PREFACE

This edited volume contains a selection of papers that are an outgrowth of the first International Conference on Game Theory and Management with a few additional contributed papers. These papers present an outlook of the current development of the theory of games and its applications to management and various domains, in particular, energy, the environment and economics.

The International Conference on Game Theory and Management, a two day conference, was held in St. Petersburg, Russia, in June 28-29, 2007. The conference was organized by Graduate School of Management of St. Petersburg University in collaboration with The International Society of Dynamic Games (Russian Chapter) and Faculty of Applied Mathematics and Control Processes SPU within the framework of a National Priority Project in Education. More than 70 participants from 14 countries had an opportunity to hear state-of-the-art presentations on a wide range of game-theoretic models, both theory and management applications.

Plenary lectures covered different areas of games and management applications. They had been delivered by Professor Robert Aumann, Hebrew University (Israel), Nobel Prize Winner in Economics in 2005; Professor Sergiu Hart, Hebrew University (Israel); Professor David W.K. Yeung, Hong Kong Baptist University (Hong-Kong) and Professor Georges Zaccour, HEC (Canada). The importance of strategic behavior in the human and social world is increasingly recognized in theory and practice. As a result, game theory has emerged as a fundamental instrument in pure and applied research. The discipline of game theory studies decision-making in an interactive environment. It draws on mathematics, statistics, operations research, engineering, biology, economics, political science and other subjects. In canonical form, a game obtains when an individual pursues an objective(s) in a situation in which other individuals concurrently pursue other (possibly conflicting, possibly overlapping) objectives and at the same time the objectives cannot be reached by individual actions of one decision-maker. The problem is then to determine each individual's optimal decision, how these decisions interact to produce equilibria, and the properties of such outcomes. The foundations of game theory were laid some sixty years ago by von Neumann and Morgenstern (1944).

Theoretical research and applications in games are proceeding apace, in areas ranging from aircraft and missile control to inventory management, market development, natural resources extraction, competition policy, negotiation techniques, macroeconomic and environmental planning, capital accumulation and investment. In all these areas game theory is perhaps the most sophisticated and fertile paradigm applied mathematics can offer to study and analyze decision-making under real world conditions.

The papers presented at this first International Conference on Game Theory and Management certainly reflect both the maturity and the vitality of modern day game theory and management science in general, and of dynamic games, in particular. The maturity can be seen from the sophistication of the theorems, proofs, methods and numerical algorithms contained in the most of the papers in these contributions. The vitality is manifested by the range of new ideas, new applications, the growing number of young researchers and the expanding world wide coverage of research centers and institutes from whence the contributions originated.

The contributions demonstrate that GTM 2007 offers an interactive program on a wide range of latest developments in game theory and management. It includes recent advances in topics with high future potential and exiting developments in classical fields.

We thank the 5th year student of the Faculty of Applied Mathematics (SPU) Takaeva Margarita for displaying extreme patience and typesetting the manuscript.

Editors, Leon A. Petrosyan and Nikolay A. Zenkevich

A Two Population Growing Model: Exogamic or Endogamic

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Abstract. We show an analytic model for a situation in which two populations are confronted in an exogamic or endogamic way. Our approach is based on Evolutionary Game Theory for Non-Symmetric Games but considering a new rule of imitation: evolutive regret when the probability of selecting the best strategy is included. The rule states to choose the actions with the best results, with a probability proportional to the expected gains. In particular, we show the relation between Dynamic Strategy and Nash equilibrium in an asymmetric game of imitation strategies.
JEL classification: C70, C72, C69.

Keywords: Imitation, replicator dynamic, stable population, stability and Nash equilibrium.

Introduction

In Evolutionary Game Theory (EGT) we deal with populations of players that are programmed for a certain strategy. Players replicate and pass on their strategy to their offsprings. The number of offsprings is directly related to the average payoff of the parent strategy. EGT can be thinking as a theory that combines the dynamics of animal behavior with population (It is a predatorprey relationship, see [Schlag, 1998]). This article focuses on the relation between dynamical equilibria and Nash equilibria by appealing to an evolutionary process (as defined by Samuelson, and J. Zhang [Samuelson, 1992]) from two populations in the long run, with evolutionary imitation strategies, to give conditions under which dynamic selection processes will yield stability requirement ensures that outcomes will be Nash equilibria. Since, imitation dynamics is more appropriate here than the replicator dynamics that is used in applications of EGT to theoretical biology. According to the imitation dynamics, players are not mortal and have no offsprings. However, every so often, a player is offered the opportunity to pick out some other player and to change his own strategy against the strategy of the other player. The probability that a certain strategy is adopted for imitation is positively correlated to the gain in average utility that is to be expected by this strategy change. So here as well as in the standard model, successful strategies will tend to spread while unsuccessful strategies die out. Moreover, exactly the same strategies are evolutionary stable under the replicator dynamics and under the imitation dynamics (see [Basar, 1982]).

An imitation game is characterized when the players show imitative behavior (see [Hines, 1980]): never imitate an individual that is performing worse than oneself, and imitate individuals doing better with a probability proportional to how much better they perform. It is shown that this behavioral rule results in an adjustment process that can be approximated by the replicator dynamics. The model that we present has a significant difference with the basic models; the asymmetric context with different populations appearing, the exogamic or endogamic situation. Hence, we delete the assumption of a single symmetric population, since interaction takes place between individuals from distinct populations. In economics, we may think of contests between the owner of a territory and an intruder, that is, the problems of migration; the analogy is a tourism economic activity in which resident attitudes toward and perceptions of tourism development efforts. Moreover, our model permits to obtain an evolutionary dynamics coinciding with the Replicator Dynamics. Hence, the replicator dynamics that we show is adequate for modeling social interactions in a population dynamics.

The paper is organized as follows. Experimentation is governed by one initial law of motion, this is explained in section 2. In section 3 we discuss our game where the friendship strategy is implemented by imitating your neighbor where players are nearby located. Subsection 3.1 discuss the dynamic and Nash equilibrium. In section 4 both populations choose their bids on the basis of their friendspast successful experiences in the same population: a successful imitation strategy.

1. A situation description

Let us assume that people at a given time on a given territory can be distinguished into two populations, residents R and migrants M. Each population can be in its turn split into two sets, or clubs, depending upon the strategy being played against the opponent. Suppose that these strategies are: to admit or not the marriage with a member of the other population. We identify these strategies respectively as m and nm. Let $x^\tau \in R^2_+$, where the vector $x^\tau = (x^\tau_m, x^\tau_{mn})$, $\tau \in \{R, M\}$ is normalized so that $x^\tau_m + x^\tau_{mn} = 1$ and each entry is the share of individuals in the respective club over the total population. Each period an individual of a given population τ playing the i-th strategy $i \in \{m, mn\}$ faces the decision whether remaining unchanged or change it with probability r_i^τ. Let p_{ij}^τ be the probability of changing from the i-th to the j-th strategy, hence of moving across clubs. Clearly, $r_i^\tau p_{ij}^\tau$ is the probability of changing from the i-th to j-th club. In the sequel, $e_m = (1, 0)$ and $e_{nm} = (0, 1)$ will indicate vectors of pure strategies, m or nm independently from the population τ.

Now let us assume that the probability to change the strategy is exclusively a function of the proportion of members belonging to each club.

For example, migrants who are considering whether to change or not their strategy, will take into account only the behaviours of those other fellows they are meeting and will totally disregard the residents decisions. Let it be the same for the latter population. Thus, the expected change in the share of players type i in the population τ they belong to, will be given on average by the probability of a j-club member moving into the i-th club multiplied by the existing share of members in their initial club less than the probability of members of the i-th to move over to the j-th club weighted in the same way. For any given population τ this hypothesis yields the following equation of motion:

$$\dot{x}_i^\tau = r_j^\tau p_{ij}^\tau x_j^\tau - r_i^\tau p_{ij}^\tau x_i^\tau, \forall i, j \in \{m, nm\}, j \neq i, \tau \in \{R, M\}. \tag{1}$$

Since (1) is a representation of the interaction of two clubs, or species, in a two-patch environment assuming that individuals behave adaptatively, hence, they maximize Darwanian fitness.

2. Imitating your neighbor

We introduce hereafter the first evolutionary exercise. This is called imitation: between choices each individual may observe the performance of one other individual (see [Hines, 1980], [Samuelson, 1992]). We make an individuals decision as to whether stick to a strategy/club or change over a function of the type of individuals in their own population they encounters. We consider the following assumptions:

(i) let us assume that the decision of an individual depends upon the utility deriving from it $u^\tau(e_i, x_{-\tau})$, where $(-\tau \in \{R, M\}, -\tau \neq \tau)$ and the share of individuals in the same club:

$$r_i^\tau = f_i^\tau(u^\tau(e_i, x_{-\tau})).$$

The function $f_i^\tau(u^\tau(e_i, x_{-\tau}))$ is reasonably interpreted as the propensity of a member of the i-th club to consider switching membership as a function of the expected utility gains from such a choice.

(ii) next, let us assume that once opted for a change, he/she will adopt the strategy chosen by the first population fellow to be encountered, her neighbor, i.e. for any $\tau \in \{R, M\}$

$$p(i \to j \setminus \text{he/she consider to change strategy}) = p_{ij}^\tau = x_j^\tau, i, j \in m, nm.$$

With the above considerations and by (1), we have:

$$\dot{x}_i^\tau = x_j^\tau f_j^\tau(u^\tau(e_j, x^{-\tau}))x_i^\tau - x_i^\tau f_i^\tau(u^\tau(e_i, x^{-\tau}))x_j^\tau, \tag{2}$$

or

$$\dot{x}_i^\tau = (1 - x_i^\tau)x_i^\tau[f_j^\tau(u^\tau(e_j, x^{-\tau})) - f_i^\tau(u^\tau(e_i, x^{-\tau}))]. \tag{3}$$

This is the general form of the dynamical system representing the evolution of a two-population four clubs structure. It is a system of four simultaneous equations with four state variables (a state variable being the share of the club members over their respective population). However, given the normalization rule $x_m^\tau + x_{mn}^\tau = 1$ for each $\tau \in \{R, M\}$, equation (3) can be reduced to two equations with two independent state variables. Taking advantage of this property, from now onwards we choose variables x_C^R and x_m^M with their respective equations.

For a first grasp of the problem, let us assume f_i^τ to be population specific, but the same across all its components independently from club membership, and furthermore to be linear in the utility levels. Thus, it makes sense to think that the propensity to switch behaviour will be decreasing in the level of the utility:

$$f_i^\tau(u^\tau(e_i, x^{-\tau})) = \alpha^\tau - \beta_\tau u^\tau(e_i, x^{-\tau}) \in [0, 1]$$

with $\alpha^\tau, \beta_\tau \geq 0$. To get a full linear form we assume:

$$u^\tau(e_i, x^{-\tau}) = e_i A^\tau x^{-\tau}, i \in \{m, nm\};$$

in other words, utility is a linear function of both variables, through a population-specific matrix of weights or constant coefficients, $A^\tau \in M_{2\times 2}$, ($\tau \in \{R, M\}$). This latter assumption implies that utility levels reflect population specific (and therefore in principle different) properties, i.e. broadly speaking preference structures over their outcomes. This reduces the previous model to a much simplified version:

$$\dot{x}_m^\tau = \beta_\tau x_m^\tau(1 - x_m^\tau)[(1, -1)A^\tau x_C^{-\tau}], \tau \in \{R, M\}, \tag{4}$$

or in full

$$\begin{cases} \dot{x}_m^R = \beta_R x_a^R(1 - x_m^R)(a^R x_m^M + b^R)); \\ \dot{x}_M^R = \beta_M x_m^M(1 - x_m^M)(a^M x_m^R + b^M)). \end{cases}, \tag{5}$$

whose coefficients a^M and b^R depend of course upon the entries of the two population-specific matrices A^M and A^R, respectively.

2.1. Dynamic stability and Nash properties

System (5) admits five stationary states or dynamical equilibria, i.e.

$(0, 0), (0, 1), (1, 0), (1, 1)$ and a positive interior equilibrium $(\bar{x}_m^R, \bar{x}_m^M)$, where

$$\bar{x}_m^R = -\frac{b^R}{a^R}, \qquad \bar{x}_m^M = -\frac{b^T}{a^T}.$$

In fact, the interesting case is when $\bar{P} = (\bar{x}_m^R, \bar{x}_m^M)$ is an equilibrium lying in the interior of the square $C = [0,1] \times [0,1]$, this happening when

$$0 < -\frac{b^R}{a^R} < 1 \qquad \text{and} \qquad 0 < -\frac{b^M}{b^M} < 1,$$

while of course the other four equilibria are the corners of the square itself. We can proceed to inquire about the stability of the five equilibria. These equilibria can be interpreted as follows:

- Clearly the trivial equilibrium is one where none of the residents is inclined to admit migrants, nor any migrant is willing to mix up with residents.
- On the other hand, there is another equilibrium at the opposite corner (where the sharing clubs involve all of their respective population): this is the case where reciprocal integration of the two populations is complete. The two remaining border equilibria show a different club dominating the two populations and in a sense a mismatch between strategies.
- Finally, of course we have the possibly interior equilibrium, this situation implies that certain percentage of each population has a good disposition to the other population, and the rest only accept marriage between persons of his own population.

Stability analysis looks at the properties of the Jacobian of the system (5):

$$J(x_m^R, x_m^M) = \begin{bmatrix} \beta_R(1-2x_m^M)(a^T x_m^R + b^M) & \beta_R a^T x_m^T (1 - x_a^M) \\ \\ \beta_T a^R x_m^R (1 - x_m^R) & \beta_T(1-2x_m^R)(m^R x_C^T + b^R) \end{bmatrix}, \qquad (6)$$

whose value is of course dependent on the population specific matrices, among other things.

- If $b^R < 0$ and $b^M < 0$, the interior equilibrium $\bar{P}$ is the unique attractor in C.
- If $b^R > 0$ and $b^T > 0$, on the contrary, $\bar{P}$ is the unique repulsor, all other four being local attractors in C. The latter is partitioned in four regions, the respective basins of attractions of the corner attractors.
- Otherwise, $\bar{P}$ is a saddle point, with the outset going through C from south-west to north-east, or the contrary. Accordingly the corner equilibria may be attractors or repulsors: in the former case $(0,0)$ and $(1,1)$ will turn up to be repulsors and $(1,0)$ and $(0, 1)$ – attractors; the opposite in the latter case.

The dynamic described above has a ready interpretation in terms of game theory, as matrices A^M and A^R are the pay-off matrices of a bi-matrix game with two players, the population of migrants M and the population R of residents.

The analysis of the relation between dynamical equilibria and Nash equilibria provides useful additional information. The following properties can be easily derived (see, e.g. [Schlag, 1998]).

- Stability is a sufficient condition for a dynamic equilibrium to be a Nash equilibrium.
- Being an interior point is sufficient for an equilibrium to be a Nash equilibrium.

Therefore, for $b^R < 0$ and $b^M < 0$, at the interior equilibrium $\bar{P}$ agents have no incentives to switch away from the chosen strategy, moreover if they happen to be shocked away from it, they will return. Thus, the relative sizes of m and nm communities within either population will be evolutionary stable. On the contrary, i.e. when $b^R > 0$ and $b^M > 0$, the repulsor $\bar{P}$ is such that any slight shock or move away from it, will initialize a path systematically diverging towards one of the corner equilibria where we have homogeneous, or single community, populations. These combined properties lend themselves to interpret some of the population development scenarios we know of. First of all, it shows how fragile can be the cooperative equilibrium associated with the population interaction.

3. Picking up the most successful strategy

Now a migrant or a resident considers changing strategy imitating the most successful strategy played by members of the same population. The rule states to choose the actions with the best results, with a probability proportional to the expected gains. In other words, a migrant (or a resident, respectively) will change over to a different strategy played by another member of the population if, and only if, the latter brings a greater benefit in terms of expected utility.

Let us assume an individual of population $\tau \in \{R, M\}$ in the community $j \in \{m, nm\}$ encountering somebody of the alternative community, $i \neq j$, and that the former will change over to the latter membership/strategy if $u^\tau(e_i, x^{-\tau}) > u^\tau(e_j, x^{-\tau})$.

Note that the utility of each individual depends on his/her own strategy and on the characteristics of the individuals of the other population. Further, we assume that there is some uncertainty in the estimated return of the alternative strategies, so that must estimate the value $u^\tau(e_i, x^{-\tau})$. Thus, the probability of a j-individual to change strategy is given by the probability of encountering a i-individual multiplied by the probability that the estimate of the return:

$$\bar{D} = u^\tau(e_i, x^{-\tau}) - u^\tau(e_j, x^{-\tau})$$

is positive. By such way, the probability p^τ_{ij} of changing from the j-th to the i-th community/strategy will be x_i times $P^\tau(\bar{D} \geq 0)$, the probability of the observing[3] $\bar{D} = u^\tau(e_i, x^{-\tau}) - u^\tau(e_j, x^{-\tau}) > 0$. Finally, $p^\tau_{ij} = x^\tau_{ij} P^\tau(\bar{D} \geq 0)$.

Let us assume that $P^\tau(\bar{D} \geq 0)$ depends upon the true value of the difference $u^\tau(e_i, x^{-\tau}) - u^\tau(e_j, x^{-\tau})$ which is unknown to the i-th individual. That is to say

$$P^\tau(\bar{D} \geq 0) = \phi^\tau(u^\tau(e_j, x^{-\tau}) - u^\tau(e_i, x^{-\tau})).$$

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[3] I.e. the probability of the event where the estimator of the utility associate to the i-th community/strategy, given the characteristics of the other population, be greater than her own strategy given the characteristics of the other population.

Therefore, the probability of a j-th individual in the population τ to observe a positive value of $\bar{D}$ increases with the true value of the difference $u^\tau(e_j, x^{-\tau}) - u^\tau(e_i, x^{-\tau})$. For the sake of simplicity (by considering an existence of an expected utility), let $u^\tau(e_i, x^{-\tau})$ be linear, i.e. $u^\tau(e_i, x^{-\tau}) = e_i A^\tau x^\tau$.

Thus, the probability of a j-th individual to change over to the i-th strategy is

$$p^\tau_{ij}(u^\tau(e_i - e_j, x^\tau)) = \phi^\tau(u^\tau(e_i - e_j, x^{-\tau}))x^\tau_i, \tag{7}$$

the change in the share of the i-players will be given by the probability of a j-player to become an i-player weighted by the relative number of j-players in its population, minus the probability of an i to become a j-player likewise weighted: $\dot{x}^\tau_j = [x^\tau_j p^\tau_{ji} - p^\tau_{ij} x^\tau_i]x^\tau_i$.

In our case the equation becomes

$$\dot{x}^\tau_j = x^\tau_j x^\tau_i[\phi^\tau(u^\tau(e_j - e_i, x^{-\tau})) - \phi^\tau(u^\tau(e_i - e_j, x^{-\tau}))], \tag{8}$$

and its first order approximation is

$$\begin{aligned} \dot{x}^\tau_j &= x^\tau_i \phi^{\tau'}(0, x^\tau)[u^\tau(e_j - e_i, x^\tau) - u^\tau(e_j - e_i, x^{-\tau})] = \\ &= 2\phi^{\tau'}(0, x^\tau)u^\tau(e_i - x^\tau, e^{-\tau})x^\tau_i. \end{aligned} \tag{9}$$

Then, in a neighborhood of an interior stationary point the dynamics is approximately represented by a replicator dynamics multiplied by a constant. Stability analysis of the local type can, therefore, be carried out using the linear part of the nonlinear system[4].

In the special case where ϕ^τ is linear: $\phi^\tau = \lambda_\tau + \mu^\tau u^\tau(e_j - e_i, x^\tau)$ with λ^τ and μ^τ:

$$0 \leq \lambda^\tau + \mu^\tau u(x^\tau, x^{-\tau}) \leq 1, \qquad x \in \left\{z \in R^2_+ : \max z_i \leq 1; i = 1, 2\right\}$$

we get the equation $\dot{x}^\tau_j = 2\mu^\tau u^\tau(e_i - x^\tau_i, x^{-\tau})x^\tau_i$. In this case stability analysis is similar to the one of the model of simple imitation.

4. Conclusion

Although the model is different from the one of simple imitation, as long as the linear approximation is mathematically valid, it permits to obtain an evolutionary dynamics of the same type. Similar conclusions apply to the issue of the relation between stable and Nash equilibria. For further characterization results on the set of Evolutionary Stable Strategies we refer the reader to [Cover, 1991], [Schlag, 1998]. The real justification for this extension of the simpler model lies in the description of an observed behaviour. Here people choose for an expected maximization of benefits, since they do what the others do.

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[4] If the equilibrium is non hyperbolic.

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Power Level Management in Wireless and DSL Networks with Transmission Cost[1]

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Abstract. We study power level management in optimization and game frameworks with assumption that there is a transmission cost. In the optimization framework there is a single decision-maker who assigns network resources and in the game framework players share the network resources according to Nash equilibrium. We study conditions for uniqueness of the Nash equilibrium. Besides we provide a closed form solution to the problems, which allows us to solve it in a finite number of operations, and we also consider a jamming plot of the game.

Keywords: Nash equilibrium, resource allocation, non-linear programming.

Introduction

In wireless networks and DSL access networks the total available power for signal transmission has to be distributed among several resources. In the context of wireless networks the resources may correspond to frequency bands (e.g. as in OFDM), or they may correspond to capacity available at different time slots. In the context of DSL access networks the resources correspond to available frequency tones. This spectrum of problems can be considered in either optimization scenario or as a result of a non-cooperativie game scenario. The optimization scenario leads to "Water Filling Optimization Problem" [Heinzelman, 2000], [Goldsmith, 1997], [Tse, 2005] and the game scenario leads to "Water Filling Game" or "Gaussian Interference Game" [Lai, 2005], [Popescu, 2003], [Rose, 2004], [Yu, 2002]. In the optimization scenario, one needs to maximize a concave function (Shannon capacity) subject to power constraints. The Lagrange multiplier corresponding to the power constraint is determined by a non-linear equation. In the previous works [Heinzelman, 2000], [Goldsmith, 1997], [Tse, 2005] it was suggested to find the Lagrange multiplier by means of a bisection algorithm, where comes the name "Water Filling Problem". Here

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[1] This work is partly supported by joint RFBR and NNSF Grant no.06-01-39005.

we show that the Lagrange multiplier and, hence, the optimal solution of the water filling problem can be found in explicit form with a finite number of operations. In the multiuser context one can view the problem in either cooperative or non-cooperative setting. If a centralized controller wants to maximize the sum of all users' rates, the controller will face a non-convex optimization problem. On the other hand, in the non-cooperative setting, the power allocation problem becomes a game problem where each user perceives the signals of the other users as interference and maximizes a concave function of the noise to interference ratio. In [Lai, 2005], [Weissing, 1991] the spectrum of available resources was continuous, here as in [Popescu, 2003], [Song, 2002] we consider the discrete spectrum of available resources. A natural approach in the non-cooperative setting is the application of the Iterative Water Filling Algorithm (IWFA) [Yu, 2002]. Recently, the authors of [Luo, 2006] proved the convergence of IWFA under fairly general conditions. In the present work we analyze the case of symmetric water filling game with two users and transmission cost. We would like to note that even the symmetric scenario allows us to obtain several important conclusions about the Gaussian Interference Game. Our main result is an explicit form for the Nash equilibrium. In addition, to its mathematical beauty, the explicit solution allows one to find the Nash equilibrium in water filling game in a finite number of operations and to study limiting cases when the crosstalk coefficient is either small or large. As a by-product, we obtain an alternative simple proof of the convergence of the Iterative Water Filling Algorithm. Furthermore, it turns out that the convergence of IWFA slows down when the crosstalk coefficient is large. Using the closed form solution, we can avoid this problem. Finally, we compare the non-cooperative approach with the cooperative approach and conclude that the cost of anarchy is small in the case of small crosstalk coefficients. Applications that can mostly benefit from decentralized non-cooperative power control are ad-hoc and sensor networks with no predefined base stations [Goldsmith, 1997], [Lin, 1997], [Kwon, 1999]. An interested reader can find more references on non-cooperative power control in [Cover, 1991], [Lai, 2005]. We would like to mention that the water filling problem and jamming games with transmission costs have been analyzed in [Altman, 2007].

The structure of the paper is as follows: in the next Section 1, we recall the single decision-maker setup of the water filling optimization problem. Then, in Section 2 we provide its explicit solution. In Sections 3 and 4 we formulate two users symmetric water filling game and characterize its Nash equilibrium. In Section 5 we give an alternative simple proof of the convergence of the iterative water filling algorithm. In Section 6 we give the explicit form of the players' strategy in the Nash equilibrium. In Section 7 we confirm our finding with the help of numerical example. In that section we also show that the cost of anarchy is small when the crosstalk coefficient is small. In Section 8 we consider a jamming plot of the game. We make conclusions in Section 9.

1. Single decision-maker

We consider the following power allocation problem in the case of a single decision-maker. There is a single decision-maker (also called "Transmitter") who wants to send information using n independent resources so as to maximize the Shannon capacity.

The strategy of Transmitter is $T = (T_1, \dots, T_n)$, such that $T_i \geq 0$ for $i \in [1, n]$ and $\sum_{i=1}^{n} T_i \leq \bar{T}$, where $\pi_i > 0$ for $i \in [1, n]$ and $\bar{T} > 0$. As the payoff to Transmitter we take the Shannon capacity minus expenses connected with transmission:

$$v(T) = \sum_{i=1}^{n} \ln\left(1 + \frac{T_i}{N_i^0}\right) - C \sum_{i=1}^{n} T_i,$$

where $N_i^0 > 0$ is the noise level in the sub-carrier i, and $C > 0$ is the cost of transmission of a unit signal. It is worth noting that the case $C = 0$ was investigated in [Avrachenkov, 2007]. Following the standard water-filling approach [Heinzelman, 2000], [Goldsmith, 1997], [Tse, 2005], which assumes application Kuhn–Tacker Theorem, we have the following result.

Theorem 1. *Let* $1/N_1^0 = \max_{i \in [1,n]} 1/N_i^0$, *and*

$$T_i(\omega) = \left[1/(C + \omega) - N_i^0\right]_+ \text{ for } i \in [1, n], \quad \text{and}$$

$H(\omega) = \sum_{i=1}^{n} T_i(\omega)$.

(i) *If* $C \geq 1/N_1^0$, *then* $T^* = (0, \dots, 0)$ *is the unique equilibrium with zero payoff.*

(ii) *If* $C < 1/N_1^0$, *then* $T(\omega^*) = (T_1(\omega^*), \dots, T_n(\omega^*))$ *is the unique equilibrium with zero payoff* $v(T(\omega^*))$, *where*

$$\omega^* = \begin{cases} 0 & \text{for } H(0) \leq \bar{T}, \\ \text{the unique root} & \\ \text{of the equation } H(\omega) = \bar{T} & \text{for } H(0) > \bar{T}. \end{cases}$$

Proof.

First, note that $\frac{\partial^2 v(T)}{T_i^2} = -\frac{1}{(N_i^0 + T_i)^2} < 0$.

Thus, v is concave on T_i and the Kuhn–Tucker Theorem implies that T^* is the optimal strategy if, and only if, there is a non-negative ω, such that

$$\frac{\partial}{\partial T_i} v(T^*) = \frac{1}{N_i^0 + T_i^*} - C \begin{cases} = \omega & \text{for } T_i^* > 0, \\ \leq \omega & \text{for } T_i^* = 0, \end{cases}$$

where

$$\omega \begin{cases} \geq 0 & \text{for } \sum_{i=1}^{n} T_i^* = \bar{T}, \\ = 0 & \text{otherwise.} \end{cases} \tag{10}$$

So,

$$T_i(\omega) = \left[1/(C+\omega) - N_i^0\right]_+ \text{ for } i \in [1,n],$$

which gives a form the optimal strategy has to have, and this form depends on a parameter ω.

Now we have to find the optimal value of ω. First, note that the function $H(\omega)$ has the following properties:

(i) $H(\omega)$ is non-negative and continuous in $(0,\infty)$,

(ii) $H(\omega)$ is positive and strictly decreasing in $(0,[1/N_1^0 - C]_+)$,

(iii) $H(\omega) = 0$ for $\omega \in ([1/N_1^0 - C]_+, \infty)$.

Thus, the equation $H(\omega) = \bar{T}$ has a root, and this root is unique if, and only if, $H(0) > \bar{T}$. This fact together with (21) imply the result.

It is interesting that the result of the previous theorem can be proved straight forward without application the Kuhn–Tacker Theorem. To complete the picture we also give this proof.

Let $T^* = (T_1^*, \dots, T_n^*)$ be the optimal strategy. Let $T^* \neq (0,\dots,0)$. Then, there is a $m \in [1,n]$, such that $T_m^* > 0$. Let ϵ be any enough small positive number and $k \neq m$. Let $T^{\epsilon,k} = (T_1^{\epsilon,k}, \dots, T_n^{\epsilon,k})$ be such that

$$T_i^{\epsilon,k} = \begin{cases} T_m^* - \epsilon & \text{for } i = m, \\ T_k^* + \epsilon & \text{for } i = k, \\ T_i^* & \text{for } i \notin \{m,k\}. \end{cases}$$

It is clear that $T^{\epsilon,k}$ also is a strategy for any enough small positive ϵ. Then, since T^* is the optimal strategy, we have that $v(T^*) \geq v(T^{\epsilon,k})$. Thus,

$$\begin{aligned} &\ln\left(1 + \frac{T_k^*}{N_k^0}\right) - CT_k^* + \ln\left(1 + \frac{T_m^*}{N_m^0}\right) - CT_m^* \geq \\ &\geq \ln\left(1 + \frac{T_k^* - \epsilon}{N_k^0}\right) - C(T_k^* - \epsilon) + \ln\left(1 + \frac{T_m^* + \epsilon}{N_m^0}\right) - C(T_m^* + \epsilon). \end{aligned}$$

So, putting $\epsilon \to 0$ we have that

$$\frac{1}{T_m^* + N_m^0} \leq \frac{1}{T_k^* + N_k^0} \text{ for any } m \neq k.$$

Thus, there is a non-negative ω, such that

$$\frac{1}{T_i^* + N_i^0} - C \quad \begin{cases} = \omega, & \text{for } T_i^* > 0, \\ \leq \omega, & \text{for } T_i^* = 0. \end{cases}$$

Then, further we just have to repeat word by word the previous proof.

2. Closed form solution for water filling problem

In the previous studies of the water filling problems it was suggested to use numerical (e.g., bisection) method to solve the equation (1). Here we propose an explicit form approach for the solution of equation (1).

Without loss of generality we can assume that the sub-carriers are arranged by the noise level as follows:

$$\frac{1}{N_1^0} \geq \frac{1}{N_2^0} \geq \ldots \geq \frac{1}{N_n^0} \geq \frac{1}{N_{n+1}^0},$$

where $N_{n+1}^0 = \infty$.

Theorem 2. *The solution of the water filling optimization problem is given by:*

(i) *if* $\frac{1}{N_1^0} \leq C$, *then* $T^* = (0, \ldots, 0)$,

(ii) *if*

$$\frac{k}{\bar{T} + \sum_{t=1}^{k} N_t^0} \leq C \tag{11}$$

for an integer k, *such that*

$$\frac{1}{N_{k+1}^0} \leq C < \frac{1}{N_k^0}, \quad \text{then} \quad T_i^* = \begin{cases} \frac{1}{C} - N_i^0, & \text{if } i \leq k, \\ 0, & \text{if } i > k, \end{cases} \tag{12}$$

(iii) *otherwise*

$$T_i^* = \begin{cases} \dfrac{\bar{T} + \sum\limits_{t=1}^{k}(N_t^0 - N_i^0)}{k}, & \text{if } i \leq k, \\ 0, & \text{if } i > k, \end{cases} \tag{13}$$

where k *can be found from the following conditions:* $\varphi_k < \bar{T} \leq \varphi_{k+1}$,

$$\varphi_t = \sum_{i=1}^{t}(N_t^0 - N_i^0) \quad \text{for} \quad t \in [1, n]. \tag{14}$$

Proof.

Consider separately two cases: (a) $\omega^* = 0$ and (b) $\omega^* > 0$.

(a) Let $\omega^* = 0$. Consider separately two subcases:

(1) If $C \geq 1/N_1^0$ then, by Theorem 1 (i), $T^* = (0, \ldots, 0)$ and (a) follows.

(2) If $C \leq 1/N_1^0$ then, by Theorem 1 (ii), $\omega^* = 0$ is the optimal one if

$$H(0) = \sum_{i=1}^{n} \left[\frac{1}{C} - N_i^0\right]_+ \leq \bar{T}. \tag{15}$$

Let k be such that

$$\frac{1}{N_{k+1}^0} \le C < \frac{1}{N_k^0}.$$

Then, the optimal strategy T^* given by (4) and (6) implies that

$$\sum_{i=1}^{k} \left(\frac{1}{C} - N_i^0 \right) = \frac{k}{C} - \sum_{i=1}^{k} N_i^0 \le \bar{T}.$$

So, (2) and (ii) follow.

(b) Let $\omega^* > 0$ and $k \in [1, n]$ be such that $\frac{1}{N_k^0} > \omega^* + C \ge \frac{1}{N_{k+1}^0}$. Then,

$$\left[\frac{1}{\omega^* + C} - N_i^0 \right]_+ = \begin{cases} \dfrac{1}{\omega^* + C} - N_i^0 & \text{for } i \in [1, k], \\ 0 & \text{for } i \in [k+1, n]. \end{cases}$$

So,

$$H(\omega^*) = \sum_{i=1}^{k} \left(\frac{1}{\omega^* + C} - N_i^0 \right).$$

Since $H(\omega^*) = \bar{T}$, we have that

$$\omega^* = \frac{k}{\bar{T} + \sum_{i=1}^{k} N_i^0} - C. \tag{16}$$

Because of strictly decreasing H in $(0, 1/N_1^0)$ we can find k from the following conditions:

$$H(1/N_k^0 - C) < \bar{T} \le H(1/N_{k+1}^0 - C).$$

Since

$$\sum_{i=1}^{k} (N_{k+1}^0 - N_i^0) = \sum_{i=1}^{k+1} (N_{k+1}^0 - N_i^0)$$

the switching sub-carrier k can be found from the following equivalent conditions:

$$\varphi_k < \bar{T} \le \varphi_{k+1}, \quad \textit{where} \quad \varphi_t = \sum_{i=1}^{t} (N_t^0 - N_i^0) \text{ for } t \in [1, n]. \tag{17}$$

So, Theorem (1), (7) and (8) imply Theorem (2).

Let us demonstrate the closed form approach by a numerical example. Take $n = 5$, $\bar{T} = 1$, $C = 0$, $N_i^0 = \kappa^{i-1}$, $\kappa = 1.7$, $\pi_i = 1/5$ for $i \in [1, 5]$. Then, as the first step we calculate φ_t for $t \in [1, 5]$. In our case we get (0, 0.14, 0.616, 1.8298, 4.58108). Then, by (5), $k = 3$. Thus, by (4), the optimal water filling strategy is $T^* = (2.53, 1.83, 0.64, 0, 0)$ with payoff 0.438.

3. Symmetric water filling game

In this section we consider game-theoretical formulation of the situation where a few users, namely, two users (Transmitters), try to send information through n resources so as to maximize the quality of the transmitted information. The strategy of Transmitter j is $T^j = (T^j_1, \dots, T^j_n)$ with $T^j_i \geq 0$ and with

$$\sum_{i=1}^{n} T^j_i \leq \bar{T}^j, \tag{18}$$

where $\bar{T}^j > 0$ for $j = 1, 2$. The payoffs to Transmitters are given as follows:

$$v^1(T^1, T^2) = \sum_{i=1}^{n} \ln\left(1 + \frac{\alpha_i T^1_i}{g_i \beta_i T^2_i + N^0_i}\right) - C_1 \sum_{i=1}^{n} T^1_i,$$
$$v^2(T^1, T^2) = \sum_{i=1}^{n} \ln\left(1 + \frac{\beta_i T^2_i}{g_i \alpha_i T^1_i + N^0_i}\right) - C_2 \sum_{i=1}^{n} T^2_i,$$

where $g_i, \alpha_i, \beta_i,\ i \in [1, n], C^1, C^2$ are some positive. These payoffs correspond to Shannon capacities. This is an instance of the Water-Filling or Gaussian Interference Game corresponding a particular case of OFDM wireless network and DSL access network.

In this work we restrict ourselves to the case of symmetric game with equal crosstalk coefficients, namely, $\alpha_i = \beta_i$, $g_i = g$, $N^0_i = N^0_i/\alpha_i$ for $i \in [1, n]$. Then,

$$v^1(T^1, T^2) = \sum_{i=1}^{n} \ln\left(1 + \frac{T^1_i}{g T^2_i + N^0_i}\right) - C_1 \sum_{i=1}^{n} T^1_i,$$
$$v^2(T^1, T^2) = \sum_{i=1}^{n} \ln\left(1 + \frac{T^2_i}{g T^1_i + N^0_i}\right) - C_2 \sum_{i=1}^{n} T^2_i.$$

This situation can correspond, for example, to the scenario when the transmitters are situated at about the same distance from the base station.

We will assume that $C^1 > 0$ and $C^2 > 0$. The case $C^1 = C^2 = 0$ was investigated in [Avrachenkov, 2007].

We shall characterize a Nash Equilibrium of this problem. The strategies (T^{1*}, T^{2*}) constitute a Nash Equilibrium, if for any strategies (T^1, T^2) the following inequalities hold:

$$v^1(T^1, T^{2*}) \leq v^1(T^{1*}, T^{2*}), \quad v^2(T^{1*}, T^2) \leq v^2(T^{1*}, T^{2*}).$$

The goal of this paper is to find Nash Equilibrium. To perform this purpose, first, note that

$$\frac{\partial^2 v^1(T^1, T^2)}{\partial {T^1_i}^2} = -\frac{1}{(N^0_i + g T^2_i + T^1_i)^2} < 0$$

and
$$\frac{\partial^2 v^2(T^1,T^2)}{\partial {T_i^2}^2} = -\frac{1}{(N_i^0 + gT_i^1 + T_i^2)^2} < 0.$$

So, v^1 and v^2 are concave on T_i^1 and T_i^2 respectively, the Kuhn–Tucker Theorem implies the following result, describing the equilibrium structure.

Theorem 3. *(T^{1*},T^{2*}) is a Nash equilibrium if, and only if, there are non-negative ω^1 and ω^2 (Lagrange multipliers) such that*

$$\frac{1}{gT_1^{2*} + T_i^{1*} + N_i^0} - C^1 \begin{cases} = \omega^1 & \text{for } T_i^{1*} > 0, \\ \le \omega^1 & \text{for } T_i^{1*} = 0, \end{cases} \tag{19}$$

$$\frac{1}{gT_1^{1*} + T_i^{2*} + N_i^0} - C^2 \begin{cases} = \omega^2 & \text{for } T_i^{2*} > 0, \\ \le \omega^2 & \text{for } T_i^{2*} = 0, \end{cases} \tag{20}$$

where
$$\omega_1 \begin{cases} \ge 0 & \text{for } \sum_{i=1}^n T_i^{1*} = \bar{T}^1, \\ = 0 & \text{otherwise,} \end{cases} \tag{21}$$

$$\omega_2 \begin{cases} \ge 0 & \text{for } \sum_{i=1}^n T_i^{2*} = \bar{T}^2, \\ = 0 & \text{otherwise.} \end{cases} \tag{22}$$

Let us introduce the following sets:

$$\begin{aligned}
I'_{00}(\omega^1,\omega^2) &= \Big\{ i \in [1,n] : \frac{1}{\omega^1 + C^1} \le N_i^0, \quad \frac{1}{\omega^2 + C^2} \le N_i^0 \Big\}, \\
I'_{10}(\omega^1,\omega^2) &= \Big\{ i \in [1,n] : N_i^0 < \frac{1}{\omega^1 + C^1} \text{ and either } N_i^0 \ge \frac{1}{\omega^2 + C^2} \\
&\qquad \text{or } N_i^0 < \frac{1}{\omega^2 + C^2} \text{ and } \frac{1}{\omega^2 + C^2} - N_i^0 \le g(\frac{1}{\omega^1 + C^1} - N_i^0) \Big\}, \\
I'_{01}(\omega^1,\omega^2) &= \Big\{ i \in [1,n] : N_i^0 < \frac{1}{\omega^2 + C^2} \text{ and either } N_i^0 \ge \frac{1}{\omega^1 + C^1} \\
&\qquad \text{or } N_i^0 < \frac{1}{\omega^1 + C^1} \text{ and } \frac{1}{\omega^1 + C^1} - N_i^0 \le g(\frac{1}{\omega^2 + C^2} - N_i^0) \Big\}, \\
I'_{11}(\omega^1,\omega^2) &= \Big\{ i \in [1,n] : 0 < \\
&\qquad < g(\frac{1}{\omega^2 + C^2} - N_i^0) < \frac{1}{\omega^1 + C^1} - N_i^0 < \frac{1}{g}(\frac{1}{\omega^2 + C^2} - N_i^0) \Big\}.
\end{aligned}$$

The next result characterizes the forms that the Nash equilibrium can take.

Lemma 1. *Let (T^{1*},T^{2*}) be a Nash equilibrium, then*

(i) *if $T_i^{1*} = 0$ and $T_i^{2*} = 0$ then $i \in I'_{00}(\omega^1,\omega^2)$,*

(ii) *if* $T_i^{1*} > 0$ *and* $T_i^{2*} = 0$ *then* $i \in I'_{10}(\omega^1, \omega^2)$ *and* $T_i^{1*} = \frac{1}{\omega^1+C^1} - N_i^0$,

(iii) *if* $T_i^{1*} = 0$ *and* $T_i^{2*} > 0$ *then* $i \in I'_{01}(\omega^1, \omega^2)$ *and* $T_i^{2*} = \frac{1}{\omega^2+C^2} - N_i^0$,

(iv) *if* $T_i^{1*} > 0$ *and* $T_i^{2*} > 0$ *then* $i \in I'_{11}(\omega^1, \omega^2)$ *and*

$$\begin{aligned} T_i^{1*} &= \frac{(\dfrac{1}{\omega^1+C^1} - N_i^0) - g(\dfrac{1}{\omega^2+C^2} - N_i^0)}{1-g^2}, \\ T_i^{2*} &= \frac{(\dfrac{1}{\omega^2+C^2} - N_i^0) - g(\dfrac{1}{\omega^1+C^1} - N_i^0)}{1-g^2}. \end{aligned} \tag{23}$$

Proof.

