability as stated by the game-theoretical concept of the core. For this purpose, we consider a model of $n$ players, where each player $i$ stands for a consumer – and in the general case also for a producer – in the cooperation.

[15] There are further stable concepts like the Epsilon-Core or the Kernel. Especially the Nucleolus, as introduced by (Schmeidler, 1969), offers the advantage of offering a unique solution. However, none of these methods are applicable for a larger set of players.

Introductorily, let us consider a simplified model of cooperation with $m$ commodities, which are all exclusively produced by player $i$. Thus, player $i$ represents a multi-output monopoly, disposing of the specific production technologies.[16]

Let $q_i^k$ be the quantity of a commodity $k$ that player $i$ delivers to the other players, and denote the price for that commodity with $p_i^k$. Let $\boldsymbol{\pi_i}$ be the vector of prices for all inputs that player $i$ purchases on the market. The requirement for all of his commodities is written as $\mathbf{q_i} = (q_i^1, \ldots, q_i^m)$, as $\mathbf{p_i} = (p_i^1, \ldots, p_i^m)$ is the vector of $i$'s prices. For any given requirement, $C_i(\mathbf{q_i}; \boldsymbol{\pi_i})$ is the value of the (monotonous and positive) cost-function of the monopoly, showing the cost-efficient production of the requirement $\mathbf{q_i}$ in $i$, $C_i(\mathbf{q_i}, \cdot) = 0$ for $\mathbf{q_i} = \mathbf{0}$ and $C_i(\mathbf{q_i}, \cdot) \geq 0$ for $\mathbf{q_i} \geq \mathbf{0}$.[17]

As a practicable method to allocate costs, we now introduce *subsidy-free prices*.[18]

**Definition 1.** Denote the requirement for the monopoly's commodities with $\overline{\mathbf{q}}_\mathbf{i}$. We call a system of prices $\mathbf{p_i}$ for all $m$ commodities of player $i$ subsidy-free, if:

$$\sum_{k=1}^{m} p_i^k \overline{q}_i^k = C_i(\overline{\mathbf{q}}_\mathbf{i}; \boldsymbol{\pi_i}) \tag{3}$$

$$\sum_{k=1}^{m} p_i^k q_i^k \leq C_i(\mathbf{q_i}; \boldsymbol{\pi_i}) \text{ for all } \mathbf{q_i} \leq \overline{\mathbf{q}}_\mathbf{i}. \tag{4}$$

$\mathbf{q_i} \leq \overline{\mathbf{q}}_\mathbf{i}$ implicates that $q_i^k \leq \overline{q}_i^k$ for all $k \in \{1, \ldots, m\}$, and that there exists at least one $l \in \{1, \ldots, m\}$ with $q_i^l < \overline{q}_i^l$.

The first property (3) defines the monopoly as a *cost-center*: it sets prices in such a way that its revenue covers its cost exactly. The player accepting to supply all other players is not to gain any profit. The second property (4) assures that there is no *cross-subsidization* between the products of the monopoly, as introduced by (Faulhaber, 1975).[19]

We illustrate the consequences for a simple example (see Fig. 2), in which player $i$ produces 100 units of each product $A$ and $B$ for the total costs of 1000. The price for both products is 5. $A$ is only produced for $j$, while $B$ is sold to all other players, which we pool in the coalition $S$. We now examine the case that $i$ solely

[16] The term monopoly is used here to exemplify the role of $i$ within the internal market of the cooperation. We assume that within the cooperation, a specific commodity is only produced by one player. It does not state that this commodity might not be available outside the boundaries of the cooperation.

[17] The cost-function $C_i(\cdot)$ is determined by cost minimization: $C_i(\cdot)$ calculates the minimal costs for $i$ to produce the required group of commodities $\mathbf{q_i}$ for given prices $\boldsymbol{\pi_i}$.

[18] Adapted from (Sharkey & Telser, 1978 p. 25), also see (Moulin, 1988 p. 98). Our analysis focuses solely on the cost side, for a comparison with the original formulation of subsidy-free prices as in (Faulhaber, 1975), anonymously equitable prices as in (Faulhaber & Levinson, 1981) or (non-cooperative) sustainable prices as in (Panzar & Willig, 1977), see (Mirman *et al.*, 1985).

[19] (Faulhaber, 1975 p. 969) formulates a test for cross-subsidization

$$\sum_{i \in S} x_i \geq c(N) - c(N \setminus S),$$

which is equivalent to (2) if it holds for all $S \subseteq N$.

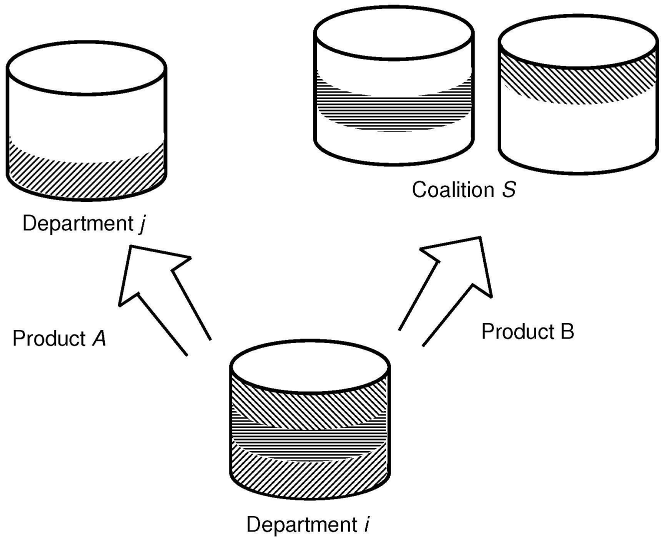


**Fig.2.** Example of cost-allocation through prices

produces $A$ and delivers it to $j$. Suppose $i$'s costs for producing solely $A$ only add up to 300, corresponding to a price of 3 per unit $A$. According to Definition 1, prices are not subsidy-free, because $B$ is subsidized by $A$, which leads to the outcome of $j$ subsidizing $S$. Because $j$ has to bear costs that other players caused, he would rather stand alone, and would not accept being served by $i$. Then, supplier $i$ would make a loss with the sale of $B$ to $S$ and the grand coalition would fall apart.

Subsidy-free prices challenge certain characteristics of the cost-function, that is, non-rising average prices. Hence, the multi-output monopoly must have a cost-function that defines it as a *natural monopoly*.[20] These cost-functions are called *supportable*.[21]

**Definition 2.** A cost function $C_i(\cdot)$ is supportable if for every $\overline{\mathbf{q}}_\mathbf{i} > 0$ there exists a subsidy-free price vector $\mathbf{p_i}$ according to Definition 1.

Necessary conditions for the supportability of a cost function are that $C_i(\cdot)$ is[22]

– subadditive, $C_i(\mathbf{q_i} + \mathbf{q'_i}; \boldsymbol{\pi_i}) \leq C_i(\mathbf{q_i}; \boldsymbol{\pi_i}) + C_i(\mathbf{q'_i}; \boldsymbol{\pi_i})$,

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[20] Again, the term monopoly is limited to the cooperation-internal market.

[21] See (Sharkey & Telser, 1978 p. 25). Similar concepts like sustainable cost functions were developed under the topic of contestable market theory. ((Panzar & Willig, 1977), (Baumol & Willig, 1977) and (Baumol *et al.*, 1982) Contestable markets, however, are interpreted as a description of market entry in non-cooperative settings, while the concept of supportability focuses on cooperative settings, which makes it the appropriate tool in the given setting, see (Sharkey & Telser, 1978 p. 24–27). All of the mentioned concepts are equal under certain assumptions, see (Sharkey, 1982) and (Mirman *et al.*, 1985).

[22] See (Sharkey & Telser, 1978 p. 27–28).

– subhomogenous, $C_i(\lambda\mathbf{q_i};\boldsymbol{\pi_i}) \leq \lambda C_i(\mathbf{q_i};\boldsymbol{\pi_i})$ for $\lambda \geq 1$,
– $\nabla C_i(\overline{\mathbf{q}}_\mathbf{i};\boldsymbol{\pi_i}) \cdot \mathbf{q_i} \leq C_i(\mathbf{q_i};\boldsymbol{\pi_i})$.[23] Non-rising average prices imply that marginal cost pricing will lead to budget deficits in the single output case. This is the generalization of this result to a multi-output monopoly.

If the production technology of a player does not allow for a natural monopoly, it might be better for one or several of the other players not to share this technology jointly. This is the case if a player would be better off accounting for his needs himself. If the production of the collective requirement is more expensive than the partitioned production of subsets of the collective requirement, a coalition can attack the player's position by rejecting his commodities and producing them by itself. Supportable cost-functions and subsidy-free prices do not allow such attacks and make entry into the monopoly unattractive for any other player or coalition and, therefore, guarantee that all other players channel their requirement onto this one technology. They ensure that the least overall costs for all players are achieved by sharing every player's requirement from the same source. Thus, subsidy-free prices effectuate the grand coalition.

The described correlation can be shown formally. One can prove that stability in a coalitional cost-sharing game is equal to the underlying cost-function being supportable, which in turn shows that subsidy-free prices exist.

**Lemma 1.** [24]
*The cost function $C_i(\cdot)$ is supportable if and only if the corresponding game in coalitional form has a nonempty core.*

## 4. Practicable Cost-Allocation in Cooperative Markets

In this section, we provide a practicable cost-allocation method for cooperative markets, which generates stable outcomes. Let us use the example of management accounting to illustrate the generalization of our model: How can the result of the previous section be of use for accounting in enterprises? If the labor division inside the company is seen as an internal market, and if every department is seen as a monopoly, trading (sharing) its commodities with all other departments, we comprehend the interrelated transfer of goods and services within the company. We see the organization as a network and simply attach the "right" prices to the already ongoing labor division. This approach is transferable to any situation in which more than two players require multiple commodities and decide to share their production.

Therefore, we generalize our approach to $n$ interconnected players, who share $m_1 + \cdots + m_n = m \geq n$ jobs by distributing them among each other, where $m_i$ denotes the number of jobs player $i$ produces. Each player (potentially) has access to all technologies; but, by labor division, the jobs are assigned in a way that each of the $m$ different jobs is exclusively assigned to a player.

[23] $\nabla C_i(\cdot)$ is the vector of partial derivatives of the cost function,

$$\nabla C_i(\cdot) = \left(\frac{\partial C_i(\cdot)}{\partial q_i^1}, \ldots, \frac{\partial C_i(\cdot)}{\partial q_i^m}\right).$$

[24] See (Sharkey & Telser, 1978 p. 28–29), (Telser, 1978) and (Moulin, 1988 p. 100–101). We will demonstrate how to construct such a corresponding game in coalitional form in Sect. 4.

Player $i$ requires certain quantities of different commodities produced within the cooperation, except the ones produced by himself.[25] Let $\mathbf{d_{ij}}$ be the required amount of products that $i$ asks for from player $j$, so that $\mathbf{d_{ij}} = (d_{ij}^1, \ldots, d_{ij}^{m_j})$ with $d_{ij}^k \geq 0, k = 1, \ldots, m_j$. Now let

$$\mathbf{d_i} = (\mathbf{d_{i1}}, \ldots \mathbf{d_{in}})$$

be the vector of $i$'s overall requirement of all products available within the cooperation, whereby $\mathbf{d_{ii}} = \mathbf{0}$. The components of the vector $\mathbf{q_j} = (q_j^1, \ldots, q_j^{m_j})$ are then determined by $q_j^k = \sum_{i \neq j} d_{ij}^k$. Due to these interdependencies, the cost-functions $C_i(\cdot)$ not only depend on the external prices $\boldsymbol{\pi_i}$, but also on the internal prices $\mathbf{p_j}$. Thus, the prices $\mathbf{p_j} = (p_j^1, \ldots, p_j^{m_j})$ of player $j \neq i$ effect the value of $C_i(\cdot)$. Henceforth, the cost-functions will be considered as $C_i(\mathbf{q_i}; \mathbf{p_{-i}}, \boldsymbol{\pi_i})$, where $\mathbf{p_{-i}} = (\mathbf{p_1}, \ldots, \mathbf{p_{i-1}}, \mathbf{p_{i+1}}, \ldots, \mathbf{p_n})$ expresses all internal prices that the cost-function of $i$ depends on. Additionally, we introduce $\mathbf{p}$ ($\mathbf{q}$) as the vector over all prices established (commodities produced) within the cooperation.

Now, without changing its validity, Definition 1 can be extended by using $C_i(\mathbf{q_i}; \mathbf{p_{-i}}, \boldsymbol{\pi_i})$ instead of $C_i(\mathbf{q_i}; \boldsymbol{\pi_i})$ and, thus, also taking the prices $\mathbf{p_{-i}}$ of those commodities into account that player $i$ requires from other players. In the generalized model, conditions (3) and (4) of Definition 1 are substituted by

$$\sum_{k=1}^{m_i} p_i^k \overline{q}_i^k = C_i(\overline{\mathbf{q}}_\mathbf{i}; \mathbf{p_{-i}}, \boldsymbol{\pi_i}) \tag{5}$$

$$\sum_{k=1}^{m_i} p_i^k q_i^k \leq C_i(\mathbf{q_i}; \mathbf{p_{-i}}, \boldsymbol{\pi_i}) \text{ for all } \mathbf{q_i} \leq \overline{\mathbf{q}}_\mathbf{i}. \tag{6}$$

First, consider the costs that arise because of the requirement of one player. If player $i$ requires $\mathbf{d_i}$, he induces costs of $C_j(\mathbf{d_{ij}}; \mathbf{p_{-j}}, \boldsymbol{\pi_j})$ to player $j$ in the case that no other player requires any of $j$'s commodities. These costs exactly mirror the costs of $i$ if he stands alone (i.e., $i$ uses technology $j$ for producing $\mathbf{d_i}$ autonomously). Since $i$ may require commodities from all other players, his stand-alone costs $c(\{i\})$ are given by

$$c(\{i\}) = \sum_{j=1}^{n} C_j(\mathbf{d_{ij}}; \mathbf{p_{-j}}, \boldsymbol{\pi_j}).$$

The actual costs, $i$ has to bear, depend on the quantities and prices of his required commodities. These costs can be written as the scalar (inner) product $\mathbf{p} \cdot \mathbf{d_i}$, which is our proposed allocation method of costs. Thereby, $\mathbf{p} \cdot \mathbf{d_i} = p_1^1 \cdot d_{i1}^1 + \ldots + p_n^{m_n} \cdot d_{in}^{m_n}$.

If several players ally in a coalition $S$, the requirement of coalition $S$ is then given by $\mathbf{d_S} = \sum_{i \in S} \mathbf{d_i}$, while $\mathbf{d_N} = \sum_{i \in N} \mathbf{d_i}$ ($\mathbf{d_N} = \mathbf{q}$) denotes the requirement of the grand coalition. The stand-alone costs of $S$ write as

$$c(S) = \sum_{i=1}^{n} C_i(\mathbf{d_{Si}}; \mathbf{p_{-i}}, \boldsymbol{\pi_i}).$$

[25] Kaplan showed that, even if a producer consumes his own services, that is, $q_{ii}^k > 0$, one can set $q_{ii}^k = 0$. The self-service costs have to be distributed among the other recipients of $i$'s commodities. If $i$'s production is distributed in the pattern of [i; j; j'] = [0.1; 0.8; 0.1], the new distribution is [0; 8/9; 1/9]. See (Kaplan, 1973).

The costs to bear for a coalition $S$ are equal to $\mathbf{p} \cdot \mathbf{d_S} = p_1^1 \cdot \sum_{i \in S} d_{i1}^1 + \ldots + p_n^{m_n} \cdot \sum_{i \in S} d_{in}^{m_n}$ and $\mathbf{p} \cdot \mathbf{d_N}$ for the grand coalition. According to the the efficiency principle (1), that is, total internal costs have to equal total internal returns, and the stand-alone principle (2), that is, stand-alone costs $c(S)$ must not be higher than the costs to bear, we derive the following conditions

$$\mathbf{p} \cdot \mathbf{d_N} = \sum_{i=1}^{n} C_i(\mathbf{d_{Ni}}; \mathbf{p_{-i}}, \boldsymbol{\pi_i}) \tag{7}$$

$$\mathbf{p} \cdot \mathbf{d_S} \leq \sum_{i=1}^{n} C_i(\mathbf{d_{Si}}; \mathbf{p_{-i}}, \boldsymbol{\pi_i}) \text{ for all } S \subseteq N, \tag{8}$$

which resemble the subsidy-free conditions (5) and (6) for a game with $n$ multi-output monopolies.[26]

In this way, we can link cost-sharing through prices to coalitional cost-allocation. Unfortunately, defining and testing for a core-allocation using (7) and (8) is also not practicable, because it generates $2^n$ constraints. (7) and (8) also model the game in coalitional form. In the following we focus on the prices to evade this problem. In fact, it can be proved that the corresponding cost-allocation game in coalitional form has a nonempty core *if and only if* subsidy-free prices exist, that is, if and only if all cost functions are supportable, see Definition 2.

**Proposition 1.** *Given the cost functions $C_i(\mathbf{q_i}; \mathbf{p_{-i}}, \boldsymbol{\pi_i})$ for all $n$ players, the core of the corresponding cost sharing game in characteristic functions $(N, c)$*

$$c(N) = \sum_{i=1}^{n} C_i(\mathbf{d_{Ni}}; \mathbf{p_{-i}}, \boldsymbol{\pi_i}) \tag{9}$$

$$c(S) = \sum_{i=1}^{n} C_i(\mathbf{d_{Si}}; \mathbf{p_{-i}}, \boldsymbol{\pi_i}) \ \forall S \subset N \tag{10}$$

*is nonempty if and only if all cost functions are supportable according to Definition 2.*

The proof is given in (Selders & Ehrhart, 2008). Its first part also proves the following proposition.

**Proposition 2.** *Consider the cost-sharing game in characteristic functions as defined in Proposition 1. Given the supportable cost functions $C_i(\mathbf{q_i}; \mathbf{p_{-i}}, \boldsymbol{\pi_i})$ for all $n$ players, the cost-allocation through subsidy-free prices $\mathbf{p_i}$ is a core-allocation.*

Since the costs of a requirement of a coalition $S$ equal the value of the characteristic function $c(S)$, and the cost-allocation $\sum_{i \in S} x_i$ consists of the internal pricing $\mathbf{p} \cdot \mathbf{d_S}$, properties (3), (5) and (7) mirror the principle of efficiency (1), and the constraints (4), (6) and (8) mirror the stand-alone principle (2).

We comment on the following important result: Proposition 2 only works one way – one can construct a core-allocation that cannot be achieved by subsidy-free

[26] Note that the calculation of $\mathbf{p}$ such that (7) holds is not trivial, but it can be shown that positive and definite prices exist at every production level $\mathbf{q}$. See (Selders & Ehrhart, 2008).

prices. Consider the example of two players sharing one single-output technology with the cost function $C_i(q_i; \mathbf{p_{-i}}, \boldsymbol{\pi_i}) = \sqrt{q_i}$ (all input prices are assumed to be 1), whereby player 1 requires 9 units and player 2 requires 16 units. The respective stand alone costs (and prices) of the players are 3 (0.33) and 4 (0.25). Since the cost function is supportable, the least cost level is the grand coalition, resulting in the subsidy-free price of 0.2 and in the allocation $\mathbf{x} = (1.8, 3.2)$, which is a core element. However, the allocation $\mathbf{x'} = (2, 3)$ is also a core-allocation. To achieve this allocation with prices, the prices of the goods have to vary according to the consumer, that is, player 1 has to pay 0.2222 and player 2 has to pay 0.1875 per unit. This is against the concept of subsidy-free prices, but not the stand-alone principle as it is defined in (2).

Since the core is a set of allocations, but subsidy-free prices are a unique allocation, one might ask how to interpret this result. Obviously, charging different prices to different players for the same commodity does not strike us as reasonable in the context of a game formulated as a market. This demonstrates that subsidy-free prices impose a notion of cost-sharing that is not only stable (in the sense of the core), but also "fair" in the sense that they pick a core-allocation that treats all players equally. Moreover, as all proportional cost sharing methods, cost-allocation through subsidy-free prices fulfills the weakest form of monotonicity.[27]

Summarizing, the fundamental characteristics of cooperative market-entry games can be extended to the case of various interactive players. Setting up a market-like model and reinterpreting entry-deterring conditions results in a cost-sharing game of the most general type.

## 5. Conclusion

The task at hand is to provide a cost-allocation method that fulfills desirable game-theoretic criteria and is still practicable to serve as a serious alternative for existing cost-sharing practices. Since we acknowledge that our (internal or external) market (in the broadest sense) is a cooperative one, we focus on stability instead of incentive compatibility.

Instead of using the characteristic function, we model the market as a network with interdependent players. One major advantage of the network model is that the amount of players and the amount of commodities are not restricted and do not diminish applicability. The model specifically includes the case that all productions and requirements are interrelated, that is, the requirements of other players and the costs of other player's goods have a direct impact on the price of player $i$'s commodities.[28]

By proving that the proposed solution is equivalent to comparable solutions of cooperative games in coalitional form, namely the core, we show that it offers the same strategic benefits of stability. It can be distinguished from game-

[27] See (Young, 1994 p. 1210). An allocation $\mathbf{x}$ fulfills the weakest form of monotonicity if for $c'(N) \leq c(N)$ and $c'(S) = c(S)$ for all $S \subset N$ the condition $x'_i \leq x_i$ holds for all $i \in N$.

[28] We limit our analysis to the cost side, that is, requirements do not depend on prices. This assumption is realistic for (internal) cooperative markets, where the decision to cooperate has already been taken, the desired scope of the cooperation is set and strategies are binary in the sense that a player either follows through with the cooperation or defects.

theoretical concepts – which are not practicable – and from accounting tools, e.g. ABC (Activity-Based-Costing) – which are not stable.[29]

Therefore, to check that indeed the proposed method of cost allocation is stable, one does not have to set up a characteristic function and examine each and every possible coalition, but one only needs to check if all prices are subsidy-free. This is essential characteristic of our approach that guarantees its practicability. Furthermore, the solution is unique and one can check that the existence of a system of positive subsidy-free prices is also guaranteed. In Selders & Ehrhart, 2008 it is additionally shown that the Aumann-Shapley price mechanism (Aumann & Shapley, 1974) serves as an applicable tool for the calculation of subsidy-free prices.

Summarizing, we introduce a practicable method for cost-allocation problems in any given setting, where any number of players can act as consumers, as producers, or as both at the same time. One can think of such diversified applications of this method as joint-ventures, management accounting, or pricing of public goods. It is possible to derive guidelines for accounting tools that allow us to understand the benefits of alternative methods, e.g. *lean accounting.*[30]

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[29] See (Wißler, 1997).

[30] See (Maskell & Baggaley, 2004).

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# On the Value Function to Differential Games with Simple Motions and Piecewise Linear Data*

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**Abstract** Positional differential games with simple dynamics are considered under assumption that at least one of two input functions (the Hamiltonian and the cost terminal function) is piecewise linear and positively homogeneous. The structure of the value function of the differential game is investigated in the framework of the theory of minimax (or/and viscosity) solutions for Hamilton–Jacobi equations. Inequalities are provided to estimate the value function. Cases of explicit formulas for the value function are pointed out.

**Keywords:** positional differential games, value function, Hamilton–Jacobi equations, minimax solutions, viscosity solutions, Hopf formulas, piecewise linear functions.

## 1. Introduction

The present paper deals with antagonistic differential games with fixed termination moment. The games are considered in the context of formalization suggested by Krasovskii and Subbotin, 1974.

In such a game the motion of a dynamical system is described by ordinary differential equations containing controls of players. The payoff functional is given. The first player minimizes the payoff, and the second player maximizes it as the system is evolving from a given initial position. The both players use feedback strategies. The value of the game is an optimal guaranteed result for the both players. The value function assigns the optimal unique result to each initial position of the game.

One of the approaches to solving differential games consists in finding the value function. The optimal feedback strategies of the players can be constructed on the basis of the value function.

It is known (Subbotin, 1991, 1995) that the value function is a minimax solution of Hamilton–Jacobi–Bellman–Isaacs equation corresponding to the considered differential game. The concept of minimax solution introduced by Subbotin is equivalent to the concept of viscosity solution introduced by Crandall and Lions (1983).

Many existence and uniqueness theorems are proved in the theory of minimax and viscosity solutions of Hamilton–Jacobi equations, but there is not an universal way to obtain the global solution exactly in general case. Successful situations when one can receive explicit formulas for solutions occur very seldom, so it is important to develop constructive methods for approximations of minimax solutions.

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To approximate solutions in general differential games it is possible to use solutions of differential games with simple motions. The corresponding local estimates have been established by Subbotin (1991).

Differential games with simple motions are simple models of conflict-controlled systems. Such games were studied by many authors, in particular, Petrosjan (1977), Ukhobotov (1991). The analysis of games from this class is based essentially on input data consisting of the Hamiltonian and the cost function. In the case when at least one of these functions is convex or concave it is possible to write out explicit formulas for the value function (Hopf, 1965; Pshenichnyi and Sagaidak, 1970; Bardi and Evans, 1984). But in general case, when the Hamiltonian and the cost function are arbitrary, explicit formulas for the value function are not known.

A finite algorithm has been worked out for an exact constructing the value function of differential game with simple dynamics (Subbotin and Shagalova, 1993; Shagalova, 1999). This algorithm is developed for the case of two-dimensional phase space. Both the Hamiltonian and the cost function are assumed to be positively homogeneous and piecewise linear functions simultaneously in the algorithm.

In the present paper differential game with simple motions is considered without any restriction on the dimension of phase space. Inequalities are obtained to estimate the value function in the cases when at least one of the input functions is positively homogeneous and piecewise linear function. We have the formula for the value function if maximins and minimaxes in the left and the right parts of these inequalities coincide. The investigations are based on Hopf formulas and quasidifferentiability of positively homogeneous piecewise linear functions in the sense of Demyanov and Rubinov (1986).

## 2. Problem Statement

### 2.1. Differential game

A differential game with simple motions is considered. The dynamics is described by equation

$$\dot{x} = f(u, v), \tag{1}$$

where $t \in [0, \theta]$, the instant $\theta > 0$ is fixed, $x \in R^n$ is the phase variable, $u$ and $v$ are respectively controls of the first and the second players, $u \in \mathrm{P} \subset R^p$, $v \in \mathrm{Q} \subset R^q$, P and Q are compact sets. It is supposed that function $f : \mathrm{P} \times \mathrm{Q} \to R^n$ is continuous and for all $s \in R^n$ satisfies the condition

$$\min_{u \in \mathrm{P}} \max_{v \in \mathrm{Q}} \langle s, f(u, v)\rangle = \max_{v \in \mathrm{Q}} \min_{u \in \mathrm{P}} \langle s, f(u, v)\rangle = H(s). \tag{2}$$

Here the symbol $\langle s, f\rangle$ denotes the scalar product of vectors $s$ and $f$.

The function $H$ defined by equality (2) is called the Hamiltonian. It follows from (2) that $H$ is positively homogeneous, i.e.

$$H(\alpha s) = \alpha H(s), \quad s \in R^n, \quad \alpha > 0. \tag{3}$$

The Hamiltonian also assumed to be Lipschitz continuous, i.e.

$$|H(s_1) - H(s_2)| \leq L||s_1 - s_2||, \quad ||s_1|| \leq 1, ||s_2|| \leq 1. \tag{4}$$

Also a terminal payoff functional is given

$$x(\cdot) \to \sigma(x(\theta)), \tag{5}$$

where the cost function $\sigma : R^n \to R$ is continuous. The first player seeks to minimize the payoff over his controls $u(\cdot)$, whereas the second player wishes to maximize it over controls $v(\cdot)$.

### 2.2. Value Function

It is known (Krasovskii and Subbotin, 1974; Subbotin, 1995) that value $w(t_0, x_0)$ exists in the differential game (1)–(5) for any initial position $(t_0, x_0) \in [0, \theta] \times R^n$. Value function $w : [0, \theta] \times R^n \to R$ is the unique minimax (and/or viscosity) solution of the Hamilton–Jacobi equation

$$\partial w(t, x)/\partial t + H(\partial w(t, x)/\partial x) = 0, \qquad t \in (0, \theta), \quad x \in R^n, \tag{6}$$

satisfying boundary condition

$$w(\theta, x) = \sigma(x), \quad x \in R^n. \tag{7}$$

Our purpose is to find the value function of the differential game (1)–(5), i.e. the minimax solution of the Cauchy problem (6)–(7).

## 3. Some Known Results

This section contains a brief review of certain known results concerning value function of the differential game (1)–(5).

### 3.1. Hopf formulas

In the case when at least one of the functions $H$ and $\sigma$ is convex or concave an explicit formulas can be written for the minimax solution of the problem (6)–(7).

If the cost function $\sigma$ is convex then the next formula is valid for the minimax solution

$$w(t, x) = \sup_{s \in R^n} \inf_{y \in R^n} [\sigma(y) + \langle s, x - y \rangle + (\theta - t)H(s)] \tag{8}$$

If the Hamiltonian $H$ is convex function then the minimax solution can be written with the help of the next formula

$$w(t, x) = \sup_{y \in R^n} \inf_{s \in R^n} [\sigma(y) - \langle s, y - x \rangle + (\theta - t)H(s)] \tag{9}$$

One can write analogous formulas for concave input functions too.

Formulas (8), (9) were received by Hopf (1965) as formulas for generalized solutions of the equation (6), i.e. as functions satisfying (6) at all points of differentiability. Later Bardi and Evans (1984) proved that functions defined by (8), (9) coincide with viscosity solutions of the Cauchy problem (6)–(7), and Subbotin (1995) independently showed that these functions are the minimax solutions. In the theory of differential games it is shown (Pshenichnyi and Sagaidak, 1970) also that the function of the form (8) coincides with the value function of the differential game with simple motions and a convex cost function.

### 3.2. Reduction of the problem in the case when the cost function is positively homogeneous

It is assumed in this subsection that the function $\sigma$ is positively homogeneous, i.e.

$$\sigma(\alpha x) = \alpha\sigma(x), \quad x \in R^n, \quad \alpha \geq 0. \tag{10}$$

From (10) one can receive the next equality for the minimax solution of the problem (6)–(7)

$$w(t,x) = (\theta - t)w\left(0, \frac{x}{\theta - t}\right), \quad t \in [0,\theta), \quad x \in R^n. \tag{11}$$

The equality (11) permits to substitute problem (6)–(7) by a reduced problem whose solution will be function

$$\varphi = \varphi(y) = w(0,y), \qquad y \in R^n.$$

So, the reduced problem is to find the minimax solution for the next partial differential equation of the first order

$$H(D\varphi(y)) + \langle D\varphi(y), y\rangle - \varphi(y) = 0, \tag{12}$$

and to satisfy the limit relation

$$\lim_{\alpha\downarrow 0} \alpha\varphi\left(\frac{y}{\alpha}\right) = \sigma(y), \qquad y \in R^n. \tag{13}$$

Here the symbol $D\varphi(y)$ denotes the gradient of the function $\varphi$ at the point $y$.

The minimax solution of the equation (12) can be defined by various ways, different in the form, but equivalent in essence (Subbotin, 1991, 1995). In particular, the continuous function $\varphi$ is the minimax solution of the equation (12) if and only if this function satisfies the pair of differential inequalities which can be written, for example, in the next form.

$$\min_{f\in R^n}[d^-\varphi(y;f) - \langle s, f-y\rangle + H(s) - \varphi(y)] \leq 0, \tag{14}$$

$$\max_{f\in R^n}[d^+\varphi(y;f) - \langle s, f-y\rangle + H(s) - \varphi(y)] \geq 0. \tag{15}$$

Inequalities (14)-(15) must be satisfied for all $y \in R^n, s \in R^n$. The symbols $d^-\varphi(y;f)$ and $d^+\varphi(y;f)$ denote respectively the lower and the upper derivatives of the function $\varphi$ on the direction $f$ at the point $y$:

$$d^-\varphi(y;f) = \lim_{\varepsilon\downarrow 0}\inf_{0<\delta<\varepsilon}\{[\varphi(y+\delta f) - \varphi(y)]\delta^{-1}\},$$

$$d^+\varphi(y;f) = \lim_{\varepsilon\downarrow 0}\sup_{0<\delta<\varepsilon}\{[\varphi(y+\delta f) - \varphi(y)]\delta^{-1}\}.$$

### 3.3. An algorithm for exact constructing the piecewise linear value function on the plane

There are not known general explicit formulas for minimax solution of the problem (6)–(7) in general case, i.e. in the case when the Hamiltonian $H$ and the function $\sigma$ are not convex or concave.

For the case when data ($H$ and $\sigma$) are piecewise linear and phase space are two-dimensional a finite algorithm for the construction of the exact minimax solution of the problem (6)–(7) have been developed and justified (Subbotin and Shagalova, 1993, Shagalova, 1999). The minimax solution that can be obtained with the help of this algorithm also turns out to be a piecewise linear function. Here we do not describe this algorithm extensively. The essence consists in successive solving certain elementary problems arising in a definite order. This order depend on the structure of the function $\sigma$ and on the singularities of the Hamiltonian. It should be emphasize also that both functions $H$ and $\sigma$ are supposed to be positively homogeneous, so the algorithm is worked out for solving the reduced problem (12)–(13).

On the base of the algorithm the computing program and the program drawing the level lines for corresponding minimax solutions were worked out. Some illustrations obtained with the help of these programs are represented on Fig. 1.

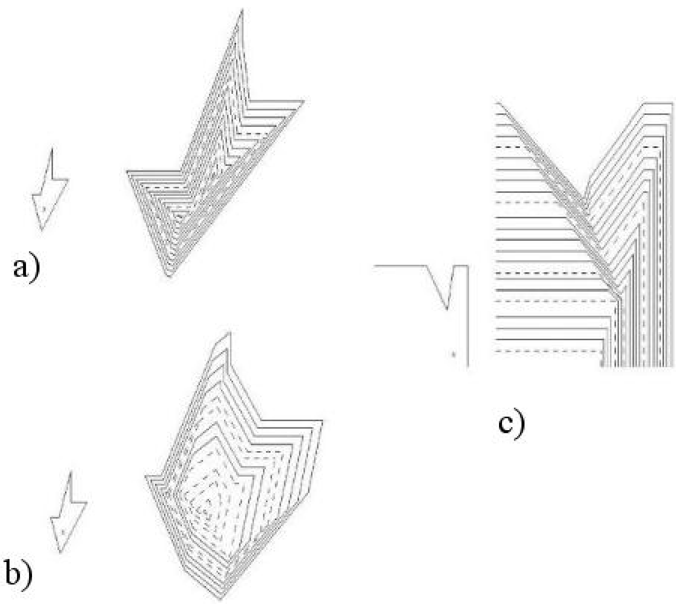


**Fig.1.** Level lines of the cost function $\sigma$ and corresponding value function.

On the left side of every picture on Fig. 1 there is the line $\{x \in R^2 | \sigma(x) = 1\}$. The origin is marked by cross. On the right side of the same picture there are level lines of the corresponding value function.

Since the specific properties of the plane is used essentially in the algorithm, attempts to extend it to phase space of more dimensions have no success yet.

## 4. Estimates for the Value Function in the Game with Piecewise Linear Data

The main results of the present paper are presented in this section. Note that we do not impose any restrictions on the dimension of the phase space here.

### 4.1. Preliminaries

The material presented in this subsection will be needed below to prove our main results.

**Quasidifferentiability of positively homogeneous piecewise linear functions.** Our results use the next theorem proved by Melzer (1986).

**Theorem 1.** *Function $g \in C(R^n)$ is a piecewise linear positively homogeneous function if and only if there exists a pair $\underline{\partial} g(0) \subset R^n$, $\bar{\partial} g(0) \subset R^n$ of convex compact polyhedrons such that*

$$g(v) = \max_{y \in \underline{\partial} g(0)} \langle y, v \rangle + \min_{z \in \bar{\partial} g(0)} \langle z, v \rangle . \tag{16}$$

Here $C(R^n)$ is the set of continuous functions defined on the space $R^n$. It should be noted that the pair $(\underline{\partial} g(0), \bar{\partial} g(0))$ in the representation (16) is Demyanov's quasidifferential of $g$ at zero (Demyanov and Rubinov, 1986). A pair of polyhedrons which make up Demyanov's quasidifferential is defined not uniquely.

**Lower and upper minimax solutions for Hamilton–Jacobi equations with different Hamiltonians and the same terminal function.** Here we present a proposition which will be needed below to obtain estimates for the value function in the case of piecewise linear Hamiltonian.

Let $H_*$ and $H^*$ be Lipschitz continuous positively homogeneous functions such that

$$H_*(s) \leq H^*(s), \quad s \in R^n. \tag{17}$$

Consider two Cauchy problems with different Hamiltonians.

$$\begin{aligned} \partial u(t,x)/\partial t + H_*(\partial u(t,x)/\partial x) = 0, \\ u(\theta, x) = \sigma(x), \quad x \in R^n. \end{aligned} \tag{18}$$

$$\begin{aligned} \partial u(t,x)/\partial t + H^*(\partial u(t,x)/\partial x) = 0, \\ u(\theta, x) = \sigma(x), \quad x \in R^n. \end{aligned} \tag{19}$$

The common function $\sigma$ assumed to be continuous.

Let $u_*(t,x)$ and $u^*(t,x)$ be minimax solutions of problems (18) and (19) respectively. It is not very difficult to prove the next assertion.

**Proposition 1.** *The function $u_*(t,x)$ is a lower minimax solution of the problem* (19)*, and function $u^*(t,x)$ is an upper minimax solution of the problem* (18)*.*

The notions of subdifferentials and superdifferentials will be used in the proof of the proposition 1. Recall the definitions of these notions (Clarke, 1983; Demyanov and Rubinov, 1986; Subbotin, 1995).

Let

$$D^-u(t,x) = \{(a,s) \in R \times R^n | \\ a\alpha + \langle s, f\rangle - d^-u(t,x;\alpha,f) \le 0, \quad \forall(\alpha,f) \in R \times R^n)\}. \tag{20}$$

$$D^+u(t,x) = \{(a,s) \in R \times R^n | \\ a\alpha + \langle s, f\rangle - d^+u(t,x;\alpha,f) \ge 0, \quad \forall(\alpha,f) \in R \times R^n)\}, \tag{21}$$

The set $D^-u(t,x)$ (the set $D^+u(t,x)$) is called the subdifferential (the superdifferential) of the function $u$ at the point $(t,x) \in (0,\theta) \times R^n$. Elements of the set $D^-u(t,x)$ (the set $D^+u(t,x)$) are called subgradients (supergradients). $D^-u(t,x)$ and $D^+u(t,x)$ are closed and convex sets (which can be empty also). If the function $u$ is differentiable at the point $(t,x) \in (0,\theta) \times R^n$ then one can verify easily that $D^-u(t,x) = D^+u(t,x) = (\partial u(t,x)/\partial t, \partial u(t,x)/\partial x)$.

*Proof (of proposition 1).* Since the function $u_*(t,x)$ is the minimax solution of the problem (18), this function is simultaneously the lower minimax solution for the same problem. So, the next inequality is valid.

$$a + H_*(s) \ge 0, \qquad (t,x) \in (0,\theta) \times R^n, \quad (a,s) \in D^+u_*(t,x). \tag{22}$$

From the inequalities (17), (22) we receive

$$a + H^*(s) \ge 0, \qquad (t,x) \in (0,\theta) \times R^n, \quad (a,s) \in D^+u_*(t,x). \tag{23}$$

The inequality (23) means that the function $u_*(t,x)$ is the lower minimax solution of the problem (19).

Since the function $u^*(t,x)$ is the minimax solution of the problem (19), this function is simultaneously the upper minimax solution for the same problem. So, the next inequality is valid.

$$a + H^*(s) \le 0, \qquad (t,x) \in (0,\theta) \times R^n, \quad (a,s) \in D^-u^*(t,x). \tag{24}$$

From(17), (24) we have

$$a + H_*(s) \le 0, \qquad (t,x) \in (0,\theta) \times R^n, \quad (a,s) \in D^-u^*(t,x). \tag{25}$$

It follows from the inequality (25) that the function $u^*(t,x)$ is the upper minimax solution of the problem (18). □

**Value functions of games with different cost functions.** Let $\Gamma_1$ and $\Gamma_2$ be two differential games of the form (1)–(5) with the common Hamiltonian $H$. Let $\sigma_1$ be the cost function in the game $\Gamma_1$ and $\sigma_2$ be the cost function in the game $\Gamma_2$. Let $w_1(t,x)$ and $w_2(t,x)$ be the corresponding value functions. The following easily proved fact is known.

**Proposition 2.** *If $\sigma_1(x) \le \sigma_2(x)$ for all $x \in R^n$ then $w_1(t,x) \le w_2(t,x)$ for all $(t,x) \in [0,\theta] \times R^n$.*

### 4.2. The case of piecewise linearity of the cost function

At first we consider the case when the function $\sigma$ is Lipschitz continuous piecewise linear function satisfying the positive homogeneity condition (10), and the Hamiltonian $H$ is an arbitrary function satisfying conditions (3)–(4). Note that the piecewise linearity of the Hamiltonian is not required in this subsection.

It is possible in this case to consider as original terminal problem (6)–(7) as reduced problem (12)–(13).

According to Proposition 1 there exists a pair of convex compact polyhedrons $A$ and $B$ such that the function $\sigma$ can be presented as

$$\sigma(x) = \max_{a \in A} \langle a, x \rangle + \min_{b \in B} \langle b, x \rangle . \tag{26}$$

Let

$$\sigma_a(x) = \langle a, x \rangle + \min_{b \in B} \langle b, x \rangle ,$$

$$\sigma^b(x) = \max_{a \in A} \langle a, x \rangle + \langle b, x \rangle .$$

Then from (26) we receive

$$\sigma_a(x) \le \sigma(x) \le \sigma^b(x), \qquad a \in A, b \in B, \quad x \in R^n. \tag{27}$$

Applying Hopf formulas for convex and concave terminal functions $\sigma^b$, $\sigma_a$ and using proposition 2, we obtain the next estimates for the value function.

$$\begin{aligned} \min_{b \in B} [\langle a + b, x \rangle + (\theta - t) H(a + b)] \le w(t, x) \le \\ \max_{a \in A} [\langle a + b, x \rangle + (\theta - t) H(a + b)], \\ t \in [0, \theta], \quad x \in R^n. \end{aligned} \tag{28}$$

For any $a \in A$ the function in the left-hand side of (28) is the lower minimax solution of the problem (6)–(7). Similarly, for any $b \in B$ the function in the right-hand side is the upper minimax solution.

It is known (Subbotin, 1995) that the upper envelope of lower minimax solutions is the lower minimax solutions. Also the lower envelope of upper minimax solutions is the upper minimax solutions. Thus we receive from (28) the following inequality.

$$\begin{aligned} \max_{a \in A} \min_{b \in B} [\langle a + b, x \rangle + (\theta - t) H(a + b)] \le w(t, x) \le \\ \min_{b \in B} \max_{a \in A} [\langle a + b, x \rangle + (\theta - t) H(a + b)], \\ t \in [0, \theta], \quad x \in R^n. \end{aligned} \tag{29}$$

So, we have proved the next theorem.

**Theorem 2.** *Let the Hamiltonian $H$ satisfy conditions* (3)–(4) *and the cost function $\sigma \in C(R^n)$ be positively homogeneous and piecewise linear. Let the representation* (26) *be valid for $\sigma$. Then for the value function $w(t, x)$ of the differential game* (1)–(5) *the inequality* (29) *is valid.*

There are lower and upper minimax solutions of the problem (6)–(7) respectively in the left and right side of the inequality (29). So, in situations when maximin is equal to minimax we will get the formula for minimax solution (value function).

### 4.3. The case when the Hamiltonian is piecewise linear

In this subsection the cost function $\sigma$ is assumed to be an arbitrary continuous function, i.e. $\sigma \in C(R^n)$, and the Hamiltonian $H$ is supposed to be piecewise linear function satisfying conditions (3)–(4).

According to Proposition 1 the Hamiltonian can be represented in the form

$$H(s) = \max_{q\in Q} \langle s, q\rangle + \min_{p\in P} \langle s, p\rangle\,, \quad s \in R^n, \tag{30}$$

where the pair of compact convex polyhedrons $Q$ and $P$ makes up the Demyanov's quasidifferential of $H$ at zero. The next inequality is valid

$$H_q(s) \le H(s) \le H^p(s), \quad s \in R^n, \tag{31}$$

where $p \in P, \quad q \in Q,$ and

$$H_q(s) = \langle q, s\rangle + \min_{p\in P} \langle p, s\rangle\,,$$
$$H^p(s) = \max_{q\in Q} \langle q, s\rangle + \langle p, s\rangle\,.$$

Using Hopf formulas for Hamilton–Jacobi equations with the convex Hamiltonian $H^p$ and the concave Hamiltonian $H_q$ we receive from inequalities (31) and Proposition 1 the next estimates for the minimax solution of the problem (6)–(7) with piecewise linear Hamiltonian

$$\begin{aligned}\min_{p\in P} \sigma(x + (\theta - t)(p + q)) \le w(t, x) \le \\ \le \max_{q\in Q} \sigma(x + (\theta - t)(p + q)).\end{aligned} \tag{32}$$

For any $q \in Q$ the function in the left–hand side of (32) is the lower minimax solution of the problem (6)–(7). For any $p \in P$ the function in the right–hand side is the upper minimax solution. So, using the known properties of the envelopes of these solutions (Subbotin, 1995), we obtain the following inequalities

$$\begin{aligned}\max_{q\in Q} \min_{p\in P} \sigma(x + (\theta - t)(p + q)) \quad \le \quad w(t, x) \quad \le \\ \min_{p\in P} \max_{q\in Q} \sigma(x + (\theta - t)(p + q)).\end{aligned} \tag{33}$$

Note that $(t, x) \in [0, \theta] \times R^n$ in (32), (33).

Thus we have proved the following assertion.

**Theorem 3.** *Let $\sigma \in C(R^n)$. Let the Hamiltonian $H$ be piecewise linear and satisfy conditions* (3)–(4). *Let the representation* (30) *be valid for $H$. Then for the value function $w(t, x)$ of the differential game* (1)–(5) *the inequality* (33) *is valid.*

### 4.4. The case when both input functions are piecewise linear

Now we can combine the results of subsections 4.2. and 4.3. Thus we get the next

**Theorem 4.** *Let the Hamiltonian $H$ be a piecewise linear function that satisfies the conditions* (3) *and* (4). *Let also $\sigma$ be a Lipschitz continuous piecewise linear and positively homogeneous function. Let the functions $H$ and $\sigma$ be representable in the forms* (30) *and* (26) *respectively. Then for the value function $w(t, x)$ of the* *differential game* (1)–(4) *(and/or for the minimax solution of the Cauchy problem* (6)–(7)) *the next relations are valid.*

$$w_-(t,x) \le w(t,x) \le w_+(t,x), \tag{34}$$

*where* $(t,x) \in [0,\theta] \times R^n$,

$$w_-(t,x) = \max\left\{ \max_{q\in Q} \min_{p\in P} \max_{a\in A} \min_{b\in B} \psi(t,x,a,b,p,q), \right.$$
$$\left. \max_{a\in A} \min_{b\in B} \max_{q\in Q} \min_{p\in P} \psi(t,x,a,b,p,q) \right\},$$

$$w_+(t,x) = \min\left\{ \min_{p\in P} \max_{q\in Q} \min_{b\in B} \max_{a\in A} \psi(t,x,a,b,p,q), \right.$$
$$\left. \min_{b\in B} \max_{a\in A} \min_{p\in P} \max_{q\in Q} \psi(t,x,a,b,p,q) \right\},$$
$$\psi(t,x,a,b,p,q) = \langle a+b, x+(\theta-t)(p+q)\rangle .$$

### 4.5. Remark

There are the lower minimax solutions of the problem (6)–(7) in the left-hand sides in (29), (33), (34). The corresponding right-hand sides are the upper minimax solutions of the problem. The value function of the differential game (1)–(5) coincides with the minimax solution of the problem (6)–(7). Thus the value function is the unique function which is simultaneously the lower minimax solution and the upper one.

## 5. Conclusion

Differential games with simple motions and fixed terminal instant were concidered provided input functions are piecewise linear. No restrictions on the dimension of phase space are imposed. Estimates are obtained for the value functions.

Results of this paper can be used for development methods of approximating value functions of differential games with general dynamics.

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# Time-consistency Problem Under Condition of a Random Game Duration in Resource Extraction


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**Abstract** We consider time-consistency problem for cooperative differential $n$-person games with random duration. It is proved, that in many cases the solution (or optimality principle) for such games is time-inconsistent. For regularization of solution the special imputation distributed procedure (IDP) is introduced and the time-consistency of the new regularized optimality principle is proved. At last we consider one game-theoretical problem of non-renewable resource extraction under condition of a random game duration. The problem of time-consistency for the Shapley Value in this example is investigated.



**Keywords:** time-consistency, random duration, Shapley Value, differential game, non-renewable resource.


## 1. Basic Model: a Game-Theoretic Model of Nonrenewable Resource Extraction with Random Game Duration

Consider one simple model of common-property nonrenewable resource extraction published in (E.J. Dockner, S. Jorgensen et al., 2000).

Let $x(t)$ and $c_i(t)$ denote respectively the stock of the nonrenewable resource such as an oil field and player $i$'s rate of extraction at time $t$. We assume that $c_i(t) \geq 0$ and that, if $x(t) = 0$, then the only feasible rate of extraction is $c_i(t) = 0$. Let the transition equation has the form

$$\dot{x}(t) = -\sum_{i=1}^{n} c_i(t); \tag{1}$$

$$\lim_{t->\infty} x(t) \geq 0; \tag{2}$$

$$x(t_0) = x_0. \tag{3}$$

The game starts at $t_0$ from $x_0$. We suppose that the game ends at the random time instant $T$ with exponential distribution $f(t) = \rho * e^{-\rho(t-t_0)}$, $t \geq t_0$.

Each player $i$ has a utility function $h(c_i)$, defined for all $c_i > 0$. The utility function depends on elasticity of marginal utility $\eta > 0$, so we have two forms of $h(c_i)$:

$$h(c_i) = \begin{cases} Aln(c_i) + B, & \text{if} \quad \eta = 1; \\ A\frac{c_i^{1-\eta}}{1-\eta} + B, & \text{if} \quad \eta \neq 1. \end{cases} \tag{4}$$

Here, $A$ is positive and $B$ is a constant which may be positive, negative or zero.

We define integral expected payoff

$$K_i(x_0, c_1, \ldots, c_n) = \int_{t_0}^{\infty} \int_{t_0}^{t} h(c_i(\tau)) \rho e^{-\rho(t-t_0)} d\tau dt, \quad i = 1, \ldots, n.$$

and consider total payoff in cooperative form of the game:

$$\max_{\{c_i\}} \sum_{i=1}^{n} K_i(x_0, c_1, \ldots, c_n) = \sum_{i=1}^{n} K_i(x_0, c_1^I, \ldots, c_n^I) =$$
$$= \int_{t_0}^{\infty} \int_{t_0}^{t} h(c_i^I) \rho e^{-\rho(t-t_0)} d\tau dt.$$

## 2. The Shapley Value

We suppose that before begining of the game players chose the Shapley Value as optimality principle. It is common knowledge, that the formula for a Shapley Value in the n-person game has the form

$$Sh_i = \sum_{\substack{S \subset N \\ i \in S}} \frac{(n-s)!(s-1)!}{n!} [V(S) - V(S \backslash \{i\})], \quad i = 1, \ldots, n. \tag{5}$$

The common way to define the characteristic function in $\Gamma(x_0)$ is as following:

$$V(S, x_0) = \begin{cases} 0, \quad S = \emptyset; \\ \max\limits_{u_S} \min\limits_{u_{N \backslash S}} \sum_{i \in S} K_i(x_0, u), \quad S \subset N, \\ \max\limits_{u} \sum_{i=1}^{n} K_i(x_0, u), \quad S = N. \end{cases} \tag{6}$$

But this approach doesn't seems to be the best in context of environmental or other problems, because unlikely that if a subset of players form a coalition to tackle an enviromental problem, then the remaining players would form an anti-coalition to harm their efforts. For enviromental problem we can use another method of characteristic function construction with assumption that left-out players stick to their feedback Nash strategies. This approach was proposed in (Petrosjan L. A., Zaccour G.,2003).