(i) follows directly from (10) where $T_i^{1*} = T_i^{2*} = 0$.

(ii) Let $T_i^{1*} > 0$ and $T_i^{2*} = 0$. Then, by (10), we have that

$$\frac{1}{T_i^{1*} + N_i^0} = \omega^1 + C^1.$$

Thus, $\dfrac{1}{\omega^1 + C^1} > N_i^0$ and $T_i^{1*} = \dfrac{1}{\omega^1 + C^1} - N_i^0$.
Then, by (10), we have that

$$\omega^2 + C^2 \geq \frac{1}{gT_i^{1*} + N_i^0} = \frac{1}{g(\dfrac{1}{\omega^1 + C^1} - N_i^0) + N_i^0}.$$

Thus,

$$g(\frac{1}{\omega^1 + C^1} - N_i^0) \geq \frac{1}{\omega^2 + C^2} - N_i^0,$$

and the result follows.

(iii) can be proved similarly to (ii).

(iv) Let $T_i^{1*} > 0$ and $T_i^{2*} > 0$. Then, by (10), we have that

$$\frac{1}{\omega^1 + C^1} > N_i^0 \quad \text{and} \quad \frac{1}{\omega^2 + C^2} > N_i^0.$$

Also, by (10), we have that T_i^{1*} and T_i^{2*} are given by (14). Then, since $T_i^{1*} > 0$ and $T_i^{2*} > 0$ we have that $i \in I'_{11}(\omega^1, \omega^2)$. This completes the proof of the lemma.

Based on Lemma 1 and Theorem we straightforward obtain the following three theorems dealing with the situations where the transmission cost is too high either for both users or at least for one of them.

Theorem 4.

$$\min\{C_1, C_2\} \geq \frac{1}{N_1^0}$$

then there is unique equilibrium (T^{1*}, T^{2*}) *and* $T^{1*} = T^{2*} = (0, \dots, 0)$.

Theorem 5. *If*

$$C_1 \geq \frac{1}{N_1^0}, \quad C_2 < \frac{1}{N_1^0}$$

then there is unique equilibrium (T^{1*}, T^{2*}). *Besides,* $T^{1*} = (0, \dots, 0)$ *and* $T^{2*} = T^{2*}(\omega^*) = (T_1^{2*}(\omega^*), \dots, T_n^{2*}(\omega^*))$ *where*

$$T_i^2(\omega) = \left[1/(C_2 + \omega) - N_i^0\right]_+ \text{ for } i \in [1, n]$$

and

$$\omega^* = \begin{cases} 0 & \text{for } H^2(0) \leq \bar{T}^2, \\ \text{the unique root} & \\ \text{of the equation } H^2(\omega) = \bar{T}^2 & \text{for } H^2(0) > \bar{T}^2. \end{cases}$$

with

$$H^2(\omega) = \sum_{i=1}^{n} T_i^2(\omega).$$

Theorem 6. *If*

$$C_2 \geq \frac{1}{N_1^0}, \quad C_1 < \frac{1}{N_1^0}$$

then there is unique equilibrium (T^{1*}, T^{2*}). *Besides,* $T^{2*} = (0, \dots, 0)$ *and* $T^{1*} = T^{1*}(\omega^*) = (T_1^{1*}(\omega^*), \dots, T_n^{1*}(\omega^*))$ *where*

$$T_i^1(\omega) = \left[1/(C_1 + \omega) - N_i^0\right]_+ \text{ for } i \in [1, n]$$

and

$$\omega^* = \begin{cases} 0 & \text{for } H^1(0) \leq \bar{T}^1, \\ \text{the unique root} & \\ \text{of the equation } H^1(\omega) = \bar{T}^1 & \text{for } H^1(0) > \bar{T}^1. \end{cases}$$

with $H^1(\omega) = \sum_{i=1}^{n} T_i^1(\omega)$. So, now we can restrict our attention to the case $\max\{C_1, C_2\} < \frac{1}{N_1^0}$.

Although the game has symmetric nature there are some non-symmetric features impacted by the fact that the Lagrange multiplies are different as a rule. This difference will allow us to simplify the structure of the sets I' and the strategies. For this purpose, first, introduce the following auxiliary notations for positive ω^1 and ω^2:

(i) if $\omega^1 + C^1 < \omega^2 + C^2$, so $1/(\omega^2 + C^2) < 1/(\omega^1 + C^1)$ then let

$$\begin{aligned}
&I_{00}(\omega^1,\omega^2) = \left\{i \in [1,n] : \frac{1}{\omega^1 + C^1} \le N_i^0\right\},\\
&I_{10}(\omega^1,\omega^2) = \left\{i \in [1,n] : \frac{\dfrac{1}{\omega^2 + C^2} - \dfrac{g}{\omega^1 + C^1}}{1-g} \le N_i^0 < \frac{1}{\omega^1 + C^1}\right\},\\
&I_{01}(\omega^1,\omega^2) = \emptyset,\\
&I_{11}(\omega^1,\omega^2) = \left\{i \in [1,n] : N_i^0 < \frac{\dfrac{1}{\omega^2 + C^2} - \dfrac{g}{\omega^1}}{1-g}\right\};
\end{aligned}$$

(ii) if $\omega^2 + C^2 < \omega^1 + C^1$, so $1/(\omega^1 + C^1) < 1/(\omega^2 + C^2)$ then let

$$\begin{aligned}
&I_{00}(\omega^1,\omega^2) = \left\{i \in [1,n] : \frac{1}{\omega^2 + C^2} \le N_i^0\right\},\\
&I_{10}(\omega^1,\omega^2) = \emptyset,\\
&I_{01}(\omega^1,\omega^2) = \left\{i \in [1,n] : \frac{\dfrac{1}{\omega^1 + C^1} - \dfrac{g}{\omega^2 + C^2}}{1-g} \le N_i^0 < \frac{1}{\omega^2 + C^2}\right\},\\
&I_{11}(\omega^1,\omega^2) = \left\{i \in [1,n] : N_i^0 < \frac{\dfrac{1}{\omega^1} - \dfrac{g}{\omega^2}}{1-g}\right\};
\end{aligned}$$

(iii) if $\omega^2 + C^2 = \omega^1 + C^1$ then let

$$\begin{aligned}
&I_{00}(\omega^1,\omega^2) = \{i \in [1,n] : \frac{1}{\omega^2 + C^2} \le N_i^0\},\\
&I_{10}(\omega^1,\omega^2) = \emptyset, \quad I_{01}(\omega^1,\omega^2) = \emptyset,\\
&I_{11}(\omega^1,\omega^2) = \{i \in [1,n] : N_i^0 < \frac{1}{\omega^2 + C^2}\}.
\end{aligned}$$

The next lemma asserts that the sets I' coincide with the sets I.

Lemma 2. *The following relations between sets I' and I hold for any non-negative ω^1 and ω^2.*

(i) $I'_{00}(\omega^1,\omega^2) = I_{00}(\omega^1,\omega^2)$,

(ii) $I'_{10}(\omega^1,\omega^2) = I_{10}(\omega^1,\omega^2)$,

(iii) $I'_{01}(\omega^1,\omega^2) = I_{01}(\omega^1,\omega^2)$,

(iv) $I'_{11}(\omega^1,\omega^2) = I_{11}(\omega^1,\omega^2)$.

Proof.

Let, for example,

$$\omega^1 + C^1 < \omega^2 + C^2. \tag{24}$$

(i) is obvious.

(ii) Let $i \in I'_{10}(\omega^1, \omega^2)$. Thus, either $\frac{1}{\omega^2+C^2} \le N_i^0 < \frac{1}{\omega^1+C^1}$ or

$$\frac{\dfrac{1}{\omega^2 + C^2} - \dfrac{g}{\omega_1}}{1-g} \le N_i^0 < \min\left\{\frac{1}{\omega^1 + C^1}, \frac{1}{\omega^2 + C^2}\right\}.$$

Then, by (15), we obtain

$$\min\left\{\frac{1}{\omega^1 + C^1}, \frac{1}{\omega^2 + C^2}\right\} = \frac{1}{\omega^2 + C^2}$$

and

$$\frac{\dfrac{1}{\omega^2 + C^2} - \dfrac{g}{\omega^1 + C^1}}{1-g} < \frac{1}{\omega^2 + C^2}.$$

Thus, $I'_{10}(\omega^1, \omega^2) = I_{10}(\omega^1, \omega^2)$.

(iii) Let $i \in I'_{01}(\omega^1, \omega^2)$. Thus, either

$$\frac{1}{\omega^1 + C^1} \le N_i^0 < \frac{1}{\omega^2 + C^2}$$

or

$$\frac{\dfrac{1}{\omega^1 + C^1} - \dfrac{g}{\omega^2 + C^2}}{1-g} \le N_i^0 < \min\left\{\frac{1}{\omega^1 + C^1}, \frac{1}{\omega^2 + C^2}\right\}.$$

Then, by (15), we obtain

$$\frac{\dfrac{1}{\omega^1 + C^1} - \dfrac{g}{\omega^2 + C^2}}{1-g} > \frac{1}{\omega^2 + C^2}.$$

So, $I'_{01}(\omega^1, \omega^2) = \emptyset = I_{01}(\omega^1, \omega^2)$.

(iv) Let $i \in I'_{11}(\omega^1, \omega^2)$. So,

$$N_i^0 < \min\Big\{\frac{\dfrac{1}{\omega^1 + C^1} - \dfrac{g}{\omega^2 + C^2}}{1-g}, \frac{\dfrac{1}{\omega^2 + C^2} - \dfrac{g}{\omega^1 + C^1}}{1-g}\Big\} = \frac{\dfrac{1}{\omega^2 + C^2} - \dfrac{g}{\omega^1 + C^1}}{1-g}.$$

Thus, $I'_{11}(\omega^1, \omega^2) = I_{11}(\omega^1, \omega^2)$. This completes the proof of Lemma 2.

Now we introduce some strategies, which the Nash Equilibrium will have the form of. Namely, for non-negative ω^1 and ω^2, such that $\omega^1 + C^1 \le \omega^2 + C^2$, and for $i \in [1, n]$ we introduce the following notations:

$$T_i^1(\omega^1, \omega^2) = \begin{cases} \dfrac{1}{1+g}\left(\dfrac{\dfrac{1}{\omega^1 + C^1} - \dfrac{g}{\omega^2 + C^2}}{1-g} - N_i^0\right) & \text{if } N_i^0 < \dfrac{\dfrac{1}{\omega^2 + C^2} - \dfrac{g}{\omega^1 + C^1}}{1-g}, \\ \dfrac{1}{\omega^1 + C^1} - N_i^0 & \text{if } \dfrac{\dfrac{1}{\omega^2 + C^2} - \dfrac{g}{\omega^1 + C^1}}{1-g} \le N_i^0 \\ & \text{and } N_i^0 < \dfrac{1}{\omega^1 + C^1}, \\ 0 & \text{if } \dfrac{1}{\omega^1 + C^1} \le N_i^0, \end{cases}$$

$$T_i^2(\omega^1, \omega^2) = \begin{cases} \dfrac{1}{1+g}\left(\dfrac{\dfrac{1}{\omega^2 + C^2} - \dfrac{g}{\omega^1 + C^1}}{1-g} - N_i^0\right) & \text{if } N_i^0 < \dfrac{\dfrac{1}{\omega^2 + C^2} - \dfrac{g}{\omega^1 + C^1}}{1-g}, \\ 0 & \text{if } \dfrac{\dfrac{1}{\omega^2 + C^2} - \dfrac{g}{\omega^1 + C^1}}{1-g} \le N_i^0; \end{cases}$$

either in the following equivalent form as follows

$$T_i^2(\omega^1, \omega^2) = \begin{cases} \dfrac{1}{1+g}\left(t^2 - N_i^0\right) & \text{if } N_i^0 < t^2, \\ 0 & \text{if } t^2 \le N_i^0, \end{cases}$$

$$T_i^1(\omega^1, \omega^2) = \begin{cases} \dfrac{1}{1+g}\left((1+g)t^1 - gt^2 - N_i^0\right) & \text{if } N_i^0 < t^2, \\ t^1 - N_i^0 & \text{if } t^2 \le N_i^0 < t^1, \\ 0 & \text{if } t^1 \le N_i^0, \end{cases}$$

where

$$t^2 = \frac{\dfrac{1}{\omega^2 + C^2} - \dfrac{g}{\omega^1 + C^1}}{1-g}, \quad t^1 = \frac{1}{\omega^1 + C^1}. \tag{25}$$

It is clear that

$$\frac{1}{\omega^1 + C^1} = t^1, \quad \frac{1}{\omega^2 + C^2} = gt^1 + (1-g)t^2. \tag{26}$$

For the case $\omega^1 + C^1 > \omega^2 + C^2$, $T_i^1(\omega^1, \omega^2)$ and $T_i^2(\omega^1, \omega^2)$ can be defined by symmetry.

The next result simplifies the form of the Nash equilibrium, given by Lemma 1, and it shows that the strategies are not so symmetric as it could be expected and their non-symmetric structure is motivated by difference in Lagrange multipliers and in the power of the signals the players have to transfer.

Theorem 7. *Each Nash equilibrium is of the form* $(T^1(\omega^1,\omega^2),T^2(\omega^1,\omega^2))$ *for some non-negative* ω^1 *and* ω^2.

The next result shows that there is a monotonous dependence between the power of the signals the players have to transfer and Lagrange multipliers.

Corollary 1. *Let* $(T^1(\omega^1,\omega^2),T^2(\omega^1,\omega^2))$ *be a Nash equilibrium and* $\omega_1\omega_2>0$. *If*

$$\bar{T}^1>\bar{T}^2 \tag{27}$$

then

$$\omega^1+C^1<\omega^2+C^2. \tag{28}$$

Proof.

Assume that (19) does not hold. Taen $\omega^1+C^1\geq\omega^2+C^2$, and so

$$\frac{1}{\omega^1+C^1}-\frac{g}{\omega^2+C^2}\leq\frac{1}{\omega^2+C^2}-\frac{g}{\omega^1+C^1}.$$

Thus,

$$T_i^1(\omega^1,\omega^2)\leq T_i^2(\omega^1,\omega^2) \text{ for } i\in[1,n].$$

Hence,

$$\bar{T}^1=\sum_{i=1}^n T_i^1(\omega^1,\omega^2)\leq\sum_{i=1}^n T_i^2(\omega^1,\omega^2)=\bar{T}^2.$$

This contradiction to (18) completes the proof of Corollary 1.

To find the equilibrium strategies we have to find ω^1 and ω^2 such that the following conditions hold:

$$H^1(\omega^1,\omega^2)=\bar{T}^1,\quad H^2(\omega^1,\omega^2)=\bar{T}^2 \text{ for } \omega^1\omega^2>0, \tag{29}$$

or

$$H^1(\omega^1,0)=\bar{T}^1,\quad H^2(\omega^1,0)\leq\bar{T}^2 \text{ for } \omega^1>0,\,\omega^2=0, \tag{30}$$

or

$$H^1(0,\omega^2)\leq\bar{T}^1,\quad H^2(0,\omega^2)\leq\bar{T}^2 \text{ for } \omega^1=0,\,\omega^2>0, \tag{31}$$

or

$$H^1(0,0)\leq\bar{T}^1,\quad H^2(0,0)\leq\bar{T}^2 \text{ for } \omega^1=\omega^2=0, \tag{32}$$

where

$$H^1(\omega^1,\omega^2)=\sum_{i=1}^n T_i^1(\omega^1,\omega^2) \text{ and } H^2(\omega^1,\omega^2)=\sum_{i=1}^n T_i^2(\omega^1,\omega^2).$$

It is clear that $H^1(\omega^1,\omega^2)$ and $H^2(\omega^1,\omega^2)$ have the following properties, collected in the next Lemma, which follow directly from the explicit formulas of the Nash Equilibrium.

Lemma 3.

(i) *$H^1(\omega^1, \omega^2)$ and $H^2(\omega^1, \omega^2)$ are nonnegative and continuous,*

(ii) *$H^1(\omega^1, \omega^2)$ is non-increasing on ω^1 and $H^2(\omega^1, \omega^2)$ is non-increasing on ω^2,*

(iii) *$H^1(\omega^1, \omega^2) = 0$ for enough big ω^1 and $H^3(\omega^1, \omega^2) = 0$ for enough big ω^3, say, for $\omega^1 \geq 1/N_1^0$ and $\omega^2 \geq 1/N_1^0$ respectively,*

(iv) *$H^1(\omega^1, \omega^2)$ is non-decreasing by ω^2 and $H^2(\omega^1, \omega^2)$ is non-decreasing by ω^1.*

Without loss of generality we can assume that

$$\bar{T}^1 \geq \bar{T}^2. \tag{33}$$

Introduce the following notations:

$$\tilde{H}^2(t^2) = \frac{1}{1+g} \sum_{\{i:t^2 > N_i^0\}} \pi_i (t^2 - N_i^0),$$

$$\tilde{H}^1(t^1, t^2) = \sum_{\{i:t^2 \leq N_i^0 < t^1\}} \pi_i (t^1 - N_i^0) + \sum_{\{i:N_i^0 < t^2\}} \pi_i \frac{1}{1+g} \left((1+g)t^1 - gt^2 - N_i^0\right).,$$

where $t^2 < t^1$.

Lemma 4. *Let (24) holds. Then the system of non-linear equations (20) has unique positive solution (ω_*^1, ω_*^2) if, and only if,*

$$\tilde{H}^2(1/C^1) > \bar{T}^2 \text{ and } \tilde{H}^2(1/C^1, t_*^2) > \bar{T}^1$$

where t_*^2 is the unique positive root of the equation:

$$\tilde{H}^2(t_*^2) = \bar{T}^2. \tag{34}$$

Proof.

Let (ω^1, ω^2) be the positive solution of (20). Then, by Corollary 1, $\omega^1 + C^1 < \omega^2 + C^2$. Thus, instead of the system of equation (20) with variables ω^1 and ω^2 we can consider the following equivalent system of equation (26) with variables t^1 and t^2 where $0 < t^2 \leq t^1 \leq 1/C^1$:

$$\tilde{H}^2(t^2) = \bar{T}^2, \quad \tilde{H}^1(t^1, t^2) = \bar{T}^1. \tag{35}$$

Consider the first equation of (26). It is clear that the function $\tilde{H}^2$ has the following properties:

(i) $\tilde{H}^2(\cdot)$ is continuous in $(0, 1/C^1)$,

(ii) $\tilde{H}^2(\tau) = 0$ for $\tau \leq N_1^0$,

(iii) $\tilde{H}^2(\cdot)$ is strictly increasing in $(N_1^0, 1/C^1)$.

Thus, if $\tilde{H}^2(1/C^1) > \bar{T}^2$ then there is the unique root t_*^2 of (25).

Now we pass on to considering the second equation of (26). It is obvious that the function $\tilde{H}^1$ has the following properties:

(i) $\tilde{H}^1(\cdot, t_*^2)$ is continuous and increasing in $(t_*^2, 1/C^1)$,

(ii) $\tilde{H}^1(t_*^2, t_*^2) = \tilde{H}^2(t_*^2) = \bar{T}^2 \le \bar{T}^1$.

So, if $\tilde{H}^1(1/C^1, t_*^2) > \bar{T}^2$ there is the unique positive t_*^1 such that

$$\tilde{H}^1(t_*^1, t_*^2) = \bar{T}^1. \tag{36}$$

So, the system (26) has the unique solution (t_*^1, t_*^2). Thus, (20) also has the unique solution and it can be found by (17). This completes the proof of Lemma 4.

Lemma 1 and proof of Lemma 4 imply the following main results describing the Nash equilibrium.

Theorem 8. *Let (24) holds and*

$$\tilde{H}^2(1/C^1) > \bar{T}^2 \text{ and } \tilde{H}^1(1/C^1, t_*^2) > \bar{T}^1.$$

The symmetric water filling game has the unique Nash equilibrium $(T^1(\omega_*^1, \omega_*^2)$, $T^2(\omega_*^1, \omega_*^2))$ *for* $g \in (0,1)$, *where* ω_*^1, ω_*^2 *can be found through* t_*^1 *and* t_*^2 *from (16) which are the unique solution of the triangular system of equations (26).*

Theorem 9. *Let (24) holds and*

$$\tilde{H}^2(1/C^1) > \bar{T}^2 \text{ and } \tilde{H}^1(1/C^1, t_*^2) < \bar{T}^1.$$

The symmetric water filling game has the unique Nash equilibrium $(T^1(0, \omega_*^2)$, $T^2(0, \omega_*^2))$ *for* $g \in (0,1)$, *where* ω_*^2 *can be found through* $t_*^1 = 1/C^1$ *and* t_*^2 *from (16).*

Theorem 10. *Let (24) holds and* $\tilde{H}^2(1/C^1) \le \bar{T}^2$. *The symmetric water filling game has the unique Nash equilibrium* $(T^1(0,0)$, $T^2(0,0))$.

4. Ununiqueness of Nash Equilibrium

The assumption that $g < 1$ is essential for the uniqueness of Nash Equilibrium as it is shown in the following Proposition.

Proposition 1. *For* $g = 1$ *and and* $C_1 = C_2 = 0$ *the symmetric water filling game has a continuum of Nash equilibria.*

Proof.

Suppose that (T^{1*}, T^{2*}) be a Nash equilibrium. Then, similarly to Lemma 1, we have to consider three cases (i)–(iii) where at least one of the components of the vector (T^{1*}, T^{2*}) is positive.

(i) Let $T_i^{1*} > 0$ and $T_i^{2*} = 0$. Then, by (10), we have that $\dfrac{1}{T_i^{1*} + N_i^0} = \omega^1$. Thus,

$$\frac{1}{\omega^1} > N_i^0 \text{ and } T_i^{1*} = \frac{1}{\omega^1} - N_i^0.$$

Then, by (10),

$$\omega^2 \geq \frac{1}{T_i^{1*} + N_i^0} = \frac{1}{\dfrac{1}{\omega^1} - N_i^0 + N_i^0} = \omega^1.$$

(ii) Let $T_i^{2*} > 0$ and $T_i^{1*} = 0$. Then, similarly to (ii), we have that

$$T_i^{2*} = \frac{1}{\omega^2} - N_i^0 \quad \text{and} \quad \frac{1}{\omega^2} > N_i^0, \quad \omega^1 \geq \omega^2.$$

(iii) Let $T_i^{1*} > 0$ and $T_i^{2*} > 0$. Then, by (10), we have that

$$\frac{1}{T_i^{1*} + T_i^{2*} + N_i^0} = \omega^1 = \omega^2.$$

Assume that $\omega^1 > \omega^2$ then (i) does not hold, so $T_i^{1*} = 0$ for each i which contradicts to (9). Similarly, the case $\omega^1 < \omega^2$ cannot hold.

Thus, $\omega^1 = \omega^2 = \omega$. So, T_i^{1*} and T_i^{2*}, $i \in [1, n]$ have to be any non-negative such that

$$T_i^{1*} + T_i^{2*} = [1/\omega - N_i^0]_+ \quad \text{and} \quad \sum_{i=1}^n T_i^{1*} = \bar{T}^1, \sum_{i=1}^n T_i^{2*} = \bar{T}^2,$$

where ω is the unique positive root of the equation

$$\sum_{i=1}^n [1/\omega - N_i^0]_+ = \bar{T}^1 + \bar{T}^2.$$

It is clear that there is a continuum of such strategies. For example, if (T^{1*}, T^{2*}) is the one of them, and let $T_k^{1*}, T_k^{2*} > 0$ and $T_m^{1*}, T_m^{2*} > 0$ for some k and m. Then, it is clear that the following strategies for any enough small positive ϵ are also optimal:

$$\tilde{T}_i^{1*} = \begin{cases} T_i^{1*} & \text{for } i \neq k, m, \\ T_i^{1*} + \epsilon & \text{for } i = k, \\ T_i^{1*} - \epsilon \pi_k / \pi_m & \text{for } i = m, \end{cases}$$

$$\tilde{T}_i^{2*} = \begin{cases} T_i^{2*} & \text{for } i \neq k, m, \\ T_i^{2*} - \epsilon & \text{for } i = k, \\ T_i^{2*} + \epsilon \pi_k / \pi_m & \text{for } i = m. \end{cases}$$

This completes the proof of Proposition 1.

5. Convergence of an IWFA

In this section we describe a version of the water filling algorithm for finding the Nash Equilibrium for $C^1 = C^2 = 0$ and supply a simple proof of its convergence based on some monotonicity properties. These properties give a simple proof of the convergence of the following water filling algorithm for finding the Nash Equilibrium:

Step 1. Let ω_0^1 and ω_0^2 be such that $H^1(\omega_0^1, \omega_0^2) = H^2(\omega_0^1, \omega_0^2) = 0$, for example $\omega_0^1 = \omega_2^0 = 1/N_1^0$.

Step 2. Let $\omega_1^2 = \omega_0^2$ and define ω_1^1 such that $H^1(\omega_1^1, \omega_1^2) = \bar{T}^1$. Such ω_1^1 exists by Lemma 3 (i)–(iii).

Step 3. Then, by Lemma 3 (i), (v), $H^2(\omega_1^1, \omega_1^2) = 0$. Let $\omega_2^1 = \omega_1^1$ and define ω_2^2 such that $H^2(\omega_2^1, \omega_2^2) = \bar{T}^2$.

Step 4. Then, by Lemma 3 (v), $H^1(\omega_2^1, \omega_2^2) \le \bar{T}^1$, and so on.

So, we have non-increasing positive sequence (ω_k^1, ω_k^2). Thus, it converges to an (ω_*^1, ω_*^2) which produces a Nash Equilibrium.

6. Closed form solution for symmetric water filling game

In this section, based on the proof of Lemma 4, we propose the solution of the two players symmetric water filling game in the closed form for $C^1 = C^2 = 0$.

Without lost of generality we can assume that $\bar{T}_1 > \bar{T}_2$. Let k_2 be such that $N_{k_2+1}^0 \ge t_*^2 > N_{k_2}^0$. Then, since $\tilde{H}^2(t_*^2) = \bar{T}^2$, we have that

$$t_*^2 = \frac{(1+g)\bar{T}^2 + \sum_{i=1}^{k_2} N_i^0}{k_2}. \tag{37}$$

Since $\tilde{H}^2(\cdot)$ is strictly increasing, k_2 can be found from the condition

$$\tilde{H}^2(N_{k^2}^0) < \bar{T}^2 \le \tilde{H}^2(N_{k^2+1}^0).$$

Hence, k^2 can be found from the following equivalent conditions:

$$\varphi_{k^2}^2 < \bar{T}^2 \le \varphi_{k^2+1}^2, \tag{38}$$

where

$$\varphi_k^2 = \frac{1}{1+g} \sum_{i=1}^{k} (N_k^0 - N_i^0) \text{ for } k \le n \text{ and } \varphi_{n+1}^2 = \infty.$$

Since t_*^1 is the root of the equation $\tilde{H}^1(\cdot, t_*^2) = \bar{T}^1$, there is $k^1 \ge k^2$ such that $N_{k^1+1}^0 \ge t_*^1 > N_{k^1}^0$. So,

(i) if $k^1 > k^2$ then

$$t_*^1 = \frac{\bar{T}^1 + \sum_{i=k^2+1}^{k^1} N_i^0 + \frac{1}{1+g} \sum_{i=1}^{k^2} (gt_2^* + N_i^0)}{k^1}, \tag{39}$$

(ii) if $k^1 = k^2$ then

$$t_*^1 = \frac{\bar{T}^1 + \frac{1}{1+g} \sum_{i=1}^{k^2} (gt_2^* + N_i^0)}{k^1}. \tag{40}$$

Thus, $k^1 \geq k^2$ can be found as follows:

(i) if $\bar{T}^1 \leq \varphi_{k^2+1}^1$ then $k^1 = k^2$,

(ii) if $\bar{T}^1 > \varphi_{k^2+1}^1$ then k^1 is given by the condition:

$$\varphi_{k^1}^1 < \bar{T}^1 \leq \varphi_{k^1+1}^1, \tag{41}$$

where

$$\varphi_k^1 = \sum_{i=k^2+1}^{k} (N_k^0 - N_i^0) + \frac{1}{1+g} \sum_{i=1}^{k^2} \big((1+g)N_k^0 - N_i^0 - gt_*^2\big) \text{ for } k \in [k_2+1, n]$$

and

$$\varphi_{n+1}^1 = \infty.$$

We can summarize the obtained results in the following theorem.

Theorem 11. *Let $\bar{T}_1 > \bar{T}_2$. Then, the Nash equilibrium strategies are given by*

$$T_i^{1*} = \begin{cases} t_*^1 - \dfrac{gt_*^2 + N_i^0}{1+g} & \textit{if } i \in [1, k^2], \\ t_*^1 - N_i^0 & \textit{if } i \in [k^2+1, k^1], \\ 0 & \textit{if } i \in [k^1+1, n], \end{cases}$$
$$T_i^{2*} = \begin{cases} \dfrac{1}{1+g}(t_*^2 - N_i^0) & \textit{if } i \in [1, k^2], \\ 0 & \textit{if } i \in [k^2+1, n], \end{cases} \tag{42}$$

where k^2, t_*^2, k^1 and t_*^1 are given by (29), (28), (32) and (30).

7. Numerical example

Let us demonstrate the closed form approach by a numerical example. Take $n = 5$, $g = 0.9$, $\bar{T}^1 = 5$, $\bar{T}^2 = 0.5$, $N_i^0 = \kappa^{i-1}$, $\kappa = 1.7$, $\pi_i = 1/5$ for $i \in [1, 5]$ and $C^1 = C^2 = 0$. Then, as the first step we calculate φ_t^2 for $t \in [1, 5]$.

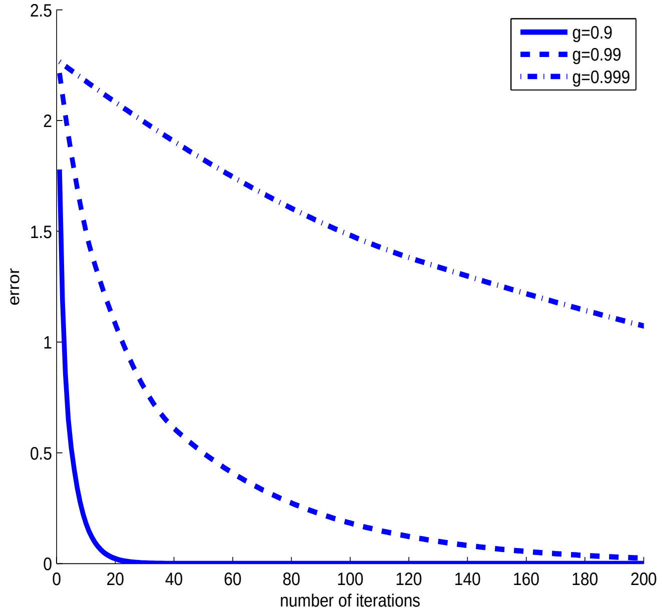

Fig. 1: Convergence of IWFA

In our case we get (0, 0.074, 0.324, 0.963, 2.411). Then, by (29), $k^2 = 3$. Thus, by (28), $t_*^2 = 3.447$. Then we calculate φ_t^1 for $t \in [4, 5]$. In our case we get (1.380, 4.131). So, by (32), $k^1 = 5$. Using (30), we find $t_*^1 = 9.221$. Thus, by (33) we have the following equilibrium strategies $T^{1*} = (7.062, 6.694, 6.067, 4.308, 0.869)$ and $T^{2*} = (1.288, 0.919, 0.293, 0, 0)$ with payoffs 0.909 and 0.062. We have run IWFA, which produced the same values for the optimal strategies and payoffs. However, we have observed that the convergence of IWFA is slow when $g \approx 1$. In Figure 1 we have plotted the total error in strategies $||T_k^1 - T^{1*}||_2 + ||T_k^2 - T^{2*}||_2$, where T_k^i are the strategies produced by IWFA on the k-th iteration and T^{i*} are the Nash equilibrium strategies. Our approach instantaneously finds the Nash equilibrium for all values of g. Also, it is interesting to note that by (33) the quantity of channels as well as the channels themselves used by weaker player (with smaller resources) is independent on behavior of the stronger player (with bigger resources) but of course each player allocating his/her resources among these channels take into account the opponent behaviour.

In Figure 2, we compare the non-cooperative approach with the cooperative approach. Specifically, we compare the transmission rates and their sum under Nash equilibrium strategies and under strategies obtained from the centralized optimization of the sum of transmitters' rates. The main conclusions are: the cost of anarchy

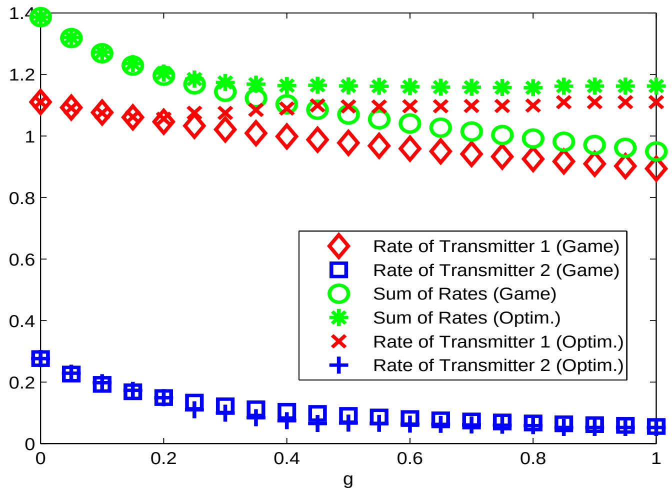

Fig. 2: Centralized Optimization vs. Game

is nearly zero for $g \in [0, 1/4]$ and then it grows up to 22% when g grows from 1/4 to 1; the transmitter with more resources gains significantly more from the centralized optimization. Hence, the non-cooperative approach results in a more fair resource distribution.

In Table 1 we give strategies of both users obtained in the case of the centralized optimization for different values of the crosstalk coefficient g. First, we observe that when the crosstalk coefficient is large, the users share different resources. The user with the larger average power takes better resources. When the crosstalk coefficient is below 0.7, the users start to share the resources. As the value of the crosstalk coefficient decreases, the 2nd user with the smaller average power begins to occupy better resources. As expected, when the crosstalk coefficient is very small, the optimal strategies start to look like strategies which are optimal in the case of no interference.

8. A jamming plot

In this section we consider a plot where two users have even antagonistic objectives. Namely, let there are two mobile terminals and one base station. Since we use the framework of game theory, we shall use the terms mobiles and players interchangably. Player (user) 1 seeks to transmit information to the base station. We shall refer to it as “Transmitter”. Player (user) 2 has an antagonistic objective: to prevent or to jam the transmissions of Player 1 to the base station. Thus, we shall call Player 2 “Jammer”. Both players have in addition a transmission cost which prevents us from using zero-sum games to model our problem.

Table 1: Centralized optimization

g	Users' strategies					
0.95	1st user	7.43	7.17	6.19	4.19	0.00
	2nd user	0.00	0.00	0.00	0.00	2.50
0.70	1st user	7.83	7.22	5.98	3.96	0.00
	2nd user	0.00	0.00	0.00	0.00	2.50
0.65	1st user	8.25	7.42	6.57	2.76	0.00
	2nd user	0.00	0.00	0.00	0.83	1.67
0.35	1st user	8.27	7.73	4.77	3.33	0.90
	2nd user	0.00	0.00	1.24	1.22	0.04
0.20	1st user	6.65	6.44	6.16	4.36	1.38
	2nd user	0.91	0.99	0.60	0.00	0.00
0.10	1st user	6.27	6.65	6.47	4.53	1.08
	2nd user	1.43	1.00	0.07	0.00	0.00
0.01	1st user	7.45	7.03	6.00	3.98	0.54
	2nd user	1.58	0.92	0.00	0.00	0.00

The pure strategy of Transmitter is $T = (T_1, \ldots, T_n)$ where $T_i \geq 0$ for $i \in [1, n]$ and $\sum_{i=1}^{n} T_i \leq \bar{T}$ where $\bar{T} > 0$.

The pure strategy of Jammer is $N = (N_1, \ldots, N_n)$ where $N_i \geq 0$ for $i \in [1, n]$ and $\sum_{i=1}^{n} N_i \leq \bar{N}$ where $\bar{N} > 0$. The payoffs to Transmitter and Jammer are given as follows

$$\begin{aligned} v_T(T, N) &= \sum_{i=1}^{n} \ln\left(1 + \frac{T_i}{gN_i + N_i^0}\right) - C_T \sum_{i=1}^{n} T_i, \\ v_N(T, N) &= -\sum_{i=1}^{n} \ln\left(1 + \frac{T_i}{gN_i + N_i^0}\right) - C_N \sum_{i=1}^{n} N_i, \end{aligned} \tag{43}$$

where N_i^0 is the power level of the uncontrolled noise of the environement at state i, $C_T > 0$ and $C_N > 0$ are the costs of power usage for Transmiter and Jammer, and $g > 0$ is fading channel gains for Transmiter and Jammer when the environement is in state i.

We shall look for a Nash equilibrium, that is, we want to find $(T^*, N^*) \in A \times B$ such that

$$\begin{aligned} v_T(T, N^*) &\leq v_T(T^*, N^*) \text{ for any } T \in A, \\ v_N(T^*, N) &\leq v_N(T^*, N^*) \text{ for any } N \in B, \end{aligned}$$

where A and B are the sets of all the strategies of Transmitter and Jammer, respectively.

In the special case when C_T and C_N are zero in (34), the game is zero-sum. As v_T is convex in T_i and concave in N_i, we can apply Sion's minimax Theorem to conclude that it has a saddle point.

Since υ_T and υ_N are concave in T and N, the Kuhn–Tucker Theorem implies the following theorem.

Theorem 12. *(T^*, N^*) is a Nash equilibrium if, and only if, there are non-negative ω and ν such that*

$$\frac{\partial}{\partial T_i}\upsilon_T(T^*, N^*) = \frac{1}{T_i^* + gN_i^* + N_i^0} - C_T \begin{cases} = \omega & \text{for } T_i^* > 0, \\ \leq \omega & \text{for } T_i^* = 0; \end{cases} \tag{44}$$

$$\frac{\partial}{\partial N_i}\upsilon_N(T^*, N^*) = \frac{gT_i^*}{(T_i^* + gN_i^* + N_i^0)(gN_i^* + N_i^0)} - C_N \begin{cases} = \nu & \text{for } N_i^* > 0, \\ \leq \nu & \text{for } N_i^* = 0, \end{cases} \tag{45}$$

where

$$\begin{aligned} &\omega \begin{cases} \geq 0 & \text{for } \sum_{i=1}^n T_i^* = \bar{T}, \\ = 0 & \text{for } \sum_{i=1}^n T_i^* < \bar{T}; \end{cases} \\ &\nu \begin{cases} \geq 0 & \text{for } \sum_{i=1}^n N_i^* = \bar{N}, \\ = 0 & \text{for } \sum_{i=1}^n N_i^* < \bar{N}. \end{cases} \end{aligned} \tag{46}$$

For non-negative ω and ν let

$$\begin{aligned} I_{00}(\omega, \nu) &= I_{00}(\omega) = \{i \in [1, n] :\ g/N_i^0 \leq g(\omega + C_T)\}, \\ I_{10}(\omega, \nu) &= \{i \in [1, n] :\ g(\omega + C_T) < g/N_i^0 \leq g(\omega + C_T) + \nu + C_N\}, \\ I_{11}(\omega, \nu) &= \{i \in [1, n] :\ g(\omega + C_T) + \nu + C_N < g/N_i^0\}, \end{aligned}$$

$$T_i(\omega, \nu) = \begin{cases} \dfrac{1}{(\omega + C_T)g + \nu + C_N} \times \dfrac{\nu + C_N}{\omega + C_T} & \text{for } i \in I_{11}(\omega, \nu), \\ \dfrac{1}{C_T + \omega} - N_i^0 & \text{for } i \in I_{10}(\omega, \nu), \\ 0 & \text{for } i \in I_{00}(\omega, \nu); \end{cases} \tag{47}$$

$$N_i(\omega, \nu) = \begin{cases} \dfrac{1}{(\omega + C_T)g + \nu + C_N} - \dfrac{N_i^0}{g} & \text{for } i \in I_{11}(\omega, \nu), \\ 0 & \text{for } i \in I_{00}(\omega, \nu). \end{cases} \tag{48}$$

Theorem 13. *Each Nash equilibrium is of the form $(T(\omega, \nu), N(\omega, \nu))$ for some nonnegative ω and ν.*

Now we go on to finding optimal ω and ν. Let

$$H_T(\omega, \nu) = \sum_{i=1}^n T_i(\omega, \nu), \quad H_N(\omega, \nu) = \sum_{i=1}^n N_i(\omega, \nu).$$

Then Theorem 13 implies that

$$H_T(\omega, \nu) = \sum_{i \in I_{10}} \left(\frac{1}{C_T + \omega} - N_i^0\right) + \frac{\nu + C_N}{\omega + C_T} \sum_{i \in I_{11}} \frac{1}{(\omega + C_T)g + \nu + C_N},$$

$$H_N(\omega,\nu) = \sum_{i\in I_{11}} \Big(\frac{1}{(\omega+C_T)g+\nu+C_N} - \frac{N_i^0}{g}\Big).$$

In the next lemma some monotonous properties of $T_i(\omega,\nu)$ and $N_i(\omega,\nu)$, $H_T(\omega,\nu)$ and $H_N(\omega,\nu)$ are obtained.