Then we have the following definition of the characteristic function:

$$V(S, x^*(\vartheta)) = \begin{cases} 0, \quad S = \emptyset; \\ W_i(x^*(\vartheta), \vartheta), \quad i = 1, \ldots, n; \quad \{i\} \in I; \\ W_K(x^*(\vartheta), \vartheta), \quad K \subseteq I, \end{cases} \tag{7}$$

where $W_i(x^*(\vartheta), \vartheta), W_K(x^*(\vartheta), \vartheta)$ are the results of the corresponding Hamilton-Jacobi-Bellman equations. Remark, that the constructed function $V(S, x^*(\vartheta))$ (13) is not superadditive in general.

If we use the characteristic function $V(S, x^*(\vartheta))$ (13) constructed by concept of Petrosjan and Zacoour there is a need to examine the superadditivity of this function.

The Hamilton-Jacobi-Bellman equation appropriates to a problem with random game duration had been derived i paper (Shevkoplyas E.V., 2005). So, we get the Hamilton-Jacobi-Bellman equation in general case of arbitrary distribution function $F(t)$ (and we suppose an existence of density probability function $f(t) = F'(t)$):

$$\frac{f(\vartheta)}{1 - F(\vartheta)} W(x, \vartheta) \;=\; \frac{\partial W(x, \vartheta)}{\partial \vartheta} + \max_u \left[ H(x(\vartheta), u(\vartheta)) + \frac{\partial W(x, \vartheta)}{\partial x} g(x, u) \right]. \quad (8)$$

Suppose that the final time instant $T$ has the exponential distribution. Let us remark that for a problem with random duration $(T - t_0) \in [0, \infty)$ the first term on the right-hand side (8) is equal to zero ($\frac{\partial W(t,x)}{\partial t} = 0$) for a case of exponential distribution, but it doesn't satisfy for arbitrary distribution. Then the Hamilton-Jacobi-Bellman equation (8) get the form:

$$\rho W(x, t) = \max_u \left\{ H(x(t), u(t)) + \frac{\partial W(x, t)}{\partial x} g(x, u) \right\}. \quad (9)$$

This equation looks like Hamilton-Jacobi-Bellman equation for the infinite time horizon problem with discount factor $\rho$ (E.J. Dockner, S. Jorgensen et al., 2000).

### 2.1. Algorithm

Thus, in the case of the random game duration we propose the following algorithm of the Shapley Value calculation.

(1) Maximize the total expected payoff of the grand coalition $I$.

$$W_I(x, \vartheta) = \max_{c_i, i \in I} \frac{1}{1 - F(\vartheta)} \sum_{i=1}^{n} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} h_i(x(\tau)) d\tau f(t) dt, \quad (10)$$
$$x(\vartheta) = x.$$

Denote $\sum_{i=1}^n h_i(\dot{)}$ by $H(\dot{)}$. Then the Bellman function $W_I(x, \vartheta)$ satisfies the HJB equation (8). Results of optimization are optimal trajectory $x^I(t)$ and optimal strategies $c^I = (c_1^I, \ldots, c_n^I)$.

(2) Calculate a feedback Nash equilibrium.
Without cooperation each player $i$ seeks to maximize his expected payoff (**??**). Thus the player $i$ solves a dynamic programming problem:

$$W_i(x, \vartheta) = \max_{c_i} \frac{1}{1 - F(\vartheta)} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} h_i(x(\tau)) d\tau f(t) dt, \quad (11)$$
$$x(\vartheta) = x.$$

Denote $h_i(\dot{)}$ by $H(\dot{)}$. In this notation $W_i(x, \vartheta)$ satisfies the HJB equations (8) for all $i \in I$.
Denote by $c^N(\cdot) = \{c_i^N(\cdot), \quad i = 1, \ldots, n\}$ any feedback Nash equilibrium of this noncooperative game $\Gamma(x_0)$. Let the corresponding trajectory be $x^N(t)$.
We calculate $W_i(x^*(\vartheta), \vartheta)$ under condition that before time instant $\vartheta$ players use their optimal strategies $c_i^I$.

(3) Compute outcomes for all remaining possible coalitions.

$$W_K(x, \vartheta) = \max_{c_i, i \in K} \frac{1}{1 - F(\vartheta)} \sum_{i \in K} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} h_i(x(\tau)) d\tau f(t) dt, \quad (12)$$
$$c_j = c_j^N \quad \text{for} \quad j \in I \setminus K,$$
$$x(\vartheta) = x.$$

Here we insert for the left-out players $i \in I \setminus K$ their Nash values (see step 2). In the notation $\sum_{i\in K} h_i(\dot{}) = H(\dot{})$ the Bellman function $W_K(x,\vartheta)$ satisfies the corresponding HJB equation (8).

(4) Define the characteristic function $V(S, x^*(\vartheta))$, $\forall S \subseteq I$ as

$$V(S, x^*(\vartheta)) = \begin{cases} 0, \quad S = \emptyset; \\ W_i(x^*(\vartheta), \vartheta), \quad i = 1, \dots, n; \quad \{i\} \in I; \\ W_K(x^*(\vartheta), \vartheta), \quad K \subseteq I. \end{cases} \tag{13}$$

(5) Test of the superadditivity for characteristic function (13).
We need to check the following inequality satisfiability:

$$V(x_0, S_1 \cup S_2) \geq V(x_0, S_1) + V(x_0, S_2).$$

If the superdditivity of the characteristic function is fulfilled, go to the next step. Otherwise we can not use the Shapley Value as optimality principle for the players, but we can consider another optimality principles like Banzaph Value and others.

(6) Calculation of the Shapley Value by formula:

$$Sh_i = \sum_{\substack{S \subset N \\ i \in S}} \frac{(n-s)!(s-1)!}{n!}[V(S) - V(S \backslash \{i\})], \quad i = 1, \dots, n.$$

### 2.2. Solution

**I.** Consider the case of logarithmic utility function:

$$h(c_i(\tau)) = Aln(c_i) + B.$$

Further we use 5 steps of our algorithm to calculate the characteristic function values.

**Step 1.** Let us consider the grand coalition $I = \{1, \dots, n\}$. Then the Bellman function is as follows:

$$W_I(x) = \max_{\{c_i, i \in I\}} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} (A * ln(c_i) + A * \sum_{j \neq i} ln(c_j) + nB) \rho * e^{-\rho(t-\vartheta)} d\tau dt. \tag{14}$$

Let us define $\sum_{i=1}^{n} h_i(c_i(\dot{})) = H(c(\dot{}))$. Then we can use the Hamilton-Jacobi-Bellman equation (9) :

$$\rho W_I(x) = \max_{c} \left( H(c) + \frac{\partial W_I(x)}{\partial x} g(x, c) \right). \tag{15}$$

Combining (15) and (14), we obtain

$$\rho W_I(x) = \max_{\{c_i\}} \left( A * ln(c_i) + A * \sum_{j \neq i} ln(c_j) + nB + \frac{\partial W_I(x)}{\partial x}(-c_i - \sum_{j \neq i} c_j) \right). \tag{16}$$

Suppose the Bellman function $W_I$ has the form

$$W_I = A_I ln(x) + B_I. \tag{17}$$

Then we get

$$\frac{\partial W_I(x)}{\partial x} = \frac{A_I}{x}. \tag{18}$$

Differentiating the right-hand side of (16) with respect to $c_i$, we obtain optimal strategies

$$c_i^I = \frac{A}{\frac{\partial W_I(x)}{\partial x}}. \tag{19}$$

Using (18 and (19), we get

$$\dot{x} = -\frac{nA}{A_I}x. \tag{20}$$

Substituting (31), (19) and (20) in (16), we have

$$\rho A_I ln(x) + \rho B_I = -nA + nB + Anln(A) - Anln(A_I) + Anln(x). \tag{21}$$

The result is:

$$A_I = \frac{An}{\rho}; \tag{22}$$
$$B_I = \frac{Bn}{\rho} - \frac{An}{\rho} - \frac{Anln(n)}{\rho} + \frac{Anln(\rho)}{\rho}.$$

From (14) and (22) it follows that

$$W_I(x) = \frac{An}{\rho}ln(x) + \frac{Bn}{\rho} - \frac{An}{\rho} - \frac{Anln(n)}{\rho} + \frac{Anln(\rho)}{\rho}. \tag{23}$$

Then we get the optimal strategies

$$c_i^I = \frac{\rho}{n}x, \qquad i = 1, \dots, n. \tag{24}$$

Finally, we have optimal trajectory and optimal controls

$$\begin{aligned} x^I(t) &= x_o * e^{-\rho(t-t_0)}; \\ c_i^I(t) &= \frac{x_0\rho}{n}e^{-\rho(t-t_0)}, \end{aligned} \tag{25}$$

and

$$\begin{aligned} &V(I, x^I(\vartheta)) = W_I(x^I(\vartheta)) = \\ &= \frac{An}{\rho}ln(x^I) + \frac{Bn}{\rho} - \frac{An}{\rho} - \frac{Anln(n)}{\rho} + \frac{Anln(\rho)}{\rho} = \\ &= \frac{An}{\rho}ln(x_0) - An(\vartheta - t_0) + \frac{Bn}{\rho} - \frac{An}{\rho} - \frac{Anln(n)}{\rho} + \frac{Anln(\rho)}{\rho}. \end{aligned} \tag{26}$$

As you can see the optimal trajectory $x^I(t)$ satisfies Lyapunov stability condition. Let $\vartheta = t_0$. Then

$$V(I, x_0) = W_I(x_0) = \frac{An}{\rho}ln(x_0) + \frac{Bn}{\rho} - \frac{An}{\rho} - \frac{Anln(n)}{\rho} + \frac{Anln(\rho)}{\rho}. \tag{27}$$

**Step 2.** Now let us find a Feedback Nash equilibrium. The Bellman function for player $i$ is as follows:

$$W_i(x) = \max_{c_i} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} (Aln(c_i(\tau)) + B)\rho * e^{-\rho(t-\vartheta)} d\tau dt. \tag{28}$$

The initial state is

$$x(\vartheta) = x^I(\vartheta). \tag{29}$$

Now the HJB equation (15) has the form

$$\rho W_i(x) = \max_{c_i} \left( Aln(c_i) + B + \frac{\partial W_I(x)}{\partial x}(-\sum_{i=1}^{n} c_i) \right). \tag{30}$$

We find $W_i$ in the form

$$W_i = A_N ln(x) + B_N. \tag{31}$$

As before we get

$$A_N = \frac{A}{\rho}; \tag{32}$$

$$B_N = \frac{B}{\rho} - \frac{An}{\rho} + \frac{Aln(\rho)}{\rho}.$$

$$c_i^N = \rho x, \qquad i = 1, \ldots, n; \tag{33}$$

$$x^N(t) = x^I(\vartheta) * e^{-n\rho(t-\vartheta)}; \tag{34}$$

$$c_i^N(t) = \rho x^I(\vartheta) * e^{-n\rho(t-\vartheta)};$$

$$V(\{i\}, x^I(\vartheta)) = W_i(x^I(\vartheta)) = \frac{A}{\rho} ln(x^I(\vartheta)) + \frac{B}{\rho} - \frac{An}{\rho} + \frac{Aln(\rho)}{\rho}. \tag{35}$$

Let $\vartheta = t_0$. Then

$$V(\{i\}, x_0) = W_i(x_0) = \frac{A}{\rho} ln(x_0) + \frac{B}{\rho} - \frac{An}{\rho} + \frac{Aln(\rho)}{\rho}. \tag{36}$$

The main results obtained by steps 1,2 firstly had been published in E.J. Dockner, S. Jorgensen et al., 2000 for a case of $A = 1$, $B = 0$.

**Step 3.** Let us consider a coalition $K \subset I$, $|K| = k$, $|I \setminus K| = n - k$. For this case we have the Bellman function:

$$W_K(x) = \max_{c_i, i \in K} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} (A \sum_{i \in K} ln(c_i(\tau)) + kB)\rho * e^{-\rho(t-\vartheta)} d\tau dt. \tag{37}$$

The initial state is

$$x(\vartheta) = x^I(\vartheta). \tag{38}$$

Let us recall, that the left-out players $i \in I \setminus K$ will use feedback Nash strategies (45).

In the same way, we get

$$x^K(t) = x^I(\vartheta) * e^{-(n-k+1)\rho(t-\vartheta)}; \tag{39}$$
$$c_i^K(t) = \frac{\rho}{k} x^I(\vartheta) * e^{-(n-k+1)\rho(t-\vartheta)};$$

$$V(K, x^I(\vartheta)) = W_K(x^I(\vartheta)) = \tag{40}$$
$$= \frac{Ak}{\rho} ln(x^I(\vartheta)) + \frac{kB}{\rho} - \frac{Ak}{\rho} - \frac{Ak(n-k)}{\rho} - \frac{Ak}{\rho} ln(k) + \frac{Akln(\rho)}{\rho}.$$

Let $\vartheta = t_0$. Then

$$V(K, x_0) = W_K(x_0) = \frac{Ak}{\rho} ln(x_0)) + \frac{kB}{\rho} - \frac{Ak}{\rho} - \frac{Ak(n-k)}{\rho} - \frac{Ak}{\rho} ln(k) + \frac{Akln(\rho)}{\rho}. \tag{41}$$

Thus we have constructed the characteristic function $V(K, x_0)$,$K \subseteq I$ (see (27),(41)).

**Proposition 1.** *Suppose the characteristic function $V(K, x_0)$,$K \subseteq I$ is given by (27),(41). Then $V(K, x_0)$ is superadditive.*

To prove this proposition, we need following lemma.

**Lemma 1.** *Let $s_1 \geq 1, s_2 \geq 1$. Then*

$$s_1 ln(s_1) + s_2 ln(s_2) + 2s_1 s_2 \geq (s_1 + s_2) ln(s_1 + s_2). \tag{42}$$

This lemma can be proved by standard methods. It is easily shown that the left-hand side is fast increasing than the right-hand side.

Now proof of Proposition1 is by direct calculations.

Finally, we get the Shapley Value in our example:

$$Sh_i(x(t)) = \frac{V(I,x)}{n} = \frac{A}{\rho} ln(x) + \frac{B}{\rho} - \frac{A}{\rho} - \frac{Aln(n)}{\rho} + \frac{Aln(\rho)}{\rho}; \tag{43}$$

$$Sh_i(x_0) = \frac{V(I,x_0)}{n} = \frac{A}{\rho} ln(x_0) + \frac{B}{\rho} - \frac{A}{\rho} - \frac{Aln(n)}{\rho} + \frac{Aln(\rho)}{\rho}. \tag{44}$$

**II.** Consider a problem with utility function

$$h(c_i) = A \frac{c_i^{1-\eta}}{1-\eta} + B, \quad \eta \neq 1.$$

We find $W$ in form $W = Ax^{1-\eta} + B$. We get the optimal strategies

$$c_i^I = \frac{\rho}{\eta n} x, \qquad i = 1, \ldots, n.$$

Then we get optimal trajectory and optimal controls

$$x^I(t) = x_o \, e^{-\frac{\rho(t-t_0)}{\eta}};$$
$$c_i^I(t) = \frac{x_0 \, \rho}{n\eta} e^{-\frac{\rho(t-t_0)}{\eta}}.$$

Then we get the values

$$V(I,x(\vartheta)) = W_i(x) = \left(\frac{n\eta}{\rho}\right)^{\eta}\frac{A}{1-\eta}x(\vartheta)^{1-\eta}+\frac{nB}{\rho};$$
$$V(I,x^I(\vartheta)) = W_I(x^I(\vartheta)) = \left(\frac{n\eta}{\rho}\right)^{\eta}\frac{A}{1-\eta}x^I(\vartheta)^{1-\eta}+\frac{nB}{\rho} =$$
$$= \left(\frac{n\eta}{\rho}\right)^{\eta}\frac{A}{1-\eta}x_0^{1-\eta}e^{-\frac{\rho(1-\eta)(\vartheta-t_0)}{\eta}}+\frac{nB}{\rho}.$$

Then we get the Nash feedback strategies and trajectory

$$c_i^N = \frac{\rho x}{(1-n+n\eta)}, \qquad i=1,\ldots,n; \tag{45}$$

$$x^N(t) = x^I(\vartheta)\,e^{-\frac{n\rho}{(1-n+n\eta)}(t-\vartheta)};$$
$$c_i^N(t) = \frac{\rho x^I(\vartheta)}{(1-n+n\eta)}\,e^{-\frac{n\rho}{(1-n+n\eta)}(t-\vartheta)}.$$

Obviously, $c_i^N > 0$ if $\eta > (1-1/n)$. So we'll consider the game under condition $\eta > (1-1/n)$. Otherwise feedback Nash strategies as rates of resource extraction make no sense.

So, we get the value

$$V(\{i\},x(\vartheta)) = W_i(x) = \left(\frac{(1-n+n\eta)}{\rho}\right)^{\eta}\frac{A}{1-\eta}x(\vartheta)^{1-\eta}+\frac{B}{\rho};$$
$$V(\{i\},x^I(\vartheta)) = W_i(x^I(\vartheta)) = \left(\frac{(1-n+n\eta)}{\rho}\right)^{\eta}\frac{A}{1-\eta}x^I(\vartheta)^{1-\eta}+\frac{B}{\rho} =$$
$$= \left(\frac{(1-n+n\eta)}{\rho}\right)^{\eta}\frac{A}{1-\eta}x_0^{1-\eta}e^{-\frac{\rho(1-\eta)}{\eta}(\vartheta-t_0)}+\frac{B}{\rho}.$$

In the same way, we get "optimal" for coalition $K$ controls

$$c_i^K = \frac{\rho(1-k+k\eta)x}{k\eta(1-n+n\eta)}, \qquad i=1,\ldots,n;$$

and the value of coalition payoff

$$V(K,x(\vartheta)) = W_K(x(\vartheta)) = \left(\frac{k\eta(1-n+n\eta)}{\rho(1-k+k\eta)}\right)^{\eta}\frac{A}{1-\eta}x(\vartheta)^{1-\eta}+\frac{kB}{\rho};$$

Thus we have constructed the characteristic function $V(K,x_0)$,$K \subseteq I$. We can prove the following proposition.

**Proposition 2.** *The characteristic function $V(K,x_0)$,$K \subseteq I$ is superadditive.*

To prove this proposition, we need following lemma.

**Lemma 2.** *Let $s_1 \geq 1, s_2 \geq 1$. Then*

$$\frac{(s_1+s_2)^{\eta}}{(1-(s_1+s_2)+(s_1+s_2)\eta)^{\eta}} \geq \frac{s_1^{\eta}}{(1-s_1+s_1\eta)^{\eta}}+\frac{s_2^{\eta}}{(1-s_2+s_2\eta)^{\eta}}. \tag{46}$$

This lemma can be proved by standard methods. It is easily shown that the left-hand side is fast increasing than the right-hand side.

We get the following components of the Shapley Value:

$$Sh_i(x(t)) = \frac{A}{1-\eta}\left(\frac{x(t)}{n}\right)^{1-\eta}\left(\frac{\rho}{\eta}\right)^{-\eta} + \frac{B}{\rho}; \tag{47}$$

$$Sh_i(x_0) = \frac{A}{1-\eta}\left(\frac{x_0}{n}\right)^{1-\eta}\left(\frac{\rho}{\eta}\right)^{-\eta} + \frac{B}{\rho}. \tag{48}$$

## 3. Time-consistency Problem

It would seem that the superadditivity of the characteristic function $V(x_0, S)$ should provide saving the cooperation of players during the game, but it is not so. Really, moving along the optimal trajectory $x^*(t)$ players enter into subgames with current initial states, in which the same player has different possibilities. Therefore, in some moment it may happen, that the solution of the current game will be not optimal in sense of originally selected optimality principle and the desire to operate jointly can expose the threat in some moment $\vartheta$. It means that the optimality principle may loose time-consistency.

Let $Sh = \{Sh_i\}$ is the Shapley Value in the whole game.

**Definition 1.** Consider vector function $\gamma(t)\} = \{\gamma_i(t)\}$, such that

$$Sh_i = \int_{t_0}^{\infty}\int_{t_0}^{t}\gamma_i(\tau)d\tau dF(t). \tag{49}$$

Vector function $\gamma(t) = \{\gamma_i(t)\} \geq 0$ is called the imputation distribution procedure (IDP).

IDP determines a rule, according to which the components of the Shapley Value are distributed on interval $[t_0, t]$, where random value $t$ is the final time instant of the game.

Payoff obtained by player $i$ on an interval $[t_0, \vartheta)$ is denoted by $\alpha_i(\vartheta)$:

$$\alpha_i(\vartheta) = \int_{t_0}^{\vartheta}\int_{t_0}^{t}\gamma_i(\tau)d\tau dF(t). \tag{50}$$

**Definition 2.** The Shapley Value is time-consistent (TCSV), if exists IDP $\{\gamma_i(\tau)\}\geq 0$, $\tau \in [t_0, \infty)$, such that the vector $\bar{S}h^{\vartheta} = \{\bar{S}h_i^{\vartheta}\}$ (expected payoff in subgame $\Gamma(x^*(\vartheta))$, calculated by the formula

$$\bar{S}h_i^{\vartheta} = \frac{1}{1-F(\vartheta)}\int_{\vartheta}^{\infty}\int_{\vartheta}^{t}\gamma_i(\tau)d\tau dF(t), \quad i = 1, \ldots, n. \tag{51}$$

belongs to the same optimality principle in the subgame , i.e. $\bar{S}h^{\vartheta}$ is the Shapley Value in subgame $\Gamma(x^*(\vartheta))$.

It can be easily seen that TCSV is equivalent to the following formula for the imputation $\bar{S}h$:

$$\bar{Sh_i} = \alpha_i(\vartheta) + (1 - F(\vartheta))\left(\int_{t_0}^{\vartheta}\gamma_i(\tau)d\tau + \bar{Sh_i}^{\vartheta}\right). \tag{52}$$

Differentiating (52) with respect to $t$ we get the formula for IDP:

$$\gamma_i(\vartheta) = \frac{F'(\vartheta)}{1-F(\vartheta)} S\bar{h}_i^{\vartheta} - (S\bar{h}_i^{\vartheta})'. \tag{53}$$

Let us consider the case of exponential distribution $f(\vartheta) = F'(\vartheta) = \rho e^{\lambda t}$. Then we get the formula

$$\gamma_i(\vartheta) = \rho S\bar{h}_i^{\vartheta} - (S\bar{h}_i^{\vartheta})'. \tag{54}$$

This formula coincides with formula for IDP for the problem with infinite time horizon and discount rate $\rho$ had been derived in Petrosjan L. A., Zaccour G.,2003. So, we had another validation of the idea that the problem with infinite time horizon and discounting of the payoffs supplies the same result as the problem with random game duration and exponential distribution.

It is important that $\gamma_i(\vartheta), \quad i = 1, \ldots, n$ should be nonnegative. As it is impossible to guarantee this in general, not all optimality principles are time- consistent.

Remark. We can change the restriction for IDP $\gamma_i(\vartheta), \quad i = 1, \ldots, n$ to be nonnegative for the problem of minimization of the total costs (but not maximization of the total payoffs), because in this case IDP means a rule of distribution of the players costs during the game.

### 3.1. New characteristic function

Define the following function

$$\bar{V}(x_0, S) = \int_{t_0}^{\infty} \int_{t_0}^{t} V(x^*(\tau), S) \frac{\sum_{i=1}^{n} h_i(x^*(\tau))}{V(x^*(\tau), N)} d\tau dF(t), \\ S \subseteq N. \tag{55}$$

It is clear that

$$\bar{V}(x_0, N) = V(x_0, N), \qquad \bar{V}(x_0, \emptyset) = 0.$$

Also we have

$$\bar{V}(x_0, S_1 \cup S_2) \geq \bar{V}(x_0, S_1) + \bar{V}(x_0, S_2).$$

It follows from the superadditivity of the characteristic function $V(x_0, S)$. Thus, following lemma is true.

**Lemma 3.** *Function $\bar{V}(x_0, S)$ is the characteristic function in the game $\Gamma(x_0)$.* □

Similarly, it is not difficult to show, that the function

$$\bar{V}(x^*(\vartheta), S) = \frac{1}{1-F(\vartheta)} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} V(x^*(\tau), S) \frac{\sum_{i=1}^{n} h_i(x^*(\tau))}{V(x^*(\tau), N)} d\tau dF(t).$$

is the characteristic function in the subgame $\Gamma(x^*(\vartheta))$.

### 3.2. Regularization

Introduce new optimality principles, which are based on classical optimality principles and are always time-consistent. Remark that the regularization will be correct only for the case of nonnegative instantaneous payoffs $h_i(\cdot) \geq 0,\ i = 1, \ldots, n$.

Consider new IDP

$$\bar{\gamma}_i(\vartheta) = {Sh_i}^{\vartheta} \frac{\sum_{i=1}^{n} h_i(x^*(\vartheta))}{V(x^*(\vartheta), N)}. \tag{56}$$

It is clear, that $\bar{\gamma}_i(\vartheta) \geq 0, \quad \forall\, \vartheta, \quad i = 1, \ldots, n.$

REMARK 1. It is easy to show, that sum of all new IDP components (56) at the intermediate time instant $\vartheta$, $\vartheta \in [t_0, \infty)$ is equal to sum of all instantaneous payoffs at the time instant $\vartheta$, i.e.

$$\sum_{i=1}^{n} \gamma_i(\vartheta) = \sum_{i=1}^{n} h_i(x^*(\vartheta)), \tag{57}$$

because $\sum_{i=1}^{n} {Sh_i}^{\vartheta} = V(x^*(\vartheta), N)$. Thus, according to new IDP at the each time instant we divide the sum of the instantaneous payoffs of the players obtained at the same time instant.

Define vector $\bar{Sh} = \{\bar{Sh_i}\}$ by formula

$$\bar{Sh_i} = \int_{t_0}^{\infty} \int_{t_0}^{t} {Sh_i}^{\tau} \frac{\sum_{i=1}^{n} h_i(x^*(\tau))}{V(x^*(\tau), N)} d\tau dF(t). \tag{58}$$

It is not difficult to show, that $\sum_{i=1}^{n} \bar{Sh_i} = V(x_0, N) = \bar{V}(x_0, N)$. Hence, $\bar{Sh} = \{\bar{Sh_1}, \ldots, \bar{Sh_n}\}$ is the allocation of the total expected payoff.

**Proposition 3.** *The allocation of the total expected payoff $\bar{Sh} = \{\bar{Sh_i}\}$ (58) is the imputation, i.e. $\bar{Sh_i} \geq \bar{V}(x_0, \{i\})$.*

To prove this proposition note, that $Sh^{\tau}$ is the imputation in the subgame $\Gamma(x^*(\tau))$, then ${Sh_i}^{\tau} \geq V(x^*(\tau), \{i\})$. Hence

$$\bar{Sh_i} = \int_{t_0}^{\infty} \int_{t_0}^{t} {Sh_i}^{\tau} \frac{\sum_{i=1}^{n} h_i(x^*(\tau))}{V(x^*(\tau), N)} d\tau dF(t) \geq$$
$$\geq \int_{t_0}^{\infty} \int_{t_0}^{t} V(x^*(\tau), \{i\}) \frac{\sum_{i=1}^{n} h_i(x^*(\tau))}{V(x^*(\tau), N)} d\tau dF(t) = \bar{V}(x_0, \{i\}).$$

**Theorem 1.** *The sregularized Shapley Value $\bar{Sh}$ is time-consistent optimality principle.*

Introduce new imputation in the subgame by formula

$$\bar{Sh_i}^{\vartheta} = \frac{1}{1 - F(\vartheta)} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} {Sh_i}^{\tau} \frac{\sum_{i=1}^{n} h_i(x^*(\tau))}{V(x^*(\tau), N)} d\tau dF(t).$$

Because of

$$\bar{Sh_i} = \int_{t_0}^{\vartheta} \int_{t_0}^{t} {Sh_i}^{\tau} \frac{\sum_{i=1}^{n} h_i(x^*(\tau))}{V(x^*(\tau), N)} d\tau dF(t) + \int_{\vartheta}^{\infty} \int_{t_0}^{t} {Sh_i}^{\tau} \frac{\sum_{i=1}^{n} h_i(x^*(\tau))}{V(x^*(\tau), N)} d\tau dF(t),$$

we get

$$\bar{Sh}_i = \int_{t_0}^{\vartheta}\int_{t_0}^{t} Sh_i{}^{\tau}\frac{\sum_{i=1}^{n} h_i(x^*(\tau))}{V(x^*(\tau),N)}d\tau dF(t) + (1-F(\vartheta))\int_{t_0}^{\vartheta} Sh_i{}^{\tau}\frac{\sum_{i=1}^{n} h_i(x^*(\tau))}{V(x^*(\tau),N)}d\tau +$$
$$+\int_{\vartheta}^{\infty}\int_{\vartheta}^{t} Sh_i{}^{\tau}\frac{\sum_{i=1}^{n} h_i(x^*(\tau))}{V(x^*(\tau),N)}d\tau dF(t). =$$
$$= \int_{t_0}^{\vartheta}\int_{t_0}^{t}\bar{\gamma}_i(\tau)d\tau dF(t) + (1-F(\vartheta))\Big(\int_{t_0}^{\vartheta}\bar{\gamma}_i(\tau)d\tau + \bar{Sh}_i{}^{\vartheta}\Big).$$

Thus, the imputation $\bar{Sh}$ has the form (52). Therefore, the imputation $\bar{Sh}^{\vartheta}$ in the subgame starting at the moment $\vartheta$ belongs to the same optimality principle.

**Theorem 2.** *The Shapley Value calculated for the characteristic function $\bar{V}(x_0,S)$ is time-consistent.*

We have the following formula for the Shapley Value:

$$Sh_i = \sum_{\substack{S\subset N\\ i\in S}} \frac{(n-s)!(s-1)!}{n!}[V(S) - V(S\backslash\{i\})], \quad i=1,\dots,n. \tag{59}$$

Therefore, the components of the Shapley Value in $\Gamma(x_0)$ for the characteristic function $\bar{V}(x_0,S)$ are computed by the formula

$$\bar{Sh}_i(x_0) = \sum_{\substack{S\subset N\\ i\in S}} \frac{(n-s)!(s-1)!}{n!}[\bar{V}(x^*(\tau),S) - \bar{V}(x^*(\tau),S\backslash\{i\})]. \tag{60}$$

Call $\bar{Sh}(x_0)$ —regularized Shapley Value (RSV). From (55) and (60) we get the following form of RSV:

$$\bar{Sh}_i(x_0) = \int_{t_0}^{\infty}\int_{t_0}^{t}\frac{(n-s)!(s-1)!}{n!}\sum_{S\subset N}[V(x^*(\tau),S) -$$
$$- V(x^*(\tau),S\backslash\{i\})]\frac{\sum_{i=1}^{n} h_i(x^*(\tau))}{V(x^*(\tau))}\,d\tau\, dF(t) =$$
$$= \int_{t_0}^{\infty}\int_{t_0}^{t} Sh_i(x^*(\tau))\frac{\sum_{i=1}^{n} h_i(x^*(\tau))}{V(x^*(\tau))}\,d\tau\, dF(t).$$

Similarly, it is not difficult to show, that the components of the Shapley Value $\bar{Sh}(x^*(\tau))$ in the subgame $\Gamma(x^*(\tau))$ are equal to

$$\bar{Sh}_i(x^*(\tau)) = \frac{1}{1-F(\vartheta)}\int_{\vartheta}^{\infty}\int_{\vartheta}^{t} Sh_i(x^*(\tau))\frac{\sum_{i=1}^{n} h_i(x^*(\tau))}{V(x^*(\tau))}\,d\tau\, dF(t).$$

Define IDP $\bar{\gamma}(\tau)$, $\tau\in[t_0,t]$ as $\bar{\gamma}_i(\tau) = Sh_i(x^*(\tau))\frac{\sum_{i=1}^{n} h_i(x^*(\tau))}{V(x^*(\tau))}$. We get

$$\bar{Sh}_i(x_0) = \int_{t_0}^{\vartheta}\int_{t_0}^{t}\bar{\gamma}_i(\tau)d\tau dF(t) + (1-F(\vartheta))\Big(\int_{t_0}^{\vartheta}\bar{\gamma}_i(\tau)d\tau + \bar{Sh}_i(x^*(\vartheta))\Big).$$

Thus, Shapley value for the characteristic function $\bar{V}(x_0,S)$ is time- consistent.

Then for regularization of the Shapley Value we may use 2 ways such that to use new IDP $\bar{\gamma}(\tau)$ or to calculate the regularized Shapley Value by new characteristic function $\bar{V}(x_0,S)$.

### 3.3. Time-consistency problem in game-theoretical problem of nonrenewable resource extraction

Let us investigate the Shapley Value (44) for logarithmic utility function on time-consistency. We have to calculate the IDP by formula (54). We get:

$$\gamma_i(\vartheta) = Aln(x_0) - A\rho(\vartheta - t_0) + Aln(\rho) - Aln(n) + B. \tag{61}$$

It is obviously we cannot guarantee the nonnegativity of the IDP in (61). Then, the Shapley Value (44) is time-inconsistent. But we can not regularized it because the instantaneous function $h_i = ln(c_i)$ is not nonnegative function.

In the case of power utility function we have the Shapley Value (47). We use the formula (54) for IDP calculation. Then we get

$$\gamma_i(\vartheta) = \rho(\frac{n\eta}{\rho})^\eta A x_0^{1-\eta} e^{-\frac{\rho(\vartheta - t_0)}{\eta}} \frac{1}{\eta(1-\eta)} + \frac{B}{\rho}. \tag{62}$$

It is clear that IDP (62) is nonnegative for $\eta \in (0;1)$ and $B \geq 0$. Then the Shapley Value is time-consistent optimality principle for $\eta \in (0;1)$ and $B \geq 0$.

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# Dynamic Game-theoretic Model of Production Planning under Competition

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## 1. Introduction

We consider the classical model of production planning under competition (Petrosjan and Zenkevich, 2006), but now we are interested in the discrete-time case of it. Considering the model as a linear-quadratic discrete-time dynamic game, we find a feedback Nash equilibrium of it. Also we study the cooperative case of this game and obtain characteristic function and the Shapley value of it.

## 2. The Model

Consider a model of duopoly in which the players are the firms. There are two competing firms producing a single homogeneous product.

If a Firm $i$ produces the quantity $q_i$ units of the good, then the demand function of the product is given by

$$g(k) = a - [q_1(k) + q_2(k)], \quad a > 0.$$

And the cost function of Firm $i$ is

$$C(q_i) = cq_i + \frac{1}{2}q_i^2, \quad c > 0.$$

For the market price we have equation

$$p(k+1) = s(a - [q_1(t) + q_2(t)] - p(k)); \quad p(0) = p_0 > 0,$$

$$s \in [0, \infty).$$

Each firm tries to maximize its profit

$$J_i(q_i) = \sum_{k=0}^{\infty} \left(\frac{1}{1+\rho}\right)^k (p(k)q_i(k) - C(q_i(k))).$$

Where $\rho$ is a discount factor. Now to obtain more convenient system introduce

$$x_1(k) = \left(\frac{1}{1+\rho}\right)^{\frac{k}{2}} (p(k) - c),$$

$$x_2(k) = (s(a-c) - c)\left(\frac{1}{1+\rho}\right)^{\frac{k+1}{2}},$$

$$u_1(k) = \left(\frac{1}{1+\rho}\right)^{\frac{k}{2}} (q_1(k) - p(k) + c),$$

$$u_2(k) = \left(\frac{1}{1+\rho}\right)^{\frac{k}{2}} (q_2(k) - p(k) + c).$$

Then our system can be rewritten as

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} -3s\left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} & 1 \\ 0 & \left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{pmatrix} -s\left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} \\ 0 \end{pmatrix} u_1(k) +$$

$$+ \begin{pmatrix} -s\left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} \\ 0 \end{pmatrix} u_2(k),$$

$$J_i(u_i) = \sum_{k=0}^{\infty} (x^T(k) \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} x(k) - \frac{1}{2} u_i^2(k)),$$

$$x_0 = \begin{pmatrix} p_0 - c \\ \left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} (s(a-c) - c) \end{pmatrix}.$$

So, we obtain a linear-quadratic discrete-time dynamic game. And we shell try to find a feedback Nash equilibrium, the characteristic function and the Shapley value of this game. To find all these decisions we have to consider some results from the theory of linear-quadratic games.

## 3. Nash Equilibrium

Consider the class of $n$-person discrete-time dynamic games which are described by the state equation

$$x(k+1) = A(k)x(k) + \sum_{i=1}^{n} B_i(k) u_i(k), \tag{1}$$

$$k \geq k_0, \quad k_0 \in \mathcal{T}_+, \quad x(k_0) = x_0.$$

Here, $x$ is the $m$-dimensional state of the system, $u_i$ is a $r$-dimensional (control) variable player $i$ can manipulate, $x(k_0) = x_0$ is the arbitrarily chosen initial state of the system, $\mathcal{T}_+$ – is a set of non-negative integer numbers. $A(k), B_i(k) \in Z(\mathcal{T}_+)$ are matrices of appropriate dimensions, where $Z(\mathcal{T}_+)$ – is a set of real bounded on $\mathcal{T}_+$ matrices. Let $N = \{1, \ldots, n\}$. The performance criterion player $i \in N$ aims to maximize is

$$J_i = \sum_{k=k_0}^{\infty} (x^T(k) P_i(k) x(k) + u_i^T(k) R_i(k) u_i(k)), \tag{2}$$

where $P_i(k), R_i(k) \in Z(\mathcal{T}_+)$, $P_i(k) = P_i^T(k)$, $R_i(k) = R_i^T(k) \quad \forall i = 1, \ldots, n$.

We will assume that the players use feedback strategies, $u_i(k, x) = M_i(k)x(k)$, to control the system.

**Definition 1.** A set of feedback strategies

$$\{u_i(k, x) = M_i(k)x(k), \quad i = 1, \ldots, n\} \tag{3}$$

is called permissible, if the following conditions are satisfied.

1)$M_i(k) \in Z(\mathcal{T}_+) \quad \forall i = 1, \ldots, n.$
2) The resulting system described by

$$x(k+1) = (A(k) + \sum_{i=1}^{n} B_i(k) M_i(k)) x(k) \tag{4}$$

is uniformly asymptotically stable. (when $k \to \infty$).

Let $Q_+(\mathcal{T}_+) \subset Z(\mathcal{T}_+)$ – a set of positive matrices and $Q_-(\mathcal{T}_+) \subset Z(\mathcal{T}_+)$ – a set of negative matrices.

**Theorem 1.** *There exists a feedback Nash equilibrium for the game $\Gamma(k_0, x_0)$ if and only if the following conditions are satisfied.*

1. *The system of matrix equations*

$$\begin{cases} (A(k) + \sum_{i=1}^{n} B_i(k) M_i^{NE}(k))^T \Theta_i(k+1)(A(k) + \sum_{i=1}^{n} B_i(k) M_i^{NE}(k)) - \\ - \Theta_i(k) - P_i(k) - M_i^{NE}(k)^T R_i(k) M_i^{NE}(k) = 0 \\ M_i^{NE}(k) = -(-R_i(k) + B_i^T(k)\Theta_i(k+1)B_i(k))^{-1} B_i^T(k)\Theta_i(k+1) \times \\ \times (A(k) + \sum_{j \neq i} B_j(k) M_j^{NE}(k)), \quad i = 1, \ldots, n \end{cases}$$

*has the solution $\{M_i^{NE}(k),\, \Theta_i(k)\} \in Z(\mathcal{T}_+)$,with dimension $r \times m$ and $m \times m$ respectively, where $\Theta_i(k)$ is symmetric for all $i \in N$.*

2. *The set of strategies*

$$\{u_i = -(-R_i(k) + B_i^T(k)\Theta_i(k+1)B_i(k))^{-1} B_i^T(k) \times$$
$$\times \Theta_i(k+1)(A(k) + \sum_{j \neq i} B_j(k) M_j^{NE}(k)) x(k), \quad i = 1, \ldots, n\} \tag{5}$$

*is permissible.*

3. $$(-R_i(k) + B_i^T(k)\Theta_i(k+1)B_i(k)) \in Q_+(\mathcal{T}_+), \quad i = 1, \ldots, n.$$

*Then the set of strategies (5) is a feedback Nash equilibrium for the game $\Gamma(k_0, x_0)$. And*

$$J_i(u^{NE}) = -x_0^T \Theta_i(k_0) x_0, \quad i = 1, \ldots, n.$$

Now return to our model. Here

$$A = \begin{pmatrix} -3s\left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} & 1 \\ 0 & \left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} \end{pmatrix}, B_1 = B_2 = \begin{pmatrix} -s\left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} \\ 0 \end{pmatrix},$$

$$P_1 = P_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, R_1 = R_2 = -\frac{1}{2}, x_0 = \begin{pmatrix} p_0 - c \\ \left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} (s(a-c) - c) \end{pmatrix}.$$

To find a feedback Nash equilibrium, we use the Theorem1 and solve the system of matrix equations.

$$\begin{cases} (A+B_1M_1^{NE}+B_2M_2^{NE})^T\Theta_1(k+1)(A+B_1M_1^{NE}+B_2M_2^{NE})-\Theta_1(k)-P_1+ \\ +\dfrac{1}{2}(M_1^{NE})^TM_1^{NE}=0 \\ (A+B_1M_1^{NE}+B_2M_2^{NE})^T\Theta_2(k+1)(A+B_1M_1^{NE}+B_2M_2^{NE})-\Theta_2(k)-P_2+ \\ +\dfrac{1}{2}(M_2^{NE})^TM_2^{NE}=0 \\ M_1^{NE}(k)=-(\dfrac{1}{2}+B_1^T\Theta_1(k+1)B_1)^{-1}B_1^T\Theta_1(k+1)(A+B_2M_2^{NE}) \\ M_2^{NE}(k)=-(\dfrac{1}{2}+B_2^T\Theta_2(k+1)B_2)^{-1}B_2^T\Theta_2(k+1)(A+B_1M_1^{NE}). \end{cases}$$

Then the set of strategies $\{u_i^{NE}(k,x)=M_i^{NE}(k)x(k), i=1,\dots,n\}$ is a feedback Nash equilibrium for our game. And, for example, if $s=1$, $\left(\frac{1}{1+\rho}\right)^{\frac{1}{2}}=\frac{1}{15}$, then

$$u_1^{NE}(k,x)=\begin{pmatrix}0,014026995 & -0,06919932097\end{pmatrix}x(k),$$

$$u_2^{NE}(k,x)=\begin{pmatrix}0,014026995 & -0,06919932097\end{pmatrix}x(k),$$

$$J_1(u^{NE})=J_2(u^{NE})=-x_0^T\begin{pmatrix}-0,5211388670 & 0,1042843108 \\ 0,1042843108 & -0,5166690544\end{pmatrix}x_0,$$

$$q_1^{NE}(k,p)=q_2^{NE}(k,p)=1.14026995p(k)-0.4613288065a-2.062927563c.$$

## 4. Cooperative Case of the Game

Let

$$J^S=\sum_{i\in S}J_i(u).$$

**Theorem 2.** *When $r=1$ there exists the unique set of strategies $\{u_i^0=M_i^0(k)x,\quad i\in S\}$, maximizing $J^S$ when the set of strategies $\{u_j=\bar{M}_j(k)x,\quad j\notin S\}$ is fixed if the following conditions are satisfied.*

1. *The system of matrix equations*

$$\begin{cases} (A(k)+\sum_{j\notin S}B_j(k)\bar{M}_j(k)+\sum_{i\in S}B_i(k)M_i^0(k))^T\Theta_S(k+1)(A(k)+ \\ +\sum_{j\notin S}B_j(k)\bar{M}_j(k)+\sum_{i\in S}B_i(k)M_i^0(k))-\Theta_S(k)-\sum_{i\in S}P_i(k)- \\ -\sum_{i\in S}M_i^0(k)^TR_i(k)M_i^0(k)=0 \\ M_i^0(k)=-(-R_i(k)+B_i^T(k)\Theta_S(k+1)B_i(k))^{-1}B_i^T(k)\Theta_S(k+1)\times \\ \times(A(k)+\sum_{j\notin S}B_j(k)\bar{M}_j(k)+\sum_{j\in S,j\neq i}B_j(k)M_j^0(k))\quad \forall i\in S \end{cases}$$

*has the solution $\{M_i^0(k),\Theta_S(k)\}\in Z(\mathcal{T}_+)$,with dimension $r\times m$ and $m\times m$ respectively, where $\Theta_S(k)$ is symmetric.*

*2. The set of strategies*

$$\{u_j = \bar{M}_j(k)x, \quad j \notin S, \quad u_i = M_i^0(k)x(k), \quad i \in S\} \tag{6}$$

*is permissible.*

*3.*

$$(-R_i(k) + \sum_{j\in R} B_j^T(k)\Theta_S(k+1)B_i(k)) \in Q_+(\mathcal{T}_+) \quad \forall R \subset S, \forall i \in S,$$
$$B_j^T(k)\Theta_S(k+1)B_i(k) \in Q_-(\mathcal{T}_+) \quad \forall j,i \in S, i \neq j.$$

*Then the set of strategies (6) maximizes $J^S$.And*

$$J^S(u^0, k_0, x_0) = -x_0^T\Theta_S(k_0)x_0.$$

We will obtain the characteristic function $v(S,k_0)$ under the following rule

$$v(S,k_0) = \max_{u_i, i\in S} J^S(u^{NE}/u^S),$$

where $(u^{NE}/u^S) = \{u_j^{NE}, j \notin S, \quad u_i, i \in S\}$. In general, it can be non-superadditive.

According to the theorem2 $v(N,k_0) = J^N(u^0,k_0,x_0) = -x_0^T\Theta_N(k_0)x_0$, where $\Theta_N(k_0)$ we can find from the following system of matrix equations

$$\begin{cases} (A + B_1M_1^N + B_2M_2^N)^T\Theta_{1,2}(k+1)(A + B_1M_1^N + B_2M_2^N) - \Theta_{1,2}(k) - P_1 - \\ - P_2 + (M_1^N)^TM_1^N = 0 \\ M_1^N(k) = -(\dfrac{1}{2} + B_1^T\Theta_{1,2}(k+1)B_1)^{-1}B_1^T\Theta_{1,2}(k+1)(A + B_2M_2^N) \\ M_2^N(k) = -(\dfrac{1}{2} + B_2^T\Theta_{1,2}(k+1)B_2)^{-1}B_2^T\Theta_{1,2}(k+1)(A + B_1M_1^N). \end{cases}$$

And, for example, if $s = 1$, $\left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} = \frac{1}{15}$, then

$$u_1^N = u_2^N = \begin{pmatrix} 0,02832460 & -0,1397247874 \end{pmatrix} x(k),$$

$$J^N = -x_0^T \begin{pmatrix} -1,042486890 & 0,2095871811 \\ 0,2095871811 & -1,038317865 \end{pmatrix} x_0,$$

$$v(N,k_0) = -x_0^T \begin{pmatrix} -1,042486890 & 0,2095871811 \\ 0,2095871811 & -1,038317865 \end{pmatrix} x_0.$$

In our game

$$v(1,k_0) = v(2,k_0) = J_1(u^{NE}) = J_2(u^{NE}) =$$

$$= -x_0^T \begin{pmatrix} -0,5211388670 & 0,1042843108 \\ 0,1042843108 & -0,5166690544 \end{pmatrix} x_0,$$

$$\varphi^{Shapley} = \left(\frac{v(N,k_0)}{2}; \frac{v(N,k_0)}{2}\right).$$

## Appendix

### 1. The proof of the Theorem 1.

*Necessity.* Let $u^{NE} = (u_1^{NE}, \dots, u_n^{NE})$ is the Nash equilibrium. Then

$$J_i(u^{NE}/u_i) \le J_i(u^{NE}), \quad \text{for all} \quad i = 1, \dots, n.$$

So $u_i^{NE}$ is the optimal control in the following problem:

$$x(k+1) = (A(k) + \sum_{j \ne i} B_j(k) M_j^{NE}(k))x(k) + B_i(k)u_i(k)$$

with initial state $x(k_0) = x_0$ and functional $J_i$. In (Smirnov, 1997) conditions for existence of the unique optimal control for such problems are provided. Then, according to (Smirnov, 1997)

$$\{u_i = -(-R_i(k) + B_i^T(k)\Theta_i(k+1)B_i(k))^{-1}B_i^T(k)\Theta_i(k+1)(A(k)+$$
$$+\sum_{j \ne i} B_j(k)M_j^{NE}(k))x(k), \quad i = 1, \dots, n\},$$

where $\Theta_i(k)$ – is symmetric bounded matrix with dimension $m \times m$,and

$$(A(k) + \sum_{j \ne i} B_j(k)M_j^{NE}(k))^T\Theta_i(k+1)(A(k) + \sum_{j \ne i} B_j(k)M_j^{NE}(k)) - \Theta_i(k)-$$
$$- P_i(k) - (A(k) + \sum_{j \ne i} B_j(k)M_j^{NE}(k))^T\Theta_i(k+1)B_i(k)(-R_i(k) + B_i^T(k)\Theta_i(k)\times$$
$$\times B_i(k))^{-1}B_i^T(k)\Theta_i(k+1)(A(k) + \sum_{j \ne i} B_j(k)M_j^{NE}(k)) = 0,$$

$$(-R_i(k) + B_i^T(k)\Theta_i(k+1)B_i(k)) \in Q_+(\mathcal{T}_+).$$

Since it holds for all $i$, it is not difficult to obtain conditions of the theorem.
*Sufficiency.* Substituting the set of strategies $\{u^{NE}/u_i\}$ in the system (1) , we obtain the system with one control :

$$x(k+1) = (A(k) + \sum_{j \ne i} B_j(k)M_j^{NE}(k))x(k) + B_i(k)u_i(k). \tag{7}$$

And for $u_i$ exist such $M_i(k)$ and $\Theta_i(k)$ that

$$(A(k) + \sum_{j \ne i} B_j(k)M_j^{NE}(k))^T\Theta_i(k+1)(A(k) + \sum_{j \ne i} B_j(k)M_j^{NE}(k)) - \Theta_i(k)-$$
$$- P_i(k) - (A(k) + \sum_{j \ne i} B_j(k)M_j^{NE}(k))^T\Theta_i(k+1)B_i(k)(-R_i(k) + B_i^T(k)\Theta_i(k)\times$$
$$\times B_i(k))^{-1}B_i^T(k)\Theta_i(k+1)(A(k) + \sum_{j \ne i} B_j(k)M_j^{NE}(k)) = 0,$$

$$\{u_i = -(-R_i(k) + B_i^T(k)\Theta_i(k+1)B_i(k))^{-1}B_i^T(k)\Theta_i(k+1)(A(k)+$$

$$+\sum_{j\neq i} B_j(k)M_j^{NE}(k))x(k)\},$$

$$(-R_i(k)+B_i^T(k)\Theta_i(k+1)B_i(k))\in Q_+(\mathcal{T}_+).$$

Then according to (Smirnov, 1997), $u_i(k)$ – is the optimal control for the system (7) with functional $J_i$, i.e.

$$J_i(u^{NE}/u_i)\leq J_i(u^{NE}).$$

Since it holds for all $i$, the set of stratgies (5) is the Nash equilibrium.