Lemma 5.

(i) *For fixed* $\omega > 0$ *and* $0 \le \nu_1 < \nu_2$ *we have:*

- **(1)** $T_i(\omega,\nu_1) \le T_i(\omega,\nu_2)$ *where strict inequality holds if, and only if,* $i \in I_{10}(\omega,\nu_1)$,
- **(2)** $N_i(\omega,\nu_1) \ge N_i(\omega,\nu_2)$ *where strict inequality holds if, and only if,* $i \in I_{10}(\omega,\nu_1)$,
- **(3)** $H_T(\omega,\nu_1) \le H_T(\omega,\nu_2)$ *where equality holds if, and only if,* $I_{10}(\omega,\nu_1) = \emptyset$,
- **(4)** $H_N(\omega,\nu_1) \ge H_N(\omega,\nu_2)$ *where equality holds if, and only if,* $I_{10}(\omega,\nu_1) = \emptyset$.

(ii) *For fixed* $\nu > 0$ *and* $0 \le \omega_1 < \omega_2$ *we have:*

- **(1)** $T_i(\omega_1,\nu) \le T_i(\omega_2,\nu)$ *where equality holds if, and only if,* $i \in I_{00}(\omega_1,\nu)$,
- **(2)** $N_i(\omega_1,\nu) \ge N_i(\omega_2,\nu)$ *where equality holds if, and only if,* $i \notin I_{10}(\omega_1,\nu)$,
- **(3)** $H_T(\omega_1,\nu_1) \ge H_T(\omega_2,\nu)$ *where equality holds if, and only if,* $I_{00}(\omega,\nu_1) = = [1,n]$,
- **(4)** $H_N(\omega_1,\nu) \ge H_N(\omega_2,\nu)$ *where equality holds if, and only if,* $I_{10}(\omega_1,\nu) = \emptyset$.

(iii) $H_T(\omega,\nu)$ *and* $H_N(\omega,\nu)$ *are non-negative and continuous in* $[0,\infty)\times[0,\infty)$.

(iv) *If* $H_N(0,0) \le \bar{N}$ *then* $H_N(\omega,\nu) < \bar{N}$ *for* $\omega > 0$ *and* $\nu > 0$.

Based on monotonous properties described in Lemma 5 we can establish the following result about the number of Nash equilibrium the game can have.

Theorem 14. *There is at most one Nash equilibrium.*
Note that

$$\begin{aligned} H_T(\omega,0) = &\sum_{i\in[1,n]:\, g(\omega+C_T)<g/N_i^0\le g(\omega+C_T)+C_N} \left(\frac{1}{C_T+\omega} - N_i^0\right) + \\ &+ \frac{C_N}{\omega+C_T} \times \sum_{i\in[1,n]:\, g(\omega+C_T)+C_N<g/N_i^0} \frac{1}{(\omega+C_T)g+C_N}. \end{aligned} \tag{49}$$

The following lemma supplying some properties of $H_T(\omega,0)$ follows straighforward from (40) and Lemma 5.

Lemma 6.

(i) $H_T(\cdot, 0)$ *is non-negative and continuous in* $(0, \infty)$,

(ii) $H_T(\omega, 0) = 0$ *for enough big* ω*, namely, for* $\omega \geq \max_i\{1/N_i^0 - C_N/g\} - C_T$,

(iii) $H_T(\omega, 0)$ *is strictly decreasing on* ω *while* $H_T(\omega, 0) > 0$.

Lemma 6 implies that if $H_T(0,0) > \bar{T}$ then there exists the unique positive ω_{10}^* such that $H_T(\omega_{10}^*, 0) = \bar{N}$ (indices 10 mean that at this moment we look for the optimal solution where $\omega > 0$ and $\nu = 0$). If $H_T(0,0) \leq \bar{T}$ then $H_T(\tau, 0) < \bar{T}$ for $\tau > 0$. Then, from Theorems 12 and 13 and Lemmas 5 (iv) and 6 we have the following theorem.

Theorem 15. Let $H_N(0,0) \leq \bar{N}$ *then*
(a) if $H_T(0,0) \leq \bar{T}$ *then* $(T(0,0), N(0,0))$ *is Nash equilibrium,*
(b) if $H_T(0,0) > \bar{T}$ *then* $(T(\omega_{10}^*, 0), N(\omega_{10}^*, 0)$ *is Nash equilibrium.* By Lemma 5 the following Lemma holds.

Lemma 7.

(a) *If* $H_N(0,0) > \bar{N}$ *then there is* ν_{01}^* *such that* $H_N(0, \nu_{01}^*) = \bar{N}$ (*subscript* 01 *signifies that we look for the optimal solution where* $\omega = 0$ *and* $\nu > 0$) and there is $\hat{\omega}$ such that $H_N(\hat{\omega}, 0) = \bar{N}$.

(b) $H_N(\omega, \nu) < \bar{N}$ for each $\omega > \hat{\omega}$ and each non-negative ν.

(c) For each $\omega \in (0, \hat{\omega}]$ there is unique non-negative $\nu(\omega)$ such that $H_N(\omega, \nu(\omega)) = \bar{N}$.

(d) $\nu(\omega)$ is continuous and strictly decreasing on ω, $\nu(0) = \nu_{01}^*$ and $\nu(\hat{\omega}) = 0$.

Thus, by Lemma 7 we can introduce the following notation:

$$\bar{H}_T(\omega) = H_T(\omega, \nu(\omega)) = \sum_{i \in I_{10}(\omega, \nu(\omega))} \pi_i \left(\frac{1}{C_T + \omega} - N_i^0 \right) +$$
$$+ \frac{\nu(\omega) + C_N}{\omega + C_T} \times \sum_{i \in I_{11}(\omega, \nu(\omega))} \pi_i \frac{g_i}{(\omega + C_T)g + (\nu(\omega) + C_N)}.$$

Then by Lemma 5 $\bar{H}_T$ is continuous and strictly decreasing in $(0, \hat{\omega})$. Thus, if $\bar{H}_T(0) \leq \bar{T}$ then $\bar{H}_T(\omega) < \bar{T}$ for $\omega \in (0, \hat{\omega})$. If $\bar{H}_T(\hat{\omega}) > \bar{T}$ then $\bar{H}_T(\omega) > \bar{T}$ for $\omega \in (0, \hat{\omega})$. If $\bar{H}_T(\hat{\omega}) < \bar{T}$ and $\bar{H}_T(0) > \bar{T}$ then there is unique $\omega_{11}^* \in (0, \hat{\omega})$ such that $\bar{H}_T(\omega_{11}^*) = \bar{T}$ (subscript 11 signifies that we look for the optimal solution where $\omega, \nu > 0$). Then, from Theorems 12 and 13 we have the following theorem.

Theorem 16. *Let* $H_N(0,0) > \bar{N}$ *then*

(a) *if* $\bar{H}_T(0) = H_T(0, \nu_{01}^*) \leq \bar{T}$ *then* $(T(0, \nu_{01}^*), N(0, \nu_{01}^*))$ *is Nash equilibrium,*

(b) *if* $\bar{H}_T(0) = H_T(0, \nu_{01}^*) > \bar{T}$ *and* $\bar{H}_T(\hat{\omega}) = H_T(\hat{\omega}, 0) > \bar{T}$ *then* $(T(\omega_{10}^*, 0), N(\omega_{10}^*, 0))$ *is Nash equilibrium,*

(c) *if* $\bar{H}_T(0) = H_T(0, \nu_{01}^*) > \bar{T}$ *and* $\bar{H}_T(\hat{\omega}) = H_T(\hat{\omega}, 0) \le \bar{T}$ *then* $(T(\omega_{11}^*, \nu(\omega_{11}^*)), N(\omega_{11}^*, \nu(\omega_{11}^*)))$ *is Nash equilibrium.*

Theorems 14–16 imply the following main result.

Theorem 17. *There is unique Nash equilibrium given by Theorems 15 and 16.*

9. Conclusion

We have considered power level management problem for wireless and DSL access networks with transmission cost in optimization and game frameworks. Closed form solutions for the water filling optimization problem and two players symmetric water filling games have been provided. Namely, now one can calculate optimal/equilibrium strategies with a finite number of arithmetic operations. We have also provided a simple alternative proof of convergence for a version of iterative water filling algorithm. It had been known before that the iterative water filling algorithm converges very slow when the crosstalk coefficient is close to one. For our closed form approach possible proximity of the crosstalk coefficient to one is not a problem. We have shown that when the crosstalk coefficient is equal to one, there is a continuum of Nash equilibria. Finally, we have demonstrated that the price of anarchy is small when the crosstalk coefficient is small and that the decentralized solution is better than centralized with respect to fairness.

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An Economic Index of Riskiness[1]

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Define the riskiness of a gamble as the reciprocal of the absolute risk aversion (ARA) of an individual with constant ARA who is indifferent between taking and not taking that gamble. We characterize this index by axioms, chief among them a "duality" axiom which, roughly speaking, asserts that less risk-averse individuals accept riskier gambles. The index is homogeneous of degree 1, monotonic with respect to first and second order stochastic dominance, and for gambles with normal distributions, is half of variance/mean. Examples are calculated, additional properties derived, and the index is compared with others in the literature.

On June 28, 2007 the Graduate School of Management hosted the lecture "Game engineering" by Prof. Robert Aumann (Hebrew University of Jerusalem, Israel), Nobel Memorial Prize in Economics, 2005.

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[1] The invited lecture was presented at the First International Conference on "Game Theory and Management" held at Graduate School of Management (St. Petersburg University) on June the 28th. Full text of the working paper "An Economic Index of Riskiness" is available on the website: http://www.ratio.huji.ac.il/dp.php

Attackers' Motivation and Security Investment

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Abstract. We model economic behavior of attackers when they are able to obtain complete information about the security characteristics of targets and when such information is unavailable. We find that when attackers are able to distinguish targets by their security characteristics and switch between multiple alternative targets, the effect of a given security measure is stronger. That is due to the fact that attackers rationally put more effort into attacking systems with low security levels. Ignoring that effect would result in underinvestment in security or misallocation of security resources. We also find that systems with better levels of protection have stronger incentives to reveal their security characteristics to attackers than poorly protected systems. Those results have important implications for security practices and policy issues.

Keywords: Economics of information systems, information system security, perceived security, investment evaluation, attacker behavior.

Introduction

The importance of developing quantitative models of computer security has been widely recognized in economics [Gordon and Loeb, 2005], computer science [Liu et al., 2005], [Schechter, 2004], and dependable computing [Avizienis et al., 2004], [Littlewood et al., 1993]. More specifically, great attention has been paid to analyzing the interaction of computer systems with the operational environment, which, from the security standpoint, includes the behavior of attackers as one of the main components. Quantitative techniques for evaluating attackers' behavior have been fruitfully applied [Avizienis et al., 2001], [Nicol et al., 2004] and several models of attackers' behavior have been proposed [Jonsson and Olovsson, 1997], [McDermott, 2005], [Ortalo et al., 1999]. However, the attacker's behavior in the aforementioned works is modeled as exogenous and the principles guiding the attackers' behavior remain unclear.

The current paper examines attackers who are assumed to behave economically (that is, choose their actions optimally based on comparison of their costs to benefits). Viewing attackers as rational agents is consistent with several theoretical and empirical studies. Some prior work has recognized that attackers act strategically either by rationally selecting their targets or in response to targets' actions [Jajodia, Miller, 1993], [NIST, 2002], [Schechter, Smith, 2003]. Leeson and Coyne (2006) make a distinction between fame-driven and profit-driven attackers[1] with the former attracted by the possible notoriety and the latter focused on gaining monetary rewards, and conclude that the two groups must be analyzed separately. In the past the stereotypical view of an attacker was mainly that of a fame-driven individual [Denning, 1990]. Even more recently the "15 minutes of fame" was claimed to be one of the biggest motivation for attackers (Curry, 2002). While fame-driven attackers are certainly still numerous, ample evidence exists that economically-minded attackers are posing a much more serious threat to corporate information security. Recently, a pronounced shift toward financially motivated intrusions [Sieberg, 2005] has resulted in an increase in the average losses caused by unauthorized access to information and theft of proprietary information [Gordon et al., 2005]. We follow the body of work that addresses the issue of the optimal amount of security by studying attackers who rationally choose their course of action based on cost-benefit analysis [Anderson, 2001], [Schechter, Smith, 2003]. We specifically focus on the role of the opportunity cost of attacking a given target, which is represented by alternatives available to an attacker.

Our results support the important role the presence of alternative targets plays in attackers' decisions. More specifically, we show that in the presence of targets with heterogeneous security characteristics the amount of effort optimally spent by an attacker on a target decreases in the target's security level. Thus any given security measure affects the frequency of intrusions through two mechanisms. One is the increased ability of a target to withstand attacks of a given intensity. This direct technical effect is commonly recognized by practitioners dealing with security threats. The other effect contributing to a reduction in intrusions occurs through a change in attackers' perception of the target in question.

In the presence of alternatives, a more secure target is less attractive for rational attackers, which eventually results in the decreased effort attackers put into attack attempts. Unfortunately, this behavioral component is largely neglected when security strategies are defined. This paper demonstrates that the behavioral effect can substantially exceed the direct effect of a security measure and discusses how taking both effects into account may help defenders choose better defense strategies. The ability to signal one's security level to an attacker is also important for successful defense. The results from two alternative specifications of our model suggest that the absence of such signals makes systems with a low security level better off and those with a high security level worse off. Besides, lack of information attackers have

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[1] We prefer the expression "economic behavior" to "profit-driven" as the more accurate and less restrictive one.

about the security characteristics of each potential target reduces the incentives for individual firms to invest in security.

We use our findings to discuss various approaches to investments in security technology and make recommendations regarding security practices of individual firms as well as policy recommendations. In particular, we argue that the Annual Loss Expectancy (ALE) [Soo Hoo, 2000] and other widely adopted approaches to information security can severely underestimate positive effects of security investments, therefore leading to underinvestment in information security or misallocation of resources.

1. Related Work

Our research belongs to the field of economics of information security. Of all the issues within that broadly defined area, we are focusing on what economic research has to say about the best strategies for investing in security technologies. The advantage of the economic approach over traditional ones is that it recognizes and accounts for the presence of a strategic interaction between different parties involved.

The literature combining economic approach with information technology issues is vast. Clemons (1991) discusses the reasons why businesses have difficulty evaluating when to use information technology. Relevant to our research is his observation that some investments should be made to limit the possibility of future losses rather than to obtain long-term additional value. This notion applies perfectly to the case of security technology investments. When companies face environmental changes, a common scenario for information security, needed investments in information technology may be diverted if such changes are not foreseen. This is the effect Clemons (1991) called the "trap of the vanishing status quo".

Despite the amount of research making a case for wider use of economic approach to information security [Anderson, 2001], [Gordon and Loeb, 2002], [Gordon et al., 2003], [Rodewald, 2005], [Schechter, 2005], little attention has been paid to those findings by security practitioners. For example, a comparative analysis of two traditional investment evaluation techniques, Return on Investment (ROI) and Net Present Value (NPV), shows that the NPV approach is more applicable to computer security issues than ROI [Gordon and Richardson, 2004]. Nevertheless, ROI is by far the most popular metric used, as documented by the 2005CSI/FBI Computer Crime and Security Survey [Gordon et al., 2005]. This point is similar to one we make in our work, where we show how traditional investment evaluation techniques can greatly underestimate the effectiveness of a security solution by not considering the strategic nature of the problem and the interdependency between attackers' and defenders' actions.

Among other attempts to develop better techniques for evaluating investment in security, Geer (2005) suggests an alternative to traditional ROI formula which involves performing a cost-effective analysis, rather than a cost-benefit analysis, in the case when costs and benefits are not commensurate. Purser (2004) proposes a modification to the ROI approach that would assign a monetary value to an increase or decrease in the risk resulting from an investment. According to that approach, higher risk results in a lower ROI and vice versa. While the idea of considering a secondary effect of security investments resulting from a modified operational environment is similar to the one explored in our paper, our approach, based on game theory, is better suited to model such an interdependent behavior.

Other work in the game theory field that is related to ours includes Cavusoglu and Raghunatan (2004), which compares decision theory and game theory approaches in the context of the configuration of detection software. Although the subject of that paper is different from ours, the approach taken in it is close to what we have done in terms of contrasting the results obtained by each of the two approaches. Several other papers have used game-theoretic models for evaluating security investments [Cavusoglu, 2004, 2005]. Each of them, however, focuses on specific security technologies, whereas we consider the more general problem of evaluating investments in computer security solutions.

The main purpose of investing in security is to defend against malicious attackers. Acquiring proper understanding of attackers' behavior is therefore a necessary step towards best security practices. Jonsson and Olovsson (1997) contributed to such understanding by performing an empirical study of attackers' behavior in a laboratory environment. While their work is descriptive in nature, we are able to use some aspects of their analysis as a starting point in setting up our model. In particular, they provide empirical evidence of several distinctive phases of an attack and hypothesize the presence of "behavioral" and "preventive" effects of security measures. We make a similar distinction in our model. Several papers model strategic defender–attacker interaction and show that a defender can influence the attacker's choice of targets by selecting its actions accordingly [Bier et al., 2007], [Schechter and Smith, 2003].

An attack in those models is a one-time decision and the security characteristics of each target are assumed to be known. While the starting premise of our analysis is similar, we are able to move further in our analysis by studying the endogenous choice of effort made by attackers and examining the role of information about the target security level in the formulation of an optimal defense strategy.

The strategic interaction between defenders and attackers has also been discussed in the context of anti-terrorism policies [Enders, Sandler, 2004]. It is shown that a defensive measure may mitigate a specific security threat through two separate mechanisms, the income effect and the substitution effect. While such an approach is conceptually similar to ours, our results are more directly applicable to information security.

Other work directly related to the issues we are interested in includes papers that introduce the concept of an interdependent security game and show that the ultimate safety of each participant may depend in a complex way on the actions of the entire population [Kearns and Ortiz, 2004], [Kuhnreuther and Heal, 2003]. In our present work we have used some elements of interdependent security models by discussing the effect of a security measure on the strategic behavior of attackers.

2. The Model

The model consists of N corporate networks that serve as targets for malicious attacks and an unspecified number of attackers all of whom are identical. An attacker targets one network at a time and the amount of effort spent on target i is x_i.

As an attack progresses, it leaves behind a stream of evidence detectable by the victim. Intrusion detection literature [Lee and Xiang, 2001], [Ning et al., 2004], [Valeur et al., 2004], [Wespi et al., 2000] recognizes that the more suspicious events are observed, the more likely is the data correlation to result in alerts, which improves the detection success rate. A successful detection of an attack may in turn lead to a punishment being imposed on the attacker. With that in mind, we assume the attacker's cost, C, to be increasing and convex in the amount of effort spent, $\frac{\partial C(x)}{\partial x} > 0$, $\frac{\partial^2 C(x)}{\partial x^2} > 0$[2].

If successful, an attack results in an intrusion. The expected benefit from an attack is $E(B(x)) = \pi(x) \cdot G$, where $\pi(x)$ is the probability of success given the amount of effort put into attacking a given target and G is the one-time payoff the attacker receives in the case of a success. The size of that payoff is assumed to be the same for all targets[3].

Attackers are maximizing their expected net payoff from attacks. They do so by deciding how much effort to spend on each target. An attacker stops attacking a target when the marginal benefit, MB, of effort no longer exceeds its marginal cost, MC.

Denote $\hat{x}$ the amount of effort that solves $MB = MC$. After x units of attacker's effort do not lead to success, the residual net benefit that he still expects to receive from the target is

$$ENB(x) = \int_{x}^{\hat{x}} \rho(\tau)(G - C(\tau) + C(x))d\tau - (1 - \pi(\hat{x} - x))(C(\hat{x}) - C(x)), \qquad (1)$$

where $\pi(\hat{x}) = \int_{0}^{\hat{x}} \rho(\tau)d\tau$ and $1 - \pi(\hat{x}) = \int_{\hat{x}}^{\infty} \rho(\tau)d\tau$.

Here, $\rho(x)$ denotes the conditional probability distribution function given no success upon spending effort x. We proceed by choosing specific functional forms for benefit and cost. The marginal cost of effort is assumed to be given by $MC = \alpha_0 + \alpha_1 x_i$, where the first term represents the opportunity cost of effort and x_i is the amount of effort spent on target i. The cumulative probability of success given effort is $\pi(x) = 1 - e^{-x/\mu}$[4]. Since $\frac{d\pi}{d\mu} < 0$, one can think of μ as the security level of the

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[2] Following the approach used in [Jonsson and Olovsson, 1997], we make no distinction between effort and time. Thus, the intensity of effort, or how much effort is exerted in a unit of time, in our model is assumed to be exogenous and constant. A more complex setup is saved for later work.

[3] A richer setup allowing for heterogeneity in the size of that payoff is saved for future work.

[4] This form is commonly used in the relevant literature [Cavusoglu and Raghunathan, 2004], [Littlewood et al., 1993]. Its validity has been confirmed empirically [Jonsson and Olovsson, 1997].

system subject to attacks, with greater values of μ corresponding to better protected systems. Due to the memorylessness property of the exponential distribution, in the current specification $\rho(x) = {}^1\!/_\mu$ for any x.

2.1. Scenario 1 – One target

We start with the simplest case in which there is only one specific target the attacker is interested in, $N = 1$. Its security level, μ, is common knowledge. The attacker's expected net benefit from attacking that target is

$$ENB(x) = G - \mu\alpha_0 - \mu\alpha_1 x - \mu^2\alpha_1(1 - e^{x/\mu}e^{-(G/\mu-\alpha_0)/\mu\alpha_1}), \quad x \geq 0. \qquad (2)$$

It is easy to show that the attacker's optimal problem, $MB = MC$, is equivalent to $ENB(x) = 0$[5] and is solved by $\hat{x} = \frac{G-\mu\alpha_0}{\mu\alpha_1}$. Thus, we have the following proposition:

Proposition 1. *The amount of effort, $\hat{x}$, an attacker optimally puts into breaching a system increases in the size of the payoff he receives in the case of intrusion, ($\frac{\partial\hat{x}}{\partial G} > 0$), decreases in the target's security level ($\frac{\partial\hat{x}}{\partial\mu} < 0$), and decreases in the cost of performing an attack ($\frac{\partial\hat{x}}{\partial\alpha_0} < 0$, $\frac{\partial\hat{x}}{\partial\alpha_1} < 0$).*

Here is a numerical example to help illustrate attacker's decisions in this case. Let $G = 1000$, $\mu = 50$, so that $\pi(x) = 1 - e^{-0.02x}$, and $\alpha_0 = 10$, $\alpha_1 = 2$. From either $ENB(x) = 0$ or $MB = MC$, $\hat{x} = \frac{G-\mu\alpha_0}{\mu\alpha_1} = 5$.

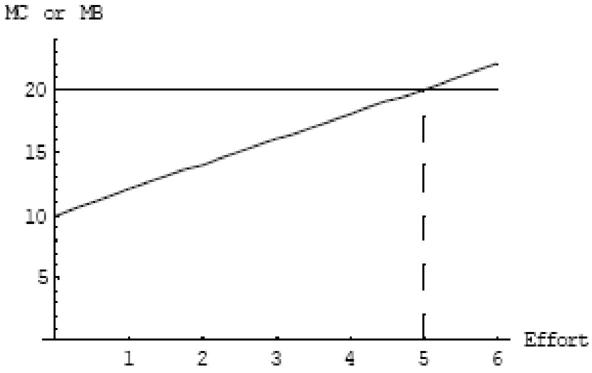

Fig. 1: Marginal cost and marginal benefit of attacker's effort

The following graphs may clarify the decision-making process and the equivalence of the two approaches to solving the optimal stopping problem.

Fig. 1 is trivial and shows the marginal benefit and marginal cost of effort. Clearly, stopping at $\hat{x} = 5$ is in the attacker's best interest.

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[5] Optimal search literature confirms this fact. See Cozzolino (1972) and others.

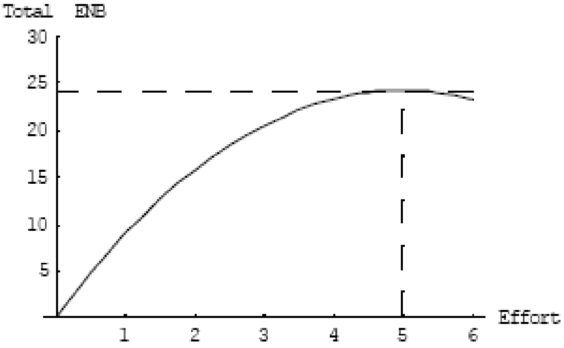

Fig. 2: Expected net benefit from attacks as a function from future effort spent, as viewed before attacks start

Figure 2 shows the overall expected net benefit that can be received from attacking the target as a function of effort that will be put in (as viewed before the start of attack attempts).

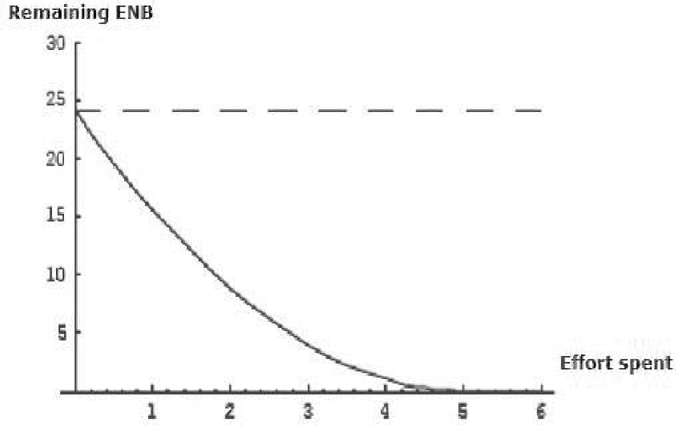

Fig. 3: Residual expected net benefit of future effort as a function of effort spent

Given the knowledge about the target's security parameter, μ, the attacker knows that the maximum value of the total expected net benefit is going to reach its maximum at $\hat{x} = 5$ $(ENB(\hat{x}) = 24.19)$ and decrease afterwards. In other words, the attacker can make an advance commitment to putting in 5 units of effort (or less if he happens to succeed sooner).

Still another way to present the stopping decision is through the residual expected net benefit from attacks after some effort has been already spent (see Fig. 3). The greater the amount of past effort, the greater is the marginal cost of effort and the smaller is the net benefit the attacker still expects to derive from future effort. In this version of the model the attacker keeps trying until $ENB(x) = 0$. We find this representation the most insightful of the three ones and will utilize it in the analysis of more advanced versions of the model.

2.2. Scenario 2 – Multiple identical targets

There are $N > 1$ potential targets with the same security level, μ, which is common knowledge. Attackers are now able to stop working on one target and switch to another at any time. Switching to a different target involves cost, C_S, which we interpret as the cost of effort put into the "learning phase" of an attack[6] in the context of the aforementioned Jonsson and Olovsson (1997). The size of C_S is assumed to be the same for each target.

One major difference between this setup and the one discussed above is that there is now an outside opportunity present that has a certain value to an attacker. Therefore the attacker will make the decision to stop attacking one target and switch to another, randomly picked, target once his remaining expected net benefit from the current target gets smaller than the net benefit he expects to get if he switches. The optimal stopping rule for this case is therefore

$$ENB(x) = ENB(0) - C_S. \tag{3}$$

Proposition 2. *The maximum amount of effort an attacker puts into attacking a target increases in the size of the switching cost,* $C_S(\frac{\partial \hat{x}}{\partial C_S} > 0)$.

For $C_S = 5$ and the parameter values used above, (2) is solved by $\hat{x} = 0.5548$ (see Fig. 4). The downward sloping line on the graph shows the residual expected net benefit from the present target, $ENB(x)$. As discussed above, it decreases in the amount of effort already spent, $\frac{\partial ENB(x)}{\partial x} < 0$. The horizontal line represents $ENB(0) - C_S$. Once $ENB(x) \leq ENB(0) - C_S$, the attacker is better off paying the one-time switching cost, C_S and moving to a different target.

Note that if $C_S = 0$, then $\hat{x} = 0$ trivially. Therefore, in the absence of switching costs and presence of multiple identical targets attackers would be switching between targets all the time. The analysis done insofar did not discuss decisions made by defenders. Those decisions are endogenized in the next modification of the model.

2.3. Scenario 3 – Heterogeneous targets

We now further advance our analysis by relaxing the assumption of target homogeneity. In the present setup, there are $(H + L)$ targets present, H of which have a high security level and L have a low security level. We will further refer to such targets as being of H-type or L-type, respectively, where $\mu_H > \mu_L$. The type a target belongs to becomes known to attackers with certainty after the reconnaissance stage

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[6] For example, we may think of usual reconnaissance operations performed to gather information on potential targets like port scanning, OS and application fingerprinting, and so forth.

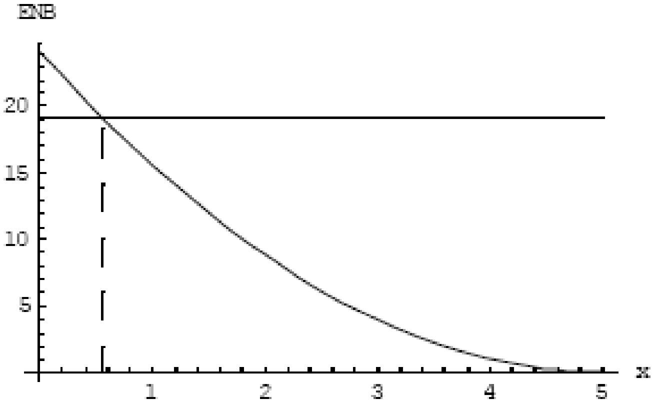

Fig. 4: Optimal stopping decision in the presence of multiple targets and switching cost

(hence upon paying the switching cost, C_S). The expected net benefit received from a target of a known type is a modification of (2) obtained earlier:

$$ENB_i(x) = G - \mu_i\alpha_0 - \mu_i\alpha_1 x - \mu_i^2\alpha_1(1 - e^{x/\mu_i}e^{-(G/\mu_i - \alpha_0)/\mu_i\alpha_1}), \quad x \geq 0, \tag{4}$$

where $i = H, L$. Switching to a different, randomly chosen target involves cost C_S and gives the attacker an expected net benefit:

$$ENB_{\text{random}} = \eta ENB_H(0) + (1-\eta)ENB_L(0) = G - \eta\mu_H\alpha_0 - \eta\mu_H^2\alpha_1(1- \\ e^{-(G/\mu_H - \alpha_0)/\mu_H\alpha_1}) - (1-\eta)\mu_L\alpha_0 - (1-\eta)\mu_L^2\alpha_1(1 - e^{-(G/\mu_i - \alpha_0)/\mu_i\alpha_1}), \tag{5}$$

where $\eta = \frac{H}{H+L}$ is the proportion of H-type systems in the population.

Applying the optimal stopping rule to this version of the model suggests that the attacker should continue putting effort into one target as long as $ENB_{\text{random}}(0) - -ENB_i(x) \leq C_S$ and switch to a different randomly chosen target when $ENB_{\text{random}}(0) -ENB_i(x) \geq C_S$, where $i = H, L$ is the type of his present target. Thus, the effort after which it is optimal to switch to a different target is given by the solution to

$$ENB_{\text{random}}(0) - ENB_i(x) = C_S. \tag{6}$$

While no closed form solution for (6) exists, we are able to get the following result by using differentiation of an implicit function.

Proposition 3. *Given the presence of targets of different security types and attackers' ability to determine the target type, the amount of effort optimally put by an attacker into a target decreases in the target's security level* μ, $\frac{d\hat{x}}{d\mu} < 0$.

This important fact drives many results of our paper and has important implications for security practices. It suggests that every security solution affects the state of security through two distinct mechanisms. One is what we call the *direct* or *technical effect*, represented by the increased ability of a system to withstand intrusion attempts given the intensity of those attempts. The direct effect is commonly recognized by security practitioners. It can be shown that, when the probability of attack success is small, the direct effect is approximately proportional to the increase in the security parameter μ:

$$\frac{\partial \pi}{\partial \mu} \times \frac{\mu}{\pi} = \frac{x e^{-x/\mu}}{\mu(1 - e^{-x/\mu})} \approx 1.$$

Thus, according to the direct effect, a 10% increase in security level results in an approximately 10% reduction in the probability that each attack will result in a successful intrusion.

There is, however, another effect as well, which we call *indirect*, or *behavioral, effect.* Ceteris paribus, a more secure system is less appealing to attackers than a less secure one. Thus, a security enhancement performed at one system diverts attackers' effort away from it, and, since $\pi(\mu, x) = 1 - e^{-x/\mu}$, less effort on attackers' part translates into a lower probability of a security incident. This fact deserves to be summarized in another proposition.

Proposition 4. *Given the heterogeneity of target types and the presence of rational attackers able to determine a target's type, any security improvement causes more than proportional reduction in the probability of success of each individual attack.*

The result in Proposition 4 can be confirmed by differentiation by the chain rule:

$$\frac{d\pi}{d\mu} = \frac{\partial \pi}{\partial \mu} + \frac{\partial \pi}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial \mu} < \frac{\partial \pi}{\partial \mu} < 0. \tag{7}$$

While the probability with which an individual attack results in an intrusion is qualitatively important, it is not directly observed by defenders. Instead, defenders are primarily concerned with the frequency with which security incidents ("intrusions" according to our terminology) occur. Moreover, the loss incurred by defenders per unit of time (ALE being one such example) is directly related to the number of security incidents per unit of time. Therefore we chose to discuss security enhancement solutions in the context of their effect on the frequency of intrusions.

The frequency of intrusions is the product of the probability that an attack results in an intrusion, $\pi_i(x_i) = 1 - e^{-\hat{x}_i/\mu_i}$, and the rate of attackers' arrival at a target. The arrival rate is the same across targets. It is proportional to the overall number of attackers, NA, and inversely proportional to the number of potential targets, NT and the average length of an attacker's stay on each target, τ. To determine τ, one needs to realize that an attacker leaves one target and starts looking for another if he either has successfully breached the system or feels the current target is no longer worth his continued effort. Once we know the solution to the optimal stopping condition for each type of system, $\hat{x}_i$, $i = H, L$, we can determine the average, or "expected", amount of effort an attacker spends on a system:

$$\tau_i = \int_0^{\hat{x}_i} xf(x)dx + \hat{x}_i e^{-\hat{x}_i/\mu_i} = \mu_i(1 - e^{-\hat{x}_i/\mu_i}). \tag{8}$$

Keep in mind that if switching cost is interpreted as the opportunity cost of the reconnaissance effort, then it has to be included in the calculation of the length of stay as τ_S. A sufficiently close approximation for it is $\tau_S = {}^{C_S}\!/_{\alpha_0}$. Thus, an attacker spends an average of $(\tau_S + \tau_H)$ units of effort on an H-type system and $(\tau_S + \tau_L)$ units of effort on an L-type system. Since effort in our model is equivalent to time, attackers return to the pool and start probing another target at $\tau = (\tau_S + \eta\tau_H + (1 - \eta)\tau_L)$ intervals.

Finally, the frequency of intrusions equals

$$\nu_i = \frac{N_A \cdot \pi(\mu_i, \hat{x}_i)}{N_T(\tau_S + \eta\tau_H + (1 - \eta)\tau_L)} = \frac{N_A(1 - e^{-\hat{x}_i/\mu_i})}{N_T\left(\tau_S + \eta\mu_H(1 - e^{-\hat{x}_H/\mu_H}) + (1 - \eta)\mu_L(1 - e^{-\hat{x}_L/\mu_L})\right)}, \tag{9}$$

where $i = H, L$.

We are now able to state the effect of a security enhancing solution on the frequency of intrusions and, therefore, on the annual loss expectancy.

Proposition 5. *Given the heterogeneity of target types and presence of rationally behaving attackers who are able to determine a target's type, any security enhancement causes more than proportional reduction in the frequency of security incidents and in the expected annual loss from attacks (ALE). The extent of that reduction,* $\xi = \left|\frac{\nu_L - \nu_H}{\mu_H - \mu_L}\right|$, *is inversely related to the size of the switching cost,* $\frac{d\xi}{dC_S} < 0$.

The specifics of the indirect behavioral effect may become clearer from the numerical simulation results. For simulations, we use the same parameters as before, $G = 1000$, $\alpha_0 = 10$, $\alpha_1 = 2$, $C_S = 5$. The security parameters of the two target types are $\mu_H = 55$ and $\mu_L = 50$. There is an equal proportion of systems of each type, $\eta = 0.5$. The number of attackers and the total number of targets are both normalized to 1.

Initially, an L-type defender with $\mu_L = 50$ suffers intrusions with frequency $\nu_L = 0.0197$. They result from attackers arriving at the system at the rate of 0.916 per unit of time, staying no more than $\hat{x}_L = 1.089$, and each attack leads to a success with probability $\pi_L = 0.0215$.

Next, a security enhancement is considered that would change the system security parameter to $\mu_H = 55$. If the rate of attackers' arrival and individual attacker's effort were assumed to remain the same, then it would be reasonable to expect the frequency of breaches to decrease to $\nu = 0.0178$. In fact, due to the amount of attacker's effort being substantially reduced (from $\hat{x}_L = 1.089$ to $\hat{x}_H = 0.094$), the frequency of intrusions also sees a significant drop to $\nu_H = 0.0017$ (see Fig. 5.)

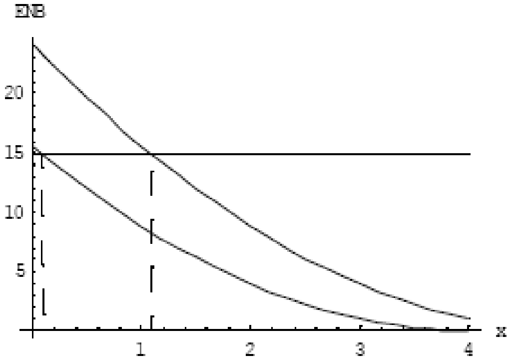

Fig. 5: Difference in the optimal attacker's effort across target types

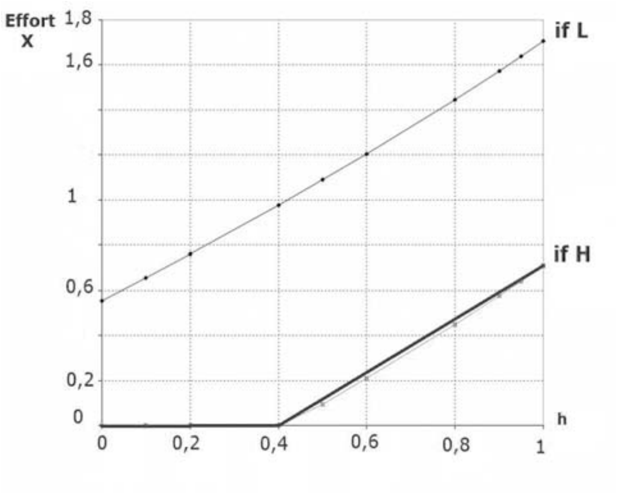

Fig. 6: Optimal attacker's effort as a function of the type of the system and the composition of the population. $G = 1000$, $\alpha_0 = 10$, $\alpha_1 = 2$, $C_S = 5$, $\mu_H = 55$, $\mu_L = 50$. Parameter η denotes the proportion of H-type systems

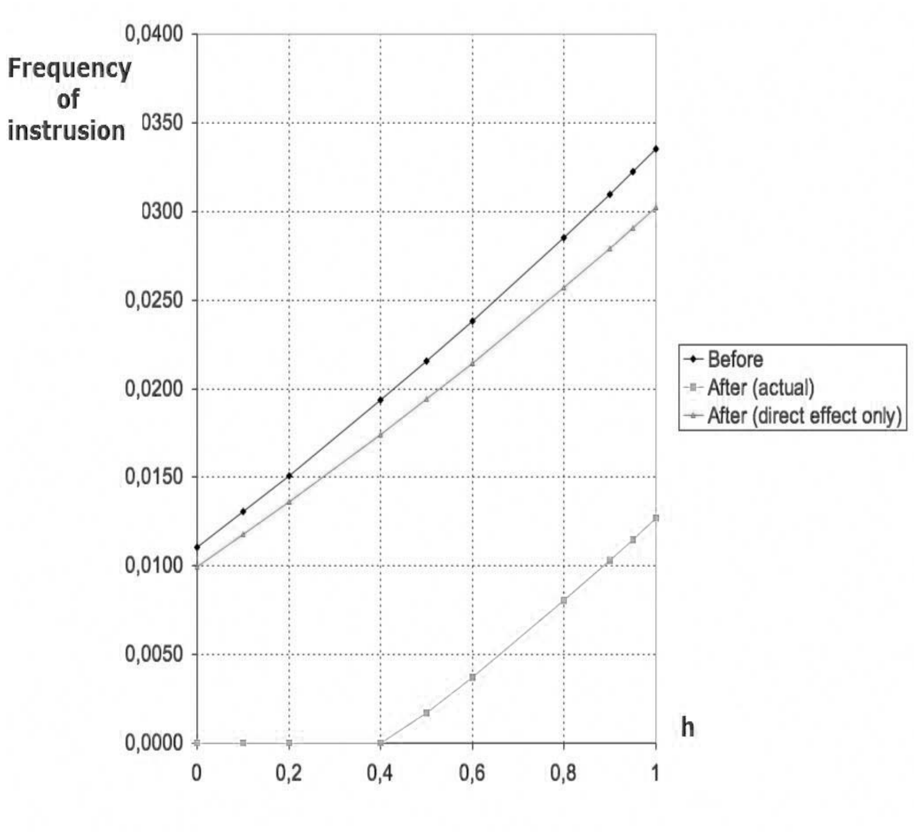

Fig. 7: Direct only and overall effects of a security enhancement on the frequency of breaches, plotted against the proportion of H-type systems. $G = 1000$, $\alpha_0 = 10$, $\alpha_1 = 2$, $C_S = 5$, $\mu_H = 55$, $\mu_L = 50$. The top curve, "Before", represents an L-type system before security enhancement. The middle one, "After (direct effect only)", shows the expected effect of upgrading to H-type, taking only the direct effect into account. The bottom line, "After (actual)", shows the frequency of intrusions that will actually occur as a result of an upgrade. It includes both the direct and the behavioral effects

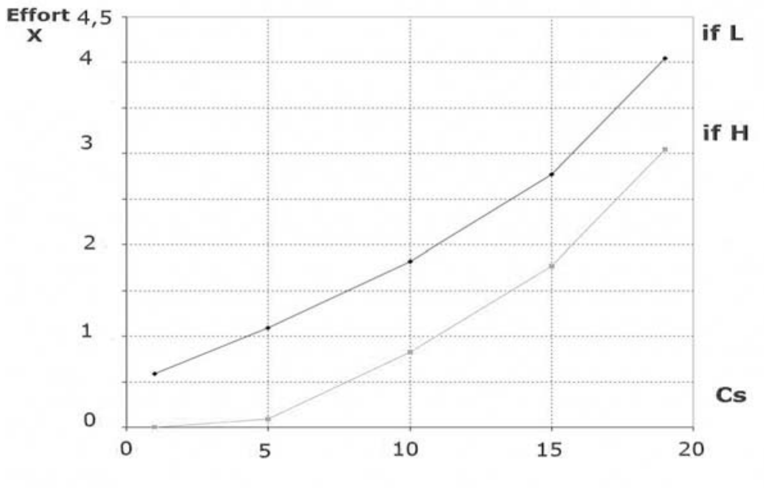

Fig. 8. The effect of the switching cost C_S on the effort put by attackers into systems of each type. $G = 1000$, $\alpha_0 = 10$, $\alpha_1 = 2$, $\mu_H = 55$, $\mu_L = 50$, $\eta = 0.5$

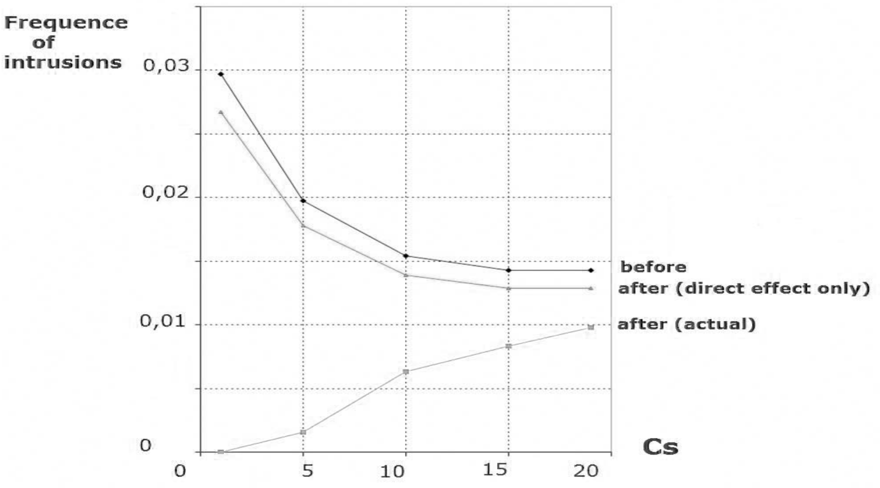

Fig. 9: Frequency of intrusions before and after a security enhancement, plotted against the switching cost

Naturally, the outcome of a security enhancing measure depends on the distribution of target types and the size of the switching cost. Both effects are illustrated by the diagrams in Figures 6 and 7.

The size of direct and indirect effects of increased security on the frequency of intrusions and, therefore, the ALE can be seen from Figure 7.

The top line represents the frequency of intrusion occurrences for an L-type system (before investment in security is made). The middle line shows the rate that is expected after its security level is raised from L to H if only the direct effect is accounted for. The bottom line is the intrusion rate after the security is raised, given the presence of both direct and behavioral effects. As the diagram indicates, the indirect effect a security enhancement may have on the frequency of breaches can substantially exceed the direct effect. It is also possible that an attacker will not find it worthwhile to spend any effort at all on an H-type target and will prefer to leave an H-type target immediately and look for an L-type target instead. Figure 9 allows us to elaborate on some specifics of the layered security approach and compartmentalized security architectures (Wells and Thrower, 2002). Figure 8 confirms the second result of Proposition 5, namely the positive correlation between the size of the switching cost and the amount of effort put into each attack on L-type and H-type systems, respectively.

Those approaches to information security have gained popularity in the last several years and have proven to be superior to more traditional ones that rely on a security perimeter only. Still, existing security guidelines rarely make a distinction between investments into different security layers. If anything, there still seems to exist a tendency toward the "secure the perimeter" philosophy, according to which security investments should focus more on preventing intrusion attempts at early stages and therefore be concentrated on system areas that are closest to the external network. That means more attention is given to exterior layers of security than to interior ones. No clear consensus on the issue exists, however.

In the context of our model, the switching cost can represent the security of the exterior layer whereas the security parameter μ is a characteristic of the interior layer. As Figure 9 clearly indicates, strengthening outer echelons of defense may be less effective in reducing the frequency of intrusions than a combination of enhancing the security of the interior layer at the same time making that enhancement evident to attackers. It also shows that the higher the security level of a system, the more reason it has to signal its security level to attackers. As discussed above, such a signal may induce attackers to switch to other targets instead of continuing the intrusion attempts. This means that, at least for well-protected systems, it may be beneficial to have some means of implicit communication with attackers that would make them able to assess the target's security level. That is strikingly different from the aforementioned "secure the perimeter" approach and the traditional preference for opacity of protected networks that limit the amount of available information about deployed security measures.

Figure 9, however, has to be interpreted with care since the switching cost there is assumed to be the same across all systems. In reality, a security professional in charge of a specific system can control only the cost to an attacker of switching to his system but not to other systems. Second, as we try to translate this theoretical result into real world security practices, it is not completely clear what can serve as a credible signal of strong inner security and not undermine that security at the same time. Therefore, we are not ready to make any recommendations for security practices an individual firm may follow based on this result. A further exploration of this issue is among our priorities for future research.

The last result presents an interesting policy issue, however. It suggests that the same change in μ causes a bigger change in the frequency of intrusions when switching costs are smaller, that is, when it is easier for attackers to determine what type of a target they are dealing with. Therefore, incentives to invest in security are stronger when switching costs for all systems are small. $G = 1000$, $\alpha_0 = 10$, $\alpha_1 = 2$, $\mu_H = 55$, $\mu_L = 50$, $\eta = 0.5$. The top curve, “Before”, represents an L-type system before security enhancement. The middle one, “After (direct effect only)”, shows the expected effect of upgrading to H-type, taking only the direct effect into account. The bottom line, “After (actual)”, shows the frequency of intrusions that will actually occur as a result of an upgrade. It includes both the direct and the behavioral effects.

2.4. Scenario 4 – Targets with unknown security level

In this final version of the model we consider the case when the defender knows its security type but attackers do not. Thus, in this case we are dealing with incomplete asymmetric information. In this case, attackers base their behavior on their beliefs about the security level of a particular target.

The attacker’s belief that he is dealing with an L-type target after effort x has not resulted in a break-in, $P(i = L|x)$ can be determined from the Bayes’ theorem,

$$P(i = L|x) = \frac{P(x|i = L) \cdot P(i = L)}{P(x)} = \frac{(1-\eta)e^{-x/\mu_L}}{(1-\eta)e^{-x/\mu_L} + \eta e^{-x/\mu_H}}. \tag{10}$$

Here, $P(x|i = L)$ is the probability that an L-type target will remain intact after effort x has been spent on it. $P(i = L)$ is derived from the known prior distribution of systems within the population, and $P(x)$ is the probability that an attack on a randomly chosen target will not lead to success after effort x has been spent.

It is easy to show that $\frac{\partial P(i=L|x)}{\partial x} < 0$, which implies that the more effort a target is able to withstand, the less likely the target is of the L-type. It also means that, unlike all the cases discussed so far, the marginal benefit of effort in this case is decreasing in effort. To see this is so, recall that in the deterministic cases discussed above $MB(x|i) = e^{-x/\mu_i} \times G/\mu_i$, where i is the system type. Given the Bayesian mechanism of forming beliefs, the marginal benefit of effort is given by

$$MB_{Bayes}(x) = P(i = L|x)MB(x|L) + P(i = H|x)MB(x|H) = \\ = \frac{G\left((1-\eta)\mu_H e^{-2x/\mu_L} + \eta\mu_L e^{-2x/\mu_H}\right)}{\mu_H\mu_L((1-\eta)e^{-x/\mu_L} + \eta e^{-x/\mu_H})} \quad (11)$$

and $\frac{\partial MB_{Bayes}(x)}{\partial x} < 0$. Even though the attacker's perception of his marginal benefit is constantly evolving, the optimal stopping rule can still be applied. Clearly, $MB_H(x) \leq MB_{Bayes}(x) \leq MB_L(x)$, which in turn implies $\hat{x}_H \leq \hat{x}_{Bayes} \leq \hat{x}_L$ (see the graph in Fig. 10).

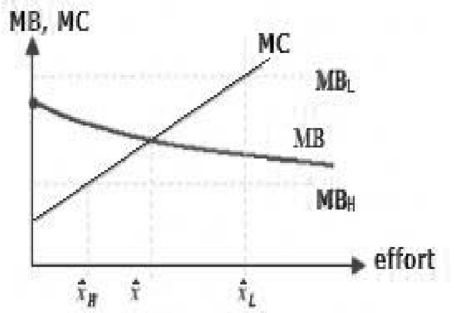

Fig. 10: Marginal benefit and the solution to the attacker's optimal stopping problem for the complete information and the incomplete asymmetric information cases. The dot represents the prior probability of success from a randomly selected target. The expected marginal benefit of attacker's effort is decreasing in effort because the more effort the attacker spends on a target with no success, the more he believes he is dealing with an H-type system

In order to preserve consistency with the preceding analysis, we state the optimal stopping rule by using the expected net benefit from future effort, which in this case equals

$$ENB_{Bayes}(x) = P(i = H|x)ENB_H(x) + P(i = L|x)ENB_L(x) = \\ = \frac{\mu_H^2\alpha_1(z_H(x)-1+e^{-z_H(x)})\eta e^{-x/\mu_H}+\mu_L^2\alpha_1(z_L(x)-1+e^{-z_L(x)})(1-\eta)e^{-x/\mu_L}}{(1-\eta)e^{-x/\mu_L}+\eta e^{-x/\mu_H}}, \quad (12)$$

where $z_i(x) = \frac{G/\mu_i - \alpha_0 - \alpha_1 x}{\mu_i\alpha_1}$, $i = H, L$.

Once again, $ENB_H(x) \leq ENB_{Bayes}(x) \leq ENB_L(x)$ for any x and $\hat{x}_H \leq \leq \hat{x}_{Bayes} \leq \hat{x}_L$.

The following two propositions summarize the results of our analysis of this case.

Proposition 6. *When targets are heterogeneous and their type cannot be determined by attackers, the optimal amount of effort put forth by an attacker does not depend* *on the type of the system. For L-type targets, that amount is smaller than in the case when the target type is known to attackers (the complete information case) whereas for H-type targets it is greater than in the complete information case.* The expression for the frequency of intrusions at a given system in the presence of uncertainty about the target type is modified accordingly:

$$\nu_{i,uncert} = \frac{N_A(1 - e^{-\hat{x}_{i,Bayes}/\mu_i})}{N_T\left(\tau_S + \eta\mu_H(1 - e^{-\hat{x}_{i,Bayes}/\mu_H}) + (1-\eta)\mu_L(1 - e^{-\hat{x}_{i,Bayes}/\mu_L})\right)}. \quad (13)$$

It can be shown that $\nu_H < \nu_{i,uncert} < \nu_L$, where ν_H and ν_L are the frequencies of intrusions in the complete information case given by (9). As was pointed out earlier, the ALE at each system is directly related with the frequency with which intrusions occur. Therefore, we have the following proposition.

Proposition 7. *Attackers' uncertainty about target types increases the annual loss expectancy of H-type systems and decreases it for L-type systems. The size of that effect on systems of a certain type is negatively correlated with the proportion of that type in the population.*

Numerical simulations were performed for the same parameters as before, $G = 1000$, $\alpha_0 = 10$, $\alpha_1 = 2$, $C_S = 5$, $\mu_H = 55$, and $\mu_L = 50$. Figure 11 confirms $\hat{x}_H \leq \hat{x}_{Bayes} \leq \hat{x}_L$.

Figure 12 shows the frequencies of intrusion in the incomplete asymmetric information case as a function of the relative proportion of H-type and L-type systems in the population. Since attackers now put the same amount of effort into attacking each system, the only difference in the frequency of intrusions across the two types (shown by the two lines in the middle) is attributed solely to the direct effect and is, therefore, proportional to the increase in the security parameter. This suggests that under incomplete asymmetric information the incentives to invest in security are substantially reduced.

The frequency of intrusions for each type in the certainty case is also provided for comparison. Clearly, attackers' uncertainty about target types makes$L$-type systems better off and H-type systems worse off. The size of each type's welfare gain or loss from information asymmetry depends on the composition of the population. The fewer H-type systems there are, the greater the loss in their welfare resulting from informational asymmetry, and therefore the more important it is for them to distinguish themselves from the rest of the population.

4. Discussion and Conclusion

Two cases were considered, one in which attackers were able to obtain information about each target's security level and the other one in which the security level was known only to defenders. In both cases, attackers could choose from among multiple alternative targets. In the first, complete information case, attackers' optimal strategy is to put more effort into attacking systems with low security level

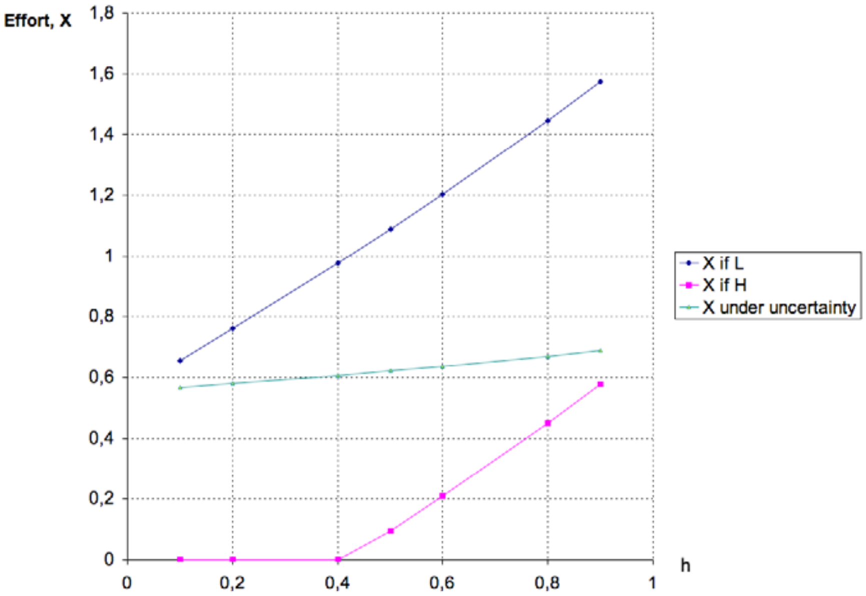

Fig. 11: Effort optimally put by attackers into each system under different scenarios (as a function of the composition of the population). The top and the bottom lines represent $\hat{x}_L$ and $\hat{x}_H$, respectively, from the complete information case, as shown in Figure 6

than into systems with high security level. As a result, any increase in the defender's security level has two effects on the frequency of security incidents. One is the *direct effect* that is attributed to technical characteristics of a system and decreases the probability of success for a given attack effort.

The second, indirect, or *behavioral, effect* decreases the amount of effort an attacker puts into intrusion attempts, thus further decreasing the frequency of security incidents and the expected loss from attacks.

Our analysis suggests that *the magnitude of the behavioral effect can greatly exceed that of the direct one.* As a result, the benefit an individual system may receive from a security enhancement may be severely underestimated if the behavioral effect is not taken into consideration. This implies that some security investments worth making will not be made, leading to either underinvestment in security at the individual system level or, at the very least, to substantial misallocation of resources.

The magnitude of the behavioral effect depends on the security levels of available targets and the attackers ability to obtain information about potential targets' security characteristics and to rank those targets based on their attractiveness. In the complete information setup, attackers were able to determine a target's security level after some reconnaissance phase, with the effort an attacker had to spend to find out

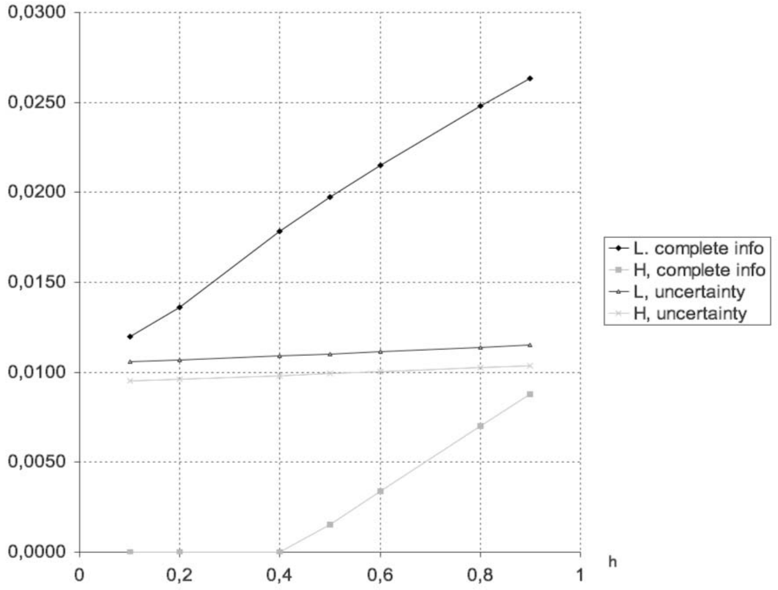

Fig. 12: Frequency of intrusions under different scenarios (as a function of the composition of the population). The top and the bottom lines represent intrusion frequencies for L-type and H-type systems, respectively, from the complete information case, provided in Figure 7. The two lines in the middle represent intrusion frequencies for the two types in the incomplete information case

the target type represented by 'switching costs'. The analysis of that case suggests that the efficacy of security investment depends on the characteristic of the environment which we call "opacity". The greater the switching costs, the more "opaque" the environment is in the sense that it gets harder for attackers to determine the type of a target. As a result, the behavioral component of the overall effect gets weaker. If, on the contrary, the environment is "transparent" and determining the type is relatively easy, then the behavioral and the overall effects of a security investment will be stronger. From the practical perspective, this means that a given security solution will be more effective if potential attackers are aware of extra tools being deployed, and the incentives for firms to invest in security in that case will also be greater.

The above result was further confirmed by the second modification of our model, namely that of incomplete asymmetric information, in which attackers never know the target type with certainty. As a result, they treat every target the same, and the behavioral effect is not present. As a result, systems whose security level is consistently low relative to the rest of the population will have preference for opacity over transparency since it gives them a better chance to disguise themselves as well-protected systems, thus reducing the amount of effort attackers put into attacking them. Well-protected systems, on the contrary, are better off in a transparent environment than in an opaque one and, therefore, have an incentive to signal their high security level to the attackers in order to separate themselves from less secure systems. This is consistent with existing theoretical research on economics of incomplete asymmetric information [Akerlof, 1970] that suggests that the ability to signal one's type (more transparency in the context of our model) benefit "high quality products" (well protected, or H-type systems) and penalize "low quality products" (poorly protected, or L-type systems). The incomplete asymmetric information version of the model also allowed us to address the effect of informational issues and the distribution of system types on the expected welfare losses from attacks. When the proportion of H-type systems in the population is small, then the benefit each of them gets from transparency (thus from identifying themselves as H-type systems) is substantial while L-type systems do not lose much since there are so many of them. When the proportion of H-type systems is large, the opposite is true. Interestingly, we did not find any significant effect of informational assumptions on the aggregate welfare since any benefit H-type systems as a group get from increased transparency was offset by a reduction in L-type systems welfare.

To a certain extent, the above discussion of opacity versus transparency is related to the debate surrounding the "security through obscurity" approach [Beale, 2000], [Perens, 1998], [Schneier, 2002], [Swire, 2004]. "Security through obscurity" is the term coined to denote technical security solutions the effectiveness of which is based on the secrecy of processes, protocols, or algorithms, which contrasts the basic rule of cryptography, the principles of the open source movement. A similar trade-off between disclosing information about the security level and keeping them secret exists. On both occasions, the resulting conclusions are controversial, although overall evidence seems to favor the rejection of the "security through obscurity" (or "security through opacity" in our case) approach. Today, when it comes to the information related to the security level of a corporate network, the practice is to keep it secret because its disclosure might favor attackers. Such a strategy is the more justified the smaller is the share of those "H-type" organizations in the population of potential targets. Thus, our analysis further underscores the importance of an accurate and timely assessment an organization's security level relative to the rest of the population.

Our results also have practical implications for the case of layered security architecture. They suggest that poorly protected systems have more reason to invest in the exterior security layer, thus increasing opacity, whereas for well protected systems investment in the security of interior layers may be more beneficial, assuming they intend to maintain their advantage in protection level over the rest of the population. As always, our analysis has some limitations. Most importantly, our model is static in the sense that it only examines the instantaneous effect of security enhancement assuming the distribution of system types and the rate of attackers' arrival at each target stayed the same. The outcome of any individual defender's decision will, however, also depend on what other defenders are doing at the same time. The interrelationship between the decisions of individual defenders and those of the rest of the population can be fully and properly understood only in a dynamic game-theoretic model. Therefore incorporating tools used in the analysis of interdependent security games is going to greatly enrich our understanding of security processes and practices.

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Dynamic Regularization of Self-Enforcing International Environmental Agreement in the Game of Heterogeneous Players

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Abstract. In the presented paper we consider a coalition formation game with heterogeneous players, where a central issue is a problem of international cooperation towards pollution control. The main concern is to provide a better insight into asymmetric pattern and characterize size and structure of a stable agreement when abatement target is succeeded over a fixed and finite period of time.

For this purpose we suppose that all nations are allocated among K groups with respect to their welfare function. To define a voluntary membership of an international environmental agreement (IEA), we apply the concept of a self-enforcing coalition from oligopoly literature and determine equilibrium abatement commitments for each nation.

We have assumed that once a self-enforcing IEA emerges, signatories decide to perform the required emission reduction uniformly. As soon as the formed coalition initiates activities on emission decrease and the first results are observable, further agreement stability can be in danger. Withdrawal of some nations from the agreement and accessing of others would imply that the coalition will undergo structural change, which in its turn causes sequential switch to another abatement goal.

Presented analyses and examples reveal the following results. Self-enforcing IEA, which performs pollution decrease according to uniform scheme, that has been myopically picked up at the initial moment, is stable only over a certain part of the path. Once abatement has reached a threshold level, external stability fails and free-riders have incentives to access the agreement. This occurs because the uniform pollution reduction scheme sets abatement targets, which differ from optimal ones both for IEA members and free-riders. To protect the coalition against free-riding, we shall continue with constructing a dynamic abatement scheme, which goes along with optimal choice and can depict agreement time-consistency.

Keywords: IEA, heterogeneous players, self-enforcing equilibrium, coalition formation, coalitional games, time-consistency, regularization mechanisms.

Introduction

A set of international environmental agreements has been developed to protect the environment and ecosystems stability, in regional and global manner. These documents, the Kyoto Protocol, in particular, prescribe pollution limitation and reduction, growth of industry effectiveness and introduction of non-hazardous and environmentally friendly raw materials and combustibles. To analyse and manage the process of fulfilling obligations required by the agreement, a game-theoretic approach [Petrosjan and Zakharov, 1997], is introduced to assist us in decision-making in conflicting situations. The theoretical literature on international environmental agreements (IEA) concludes that IEAs suffer from free-riding, since reduced emissions are a global public good, [Breton, 2006]. Examples of IEA formations can be found in [Barrett, 1994], [Carraro and Siniscalco, 1993, 1998], [Diamantoudi and Sartzetakis, 2002], [Finus, 2003], [Finus and Rundshagen, 2001], [Carraro and Marchiori, 2003].

For instance, in [Barrett, 1994] coalition formation game among identical players is considered. It has been obtained that free-riding problem becomes more severe as the potential gains to IEA increase. Similar approach has been used in [Carraro and Siniscalco, 1993], and [Diamantoudi and Sartzetakis, 2002].

In the considered papers coalition formation game with identical players under perfect information was studied. As a result of the previous research stable coalition size of $2, 3$ and 4 players in [Diamantoudi and Sartzetakis, 2002] and 2 and 3 players in [Carraro and Siniscalco, 1993, 1998], regardless of specification of participants, were determined. [Barrett, 1994] finds that the number of coalition members can be very high but only when gains to cooperation are very small.

In the presented paper we consider coalition formation game with heterogeneous players. The main concern is to provide a better insight into asymmetric behavior and to characterize structure of a stable agreement directed to environment protection and decrease in pollution. Asymmetric approach, performed by numerical simulations with 2 groups of players, has been presented in [Barrett, 1997, 2001]. In [McGinty, 2006] N asymmetric player game is presented. Such an advanced approach allows to study pollution transfers and rule of fair surplus allocation among IEA signatories, though it makes it rather difficult to provide estimations of IEA size and structure. To derive a pattern of a stable IEA, we go over to the case when all N players can be allocated among K $(K \leq N)$ types each, composed of N_i $(N_i > 0)$ participants, $i = 1, \ldots, K$. We suppose that players net benefits are presented as difference between polynomial benefit function and quadratic cost function and that all players are familiar with benefits of others.

To define a voluntary membership of the IEA, we have applied the concept of a self-enforcing agreement from oligopoly literature [D'Aspremont, 1983].

This notion of stability means that no non-signatory has an incentive to join IEA and no signatory has an incentive to leave. The goal we posing is to determine optimal abatement levels, to evaluate optimal parameters of cooperation within the coalition and to characterize structure of a stable coalition when players are asymmetric. If players are split into two groups, it may be interpreted as belonging to Annex B countries (Australia, Austria, Belgium, Bulgaria, Canada, Croatia, Czech Republic, Denmark, Estonia, Finland, France (including Monaco), Germany, Greece, Hungary, Iceland, Ireland, Italy (including San Marino), Japan, Latvia, Lithuania, Luxembourg, Netherlands, New Zealand, Norway, Poland, Portugal, Romania, Russian Federation, Slovakia, Slovenia, Spain, Sweden, Switzerland (including Liechtenstein), Ukraine, United Kingdom, United States of America) and non-Annex countries of the Kyoto Protocol.

If we consider three different groups, then it means that we distinguish group of industrialized countries, like USA, European Union, Japan, without pollution permits, group of rapidly developing countries, like Russia, China (whose pollution permits stock is big enough but is expected to be fully exploited internally to compensate extra emission discharge caused by industrial growth) and group of agricultural countries with low abatement cost and large pollution permits stock. As soon as the formed coalition initiates activities on emission decrease and the first results are observable, further agreement stability can be in danger.

Withdrawal of some nations from the agreement and accessing of others would imply that the coalition will undergo structural change, which in its turn causes sequential switch to another abatement goal.

To protect the coalition against free-riding, we suggest constructing a suitable abatement scheme that can depict agreement time-consistency (see [Petrosjan, 1977], [Strotz, 1956]). Assuming stepwise emission reduction over a fixed and finite period $[\tau_0, \ldots, \tau_m]$, we focus attention on the coalition stability in relation to remaining part of abatement commitment. We suggest that the self-enforcing agreement is time-consistent under the proposed abatement scheme, if according to this scheme for every moment τ_j amount of emission to be reduced over the remaining interval $[\tau_j, \tau_m]$ should confirm stability[1] of the formed coalition. Uniform, decreasing and increasing plans for emission reduction can be considered as clear examples of an abatement scheme. Since it is necessary to check if such simple rules fit our purposes, we are going to study relationship between model parameters and ability of the agreement

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[1] In a sense of self-enforcement.

to keep its initial size and structure during stepwise emission reduction, assuming that a certain abatement plan has been adopted. The presented paper is arranged as follows. In the first section description of an IEA model is given. After having described cost and benefit functions of asymmetric nations, we come over the two level game and determine optimal abatement strategies. After that principle of self-enforcement is generalized and employed to depict the pattern of the stable IEA. The second section is mainly devoted to dynamic performance of the chosen abatement commitments and time-consistency of the optimality principle. Finally, to sum up our analysis we present numerical examples to illustrate self-enforcement under the uniform abatement scheme. A more complex setup with stepwise abatement re-optimization is saved for future work.

1. The model of an IEA

1.1. Linear marginal abatement benefits and costs

Consider a world of $N = \sum_{i=1}^{K} N_i$ countries, each of which emits pollutant that damages a shared environment resource. We assume that current abatement benefit $B(E)$ of a country depends on current total abatement $E = \sum_{j=1}^{N} e_j$ as follows[2]

$$B(E) = \frac{b}{N}(aE - E^2/2), \tag{1}$$

where a and b are positive parameters, and E is global abatement. Since there is no strict rule of how abatement benefit is determined, we assume that it can equally be allocated among all countries and expressed by (1).

According to (1) individual marginal benefit function (which can be found as the derivative of $B(E)$) is $MB(E) = \frac{b}{N}(a - E)$ and global marginal benefit function (which can be found as the derivative of global benefit function $\sum_{j=1}^{N} B(E) = \sum_{i=1}^{K} N_i B(E)$) is

$$\sum_{j=1}^{N} MB(E) = b(a - E).$$

Each country's abatement costs are assumed to depend on its own abatement level and no one else's. We make simplifying assumption that countries can generally be split into K groups of N_i, $i = 1, \ldots, K$, nations by their abatement costs[3].

For a country j, $j = 1, \ldots, N$ of type i, $i = 1, \ldots, K$ the abatement cost function $C_i(e_j)$ is assumed to be given by

$$C_i(e_j) = \frac{1}{2} c_i e_j^2, \tag{2}$$

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[2] Forms of benefit and cost functions were borrowed from Barret (1994).

[3] It is noteworthy that we assume nations to be symmetric in their benefit functions and heterogeneous in their costs.

where e_j is country j's abatement and parameter c_i presents the slope of each country's marginal abatement cost curve. Marginal cost function is

$$MC_i(e_j) = c_i e_j.$$

Let us denote $\mathbf{e} = (e_1, \dots, e_j, \dots, e_N)$ as a vector of abatements and

$$\pi_j(e_j, E) = B(E) - C_i(e_j), \tag{3}$$

the j-th country from group i net benefit. Global net benefit will be

$$\Pi(\mathbf{e}) = \sum_{j=1}^{N} \pi_j(e_j, E) = \sum_{i=1}^{K} \left(N_i B(E) - \sum_{l}^{N_i} C_i(e_l) \right). \tag{4}$$

1.2. IEA formation

Let us consider that some nations decide to play non-cooperatively and study their own interests, when other choose collaborative abatement. Let n_i denote number of countries from group i, which sign the IEA. There are then $\sum_{i=1}^{K} n_i$ signatories and $\sum_{i=1}^{K}(N_i - n_i)$ free-riders. Let π_i^s be the net benefit of each signatory of type i and π_i^f that of each free-rider of type i. The equilibrium $\mathbf{n} = (n_1, \dots, n_i, \dots, n_K)$ of countries participating in an IEA can be derived by applying the notions of internal and external stability of a coalition as was originally developed in D'Aspremont *et al.* (1983) and extended to IEAs in Barrett (1994, 1997), Carraro and Siniscalco (1993).

Definition 1. *An IEA consisting of* $\mathbf{n}$ *signatories of* K *(*$K \leq N$*) types is self-enforcing if for each type* i *it holds that*[4]

$$\pi_i^f(n_1, \dots, n_i - 1, \dots, n_K) \leq \pi_i^s(\mathbf{n}), \tag{5}$$

$$\pi_i^f(\mathbf{n}) \geq \pi_i^s(n_1, \dots, n_i + 1, \dots, n_K). \tag{6}$$

Inequality (5) sets condition of internal stability of coalition, *i.e.* no member prefers to withdraw. Condition (6) of external stability guarantees that none of non-members would like to access the coalition. Stability conditions ensure that no player wants to unilaterally deviate, a condition that must hold in any Nash Equilibrium.

Signatories of IEA reduce e_i^s of their emissions, and total abatement undertaken by coalition is $E_s = \sum_{i=1}^{K} e_i^s n_i$. Having chosen non-cooperation, free-riding countries abate e_i^f each and $E_f = \sum_{i=1}^{K} e_i^f (N_i - n_i)$ together.

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[4] Individual net benefit function depends on total abatement E of signatories and free-riders (E_s and E_f respectively, and on individual abatement, e_i^s if it is signatory, or e_i^f if it is free-rider. It is necessary to point out that individual abatements depend on coalition structure given by vector $\mathbf{n} = (n_1, \dots, n_K)$, so $\mathbf{e} = \mathbf{e}(\mathbf{n})$. Thus, it follows that $\pi_i(e_i^{s(f)}, E) = \pi_i(e_i^{s(f)}(\mathbf{n}), E(\mathbf{n})) = \pi_i^{s(f)}(\mathbf{n})$.

We consider two level game, the 1st level is the leader, coalition of IEA, and the 2nd level is the followers, free-riders.

Free-riders adjust their abatement levels after having observed the choice of signatories. Every free-rider maximizes its net benefit non-cooperatively

$$\left\{ \max_{e_i^f} \pi_i(e_i^f, E), \quad i = 1, \dots, K, \right. \tag{7}$$

where $\pi_i(e_i^f, E)$, according to (3), is

$$\pi_i(e_i^f, E) = \frac{b}{N}(a(E_s + E_f) - \frac{1}{2}(E_s + E_f)^2) - \frac{1}{2}c_i \left(e_i^f\right)^2.$$

The first order conditions

$$\frac{\pi_i(e_i^f, E)}{\partial e_i^f} = 0, \quad i = 1, \dots, K,$$

deliver

$$\begin{cases} c_1 e_1^f = \frac{b}{N}\left(a - (E_s + \sum_{l=1}^K e_l^f (N_l - n_l))\right), \\ \dots \\ c_i e_i^f = \frac{b}{N}\left(a - (E_s + \sum_{l=1}^K e_l^f (N_l - n_l))\right), \\ \dots \\ c_K e_K^f = \frac{b}{N}\left(a - (E_s + \sum_{l=1}^K e_l^f (N_l - n_l))\right). \end{cases} \tag{8}$$

It is practical to introduce the following notations:

- Let $\lambda = (\lambda_1, \dots, \lambda_i, \dots, \lambda_K)$ be a vector, where
$$\lambda_i = \frac{b}{c_i}. \tag{9}$$
- $\mathbf{N} = (N_1, \dots, N_i, \dots, N_K)$.
- Let $\bar{\mathbf{1}} = (1, \dots, 1)$ be a vector of units.
- For two given vectors $\mathbf{x} = (x_1, \dots, x_r)$ and $\mathbf{y} = (y_1, \dots, y_r)$ expression $(\mathbf{x}, \mathbf{y})$ means their scalar product $\sum_{i=1}^r x_i y_i$.

Using notations described above, individual abatements of free-riders can be expressed in terms of reaction function of signatories' abatement, *i.e.*

$$e_i^f = \frac{\lambda_i(a - E_s)}{N + \sum_{l=1}^K \lambda_l (N_l - n_l)} = \frac{\lambda_i(a - E_s)}{(\bar{1} + \lambda, \mathbf{N}) - (\lambda, \mathbf{n})}. \tag{10}$$

Abatement of all non-signatories is

$$E_f = \sum_{i=1}^K (N_i - n_i) e_i^f = \frac{(\lambda, \mathbf{N} - \mathbf{n})}{(\bar{1} + \lambda, \mathbf{N}) - (\lambda, \mathbf{n})}(a - E_s) = g(a - E_s), \tag{11}$$

where

$$g = \frac{(\lambda, \mathbf{N} - \mathbf{n})}{(\bar{1} + \lambda, \mathbf{N}) - (\lambda, \mathbf{n})}, \tag{12}$$

and $E_s = E_s(\mathbf{n})$, and consequently $E_f = E_f(\mathbf{n})$. Signatories choose their abatement level E_s by maximizing their collective net benefit while taking into account behavior of non-signatories

$$\sum_{i=1}^{K} \pi_i(e_i^s, E) = \frac{\sum_{i=1}^{K} n_i}{N} b \left(a(E_s + g(a - E_s)) - \frac{1}{2}(E_s + g(a - E_s))^2 \right)$$

$$-\frac{1}{2} \sum_{i=1}^{K} n_i c_i \left(e_i^s\right)^2 .$$

Abatement E_s can be chosen by solving the problem

$$\max \textstyle\sum_{e_i^s} \pi_i(e_i^s, E), \quad \text{subject to (11)}, \tag{13}$$

that leads to a system of partial differential equations

$$\left\{ \tfrac{\partial \pi_i(e_i^s, E)}{\partial e_i^s} = 0, \quad i = 1, \ldots, K. \right.$$

Thus, we come to the following system

$$\begin{cases} c_1 e_1^s = b\left(a(1-g)^2 - E_s(1-g)^2\right) (\sum_{l=1}^{K} n_l)/N, \\ \ldots \\ c_i e_i^s = b\left(a(1-g)^2 - E_s(1-g)^2\right) (\sum_{l=1}^{K} n_l)/N, \\ \ldots \\ c_K e_K^s = b\left(a(1-g)^2 - E_s(1-g)^2\right) (\sum_{l=1}^{K} n_l)/N. \end{cases}$$

Individual abatement of a signatory of type i is

$$e_i^s = \frac{a\lambda_i (1-g)^2 (\bar{1}, \mathbf{n})}{(\bar{1}, \mathbf{N}) + (1-g)^2 (\bar{1}, \mathbf{n})(\lambda, \mathbf{n})}, \quad i = 1, \ldots, K, \tag{14}$$

and the total abatement of signatories would be

$$E_s = \frac{a(1-g)^2 (\bar{1}, \mathbf{n})(\lambda, \mathbf{n})}{(\bar{1}, \mathbf{N}) + (1-g)^2 (\bar{1}, \mathbf{n})(\lambda, \mathbf{n})}. \tag{15}$$

Substituting (15) into (11) and (10), total and individual abatement of non-signatories would be, respectively

$$E_f = \frac{ag(\bar{1}, \mathbf{N})}{(\bar{1}, \mathbf{N}) + (1-g)^2 (\bar{1}, \mathbf{n})(\lambda, \mathbf{n})},$$

$$e_i^f = \frac{\lambda_i a(\bar{1}, \mathbf{N})}{[(\bar{1}, \mathbf{N}) + (1-g)^2(\bar{1}, \mathbf{n})(\lambda, \mathbf{n})][(\bar{1}+\lambda, \mathbf{N}) - (\lambda, \mathbf{n})]}. \tag{16}$$

Proposition 1. *On the assumption that model parameters a, b and c_i, $i = 1, \dots, K$, are positive, each nation accessed IEA has finite and positive abatement commitments, determined in (14). Individual abatements of free-riders, given by (16), are also finite and positive.*

Proof.

The validity of the statement directly follows from expressions (14) and (16). If $a, b, c_i > 0$ holds, the numerator of fraction (14): $a\lambda_i(1-g)^2(\bar{1}, \mathbf{n})$ and the term of fraction

$$(\bar{1}, \mathbf{N}) + (1-g)^2(\bar{1}, \mathbf{n})(\lambda, \mathbf{n})$$

are both also positive. The numerator of (16) $\lambda_i a(\bar{1}, \mathbf{N})$ and the term of fraction

$$[(\bar{1}, \mathbf{N}) + (1-g)^2(\bar{1}, \mathbf{n})(\lambda, \mathbf{n})][(\bar{1}+\lambda, \mathbf{N}) - (\lambda, \mathbf{n})]$$

are both positive.

In this section we have considered formation of international agreement aiming at coordinate activities of different nations on emission reduction. In relation to IEA players have split to signatories and free-riders and committed to certain abatement levels.

Real world examples show that a seemingly stable agreement might get in danger as soon as activities on emission decrease have been initiated and the first results are observable. Withdrawal of some nations from the agreement and accessing of others would imply that the coalition will undergo structural change, which in its turn causes sequential switch to another abatement goal.

2. Abatement dynamics

Since one of the possible coalitions has been formed, each nation needs to develop an optimal abatement scheme, according to which it would stepwise reduce its emissions over a finite and discrete time period $[t_0, \dots, t_m]$, $m > 1$. A scheme should be designed in a way that for any moment t_j, $j = 0, \dots, m$ remaining part of abatement commitment, which is supposed to be fulfilled during $[t_j, t_m]$, must deliver stability of the formed coalition. In other words, self-enforcing coalition must be time-consistent under proposed abatement scheme.

Time-consistency (see *e.g.* [Petrosjan, 1977, 1993]) of optimality principle means that any segment of the optimal trajectory determines the optimal motion with respect to relevant initial states of the process. This property holds for the overwhelming majority of classical optimal control problems and follows from the Bellman optimality principle [Bellman, 2003]. Absence of time-consistency (see [Strotz, 1955–1956]) involves the possibility that the previous "optimal" decision is abandoned at some current moment of time, thereby making meaningless the problem for seeking an optimal control.

We say that self-enforcing coalition is time-consistent if any of the signatories/free-riders has no incentive to change plans for future abatement and leave/access coalition, once current abatement decisions are made. It implies that if at the initial moment t_0 of the game players agreed that they would form a coalition according to the self-enforcing optimality principle (see Definition 1) then at each current moment t_j $(t_0 \leq t_j \leq t_m)$ the formed coalition must remain stable and satisfy conditions of internal (5) and external stability (6) required by self-enforcement.

Let us present an abatement scheme as follows. A division of time period is given as $t_0 < t_1 < \ldots < t_m$ and an abatement scheme

$$^5\left\{e_i(S, t_j)\right\}, \quad i = 1, \ldots, K, j = 0 \ldots, m.$$

For the sake of simplicity, we introduce some assumptions and notations.