**1. The proof of the Theorem 2.**

Let

$$u(k,x):\{u_j=\bar{M}_j(k)x, j\notin S,\quad u_i=M_i(k)x(k), i\in R\subset S,$$

$$u_i=M_i^0(k)x(k), i\in S\setminus R\}$$

– permissible set of strategies.

Let

$$\widetilde{A}(k)=A(k)+\sum_{j\notin S} B_j(k)\bar{M}_j(k),$$

$$\widetilde{F}(k,M^0)=\widetilde{A}(k)+\sum_{i\in S} B_i(k)M_i^0(k),$$

$$V_0(k,x(k))=x^T(k)\Theta_S(k)x(k),$$

$$z_0(k,x(k),u(k,x))=V_0(k+1,x(k+1))-V_0(k,x(k))-\sum_{i\in S} w_i(k,x(k),u(k,x)).$$

Since

$$z_0(k,x(k),u_0(k,x))=x^T(k)(A(k)+\sum_{j\notin S} B_j(k)\bar{M}_j(k)+\sum_{i\in S} B_i(k)M_i^0(k))^T\times$$

$$\times\Theta_S(k+1)(A(k)+\sum_{j\notin S} B_j(k)\bar{M}_j(k)+\sum_{i\in S} B_i(k)M_i^0(k))-\Theta_S(k)-\sum_{i\in S} P_i(k)-$$

$$-\sum_{i\in S} M_i^0(k)^T R_i(k)M_i^0(k))x(k),$$

then

$$z_0(k,x(k),u_0(k,x))=0.$$

Let

$$M_i^{'}=M_i-M_i^0.$$

$$z_0(k,x(k),u(k,x))=z_0(k,x(k),u_0(k,x))+x^T(k)(\sum_{i\in R}(M_i^{'T}B_i^T(k))\Theta_S(k+1)\times$$

$$\times\sum_{i\in R}(B_i(k)M_i^{'})-\sum_{i\in R} M_i^{'T}R_i(k)M_i^{'}))x(k)+2x^T(k)\sum_{i\in R}(M_i^{'T}(B_i^T(k)\Theta_S(k+1)\times$$

$$\times\widetilde{F}(k,M^0)-R_i(k)M_i^0))x(k),$$

and since

$$(\widetilde{F}^T(k, M^0)\Theta_S(k+1)B_i(k) - (M_i^0)^T(k)R_i(k)) \equiv 0 \forall i \in S,$$

then

$$z_0(k, x(k), u(k)) = x^T(k)(\sum_{i\in R}(M_i^{'T}B_i^T(k))\Theta_S(k+1)\sum_{i\in R}(B_i(k)M_i^{'})-$$

$$-\sum_{i\in R}M_i^{'T}R_i(k)M_i^{'}))x(k).$$

Notice that

$$x^T(k)(\sum_{i\in R}(M_i^{'T}B_i^T(k))\Theta_S(k+1)\sum_{i\in R}(B_i(k)M_i^{'}) - \sum_{i\in R}M_i^{'T}R_i(k)M_i^{'}))x(k) =$$

$$= x^T(k)(\sum_{i\in R}M_i^{'T}(\sum_{j\in R}B_j^T(k)\Theta_S(k+1)B_i(k) - R_i(k))M_i^{'}) - \sum_{i\in R}(M_i^{'T}\times$$

$$\times \sum_{j\in R, i\neq j}(B_j^T(k))\Theta_S(k+1)B_i(k)M_i^{'}) + \sum_{i,j\in R, i\neq j}(M_i^{'T}B_i^T(k)\Theta_S(k+1)B_j(k)M_j^{'}))x(k).$$

Where

$$x^T(k)(-\sum_{i\in R}(M_i^{'T}\sum_{j\in R, i\neq j}(B_j^T(k))\Theta_S(k+1)B_i(k)M_i^{'})+$$

$$+\sum_{i,j\in R, i\neq j}(M_i^{'T}B_i^T(k)\Theta_S(k+1)B_j(k)M_j^{'}))x(k) = -\sum_{i,j\in R, i\neq j}(x^T(k)(M_i - M_j)^{'T}\times$$

$$\times B_i^T(k)\Theta_S(k+1)B_j(k)(M_i - M_j)x(k)).$$

I.e.

$$z_0(k, x(k), u(k,x)) = x^T(k)(\sum_{i\in R}M_i^{'T}(\sum_{j\in R}B_j^T(k)\Theta_S(k+1)B_i(k) - R_i(k))M_i^{'})x(k)-$$

$$-\sum_{i,j\in R, i\neq j}(x^T(k)(M_i - M_j)^{'T}B_i^T(k)\Theta_S(k+1)B_j(k)(M_i - M_j)x(k)).$$

Taking into account, that

$$(-R_i(k) + \sum_{j\in R}B_j^T(k)\Theta_S(k+1)B_i(k)) \in Q_+(\mathcal{T}_+) \quad \forall R \subset S, \forall i \in S,$$

$$B_j^T(k)\Theta_S(k+1)B_i(k) \in Q_-(\mathcal{T}_+) \quad \forall j, i \in S, i \neq j,$$

we obtain

$$z_0(k, x(k), u(k,x)) > 0.$$

Sum these inequalities from $k_0$ to $k_0 + l$. And passing to the limit where $l \to \infty$ we obtain

$$-\sum_{k=k_0}^{k_0+l}\sum_{i\in S}w_i(k, x(k), u(k,x)) + V_0(k_0 + l + 1, x(k_0 + l + 1)) - V_0(k_0, x_0) > 0.$$

I.e.

$$-J^S(u, k_0, x_0) + J^S(u_0, k_0, x_0) > 0,$$

$$J^S(u, k_0, x_0) < J^S(u_0, k_0, x_0).$$

Since $R \subset S$ is the arbitrarily chosen, then $J^S$ achieves the maximum on $u^0(k)$.

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# Cooperative Game-theoretic Mechanism Design For Optimal Resource Use

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**Abstract** Economic analysis no longer treats the economic system as given since the appearance of Leonid Hurwicz's pioneering work on mechanism design. The failure of the market to provide an effective mechanism for optimal resource use will arise if there exist imperfect market structure, externalities, imperfect information or public goods. These phenomena are prevalent in the current global economy. As a result inefficient outcomes continue to emerge under the conventional market system. Cooperative game theory suggests the possibility of socially optimal and group efficient solutions to decision problems involving strategic actions. This lecture focuses on cooperative game-theoretic design of mechanisms for optimal resource use. Crucial features that are essential for a successful mechanism – individual rationality, group optimality, dynamic consistency, distribution procedures, budget balance, financing, incentives to cooperate and practicable institutional arrangements – are considered. Finally, cooperative game-theoretic mechanism design is used to establish the foundation for an effective policy menu to tackle sub-optimal resource use problems which the conventional market mechanism fails to resolve.

**Keywords:** Cooperative differential games, mechanism design, optimal resource use

Emile de Laveleye (1882): " *Political economy may... be defined as the science which determines what laws men ought to adopt in order that they may, with the least possible exertion, procure the greatest abundance of things useful for the satisfaction of their wants, may distribute them justly and consume them rationally.*"

## 1. Introduction

Economic analysis no longer treats the economic system as given since the appearance of Leonid Hurwicz's (1973) pioneering work on mechanism design. The term "design" stresses that the structure of the economic system is to be regarded as an unknown. New mechanisms are like synthetic chemicals: even if not usable for practical purposes, they can be studied in a pure form and so contribute to the understanding of the difficulties and potentialities of design. The design point of view enlarges our vision and helps economics avoid a narrow focus on existing institutions. The failure of the market to provide an effective mechanism for optimal

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resource use will arise if there exist imperfect market structure, externalities, imperfect information or public goods. These phenomena are prevalent in the current global economy. As a result not only inefficient outcomes like over-extraction of natural resources had appeared but gravely detrimental events like catastrophe-bound industrial pollution had also emerged under the conventional market system.

As Hurwicz (1973) pointed out that a major impetus was given to the design of mechanisms by the developments of:

i) activity analysis and linear programming (including the simplex method)—Dantzig, Kantorovitch, Koopmans;

ii) game theory, including the iterative solution procedures—von Neumann and Morgenstern, George Brown, Julia Robinson;

iii) discoveries concerning the relationships connecting programming (linear or nonlinear), two-person zero sum games, and the long known Lagrange multipliers—Gale, Kuhn, Tucker.

Moreover, he stated: "While in economics one deals with goal conflicts due to multiplicity of consumers, linear and nonlinear programming models usually presuppose a single well-defined objective function to be, say, maximized, i.e., a situation corresponding to an economy with a single consumer. So it is not surprising that the mechanisms designed under the influence of programming theory dealt to a large extent with one-objective-function problems and thus failed to face the crucial issue of goal conflict."

Hurwicz continued to say: "it is evident that the incentive structure is largely determined by what the participants can achieve for themselves by their free actions; this in turn depends on such institutional phenomena as private property, rules for the distribution of profits, or the freedom not to trade. A tool appropriate for the analysis of such phenomena is the characteristic function of a game defined by von Neumann and Morgenstern. Shapley and Shubik carried out a study of different institutional property arrangements, including feudalism, sharecropping, and the village commune, by constructing the corresponding characteristic functions and exploring the different versions of game solutions (von Neumann-Morgenstern solutions, the core, the Shapley value). Thus a significant step is taken toward a formalization of the distributional aspects of the economic system."

Cooperative games suggest the possibility of socially optimal and group efficient solutions to decision problems involving strategic actions. This lecture focuses on cooperative game-theoretic design of mechanisms for optimal resource use. Since resource use is often a dynamic process we concentrate on the design of mechanisms involving an intertemporal framework. Crucial features that are essential for a successful mechanism – individual rationality, group optimality, dynamic consistency, distribution procedures, budget balance, financing, incentives to cooperate and practicable institutional arrangements – are considered. Finally, cooperative game-theoretic mechanism design is used to establish the foundation for an effective policy menu to tackle sub-optimal resource use problems which the conventional market mechanism fails to resolve.

The lecture is organized as follows. To formulate dynamic cooperative game-theoretic mechanism design, we first present the basic setting of cooperative differential Games, and the notions of group optimality and individual rationality. This is done in Section 2. The concepts of dynamic stability which is essential to the sustainability of a mechanism design are discussed in Section 3. The corresponding payoff distribution procedures leading to the realization of dynamic stability solutions are derived in Section 4. Section 5 considers noncooperative equivalence imputation as a benchmark of allocation in optimal resource use mechanism design. Mechanism design for global environmental management is discussed in Section 6. The impact of financial constraint and irrational behavior on mechanism design is explored in Section 7. Concluding remarks are given in Section 8.

## 2. Cooperative Differential Games, Group Optimality and Individual Rationality

To formulate dynamic cooperative game-theoretic mechanism design, we first have to present the basic setting of cooperative differential Games, and the notions of group optimality and individual rationality.

### 2.1. Basic Settings of Cooperative Differential Games

Differential games study a class of decision problems, under which the evolution of the state is described by a differential equation and the players act throughout a time interval. In particular, in the general $n$-person differential game, player $i$ seeks to maximize its objective:

$$\max_{u_i} \Big\{ \int_{t_0}^{T} g^i[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]ds + q^i(x(T)) \Big\}, \tag{2.1}$$

$$\text{for} \quad i \in N = \{1, 2, \ldots, n\},$$

subject to the deterministic dynamics

$$\dot{x}(s) = f[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)], \quad x(t_0) = x_0, \tag{2.2}$$

where $x(s) \in X \subset R^m$ denotes the state variables of game, and $u_i \in U^i$ is the control of player $i$, for $i \in N$.

In the case when the terminal horizon $T$ approaches infinity an autonomous game structure with constant discounting will replace (2.1) and (2.2). In particular, the game becomes:

$$\max_{u_i} \int_{t_0}^{\infty} g^i[x(s), u_1(s), u_2(s), \cdots, u_n(s)] \times \exp[-r(s - t_0)]ds, \quad \text{for} \quad i \in N, \tag{2.3}$$

subject to the deterministic dynamics

$$\dot{x}(s) = f[x(s), u_1(s), u_2(s), \cdots, u_n(s)], \quad x(t_0) = x_0, \tag{2.4}$$

where $r$ is a constant discount rate.

Basar and Olsder (1995) provided a comprehensive overview of the analysis of zero-sum and non zero-sum noncooperative differential games.

Consider the case when the players agree to act cooperatively and play a cooperative game. The agreements on how to act cooperatively and allocate the cooperative payoffs constitute the solution optimality principle of a cooperative scheme. In particular, the solution optimality principle for a cooperative differential game includes:

(i) an agreement on a set of cooperative strategies/controls $\{u_1^*(s), u_2^*(s), \cdots, u_n^*(s)\}$ for $s \in [t_0, T]$, which would also determine the players' payoffs in the case when payoffs are nontransferable; and

(ii) a mechanism to distribute total payoff among players with the players' cooperative payoffs being $\{\xi^1(s), \xi^2(s), \cdots, \xi^n(s)\}$ for $s \in [t_0, T]$, in the case when payoffs are transferable.

### 2.2. Group Optimality and Individual Rationality

An essential element of a cooperative solution is group optimality, which ensures that all potential gains from cooperation are captured. Failure to guarantee group optimality leads to condition where there would be incentive to deviate from the agreed upon solution plan in order to extract the unexploited gains. Consider first the cooperative differential game with transferable payoffs (2.1) and dynamics (2.2) which solutions are based on group optimality. To secure group optimality the players seek to maximize their joint payoff by solving the following optimal control problem:

$$\max_{u_1, u_2, \cdots, u_n} \left\{ \int_{t_0}^{T} \sum_{j=1}^{n} g^j[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]\, ds + \sum_{j=1}^{n} q^j(x(T)) \right\}, \quad (2.5)$$

subject to (2.2).

Let $\{\, [\psi_1^*(t,x), \psi_2^*(t,x), \cdots, \psi_n^*(t,x)]$, for $t \in [t_0, T]\,\}$ denote a set of controls that provides an optimal solution (assuming its existence) to the control problem (2.5) – (2.2).

Substituting this set of control into (2.2) yields the dynamics of the optimal (cooperative) trajectory as;

$$\dot{x}(s) = f[s, x(s), \psi_1^*(s, x(s)), \psi_2^*(s, x(s)), \cdots, \psi_n^*(s, x(s))], \quad x(t_0) = x_0. \quad (2.6)$$

Let $x^*(t)$ denote the solution to (2.6). The optimal trajectory $\{x^*(t)\}_{t=t_0}^{T}$ can be expressed as:

$$x^*(t) = x_0 + \int_{t_0}^{t} f[s, x^*(s), \psi_1^*(s, x^*(s)), \psi_2^*(s, x^*(s)), \cdots, \psi_n^*(s, x^*(s))]ds. \quad (2.7)$$

For notational convenience, we use the terms $x^*(t)$ and $x_t^*$ interchangeably in case where there is no ambiguity. The optimal level of joint payoff along the cooperative trajectory can be expressed as:

$$\begin{aligned} W(t, x_t^*) = \int_t^T \sum_{j=1}^{n} g^j[s, x^*(s), \psi_1^*(s, x^*(s)), \psi_2^*(s, x^*(s)), \cdots, \psi_n^*(s, x^*(s))]\, ds \\ + \sum_{j=1}^{n} q^j(x^*(T)). \end{aligned} \quad (2.8)$$

Note that group optimality will be guaranteed only if the agreed upon optimal strategies $[\psi_1^*(t,x), \psi_2^*(t,x), \cdots, \psi_n^*(t,x)]$ are adopted throughout the game horizon $[t_0, T]$. Dockner and Jørgensen (1984), Dockner and Long (1993), Tahvonen (1994), Maler and de Zeeuw (1998) and Rubio and Casino (2002) presented cooperative solutions satisfying group optimality in differential games.

Individual rationality is another essential element of a cooperative game solution. If the players in a game wish to make an agreement to share the benefits of cooperation, the axiom of individual rationality states that no player is willing to accept an agreement that will give him less payoff than what he could obtain by rejecting to participate in the cooperative solution. In games evolving over time, resolving the problem of individual rationality may not be easy. The reason is that individual rationality may fail to apply when the game has reached a certain position, despite the fact that it was satisfied at the outset. Maintaining individual rationality and group optimality throughout the game horizon are two essential factors for dynamic stability in cooperation to arise.

Consider the transferable payoff cooperative game (2.5)–(2.6). Along the optimal trajectory $\{x^*(t)\}_{t=t_0}^{T}$ in (2.7), player $i$ would receive over the time interval an amount equaling

$$\int_t^T g^i[s, x^*(s), \psi_1^*(s, x^*(s)), \psi_2^*(s, x^*(s)), \cdots, \psi_n^*(s, x^*(s))]\, ds + q^i(x^*(T)). \quad (2.9)$$

To secure individual rationality an instantaneous transfer or side payment $\pi(s)$ at time $s$ is given to player $i$ so that the actual cooperative payoff offered to player $i$ over the duration $[t, T]$ becomes

$$\begin{aligned}\xi^i(t, x_t^*) = \int_t^T \{ & g^i[s, x^*(s), \psi_1^*(s, x^*(s)), \psi_2^*(s, x^*(s)), \cdots, \psi_n^*(s, x^*(s))] \\ & +\pi(s)\} ds + q^i(x^*(T)).\end{aligned} \quad (2.10)$$

Individual rationality requires the cooperative payoff for each player to be no less than his noncooperative payoff throughout the game horizon, that is

$$\xi^i(t, x_t^*) \geq V^i(t, x_t^*), \quad \text{for} \quad i \in N \quad \text{and} \quad t \in [t_0, T], \quad (2.11)$$

where $V^i(t, x_t^*)$ is player $i$'s payoff value function in the noncooperative differential game:

$$\max_{u_i} \left\{ \int_t^T g^i[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]\, ds + q^i(x(T)) \right\}, \quad \text{for} \quad i \in N,$$

subject to

$$\dot{x}(s) = f[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)], \quad x(t) = x_t^*. \quad (2.12)$$

For group optimality to be realized, it is required that

$$\sum_{j=1}^{n} \xi^j(t, x_t^*) = W(t, x_t^*), \quad (2.13)$$

where $W(t, x_t^*)$ is the value function of the optimal control problem:

$$\max_{u_1, u_2, \cdots, u_n} \left\{ \int_t^T \sum_{j=1}^{n} g^j[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]\, ds + \sum_{j=1}^{n} q^j(x(T)) \right\}, \quad (2.14)$$

subject to (2.10).

An optimality principle under which the players agree to act to maximize joint profit (2.5) and allocate cooperative payoff $\{\xi^1(t, x_t^*), \xi^2(t, x_t^*), \cdots, \xi^n(t, x_t^*)\}$ fulfilling (2.11) and (2.13) yields a cooperative solution which satisfies group optimality and individual rationality.

The majority of cooperative differential games adopt solutions satisfying the essential criteria for dynamic stability – group optimality and individual rationality. Haurie and Zaccour (1986 and 1991), Kaitala and Pohjola (1988, 1990 and 1995), Kaitala et al (1995) and Jørgensen and Zaccour (2001) presented classes of transferable-payoff cooperative differential games with solutions which are required to satisfy group optimality and individual rationality are satisfied.

One way to incorporate stochastic elements in differential games is to introduce stochastic dynamics. A stochastic formulation for differential games of prescribed duration involves a vector-valued stochastic differential equation

$$dx(s) = f[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]\, ds + \sigma[s, x(s)]dz(s). \quad x(t_0) = x_0. \tag{2.15}$$

which describes the evolution of the state, and $n$ objective functions

$$E_{t_0}\left\{ \int_{t_0}^{T} g^i[s, x(s), u_1(s), u_2(s), ..., u_n(s)] \times \exp[-\int_{t_0}^{s} r(y)dy], ds \right. \\ \left. + \exp[-\int_{t_0}^{T} r(y)dy]\, q^i(x(T)) \right\}, \quad \text{for} \quad i \in N, \tag{2.16}$$

with $E_{t_0}\{\cdot\}$ denoting the expectation operation taken at time $t_0$, $\sigma[s, x(s)]$ is a $m\times\ m_1$ matrix and $z(s)$ is a $m_1$-dimensional Wiener process and the initial state $x_0$ is given. Let $\Omega[s,\ x(s)] = \sigma[s,\ x(s)]\ \sigma[s,\ x(s)]'$ denote the covariance matrix with its element in row $h$ and column $\zeta$ denoted by $\Omega^{h\zeta}[s, x(s)]$. Moreover, $E[dz_\omega] = 0$ and $E[dz_\omega dt] = 0$ and $E[(dz_\omega)^2] = dt$, for $\omega \in [1, 2, \cdots, m_1]$; and $E[dz_\omega dz_k] = 0$, for $\omega \in [1, 2, \cdots, m_1]$, $k \in [1, 2, \cdots, m_1]$ and $\omega \neq k$.

### 2.3. Games with Non-transferable Payoffs

In order to maintain Pareto (1909) optimality when payoffs are not transferable the players seek to maximize the payoff (see Leitmann (1974)):

$$\max_{u_1, u_2, \cdots, u_n} \left\{ \int_{t_0}^{T} \sum_{j=1}^{n} \alpha_j g^j[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]\, ds + \right. \\ \left. \sum_{j=1}^{n} \alpha_j q^j(x(T)) \right\} \tag{2.17}$$

subject to (2.2), where $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is a vector of weights.
A necessary condition for individual rationality to hold is that at the initial time $t_0$, $\xi^{(\alpha)i}(t_0, x_0)$ must be greater than $V^i(t_0, x_0)$ for all players $i \in N$ for the chosen weights $\alpha$. Let $\{\,[\psi_1^\alpha(t, x), \psi_2^\alpha(t, x), \cdots, \psi_n^\alpha(t, x)]$, for $t \in [t_0, T]\,\}$ denote a set of controls that provides an optimal solution to the control problem (2.17) – (2.2), if the vector of weights $\alpha$ is chosen. Once again group optimality will be guaranteed only if the agreed upon optimal strategies $[\psi_1^\alpha(t, x), \psi_2^\alpha(t, x), \cdots, \psi_n^\alpha(t, x)]$ are adopted throughout the game horizon $[t_0, T]$. The optimal trajectory $\{x^\alpha(t)\}_{t=t_0}^{T}$ can then be expressed as:

$$x^\alpha(t) = x_0 + \int_{t_0}^{t} f[s, x^\alpha(s), \psi_1^\alpha(s, x^\alpha(s)), \psi_2^\alpha(s, x^\alpha(s)), \cdots, \psi_n^\alpha(s, x^\alpha(s))]\, ds. \tag{2.18}$$

For notational convenience, we use the terms $x^\alpha(t)$ and $x_t^\alpha$ interchangeably.

Along the optimal trajectory $\{x^\alpha(t)\}_{t=t_0}^T$, the payoff of player $i$ receive over the time interval $[t,T]$, for $t \in [t_0,T]$, becomes

$$\xi^{(\alpha)i}(t,x_t^\alpha) = \int_t^T g^i[s,x^\alpha(s),\psi_1^\alpha(s,x^\alpha(s)),\psi_2^\alpha(s,x^\alpha(s)),\cdots,\psi_n^\alpha(s,x^\alpha(s))]\,ds$$

$$+q^i(x^\alpha(T)). \tag{2.19}$$

While at the initial time $t_0$, $\xi^{(\alpha)i}(t_0,x_0)$ is greater than $V^i(t_0,x_0)$ for all players there is no guarantee that $\xi^{(\alpha)i}(t,x_t^\alpha)\geq V^i(t,x_t^\alpha)$, for all $i \in N$ and $t \in [t_0,T]$ along the optimal trajectory $\{x^\alpha(t)\}_{t=t_0}^T$. Most existing cooperative differential games with nontransferable payoffs offer solutions which satisfy group optimality throughout the game horizon but not individual rationality. Threats and monitoring schemes are used to deter players deviating from the cooperative strategies as the game proceeds. Leitmann (1974 and 1975), Tolwinski et al (1986), Hamalainen et. al (1986), Haurie and Pohjola (1987), Gao et al (1989), Haurie (1991) and Haurie et al (1994) presented solutions satisfying group optimality and individual rationality at the initial time to cooperative differential games with nontransferable payoffs. In addition, threats are sometimes used to ensure that no players will deviate from the agreed-upon cooperative strategies throughout the game horizon (see Hamalainen et al (1986) and Tolwinski et al (1986)).

In particular, an optimality principle under which the players agree to act to choose a vector $\alpha$ throughout the game horizon to maximize (2.17) and the chosen vector of weights $\alpha$ leads to the satisfaction of

$$\xi^{(\alpha)i}(t,x_t^\alpha) \geq V^i(t,x_t^\alpha), \quad \text{for} i \in N \quad \text{and} \quad t \in [t_0,T], \tag{2.20}$$

yields a cooperative solution which satisfies group optimality and individual rationality throughout the game horizon.