- Model parameters, such as c_i, $i = 1, \ldots, K$, are considered to be constant over the time $\{t_0, t_1, \ldots, t_m\}$.

- Multistage coalitional game we consider is described as $(S, \Delta(S))$, where $S \subseteq \mathcal{N}$ determines set of players which belong to self-enforcing, IEA and the function

$$\Delta(S) = \sum_{\{i\} \in S} \Delta_i(S)$$

  is the total payoff available to the members of set S, which is formed according to the optimality principle, where $\Delta_i(S)$ is individual surplus of the coalition member. We assume that side payments are not allowed inside the coalition. Hence, the only motivating factor for signatory to access the coalition is individual surplus from participating, determined as

$$\Delta_i(S) = \pi_i^s(S) - \pi_i^f(S \setminus \{i\}),$$

  where $\pi_i^s(S)$ is individual net benefit of player of type i from coalition S, and $\pi_i^f(S \setminus \{i\})$ is individual net benefit of player i if it decides to leave the coalition. Hence,

$$\Delta_i(S) = \frac{b}{N}\left(aE(S) - \frac{1}{2}(E(S)^2\right) - \frac{1}{2}c_i\left(e_i^s(S)\right)^2$$

$$-\frac{b}{N}\left(aE(S \setminus \{i\}) - \frac{1}{2}(E(S \setminus \{i\})^2\right) + \frac{1}{2}c_i\left(e_i^f(S \setminus \{i\})\right)^2,$$

  According to the optimality principle, $\Delta_i(S)$ is always nonnegative, if coalition S is self-enforcing.

---

[5] Index S corresponds to the current coalition structure, and $e_i(S, t_m) = 0$ for all $i = 1, \ldots, K$ means that nation of type i has no commitment left by moment t_m.

- Value $e_i^{s(f)}(S,t_j)$ denotes remainder of nation i's abatement to be undertaken during $[t_j,t_m]$. In other words, it means how much emission should be reduced by nation i by the end of the game. At the beginning of the game (moment t_0) player i has to abate $e_i^{s(f)}(S,t_0)=e_i^{s(f)}$. During the period $[t_0,t_1]$ player i abates $e_i^{s(f)}(S,t_0)-e_i^{s(f)}(S,t_1)$, and at the moment t_1 its abatement commitment is $e_i^{s(f)}(S,t_1)$, *etc.*

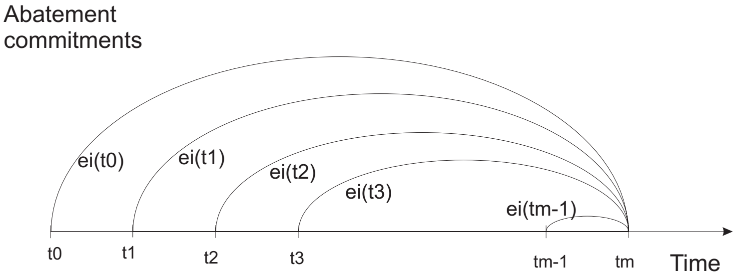

Fig. 1: Graphical explanation of an abatement scheme

Total abatement commitment at t_j is denoted $E(S,t_j)=E_s(S,t_j)+E_f(S,t_j)$ and calculated as a sum of commitments of signatories and free-riders. Recalling that free-riders' reaction to signatories' abatement is given by (11), it leads to[6]

$$E(S,t_j)=(1-g)\times E_s(S,t_j)+a(t_j)g,$$

where $a(t_j)=a-(E(S,t_0)-E(S,t_j))$ and $E(S,t_0)=E(S)$.

In the general case, self-enforcing is determined according to Definition 1. We remind here that it is such an optimality principle which requires satisfaction of conditions of internal and external stability

$$\pi_i^f(S\setminus\{i\})\le\pi_i^s(S),\qquad \{i\}\in S$$

$$\pi_i^f(S)\ge\pi_i^s(S\cup\{i\}),\qquad \{i\}\in\mathcal{N}\setminus S$$

for certain set of signatories S.

Definition 2. *Self-enforcing coalition is time-consistent under a certain abatement scheme if for every moment t_j, $j=0,\dots,m-1$,*

1. *condition of internal stability holds*

$$\frac{b}{N}\left(a(t_j)E(S\setminus\{i\},t_j)-\frac{1}{2}(E(S\setminus\{i\},t_j))^2\right)-\frac{1}{2}c_i\left(e_i^f(S\setminus\{i\},t_j)\right)^2$$

---

[6] Value g is determined by (12) and presents itself a function on coalition structure. For simplicity we omit $g=g(\mathbf{n})$, except special cases where it is necessary.

$$\leq \frac{b}{N}\left(a(t_j)E(S,t_j) - \frac{1}{2}(E(S,t_j))^2\right) - \frac{1}{2}c_i\left(e_i^s(S,t_j)\right)^2,$$

where set $S \setminus \{i\}$ reflects situation when one player of type i abandons former coalition S, and

2. *condition of external stability holds*

$$\frac{b}{N}\left(a(t_j)E(S\cup\{i\},t_j) - \frac{1}{2}(E(S\cup\{i\},t_j))^2\right) - \frac{1}{2}c_i\left(e_i^s(S\cup\{i\},t_j)\right)^2$$

$$\leq \frac{b}{N}\left(a(t_j)E(S,t_j) - \frac{1}{2}(E(S,t_j))^2\right) - \frac{1}{2}c_i\left(e_i^f(S,t_j)\right)^2,$$

for external stability, where set $S\cup\{i\}$ means that one player of type t accesses former coalition S.

2.1. Uniform abatement scheme

It seems natural to start analysis of the problem from considering a simple case, assuming that emission reduction commitments are uniformly split along the accounted period.

As we have already mentioned above, this section is devoted to specification of uniform division rule and correspondent pollution reduction scheme, and analysis of time-consistency of self-enforcing coalition under such a rule.

Uniform division rule (see Fig. 2 and 3) delivers the following individual pollution reduction scheme

$$e_i(S,t_j) = \frac{m-j}{m}e_i(S), \qquad j = 0,\ldots,m, \tag{17}$$

hence, coalitional abatement scheme is

$$E_s(S,t_j) = \frac{m-j}{m}E_s(S), \qquad j = 0,\ldots,m.$$

It is important to point out here, that according to formulaes (14) and (15), values $e_i^{s(f)}(S,t_j)$ and $E_s(S,t_j)$ depend on size and structure of self-enforcing coalition S characterized by the structure vector $\mathbf{n} = (n_1,\ldots,n_K)$.

Proposition 2. *The abatement scheme given as (17) satisfies two properties:*

1. *under such a scheme pollution reduction is undertaken uniformly,*
2. *total abatement under such a scheme is equal to abatement commitment required by the agreement.*

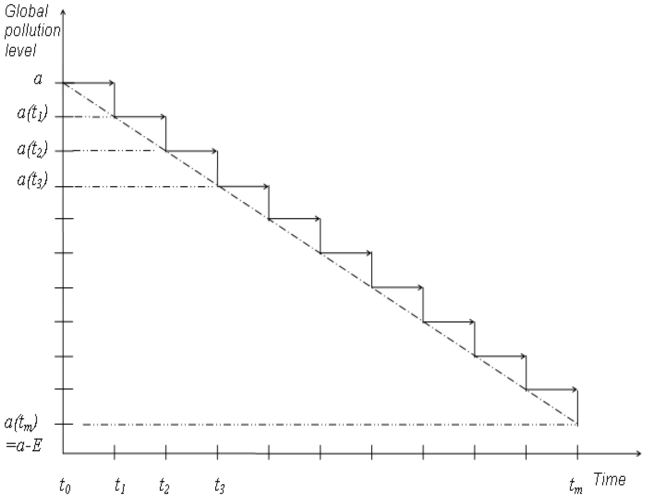

Fig. 2: Graphical explanation of the uniform reduction of the global pollution

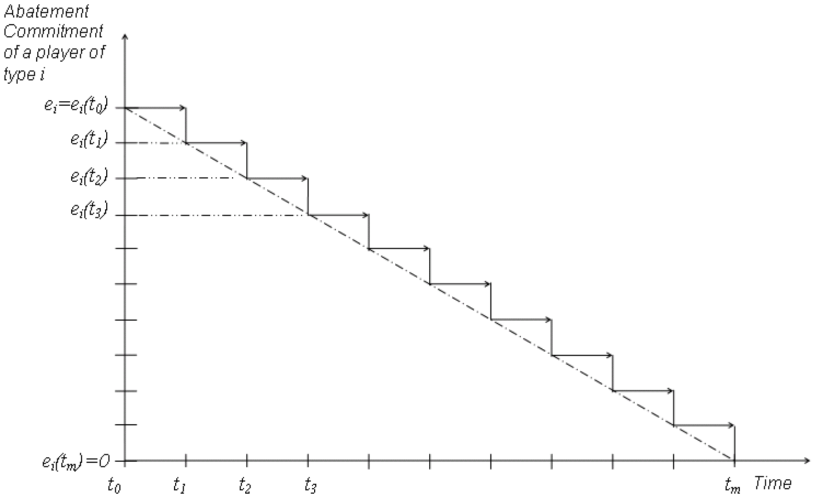

Fig. 3: Graphical explanation of the uniform abatement dynamics

Proof.

To prove the first statement let us consider emission reductions over the interval $[t_j, t_{j+1}]$ and see that it does not depend on moment t_j, $j = 0, \ldots, m-1$.

$$(e_i(S, t_j) - e_i(S, t_{j+1})) = \frac{m-j}{m} e_i(S) - \frac{m-j-1}{m} e_i(S) = \frac{1}{m} e_i(S).$$

The second statement directly follows from the following expression

$$\sum_{j=0}^{m-1} (e_i(S,t_j) - e_i(S,t_{j+1})) = \sum_{j=0}^{m-1} \frac{1}{m} e_i(S) = e_i(S),$$

which demonstrate that whole required abatement is undertaken by every player.

The following statement provides conditions on the model parameters and the IEA structure, which guarantee that the self-enforcing coalition is time-consistent under the uniform abatement scheme.

Proposition 3. *Self-enforcing coalition S is time-consistent under the uniform abatement scheme (17), $m > 1$, if the following inequalities hold for all $j = 1, \ldots, m-1$,*

$$1. \quad \frac{j}{m-j} \cdot \frac{b}{N} \cdot (E(S \setminus \{i\}) - E(S))\,(a - E(S)) \leq \Delta_i(S),$$

where $S \setminus \{i\}$ means that one signatory of type i abandons former coalition S,

$$2. \quad \frac{j}{m-j} \times \frac{b}{N} \times (E(S) - E(S \cup \{i\}))\,(a - E(S)) \geq \Delta_i(S \cup \{i\}),$$

$S \cup \{i\}$ means that one free-rider of type i accesses former coalition S.

Proof.

Let us check time-consistency of self-enforcing coalition under accepted uniform abatement scheme, preliminary introducing practical notations

$$E = E(S), \quad E' = E(S \setminus \{i\}), \quad E'' = E(S \cup \{i\}),$$

$$e_s = e_i^s(S), \quad e_f = e_i^s(S), \quad e'_f = e_i^f(S \setminus \{i\}), \quad e''_s = e_i^s(S \cup \{i\}).$$

First we examine internal stability

$$\frac{b}{N}\left(a(t_j)E(S,t_j) - \frac{1}{2}(E(S,t_j))^2\right) - \frac{1}{2}c_i\left(e_i^s(S,t_j)\right)^2$$

$$-\frac{b}{N}\left(a(t_j)E(S \setminus \{i\},t_j) - \frac{1}{2}(E(S \setminus \{i\},t_j))^2\right) + \frac{1}{2}c_i\left(e_i^f(S \setminus \{i\},t_j)\right)^2 \geq 0.$$

We call attention to that $a(t_j) = a - (E(t_0) - E(t_j))$, where $E(t_0) = E(S)$ since IEA was self-enforcing at the moment t_0, value $E(t_j)$ depends on the coalition structure at the current moment.

$$\frac{b}{N}\left(\frac{m-j}{m}(a - E + E\frac{m-j}{m})E - \frac{1}{2}(\frac{m-j}{m}E)^2\right) - \frac{1}{2}c_i\left(\frac{m-j}{m} - e_s\right)^2$$

$$-\frac{b}{N}\left(\frac{m-j}{m}(a - E + E\frac{m-j}{m})E' - \frac{1}{2}(\frac{m-j}{m}E')^2\right) + \frac{1}{2}c_i\left(\frac{m-j}{m}e'_f\right)^2 \geq 0,$$

$$\left(\frac{m-j}{m}\right)^2 \frac{b}{N}\left[a\,(E-E') - \frac{1}{2}(E^2 - E'^2) - \frac{1}{2}c_i(e_s^2 - {e'_f}^2)\right] +$$
$$+\frac{(m-j)j}{m^2} \times \frac{b}{N}(E-E')(a-E) \geq 0,$$
$$\Delta_i(S) + \frac{j}{m-j} \times \frac{b}{N} \times (E-E')\,(a-E) \geq 0,$$

for every $j = 1, \ldots, m-1$.

The sign of the first term depends on the sign of $\Delta_i(S)$. According to *a priori* condition about self-enforcement of coalition S at the moment t_0, it follows that condition of internal stability holds at moment t_0

$$\pi_i^s(S) \geq \pi_i^f(S \setminus \{i\}),$$

and thus $\Delta_i(S) \geq 0$, where $\Delta_i(S) = \pi_i^s(S) - \pi_i^f(S \setminus \{i\})$. Hence, the first term is non-negative. The sign of the second term is indicated by $E - E'$, the difference between total abatement in cases when agreement is composed of players from set S and when it consists of $S \setminus \{i\}$. Its sign fully depends on the model parameters.

Thus, internal stability of coalition S is time-consistent if for every $j =, \ldots, m-1$

$$\frac{j}{m-j} \cdot \frac{b}{N} \cdot (E(S \setminus \{i\}) - E(S))\,(a - E(S)) \leq \Delta_i(S),$$

where $\{i\} \in S$.

Now we turn our attention to external stability:

$$\frac{b}{N}\left(a(t_j)E(S \cup \{i\}, t_j) - \frac{1}{2}(E(S \cup \{i\}, t_j))^2\right) - \frac{1}{2}c_i\left(e_i^s(S \cup \{i\}, t_j)\right)^2 -$$
$$-\frac{b}{N}\left(a(t_j)E(S, t_j) - \frac{1}{2}(E(S, t_j))^2\right) + \frac{1}{2}c_i\left(e_i^f(S, t_j)\right)^2 \leq 0.$$

Similar reasoning shows that

$$\Delta_i(S \cup \{i\}) + \frac{b}{N} \times \frac{j}{m-j} \times (E'' - E)(a-E) \leq 0.$$

The sign of the first term depends on the sign of $\Delta_i(S \cup \{i\})$. According to *a priori* condition about self-enforcement of coalition S at the moment t_0, it follows that condition of external stability holds at moment t_0

$$\pi_i^f(S) \geq \pi_i^s(S \cup \{i\}),$$

and thus

$$\Delta_i(S \cup \{i\}) \leq 0,$$

where

$$\Delta_i(S \cup \{i\}) = \pi_i^s(S \cup \{i\}) - \pi_i^f(S), \qquad \{i\} \in \mathcal{N} \setminus S.$$

Hence, the first term is non-positive. The sign of the second term is indicated by $E'' - E$, the difference between total abatement in cases when agreement is composed of players from set S and when it consists of $S \cup \{i\}$.

Thus, external stability of coalition S is time-consistent if for every $j = 1, ..., m-1$

$$\frac{j}{m-j} \cdot \frac{b}{N} \cdot (E(S \cup \{i\} - E(S)))\,(a - E(S)) + \Delta_i(S \cup \{i\}) \leq 0.$$

Thus, both statements of Proposition 3 holds.

2.2. Numerical Simulations on Time-Consistency

Let us now turn to the numerical example[7] and test it for time-consistency. The model parameters are assumed to be as in Table 1.

Table 1: Parameters of the model

	Type 1	*Type* 2
$b = 1$	$c_1 = 0.5$	$c_2 = 0.8$
$a = 100$	$N_1 = 5$	$N_2 = 10$

Undertaken analysis in Section 1.2 allows us to depict two possible self-enforcing IEAs, see Table 2. Let us suppose that one of the possible coalitions has been formed, for instance, $S = \{'1', '2', '2'\}$ with one player of type 1 and two players of type 2.

Table 2: Self-Enforcing Coalitions

	1	2
Structure	$n_1 = 0 \quad n_2 = 3$	$n_1 = 1 \quad n_2 = 2$
Coalition members'	–	$\pi_1 = 272.226$
net benefit	$\pi_2 = 275.986$	$\pi_2 = 276.779$
Emission reduction	$E = 61.290$	$E = 61.673$

Once the coalition has emerged, nations belonging to IEA begin stepwise to reduce their emissions over a fixed and finite period $[t_0, \ldots, t_m]$, applying uniform abatement scheme.

We are going to undertake a set of numerical tests to check if the self-enforcing coalition is time-consistent under the adopted abatement plan. Proposition 3 serves this purpose well since it provides sufficient conditions on the model parameters and the IEA structure, which guarantee that the self-enforcing coalition is time-consistent under the uniform abatement scheme.

First, we consider coalition dynamics relative to players of type 1. To proceed with calculations, first, we should determine static parameters of the expressions

$$E(S \setminus \{1\}) = 59.615, \qquad E(S) = 61.673, \qquad E(S \cup \{1\}) = 65.581,$$

---

[7] Examples data are provided by Windows application, specially designed for examined model. It was coded in Delphi by Yu. Pavlova. Graphical figures were prepared in Maple 9.5.

$$E(S\setminus\{2\})=59.565,\qquad E(S)=61.673,\qquad E(S\cup\{2\})=64.802,$$
$$\Delta_{'1'}(S)=0.505,\qquad \Delta_{'1'}(S\cup\{1\})=-3.717,$$
$$\Delta_{'2'}(S)=2.485,\qquad \Delta_{'1'}(S\cup\{2\})=-0.539.$$

Inequalities 1 and 2 of Proposition 3 are as follows

$$1_1.\qquad -5.256\frac{j}{m-j}\leq 0.505,$$

$$1_2.\qquad -5.383\frac{j}{m-j}\leq 2.485,$$

$$2_1.\qquad -9.985\frac{j}{m-j}\leq -3.717,$$

$$2_2.\qquad -7.998\frac{j}{m-j}\leq -0.539.$$

As one may see, the first two inequalities (responsible for internal stability) hold for all $j=1,\ldots,m-1$ for any m. Responsible for external stability, inequalities 2_1 and 2_2 hold only for certain j.

We alter parameter m, which determines number segments on the observing interval. It means that we would change number of 'check-points'.

- Let us assume that $m=1000$. Condition of external stability holds only until $j=64$.[8]
- Now let $m=100$. External stability fails after the first step and does not hold since $j\geq 7$.
- Finally, when $m=10$ external stability does not hold.

If we test the second possible coalition $S=\{'2','2','2'\}$. Inequalities 1 and 2 of Proposition 3 are as follows

$$1.\qquad -4.323\frac{j}{m-j}\leq 1.547,$$

$$2.\qquad -7.136\frac{j}{m-j}\geq -1.189.$$

- Let us assume that $m=1000$. Internal stability holds for all feasible j. Condition of external stability holds only until $j=143$.
- Now let $m=100$. Internal stability holds for all j and external stability fails after the first step and does not hold since $j\geq 15$.

---

[8] As you may see in Figures 4 and 5, net benefit functions $\pi_i^f(S,t_j)$, given by a solid line, and $\pi_i^s(S\cup\{i\},t_j)$, given by a dash line, intersect at the interval $(63,64)$. After that $\pi_i^f(S,t_j)<\pi_i^s(S\cup\{i\},t_j)$, $j\geq 64$, which implies that a former outsider of type i decides to access the agreement.

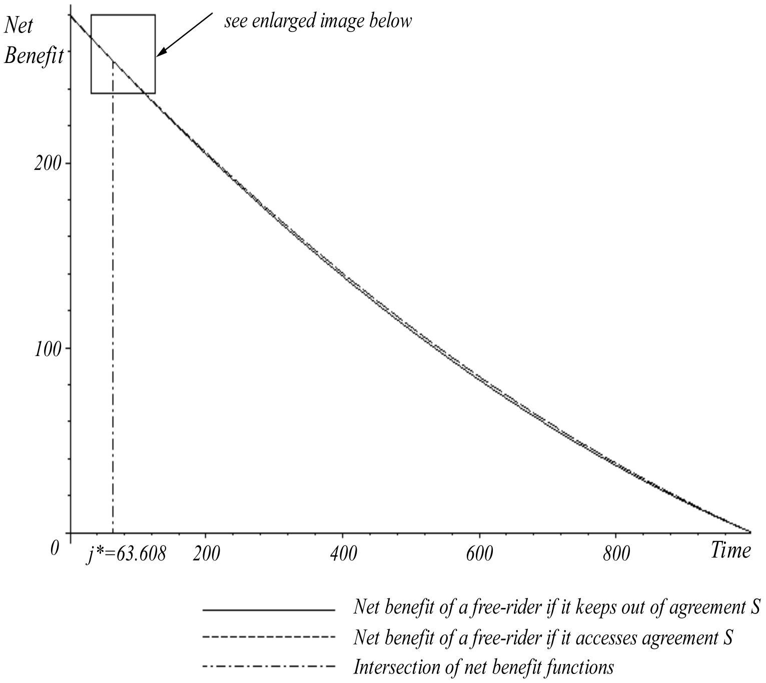

Fig. 4: Net benefit functions of a free-rider of type i

- ○ Finally, when $m = 10$ external stability holds only for $j = 1, 2$. As before, internal stability holds for all feasible j.

Appendix. Other types of abatement schemes

Decreasing abatement scheme

To present decreasing (and non-decreasing) distribution we adopt polynomial expression. Decreasing division rule delivers the following individual pollution reduction scheme

$$e_i(S, t_j) = \left(\frac{m-j}{m}\right)^2 e_i(S), \qquad j = 0, \dots, m, \tag{18}$$

hence coalitional abatement scheme is

$$E_s(S, t_j) = \left(\frac{m-j}{m}\right)^2 E_s(S), \qquad j = 0, \dots, m.$$

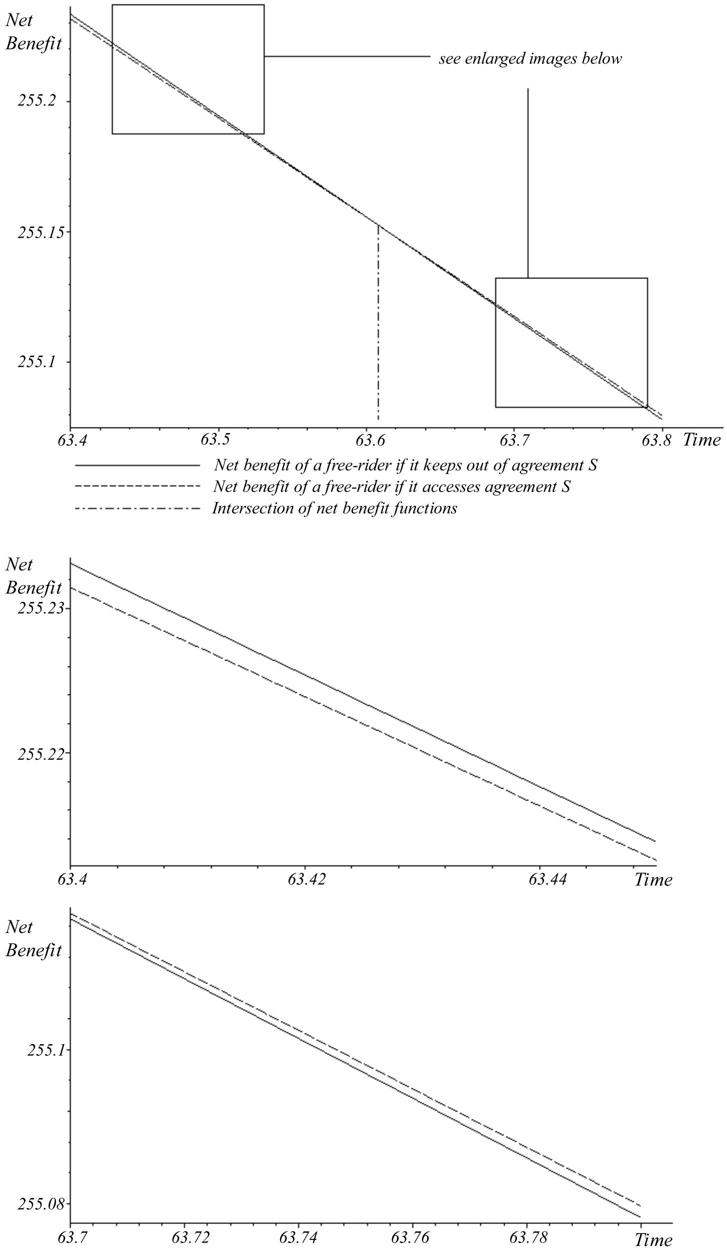

Fig. 5: External stability fails at t_j, $j \geq 64$

It is easy to notice that the given, thus, abatement scheme would be described by the concave function. In much the same way as in the previous section, we formulate the following statements.

Proposition 4. *The abatement scheme given as (18) satisfies two properties:*

1. *under such a scheme, quantity of reduced emissions at* $[t_j, t_{j+1}]$, $j = 0, \dots, m-1$ *is diminishing in time,*
2. *total abatement under such a scheme is equal to abatement commitment required by the agreement.*

Proposition 5. *Self-enforcing coalition* S *is time-consistent under the decreasing abatement scheme (18) if the following conditions hold:*

$$1. \quad E(S \setminus \{t\}) - E(S) \leq \frac{1}{m^2-1}\frac{N}{ab}\Delta_t(S), \qquad t \in S,$$

$$2. \quad E(S) - E(S \cup \{t\}) \geq \frac{1}{m^2-1}\frac{N}{ab}\Delta_t(S \cup \{t\}), \qquad t \in \mathcal{N} \setminus S.$$

Proof.

To prove Proposition 5, we adopt approach employed for reasoning of Proposition 3. It reveals that internal stability of coalition S is time-consistent under decreasing abatement scheme (18) if for every $j = 1, \dots, m-1$

$$E(S \setminus \{t\}) - E(S) - \frac{(m-j)^2}{j(2m-j)}\frac{N}{ab}\Delta_t(S) \leq 0, \quad t \in S, \tag{19}$$

and external stability of coalition S is time-consistent if for every $j = 1, \dots, m-1$

$$E(S) - E(S \cup \{t\}) - \frac{(m-j)^2}{j(2m-j)}\frac{N}{ab}\Delta_t(S \cup \{t\}) \geq 0, \tag{20}$$

where $\Delta_t(S) \geq 0$ and $\Delta_t(S \cup \{t\}) \leq 0$.

Again we introduce intermediate function

$$z(j) = w - \frac{(m-j)^2}{j(2m-j)}v, \tag{21}$$

where w and v are real-valued.

For $j = 1, \dots, m-1$ and positive m function $z(j)$ is monotonously increasing if v is positive and monotonously decreasing if v is negative (value zero is not interesting for analysis since it can hardly occur and would bring us to trivial case), since

$$z'(j) = \frac{2m^2(m-j)}{j^2(2m-j)^2}v.$$

Thus, it is only necessary to guarantee that inequalities (19) and (20) of internal and external stability respectively hold for $j = m - 1$. Plugging $j = m - 1$ in (19) and (20), we prove the statement.

Rearranging inequalities in statement of Proposition 5 allows us to formulate the corollary.

Corollary 1. *Self-enforcing coalition S is time-consistent under the decreasing abatement scheme (18) if the following conditions hold for length of the abating period m:*

1. $m \leq \sqrt{\alpha}$ *if* $E(S \setminus \{t\}) - E(S) > 0$, *where*
$$\alpha = \frac{N}{ab} \frac{\Delta_t(S)}{E(S \setminus \{t\}) - E(S)},$$
2. $m \leq \sqrt{1+\beta}$ *if* $E(S) - E(S \cup \{t\}) < 0$, *where*
$$\beta = \frac{N}{ab} \frac{\Delta_t(S \cup \{t\})}{E(S) - E(S \cup \{t\})}.$$
3. $m > 1$.

Increasing abatement scheme

According to this rule, the individual pollution reduction scheme would be

$$e_i(S, t_j) = \left(\frac{m^2 - j^2}{m^2}\right) e_i(S), \qquad j = 0, \dots, m, \tag{22}$$

hence, coalitional abatement scheme is

$$E_s(S, t_j) = \left(\frac{m^2 - j^2}{m^2}\right) E_s(S), \qquad j = 0, \dots, m.$$

The presented abatement scheme would be described by the convex function.

Proposition 6. *The abatement scheme given as (22) satisfies two properties*

1. *under such a scheme, quantity of reduced emissions at $[t_j, t_{j+1}]$, $j = 0, \dots, m-1$ is increasing in time,*
2. *total abatement under such a scheme is equal to abatement commitment required by the agreement.*

Proposition 7. *Self-enforcing coalition S is time-consistent under the non-decreasing abatement scheme (22) if the following conditions hold:*

$$1. \quad E(S \setminus \{t\}) - E(S) \leq \frac{2m-1}{(m-1)^2} \frac{N}{ab} \Delta_t(S), \qquad t \in S,$$

$$2.\quad E(S) - E(S \cup \{t\}) \geq \frac{2m-1}{(m-1)^2}\frac{N}{ab}\Delta_t(S \cup \{t\}), \qquad t \in \mathcal{N} \setminus S.$$

Proof.

Self-enforcing coalition S is time-consistent under non-decreasing abatement scheme (22) if for every $j = 1, \ldots, m-1$ conditions of internal and external stability hold:

$$E(S \setminus \{t\}) - E(S) - \frac{m^2 - j^2}{j^2}\frac{N}{ab}\Delta_t(S) \leq 0, \quad t \in S, \tag{23}$$

$$E(S) - E(S \cup \{t\}) - \frac{m^2 - j^2}{j^2}\frac{N}{ab}\Delta_t(S \cup \{t\}) \geq 0, \quad t \in \mathcal{N} \setminus S, \tag{24}$$

where $\Delta_t(S) \geq 0$ and $\Delta_t(S \cup \{t\}) \leq 0$. To prove that inequalities (23) and (24) hold for $j = 1, \ldots, m-1$, it is necessary only to guarantee that they hold for $j = m-1$. Plugging $j = m-1$ in (23) and (24), we prove the statement.

Corollary 2. *Self-enforcing coalition S is time-consistent under the non-decreasing abatement scheme (22) if the following conditions hold for length of the abating period m:*

1. *if* $\varepsilon = E(S \setminus \{t\}) - E(S) > 0$ *then*

$$1 + \frac{\xi - \sqrt{\xi^2 + \xi\varepsilon}}{\varepsilon} \leq m \leq 1 + \frac{\xi + \sqrt{\xi^2 + \xi\varepsilon}}{\varepsilon},$$

where

$$\xi = \frac{N}{ab}\Delta_t(S),$$

2. *if* $\varepsilon < 0$ *and* $\varepsilon + \xi > 0$ *then*

$$m \geq 1 + \frac{\xi - \sqrt{\xi^2 + \xi\varepsilon}}{\varepsilon},$$

3. *if* $\epsilon = E(S) - E(S \cup \{t\}) < 0$ *then*

$$1 + \frac{\zeta + \sqrt{\zeta^2 + \zeta\epsilon}}{\epsilon} \leq m \leq 1 + \frac{\zeta - \sqrt{\zeta^2 + \zeta\epsilon}}{\epsilon},$$

where

$$\zeta = \frac{N}{ab}\Delta_t(S \cup \{t\}),$$

4. $m > 1$.

$$\mathbf{n}^- \equiv (n_1^-, \ldots, n_t^-, \ldots, n_k^-),$$

- ○ is a vector of invited players, which will be better off staying outside of IEA, $n_t + n_t^- \leq N_t$ for $t = 1, \ldots, k$;

- stable IEA delivers surplus

  $$\Delta^+(\mathbf{n}) = \sum_{t=1}^{k} \Delta_t^+(\mathbf{n}) n_t,$$

  where

  $$\Delta_t^+(\mathbf{n}) = \pi_t^s(\mathbf{n}) - \pi_t^f(\{n_1, \ldots, n_t - 1, \ldots, n_k\});$$

- invited players cause non-positive surplus

  $$\Delta^-(\mathbf{n}, \mathbf{n}^-) = \sum_{t=1}^{k} \Delta_t^-(\mathbf{n}, \mathbf{n}^-) n_t^-,$$

  where

  $$\Delta_t^-(\mathbf{n}, \mathbf{n}^-) = \pi_t^s(\{n_1 + n_1^-, \ldots, n_t + n_t^-, \ldots, n_k + n_k^-\}) -$$
  $$-\pi_t^f(\{n_1 + n_1^-, \ldots, n_t + n_t^- - 1, \ldots, n_k + n_k^-\}).$$

  It is easy to notice that $\mathbf{n}^-$ can be not unique, but it is necessary and sufficient that it is such a vector that satisfies both conditions:

  $$\begin{aligned} &\Delta^+(\mathbf{n}) + \Delta^-(\mathbf{n}, \mathbf{n}^-) \geq 0, \\ &\Delta^+(\mathbf{n}) + \Delta^-(\mathbf{n}, \mathbf{n}^- + \{1\}) < 0. \end{aligned}$$

  These conditions guarantees that vector $\mathbf{n}^-$ presents maximum number of invited players, whose negative impact to coalition welfare could be covered by surplus of stable IEA.

Conclusion

In the presented paper we have considered coalition formation game with heterogeneous players, where a central issue is a problem of international cooperation towards pollution control. The main concern was to provide a better insight into asymmetric behavior and to characterize structure of a stable agreement when abatement target is succeeded over a fixed and finite period of time.

For this purpose we split the world of N nation into those, who prefer to join the IEA, and those, who decide to be free-riders. To define a voluntary membership of the IEA we have applied the concept of a self-enforcing agreement from oligopoly literature. We have determined optimal levels of abatement commitment and characterized size and structure of the stable coalition. We have suggested that once a self-enforcing IEA emerges, signatories decide to perform the required emission reduction uniformly.

As soon as the formed coalition initiates activities on emission decrease and the first results are observable, further agreement stability can be in danger. It is necessary to check time-consistency of the formed agreement. Otherwise, withdrawal of some nations from the agreement and accessing of others would cause structural change and sequential switch to another abatement goal. Presented analyses and examples reveal the following results. Self-enforcing IEA, which performs pollution decrease according to uniform scheme, that has been myopically picked up at the initial moment, is stable only over a certain part of the path. Once abatement has reached a threshold level, external stability fails and free-riders have incentives to access the agreement.

This occurs because the uniform pollution reduction scheme sets abatement targets, which differ from optimal ones both for IEA members and free-riders. Dynamics preassigned in a way, which does not somehow go along with optimal choice, can hardly be time-consistent. To protect the coalition against free-riding, we shall continue with constructing a dynamic abatement scheme that can depict agreement time-consistency.

Acknowledgements

M. Dementieva is grateful for funding from Dutch Russian Research Cooperation Program 2004 (NWO project No. 047-017-017) and Academy of Finland (No. 120666).

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Multistage Biddings with Risky Assets: the Case of Countable Set of Possible Liquidation Values [1]

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Abstract. This paper is concerned with multistage bidding models introduced by De Meyer and Moussa Saley (2002) to analyze the evolution of the price system at finance markets with asymmetric information. The zero-sum repeated games with incomplete information are considered modelling the biddings with countable sets of possible prices and admissible bids, unlike the above-mentioned paper, where two values of price are possible and arbitrary bids are allowed.

It is shown that, if the liquidation price of a share has a finite dispersion, then the sequence of values of n-step games is bounded and converges to the value of the game with infinite number of steps. We construct explicitly the optimal strategies for this game.

The optimal strategy of Player 1 (the insider) generates a symmetric random walk of posterior mathematical expectations of liquidation price with absorption. The expected duration of this random walk is equal to the initial dispersion of liquidation price. The guaranteed total gain of Player 1 (the value of the game) is equal to this expected duration multiplied with the fixed gain per step.

Keywords: Multistage biddings, asymmetric information, repeated games, optimal strategy.

Introduction

The Wiener process and its discrete analogues, random walks, are often used to model the evolution of price systems at finance markets. The random fluctuations of

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[1] This study was supported by the grant 07-06-00174a of Russian Foundation of Basic Research which is gratefully acknowledged.

prices are usually motivated by the effect of multiple exogenous factors subjected to accidental variations.

A different strategic motivation for these phenomena is proposed in the work of De Meyer and Saley (2002). The authors assert that the Brownian component in the evolution of prices on the stock market may originate from asymmetric information of stockbrokers on events determining market prices. “Insiders” are not interested in immediate revelation of their private information. This forces them to randomize their actions and results in the appearance of the oscillatory component in price evolution.

De Meyer and Saley demonstrate this idea by help of a simplified model of multi-stage biddings between two agents for risky assets (shares). A liquidation value of one share depends on a random “state of nature”. Before the biddings start a chance move determines the “state of nature” and, therefore, the liquidation value of one share once for all. Player 1 is informed on the “state of nature”, Player 2 is not. Both players know probabilities of chance move. Player 2 knows that Player 1 is an insider.

At each subsequent step $t = 1, 2, \ldots, n$ both players simultaneously propose their prices for one share. The maximal bid wins and one share is transacted at this price. If the bids are equal, no transaction occurs. Each player aims to maximize the value of his final portfolio (money plus liquidation value of obtained shares).

In this model the uninformed Player 2 should use the history of the informed Player 1 moves to update his beliefs about the state of nature. In fact, at each step Player 2 may use the Bayes rule to reestimate the posterior probabilities of chance move outcome, or, at least, the posterior mathematical expectations of liquidation value of a share. Player 1 could control these posterior probabilities.

Thus, Player 1 faces a problem of how best to use his private information without revealing it to Player 2. Using a myopic policy – bid the high price if the liquidation value is high, the low price if this value is low – is not optimal for Player 1, as it fully reveals the state of nature to Player 2. On the other hand, a strategy that does not depend on the state of nature reveals no information to Player 2, but does not allow Player 1 to take any advantage of his superior knowledge. Thus, Player 1 must maintain a delicate balance between taking advantage of his private information and concealing it from Player 2.

De Meyer and Saley consider the model where a liquidation price of a share takes only two values and players may make arbitrary bids. They reduce this model to a zero-sum repeated game with lack of information on one side, as introduced by Aumann, Maschler (1995), but with continual action sets. De Meyer and Saley show that these n-stage games have the values (i.e. the guaranteed gains of Player 1 are equal to the guaranteed losses of Player 2). They find these values and the optimal strategies of players. As n tends to infinity, the values infinitely grow up with rate $\sqrt{n}$. It is shown that Brownian Motion appears in the asymptotics of transaction prices generated by these strategies.

It is more natural to assume that players may assign only discrete bids proportional to a minimal currency unit. In our previous papers [Domansky, 2007], [Domansky and Kreps, 2007] we investigate the model with two possible values of liquidation price and discrete admissible bids. We show that, unlike the model [De Meyer and Saley, 2002], as n tends to ∞, the sequence of guaranteed gains of insider is bounded from above and converges. It makes reasonable to consider the biddings with infinite number of steps. We construct the optimal strategies for corresponding infinite games. We write out explicitly the random process formed by the prices of transactions at sequential steps. The transaction prices perform a symmetric random walk over the admissible bids between two possible values of liquidation price with absorbing extreme points. The absorption of transaction prices means revealing of the true value of share by Player 2.

Here we consider the model where any integer non-negative bids are admissible. The liquidation price of a share $C_{\mathbf{p}}$ may take any non-negative integer values $k = 0, 1, 2, \ldots$ according to a probability distribution $\mathbf{p} = (p_0, p_1, p_2, \ldots)$. This n-stage model is described by a zero-sum repeated game $G_n(\mathbf{p})$ with incomplete information of Player 2 and with countable state and action spaces. The games considered in [Domansky, 2007], [Domansky and Kreps, 2007] represent a particular case of these games corresponding to probability distributions with two-point supports.

We show that if the random variable $C_{\mathbf{p}}$ determining the liquidation price of a share has a finite mathematical expectation $\mathbf{E}[C_{\mathbf{p}}]$, then the values $V_n(\mathbf{p})$ of n-stage games $G_n(\mathbf{p})$ exist (i.e. the guaranteed gain of Player 1 is equal to the guaranteed loss of Player 2). If the dispersion $\mathbf{D}[C_{\mathbf{p}}]$ is infinite, then, as n tends to ∞, the sequence $V_n(\mathbf{p})$ diverges.

On the contrary, if the dispersion $\mathbf{D}[C_{\mathbf{p}}]$ is finite, then, as n tends to ∞, the sequence of values $V_n(\mathbf{p})$ of the games $G_n(\mathbf{p})$ is bounded from above and converges. The limit $H(\mathbf{p})$ is a continuous, concave, piecewise linear function with countable number of domains of linearity. The sets $\Theta(k)$, $k = 1, 2, \ldots$ of distributions $\mathbf{p}$ with integer mathematical expectation $\mathbf{E}[C(\mathbf{p})] = k$ form its domains of nondifferentiability. If $\mathbf{E}[C_{\bar{p}}]$ is an integer, then $H(\mathbf{p}) = \mathbf{D}[C_{\mathbf{p}}]/2$. If $\mathbf{E}[C_{\mathbf{p}} = k + \alpha$, where k is an integer, $\alpha \in [0, 1]$, then $H(\mathbf{p}) = (\mathbf{D}[C_{\mathbf{p}}] - \alpha(1 - \alpha))/2$.

As the sequence $V_n(\mathbf{p})$ is bounded from above, it is reasonable to consider the games $G_\infty(\mathbf{p})$ with infinite number of steps. We show that the value $V_\infty(\mathbf{p})$ is equal to $H(\mathbf{p})$. We construct explicitly the optimal strategies for these games.