In order to verify that individual rationality in a cooperative scheme holds along the optimal trajectory, we have to derive individual player's payoff functions under cooperation
$\xi^{(\alpha)i}(t,x_t^\alpha)$ for $i \in N$ and $t \in [t_0,T]$. One way is evaluate the integral in (2.19). Another way is to obtain an analytic solution of $\xi^{(\alpha)i}(t,x_t^\alpha)$. Yeung (2004) showed the derivation of such analytic solutions by noting that for $\Delta t \to 0$, we can write:

$$\xi^{(\alpha)i}(t,x_t^\alpha) = \int_t^{t+\Delta t} g^i[s,x^\alpha(s),\psi_1^\alpha(s,x^\alpha(s)),\psi_2^\alpha(s,x^\alpha(s)),\cdots,\psi_n^\alpha(s,x^\alpha(s))]\,ds$$

$$+\varsigma^i(t+\Delta t,x_t^\alpha+\Delta x_t^\alpha), \tag{2.21}$$

where $\Delta x_t^\alpha = f[s,x^\alpha(s),\psi_1^\alpha(s,x^\alpha(s)),\psi_2^\alpha(s,x^\alpha(s)),\cdots,\psi_n^\alpha(s,x^\alpha(s))]\,\Delta t$.
Applying Taylor's Theorem, we have

$$\xi^{(\alpha)i}(t,x_t^\alpha) = g^i[t,x_t^\alpha,\psi_1^\alpha(t,x_t^\alpha),\psi_2^\alpha(t,x_t^\alpha),\cdots,\psi_n^\alpha(t,x_t^\alpha)]\Delta t + \varsigma^i(t,x_t^\alpha)$$

$$+\varsigma_t^i(t,x_t^\alpha)\Delta t + \varsigma_{x_t^\alpha}^i(t,x_t^\alpha) f[s,x^\alpha(s),\psi_1^\alpha(s,x^\alpha(s)),\psi_2^\alpha(s,x^\alpha(s)),\cdots,$$

$$\psi_n^\alpha(s,x^\alpha(s))]\,\Delta t.$$

Canceling terms, performing the expectation operator, dividing throughout by $\Delta t$ and taking $\Delta t \to 0$, we obtain:

$$-\xi_t^{(\alpha)i}(t, x_t^\alpha) = g^i[t, x_t^\alpha, \psi_1^\alpha(t, x_t^\alpha), \psi_2^\alpha(t, x_t^\alpha), \cdots, \psi_n^\alpha(t, x_t^\alpha)] + \xi_{x_t^\alpha}^{(\alpha)i}(t, x_t^\alpha) \times f[s, x^\alpha(s), \psi_1^\alpha(s, x^\alpha(s)), \psi_2^\alpha(s, x^\alpha(s)), \cdots, \psi_n^\alpha(s, x^\alpha(s))]\,. \tag{2.22}$$

Boundary conditions require:

$$\xi^{(\alpha)i}(T, x_T^\alpha) = q^i(x_T^\alpha). \tag{2.23}$$

Therefore if there exist continuously differentiable functions

$$\xi^{(\alpha)i}(t, x_t^\alpha) : [t_0, T] \times R^n \to R \quad \text{satisfying} \quad (2.22) \quad \text{and} \quad (2.23)$$

then $\xi^{(\alpha)i}(t, x_t^\alpha)$ gives player $i$'s cooperative payoff over the interval $[t, T]$ with $\alpha$ being the cooperative weight .
Given explicit functions of $\xi^{(\alpha)i}(t, x_t^\alpha)$ for $i \in N$, one can verify individual rationality readily by checking the condition $\xi^{(\alpha)i}(t, x_t^\alpha) \geq V^i(t, x_t^\alpha)$, for $i \in N$ and $t \in [t_0, T]$. Yeung and Petrosyan (2006a) presented solutions satisfying group optimality and individual rationality throughout the entire game horizon to cooperative differential games.

Finally, details on necessary conditions for Pareto optimal controls in cooperative differential games can be found in the work developed by Vincent and Leitmenn (1970), Leitmann et al (1972), Stalford (1972), Blaquiere et al (1972) and Leitmann and Schmitendorf (1974). Particulars concerning sufficient conditions are depicted in Leitmann and Schmitendorf (1963), Leitmann and Stalford (1971) and Leitmann (1974).

## 3. Dynamic Stability and Mechanism Design

Though group optimality and individual rationality constitute two essential conditions for dynamically stable cooperation, a stringent condition - *time consistency* - is required to achieve dynamic stability. Under a time-consistent solution, the solution optimality principle determined at the outset must be maintained and remain optimal throughout the game along the optimal state trajectory. In other words, time consistency of solutions to any cooperative differential game involved the property that, as the game proceeds along an optimal trajectory, players are guided by the same optimality principle at each instant of time, and hence do not possess incentives to deviate from the previously adopted optimal behavior throughout the game.

The question of dynamic stability in differential games has been rigorously explored in the past three decades. Haurie (1976) raised the problem of instability when the Nash bargaining solution is extended to differential games. Petrosyan (1977) formalized the notion of dynamic stability in solutions of differential games. Petrosyan and Danilov (1979 and 1982) introduced the notion of “imputation distribution procedure” for cooperative solution. Petrosyan (1991 and 1993b)) studied the time consistency of optimality principles in non-zero sum cooperative differential games. Petrosyan (1993a) and Petrosyan and Zenkevich (1996) presented a detailed analysis of dynamic stability in cooperative differential games, in which the method of regularization was introduced to construct time consistent solutions. Yeung and Petrosyan (2001) designed time consistent solutions in differential games and characterized the conditions that the allocation-distribution procedure must satisfy. Petrosyan (1995b and 2003) employed the regularization method to construct time consistent bargaining procedures.

Dynamic stability is essential to the sustainability of a mechanism design. In this section, we review the notion of time consistency in cooperative differential games with transferable payoffs and survey transferable payoffs games with time consistent solutions.

### 3.1. Time-consistent Solutions in Games with Transferable Payoffs

For the sake of notational uniformity in the following exposition we identify discounting explicitly and express the objective of player $i$ in (2.1) as:

$$\max_{u_i}\Big\{\int_{t_0}^{T} g^i[s,x(s),u_1(s),u_2(s),\cdots,u_n(s)] \;\times \exp[-\int_{t_0}^{s} r(y)dy]\,ds \\ +\exp[-\int_{t_0}^{T} r(y)dy]\,q^i(x(T))\Big\} \quad \text{for} \quad i\in N, \tag{3.1}$$

where $r(s)$ is the discount rate at time $s$. In the case where there is no discounting $r(y)=0$ for $y\in[t_0,T]$.

At time $t_0$ with the state being $x_0$, the players are facing the cooperative game (3.1)–(2.2), which is denoted by $\Gamma(x_0,T-t_0)$. In the start of the game, that is at time $t_0$, the players agree to adopt a solution optimality principle $\Theta(x_0,T-t_0)$ under which the players would maximize joint cooperative payoff and allocate to the player $i$ an agreed-upon cooperative payoffs over the period $[t_0,T]$.

After the cooperative game has been played for some time and at time instant $t\in(t_0,T]$, the players are facing the game

$$\max_{u_i}\Big\{\int_{t}^{T} g^i[s,x(s),u_1(s),u_2(s),\cdots,u_n(s)] \;\times \exp[-\int_{t}^{s} r(y)dy]\,ds \\ +\exp[-\int_{t}^{T} r(y)dy]\,q^i(x(T))\Big\} \quad \text{for} \quad i\in N, \tag{3.2}$$

subject to

$$\dot{x}(s)=f[s,x(s),u_1(s),u_2(s),\cdots,u_n(s)], \quad x(t)=x_t^*. \tag{3.3}$$

We use $\Gamma(x_t^*,T-t)$ to denote the cooperative game (3.2) – (3.3).
Adopting the agreed upon solution principle at time $t$ yields $\Theta(x_t^*,T-t)$ for the game $\Gamma(x_t^*,T-t)$.

At time $t_0$ with the state being $x_\tau^*$, according to optimality principle $\Theta(x_0,T-t_0)$ the payoff over the period $[t_0,T]$ imputed to the player $i$ is $\xi^{(t_0)i}(t_0,x_0)$, for $i\in N$. At time $t$ with the state being $x_t^*$, if the players apply the originally agreed upon solution optimality principle $\Theta(x_t^*,T-t)$ in the game $\Gamma(x_t^*,T-t)$ the payoff over the period $[t,T]$ imputed to the player $i$ is:

$$\xi^{(t)i}(t,x_t^*) \quad \text{for} \quad i\in N \quad \text{and} \quad t\in[t_0,T]. \tag{3.4}$$

The vectors $\xi^{(t)}(t,x_t^*)=[\xi^{(t)1}(t,x_t^*),\ \xi^{(t)2}(t,x_t^*),\cdots,\xi^{(t)n}(t,x_t^*)]$, for $t\in[t_0,T]$ is a valid imputation vector which satisfies the following necessary condition.

***Condition 3.1***

(i) $\sum_{j=1}^{n} \xi^{(t)j}(t, x_t^*) = W(t, x_t^*)$,

(ii) $\xi^{(t)i}(t, x_t^*) \geq V^{(t)i}(t, x_t^*)$, for $i \in N$ and $t \in [t_0, T]$. □

Note that (i) ensures Pareto optimality while (ii) guarantees individual rationality.

A payoff distribution over time must be formulated so that the solution imputations in (3.4) can be realized. Let the vectors $B^\tau(s) = [B_1^\tau(s), B_2^\tau(s), \cdots, B_n^\tau(s)]$ denote the "instantaneous" rate of payments to the players at time instant $s \in [\tau, T]$ in the cooperative game $\Gamma(x_\tau^*, T-\tau)$. A terminal value of $q^i(x_T^*)$ is received by player $i$ at time $T$.

In particular, $B_i^\tau(s)$ and $q^i(x_T^*)$ constitute a payoff distribution for the game $\Gamma(x_\tau^*, T-\tau)$ in the sense that $\xi^{(\tau)i}(\tau, x_\tau^*)$ equals:

$$\left\{ \left( \int_\tau^T B_i^\tau(s) \exp\left[-\int_\tau^s r(y)dy\right] ds + q^i(x_T^*) \exp\left[-\int_\tau^T r(y)dy\right] \right) \middle| \, x(\tau) = x_\tau^* \right\}, \tag{3.5}$$

for $i \in N$ and $\tau \in [t_0, T]$.
Moreover, for $i \in N$ and $t \in [\tau, T]$, we use the term $\xi^{(\tau)i}(t, x_t^*)$ which equals

$$\left\{ \left( \int_t^T B_i^\tau(s) \exp\left[-\int_\tau^s r(y)dy\right] ds + q^i(x_T^*) \exp\left[-\int_\tau^T r(y)dy\right] \right) \middle| \, x(t) = x_t^* \right\}, \tag{3.6}$$

to denote the present value of player $i$' cooperative payoff over the time interval $[t, T]$, given that the state is $x_t^*$ at time $t \in [\tau, T]$, for the game $\Gamma(x_\tau^*, T-\tau)$.

From (3.5) and (3.6) one can readily observe that the solution optimality principle $\Theta(x_t^*, T-t)$ would remain in effect only it assigns an imputation vector $\xi^{(\tau)}(\tau, x_\tau^*)$=$[\xi^{(\tau)1}(\tau, x_\tau^*)$,$\xi^{(\tau)2}(\tau, x_\tau^*)$, $\cdots$ ,$\xi^{(\tau)n}(\tau, x_\tau^*)]$ to the subgame $\Gamma(x_\tau^*, T-\tau)$ satisfies the condition that

$$\xi^{(t_0)i}(\tau, x_\tau^*) \exp\left[\int_{t_0}^{\tau} r(y)dy\right] = \xi^{(\tau)i}(\tau, x_\tau^*), \tag{3.7}$$

$$\text{for} \quad i \in N \quad \text{and} \quad \tau \in [t_0, T].$$

Crucial to the analysis is the formulation of a payment distribution mechanism that would lead to a time consistent solution. Section 4 will give an account on the derivation of such payment distribution mechanisms.

### 3.2. Time-consistent Solutions with Nontransferable Payoffs

In the case when payoffs are nontransferable, achieving time consistency is a much more difficult task. Haurie (1976) pointed out the problem of instability when the Nash bargaining solution is extended to differential games. Consider the nontransferable payoff version of the game 3.1-2.2, which we denote by $\hat{\Gamma}(x_0, T-t_0)$.

In the game $\hat{\Gamma}(x_0, T-t_0)$, let $\alpha^0$ be a vector of weights selected according to an agreed upon optimality principle $\Theta(x_0, T-t_0)$. The optimal trajectory

$\left\{ x^{\alpha^0}(t) \right\}_{t=t_0}^{T}$ can be expressed as:

$$x^{\alpha^0}(t) = x_0 +$$

$$+ \int_{t_0}^{t} f[s, x^{\alpha^0}(s), \psi_1^{(t_0)\,\alpha^0}(s, x^{\alpha^0}(s)), \psi_2^{(t_0)\,\alpha^0}(s, x^{\alpha^0}(s)), \cdots, \psi_n^{(t_0)\,\alpha^0}(s, x^{\alpha^0}(s))]\, ds. \tag{3.8}$$

Along the optimal trajectory $\{x^{\alpha^0}(t)\}_{t=t_0}^{T}$, the payoff of player $i$ receive over the time interval $[t, T]$, for $t \in [t_0, T]$, becomes

$$\varsigma_i^{\alpha^0(t_0)}(t, x_t^{\alpha^0}) =$$

$$= \int_t^T g^i[s, x^{\alpha^0}(s), \psi_1^{(t_0)\alpha^0}(s, x^{\alpha^0}(s)), \psi_2^{(t_0)\alpha^0}(s, x^{\alpha^0}(s)), \cdots, \psi_n^{(t_0)\alpha^0}(s, x^{\alpha^0}(s))]$$

$$\times \exp\left[-\int_{t_0}^{s} r(y)dy\right] ds + \exp\left[-\int_{t_0}^{T} r(y)dy\right] q^i(x^{\alpha^0}(T)), \quad \text{for} \quad i \in N. \tag{3.9}$$

The chosen vector of weights $\alpha^0$ must satisfy individual rationality so that:

$$\varsigma_i^{\alpha^0(t_0)}(t, x_t^{\alpha^0}) \geq V^{(t_0)i}(t, x_t^{\alpha^0}), \quad \text{for} \quad i \in N \quad \text{and} \quad t \in [t_0, T].$$

After cooperating for some time, at time instant $\tau$, the state becomes $x_\tau^{\alpha^0}$. In the game $\hat{\Gamma}(x_\tau^{\alpha^0}, T-\tau)$, let $\alpha^\tau$ be the vector of selected weights according the original agreed upon optimality principle $\Theta(x_\tau^{\alpha^0}, T-\tau)$.

The payoff of player $i$ receive over the time interval $[t, T]$, for $t \in [\tau, T]$, becomes

$$\varsigma_i^{\alpha^\tau(\tau)}(t, x_t^{\alpha^\tau}) =$$

$$= \int_t^T g^i[s, x^{\alpha^\tau}(s), \psi_1^{(\tau)\alpha^\tau}(s, x^{\alpha^\tau}(s)), \psi_2^{(\tau)\alpha^\tau}(s, x^{\alpha^\tau}(s)), \cdots, \psi_n^{(\tau)\alpha^\tau}(s, x^{\alpha^\tau}(s))]$$

$$\times \exp\left[-\int_{\tau}^{s} r(y)dy\right] ds + \exp\left[-\int_{\tau}^{T} r(y)dy\right] q^i(x^{\alpha^\tau}(T)), \quad \text{for} \quad i \in N, \tag{3.10}$$

where

$$x^{\alpha^\tau}(t) = x_\tau^{\alpha^0} +$$

$$+ \int_\tau^t f[s, x^{\alpha^\tau}(s), \psi_1^{(\tau)\,\alpha^\tau}(s, x^{\alpha^\tau}(s)), \psi_2^{(\tau)\,\alpha^\tau}(s, x^{\alpha^\tau}(s)), \cdots, \psi_n^{(\tau)\,\alpha^\tau}(s, x^{\alpha^\tau}(s))]\, ds. \tag{3.11}$$

Similar to the case when payoffs are transferable, if a time consistent solution assigns an imputation vector $\varsigma^{\alpha^\tau(\tau)}(\tau, x_\tau^{\alpha^\tau})$=$[\varsigma_1^{\alpha^\tau(\tau)}(\tau, x_\tau^{\alpha^\tau})$, $\varsigma_2^{\alpha^\tau(\tau)}(\tau, x_\tau^{\alpha^\tau})$, $\cdots, \varsigma_n^{\alpha^\tau(\tau)}(\tau, x_\tau^{\alpha^\tau})]$ to the subgame $\hat{\Gamma}(x_\tau^{\alpha^0}, T-\tau)$, for $\tau \in [t_0, T]$, the following condition must be satisfied:

$$\begin{gathered}\varsigma_i^{\alpha^0(t_0)}(\tau, x_\tau^{\alpha^0}) \exp\left[\int_{t_0}^{\tau} r(y)dy\right] = \varsigma_i^{\alpha^\tau(\tau)}(\tau, x_\tau^{\alpha^0}), \\ \text{for} \quad i \in N \quad \text{and} \quad \tau \in [t_0, T].\end{gathered} \tag{3.12}$$

For group optimality to be achievable, the cooperative controls $[\psi_1^{(t_0)\,\alpha^0}(s, x^{\alpha^0}(s)), \psi_2^{(t_0)\,\alpha^0}(s, x^{\alpha^0}(s)), \cdots, \psi_n^{(t_0)\,\alpha^0}(s, x^{\alpha^0}(s))]$ must be adopted throughout time interval $[t_0, T]$. Hence a subgame consistent solution optimality principle $\Theta(x_0, T - t_0)$ which chooses $\alpha^0$ in the game $\hat{\Gamma}(x_0, T - t_0)$ will choose $\alpha^0$ again in any subgame $\hat{\Gamma}(x_\tau^{\alpha^0}, T - \tau)$ the weight $\alpha^0$ chosen according to the solution optimality $\Theta(x_\tau^{\alpha^0}, T - \tau)$. Yeung and Petrosyan (2006a) showed that a time consistent solution to the nontransferable payoffs game $\hat{\Gamma}(x_0, T - t_0)$ requires then satisfaction of the following theorem.

**Theorem 3.1.** *A solution optimality principle under which the players agree to choose the same weight $\alpha^\tau = \alpha^0$ in all the subgames $\hat{\Gamma}(x_\tau^{\alpha^0}, T - \tau)$ for $\tau \in [t_0, T]$ and*

$$\varsigma_i^{\alpha^0(\tau)}(\tau, x_\tau^{\alpha^0}) \geq V^{(\tau)i}(\tau, x_\tau^{\alpha^0}), \quad \text{for} \quad i \in N \quad \text{and} \quad \tau \in [t_0, T],$$

*yields a time consistent solution to the cooperative game $\hat{\Gamma}(x_0, T - t_0)$.*

*Proof.* Note that any solution optimality principle as that in Theorem 3.1 satisfies (i) group optimality, (ii) individual rationality. Moreover, with $\alpha^0$ being chosen in all the subgames $\hat{\Gamma}(x_\tau^{\alpha^0}, T - \tau)$ for $\tau \in [t_0, T]$, we have

$$\varsigma_i^{\alpha^0(t_0)}(\tau, x_\tau^{\alpha^0}) \exp\left[\int_{t_0}^{\tau} r(y)dy\right] = \varsigma_i^{\alpha^\tau(\tau)}(\tau, x_\tau^{\alpha^0}) = \varsigma_i^{\alpha^0(\tau)}(\tau, x_\tau^{\alpha^0})$$

$$= \int_\tau^T g^i[s, x^{\alpha^0}(s), \psi_1^{(t_0)\alpha^0}(s, x^{\alpha^0}(s)), \psi_2^{(t_0)\alpha^0}(s, x^{\alpha^0}(s)), \cdots, \psi_n^{(t_0)\alpha^0}(s, x^{\alpha^0}(s))]$$

$$\times \exp\left[-\int_\tau^s r(y)dy\right] ds + \exp\left[-\int_\tau^T r(y)dy\right] q^i(x^{\alpha^0}(T)), \quad \text{for} \quad i \in N.$$

Hence (3.12) is satisfied. The solution optimality principle given in Theorem 3.1 is indeed time consistent. □

Yeung and Petrostyan (2006a) have provided a method to derive individual cooperative payoffs in an analytically tractable form. Specific solution optimality principles leading to time consistent solutions in a few classes of cooperative differential games with nontransferable payoffs can be found in Yeung and Petrosyan (2006a). The deterministic versions of Yeung and Petrosyan (2006a) and Yeung et al (2006) are also examples of nontransferable payoff differential games with time-consistent solutions.

### 3.3. Subgame Consistency Solutions in Cooperative Stochastic Differential Games

As one can imagine the derivation of tractable solutions in cooperative stochastic differential games could be rather difficult because of the games' complexity. Haurie *et al* (1994) derived cooperative equilibria in a stochastic differential game of fishery with the use of monitoring and memory strategies. In the presence of stochastic elements, a more stringent condition – *subgame consistency* – is required for a dynamically stable cooperative solution.

The notion of subgame consistency in cooperative differential games was first examined by Yeung and Petrosyan (2004). In particular, a cooperative solution is subgame-consistent if an extension of the solution policy to a subgame at a later starting time and any feasible state brought about by prior optimal behavior would remain optimal. Conditions ensuring subgame consistency in cooperative solutions of stochastic differential games generally are more analytically intractable than those ensuring time consistency in cooperative solutions of differential games.

Let $\varGamma_s(x_0, T-t_0)$ denote the cooperative stochastic differential games (2.15)-(2.16) with transferable payoffs. To achieve group optimality the players would seek to maximize the expected joint payoff

$$E_{t_0}\left\{ \int_{t_0}^{T} \sum_{j=1}^{n} g^j[s, x(s), u_1(s), u_2(s), ..., u_n(s)] \ \exp[-\int_{t_0}^{s} r(y)dy]\, ds + \sum_{j=1}^{n} \exp[-\int_{t_0}^{T} r(y)dy]\, q^j(x(T)) \right\} \tag{3.13}$$

subject to (2.15).

Let $\left\{[\varphi_1^{(t_0)^*}(t,x), \varphi_2^{(t_0)^*}(t,x), \cdots, \varphi_n^{(t_0)^*}(t,x), for\ t \in [t_0, T]\right\}$, denote a set of controls (if exists) that provides an optimal solution to the stochastic control problem (3.13)-(2.15).

Substituting this set of control into (2.15) and solving yields the optimal (cooperative) trajectory in the form of a stochastic path $\{x^*(t)\}_{t=t_0}^{T}$ where

$$x^*(t) = x_0 + \int_{t_0}^{t} f[s, x^*(s), \varphi_1^{(t_0)^*}(s, x^*(s)), \varphi_2^{(t_0)^*}(s, x^*(s)), \cdots, \varphi_n^{(t_0)^*}(s, x^*(s))]ds + \int_{t_0}^{t} \sigma[s, x^*(s)]dz(s). \tag{3.14}$$

We use $X_t^*$ to denote the set of realizable values of $x^*(t)$ at time $t$ generated by (3.14). The term $x_t^*$ is used to denote an element belonging to the set $X_t^*$. We use $\varGamma_s(x_\tau^*, T-\tau)$ to denote the cooperative stochastic differential game with payoffs (2.16) and dynamics (2.15) starting at time $\tau$ given the state at time $\tau$ is $x_\tau^* \in X_t^*$.

In the start of the game, the players agree to adopt a solution optimality principle $\varTheta(x_t^*, T-t)$ which states that at current time $t$ with current state $x_t^*$ the players would maximize the expected joint cooperative payoff and allocate to the player $i$ a cooperative payoffs over the period $[t, T]$ equaling

$$\xi^{(\tau)i}(\tau, x_\tau^*), \quad \text{for} \quad i \in N \quad \text{and} \quad \tau \in [t_0, T]. \tag{3.15}$$

The vectors $\xi^{(\tau)}(\tau, x_\tau^*)$=$[\xi^{(\tau)1}(\tau, x_\tau^*)$,$\xi^{(\tau)2}(\tau, x_\tau^*)$,$\cdots$,$\xi^{(\tau)n}(\tau, x_\tau^*)]$, for $\tau \in [t_0, T]$, are valid imputations if the following conditions are satisfied.

***Condition 3.2***

(i) $\xi^{(\tau)}(\tau, x_\tau^*)$, for $\tau \in [t_0, T]$ and $x_\tau^* \in X_\tau^*$, is a Pareto optimal imputation vector,

(ii) $\xi^{(\tau)i}(\tau, x_\tau^*) \geq V^{(\tau)i}(\tau, x_\tau^*)$, for $i \in N$, for $\tau \in [t_0, T]$ and $x_\tau^* \in X_\tau^*$,

where $V^{(\tau)i}(\tau, x_\tau^*)$ is the payoff to player $i$ in the noncooperative version of the game $\varGamma_s(x_\tau^*, T-\tau)$. □

In particular, part (i) of Condition 3.2 ensures Pareto optimality, while part (ii) guarantees individual rationality.

Once again, one can readily observe that the solution optimality principle $\Theta(x_t^*,\ T-t)$ would remain in effect only it assigns an imputation vector $\xi^{(\tau)}(\tau, x_\tau^*)$=$[\xi^{(\tau)1}(\tau,\ x_\tau^*), \xi^{(\tau)2}(\tau, x_\tau^*), \cdots, \xi^{(\tau)n}(\tau, x_\tau^*)]$ to the subgame $\Gamma(x_\tau^*, T-\tau)$ satisfies the condition that

$$\xi^{(t_0)i}(\tau, x_\tau^*) \exp\left[\int_{t_0}^{\tau} r(y)dy\right] = \xi^{(\tau)i}(\tau, x_\tau^*), \tag{3.16}$$

for $i \in N$ and $\tau \in [t_0, T]$.

## 4. Payoff Distribution Procedures Leading to Dynamically Stable Solutions

In this section, we consider payoff distribution procedures leading the dynamically consistent solutions.

### 4.1. Payoff Distribution Procedures for Cooperative Differential Games

A payoff distribution procedure (PDP) for cooperative differential games (as proposed in Petrosyan (1997) and Yeung and Petrosyan (2006a)) must be now formulated so that the agreed-upon imputations can be realized. For the condition in (3.7) to hold, it is required that $B_i^\tau(s) = B_i^t(s)$, for $i \in [1{,}2]$ and $\tau \in [t_0, T]$ and $t \in [t_0, T]$ and $\tau \neq t$. Adopting the notation $B_i^\tau(s) = B_i^t(s) = B_i(s)$ and applying Definition 5.1, the PDP of the subgame consistent imputation vectors $\xi^{(\tau)}(\tau, x_\tau)$ has to satisfy the following condition.