For the case $\mathbf{p} \in \Theta(k)$ of integer mathematical expectation of liquidation value of a share, the insider optimal strategy is to generate a symmetric random walk of posterior mathematical expectations over domains $\Theta(l)$. The expected duration of this random walk is equal to the dispersion of the liquidation price of a share. The value of infinite game is equal to the expected duration of this random walk multiplied by the constant one-step gain 1/2 of informed Player 1.

Let $\mathbf{p} \in \Theta(k)$. If the random variable $C_{\mathbf{p}}$ takes the value k, then the "approximate" information of Player 2 turns to be the exact one and in fact the information advantage of Player 1 disappears. Hence, the gain of Player 1 is equal to zero, and he can stop the game without any loss for himself. Otherwise, the first optimal move of Player 1 makes use of actions $k-1$ and k with equal total probabilities and with posteriors $\mathbf{p}^{k-1} \in \Theta(k-1)$, $\mathbf{p}^k \in \Theta(k+1)$. For these posteriors the equalities $\bar{p}_k^{k-1} = \bar{p}_k^k = 0$ hold.

1. Repeated games with one-sided information modelling the multistage biddings

We consider the repeated games $G_n(\mathbf{p})$ with incomplete information on one side (Aumann and Maschler, 1995) modelling the biddings described in introduction. Two players with opposite interests have money and single-type shares. The liquidation price of a share may take any non-negative integer values $s \in S = Z_+ = \{0, 1, 2, \ldots\}$.

At stage 0 a chance move determines the liquidation value of a share for the whole period of biddings n according to the probability distribution $\mathbf{p} = (p_0, p_1, p_2, \ldots)$ over S known to both Players. Player 1 is informed about the result of chance move s, Player 2 is not. Player 2 knows that Player 1 is an insider. At each subsequent stage $t = 1, \ldots, n$ both Players simultaneously propose their prices for one share, $i_t \in I = Z_+$ for Player 1 and $j_t \in J = Z_+$ for Player 2. The pair (i_t, j_t) is announced to both Players before proceeding to the next stage. The maximal bid wins, and one share is transacted at this price. Therefore, if $i_t > j_t$, Player 1 gets one share from Player 2 and Player 2 receives the sum of money i_t from Player 1. If $i_t < j_t$, Player 2 gets one share from Player 1 and Player 1 receives the sum j_t from Player 2. If $i_t = j_t$, then no transaction occurs. Each player aims to maximize the value of his final portfolio (money plus liquidation value of obtained shares).

This n-stage model is described by a zero-sum repeated game $G_n(\mathbf{p})$ with incomplete information of Player 2 and with countable state space $S = Z_+$ and with countable action spaces $I = Z_+$ and $J = Z_+$. One-step gains of Player 1 are given with the matrices $A^s = [a^s(i,j)]_{i \in I, j \in J}$, $s \in S$,

$$a^s(i,j) = \begin{cases} j - s, & \text{for } i < j; \\ 0, & \text{for } i = j; \\ -i + s, & \text{for } i > j. \end{cases}$$

At the end of the game Player 2 pays to Player 1 the sum

$$\sum_{t=1}^{n} a^s(i_t, j_t).$$

This description is common knowledge to both Players.

At step t it is enough for both Players to take into account the sequence $(i_1, \ldots, i_{t-1})$ of Player 1's previous actions only. Thus, a strategy σ for Player 1 informed on the state is a sequence of moves

$$\sigma = (\sigma_1, \ldots, \sigma_t, \ldots),$$

where the move $\sigma_t = (\sigma_t(s))_{s\in S}$ and $\sigma_t(s) : I^{t-1} \to \Delta(I)$ is the probability distribution used by Player 1 to select his action at stage t, given the state k and previous observations. Here $\Delta(\cdot)$ is the set of probability distributions over $(\cdot)$.

A strategy τ for uninformed Player 2 is a sequence of moves

$$\tau = (\tau_1, \ldots, \tau_t, \ldots),$$

where $\tau_t : I^{t-1} \to \Delta(J)$.

Observe that here we define infinite strategies fitting for games of arbitrary duration. A pair of strategies (σ, τ) induces a probability distribution $\Pi_{(\sigma,\tau)}$ over $(I\times J)^\infty$. The payoff function of the game $G_n(\mathbf{p})$ is

$$K_n(\mathbf{p}, \sigma, \tau) = \sum_{s\in S} p_s h_n^s(\sigma, \tau),$$

where

$$h_n^s(\sigma, \tau) = \mathbf{E}_{(\sigma,\tau)}[\sum_{t=1}^{n} a^s(i_t, j_t)]$$

is the s-component of the n-step vector payoff $h_n(\sigma, \tau)$ for the pair of strategies (σ, τ). Here the expectation is taken with respect to the probability distribution $\Pi_{(\sigma,\tau)}$.

For the initial probability $\mathbf{p}$ the strategy σ ensures the n-step payoff

$$w_n(p, \sigma) = \inf_{\tau} K_n(p, \sigma, \tau).$$

The strategy τ ensures the n-step vector payoff $h_n(\tau)$ with the components

$$h_n^s(\tau) = \sup_{\sigma(s)} h_n^s(\sigma(s), \tau).$$

Now we describe the recursive structure of $G_{n+1}(\mathbf{p})$. A strategy σ may be regarded as a pair $(\sigma_1, (\sigma(i))_{i\in I})$, where $\sigma_1(i|s)$ is a probability on I depending on s, and $\sigma(i)$ is a strategy depending on the first action $i_1 = i$.

Analogously, a strategy τ may be regarded as a pair $(\tau_1, (\tau(i))_{i\in I})$, where τ_1 is a probability on J.

A pair $(\mathbf{p}, \sigma_1)$ induces the probability distribution π over $S\times I$, $\pi(s, i) = p(s)\sigma_1(i|s)$. Let

$$\mathbf{q} \in \Delta(I), \quad q_i = \sum_{S} p_s \sigma_1(i|s),$$

be the marginal distribution of π on I (total probabilities of actions), and let

$$\mathbf{p}(i) \in \Delta(S), p_s(i) = p_s\sigma_1(i|s)/q_i$$

be the conditional probability on S given $i_1 = i$ (a posterior probability).

Conversely, any set of total probabilities of actions $\mathbf{q} \in \Delta(I)$ and posterior probabilities $(\mathbf{p}(i) \in \Delta(S))_{i \in I}$, satisfying the equality

$$\sum_{i \in I} q_i \mathbf{p}(i) = \mathbf{p},$$

define a certain random move of Player 1 for the current probability $\mathbf{p}$. The posterior probabilities contain the whole of essential for Player 1 information about the previous history of the game. Thus, to define a strategy of Player 1 it is sufficient to define the random move of Player 1 for any current posterior probability.

The following recursive representation for the payoff function corresponds to the recursive representation of strategies:

$$K_{n+1}(\mathbf{p}, \sigma, \tau) = K_1(\mathbf{p}, \sigma_1, \tau_1) + \sum_{i \in I} q_i K_n(\mathbf{p}(i), \sigma(i), \tau(i)).$$

Let, for all $i \in I$, the strategy $\sigma(i)$ ensure the payoff $w_n(\mathbf{p}(i), \sigma(i))$ in the game $G_n(\mathbf{p}(i))$. Then the strategy $\sigma = (\sigma_1, (\sigma(i))_{i \in I})$ ensures the payoff

$$w_{n+1}(\mathbf{p}, \sigma) = \min_{j \in J} \sum_{i \in I} [\sum_{s \in S} p_s \sigma_1(i|s) a(s, i, j) + q_i w_n(\mathbf{p}(i), \sigma(i))]. \tag{25}$$

Let, for all $i \in I$, the strategy $\tau(i)$ ensure the vector payoff $\mathbf{h}_n(\tau(i))$. Then the strategy $\tau = (\tau_1, (\tau^n(i))_{i \in I})$ ensures the vector payoff $\mathbf{h}_{n+1}(\tau)$ with the components

$$h^s_{n+1}(\tau) = \max_{i \in I} \sum_{j \in J} \tau_1(j) \, (a(s, i, j) + h^s_n(\tau(i))) \quad \forall s \in S. \tag{26}$$

The game $G_n(\mathbf{p})$ has a value $V_n(\mathbf{p})$ if

$$\inf_{\tau} \sup_{\sigma} K_n(\mathbf{p}, \sigma, \tau) = \sup_{\sigma} \inf_{\tau} K_n(\mathbf{p}, \sigma, \tau) = V_n(\mathbf{p}).$$

Players have optimal strategies σ^* and τ^* if

$$V_n(\mathbf{p}) = \inf_{\tau} K_n(\mathbf{p}, \sigma^*, \tau) = \sup_{\sigma} K_n(\mathbf{p}, \sigma, \tau^*),$$

or, in above introduced notation,

$$V_n(\mathbf{p}) = w_n(\mathbf{p}, \sigma^*) = \sum_{s \in S} p_s h^s_n(\tau^*).$$

For probability distributions $\mathbf{p}$ with finite supports, the games $G_n(\mathbf{p})$, as games with finite state and action spaces, have values $V_n(\mathbf{p})$. The functions V_n are continuous and concave in $\mathbf{p}$. Both players have optimal strategies σ^* and τ^*.

Consider the set M^1 of probability distributions $\mathbf{p}$ with finite first moment $m^1[\mathbf{p}] = = \sum_{s=0}^{\infty} p_s \times s < \infty$. For $\mathbf{p} \in M^1$ the random variable $C_{\mathbf{p}}$, determining the liquidation price of a share, has a finite mathematical expectation $\mathbf{E}[C_{\mathbf{p}}] = m^1[\mathbf{p}]$. The set M^1 is a convex subset of Banach space $L^1(\{s\})$ of sequences $\mathbf{l} = (l_s)$ with a norm

$$||\mathbf{l}||_s^1 = \sum_{s=0}^{\infty} |l_s| \times s.$$

Let $\mathbf{p}_1, \mathbf{p}_2 \in M^1$. Then, for "reasonable" strategies σ and τ,

$$|K_n(p_1, \sigma, \tau) - K_n(p_2, \sigma, \tau)| < n||\mathbf{p}_1 - \mathbf{p}_2||_s^1.$$

Therefore, the payoff of game $G_n(\mathbf{p})$ with $\mathbf{p} \in M^1$ can be approximated by the payoffs of games $G_n(\mathbf{p}_k)$ with probability distributions $\mathbf{p}_k$ having finite support. Next theorem follows immediately from this fact.

Theorem 1. *If $\mathbf{p} \in M^1$ then games $G_n(\mathbf{p})$ have values $V_n(\mathbf{p})$. The values $V_n(\mathbf{p})$ are positive and do not decrease as the number of steps n increases.*

Remark 1. If the random variable $C_{\mathbf{p}}$ does not belong to L^2, then, as n tends to ∞, the sequence $V_n(\mathbf{p})$ diverges.

2. Upper bound for values $V_n(\mathbf{p})$

Here we consider the set M^2 of probability distributions $\mathbf{p}$ with finite second moment

$$m^2[\mathbf{p}] = \sum_{s=0}^{\infty} p_s timess^2 < \infty.$$

For $\mathbf{p} \in M^2$, the random variable $C_{\mathbf{p}}$, determining the liquidation price of a share, belongs to L^2 and has a finite dispersion $\mathbf{D}[C_{\mathbf{p}}] = m^2[\mathbf{p}] - (m^1[\mathbf{p}])^2$.

The set M^2 is a closed convex subset of Banach space $L^1(\{s^2\})$ of mappings $\mathbf{l} : Z_+ \to R$ with a norm

$$||\mathbf{l}||_s^1 = \sum_{s=0}^{\infty} |l_s| \times s^2.$$

The main result of this section is that for $\mathbf{p} \in M^2$, as $n \to \infty$, the sequence $V_n(\mathbf{p})$ of values remains bounded.

To prove this we define recursively the set of infinite "reasonable" strategies $\tau^m, m = 0, 1, \ldots$ of Player 2, suitable for the games $G_n(\mathbf{p})$ with arbitrary n.

Definition 1. *The first move τ_1^m is the action $m \in J$. The moves τ_t^m for $t > 1$ depend on the last observed pair of actions (i_{t-1}, j_{t-1}) only:*

$$\tau_t^m(i_{t-1}, j_{t-1}) = \begin{cases} j_{t-1} - 1 \text{ for } i_{t-1} < j_{t-1}; \\ j_{t-1}, \text{ for } i_{t-1} = j_{t-1}; \\ j_{t-1} + 1, \text{ for } i_{t-1} > j_{t-1}. \end{cases}$$

Remark 2. The definition of strategies τ^m includes the previous actions of both players. In fact, these strategies can be implemented on the basis of Player 1's previous actions only.

Proposition 1. *The strategies* τ^m *ensure the vector payoffs* $\mathbf{h_n}(\tau^{\mathbf{m}}) \in \mathbf{R}_+^{\mathbf{S}}$ *with components given by*

$$h_n^s(\tau^m) = \sum_{l=0}^{n-1}(m-s-l)^+, \tag{27}$$

for $s \leq m$*,*

$$h_n^s(\tau^m) = \sum_{l=0}^{n-1}(s-m-1-l)^+, \tag{28}$$

for $s > m$*, where* $(a)^+ := \max\{0, a\}$*.*

Proof.

The proof is by induction on the number of steps n.

$\mathbf{n = 1}$. For $s < m$ Player 1's best reply is any action $k < m$, and

$$h_1^s(\tau^m) = \max_i a_{i,m}^s = a_{k,m}^s = m - s.$$

For $s = m$ Player 1's best reply is any action $k \leq m$ and

$$h_1^m(\tau^m) = \max_i a_{i,m}^m = a_{k,m}^m = 0.$$

For $s = m+1$ Player 1's best replies are actions m and $m+1$ and

$$h_1^{m+1}(\tau^m) = \max_i a_{i,m}^{m+1} = a_{m,m}^{m+1} = a_{m+1,m}^{m+1} = 0.$$

For $s > m+1$ Player 1's best reply is action $m+1$ and

$$h_1^s(\tau^m) = \max_i a_{i,m}^s = a_{m+1,m}^s = (s-m-1).$$

Therefore,

$$\mathbf{h}_1(\tau^k) = (k, k-1, \ldots, 1, 0, 0, 1, \ldots).$$

This proves Proposition 1 for $n = 1$.

$\mathbf{n \to n+1}$. Assume that the vector payoffs $h_n(\tau^k)$ are given with (3) and (4). We have according to (2)

$$h_{n+1}^s(\tau^m) = \max_i \begin{cases} a_{i,m}^s + h_n^s(\tau^{m-1}), & \text{for } i < m; \\ a_{i,m}^s + h_n^s(\tau^m), & \text{for } i = m; \\ a_{i,m}^s + h_n^s(\tau^{m+1}), & \text{for } i > m. \end{cases}$$

For $s < m$, the first move of Player 1's best reply is any action $i < m$. It results in

$$h^s_{n+1}(\tau^m) = a^s_{i,m} + h^s_n(\tau^{m-1}) = (m-s) + \sum_{l=0}^{n-1}(m-s-1-l)^+ = \sum_{l=0}^{n}(m-s-l)^+.$$

For $s = m$, the first move of Player 1's best reply is any action $i < m$, and $i = m$. It results in

$$h^m_{n+1}(\tau^m) = a^m_{i,m} + h^m_n(\tau^{m-1}) = a^m_{m,m} + h^m_n(\tau^m) = 0.$$

For $s = m+1$ the first moves of Player 1's best replies are actions m, and $m+1$. It results in

$$h^{m+1}_{n+1}(\tau^m) = a^{m+1}_{m,m} + h^{m+1}_n(\tau^m) = a^{m+1}_{m+1,m} + h^{m+1}_n(\tau^{m+1}) = 0.$$

For $s > m+1$ the first move of Player 1's best reply is action $m+1$. It results in

$$h^s_{n+1}(\tau^m) = a^s_{m+1,m} + h^s_n(\tau^{m+1}) = (s-m-1) + \sum_{l=0}^{n-1}(s-m-2-l)^+ = \sum_{l=0}^{n}(s-m-1-l)^+.$$

This proves Proposition 1 for $n+1$.

Theorem 2. *For* $\mathbf{p} \in M^2$ *the values* $V_n(\mathbf{p})$ *are bounded from above by a continuous, concave, and piecewise linear function* $H(\mathbf{p})$ *over* P. *Its domains of linearity are*

$$L(k) = \{\mathbf{p} : \mathbf{E}[\mathbf{p}] \in [k, k+1]\}, \quad k = 0, 1, \ldots .$$

Its domains of nondifferentiability are $\Theta(k) = \{\mathbf{p} : \mathbf{E}[\mathbf{p}] = k\}$. *The equality holds*

$$H(\mathbf{p}) = 1/2\mathbf{D}[\mathbf{p}] - 1/2\delta(\mathbf{p})(1 - (\delta\mathbf{p})),$$

where $\delta(\mathbf{p}) = \mathbf{E}[\mathbf{p}] - ent[\mathbf{E}[\mathbf{p}]]$ *and* $ent[x]$, $x \in R^1$ *is the integer part of* x.
Proof.

It is easy to see that

$$\lim_{n\to\infty} h^s_n(\tau^m) = h_\infty(\tau^m) = (s-m)(s-m-1)/2.$$

Thus, there is the following not depending on n upper bound for $V_n(\mathbf{p})$:

$$V_n(\mathbf{p}) \le \min_m \sum_{s=0}^{\infty} p_s(s-m)(s-m-l)/2, \quad m = 0, 1, \ldots . \tag{29}$$

Observe that for $\mathbf{E}[\mathbf{p}] \in [k, k+1]$ the minimum in formula (5) is attained on the k-th vector payoff. Consequently, for $\mathbf{E}[\mathbf{p}] = k + \alpha$,

$$V_n(\mathbf{p}) \le \sum_{s=0}^{\infty} p_s(s-k)(s-k-l)/2 = [(k^2+k) - (2k+1)\sum_{s=0}^{\infty} p_s s + \sum_{s=0}^{\infty} p_s s^2]/2 =$$

$$= [\sum_{s=0}^{\infty} p_s s^2 - (k+\alpha)^2 - \alpha + \alpha^2]/2 = [\mathbf{D}[\mathbf{p}] - \alpha(1-\alpha)]/2 = H(\mathbf{p}).$$

In particular, for $\mathbf{p} \in \Theta(k)$ ($\mathbf{E}[\mathbf{p}] = k$),

$$V_n(\mathbf{p}) \le \sum_{s=0}^{\infty} p_s(s-k)(s-k-l)/2 = \sum_{s=0}^{\infty} p_s(s-k)(s-k+l)/2 = \mathbf{D}[\mathbf{p}]/2. \qquad (30)$$

Corollary 1. *The strategies $\tau^m, m = 0, 1, \dots$ guarantee the same upper bound $H(\mathbf{p})$ for the upper value of the infinite game $G_\infty(\mathbf{p})$.*

Further we give another representation of the function $H(\mathbf{p})$ over $\Theta(r)$. The set $\Theta(r)$ is a closed convex subset of Banach space $L^1(\{s^2\})$. The extreme points of this set are distributions $\mathbf{p}^r(k,l) \in \Theta(r)$ with two-point supports $\{r-l, r+k\}$

$$p^r_{r-l}(k,l) = \frac{k}{k+l}, \quad p^r_{r+k}(k,l) = \frac{l}{k+l}, \qquad (31)$$

$k = 0, 1, 2, \dots, l = 0, 1, \dots, r,\ k + l > 0$. Note that $\mathbf{p}^r(0,l) = \mathbf{p}^r(k,0) = \mathbf{e}^r$, where $\mathbf{e}^r$ is the degenerate distribution with one-point support $e^r_r = 1$.

Any $\mathbf{p} \in \Theta(r)$ has the following unique representation as a convex combination of extreme points (7) of this set:

$$\mathbf{p} = p_r \times \mathbf{e}^r + \sum_{k=1}^{\infty} \sum_{l=1}^{r} \alpha_{kl}(\mathbf{p}) \times \mathbf{p}^r(k,l), \qquad (32)$$

with the coefficients

$$\alpha_{kl}(\mathbf{p}) = \frac{k+l}{\sum_{t=1}^{r} t p_{r-t}} p_{r-l} p_{r+k}. \qquad (33)$$

Consequently, the continuous linear function H over $\Theta(r)$, equal to zero at $\mathbf{e}^r$, has the following unique representation as convex combination of values at extreme points $H(\mathbf{p}^r(k,l)) = kl/2$ corresponding to decomposition (8):

$$H(\mathbf{p}) = \sum_{k=1}^{\infty} \sum_{l=1}^{r} \alpha_{kl}(\mathbf{p}) \times k \times l/2 \qquad (34)$$

with the coefficients $\alpha_{kl}(\mathbf{p})$ given by (9).

4. Asymptotics of values $V_n(\mathbf{p})$

In this section we show that, for $\mathbf{p} \in M^2$, as n tends to ∞, the sequence of values $V_n(\mathbf{p})$ of the games $G_n(\mathbf{p})$ converges to $H(\mathbf{p})$.

To prove this, we construct lower bounds for the values $V_n(\mathbf{p})$ of the games $G_n(\mathbf{p})$. These lower bounds have the same structure as the upper bounds of Theorem 1. For any $\mathbf{p} \in M^2$ we define the strategy of Player 1 ensuring these lower bounds.

Definition 2. *Here we define a sequence of continuous, concave, and piecewise linear functions* B_n *over* M^2*. Its domains of linearity are* $L(r), r = 0, 1, \ldots,$ *and its domains of nondifferentiability are* $\Theta(r)$*.*

For the extreme points $\mathbf{p}^r(k,l)$ *of the set* $\Theta(r)$ *the values* $B_n(\mathbf{p}^r(k,l))$ *are given with the recurrent equalities*

$$B_n(\mathbf{p}^r(k,l)) = [1 + B_{n-1}(\mathbf{p}^{r+1}(k-1,l+1)) + B_{n-1}(\mathbf{p}^{r-1}(k+1,l-1))]/2, \quad (35)$$

with the boundary conditions $B_{n-1}(\mathbf{p}^{r+k}(0,l+k)) = B_{n-1}(\mathbf{p}^{r-l}(k+l,0)) = 0,$ *and the initial condition* $B_0(\mathbf{p}^r(k,l)) = 0$*.*

For the interior points $\mathbf{p} \in \Theta(r)$ *the values* $B_n(\mathbf{p})$ *are convex combinations of its values at extreme points with the coefficients* $\alpha_{kl}(\mathbf{p})$ *given by (9).*

For the interior points $\mathbf{p} \in L(r)$*, the values* $B_n(\mathbf{p})$ *are convex combinations of its values at boundary points* $\mathbf{p}^r \in \Theta(r)$ *and* $\mathbf{p}^{r+1} \in \Theta(r+1)$ *such that* $\mathbf{p}{=}\alpha\mathbf{p}^r{+}(1{-}\alpha)\mathbf{p}^{r+1}$*.*

Definition 3. *For any* $\mathbf{p} \in M^2$*, we define the strategy* $\sigma(\mathbf{p})$ *of Player 1.*

Let $\mathbf{p} \in \Theta(r)$*. If the random variable* $C_{\mathbf{p}}$ *takes the value* r *then the strategy* $\sigma(\mathbf{p})$ *stops the game. Otherwise, the first move of the strategy* $\sigma(\mathbf{p})$ *makes use of two actions* $r-1$ *and* r*. These actions occur with total probabilities* $q_{r-1} = q_r = 1/2$*.*

For action $r-1$ *the posterior probability distribution is*

$$\mathbf{p}(r-1) = \mathbf{p}^- \in \Theta(r-1),$$

where

$$p_s^- = \begin{cases} p_s \frac{\sum_{j=0}^{r-1}(r-1-j)p_j}{\sum_{j=0}^{r-1}(r-j)p_j}, & \text{for } s > r; \\ 0, & \text{for } \ s = r; \\ p_s \frac{\sum_{j=r+1}^{\infty}(j-r+1)p_j}{\sum_{j=0}^{r-1}(r-j)p_j}, & \text{for } s < r. \end{cases} \quad (36)$$

For action r *the posterior probability distribution is*

$$\mathbf{p}(r) = \mathbf{p}^+ \in \Theta(r+1),$$

where

$$p_s^+ = \begin{cases} p_s \frac{\sum_{j=0}^{r-1}(r+1-j)p_j}{\sum_{j=0}^{r-1}(r-j)p_j}, & \text{for } \ s > r; \\ 0, & \text{for } \ s = r; \\ p_s \frac{\sum_{j=r+1}^{\infty}(j-r-1)p_j}{\sum_{j=0}^{r-1}(r-j)p_j} & \text{for } \ s < r. \end{cases} \quad (37)$$

For interior points $\mathbf{p} \in L(r)$ *with* $\mathbf{E}[\mathbf{p}] = r + \alpha$ *first moves of strategies* $\sigma(\mathbf{p})$ *are convex combinations of the first moves at boundary points* $\mathbf{p}^r \in \Theta(r)$ *and* $\mathbf{p}^{r+1} \in \Theta(r+1)$ *such that* $\mathbf{p} = \alpha\mathbf{p}^{r+1} + (1-\alpha)\mathbf{p}^r$*.*

Remark 3. It follows from Theorem 2 that for $\mathbf{p} \in \Theta(k)$, if the random variable $C_{\mathbf{p}}$ takes the value k, then the gain of Player 1 is equal to zero, and he can stop the game without any loss for himself.

Proposition 2. *For the game $G_n(\mathbf{p})$ the strategy $\sigma(\mathbf{p})$ ensures the payoff*

$$w_n(\mathbf{p}, \sigma(\mathbf{p})) = B_n(\mathbf{p}).$$

Proof.

It is sufficient to prove Proposition for the games $G_n(\mathbf{p}^r(k,l))$ corresponding to extreme points $\mathbf{p}^r(k,l)$ of the sets $\Theta(r)$, $r = 1, 2, \ldots$. The proof is by induction on n. *$n = 1$.* The best answer of Player 2 to the first move of the strategy $\sigma(\mathbf{p}^r(k,l))$ is any action l with $l \leq r$. The resulting immediate gain of Player 1 is equal to 1/2. Thus, the strategy $\sigma(\mathbf{p}^r(k,l))$ ensures the payoff $B_1(\mathbf{p}^r(k,l)) = 1/2$ in the one-step game $G_1(\mathbf{p}^r(k,l))$. *$n \to n+1$.*

Assume that the strategies $\sigma(\mathbf{p}^r(k,l))$ ensure the payoffs $B_n(\mathbf{p}^r(k,l))$ in the games $G_n(\mathbf{p}^r(k,l))$.

The first move of the strategy $\sigma(\mathbf{p}^r(k,l))$ has immediate gain equal to 1/2. Its posterior probability distributions are $\mathbf{p}^{r-1}(k+1, l-1)$ and $\mathbf{p}^{r+1}(k-1, l+1)$, and both of them occur with probabilities 1/2.

According to the induction assumption and formulas (1), (6), the resulting total gain of Player 1 is equal to

$$[1 + B_n(\mathbf{p}^{r-1}(k+1, l-1)) + B_n(\mathbf{p}^{r+1}(k-1, l+1))]/2 = B_{n+1}(\mathbf{p}).$$

Thus, the strategy $\sigma(\mathbf{p})$ ensures the payoff $B_{n+1}(\mathbf{p})$ in the games $G_{n+1}(\mathbf{p})$ with $\mathbf{p} = \mathbf{p}^r(k,l)$. It is easy to extend this result to all $\mathbf{p} \in M^2$.

Theorem 3. *For $\mathbf{p} \in M^2$, the following equalities hold:*

$$\lim_{n\to\infty} V_n(\mathbf{p}) = H(\mathbf{p}).$$

Proof.

According to Theorem 2 and Proposition 2 the following inequalities hold:

$$B_n(\mathbf{p}) \leq V_n(\mathbf{p}) \leq H(\mathbf{p}), \quad \forall \mathbf{p} \in M^2.$$

The functions B_n and H are continuous, concave, and piecewise linear with the same domains of linearity $L(r)$, $r = 0, 1, \ldots$. Such functions are completely determined with its values at the domains of nondifferentiability $\Theta(r), r = 1, 2, \ldots$.

Because of continuity and concavity of the functions B_n and H, to prove that the sequence B_n converges to H as n tends to ∞, it is enough to show this for $\mathbf{p} \in \Theta(r)$, $r = 1, 2, \ldots$.

The increasing sequence of continuous linear functions B_n over $\Theta(r)$ is bounded from above with the continuous linear function H. Consequently, it has a continuous linear limit function B_∞. To prove Theorem 3 for $\mathbf{p} \in \Theta(r)$ it is enough to show that

$$\lim_{n\to\infty} B_n(\mathbf{p}^r(k,l)) = B_\infty(\mathbf{p}^r(k,l)) = H(\mathbf{p}^r(k,l)) = k \times l/2, \quad \forall k, l.$$

It follows from (11) that the limits $B_\infty(\mathbf{p}^r(k,l))$ should satisfy the equality

$$B_\infty(\mathbf{p}^r(k,l)) = [1 + B_\infty(\mathbf{p}^{r+1}(k-1,l+1)) + B_\infty(\mathbf{p}^{r-1}(k+1,l-1))]/2 \qquad (38)$$

with the boundary conditions $B_\infty(\mathbf{p}^{r+k}(0,l+k)) = B_\infty(\mathbf{p}^{r-l}(k+l,0)) = 0$.

Solving the system of $k+l-1$ linear equations (14) connecting $k+l-1$ values $B_\infty(\mathbf{p}^{r+m}(k-m,l+m))$, $m = -l+1, -l+2, \ldots, k-1$, for distributions with the same two-point support $\{r-l, r+k\}$, we obtain that

$$B_\infty(\mathbf{p}^r(k,l)) = k \times l/2 = H(\mathbf{p}^r(k,l)).$$

According to (10) this proves Theorem 3 for $\mathbf{p} \in \Theta(r), r = 0, 1, \ldots$. Because of the continuity and concavity of the functions V_n it is true for all $\mathbf{p} \in M^2$.

Corollary 2. *It follows from the proof that the strategy $\sigma(\mathbf{p})$ ensures the payoff $H(\mathbf{p})$ in the infinite game $G_\infty(\mathbf{p})$. The strategy $\sigma(\mathbf{p})$ is not optimal in any finite game $G_n(\mathbf{p})$ with $n < \infty$.*

5. Solutions for the games $G_\infty(\mathbf{p})$ and random walks

For $\mathbf{p} \in M^2$, as the values $V_n(\mathbf{p})$ are bounded from above, the consideration of games $G_\infty(\mathbf{p})$ with infinite number of steps becomes reasonable.

We restrict the set of Player 1's admissible strategies in these games to the set Σ^+ of strategies employing only the moves ensuring him a non-negative one-step gain against any action of Player 2. Consequently, the payoff functions $K_\infty(\mathbf{p}, \sigma, \tau)$ of the games $G_\infty(\mathbf{p})$ become definite (may be infinite) in all cases.

We show that the infinite game $G_\infty(\mathbf{p})$ has a value, and this value is equal to $H(\mathbf{p})$.

The existence of values for these games does not follow from common considerations and has to be proved. We prove it by providing the optimal strategies explicitly.

Theorem 4. *For $\mathbf{p} \in M^2$ the game $G_\infty(\mathbf{p})$ has a value $V_\infty(\mathbf{p}) = H(\mathbf{p})$. Both Players have optimal strategies. The optimal strategy of Player 1 is the strategy $\sigma(\mathbf{p})$, given by Definition 3.*

For $\mathbf{p} \in L(r)$, $r = 0, 1 \ldots$, the optimal strategy of Player 2 is the strategy τ^r, given by Definition 1. For $\mathbf{p} \in \Theta(r), r = 1, 2, \ldots$ any convex combination of the strategies τ^{r-1} and τ^r is optimal.

Proof.

According to Corollary 2, the strategy $\sigma(\mathbf{p}) \in \Sigma^+$ ensures the payoff $H(\mathbf{p})$ in the game $G_\infty(\mathbf{p})$. Thus, for any $\mathbf{p} \in M^2$

$$\sup_{\Sigma^+} \inf_{T} K_\infty(\mathbf{p}, \sigma, \tau) \geq H(\mathbf{p}), \qquad (39)$$

and the function H is the lower bound for the lower value of the game G_∞.

On the other hand, according to Corollary 1, the strategies τ^r, $r = 0, 1, \ldots$, ensure the payoff $H(\mathbf{p})$ in the infinite game $G_\infty(\mathbf{p})$. Thus, for any $\mathbf{p} \in M^2$

$$\inf_T \sup_{\Sigma^+} K_\infty(\mathbf{p}, \sigma, \tau) \leq H(\mathbf{p}), \tag{40}$$

and the function H is the upper bound for the upper value of the game G_∞.

As the lower value is always less or equal to the upper value, it follows from (15) and (16) that

$$\sup_{\Sigma^+} \inf_T K_\infty(\mathbf{p}, \sigma, \tau) = \inf_T \sup_{\Sigma^+} K_\infty(\mathbf{p}, \sigma, \tau) = H(\mathbf{p}) = V_\infty(\mathbf{p}).$$

The strategies $\sigma(\mathbf{p}) \in \Sigma^+$ and τ^r, $r = 0, 1, \ldots$ ensure the value $H(\mathbf{p}) = V_\infty(\mathbf{p})$ in the infinite game $G_\infty(\mathbf{p})$.

6. The probabilistic interpretation of the results

For the initial probability distribution $\mathbf{p} \in Theta(r)$, $r = 1, 2, \ldots$ the random sequence of posterior probability distributions, generated with the optimal strategy $\sigma(\mathbf{p})$ of Player 1, is the symmetric random walk $(\mathbf{p}_t)_{t=1}^\infty$ over domains $\Theta(l)$. Probabilities of jumps to each of adjacent domains $\Theta(l-1)$ and $\Theta(l+1)$ are equal to $(1-p_l)/2$, and probability of absorption is equal to p_l. This is the Markov chain with the state space $\cup_{l=0}^\infty Theta(l)$, and with the transition probabilities, for $\mathbf{p} \in Theta(l)$,

$$\Pr(\mathbf{p}, \mathbf{e}^l) = p_l; \quad \Pr(\mathbf{p}, \mathbf{p}^-) = \Pr(\mathbf{p}, \mathbf{p}^+) = (1 - p_l)/2,$$

where $\mathbf{p}^-$ and $\mathbf{p}^+$ are given with (12) and (13).

Next arising posterior probability distributions $\mathbf{p}^-$ and $\mathbf{p}^+$ have $p_l = 0$ and, thus, for any subsequent visit to the domain $\Theta(l)$, the probability of absorption becomes equal to zero.

For the random walk $(\mathbf{p}_t)_{t=1}^\infty$ with the initial probability distribution $\mathbf{p} \in Theta(r)$, let $\theta\mathbf{p}$ be the random Markov time of absorption, i.e.

$$\theta(\mathbf{p}) = \min\{t : \mathbf{p}_t = \mathbf{e}^l\} - 1.$$

The Markov time $\theta\mathbf{p}$ of absorption of posterior probabilities represents the time of revelation the "true" value of share by Player 2 and, generally speaking, the time of bidding termination.

Proposition 3. *For the random walk $(\mathbf{p}_t)_{t=1}^\infty$ with the initial probability distribution $\mathbf{p} \in Theta(r)$, the expected duration $\mathbf{E}[\theta(\mathbf{p})]$ of this random walk is equal to the dispersion $\mathbf{D}[\mathbf{p}]$ of the liquidation price of a share.*

Proof.

For the random walk $(\mathbf{p}_t)_{t=1}^\infty$ with the initial probability distribution $\mathbf{p} \in Theta(r)$, the transition probabilities are linear functions over $Theta(r)$. Consequently, the expected duration $\mathbf{E}[\theta(\mathbf{p})]$ of this random walk is a linear function over $Theta(r)$ as well.

The continuous linear function $\mathbf{E}[\theta(\mathbf{p})]$ over $\Theta(r)$, equal to zero at $\mathbf{e}^r$, has the following unique representation as convex combination of values at extreme points $\mathbf{E}[\theta(\mathbf{p}^r(k,l))]$:

$$H(\mathbf{p}) = \sum_{k=1}^{\infty} \sum_{l=1}^{r} \alpha_{kl}(\mathbf{p}) \times \mathbf{E}[\theta(\mathbf{p}^r(k,l))],$$

with the coefficients $\alpha_{kl}(\mathbf{p})$ given by (9).

It is well known that

$$\mathbf{E}[\theta(\mathbf{p}^r(k,l))] = k(m-k) = \mathbf{D}[\mathbf{p}^r(k,l)].$$

As the dispersion $\mathbf{D}[\mathbf{p}]$ is a linear function over $Theta(r)$, we obtain the assertion of Proposition 4.

Remark 4. The result of Theorem 4 turns to be rather intuitive. The value of infinite game is equal to the expected duration of random walk of posterior probability distributions, multiplied by the constant one-step gain 1/2 of informed Player 1.

7. Conclusion

The obtained results on the biddings with countable sets of possible prices and admissible bids demonstrate that the Brownian component in the evolution of prices on the stock market may originate from asymmetric information of stockbrokers on events determining market prices.

Acknowlegments

The authors express their gratitude to B. De Meyer for useful discussion on the subject.

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On Quasi-cores, the Shapley Value and the Semivalues

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Abstract. The aim of the present paper is that of introducing a new concept of coalitional rationality for values of cooperative TU games, called $w-$coalitional rationality, such that this becomes the usual coalitional rationality in the case of an efficient value. As a motivation for our new concept, the class of Semivalues, which are in general non efficient values, was considered, and we proved necessary and sufficient conditions for the $w-$coalitional rationality of Semivalues. It is well known that the only efficient Semivalue is the Shapley value. The basic idea to be followed here is the fact that the Semivalues are not connected to the core, because the efficiency is missing, but may be connected to some quasi-core. (in particular, to the Shapley–Shubik weak $\varepsilon-$core, from which the w is borrowed). Shapley and Shubik (1966) have introduced the quasi-cores in connection with the market games and have discussed the non-emptiness of two types of quasi-cores (see the paper by Kannai in Handbook of Game Theory, vol.I, 1992).

Keywords: Coalitional rationality, Shapley value, Semivalues, Average per capita formulas, Quasi-cores.

Introduction

In the first section we introduced a more general concept of quasi-core and the symmetric quasi-core, which will be our basic tool; as a byproduct, we provided also necessary and sufficient conditions for the non-emptiness of the symmetric quasi-core.

The Semivalues of a game are Shapley values of an easily obtained auxiliary game, as shown in a previous paper [Dragan, 2005]; this was proved by using the fact that any Least Square value is a Shapley value [Dragan, 2006], and any Semivalue is a Least Square value [Ruiz et al.,1998]. Therefore, in the second section we proved some necessary and sufficient conditions for the appurtenance of the Shapley value to the symmetric quasi-core, by using a proof suggested by the case of the appurtenance of the Shapley value to the core.

After introducing a new concept of coalitional rationality for any value, the connection between Semivalues and the Shapley value was allowing us to reduce the problem of finding necessary and sufficient conditions of coalitional rationality for a Semivalue, to the problem of appurtenance of the Shapley value to a symmetric quasi-core. To be able to do this, in the third section we proved an Average per capita formula for the Efficient normalization of a Semivalue. This was possible due to previous results obtained in a joint paper by Dragan and Martinez-Legaz (2001), where an Average per capita formula for the Semivalues was obtained. Note that in that paper an alternative definition of coalitional rationality was given, and the case of Semivalues was also considered, following that definition.

The necessary and sufficient conditions for w-coalitional rationality of a Semivalue, are given in the last section. Some examples discuss the application of the results to the Semivalues of a three-person cooperative game, and compare the findings to the previously found similar conditions for the Shapley value. Throughout the paper we sketched the proofs of the previous results needed.

1. Quasi-cores

A cooperative transferable utility game (TU-game) is a pair (N, v), where N is a finite set of players, $n = |N|$, and $v : 2^N \rightarrow R$ is a real function defined on the set of subsets of N, denoted by 2^N, with $v(\emptyset) = 0$. Any non-empty subset S of N is called a coalition and, for all $S \subseteq N$, the number $v(S)$ is thought to represent the outcome available to S, independent of the actions of players not in S, in case that S is formed. This way of thinking implies that any vector $x = (x_i)$ of outcomes, where x_i is the win of player i, is not acceptable to some coalition S, if we have $x(S) < v(S)$. (it is used to denote by $x(S)$ the sum of all components for $i \in S$). Indeed, if this is the case, then the players in S would break the coalition(s) in which they belong and will form the coalition S in which they can win more. The opposite situation, when we have

$$x(S) \geq v(S), \quad \forall S \subseteq N, \tag{1}$$

for the outcome vector $x \in R^n$, is described by saying that x is coalitionally rational. For the grand coalition there are no players outside N, so that the outcome should satisfy

$$x(N) = v(N). \tag{2}$$

This is called the Pareto optimality condition, or efficiency condition, while (1) for singletons are called individual rationality conditions. The set of outcomes

$$CO(N, v) = \{x \in R^N : x(N) = v(N), x(S) \geq v(S), \forall S \subset N\}, \tag{3}$$

is the Core of the game (N, v), due to D.B. Gillies (1953, 1959). The Core is considered a solution of the game, in the sense that every element x in the Core is acceptable all players, because no one alone or in a group can improve upon the outcome x. However, it is clear that sometimes the Core may be empty. For example, if $B = \{S_j \subseteq N : j = 1, 2, \ldots, k\}$ is a partition of N, and the core is non-empty, then for each block S_j we should have $x(S_j) \geq v(S_j)$, so that by summing up these inequalities and using $x(N) = v(N)$ we should have

$$v(N) \geq \sum_{j=1}^{j=k} v(S_j). \tag{4}$$

Therefore, if one of such inequalities does not hold for some partition, then the Core should be empty. Moreover, (4) are only necessary conditions for non-emptiness.

The characterization of games with non-empty cores was given by O. Bondareva (1963) and L.S. Shapley (1967), and it is based upon the concept of balanced family of coalitions. A collection of coalitions $\{S_1, S_2, \ldots, S_k\}$, not necessarily a partition, which has a set of non-negative numbers $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ called the weights, such that

$$\sum_{S_j : i \in S_j} \lambda_j = 1, \quad \forall i \in N, \tag{5}$$

is called a balanced family of coalitions. The Bondareva–Shapley theorem (see [Bondareva, 1963], [Shapley, 1967]) says that a game has a non-empty Core if, and only if, for any collection of balanced sets $\{S_1, S_2, \ldots, S_k\}$ with the set of weights $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ we have

$$\sum_{j=1}^{j=k} \lambda_j v(S_j) \leq v(N). \tag{6}$$

The excellent survey by Y. Kannai (1992) is discussing the existence of the Core for TU-games and more general classes of games, as well as the connected results. A TU-game with a non-empty core is called a balanced game. The concept of balanced game and the Bondareva–Shapley theorem stated above is needed in the following.

Now, consider a cooperative TU-game (N, v), and a set of $2^N - 2$ nonnegative weights

$$\{\delta_S \in R : S \subset N, \delta_S \in [\alpha_S, \beta_S\ ]\} \tag{7}$$

with $\alpha_S \geq 0$, where each interval $[\alpha_S, \beta_S]$ contains the number 1. By extension of the way of thinking discussed above, where all δ_S were one, a coalition $S \subset N$ may consider that a vector $x = (x_i)$ of outcomes is not acceptable for S if the total win $x(S)$ does not exceed a fraction δ_S of $v(S)$, that is

$$x(S) < \delta_S v(S). \tag{8}$$

Of course, S would prefer those vectors $x \in R^n$ of outcomes for which

$$x(S) \geq \delta_S v(S), \quad \forall S \subset N. \tag{9}$$

Such a payoff vector x, satisfying (9), will be called *δ-coalitionally rational.* For the grand coalition, there are no players outside N, so that the outcome should still satisfy the condition (2).

Definition 1. *The set of outcomes*

$$CO_\delta(N, v) = \{x \in R^n : x(N) = v(N), x(S) \geq \delta_S v(S), \forall S \subset N\}, \tag{10}$$

where $x(S)$ is the sum of components for players in S, will be called the $\delta-$quasi-core of the game (N, v)

Note that for a set of weights and a game (N, v) we can define a new associate TU-game (N, w_δ) by

$$w_\delta(S) = \delta_S v(S), \quad \forall S \subseteq N, \tag{11}$$

where $\delta_N = 1$ and the other components are satisfying (7); obviously, the δ-quasi-core of (N, v) is the usual core of (N, w_δ). After a careful reading of Shapley–Shubik paper [Shapley, 1966], as well as of the Kannai survey [Kannai, 1992], the above definition of a quasi-core is not surprising. Indeed, if for a game with all $v(S) > 0$ we take in (10)

$$\delta_S = 1 - \frac{\varepsilon}{v(S)}, \forall S \subset N, \tag{12}$$

where ε is a positive number, we get

$$CO_\delta(N, v) = \{x \in R^n : x(N) = v(N), x(S) \geq v(S) - \varepsilon, \ \forall S \subset N\}, \tag{13}$$

that is the Shapley–Shubik strong $\varepsilon-$core (see [20], p.812). Similarly, if for a positive game we take in (10)

$$\delta_S = 1 - \frac{\varepsilon \, |S|}{v(S)}, \ \forall S \subset N, \tag{14}$$

where $\varepsilon > 0$ and $|S|$ is the cardinality of S, we get

$$CO_\delta(N, v) = \{x \in R^n : x(N) = v(N), x(S) \geq v(S) - \varepsilon \, |S| \, , \forall S \subset N\}, \tag{15}$$

that is the Shapley–Shubik weak $\varepsilon-$core (see [Shapley, 1966], p. 812). Moreover, the Shapley–Shubik paper gives reasonable meanings for these types of quasi-cores and is discussing the existence of the strong ε-core, ([Shapley, 1966], section 12), and the existence of the weak ε-core, ([Shapley, 1966], section 8), for market games. Accordingly, taking into account the above remark, inspired by the introduction of the game (N, w_δ) in (11), the non-emptiness of the δ-quasi-core of the game (N, v), defined in (10), can be reduced to the non- emptiness of the usual core of the game (N, w_δ). Therefore, a byproduct of this idea and of the Bondareva–Shapley theorem, is the following result:

Theorem 1. *A game (N, v) has a non-empty δ-quasi-core, associated with a set of weights (7), if and only if for any balanced collection $\{S_1, S_2, \ldots, S_k\}$ with the set of balancing weights $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, we have*

$$\sum_{j=1}^{k} \lambda_j \delta_{S_j} v(S_j) \leq v(N). \tag{16}$$

Example 1. A 3-person cooperative TU-game has 5 minimal balanced sets with the following balancing weights

$$\begin{array}{lll} B_1 = \{(1),(2),(3)\} & \rightarrow & \lambda_1 = \lambda_2 = \lambda_3 = 1, \\ B_2 = \{(1),(2,3)\} & \rightarrow & \lambda_1 = \lambda_{23} = 1, \\ B_3 = \{(2),(1,3)\} & \rightarrow & \lambda_2 = \lambda_{13} = 1, \\ B_4 = \{(3),(1,2)\} & \rightarrow & \lambda_3 = \lambda_{12} = 1, \\ B_5 = \{(1,2),(1,3),(2,3)\} & \rightarrow & \lambda_{12} = \lambda_{13} = \lambda_{23} = \dfrac{1}{2}. \end{array} \tag{17}$$

(see [G. Owen, 1992], p. 230, for a complete discussion, showing that we can confine ourselves in (16) to minimal balanced collections).

The δ-quasi-core associated with the set of weights $\{\delta_1, \delta_2, \delta_3, \delta_{12}, \delta_{13}, \delta_{23}\}$ is non-empty for a game

$$v(1) = v(2) = v(3) = 0, \quad v(1,2) = a, v(1,3) = b, v(2,3) = c, \quad v(1,2,3) = 1, \tag{18}$$

if, and only if, we have

$$\max(\delta_{12}a, \delta_{13}b, \delta_{23}c) \leq 1, \quad a\delta_{12} + b\delta_{13} + c\delta_{23} \leq 2. \tag{19}$$

Note that if we have $\delta_1 = \delta_2 = \delta_3 = \delta(1)$, and $\delta_{12} = \delta_{13} = \delta_{23} = \delta(2)$, then for the same game the non-emptiness conditions become

$$\max(a,b,c) \leq \frac{1}{\delta(2)}, \quad a + b + c \leq \frac{2}{\delta(2)}. \tag{20}$$

In the next section we consider only the δ-quasi-cores associated with weights depending on the size of the coalition, that is

$$\delta_S = \delta(|S|), \forall S \subset N, \tag{21}$$

because such quasi-cores will be interesting in the present paper. Of course, in this case a weight vector $\delta \in R^n$, called a *symmetric weight vector,* would give all weights by (21), and we take $\delta_N = 1$. For this type of δ-quasi-core we shall consider the problem: find out necessary and sufficient conditions for the appurtenance of the Shapley value of a TU-game (N, v) to the δ-quasi-core associated with a system of weights satisfying (21); this will be done in the next section. There this type of quasi-cores will be called *symmetric δ-quasi-cores.* Notice that the strong ε-core and the weak ε-core are not symmetric δ-quasi-cores, because in (12) and (14), respectively, the weight vector δ has components depending on the coalitions, not only on the sizes of coalitions; therefore, a comparison of our results to those given in [Kannai, 1992], or in [Shapley, 1966], can not be made. The conditions to be found in the next section will be further used in the last two sections, in order to derive necessary and sufficient conditions of "coalitional rationality" for a Semivalue, a concept to be defined also in the following.

2. Symmetric quasi-cores and the Shapley value

A game (N, v), and a set of symmetric weights $\delta \in R^n$, with $\delta_N = 1$, are given. Recall that the δ-quasi-core of (N, v) was defined as

$$CO_\delta(N,v) = \{x \in R^n : x(N) = v(N), x(T) \geq \delta(t)v(T), \forall T \subset N\}, \tag{22}$$

where $x(T) = \sum\limits_{i \in T} x_i, \forall T \subseteq N$. Let us introduce the notations

$$v_s = \frac{\sum\limits_{|S|=s} v(S)}{\begin{pmatrix} n \\ s \end{pmatrix}}, \quad v_s^i = \frac{\sum\limits_{|S|=s, i \notin S} v(S)}{\begin{pmatrix} n-1 \\ s \end{pmatrix}}, \forall i \in N, \quad s = 1, 2, \ldots, n-1, \tag{23}$$

i.e. v_s is the average worth of all coalitions of size s, and v_s^i is the average worth of all coalitions of size s which do not contain player i. A formula for the well known concept of solution, the Shapley value [Shapley, 1953], which was called an Average per capita formula, is expressing the value in terms of the above averages as follows:

Theorem 2 [Dragan]. *For all players $i \in N$ we have*

$$SH_i(N,v) = \frac{v(N)}{n} + \sum_{s=1}^{n-1} \frac{v_s - v_s^i}{s}, \tag{24}$$

where SH is the well known linear operator called the Shapley value, and the numbers v_s and $v_s^i, \forall i \in N$ are defined by (10) for $s = 1, 2, \ldots, n-1$.

In the joint work [Dragan, 2001], the problem of the appurtenance of a Semivalue to what was called the Power Core was considered. As the Shapley value is a Semivalue, and the Shapley value is efficient, from the necessary and sufficient conditions obtained, we derived necessary and sufficient conditions for the appurtenance of the Shapley value to the Core (see [Dragan, 2001], p.135), that is the δ-quasi-core with unitary weights. Now, we consider the problem of the appurtenance of the Shapley value to the symmetric δ-quasi-core, which is an interesting problem by itself, especially in the cases when the Core of the game is empty. As the Shapley value is efficient, we should have to satisfy in (9) only the inequalities

$$x(T) \geq \delta(t)v(T), \forall T \subset N. \tag{25}$$

However, the strategy used for the proof in [Dragan, 2001] still works, and we shall give it here below in the proof of the following:

Theorem 3. The Shapley value of the TU game (N, v) belongs to the symmetric δ-quasi-core of (N, v)associated with the weight vector $\delta \in R^n$ with $\delta(n) = 1$, if and only if we have

$$\frac{v(N)}{n} + \sum_{s=1}^{n-1} \left(st \left(\begin{array}{c} n \\ s \end{array} \right) \right)^{-1} \sum_{|S|=s} \frac{|S \cap T|\, n - st}{n-s} v(S) \geq \frac{\delta(t)}{t} v(T), \forall T \subset N. \quad (26)$$

Proof.

Consider any coalition $T \subset N$ and let us write (12) for the Shapley value by using the Average per capita formula (11). For any coalition $T \subset N$ we get

$$\sum_{i \in T} SH_i(N, v) = t\frac{v(N)}{n} + \sum_{s=1}^{n-1} \frac{tv_s - \sum\limits_{i \in T} v_s^i}{s} \geq \delta(t)v(T). \quad (27)$$

Now, to return to the coalitional form, use (10) and

$$\sum_{i \in T} v_s^i = \left(\begin{array}{c} n-1 \\ s \end{array} \right)^{-1} \sum_{|S|=s} (t - |S \cap T|)v(S) \quad (28)$$

in (14), and it is easy to see that we obtain (13). Obviously, the theorem holds in more general cases, if $\delta(t)$ is replaced by δ_T.

Note that if we check the appurtenance of the Shapley value to a symmetric δ-quasi-core by using Theorem 3, then in (13) for each fixed $T \subset N$ we should take for each $s = 1, 2, \ldots, n-1$, all coalitions of size s, even though for some we may have $|S \cap T| = 0$, because $S \cap T = \emptyset$, so that only the second term of the numerator occurs.

Note that Theorem 3 will be used in the last section in order to derive necessary and sufficient conditions of "'coalitional rationality" for a Semivalue. However, for this purpose a different equivalent form of (13) is more appropriate. The formula (13) of Theorem 3 was written such that it is pretty close to the corresponding formula of our work [Dragan, 2001]. If we use the simple equality

$$\left(st \left(\begin{array}{c} n \\ s \end{array} \right) \right)^{-1} . \frac{1}{n-s} = \frac{1}{nt} . \frac{(s-1)!(n-s-1)!}{(n-1)!} = \frac{1}{nt} \gamma_s^{n-1}, \quad (29)$$

where γ_s^{n-1} are the Shapley coefficients for the marginal contributions of coalitions of size s in an $n-1$-person cooperative TU-game, then (13) becomes

$$\frac{v(N)}{n} + \frac{1}{t} \sum_{s=1}^{n-1} \gamma_s^{n-1} [\sum_{|S|=s} (|S \cap T| - \frac{s}{n} t)v(S)] \geq \frac{\delta(t)}{t} v(T), \forall T \subset N. \quad (30)$$

These will be the conditions to be used in our last section to prove the main result (Theorem 8) and to compare the necessary and sufficient conditions of appurtenance of a Semivalue to the Core with the necessary and sufficient conditions of appurtenance of the Shapley value to the Core, obtained from (17) for $\delta(t) = 1, \forall T \subset N$.

On the other hand, it is clear that (13) of Theorem 2.2, or (17), is a sufficient condition for the non-emptiness of the symmetric δ-quasi-core. Of course, an interesting exercise would be to prove that (17) implies (16) of Theorem 1.

Example 2. Return to the game considered in Example 1 and let us write the coalitional rationality conditions for the symmetric δ-quasi-core, that is (12), when the outcome is the Shapley value. By Theorem 3, we can use the equivalent conditions (13). For coalitions T, which are singletons, we obtain

$$-a - b + 2c \leq 2, \quad -a + 2b - c \leq 2, \quad 2a - b - c \leq 2, \tag{31}$$

and for coalitions T with $|T| = 2$, we obtain

$$-a - b + 2c \geq 6\delta(2)c - 4, \quad -a + 2b - c \geq 6\delta(2)b - 4, \quad 2a - b - c \geq 6\delta(2)a - 4. \tag{32}$$

Note that $\delta(1)$ does not occur in (18) because the worth of each singleton equals zero.

Obviously, if the Shapley value is in the symmetric δ-quasi-core, then this is non-empty. For the game (18) of Example 1, we show easily that (18) and (19) imply (20). Indeed, by pairing the inequalities (18) with the corresponding inequalities (19), we get $\delta(2)\max(a, b, c) \leq 1$, and by adding up the inequalities (17) we get $\delta(2)(a + b + c) \leq 2$. Of course, we may also write (19) to get the same results, where the Shapley coefficients will appear and to explain the form of conditions (18).

In the next section, after introducing a new concept of coalitional rationality for a value of a TU-game, in order to apply this concept to a Semivalue, we have to derive an Average per capita formula for the value called Efficient normalization of a Semivalue due to Ruiz et al. (1998). They introduced the Efficient normalization of a Semivalue, in order to establish the relationship between the Semivalues and the new family of values called the Least Square Values introduced in their paper, (see also [Dragan, 2006]. Here, we have a different purpose, namely, we intend to derive a relationship between the Efficient normalization of a Semivalue, and the Shapley value, with no need of Least Square Values. Such an Average per capita formula for the Efficient normalization of a Semivalue, appeared in a different form in [Dragan, 2005], where we discussed the possibility of computation of a Semivalue by computing a Shapley value. To make the paper self-contained we show shortly the corresponding proof, after taking the Average per capita formula for Semivalues from the joint work [Dragan, 2001].

3. A concept of coalitional rationality. The Efficient normalization of a Semivalue.

Let G^N be the space of cooperative TU-games with the set of players N. If $n = |N|$, then it is well known that this space is identified with the Euclidian space R^{2^n-1}. Let G be the union of all spaces G^N for different sets of players. A value is a functional Ψ which for each game $(N, v) \in G^N$ gives an outcome vector $\Psi(N, v) \in R^n$. The value is efficient, if we have for all functionals v defined on 2^N the equality

$$\sum_{j \in N} \Psi_j(N, v) = v(N). \tag{33}$$

In general, the values are not efficient and there are several methods to make the value efficient.

Definition 2. *The Efficient normalization of a value Ψ is the value $E\Psi$ given by*

$$E\Psi_i(N, v) = \Psi_i(N, v) + \alpha(N, v), \forall i \in N, \tag{34}$$

where

$$\alpha(N, v) = \frac{1}{n}[v(N) - \sum_{j \in N} \Psi_j(N, v)]. \tag{35}$$

Of course, the normalization term $\alpha(N, v)$ depends on Ψ, but it does not depend on i; however, we do not think that any confusion may occur if we do not mention it.

In words, after giving to each player $j \in N$ his payoff provided by $\Psi_j(N, v)$, the reminder is shared equally by the players, if α is positive, or returned in equal shares if α is negative. Obviously, if Ψ is efficient, then $\alpha = 0$. Note that there are other methods to derive an efficient value from a non-efficient one. Now, we introduce a new concept of coalitional rationality for a value.

Definition 3. *A value Ψ, which is not necessarily efficient, is w-coalitionally rational, if we have*

$$\sum_{j \in S} E\Psi_j(N, v) \geq v(S), \quad \forall S \subset N. \tag{36}$$

In words, the new concept of coalitional rationality is requiring that the Efficient normalization of Ψ belongs to the Core of the game (N, v). The efficiency was not mentioned in Definition 3 because it is automatically satisfied for the Efficient normalization. Note that if the value Ψ is efficient, then the w-coalitional rationality reduces to the usual coalitional rationality. Note also that the conditions (36), which define the w-coalitional rationality, based upon (35), may be rewritte nunder the form

$$\sum_{j \in S} \Psi_j(N, v) \geq v(S) - |S|.\alpha(N, v), \quad \forall S \subset N.$$

These are the coalitional rationality conditions imposed to a value which belongs to the weak α-core (if α is positive), as it was shown by (15) in the first section. This was the reason of calling the new concept a w-coalitional rationality. The reader may compare this concept of coalitional rationality to the concept introduced in the previous joint paper [Dragan, 2001]. We consider that both have the same merit, because they are reduced to the usual concept of coalitional rationality for efficient values. As we have seen in that paper, the concept was successfully used to derive necessary and sufficient conditions of coalitional rationality for Semivalues, and the same thing will happen here, as it will be seen below. However, to write the conditions (36) for a Semivalue, we need an Average per capita formula for the Efficient normalization of a Semivalue; this is our aim here below.

The Semivalues, introduced axiomatically in a more general framework by Dubey et al. (1981), may be defined on G^N by the formula

$$SE_i(N,v) = \sum_{S:i\in S\subseteq N} p_s^n[v(S) - v(S - \{i\})], \quad \forall i \in N, \tag{37}$$

where $p^n = (p_1^n, p_2^n, \ldots, p_n^n)^T$ is a non-negative weight vector satisfying the normalization condition

$$\sum_{s=1}^{s=n} \left(\begin{array}{c} n-1 \\ s-1 \end{array} \right) p_s^n = 1. \tag{38}$$

Note that (37) is also the definition of a Semivalue for arbitrary sets N, hence for sets of players of different sizes we should have different weight vectors. In other words, to define a Semivalue on the union of all spaces G^N, a sequence of weight vectors $p^1, p^2, \ldots, p^n, \ldots$ should be given, all of them satisfying normalization conditions like (38). Moreover, following the authors mentioned above, we assume that recursive relationships connect these vectors

$$p_s^{n-1} = p_s^n + p_{s+1}^n, \quad s = 1, 2, \ldots, n-1. \tag{39}$$

We call (39) the inverse Pascal triangle relations. Obviously, from (38) and (39) it follows that the normalization conditions are satisfied by p^t, for all $t \leq n-1$ as soon as they are satisfied for $t = n$; moreover, if p^n satisfying (38) is given, then all $p^t, t \leq n$, are uniquely determined by (39). Now, the Semivalues are defined on G by the same formula (37), where (N, v) is arbitrary in G. Obviously, the Shapley value [Shapley, 1953] belongs to the family of the Semivalues, for $p_s^n = (n!)^{-1}(s-1)!(n-s)!$, $s = 1, 2, \ldots, n$.

In the previous work [Dragan and Martinez, 2001], we proved an Average per capita formula for Semivalues, which will be given here without a proof. Consider again the notations (10) for the average worth of coalitions of size s and the average worth of coalitions of size s which do not contain the player i, for $s = 1, 2, \ldots, n-1$, and all $i \in N$. We use also $v_n = v(N)$. Then, we have:

Theorem 4 [Th. 1]. *The Semivalue associated with the non-negative weight vector p^n satisfying (38) is given by*

$$SE_i(N,v) = q_n^n \frac{v_n}{n} + \sum_{s=1}^{n-1} \frac{q_s^n v_s - q_s^{n-1} v_s^i}{s}, \forall i \in N, \tag{40}$$

where the non-negative vector q is defined by

$$q_s^n = \frac{p_s^n}{\gamma_s^n}, \text{ with } \gamma_s^n = \frac{(s-1)!(n-s)!}{n!}, s = 1, 2, \ldots, n. \tag{41}$$

Note that in (40) we had $q_s^{n-1} = \frac{p_s^{n-1}}{\gamma_s^{n-1}}$, where p_s^{n-1} and γ_s^{n-1} were given by (37); also, γ_s^n are the weights which give in (39) the Shapley value, and satisfy (39). Note that from (40) for these weights, that is $p_s^n = \gamma_s^n, s = 1, 2, \ldots, n$, we get (11). However, (40) applies also to other Semivalues, for example, to the Banzhaf value (J.F. Banzhaf, III, 1965), obtained for other weight vectors p^n satisfying (38) and (39), precisely $p_s^n = 2^{1-n}, s = 1, 2, \ldots, n$. It is clear that the new weight vectors q^n satisfy other normalization conditions and other inverse Pascal triangle relations.

In the same paper was proved a formula for the computation of what was called the Power game, which may be used to compute the worth of the grand coalition in that game. Calvo and Santos (1997) and Sanchez (1997) call this Power game an auxiliary game. We defined the Power game, (N, π_v^Ψ), of the given game (N, v), by

$$\pi_v^\Psi(T) = \sum_{j \in T} \Psi_j(T, v), \quad \forall T \subseteq N. \tag{42}$$

The value Ψ was considered coalitional rational in [5] if $\Psi(N, v)$ belongs to the Core of the Power game of Ψ. Here, we need the worth of the grand coalition of the Power game, but we do not use that formula which will be obtained independently, from (40) shown above.

To derive from the new concept, called w-coalitional rationality, introduced above by Definition 3, necessary and sufficient conditions of w-coalitional rationality for a Semivalue, we need an Average per capita formula for the Efficient normalization of a Semivalue.

Consider the Average per capita formula (40) of Theorem 4. After adding up all components in (40), we get

$$\sum_{j \in N} SE_j(N, v) = q_n^n v_n + n \sum_{s=1}^{n-1} \frac{q_s^n v_s}{s} - \sum_{s=1}^{n-1} \frac{q_s^{n-1}}{s} (\sum_{j \in N} v_s^j). \tag{43}$$

Now, by using the equality $\sum_{j \in N} v_s^j = n v_s$, valid for all $s = 1, 2, \ldots, n-1$, we proved

Theorem 5. *Let SE be the Semivalue associated with the non-negative weight vector p^n, and q^n be the new weight vector introduced by (39). Then we have*

$$\pi_v^{SE}(N) = \sum_{j \in N} SE_j(N, v) = q_n^n v_n + n \sum_{s=1}^{n-1} \frac{(q_s^n - q_s^{n-1}) v_s}{s}. \tag{44}$$

Note that a similar formula would give the worth of the other coalitions in the Power game, but the averages should be taken in the subgames of (N, v); we do not need them here. From (44) we intend to compute the Average per capita formula for the Efficient normalization of a Semivalue. Taking into account Theorem 5, we have to compute only the efficiency term $\alpha(N, v)$, as seen in the proof of the following

Theorem 6. *Let SE be a Semivalue associated with a non-negative weight vector $p^n, n \geq 2$, and q^n be the non-negative weight vector defined by (41). Then, the Efficient normalization of the Semivalue, denoted by ESE, is given by*

$$ESE_i(N, v) = \frac{v_n}{n} + \sum_{s=1}^{n-1} q_s^{n-1} \frac{v_s - v_s^i}{s}, \quad \forall i \in N. \tag{45}$$

Proof.

From (35) and (44) we compute the efficiency term

$$\alpha = \frac{(1 - q_n^n)v_n}{n} - \sum_{s=1}^{n-1} \frac{(q_s^n - q_s^{n-1})v_s}{s}. \tag{46}$$

Now, by using (34) for $\Psi = SE$, from (35) and (46) we get (45).

Formula (45) is the Average per capita formula for the Efficient normalization of a Semivalue, and it will be used, together with Theorem 3, and another result about the relationship of a Semivalue with the Shapley value, in the last section, in order to derive necessary and sufficient conditions of $w-$coalitional rationality for a Semivalue.

4. Necessary and sufficient conditions for w-coalitional rationality

A relationship between the Efficient normalization of a Semivalue and the Shapley value is easy to obtain by looking at the Average per capita formulas of these two values, namely (11) and (45). It is given by

Theorem 7. *Consider a TU game (N, v), and a nonnegative weight vector $p^n, n \geq 2$. Let q^{n-1} be the non-negative weight vector defined by*

$$q_s^{n-1} = \frac{p_s^n + p_{s+1}^n}{\gamma_s^n + \gamma_{s+1}^n}, \quad s = 1, 2, \ldots, n-1, \tag{47}$$

where γ_s^n are the coefficients of terms in the formula of the Shapley value.Then, the Efficient normalization of a Semivalue for the game (N, v) is the Shapley value of the game (N, w) obtained from (N, v) by means of the rescaling:

$$w(S) = q_s^{n-1} v(S), \forall S \subset N, \quad w(N) = v(N). \tag{48}$$

Proof.

From (48) and (10) we have

$$v_n = w_n, \quad q_s^{n-1} v_s = w_s, \quad q_s^{n-1} v_s^i = w_s^i, \quad \forall i \in N, \quad s = 1, 2, \ldots, n-1. \tag{49}$$

These equalities used in (45) and the Average per capita formula for the Shapley value (10) show that we have

$$ESE(N, v) = SH(N, w), \tag{50}$$

the result stated in Theorem 7.

Returning to the w-coalitional rationality introduced by Definition 2, we see that the following result follows:

Corollary. *The Semivalue SE of the TU game (N, v) associated with the nonnegative weight vector p^n is w-coalitionally rational if, and only if, the Shapley value of the game (N, w) defined by (49) belongs to the symmetric δ-quasi-core of the game (N, w), where $\delta \in R^n$ is given by $\delta(s) = (q_s^{n-1})^{-1}, s = 1, 2, \ldots, n-1$, and $\delta(n) = 1$, and the weight vector q^{n-1} is given in terms of p^n by (47).*

Proof.

From Definition 2 with $\Psi = SE$,taking into account Theorem 7, expressed by (50), and the equalities (48), we get

$$\sum_{j \in S} SH_j(N, w) \geq \frac{1}{q_s^{n-1}} w(S), \forall S \subset N, \tag{51}$$

that is the fact that the Shapley value of the game (N, w) belongs to the symmetric δ-quasi-core with δ shown in (51).

Now, to obtain necessary and sufficient conditions of $w-$coalitional rationality it is enough to write the necessary and sufficient conditions of Theorem 3 for the appurtenance of the Shapley value of the game (N, w) to the symmetric $\delta-$quasi-core of this game, with $\delta(s) = \frac{1}{q_s^{n-1}}, s = 1, 2, \ldots, n-1$, and $\delta(n) = 1$. Hence, it is enough to write (17) for the game (N, w), with this weight vector δ and replace w in terms of v, as shown by (48). By Corollary, we obtain necessary and sufficient conditions for the $w-$coalitional rationality of a Semivalue, i.e. we proved the main result of this paper:

Theorem 8. *The Semivalue SE associated with the nonnegative weight vector p^n is w-coalitionally rational, if, and only if, for all coalitions $T \subset N$ we have*

$$\frac{v(N)}{n} + \frac{1}{t} \sum_{s=1}^{n-1} p_s^{n-1} [\sum_{|S|=s} (|S \cap T| - \frac{s}{n} t) v(S)] \geq \frac{1}{t} v(T), \ \forall T \subset N, \tag{52}$$

where the weight vector p^{n-1} is given in terms of p^n by the inverse Pascal triangle relationships.

Proof.

By Corollary, the necessary and sufficient conditions (13) for the appurtenance of the Shapley value of the game (N, w) to the symmetric δ-quasi-core, written under the form (17), are

$$\frac{w(N)}{n} + \frac{1}{t}\sum_{s=1}^{n-1}\gamma_s^{n-1}[\sum_{|S|=s}(|S\cap T| - \frac{s}{n}t)w(S)] \geq \frac{1}{tq_s^{n-1}}w(T), \forall T \subset N, \tag{53}$$

and by using (48) in (53), where (41) have also been used, we obtain (52).

Note that (52) is looking very similar to the necessary and sufficient condition for the appurtenance of the Shapley value to the Core, obtained from (17) by taking all $\delta(t) = 1$, for $t = 1, 2, \ldots, n-1$. Indeed, that condition is the particular case of (52) for $p_s^{n-1} = \gamma_s^{n-1}, s = 1, 2, \ldots, n-1$ (see also [5], formula (29), where γ_s^{n-1} have been replaced by their formula shown in (41)).

Example 3. Return to the 3-person game considered in the Example 1. Suppose that we would like to find the necessary and sufficient conditions for the w-coalitional rationality of a Semivalue defined by the weight vector $p^3 = (p_1^3, p_2^3, p_3^3)^T$, where $p_1^3 + 2p_2^3 + p_3^3 = 1$, and the weight vector $p^2 = (p_1^2, p_2^2)^T$ is derived from p^3 by the inverse Pascal triangle relations (39). Then, by using (52) we get the conditions

$$a + b - 2c \geq -\frac{1}{p_2^2}, \quad a - 2b + c \geq -\frac{1}{p_2^2}, \quad -2a + b + c \geq -\frac{1}{p_2^2},$$

for coalitions T with $|T| = 1$, and

$$a - b - c \geq \frac{3a-2}{p_2^2}, \quad -a + 2b - c \geq \frac{3b-2}{p_2^2}, \quad -a - b + 2c \geq \frac{3c-2}{p_2^2},$$

for coalitions T with $|T| = 2$. Now, if the Semivalue is the Shapley value, then the w-coalitional rationality conditions are the usual coalitional conditions, that is the conditions of the appurtenance of the Shapley value to the Core, because the Shapely value is efficient. In this case, we have $p_2^2 = \frac{1}{2}$, so that from the above inequalities we get the conditions

$$\begin{gathered} -a - b + 2c \leq 2, \quad -a + 2b - c \leq 2, \quad 2a - b - c \leq 2, \\ a + b + 4c \leq 4, \quad a + 4b + c \leq 4, \quad 4a + b + c \leq 4, \end{gathered}$$

which can also be obtained from the conditions proved in the second section for $\delta(2) = 1$.

Notice that (52) may be used for other Semivalues, for example the Banzhaf value. In this case, $p_1^3 = p_2^3 = p_3^3 = \frac{1}{4}$, so that again we have $p_2^2 = \frac{1}{2}$. It follows that the conditions are the same as for the Shapley value, but the Banzhaf value is not efficient, we can compute $\alpha(N, v) = \frac{1}{12}(1 - a - b - c)$, which in general is not zero; further, we can compute the Efficient normalization of the Banzhaf value and check that this value happens to be equal to the Shapley value for such games. This explains why the conditions are the same, even though they represent the conditions for the appurtenance of the Efficient normalization to the Core, in the last case. Note that we may also take a Semivalue defined by the weight vector $p^3 = (\frac{1}{8}, \frac{1}{8}, \frac{5}{8})^T$, for which we have $p^2 = (\frac{1}{4}, \frac{3}{4})^T$, so that the conditions for w-coalitional rationality will be different. Obviously, this will be a linear system of inequalities in three unknowns which may be consistent, or not. The fact that the system may not be consistent follows from the well known fact that sometimes the Shapley value, which is a Semivalue, does not belong to the Core.

Aknowledgement

This work has been done during the visits of I.Dragan at University Autonoma of Barcelona, Department of Economics, in 2003 and 2005, where helpful discussions have been carried out with our previous coauthor Professor Juan Enrique Martinez-Legaz.

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Secretary Problem with Group Choice

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Keywords: Secretary problem, arbitration procedure.

Introduction

The no-information, non-cooperative three-person best-choice game is considered. Three members of the committee want jointly to hire the secretary. There are n applicants for a position. Applicants are interviewed sequentially one by one in random order, all $n!$ permutations being equally likely. For each player applicants are ranked (absolute rank). Rank 1 is the best, n is the worst. Each player observes the relative rank of the interviewed applicant and decides either to accept or to reject the applicant. The relative ranks for different players are independent. The applicant is hired and the game ends if the decisions are made according to the rule. We consider two cases of decision-making rule: by voting and by arbitration. Otherwise the applicant is rejected and the next is interviewed. At the n-th stage the last applicant is accepted. An applicant once rejected cannot be recalled later. Each player aims to minimize the expected absolute rank of the applicant selected. This paper deals with the secretary problem. In classical secretary problem there is only one decision-maker. The aim of the decision-maker is to maximize the probability of selecting the best applicant. No-information case was considered in [Dynkin, 1967], and full-information case was studied in [Gilbert, 1966]. The other criterion of optimality is to minimize the expected rank of selected applicant for no-information case [Chow, 1964] and maximize the expected value of selected applicant ability [Gilbert, 1966]. The case of the two players was considered in [Sakaguchi, 2004]. If players choose different choices, then arbitration comes in and forces them to take the same choices as I's (II's) with probability p $(1-p)$. The full-information three-person game with arbitration was studied in [Sakaguchi, 2007]. Some approaches to three- and n-person game are found in [Ferguson, 2005], [Mazalov, 2003], [Mazalov, 2006]. In this paper the optimal payoffs and thresholds are obtained. The numerical results for cases of voting and arbitration are given. The work is supported by Russian Found for Basic Research, project 06-01-00128-a.

1. Model I: Voting

Let X_i, Y_i, Z_i are relative ranks of the i-th applicant for the players I, II, III respectively. The sequence $\{(X_i, Y_i, Z_i)\}_{i=1}^{n}$ of independent random variables has probability distribution $P\{X_i = x, Y_i = y, Z_i = z\} = \frac{1}{i^3}$ for $x = 1, ..., i$, $y = 1, ..., i$, $z = 1, ..., i$.The i-th applicant is hired and game ends, if greater than or equal to two players accept one. Then each player receives $Q(i, x)$, $Q(i, y)$, $Q(i, z)$,

$$Q(i,x) = \sum_{r=x}^{n-(i-x)} r\frac{C_{r-1}^{x-1}C_{n-r}^{i-x}}{C_n^i} = \frac{n+1}{i+1}x,$$

i.e. expected absolute rank for the i-th applicant, under the condition that the relative rank to those who have already seen is x. If all $n-1$ applicants are rejected, then at the n-th stage the last applicant is accepted. Each player aims to minimize the expected rank of the applicant selected.

Denote u_i, v_i, w_i the expected payoffs of the players after the first i applicants have been rejected.

At the i-th stage the game is presented by matrix $M_i(x, y, z)$,

$$M_i(x,y,z)=\begin{array}{cc}
\mathrm{A} & \left(\begin{array}{ccc} & \mathrm{A} & \mathrm{R} \\ \mathrm{A} & \big(Q(i,x),Q(i,y),Q(i,z)\big) & \big(Q(i,x),Q(i,y),Q(i,z)\big) \\ \mathrm{R} & \big(Q(i,x),Q(i,y),Q(i,z)\big) & \big(u_i,v_i,w_i\big) \end{array}\right) \\
 & \\
\mathrm{R} & \left(\begin{array}{ccc} & \mathrm{A} & \mathrm{R} \\ \mathrm{A} & \big(Q(i,x),Q(i,y),Q(i,z)\big) & \big(u_i,v_i,w_i\big) \\ \mathrm{R} & \big(u_i,v_i,w_i\big) & \big(u_i,v_i,w_i\big) \end{array}\right)
\end{array}$$

where strategies of players are A — to accept, R — to reject.

By the matrix we see that the optimal strategies for the players I, II, III are to accept the i-th applicant if $Q(i, x) \le u_i$, $Q(i, y) \le v_i$, $Q(i, z) \le w_i$. Then

$$u_{i-1} = \frac{1}{i^3}\sum_{x,y,z=1}^{i} Q(i,x)\Big[I\{Q(i,x)\le u_i, Q(i,y)\le v_i, Q(i,z)\le w_i\} + I\{Q(i,x)\le$$

$$\le u_i, Q(i,y)\le v_i, Q(i,z) > w_i\} + I\{Q(i,x)\le u_i, Q(i,y) > v_i, Q(i,z)\le w_i\} +$$

$$+ I\{Q(i,x) > u_i, Q(i,y)\le v_i, Q(i,z)\le w_i\}\Big] + \frac{1}{i^3}u_i\sum_{x,y,z=1}^{i}\Big[I\{Q(i,x) > u_i, Q(i,y) >$$

$$> v_i, Q(i,z) > w_i\} + I\{Q(i,x) > u_i, Q(i,y) \leq v_i, Q(i,z) > w_i\} + I\{Q(i,x) \leq u_i,$$
$$Q(i,y) > v_i, Q(i,z) > w_i\} + I\{Q(i,x) > u_i, Q(i,y) > v_i, Q(i,z) \leq w_i\}\Bigg],$$

where $i = 1, \ldots, n-1$; $u_{n-1} = \frac{1}{n}\sum\limits_{x=1}^{n} x = \frac{n+1}{2}$; $\{A\}$ is an indicator of event A.

By symmetry $u_i = v_i = w_i$. Then the optimal thresholds are equal to $x_i = u_i \frac{i+1}{n+1}$. Therefore,

$$x_{i-1} = \frac{1}{2i^2(i+1)}\Bigg[[x_i]^2\Big(2([x_i]+1)(i-[x_i]) + i(i+1)\Big) + 2x_i(i+2[x_i])(i-[x_i])^2\Bigg], \quad (1)$$

where $i = 1, ..., n-1$, $x_{n-1} = \frac{n}{2}$, $[x_i]$ – integer part of x_i.

Sakaguchi and Mazalov [Sakaguchi, 2004] have been considered case of two players with arbitration. If players make different choices, then arbitration forces players to take the same choices with probability p.

Table 1 shows the numerical results for cases of one player ($m = 1$), two players ($m = 2$) with arbitration ($p = \frac{1}{2}$) and three players ($m = 3$) with voting.

Table 1: Expected payoffs

	$m = 1$	$m = 2$	$m = 3$
i	x_i	x_i	x_i
100			
99	50	50	50
98	37.250	43.375	43.376
97	30.053	39.625	39.532
96	25.291	37.181	36.906
...	...	...	...
10	0.392	3.826	3.339
9	0.357	3.458	3.059
8	0.321	3.125	2.806
7	0.285	2.819	2.479
6	0.250	2.450	2.178
5	0.214	2.114	1.907
4	0.178	1.807	1.577
3	0.143	1.434	1.264
2	0.107	1.092	0.980
1	0.071	0.781	0.654
	$u_0 = 3.603$	$u_0 = 39.425$	$u_0 = 33.002$

We see that the payoffs for the case of three players is greater (the expected absolute rank of the accepted applicant is less) than the payoffs for the case of two players.

Lemma 1. *As $n \geq 19$ the optimal payoff in the secretary problem with voting for three players is greater than in the problem for two players with arbitration ($p = \frac{1}{2}$).*
Proof.

We prove that as $n \geq 19$ for expression (1) the following inequality holds: $\frac{i+2}{4} < x_i < \frac{i-1}{2}$ for $8 \leq i \leq n-2$.

The proof is conducted by induction. By inequality $\frac{i+2}{4} < x_i < \frac{i-1}{2}$ for $8 \leq i \leq n-2$ and $n \geq 19$ we obtain

$$\begin{gathered} 2.5 < x_8 < 3.5 \\ 2.25 < x_7 < 3.002 \\ 2.017 < x_6 < 2.603 \\ 1.804 < x_5 < 2.117 \\ 1.501 < x_4 < 1.816 \\ 1.213 < x_3 < 1.426 \\ 0.952 < x_2 < 1.070 \\ 0.635 < x_1 < 0.773 \\ 0.317 < x_0 < 0.3867 \end{gathered}$$

In the game with two players [Sakaguchi, 2004] $0.3875 \leq x_0 \leq 0.404$. We obtain that the thresholds in the case of three players are less than in the case of two players. The proof is complete.

2. Model II: Arbitration

In this section we cosider the game of three players, where common decision makes by arbitration. If players different choices, then the arbitration comes in. If choices are AAR, ARA or RAA, then the applicant is accepted (rejected) with probability $\frac{2}{3}$ $\left(\frac{1}{3}\right)$. If choices are RRA, RAR or ARR, then the applicant is accepted (rejected) with probability $\frac{1}{3}$ $\left(\frac{2}{3}\right)$. If the i-th applicant is hired, game ends.

Then each player receives $Q(i,x)$, $Q(i,y)$, $Q(i,z)$. Let u_i, v_i, w_i be the expected payoffs of the players after the first i applicants have been rejected.