***Condition 4.1*** The PDP with $B(s)$ and $q(x^*(T))$ corresponding to the time consistent imputation vectors $\xi^{(\tau)}(\tau, x_\tau^*)$ must satisfy the following conditions:

(i) $\sum_{j=1}^{2} B_i(s) = \sum_{j=1}^{2} g^j[s, x_s^*, \psi_1^{(\tau)^*}(s, x_s^*), \psi_2^{(\tau)^*}(s, x_s^*)]$, for $s \in [t_0, T]$;

(ii) $\int_\tau^T B_i(s) \exp\left[-\int_\tau^s r(y)dy\right] ds$+$q^i(x^*(T))$exp$\left[-\int_\tau^T r(y)dy\right] \geq V^{(\tau)i}(\tau, x_\tau^*)$, for $i \in \{1, 2\}$ and $\tau \in [t_0, T]$; and

(iii) $\xi^{(\tau)i}(\tau, x_\tau^*)$=$\int_\tau^{\tau+\Delta t} B_i(s) \exp\left[-\int_\tau^s r(y)dy\right] ds$+exp$\left[-\int_\tau^{\tau+\Delta t} r(y)dy\right]$
$\times\xi^{(\tau+\Delta t)i}(\tau+\ \Delta t, x_\tau^* + \Delta x_\tau^*)$, for $\tau \in [t_0, T]$ and $i \in \{1, 2\}$;
where

$$\Delta x_\tau^* = f[\tau, x_\tau^*, \psi_1^{(\tau)^*}(\tau, x_\tau^*, \psi_2^{(\tau)^*}(\tau, x_\tau^*)]\Delta t + o(\Delta t),$$

and $o(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$. □

Consider the following condition concerning $\xi^{(\tau)}(t, x_t^*)$, for $\tau \in [t_0, T]$ and $t \in [\tau, T]$:

***Condition 4.2*** For $i \in \{1, 2\}$ and $t \geq \tau$ and $\tau \in [t_0, T]$, the terms $\xi^{(\tau)i}(t, x_t^*)$ are functions that are continuously twice differentiable in $t$ and $x_t^*$. □

If the imputations $\xi^{(\tau)}(t, x_t^*)$, for $\tau \in [t_0, T]$, satisfy Condition 4.2, one can obtain the following relationship:

$$\begin{aligned} &\int_\tau^{\tau+\Delta t} B_i(s) \exp\left[-\int_\tau^s r(y)dy\right] ds \\ &\qquad = \xi^{(\tau)i}(\tau, x_\tau^*) - \exp\left[-\int_\tau^{\tau+\Delta t} r(y)dy\right] \xi^{(\tau+\Delta t)i}(\tau + \Delta t, x_\tau^* + \Delta x_\tau^*) \\ &\qquad = \xi^{(\tau)i}(\tau, x_\tau^*) - \xi^{(\tau)i}(\tau + \Delta t, x_\tau^* + \Delta x_\tau^*), \\ &\qquad\quad \text{for all} \quad \tau \in [t_0, T] \quad \text{and} \quad i \in \{1, 2\}\,. \end{aligned} \tag{4.1}$$

With $\Delta t \to 0$, condition (4.1) can be expressed as:

$$\begin{aligned} B_i(\tau)\Delta t = &- \left[\xi_t^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] \Delta t - \left[\xi_{x_t^*}^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] \\ &\times f[\tau, x_\tau^*, \psi_1^{(\tau)^*}(\tau, x_\tau^*), \psi_2^{(\tau)^*}(\tau, x_\tau^*)]\Delta t - o(\Delta t). \end{aligned} \tag{4.2}$$

Dividing (4.2) throughout by $\Delta t$, with $\Delta t \to 0$, yield

$$B_i(\tau) = -[\xi_t^{(\tau)i}(t, x_t^*)|_{t=\tau}] - [\xi_{x_t^*}^{(\tau)i}(t, x_t^*)|_{t=\tau}] f[\tau, x_\tau^*, \psi_1^{(\tau)^*}(\tau, x_\tau^*), \psi_2^{(\tau)^*}(\tau, x_\tau^*)]. \tag{4.3}$$

Therefore, one can establish the following theorem.

**Theorem 4.1.** *If the solution imputations $\xi^{(\tau)i}(\tau, x_\tau^*)$, for $i \in \{1, 2\}$ and $\tau \in [t_0, T]$, satisfy Condition 4.1 and Condition 4.2, a PDP with a terminal payment $q^i(x_T^*))$ at time $T$ and an instantaneous payment at time $\tau \in [t_0, T]$:*

$$B_i(\tau) = -[\xi_t^{(\tau)i}(t, x_t^*)|_{t=\tau}] - [\xi_{x_t^*}^{(\tau)i}(t, x_t^*)|_{t=\tau}] f[\tau, x_\tau^*, \psi_1^{(\tau)^*}(\tau, x_\tau^*), \psi_2^{(\tau)^*}(\tau, x_\tau^*)],$$

*for $i \in \{1, 2\}$,*
*yields a time consistent solution to the cooperative game $\Gamma_c(x_0, T - t_0)$.*

### 4.2. PDP under Specific Optimality Principles

Consider a cooperative game $\Gamma_c(x_0, T - t_0)$ in which the players agree to maximize the sum of their payoffs and divide the total cooperative payoff equally. The imputation scheme has to satisfy:

**Proposition 4.1.** *In the game $\Gamma_c(x_0, T - t_0)$, an imputation*

$$\xi^{(t_0)i}(t_0, x_0) = V^{(t_0)i}(t_0, x_0) + \frac{1}{2}\left[W^{(t_0)}(t_0, x_0) - \sum_{j=1}^{2} V^{(t_0)j}(t_0, x_0)\right],$$

*is assigned to player $i$, for $i \in \{1, 2\}$;*
*and in the subgame $\Gamma_c(x_\tau^*, T - \tau)$, for $\tau \in (t_0, T]$, an imputation*

$$\xi^{(\tau)i}(\tau, x_\tau^*) = V^{(\tau)i}(\tau, x_\tau^*) + \frac{1}{2}\left[W^{(\tau)}(\tau, x_\tau^*) - \sum_{j=1}^{2} V^{(\tau)j}(\tau, x_\tau^*)\right],$$

*is assigned to player i, for* $i \in \{1, 2\}$. □

One can readily verify that $\xi^{(\tau)i}(\tau, x_\tau^*)$ satisfies Conditions 4.1 and 4.2. Using Theorem 4.1 a PDP with a terminal payment $q^i(x(T))$ at time $T$ and an instantaneous imputation rate at time $\tau \in [t_0, T]$:

$$B_i(\tau) =$$

$$\frac{-1}{2}\left[\ \left[V_t^{(\tau)i}(t, x_t)\Big|_{t=\tau}\right] + \left[V_{x_t}^{(\tau)i}(t, x_t)\Big|_{t=\tau}\right] f[\tau, x_\tau, \psi_1^{(\tau)^*}(\tau, x_\tau), \psi_2^{(\tau)^*}(\tau, x_\tau)]\ \right]$$

$$-\frac{1}{2}\left[\ \left[W_t^{(\tau)}(t, x_t)\Big|_{t=\tau}\right] + \left[W_{x_t}^{(\tau)}(t, x_t)\Big|_{t=\tau}\right] f[\tau, x_\tau, \psi_1^{(\tau)^*}(\tau, x_\tau), \psi_2^{(\tau)^*}(\tau, x_\tau)]\ \right]$$

$$+\frac{1}{2}\left[\ \left[V_t^{(\tau)j}(t, x_t)\Big|_{t=\tau}\right] + \left[V_{x_t}^{(\tau)j}(t, x_t)\Big|_{t=\tau}\right] f[\tau, x_\tau, \psi_1^{(\tau)^*}(\tau, x_\tau), \psi_2^{(\tau)^*}(\tau, x_\tau)]\ \right], \tag{4.4}$$

for $i, j \in \{1, 2\}$ and $i \neq j$,
yields a time consistent solution to the cooperative game $\Gamma_c(x_0, T - t_0)$, in which the players agree to divide their cooperative gains according to Proposition 4.1.

### 4.3. Payoff Distribution Procedures for Cooperative Stochastic Differential Games

The PDP with $B(s)$ and $q(x(T))$ corresponding to the subgame consistent imputation vectors $\xi^{(\tau)}(\tau, x_\tau)$ must satisfy the following conditions:

***Condition 4.3***

(i) $\sum_{j=1}^{2} B_i(s) = \sum_{j=1}^{2} g^j[s, x_s, \psi_1^{(\tau)^{**}}(s, x_s), \psi_2^{(\tau)^{**}}(s, x_s)]$, for $s \in [t_0, T]$;

(ii) $E_\tau\left\{\left(\ \int_\tau^T B_i(s)\exp\left[-\int_\tau^s r(y)dy\right]ds + q^i(x(T))\exp\left[-\int_\tau^T r(y)dy\right]\ \right)\ \Bigg|\ x(\tau) = x_\tau\right\} \geq V^{(\tau)i}(\tau, x_\tau)$, for $i \in$ [1,2] and $\tau \in [t_0, T]$; and

(iii) $\xi^{(\tau)i}(\tau, x_\tau) = E_\tau\left\{\left(\ \int_\tau^{\tau+\Delta t} B_i(s)\exp\left[-\int_\tau^s r(y)dy\right]ds + \exp\left[-\int_\tau^{\tau+\Delta t} r(y)dy\right] \times \xi^{(\tau+\Delta t)i}(\tau + \Delta t, x_\tau + \Delta x_\tau)\ \right)\ \Bigg|\ x(\tau) = x_\tau\right\}$, for $\tau \in [t_0, T]$ and $i \in [1, 2]$;

where

$$\Delta x_\tau = f[t, x_\tau, \psi_1^{(\tau)**}(\tau, x_\tau, \psi_2^{(\tau)^{**}}(\tau, x_\tau)]\Delta t + \sigma[\tau, x_\tau]\Delta z_\tau + o(\Delta t),$$

$x(\tau) = x_\tau \in X_\tau^{(t_0)^*}$, $\Delta z_\tau = z(\tau + \Delta t) - z(\tau)$, and $E_\tau[o(\Delta t)]/\Delta t \to 0$ as $\Delta t \to 0$. □

Consider the following condition concerning subgame consistent imputations $\xi^{(\tau)}(\tau, x_\tau)$, for $\tau \in [t_0, T]$:

***Condition 4.4*** For $i \in [1,2]$ and $t \geq \tau$ and $\tau \in [t_0, T]$, the terms $\xi^{(\tau)i}(t, x_t)$ are functions that are continuously twice differentiable in $t$ and $x_t$. □

If the subgame consistent imputations $\xi^{(\tau)}(\tau, x_\tau)$, for $\tau \in [t_0, T]$, satisfy Condition 4.4, a PDP with $B(s)$ and $q(x(T))$ will yield the relationship:

$$E_\tau \left\{ \int_\tau^{\tau+\Delta t} B_i(s) \exp\left[ -\int_\tau^s r(y)dy \right] ds \,\middle|\, x(\tau) = x_\tau \right\}$$

$$E_\tau \left\{ \xi^{(\tau)i}(\tau, x_\tau) - \exp\left[ -\int_\tau^{\tau+\Delta t} r(y)dy \right] \xi^{(\tau+\Delta t)i}(\tau + \Delta t, x_\tau + \Delta x_\tau) \,\middle|\, x(\tau) = x_\tau \right\}$$

$$E_\tau \left\{ \xi^{(\tau)i}(\tau, x_\tau) - \xi^{(\tau)i}(\tau + \Delta t, x_\tau + \Delta x_\tau) \,\middle|\, x(\tau) = x_\tau \right\}, \tag{4.5}$$

or all $\tau \in [t_0, T]$ and $i \in [1,2]$.

With $\Delta t \to 0$, equation (4.5) can be expressed as:

$$\begin{aligned} &E_\tau \left\{ B_i(\tau)\Delta t + o(\Delta t) \right\} = \\ &\quad \Delta t - \left[ \xi_{x_t}^{(\tau)i}(t, x_t)\Big|_{t=\tau} \right] f[\tau, x_\tau, \psi_1^{(\tau)^{**}}(\tau, x_\tau), \psi_2^{(\tau)^{**}}(\tau, x_\tau)]\Delta t \\ &\quad \left[ \xi_{x_t^h x_t^\zeta}^{(\tau)i}(t, x_t)\Big|_{t=\tau} \right] \Delta t - \left[ \xi_{x_t}^{(\tau)i}(t, x_t)\Big|_{t=\tau} \right] \sigma[\tau, x_\tau]\Delta z_\tau - o(\Delta t) \Bigg\}. \end{aligned} \tag{4.6}$$

Taking expectation and dividing (4.6) throughout by $\Delta t$, with $\Delta t \to 0$, yield

$$\begin{aligned} &B_i(\tau) = \\ &- \left[ \xi_t^{(\tau)i}(t, x_t)\Big|_{t=\tau} \right] - \left[ \xi_{x_t}^{(\tau)i}(t, x_t)\Big|_{t=\tau} \right] f[\tau, x_\tau, \psi_1^{(\tau)^{**}}(\tau, x_\tau), \psi_2^{(\tau)^{**}}(\tau, x_\tau)] \\ &- \frac{1}{2} \sum_{h,\zeta=1}^n \Omega^{h\zeta}(\tau, x_\tau) \left[ \xi_{x_t^h x_t^\zeta}^{(\tau)i}(t, x_t)\Big|_{t=\tau} \right]. \end{aligned} \tag{4.7}$$

Therefore, one can establish the following theorem.

**Theorem 4.2 (Yeung and Petrosyan (2004)).** *If the solution imputations $\xi^{(\tau)i}(\tau, x_\tau)$, for $i \in [1,2]$ and $\tau \in [t_0, T]$, satisfy Conditions 4.3 and 4.4, a PDP with a terminal payment $q^i(x(T))$ at time $T$ and an instantaneous imputation rate at time $\tau \in [t_0, T]$:*

$$B_i(\tau) = - \left[ \xi_t^{(\tau)i}(t, x_t)\Big|_{t=\tau} \right] - \left[ \xi_{x_t}^{(\tau)i}(t, x_t)\Big|_{t=\tau} \right] f[\tau, x_\tau, \psi_1^{(\tau)^{**}}(\tau, x_\tau), \psi_2^{(\tau)^{**}}(\tau, x_\tau)]$$

$$- \frac{1}{2} \sum_{h,\zeta=1}^n \Omega^{h\zeta}(\tau, x_\tau) \left[ \xi_{x_t^h x_t^\zeta}^{(\tau)i}(t, x_t)\Big|_{t=\tau} \right], \quad \text{for} \quad i \in [1,2],$$

*yields a subgame consistent solution to the cooperative game $\Gamma_c(x_0, T - t_0)$.* □

## 5. Noncooperative Equivalence Imputation and Mechanism Design

In this section, we examine noncooperative equivalence imputation as a benchmark of allocation in optimal resource use mechanism design. Consider a dynamic market situation in which there are $n$ economic agents with initial state $x_0$ and duration $T - t_0$. The state space of the game is $\mathrm{X} \in R^m$, with permissible state trajectories citation $\{x(s), t_0 \leq s \leq T\}$. The state dynamics of the game is characterized by the vector-valued differential equations:

$$\dot{x}(s) = f[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)], \quad x(t_0) = x_0, \tag{5.1}$$

where $u_i(s) \in R^{m_i}$ is the control vector of agent $i$.

The objective of agent $i$ is

$$\begin{array}{c} \int_{t_0}^{T} g^i[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]\, ds + q^i(x(T)), \\ \text{for} \quad i \in \{1, 2, \cdots, n\} \equiv N, \end{array} \tag{5.2}$$

and $g^i[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]$ and $q^i(x(T))$are non-negative. The agents' payoffs are transferable.

Invoking the work of Isaacs (1965) and Bellman (1957) a feedback Nash equilibrium of the game can be characterized the following well-known theorem:

**Theorem 5.1.** *An n-tuple of strategies $\{\phi_i(t, x)$, for $i \in N\}$ provides a feedback Nash equilibrium solution to the game (5.1)-(5.2) if there exist continuously differentiable functions $V^i(t, x) : [t_0, T] \times R^m \to R$, $i \in N$, satisfying the following set of partial differential equations:*

$$-V_t^i(t, x) =$$

$$\max_{u_i} \{ g^i[t, x, \phi_1(t, x), \phi_2(t, x), \cdots, \phi_{i-1}(t, x), u_i, \phi_{i+1}(t, x), \cdots, \phi_n(t, x)]$$

$$+ V_x^i(t, x) f[t, x, \phi_1(t, x), \phi_2(t, x), \cdots, \phi_{i-1}(t, x), u_i, \phi_{i+1}(t, x), \cdots, \phi_n(t, x)] \}$$

$$V^i(T, x) = q^i(x), \quad i \in N.$$

The noncooperative payoff of agent $i$ at time $t$ given that $x(t) = t$ is given by the continuously differentiable function $V^i(t, x)$.

Now consider the attempt to bring about group optimality. To achieve group optimality one has to maximize the agents' joint payoff:

$$\int_{t_0}^{T} \sum_{j=1}^{n} g^j[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]\, ds + \sum_{j=1}^{n} q^j(x(T)) \tag{5.3}$$

subject to (5.1).

Let $\{\psi_i(s, x)$, for $i \in N\}$ denote a set of strategies leading to an optimal control solution of the problem (5.1) and (5.3) the total payoff under the group optimal scheme over the interval $[t, T]$ where $t \in [t_0, T]$ is:

$$W(t, x_t^*) = \int_{t}^{T} \sum_{j=1}^{n} g^j[s, x^*(s), \psi_1(s, x^*(s)), \psi_2(s, x^*(s)), \cdots, \psi_n(s, x^*(s))]\, ds +$$

$$+\sum_{j=1}^{n} q^j(x^*(T)). \tag{5.4}$$

The state dynamics under cooperation is:

$$\dot{x}^*(s) = f[s, x^*(s), \psi_1(s, x^*(s)), \psi_2(s, x^*(s)), \cdots, \psi_n(s, x^*(s))], \quad x(t_0) = x_0. \tag{5.5}$$

The corresponding optimal trajectory can be expressed as:

$$x^*(t) = x_0 + \int_{t_0}^{t} f[s, x^*(s), \psi_1^*(s, x^*(s)), \psi_2^*(s, x^*(s)), \cdots, \psi_n^*(s, x^*(s))]\, ds. \tag{5.6}$$

For notational convenience, we use the terms $x^*(t)$ and $x_t^*$ interchangeably.

### 5.1. Noncooperative-Equivalent Imputation Formula

The minimum requirement to induce the agents to adopt the optimal strategies $\{\psi_i(s,x)$, for $i \in N\}$ is that each agent would receive a payoff at least be equal to his noncooperative payoff. Moreover, to maintain individual rationality throughout the duration $T-t_0$, each agent's imputation must at least be equal to his noncooperative payoff at each time instant $t \in [t_0, T]$ along the optimal state path $\{x^*(t)\}$.

Let $\xi^i(\tau, x_\tau^*)$ denote the imputation to agent $i$ under the group optimal scheme over the time interval $[\tau, T]$ along the optimal path $\{x_\tau^*\}_{\tau=t_0}^{T}$ for $\tau \in [t_0, T]$. An imputation distribution procedure as in Petrosyan and Danilov (1982) and Yeung and Petrosyan (2004 and 2006a) has to be formulated so that the cooperative imputation $\xi^i(\tau, x_\tau^*) = V^i(\tau, x_\tau^*)$ can be realized along the optimal path. To do this we let $B_i(s, x^*(s))$ denote the instantaneous rate of payment received by player $i$ at time $s$. In particular,

$$\xi^i(\tau, x_\tau^*) = V^i(\tau, x_\tau^*) = \int_{\tau}^{T} B_i(s, x^*(s))\, ds + q^i(x_T^*), \quad \text{for} \quad \tau \in [t_0, T]. \tag{5.7}$$

**Theorem 5.2.** *A payment scheme with a terminal payment $q^i(x_T^*)$ at time $T$ and an instantaneous rate of payment at time $\tau \in [t_0, T]$ along the cooperative trajectory $\{x_\tau^*\}_{\tau=t_0}^{T}$ being*

$$\begin{aligned} &B_i(\tau, x_\tau^*) = \\ &-V_\tau^i(\tau, x_\tau^*) - V_{x_\tau^*}^i(\tau, x_\tau^*) f[\tau, x_\tau^*, \psi_1(\tau, x_\tau^*), \psi_2(\tau, x_\tau^*), \cdots, \psi_n(\tau, x_\tau^*)], \end{aligned} \tag{5.8}$$

*yield the noncooperative-equivalent imputation*
$\xi^i(\tau, x_\tau^*) = \int_\tau^T B_i(s, x^*(s))\, ds + q^i(x_T^*) = V^i(\tau, x_\tau^*)$, *for* $\tau \in [t_0, T]$. □

*Proof.* Using (5.7) one can obtain the identity:

$$\int_{t_0}^{\tau} B_i(s, x^*(s))\, ds + V^i(\tau, x_\tau^*) \equiv V^i(t_0, x_0). \quad \text{for} \quad \tau \in [t_0, T]. \tag{5.9}$$

Differentiating (5.8) with respect to $\tau$ yields

$$B_i(\tau) = -dV^i(\tau, x_\tau^*)/d\tau = -V_\tau^i(\tau, x_\tau^*) - V_{x_\tau^*}^i(\tau, x_\tau^*)\dot{x}^*(\tau). \tag{5.10}$$

Invoking (5.6) we obtain

$$B_i(\tau) = -V_\tau^i(\tau, x_\tau^*) - V_{x_\tau^*}^i(\tau, x_\tau^*) f[\tau, x_\tau^*, \psi_1(\tau, x_\tau^*), \psi_2(\tau, x_\tau^*), \cdots, \psi_n(\tau, x_\tau^*)]. \tag{5.11}$$

Hence Theorem 3.1 follows. □

Theorem 5.1 yields a distribution formula for noncooperative-equivalent imputation in an group optimal scheme. Such a formula can be obtained in closed form for any explicitly solvable games.

### 5.2. An Economic Exegesis of the Formula

An economic exegesis of the rationale for the noncooperative-equivalent imputation formula (5.11) can be obtained. Note that the Isaacs-Bellman equation in Theorem 5.1 for a feedback Nash equilibrium in the noncooperative game (5.1) and (5.2) leads to

$$\begin{aligned} -V^i_\tau(\tau,x^*_\tau) &= g^i[\tau,x^*_\tau,\phi_1(\tau,x^*_\tau),\phi_2(\tau,x^*_\tau),\cdots,\phi_n(\tau,x^*_\tau)] \\ &+V^i_{x^*_\tau}(\tau,x^*_\tau)f[\tau,x^*_\tau,\phi_1(\tau,x^*_\tau),\phi_2(\tau,x^*_\tau),\cdots,\phi_n(\tau,x^*_\tau)], \quad \text{for} \quad \tau\in[t_0,T]. \end{aligned} \tag{5.12}$$

Using (5.12) the distribution formula in (5.10) can be expressed as:

$$\begin{aligned} B_i(\tau,x^*_\tau) = g^i&[\tau,x^*_\tau,\phi_1(\tau,x^*_\tau),\phi_2(\tau,x^*_\tau),\cdots,\phi_n(\tau,x^*_\tau)] \\ &+ V^i_{x^*_\tau}(\tau,x^*_\tau)\left\{f[\tau,x^*_\tau,\phi_1(\tau,x^*_\tau),\phi_2(\tau,x^*_\tau),\cdots,\phi_n(\tau,x^*_\tau)]\right. \\ &- \left. f[\tau,x^*_\tau,\psi_1(\tau,x^*_\tau),\psi_2(\tau,x^*_\tau),\cdots,\psi_n(\tau,x^*_\tau)]\right\}. \end{aligned} \tag{5.13}$$

However, along the optimal path $\{x^*_\tau\}^T_{\tau=t_0}$, the instantaneous rate of payoff to agent $i$ is

$$g^i[\tau,x^*_\tau,\psi_1(\tau,x^*_\tau),\psi_2(\tau,x^*_\tau),\cdots,\psi_n(\tau,x^*_\tau)] \quad \text{at time instant} \quad \tau. \tag{5.14}$$

In order for agent $i$ to realize an instantaneous rate of payoff equaling $B_i(\tau,x^*_\tau)$ a *noncooperative-equivalent compensation formula* can be obtained as

$$\vartheta_i(\tau,x^*_\tau) = B_i(\tau,x^*_\tau) - g^i[\tau,x^*_\tau,\psi_1(\tau,x^*_\tau),\psi_2(\tau,x^*_\tau),\cdots,\psi_n(\tau,x^*_\tau)],$$

or:

$$\begin{aligned} \vartheta_i(\tau,x^*_\tau) = g^i&[\tau,x^*_\tau,\phi_1(\tau,x^*_\tau),\phi_2(\tau,x^*_\tau),\cdots,\phi_n(\tau,x^*_\tau)] \\ &- g^i[\tau,x^*_\tau,\psi_1(\tau,x^*_\tau),\psi_2(\tau,x^*_\tau),\cdots,\psi_n(\tau,x^*_\tau)] \\ &+ V^i_{x^*_\tau}(\tau,x^*_\tau)\left\{f[\tau,x^*_\tau,\phi_1(\tau,x^*_\tau),\phi_2(\tau,x^*_\tau),\cdots,\phi_n(\tau,x^*_\tau)]\right. \\ &- \left. f[\tau,x^*_\tau,\psi_1(\tau,x^*_\tau),\psi_2(\tau,x^*_\tau),\cdots,\psi_n(\tau,x^*_\tau)]\right\}. \end{aligned} \tag{5.15}$$

In formula (5.15) the term

$$g^i[\tau,x^*_\tau,\phi_1(\tau,x^*_\tau),\phi_2(\tau,x^*_\tau),\cdots,\phi_n(\tau,x^*_\tau)] \\ -g^i[\tau,x^*_\tau,\psi_1(\tau,x^*_\tau),\psi_2(\tau,x^*_\tau),\cdots,\psi_n(\tau,x^*_\tau)]$$

yields the difference between agent $i$'s rate of instantaneous payoffs when he uses the noncooperative strategy and that when he adopts the group optimal strategy. The term $V^i_{x^*_\tau}(\tau,x^*_\tau)$ reflects the marginal effects of a change in the state variables on agent $i$'s noncooperative payoff. The term $f[\tau,x^*_\tau,\phi_1(\tau,x^*_\tau),\phi_2(\tau,x^*_\tau),\cdots,\phi_n(\tau,x^*_\tau)]$

yields the instantaneous change of the states over time if the agents act noncooperatively, while the term $f[\tau, x_\tau^*, \psi_1(\tau, x_\tau^*), \psi_2(\tau, x_\tau^*), \cdots, \psi_n(\tau, x_\tau^*)]$ yields the instantaneous change of the states over time if the agents act group optimally.
Hence, the expression

$$V_{x_\tau^*}^i(\tau, x_\tau^*)\,\{f[\tau, x_\tau^*, \phi_1(\tau, x_\tau^*), \phi_2(\tau, x_\tau^*), \cdots, \phi_n(\tau, x_\tau^*)] \\ -f[\tau, x_\tau^*, \psi_1(\tau, x_\tau^*), \psi_2(\tau, x_\tau^*), \cdots, \psi_n(\tau, x_\tau^*)]\}$$

represents the compensation to agent $i$ when the change in the state variable follows the group optimal trajectory instead of the noncooperative path.

To sum up, at time instant $\tau$ the compensation to agent $i$ leading to the noncooperative-equivalent instantaneous rate of payoff $B_i(\tau, x_\tau^*)$ consists of

(i) the compensation on the difference between agent $i$'s rate of instantaneous payoffs when he uses the noncooperative strategy and that when he adopts the group optimal strategy, and

(ii) the compensation to agent $i$ for the difference in the change in the state variable on the group optimal trajectory and that on the noncooperative path.

Finally if a payment $\Lambda_i$, where $\sum_{j=1}^{n} \Lambda_j < W(t_0, x_0) - \sum_{j=1}^{n} V^j(t_0, x_0)$, is given to agent $i \in N$ at terminal time $T$ and all the players are willing to adopt the group optimal strategies, optimal resource use will result and all agents and the planning body (government) will be better off under the group optimal scheme.

## 6. Mechanism Design for Global Environmental Management

After several decades of rapid technological advancement and economic growth, alarming levels of pollutions and environmental degradation are emerging all over the world. Though cooperation in environmental control holds out the best promise of effective action, limited success has been observed. Existing multinational joint initiatives like the Kyoto Protocol can hardly be expected to offer a long-term solution because (i) the plans are limited to a confined set of controls like gas emissions and permits which is unlikely be able to offer an effective mean to reverse the accelerating trend of environmental deterioration, and (ii) there is no guarantee that participants will always be better off and hence be committed within the entire duration of the agreement.

To create a cooperative solution a comprehensive set of environmental policy instruments including taxes, subsidies, technology choices, pollution abatement activities, pollution legislations and green technology R&D has to be taken into consideration. The implementation of such a scheme would inevitably bring about different implications in cost and benefit to each of the participating nations. To construct a cooperative solution that every party would commit to from beginning to end, the proposed arrangement must guarantee that every participant will be better-off and the originally agreed upon arrangement remain effective at any time within the cooperative period for any feasible state brought about by prior optimal behavior.

### 6.1. An Analytical Framework

Consider a global economy which is comprised of $n$ nations. At time instant $s$ the demand system of the outputs of the nations is

$$P_i(s) = f^i[q_1(s), q_2(s), \cdots, q_n(s), s], \quad i \in N \equiv \{1, 2, \cdots, n\}, \tag{6.1}$$

where $P_i(s)$ is the price vector of the output vector of nation $i$ and $q_j(s)$is the output of nation $j$. The demand system (6.1) shows that the world economy is a form of generalized differentiated products oligopoly.
Industrial profits of nation $i$ at time $s$ can be expressed as:

$$f^i[q_1(s), q_2(s), \cdots, q_n(s), s]q_i(s) - c^i[q_i(s), v_i(s)], \quad \text{for} \quad i \in N, \tag{6.2}$$

where $v_i(s)$ is the set of environmental policy instruments of government $i$ and $c^i[q_i(s), v_i(s)]$ is the cost of producing $q_i(s)$ under policy $v_i(s)$.

Profit maximization by the industrial sectors yields a market equilibrium in which nation $i$'s instantaneous output as:

$$q_i^*(s) = \hat{q}^i[v_1(s), v_2(s), \cdots, v_n(s), s] \equiv \hat{q}^i[v(s), s], \quad \text{for} \quad i \in N. \tag{6.3}$$

One can readily observe from (6.3) that each nation's output decision depends on government environmental policies.

Let $x(s) \subset R^m$ denote the level of pollution at time $s$, the dynamics of pollution stock is governed by the stochastic differential equation:

$$\begin{gathered} dx(s) = \left[ \ \sum_{j=1}^n a_j[q_j(s), v_j(s)] - \sum_{j=1}^n b_j[u_j(s), x(s)] - \delta\,[x(s)]x(s) \ \right] ds + \\ +\sigma\,[x(s)]dz(s), \\ x(t_0) = x_{t_0}, \end{gathered} \tag{6.4}$$

where $\sigma$ is a noise parameter and $z(s)$ is a Wiener process,
$a_j[q_j(s), v_j(s)]$ is the amount of pollution created by $q_j(s)$ amount of output produced under policy $v_i(s)$,
$u_j(s)$ is the pollution abatement effort of nation $j$,
$b_j[u_j(s), x(s)]$ is the amount of pollution removed by $u_j(s)$ unit of abatement effort of nation $j$, and $\delta[x(s)]$ is the natural rate of decay of the pollutants.

The governments have to promote business interests and at the same time handle the financing of the costs brought about by pollution. In particular, each government maximizes the gains in the industrial sector plus tax revenue minus expenditures on pollution abatement and damages from pollution. The instantaneous objective of government $i$ at time $s$ can be expressed as:

$$\begin{aligned} f^i[q_1(s), q_2(s), \cdots, q_n(s), s]q_i(s) - c^i[q_i(s), v_i(s)] \\ -c_i^P[v_i(s)] - c_i^a[u_i(s)] - h_i[x(s)], \quad i \in N, \end{aligned} \tag{6.5}$$

where $c_i^P[v_i(s)]$ is the cost of implementing the vector policy instrument $v_i(s)$, $c_i^a[u_i(s)]$ is the cost of employing $u_i$amount of pollution abatement effort, and $h_i[x(s)]$ is the value of damage to country $i$ from $x(s)$ amount of pollution.

The governments' planning horizon is $[t_0, T]$. It is possible that $T$ may be very large. The discount rate is $r$. At time $T$, the terminal appraisal of pollution damage is $g^i[x(T)]$ where $\partial g^i/\partial x < 0$.

Substitute $q_i(s)$, for $i \in N$, from (6.3) into (6.4) and (6.5) one obtains a stochastic differential game in which government $i \in N$ seeks to:

$$\max_{v_i(s),u_i(s)} E_{t_0}\left\{ \int_{t_0}^{T} \left[ f^i\{\hat{q}^1[v(s),s], \hat{q}^2[v(s),s], \cdots, \hat{q}^n[v(s),s], s\}\hat{q}^i[v(s),s] \right.\right.$$
$$\left. -c^i\{\hat{q}^i[v(s),s], v_i(s)\} - c_i^P[v_i(s)] - c_i^a[u_i(s)] - h_i[x(s)] \right] e^{-r(s-t_0)}ds$$
$$\left. +g^i[x(T)]e^{-r(T-t_0)} \right\} \tag{6.6}$$

subject to

$$dx(s) = \left[ \sum_{j=1}^{n} a_j\{\hat{q}^j[v(s),s], v_j(s)\} - \sum_{j=1}^{n} b_j[u_j(s), x(s)] - \delta[x(s)]\,x(s) \right] ds+$$
$$+\sigma[x(s)]dz(s),$$
$$x(t_0) = x_{t_0}, \tag{6.7}$$

Explicit illustrative examples of the above theoretical framework can be found in Yeung (2007) and Yeung and Petrosyan (2008).

### 6.2. International Cooperation in Environmental Control

Now consider the case when all the nations want to cooperate and agree to act so that an international optimum could be achieved. Since nations are asymmetric and the number of nations may be large, a reasonable solution optimality principle for gain distribution is to share the expected gain from cooperation proportional to the nations' relative sizes of expected noncooperative payoffs. Hence the solution imputation scheme $\{\xi^{(\tau)i}(\tau, x_\tau^*)$; for $i \in N\}$ has to satisfy:

***Condition 6.1.***

$$\xi^{(\tau)i}(\tau, x_\tau^*) = V^{(\tau)i}(\tau, x_\tau^*) + \frac{V^{(\tau)i}(\tau, x_\tau^*)}{\sum\limits_{j=1}^{n} V^{(\tau)j}(\tau, x_\tau^*)} \left[ W^{(\tau)}(\tau, x_\tau^*) - \sum_{j=1}^{n} V^{(\tau)j}(\tau, x_\tau^*) \right] =$$

$$= \frac{V^{(\tau)i}(\tau, x_\tau^*)}{\sum\limits_{j=1}^{n} V^{(\tau)j}(\tau, x_\tau^*)} W^{(\tau)}(\tau, x_\tau^*), \tag{6.8}$$

for $i \in N$, $x_\tau^* \in X_\tau^*$ and $\tau \in [t_0, T]$. □

A distribution scheme with a terminal payment $-g^i[x_T^* - \bar{x}^i]$ at time $T$ and an instantaneous payment at time $\tau \in [t_0, T]$:

$$\begin{aligned} B_i(\tau, x_\tau^*) = &- \left[ \xi_t^{(\tau)i}(t, x_t^*)\Big|_{t=\tau} \right] - \frac{1}{2} \sum_{k,j} \sigma^{kj}(x_t^*)\ \xi_{x_k^* x_j^*}^{(t_0)i}(t, x_t^*)\Big|_{t=\tau} \\ &- \left[ \xi_{x_t^*}^{(\tau)i}(t, x_t^*)\Big|_{t=\tau} \right] \left[ \sum_{j=1}^n a_j\{\hat{q}^j[\psi(\tau, x_\tau^*), \tau], \psi_j(\tau, x_\tau^*)\} \right. \\ &\left. - \sum_{j=1}^n b_j[\varpi_j(\tau, x_\tau^*), x_\tau^*] - \delta(x_\tau^*)\, x_\tau^* \right], \quad \text{for} \quad i \in N, \end{aligned} \tag{6.9}$$

yield Condition 6.1.

### 6.3. Policy Design

Using the above analysis as a policy guide, a grand coalition of all nations should be formed to pursue a comprehensive cooperative scheme of industrial pollution abatement. In particular, the entire set of policy instruments available – including environmental taxes and charges, adoption of environment-friendly production technology, subsidy to the replacement of polluting techniques, joint research and development in clean technology, restoration and preservation of the natural ecosystem, and legislations to outlaw environmentally unacceptable practices – will be used achieve an optimal cooperative outcome. A payment distribution mechanism has to be formulated so that cooperative gains will be shared according to the proportions of the nations' relative sizes of expected noncooperative payoffs throughout the planning horizon. Appropriate policy coordination will lead to the enhancement of economic performance and the realization of a cleaner environment.

A particularly relevant mechanism design would be the formation of a United Nations Agency to coordinate international cooperative actions on pollution and climate change. The Agency is would be comprised of three divisions. An executive branch would be established to coordinate adoption and development of clean technology, pollution abatement activities, use of materials, waste disposal, mode of resource extraction and cooperation in environmental R&D. A financial branch (or FUND) would be set up to handle pollution charges, clean technology subsidies, financial aids and payoff distributions so that the agreed upon optimality principle will be realized throughout the cooperative period. Lastly, a legislative body would be in place to enact regulations on the use of dirty technologies, toxic disposal, pollutant emissions, activities damaging the environment and violation of the cooperative agreement.

## 7. Financial Constraint and Irrational-Behavior-Proof

In this section we explore the effects of financial constraint and irrational behavior on mechanism design involving cooperative differential games.

### 7.1. Financial Constraint and Mechanism Design

Consider the $n-$person nonzero-sum differential game with initial state $x_0$ and duration $T - t_0$. The state space of the game is $\mathrm{X} \in R^m$, with permissible state trajectories citation$\{x(s), \quad t_0 \leq s \leq T\}$. The state dynamics of the game is characterized by the vector-valued differential equations:

$$\dot{x}(s) = f[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)], \quad x(t_0) = x_0, \tag{7.1}$$

where $u_i(s) \in R^{m_i}$ is the control vector of player $i$.
The objective of player $i$ is

$$\int_{t_0}^{T} g^i[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]\, e^{-r(s-t_0)} ds + e^{-r(T-t_0)} q^i(x(T)),$$

for $i \in \{1, 2, \cdots, n\} \equiv N$
Let $\{\phi_i(s,x)$, for $i \in N\}$ denote a set of strategies leading to a feedback Nash equilibrium, the game equilibrium strategies can be obtained as:

$$\dot{x}(s) = f\{s, x(s), \phi_1[s, x(s)], \phi_2[s, x(s)], \cdots, \phi_n[s, x(s)]\}, \quad x(t_0) = x_0. \tag{7.2}$$

We denote the solution to (7.2) by $\{\hat{x}(s)\}_{s=t_0}^{T}$, and use the terms $\hat{x}(s)$ and $\hat{x}_s$interchangeably. The noncooperative payoff of player $i$ over the interval $[t, T]$ where $t \in [t_0, T]$ is:

$$\begin{aligned} V^i(t, \hat{x}_t) = \int_t^T & g^i[s, \hat{x}(s), \phi_1(s, \hat{x}(s)), \phi_2(s, \hat{x}(s)), \cdots, \phi_n(s, \hat{x}(s))]\, e^{-r(s-t_0)} ds \\ & + e^{-r(T-t_0)} q^i(\hat{x}(T)), \end{aligned} \tag{7.3}$$

for $i \in \{1, 2, \cdots, n\} \equiv N$, and $x(t) = \hat{x}_t$.

Under cooperation group rationality required the players to maximize their joint payoff

$$\int_{t_0}^{T} \sum_{j=1}^{n} g^j[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)] e^{-r(s-t_0)}\, ds + \sum_{j=1}^{n} e^{-r(T-t_0)} q^j(x(T)) \tag{7.4}$$

subject to (7.1).

Let $\{\psi_i(s,x)$, for $i \in N\}$ denote a set of strategies leading to an optimal control solution of the problem (7.1) and (7.3) and let $\{x^*(s)\}_{s=t_0}^{T}$ denote the optimal cooperative path, the total cooperative payoff over the interval $[t, T]$ where $t \in [t_0, T]$ is:

$$\begin{aligned} W(t, x_t^*) = \int_t^T \sum_{j=1}^{n} & g^j[s, x^*(s), \psi_1(s, x^*(s)), \psi_2(s, x^*(s)), \cdots, \psi_n(s, x^*(s))] \\ & \times e^{-r(s-t_0)} ds + \sum_{j=1}^{n} e^{-r(T-t_0)} q^j(x^*(T)). \end{aligned} \tag{7.5}$$

Let $\xi^i(\tau, x_\tau^*) = \int_\tau^T B_i(s)\, e^{-r(s-\tau)} ds + q^i(x_T^*)$ denote the imputation to player $i$ under cooperation over the time interval $[\tau, T]$ along the cooperative path $\{x_\tau^*\}_{\tau=t_0}^{T}$ for $\tau \in [t_0, T]$.

Since an imputation satisfies group and individual rationalities, we have:

(i) $W(\tau, x_\tau^*) = \sum_{j=1}^{n} \xi^j(\tau, x_\tau^*)$, and

(ii) $\xi^i(\tau, x_\tau^*) \geq V^i(\tau, x_\tau^*)$, for $i \in N$.

In a noncooperative equilibrium, the payoff received by player $i$ in the interval $[t_0, \tau]$ can be expressed as:

$$\begin{aligned} \int_{t_0}^{\tau} g^i[s, \hat{x}(s), \phi_1(s, \hat{x}(s)), \phi_2(s, \hat{x}(s)), \cdots, \phi_n(s, \hat{x}(s))]\, e^{-r(s-t_0)} ds \\ = V^i(t_0, x_0) - V^i(\tau, \hat{x}_\tau). \end{aligned} \tag{7.6}$$

The cooperative payoff received by player $i$ in the interval $[t_0, \tau]$ can be expressed as:

$$\xi^i(\tau, x_\tau^*) = \int_{t_0}^{\tau} B_i(s)\, e^{-r(s-t_0)} ds = \xi^i(t_0, x_0) - \xi^i(\tau, x_\tau^*). \tag{7.7}$$

If player $i$'s cooperative payoff in the in the interval $[t_0, \tau]$ is smaller than his noncooperative payoff in the in the interval $[t_0, \tau]$ and he cannot finance the deficit $\xi^i(t_0, x_0) - \xi^i(\tau, x_\tau^*) - [V^i(t_0, x_0) - V^i(\tau, \bar{x}_\tau)]$, he would reject the optimality principle leading to the imputation $\xi^i(\tau, x_\tau^*)$. Therefore, financial constraints often posted a problem in cooperation.

Consider the analysis in Section 6, sharing the expected gain from cooperation proportional to the nations' relative sizes of expected noncooperative payoffs is reasonable and acceptable to a large number of asymmetric nations. However, the failure of some (developing) nations to finance the deficit

$$\xi^i(t_0, x_0) - \xi^i(\tau, x_\tau^*) - [V^i(t_0, x_0) - V^i(\tau, \bar{x}_\tau)]$$

may create severe strain on the cooperative scheme. Often these nations would request to be exempted from carrying out the optimal strategies (as in the case of the Kyoto Protocol). This is certainly a suboptimal arrangement and could reduce the gain from cooperation substantially. As proposed before, financial aid should be given to these participants (with repayment made later) so that they can carry out the optimal strategies.

### 7.2. Irrational-Behavior-Proof Condition

We have shown that given subgame-consistent imputations satisfy group and individual rationalities throughout the cooperative trajectory, no rational players will deviate from the cooperative path. However, in reality irrational behavior may appear for various reasons. For instance, a player may use 'irrational' acts to extort additional gains if later circumstances allow. Refusal of other players to yield to his extortion would result in the dissolution of the cooperative scheme.

Once again an imputation satisfies group and individual rationalities, and therefore:

(i) $W(\tau, x_\tau^*) = \sum_{j=1}^{n} \xi^j(\tau, x_\tau^*)$, and

(ii) $\xi^i(\tau, x_\tau^*) \geq V^i(\tau, x_\tau^*)$, for $i \in N$.

Moreover, one can write

$$\xi^i(\tau, x_\tau^*) = \int_{\tau}^{T} B_i(s)\, e^{-r(s-t_0)} ds + e^{-r(T-t_0)} q^i(x_T^*), \quad \text{for} \quad i \in N. \tag{7.8}$$

Consider the case where the cooperative scheme has proceeded up to time $\tau$ and some players behave irrationally leading to the dissolution of the scheme.

At time $\tau$ if the condition

$$\int_{t_0}^{\tau} B_i(s)\, e^{-r(s-t_0)} ds + V^i(\tau, x_\tau^*) \geq V^i(t_0, x_0) \tag{7.9}$$

is satisfied player $i$ is irrational-behavior-proof (I-B-P) because irrational actions leading to the dissolution of cooperative scheme will not bring his resultant payoff below his initial noncooperative payoff. On the other hand, if

$$\int_{t_0}^{\tau} B_i(s)\, e^{-r(s-t_0)} ds + V^i(\tau, x_\tau^*) < V^i(t_0, x_0), \tag{7.10}$$

player $i$ is not proofed from irrational-behavior-proof of other player.

To check whether the irrational-behavior-proof condition holds, one can invoke (7.8) and express the I-B-P condition in (7.9) as:

$$\xi^i(t_0, x_0) - \xi^i(\tau, x_\tau^*) + V^i(\tau, x_\tau^*) - V^i(t_0, x_0) \geq 0, \quad \text{for} \quad i \in N. \tag{7.11}$$

In an explicitly solvable game, given a time-consistent and hence continuously differentiable imputation $\xi^i(\tau, x_\tau^*)$ one can verify whether the I-B-P condition is satisfied along the cooperative path $\{x_\tau^*\}_{\tau=t_0}^T$. In particular, the I-B-P condition may or may not be satisfied throughout the game horizon. Given in exact analytical form, the satisfaction of it along the cooperative trajectory could be obtained for explicitly solvable cooperative differential game with time-consistent and differentiable imputations (like those in Jrgensen and Zaccour (2001) and Yeung and Petrosyan (2004, 2006a)).

### 7.3. Properties of the I-B-P Condition

Several interesting properties of the I-B-P condition worth noting.

**Property 1.** At terminal time $T$ and at initial time $t_0$ the I-B-P condition holds.

*Proof.* At time $T$ the I-B-P condition becomes:

$$\xi^i(t_0, x_0) - \xi^i(T, x_T^*) + V^i(T, x_T^*) - V^i(t_0, x_0) \geq 0. \tag{7.12}$$

From (2.3), we have $V^i(T, x_T^*)=q^i(x_T^*)$ and from (7.8) we have $\xi^i(T, x_T^*)=q^i(x_T^*)$. Substituting these into (7.12) yields $\xi^i(t_0, x_0)-V^i(t_0, x_0) \geq 0$ and therefore the I-B-P condition holds.
At initial time $t_0$,

$$\xi^i(t_0, x_0) - \xi^i(t_0, x_0) + V^i(t_0, x_0) - V^i(t_0, x_0) = 0. \tag{7.13}$$

Hence the I-B-P condition again holds. □

**Property 2.** There exists a time interval $[t_i, T]$ for player $i \in N$ such that the I-B-P condition holds.

*Proof.* This is a direct result of Property 1. □

**Property 3.** A sufficient condition for the I-B-P condition to hold for player $i$ throughout the game interval $[t_0, T]$ is that his instantaneous rate of cooperative payment

$$B_i(\tau) \geq -dV^i(\tau, x_\tau^*)/d\tau \quad \text{at all} \quad \tau \in [t_0, T] \tag{7.14}$$

along the cooperative path.

*Proof.* Recall the I-B-P condition in (7.9)

$$\int_{t_0}^{\tau} B_i(s)\, e^{-r(s-t_0)} ds + V^i(\tau, x_\tau^*) \geq V^i(t_0, x_0). \tag{7.15}$$

Note that the equality sign in (7.16) will hold at $\tau = t_0$. If the time derivative on the left-hand-side of inequality (7.16) exceeds that on the right-hand-side all the time, that is

$$B_i(\tau)\, e^{-r(\tau - t_0)} + dV^i(\tau, x^*_\tau)/d\tau \geq 0, \quad \text{for all} \quad \tau \in [t_0, T], \tag{7.16}$$

the I-B-P condition of player $i$will hold throughout the game interval $[t_0, T]$. Hence Property 3 follows. □

Finally, the condition can also be applied to cooperative differential games with nontransferable payoffs. Let $\xi^{(\alpha)i}(\tau, x^\alpha_\tau)$ denote the imputation to player $i$ under cooperation over the time interval $[\tau, T]$ along the cooperative path $\{x^\alpha_\tau\}_{\tau=t_0}^T$ where $\alpha$ is the agreed-upon cooperative weights. Using Yeung's (2004) technique to obtain $\xi^{(\alpha)i}(\tau, x^\alpha_\tau)$ one can arrive at an I-B-P condition for nontranferable payoffs game as:

$$\xi^{(\alpha)i}(t_0, x_0) - \xi^{(\alpha)i}(\tau, x^\alpha_\tau) + V^i(\tau, x^\alpha_\tau) - V^i(t_0, x_0) \geq 0, \quad \text{for} \quad i \in N. \tag{7.17}$$

One can readily show that properties similar to Property 1 and Property 2 hold. The satisfaction of condition (7.17) along the cooperative trajectory can be verified for explicitly solvable cooperative differential game with nontransferable payoffs (like those in Yeung and Petrosyan (2005)). Markovkin (2006) analyzed the I-B-P condition in linear quadratic differential games.

## 8. Concluding Remarks

Economic analysis no longer treats the economic system as given since the appearance of Leonid Hurwicz's pioneering work on mechanism design. The design point of view enlarges our vision and helps economics avoid a narrow focus on existing institutions. The failure of the market to provide an effective mechanism for optimal resource use will arise if there exist imperfect market structure, externalities, imperfect information or public goods. These phenomena are prevalent in the current global economy. As a result not only inefficient outcomes like over-extraction of natural resources had appeared but gravely detrimental events like catastrophe-bound industrial pollution had also emerged under the conventional market system.

Cooperative games suggest the possibility of socially optimal and group efficient solutions to decision problems involving strategic actions. This lecture focuses on cooperative game-theoretic design of mechanisms for optimal resource use. Since resource use is often a dynamic process we concentrate on mechanism design involving an intertemporal framework. Crucial features that are essential for a successful mechanism – individual rationality, group optimality, dynamic consistency, distribution procedures, budget balance, financing, incentives to cooperate and practicable institutional arrangements – are considered. Cooperative game-theoretic mechanism design is used to establish the foundation for an effective policy menu to tackle sub-optimal resource use problems which the conventional market mechanism fails to resolve.

Large scale design of mechanisms is in order for research in optimal resource use involving cooperative dynamic game theory. This analysis is expected to open up a policy forum for the design of new economic institutions and contribute new directions of research in the field of neo-institutional economics.

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