The matrix of the game is following

$$\begin{array}{cccc} & & \text{A} & \text{R} \\ & \text{A} & Q(i,x), Q(i,y), Q(i,z) & \frac{2}{3}Q(i,x)+\frac{1}{3}u_i,\ \frac{2}{3}Q(i,y)+\frac{1}{3}v_i,\ \frac{2}{3}Q(i,z)+\frac{1}{3}w_i \\ \text{A} & & & \\ & \text{R} & \frac{2}{3}Q(i,x)+\frac{1}{3}u_i,\ \frac{2}{3}Q(i,y)+\frac{1}{3}v_i,\ \frac{2}{3}Q(i,z)+\frac{1}{3}w_i & \frac{1}{3}Q(i,x)+\frac{2}{3}u_i,\ \frac{1}{3}Q(i,y)+\frac{2}{3}v_i,\ \frac{1}{3}Q(i,z)+\frac{2}{3}w_i \end{array}$$

$$\begin{array}{cccc} & & \text{A} & \text{R} \\ & \text{A} & \frac{2}{3}Q(i,x)+\frac{1}{3}u_i,\ \frac{2}{3}Q(i,y)+\frac{1}{3}v_i,\ \frac{2}{3}Q(i,z)+\frac{1}{3}w_i & \frac{1}{3}Q(i,x)+\frac{2}{3}u_i,\ \frac{1}{3}Q(i,y)+\frac{2}{3}v_i,\ \frac{1}{3}Q(i,z)+\frac{2}{3}w_i \\ \text{R} & & & \\ & \text{R} & \frac{1}{3}Q(i,x)+\frac{2}{3}u_i,\ \frac{1}{3}Q(i,y)+\frac{2}{3}v_i,\ \frac{1}{3}Q(i,z)+\frac{2}{3}w_i & u_i, v_i, w_i \end{array}$$

The optimal strategies for the players I, II, III are to accept the i-th applicant if $Q(i,x) \leq u_i$, $Q(i,y) \leq v_i$, $Q(i,z) \leq w_i$. The optimal payoff for the player I is following: $u_{i-1} =$

$$= \frac{1}{3i}\left[\sum_{x=1}^{i} Q(i,x) I\{Q(i,x) \leq u_i\} + \sum_{y=1}^{i} \frac{n+1}{2} I\{Q(i,y) \leq v_i\} + \sum_{z=1}^{i} \frac{n+1}{2} I\{Q(i,z) \leq w_i\}\right] +$$

$$+\frac{1}{3i}u_i\left[\sum_{x=1}^{i} I\{Q(i,x) > u_i\} + \sum_{y=1}^{i} I\{Q(i,y) > v_i\} + \sum_{z=1}^{i} I\{Q(i,z) > w_i\}\right],$$

where $i = 1, \dots, n-1$; $u_{n-1} = \frac{1}{n}\sum_{x=1}^{n} x = \frac{n+1}{2}$; $I\{C\}$ is indicator of the event C.

By symmetry $u_i = v_i = w_i$. Let $x_i = \frac{i+1}{n+1}u_i$ be the threshold for the acceptance of the i-th applicant. Then the optimal thresholds are equal to

$$x_{i-1} = x_i + \frac{[x_i]}{3} - \frac{1}{i+1}\Big(1 + [x_i]\Big)\Big(x_i - \frac{[x_i]}{6}\Big), \tag{2}$$

where $i = 1, ..., n-1$, $x_{n-1} = \frac{n}{2}$.

In table 2 the optimal payoffs for $n = 100$ are presented.

Table 2: Equilibrium values for $n = 100$

	u_0
$m = 2$ $p = \frac{1}{2}$	39.425
$m = 3$ voting	33.002
$m = 3$ arbitration	43.701

Lemma 2. *The optimal thresholds in the secretary problem with arbitration ($p = \frac{2}{3}$) for three players are greater than or equal to thresholds in the problem for two players with arbitration ($p = \frac{1}{2}$).*

Proof.

In the game with two players and $p = \frac{1}{2}$ the threshold is equal to

$$y_{i-1} = y_i + \frac{[y_i]}{4} - \frac{1}{i+1}\Big(1 + [y_i]\Big)\Big(y_i - \frac{[y_i]}{4}\Big), \tag{3}$$

where $i = 1, ..., n-1$, $y_{n-1} = \frac{n}{2}$. By induction we obtain from (2) and (3) $x_{n-1} = y_{n-1} = n/2$, $x_{n-2} \geq y_{n-2}$.

Let $x_i \geq y_i$. Then we compare two expessions

$$\frac{[x_i]}{3} + \frac{1}{i+1}\Big(1 + [x_i]\Big)\frac{[x_i]}{6} \geq \frac{[y_i]}{4} + \frac{1}{i+1}\Big(1 + [y_i]\Big)\frac{[y_i]}{4}.$$

The proof is complete.

By Lemma 2 we obtain $\frac{i}{2} \leq x_i \leq i$, $i = 1, ..., n-1$, $x_{n-1} = \frac{n}{2}$.

3. Model III: General case

In this section we consider the general case, where common decision makes by arbitration with probability p. If players choose different choices, then the arbitration comes in. If choices are AAR, ARA or RAA, then the applicant is accepted (rejected) with probability p ($\bar{p}$), $\frac{1}{2} \le p \le 1$. If choices are RRA, RAR or ARR, then the applicant is accepted (rejected) with probability $\bar{p}$ (p). The voting corresponds to the arbitration with probability $p = 1$.

If the i-th applicant is hired, game ends. Then each player receives $Q(i,x)$, $Q(i,y)$, $Q(i,z)$.

Let u_i, v_i, w_i are the expected payoffs of the players after the first i applicants have been rejected.

The matrix of the game is following

$$\begin{array}{ccll}
 & & \text{A} & \text{R} \\
 & \text{A} & Q(i,x), Q(i,y), Q(i,z) & pQ(i,x)+\bar{p}u_i, pQ(i,y)+\bar{p}v_i, pQ(i,z)+\bar{p}w_i \\
\text{A} & & & \\
 & \text{R} & pQ(i,x)+\bar{p}u_i, pQ(i,y)+\bar{p}v_i, pQ(i,z)+\bar{p}w_i & \bar{p}Q(i,x)+pu_i, \bar{p}Q(i,y)+pv_i, \bar{p}Q(i,z)+pw_i
\end{array}$$

$$\begin{array}{ccll}
 & & \text{A} & \text{R} \\
 & \text{A} & pQ(i,x)+\bar{p}u_i, pQ(i,y)+\bar{p}v_i, pQ(i,z)+\bar{p}w_i & \bar{p}Q(i,x)+pu_i, \bar{p}Q(i,y)+pv_i, \bar{p}Q(i,z)+pw_i \\
\text{R} & & & \\
 & \text{R} & \bar{p}Q(i,x)+pu_i, \bar{p}Q(i,y)+pv_i, \bar{p}Q(i,z)+pw_i & u_i, v_i, w_i
\end{array}$$

By the matrix we see that the optimal strategy for each player is to accept the i-th applicant if $Q(i,x) \le u_i$, $Q(i,y) \le v_i$, $Q(i,z) \le w_i$.

Then

$$u_{i-1}=\frac{1}{i^3}\sum_{x,y,z=1}^{i} Q(i,x)\Big[J_3+pJ_2+\bar{p}J_1\Big]+\frac{1}{i^3}u_i\sum_{x,y,z=1}^{i}\Big[\bar{p}J_2+pJ_1+J_0\Big]$$

where $i = 1, 2, ..., n-1$; $u_{n-1} = \frac{1}{n}\sum_{x=1}^{n} x = \frac{n+1}{2}$; J_l – the number of events, if l players make decisions to accept the applicant, $l = 0, 1, 2, 3$.

By symmetry $u_i = v_i = w_i$. Let $x_i = \frac{i+1}{n+1}u_i$ be the threshold for the acceptance of the i-th applicant. Then the optimal thresholds are equal to

$$\begin{aligned}
x_{i-1}=\frac{1}{2i^2(i+1)}\Bigg[&[x_i]\Bigg[([x_i]+1)\Bigg([x_i]^2+3p[x_i](i-[x_i])+3(1-p)(i-[x_i])^2\Bigg)\\
&+pi[x_i](i-[x_i])+2(1-p)i(i-[x_i])^2\Bigg]+2x_i(i-[x_i])\times\\
&\times\Bigg(3(1-p)[x_i]^2+3p[x_i](i-[x_i])+(i-[x_i])^2\Bigg)\Bigg],
\end{aligned}$$

where $i = 1, ..., n-1$, $x_{n-1} = \frac{n}{2}$.

Table 3: Optimal thresholds for $n = 100$

p	$0,5$	$0,6$	$0,7$	$0,8$	$0,9$	1
x_0	$0,450$	$0,440$	$0,429$	$0,412$	$0,396$	$0,327$

In table 3 the numerical results for $n = 100$ and different p are given. The best case is the decision-making by voting. If the probability p decreases then the thresholds also decrease.

In the paper [Sakaguchi, 2007] the full-information three-person game with arbitration is considered. The obtained results agree with conclusions of paper [Sakaguchi, 2007].

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Quantum Games of Macroscopic Partners

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Abstract. Examples where quantum Nash Equilibrium has been found are considered. It turned out that under certain conditions opportunist behavior leads to a greater gain than non-opportunist one. Besides, the analysis of the structure of the quantum game has shown that it is isomorphic to a sum of two inter-connected classical games. This link named "quantum cooperation" is expressed by means of a non-linear relation between the probabilities of the choice of corresponding pure strategies. The difference between the quantum cooperation and the usual correlation has been demonstrated. Mixed strategies for participants of these games are calculated using probability amplitudes according to the rules of quantum mechanics in spite of the macroscopic nature of the game and absence of the Planck's constant. Possible role of quantum logical lattices for existence of macroscopic quantum equilibria is discussed. The games modelling opportunist behavior have the structure that is characteristic of the description of microparticles with a spin equal to $\frac{1}{2}$ and 1 when additional variables are measured.

Introduction

It often happens that mathematical structures find natural applications somewhere out of their origination. The formalism of quantum mechanics is not an exception of this rule. Applied initially to the microworld, this formalism can be used for modeling some macroscopic interactions with an element of indeterminacy. Quantum game theory based on quantum theory of micro-particles described by quantum physics is a well-developed theory today. The concept of quantum strategy proposed by J. Eisert, M. Wilkens, M. Lewenstein [Eiser, 1999] and D. Deutsch [Deutsh, 1999] has been effectively applied to the game theory [Barnum, 1999], [Piotrowsky, 1999], [Polley, 1999], [Flitney, 2002]. This approach has enriched our understanding of the Nash equilibrium and made it possible to solve the Prisoner's Dilemma [Dilemma, 2004], as well as the problem of the choice of equilibrium in some games with multiple Nash Equilibrium [Marinatto, 2000].

However, one can say that the class of phenomena finding their natural explanation in terms of principles of quantum mechanics is much wider. In the physics of the micro-world, non-distributivity has an objective status and must be present in principle. For macroscopic systems, the non-distributivity of random events expresses some specific case of the observer's 'ignorance' different from the standard probabilistic interpretation. Then it is possible due to the Luders rule to make calculations of the probabilities of definite answers for physical questions concerning properties of the quantum system. These questions are described by self-conjugate operators or in more simple cases by 'yes-no' questions, being elements of the Boolean (distributive) sublattices of the general non-Boolean lattice of properties.

As to the application of our results, we look for them not in physics but in economics and social sciences where some similarities with quantum physics can occur. In our examples, we look for macroscopic imitation of only some quantum properties arising due to non-distributivity of the lattice or non-commutativity of operators leading to complementarity. By now, a relatively complete formalism of quantization of classical games has been elaborated. It is based mainly on the approach adopted in theoretical physics which relies on the notion of a microparticle which behavior is described by a complex probability amplitude.

However, the notion of probability amplitude is also applicable in the macroworld, in the context that extends far beyond purely physical phenomena, including such fields as economics, sociology, psychology and law. In this paper we consider the examples of interactions governed not by classical Boolean logic, but by "quantum logic" (non-distributive ortholattices). Although, because of distributivity breaking, Kolmogorovian probabilities can no longer be used, the behavior of the partners are adequately described in terms of the probability amplitude.

Opportunistic use of information asymmetry in economic interactions is a typical example of breaking of distributivity. For example, one of the participants in a quantum game (the observer) tries to obtain information on the intentions of his partner by asking questions. If it is to advantage of his partner to change his current state and if such possibility is available, he gives a positive answer, thus demonstrating opportunist behavior. If such possibility is unavailable a negative answer is given. The observer is interested only in negative answers. However, it is not a Boolean algebra that is formed, but a non-distributive ortholattice which excludes application of the probability theory in general and the notion of mixed strategy in particular. In this case the behavior of both partners can be described in terms of probability amplitudes. Payoff functions are calculated according to the rules of quantum mechanics as the mean value of a sum of non-commuting operators.

1. The Stern–Gerlach Quantum Game

It is very easy to organize the quantum game using some well-known experiments with quantum microparticles. To do this one must write some payoff matrix showing what sums of money one partner must pay to the other depending on the results of the experiments. The advantage of the quantum games in comparison with the classical ones is the 'objective' nature of chance in it. In classical games chance occurs due to some ignorance, and that is why it is always possible for one of the partners with more exact information to have the privilege over the other one. In the quantum game based on measurements of some complementary observables, the result of the individual measurement is unpredictable in principle and only some average values can be predicted if the wavefunction is known to the participants of the game. Any game supposes the possibility of participants making some choices dependent on their abilities. So in quantum games participants have the freedom of preparation of the wavefunctions or density matrices for microparticles, and their profit will depend on their skill. In quantum games one has the combination of two different choices: the first choice is the manifestation of the free will of the human participant in the preparation procedure, and the second is the free choice of nature manifested in the result of measurements.

Let us consider the example of what can be called the Stern–Gerlach quantum game based on the well-known Stern–Gerlach experiment. There are two different beams of silver ions in different experimental set-ups which can be located at different places. The participants called Alice and Bob prepare their atoms in the state with some wavefunctions, so that every particle in one beam has the same wavefunction. In different beams the wavefunctions are different. Call them ψ_A and ψ_B. Then Alice and Bob measure using Stern–Gerlach magnets at first one projection of the spin and then the other one. Spin projections can be different for different participants but they are fixed. The only freedom for the participants is in change of the wave functions. The payoff matrix can be such that if Alice obtained some definite result for one projection and Bob for the other one fixed by this matrix then Bob pays to Alice some money. However, these results can be obtained only with some probabilities. The average profit of Alice is calculated then by the rule of the quantum physics as the expectation value of some combination of spin operators for the two particle system. If the beams are different as they are supposed in our game then there is no symmetrization of wave functions.

The average profit is different for different choice of wave functions. The aim of Alice is to get the maximal average profit. She can control her wave function but not that of Bob. For some choice of both partners it leads to Nash equilibrium, i.e. the choice optimal for both partners. This means that for the antagonistic game if one partner gets the maximal profit the other one has the minimal loss. An interesting feature of the quantum game is that in spite of the fact that the wave function is the description of the pure state the expectation value defining the average profit from the point of view of the game theory corresponds to the mixed strategy used by Nature. So, at first, we consider the quantum game when only two spin projections are measured. Let the payoff matrix be given by the table 1. Alice measures some values of the spin projections S_z and S_θ so that to definite eigenvalues of one spin projection operator correspond two orthogonal projector operators, call them A_1 and A_3 for one projection and A_2, A_4 for the other. The same is valid for Bob. But his projectors will be called B_1, B_2, B_3, B_4:

Table 1: Payoff matrix

	Bob			
Alice	**1**	**2**	**3**	**4**
1	0	0	c_3	0
2	0	0	0	c_4
3	c_1	0	0	0
4	0	c_2	0	0

$$A_1 + A_3 = I, \quad A_2 + A_4 = I, \qquad B_1 + B_3 = I, \quad B_2 + B_4 = I.$$

The meaning of the payoff matrix for our quantum game is that if Alice gets the result of her measurement **1** and Bob gets **3** then Bob pays to Alice the sum. If Alice gets **2** and Bob **4** then he pays, and so on. Alice in the result of the game gets the average profit calculated by the rule of quantum mechanics as the expectation value of the "profit operator"

$$H = c_3 A_1 \otimes B_3 + c_1 A_3 \otimes B_1 + c_4 A_2 \otimes B_4 + c_2 A_4 \otimes B_2.$$

So, the average profit occurs to be

$$\langle H \rangle = \langle \psi_A | \langle \psi_B | H | \psi_B \rangle | \psi_A \rangle = c_3 p_1 q_3 + c_1 p_3 q_1 + c_4 p_2 q_4 + c_2 p_4 q_2,$$

where ψ_A, ψ_B normalized vectors of states expressing the mixed strategies of Alice and Bob, $p_i = \langle \psi_A | A_i | \psi_A \rangle$, $q_j = \langle \psi_B | B_j | \psi_B \rangle$ the squares of the probability amplitudes given by the Luders rule

$$p_1 + p_3 = 1, \quad p_2 + p_4 = 1, \qquad q_1 + q_3 = 1, \quad q_2 + q_4 = 1.$$

In this game Alice gets money and Bob is paying her. So Alice is interested to get the maximal profit and Bob to pay the minimal sum. This leads to the idea of the Nash equilibrium, i.e. to the choice of such wave functions that the expectation value is maximal for one variable depending on the choice of Alice and minimal for the other one depending on Bob's choice.

Our quantum game can be compared with the "classical" game with the same payoff matrix but with mixed strategies. The payoff function for this game is written as

$$h = c_3 \alpha_1 \beta_3 + c_1 \alpha_3 \beta_1 + c_4 \alpha_2 \beta_4 + c_2 \alpha_4 \beta_2,$$

where α_i, β_j – some Boolean variables with values 0 or 1 depending on the use of the corresponding strategy. So,

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1, \qquad \beta_1 + \beta_2 + \beta_3 + \beta_4 = 1.$$

The average profit for the unlimited repeating of the game is calculated by the classical von Neumann expression:

$$\langle h \rangle = c_3 p_1 q_3 + c_1 p_3 q_1 + c_4 p_2 q_4 + c_2 p_4 q_2,$$

$$p_1 + p_2 + p_3 + p_4 = 1, \qquad q_1 + q_2 + q_3 + q_4 = 1.$$

The comparison of the expression for the quantum game average profit with the classical one shows that the quantum game can be obtained from the classical one by the "quantization" procedure. The Boolean variables α_i, β_j are transformed into projector operators so that some of them don't commute. These projectors form the structure of the non-distributive ortholattice (Fig. 1) called the quantum logical lattice. After writing the average profit in terms of probabilities one can see that the main difference between the quantum game and the classical one is in normalization of probabilities. For the classical case the probabilities are normalized on 1, for the quantum case due to Luders rule they are normalized on 2 if two non-commuting observables are measured. This corresponds to existence of two Boolean sublattices of the non-Boolean quantum logical lattice. Kolmogorovian probability measure can exist only on these sublattices, on non-Boolean lattice only the so, called quantum measure defined by the wave function can be defined. Stern–Gerlach quantum game can be generalized for the cases when three and more non-commuting spin projections are measured. Then the average profit of Alice will be constructed as the sum of three or more expectation values of the corresponding observables which can be written as the expectation value of the profit operator being the sum of non-commuting operators. For the spin one half Stern–Gerlach quantum game if two non-commuting spin operators are measured one can consider for simplicity real two dimensional space and take two-dimensional vectors in it as wave functions. Then our projectors

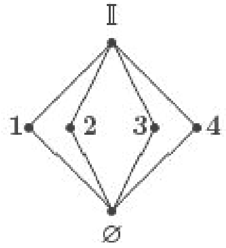

Fig. 1: Ortholattice of projectors A_1, A_2, A_3, A_4

can be defined as projectors on two vectors on the plane with some angle between them. So, in this simple case there are two different angles: one parameterizing the wave function, the other one the spin projections. In our game the angle between spin projections is considered to be fixed while the angle defining the wave function can be varied expressing, thus, the freedom of participants of the game to prepare their wave functions in different ways.

2. Macroscopic quantum games

To look for macroscopic examples of games described by the mathematical formalism of quantum physics here we consider the simple case based on the Luders rule understood as some dependence of probability measures for different experiments.

If some macroscopic player Alice is playing two games at once using for her strategies probabilities different for different games where the difference is described just by the quantum Luders rule then this will be our quantum game. The average profit is calculated as the sum of profits in two games and it is calculated as the quantum expectation value. Nash equilibrium for this combination of two games considered as one game can be found as in microscopic quantum game by varying the angle defining the wave function. However, in our macroscopic case there is no necessity to use the notion of the wave function. In macroscopic situations quantum games occur due to special form of dependence of strategies in different classical games. This dependence can be due to some asymmetry in acts of the player simultaneously playing different classical games. For example, he (she) cannot have the same frequency for acts done by the right or left hand etc. For the quantum game when three non-commuting spin observables are measured this dependence can be manifested in Heisenberg uncertainty relations for spin written in the form of some relations for frequencies in three classical games. Luders rule gives for the probabilities of getting definite answers for spin projections expressions:

$$p_1 = \cos^2 \alpha, \quad p_2 = \cos^2(\alpha - \theta), \quad q_1 = \cos^2 \beta, \quad q_2 = \cos^2(\beta - \tau). \tag{1}$$

Let Alice is playing the games on two desks: one called "even", the other one "odd". The same is for Bob. The average profits for Alice in each of the parallel games are

$$\langle H \rangle_{odd} = c_3 p_1 q_3 + c_1 p_3 q_1, \qquad \langle H \rangle_{even} = c_4 p_2 q_4 + c_2 p_4 q_2.$$

So, for the average profit in two games one obtains

$$\langle H \rangle = c_3 p_1 q_3 + c_1 p_3 q_1 + c_4 p_2 q_4 + c_2 p_4 q_2.$$

The important feature of these classical games making them different from well known situations is the existence of "quantum cooperation" given by formulas (4) (see Fig 2.) with fixed $0 < \theta < 90°$, $0 < \tau < 90°$. This cooperation can be written in more symmetric form as some equation for p_1, p_2. To do this one can introduce new variables

$$\xi = -1 + p_1 + p_2, \qquad \eta = -p_1 + p_2.$$

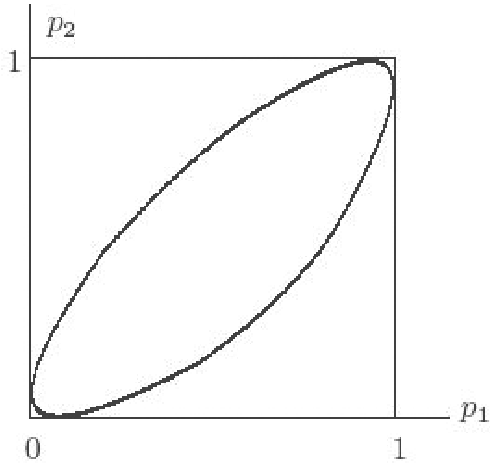

Fig. 2: Quantum cooperation for "odd" and "even" strategies

So, that by use of (4) after simple trigonometric operations one obtains

$$\frac{\xi^2}{\cos^2\theta} + \frac{\eta^2}{\sin^2\theta} = 1,$$

i.e. equation of the ellipse with axes defined by $\cos\theta$, $\sin\theta$. The same equation with angle τ one obtains for Bob. So Luders rule in our case means the existence of specific "quantum correlation". The existence of this correlation is the new feature of our games, making possible to consider it as one macroscopic quantum game.

Let us note that the "quantum correlation" arising due to the existence of the wave function, and Luders rule is not the same as classical correlation. Really if one considers two games as one antagonistic classical game the possible strategies can be considered as "$1 \star 2$", "$3 \star 2$", "$1 \star 4$", "$3 \star 4$" The same is for Bob. Then introducing

Table 2: Payoff-matrix of Alice in binary game

$A \diagdown B$	$1 \star 2$	$1 \star 4$	$3 \star 2$	$3 \star 4$
$1 \star 2$	0	c_4	c_3	$c_3 + c_4$
$1 \star 4$	c_2	0	$c_2 + c_3$	c_3
$3 \star 2$	c_1	$c_1 + c_4$	0	c_4
$3 \star 4$	$c_1 + c_2$	c_1	c_2	0

mixed strategies of Alice and Bob in this classical matrix game as p_{ik}, q_{ik} one has

$$p_1 = p_{12} + p_{14}, \quad p_3 = p_{32} + p_{34}, \quad p_2 = p_{12} + p_{32}, \quad p_4 = p_{14} + p_{34}.$$

It is evident that $p_1 + p_3 = 1$, $p_2 + p_4 = 1$. Acts of Alice in two games can be independent, i.e.

$$p_{12} = p_1 p_2, \quad p_{32} = p_3 p_2, \quad p_{14} = p_1 p_4, \quad p_{34} = p_3 p_4 \tag{2}$$

But equation for correlation can be still valid. On the contrary classical correlation means breaking of (8). One can see the other sense of "quantum cooperation". Quantum cooperation means that if $\alpha = \theta$ then the "even" game is deterministic but the "odd" game for $\theta \neq 0$ cannot be deterministic. For $\alpha = 0$ the "odd" game is deterministic but then the "even" is random. This is manifestation of complementarity due to non-commutativity of the corresponding operators. Can one look for such situations in economics, politics? It seems that the answer is positive.

3. Quantum Nash equilibrium

The definition of the Nash equilibrium $\psi_A^0 \otimes \psi_B^0$ for the quantum case is not much different from the classical case

$$\langle H\rangle(\psi_A \otimes \psi_B^0) \leqslant \langle H\rangle(\psi_A^0 \otimes \psi_B^0) \leqslant \langle H\rangle(\psi_A^0 \otimes \psi_B)$$

where ψ_A, ψ_B, normalized vectors of states expressing the mixed strategies of Alice and Bob: $p_i = \langle\psi_A|A_i|\psi_A\rangle$, $q_j = \langle\psi_B|B_j|\psi_B\rangle$ – the squares of the probability amplitudes. Introduce the variables x_1, x_2 for Alice and y_1, y_2 for Bob as

$$x_1 = \frac{-1+p_1+p_2}{\cos\theta}, \;\; x_2 = \frac{-p_1+p_2}{\sin\theta}, \qquad y_1 = \frac{-1+q_1+q_2}{\cos\tau}, \;\; y_2 = \frac{-q_1+q_2}{\sin\tau}$$

Then the equations for "quantum cooperation" become

$$x_1^2 + x_2^2 = 1, \qquad y_1^2 + y_2^2 = 1$$

So, the strategy of the participant of the game is defined by a point on the unit circle and to the game situation corresponds the point on the two dimensional torus. Back transformations from vectors to probability distributions written as $2p = M_\theta x + e,\ 2q = M_\tau y + e$, where

$$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \cos\theta & \sin\theta \end{pmatrix}, \quad e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Introduce notations:

$$n = c_1 + c_3,\ m = c_2 + c_4, \quad C = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}, \quad a = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad b = \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}.$$

Then the average payoff function can be written in matrix form as

$$\langle H\rangle = \tfrac{1}{4}[g(x,y) + \langle e, Ce\rangle],$$

where

$$g(x,y) = -\langle x, Ay\rangle + \langle x, u\rangle - \langle v, y\rangle$$

with

$$A = M_\theta^\dagger C M_\tau, \qquad u = M_\theta^\dagger \omega, \qquad v = M_\tau^\dagger \omega.$$

The vector variables x, y satisfy the limitations $|x| = 1$, $|y| = 1$. So, one has the problem of Nash equilibrium of g on torus $T^2 = S^1 \times S^1$.

It is interesting that for the classical game Nash equilibrium always exists in mixed strategies while for quantum game it is not always so. Solving of equilibrium problem based on some general properties: x_{min}, x_{max} *are the points of minimum and maximum of a function* $f(x) = \langle k, x\rangle + b$ *on the circle* $|x| = 1$ *if and only if for some non-negative* λ *the equalities* $k = \lambda x_{max}$, $k = -\lambda x_{min}$ *are valid.*

Using this properties one obtains [Grib et.al., 2002]:

Proposition 1. *The point* (x_o, y_o) *is the Nash equilibrium for the function* $g(x, y)$ *if and only if nonnegative numbers* λ *and* μ *exist satisfying equalities*

$$-Ay_o + u = \lambda x_o, \qquad A^\dagger x_o + v = \mu y_o. \tag{3}$$

Corollary 1. *If the point* (x_o, y_o) *is the Nash equilibrium for the game on torus with the payoff function* $g(x, y)$ *then for some nonnegative* λ, μ *the following equations are valid*

$$(AA^\dagger + \lambda\mu I)x_o = \mu u - Av, \qquad (A^\dagger A + \lambda\mu I)y_o = \lambda v + A^\dagger u.$$

Corollary 2. *If* $a = b$ *then the Nash equilibrium is impossible.*

Examine some special cases of existence of Nash equilibrium for the quantum game. Let ω is not equal to zero. Then $u = M_\theta^\dagger \omega$, $v = M_\tau^\dagger \omega$ are also not zero.

Proposition 2. *If* $m = n$, $\theta = \tau = 45^\circ$ *and* $n^2 \leqslant \omega_1^2 + \omega_2^2$ *then there exists one point of Nash equilibrium* (x, y), *such that*

$$x = y = \frac{1}{\sqrt{2(\omega_1^2 + \omega_2^2)}} \begin{pmatrix} \omega_2 - \omega_1 \\ \omega_2 + \omega_1 \end{pmatrix}.$$

The probabilities are equal to

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{2|\omega|} \begin{pmatrix} c_3 - c_1 + |\omega| \\ c_4 - c_2 + |\omega| \end{pmatrix}, \quad \text{with} \quad |\omega| = \sqrt{\omega_1^2 + \omega_2^2}.$$

Then the optimal value of the profit is $\langle H\rangle = n/4$.

Examples ($m = n$, $\theta = \tau = 45^\circ$)

► For $c_1 = 2$, $c_2 = 1$, $c_3 = 8$, $c_4 = 9$ the optimal strategies of Alice and Bob are $p_1 = q_1 = 0.8$, $p_2 = q_2 = 0.9$, $p_3 = q_3 = 0.2$, $p_4 = q_4 = 0.1$.

► For $c_1 = 8$, $c_2 = 9$, $c_3 = 2$, $c_4 = 1$ the optimal strategies of Alice and Bob are $p_1 = q_1 = 0.2$, $p_2 = q_2 = 0.1$, $p_3 = q_3 = 0.8$, $p_4 = q_4 = 0.9$.

► For $c_1 = 1$, $c_2 = 2$, $c_3 = 9$, $c_4 = 8$ the optimal strategies of Alice and Bob are $p_1 = q_1 = 0.9,\ \ p_2 = q_2 = 0.8,\ \ p_3 = q_3 = 0.1,\ \ p_4 = q_4 = 0.2.$

These results are in agreement with the properties of quantum logic (see Fig. 1). Really the second set of payoffs is obtained from the first by permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix},$$

expressing the automorphism of the lattice changing the elements on their orthocomplements. The third set of payoffs is obtained from the first one by the automorphism of the lattice

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

The average profit for all three cases will be the same and is equal to $\langle H \rangle = 2.5$. This can be compared with the average profit for the Nash equilibrium for the classical game (without quantum cooperation):

$$h = (c_1^{-1} + c_3^{-1})^{-1} + (c_2^{-1} + c_4^{-1})^{-1} = 2.5$$

So, in this case it has the same value for the quantum case and the classical one. However ,such a coincidence is not necessary. This can be shown by the following example:

For $c_1 = 1$, $c_2 = 9$, $c_3 = 10$, $c_4 = 2$ the optimal strategies of Alice and Bob are

$$p_1 = q_1 = \frac{130 + 9\sqrt{130}}{260} \approx 0.895, \quad p_2 = q_2 = \frac{130 - 7\sqrt{130}}{260} \approx 0.193$$

The optimal profit in the classical game is smaller than in the quantum one: $\langle h \rangle = 28/11$, $\langle H \rangle = 11/4$. Now let us consider another special case which can be called the case of "eigenequilibrium".

Proposition 3. *Let ω be the general eigenvector of operators $CM_\theta M_\theta^\dagger$ and $CM_\tau M_\tau^\dagger$, s is the eigenvalue of the operator $CM_\theta M_\theta^\dagger$. Pair of strategies $x = u/|u|$, $y = v/|v|$ defines the Nash equilibrium if and only if $s < |u|$.*

Corollary 3. *Let both components of ω be different from zero, and there are inequalities*

$$m^2\omega_1^4 - n^2\omega_2^4 > m^2 n^2(\omega_1^2 - \omega_2^2), \qquad (\omega_1^2 - \omega_2^2)(m^2\omega_1^2 - n^2\omega_2^2) > 0.$$

Then the condition

$$\cos 2\tau = \frac{(n-m)\omega_1\omega_2}{m\omega_1^2 - n\omega_2^2} = \cos 2\theta$$

is necessary and sufficient for the existence of equilibrium. Thus, the equilibrium probabilities are equal to

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{2}\left(\sqrt{\frac{\omega_1^2 - \omega_2^2}{m^2\omega_1^4 - n^2\omega_2^4}}\begin{pmatrix} m\omega_1 \\ n\omega_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

Example. Let $\theta = \tau = 45^\circ$ and $c_1 = 0$, $c_2 = 1$, $c_3 = 1$, $c_4 = 1$ then the optimal strategies of Alice and Bob are

$$p_1 = q_1 = 1, \quad p_2 = q_2 = 0.5, \quad p_3 = q_3 = 0, \quad p_4 = q_4 = 0.5.$$

In this case the profit has the same value for the quantum case and the classical one: $\langle H \rangle = \langle h \rangle = 0.5$.

4. Macroscopic quantum games and quantum logics

In the previous part we discussed the idea of the macroscopic quantum game as the system of classical games with special condition on the strategies. However, we did not consider the origin of this condition, i.e. in what situations such conditions necessarily arise.

Here we give some examples when this is so. These examples are based on the connection first mentioned by D. Finkelstein and then developed in the works of A.A.Grib and R.R. Zapatrin [Grib, Zapatrin,1990] between quantum logical lattices and graphs. These examples are taken from publications [Parfionov, 2005]. Considering the idea of existence of macroscopic situations described by the formalism of quantum physics one must also mention the publications of D. Aerts [Aerts, 1995].

It was J. von Neumann who in his paper with G. Birkhoff [Birkhoff, 1936] was the first man to see that the structure of properties of the quantum system for simple spin one half system is the structure of the orthocomplemented non distributive lattice. Non-distributivity leads to non-commutativity of projector operators representing the abstract lattice. These lattices were called quantum logical lattices, or simply "quantum logics". Later the ideas of von Neumann were developed by Jauch and Piron [Piron, 1976] for more general cases and now form the basis of the axiomatic of the quantum physics. Non-distributivity means that if there are properties A, B, C then using notation $\wedge$ for "and", notation $\vee$ for "or", then

$$(A \vee B) \wedge C \neq (A \wedge C) \vee (B \wedge C).$$

Breaking of the distributivity means that operations $\wedge$, $\vee$ cannot be understood as usual conjunctions and disjunctions of the set theory. The structure of the non-distributive lattice is not Boolean, and one cannot define on such structures the standard Kolmogorovian probability measure. Besides quantum mechanics non-Boolean lattices arise for topologies (see [Zapatrin, 1992]) so that if topologies are considered as random one also cannot define for them the standard probability measure. However, for quantum mechanical examples one can define the probability amplitude or "quantum probability measure" represented by some vector in Hilbert space.

In [Parfionov, 2005] the game called "wise Alice" was considered. Let Alice and Bob play the following game. Alice and Bob have two quadrangles, one for Alice and one for Bob. Let Bob puts some ball to the vertex of the quadrangle and Alice must guess to what vertex he did that. She asks Bob the question: "Did you put it

Fig, 3: Binary game of Alice and Bob

into **1**?". The rule of the game is such that Bob always answers "yes" if he is in **1**, but he gives the same answer if the ball was in **2** or **4**, i.e. in the vertices connected with **1** by one arc. However, it is prohibited for Bob to move by two steps from **3** to **1**, and so, if he is in **3** then he always answers to her question "no". The same rule is valid for any vertex. Alice, however, knows this property of "accommodation" of Bob to her questions. This leads to specific logic of Alice – she pays no attention to affirmative answers of Bob and notices only his negative answers.

Then it is easy to see that different positions of the ball of Bob will be described due to negative logic as disjunctive, i.e. $\mathbf{1}\wedge\mathbf{2} = \mathbf{1}\wedge\mathbf{3} = \mathbf{1}\wedge\mathbf{4} = \mathbf{3}\wedge\mathbf{4} = \mathbf{2}\wedge\mathbf{3} = \mathbf{2}\wedge\mathbf{4} = \varnothing$ but the disjunction is now not unique $\mathbf{1} \vee \mathbf{2} = \mathbb{I}$. Here $\mathbb{I}$ means "always true", $\varnothing$ – means "false". From the structure of the graph and the rules of the game it is easy to see that due to Bob's "accommodation" there is no difference for Alice between the situation $\mathbf{1} \vee \mathbf{2}$ and $\mathbf{1} \vee \mathbf{2} \vee \mathbf{3} \vee \mathbf{4}$. On the Fig. 4 we show the connection of the graph and the quantum logical lattice. This lattice is a well known lattice for Stern–Gerlach experiment when two different spin projections are measured. Lines going "up" intersect at $\vee$ ("or"), lines going down intersect at $\wedge$ ("and"). Lower drawings is called the Hasse diagram [Birkhoff, 1993].

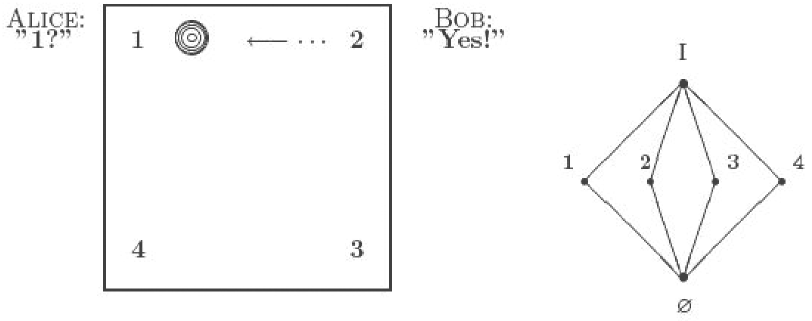

Fig. 4: Graph and Lattice of Alice's questions and Bob's answers

However, to simulate the Stern–Gerlach quantum game considered in the first part of this paper one must do the game symmetric for both partners. This means that the same rule is valid for Bob guessing to what vertex of her quadrangle Alice put her ball. So, Bob also comes to the same quantum logical lattice. Asking questions one to another Alice and Bob obtain some numbers of truly guessed due to negative answers positions of the balls and neglecting all “yes” answers.

Let these numbers be for Alice N_1, N_3 and N_2, N_4 for opposite vertices of the graph. Similar numbers are obtained by Bob. To transform these numbers into probabilities

$$\frac{N_1}{N_1+N_3}, \quad \frac{N_3}{N_1+N_3}, \quad \frac{N_2}{N_2+N_4}, \quad \frac{N_4}{N_2+N_4}$$

as it is in the quantum Stern–Gerlach game one can do the following.

Let the game consists of two parts as it was proposed in [Piron, 1976] preparation and measurement. Defining the numbers N means preparation. The second part – measurement – corresponds to the changed situation: Alice and Bob cannot accommodate one to another and now in N_1 cases Bob will be in **1**, in N_3 cases in **3** etc., but Alice every time does not know exactly if he is in **1** or **3**.

Instead of one game with quadrangle there are two games with two diagonals of the quadrangle. The strategies of Alice are defined by the probabilities obtained from the first stage due to her knowledge of the numbers N. These probabilities due to the properties of the quantum logical lattice satisfy limitations defined by the wave function for spin one half system. The same rule is valid for Bob, and he also plays two games with strategies defined by numbers obtained in the first part. The payoff is made according to the payoff matrix (Tab. 1) defined in section 1 and the average profit of one of the partners is calculated according to the quantum rule.

The necessity of going to the second part-measurement is motivated by the fact that it is only for Boolean sublattices of the non Boolean lattice that one can define probabilities. The difference of our macroscopic non-Boolean game from the microscopic Stern–Gerlach game is due to the fact that in macroscopic case Alice and Bob necessarily put their ball into position defined by the question of the partner while in microscopic case there is indeterminism, so that there is no such necessity. However, this freedom of choice is simulated in macroscopic game by the freedom of choice of the players in the second part to put their balls to any vertices with prescribed probabilities. The other difference is that the abstract quantum logical lattice which is the same for the microscopic and macroscopic cases has many different representations in terms of projectors. This means that the angle between projectors is not fixed by the lattice and can be any. In microscopic Stern Gerlach experiment this angle is chosen by the will of the experimentalist choosing the direction of the magnetic field in his magnet. In macroscopic case the angle is fixed by the ratio of probabilities for different games on the second stage. Connection between quantum logical lattices and graphs leads to the possibility of certain classification of macroscopic quantum games. For example, one can consider three classical games with strategies defined by the probabilities satisfying limitations due to the existence of the wave function, i.e. Stern–Gerlach experiment with three different spin projections for spin one half system being measured. The graph and the quantum logical lattice are shown on Fig. 5. The payoff matrices and the average profit for cases on Fig. 6. were considered in [Grib, 2003].

Fig. 5: Graph and ortholattice for spin $\frac{1}{2}$ with three spin projections measured

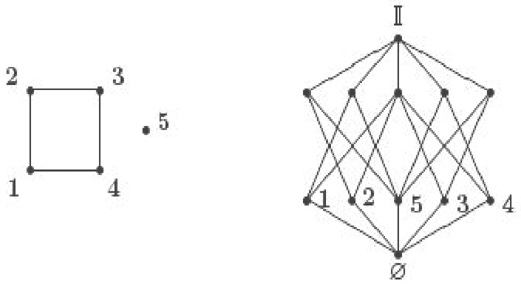

Fig. 6: Graph and ortholattice for spin 1 with two spin projections measured

What is the sense of Nash equilibrium for macroscopic quantum games based on quantum logics? The angle between two "projections" is defined by the ratio of probabilities in two classical games corresponding to Boolean sublattices of the non-Boolean lattice. Nash equilibrium for fixed angle for projections corresponds to some "patterns" of stability for players. One of the partners receives the maximal profit and the other one has the minimal loss in this situation, so the partners can come to mutual agreement on their behavior after experiencing many games of this type. This can have meaning for some economical situations.

One can construct some generalization of the macroscopic quantum game based on the use of quantum logic. It is possible to change the second part of the game so that only the angles between observables are defined in the first part for Alice and Bob. They have the possibility to choose any wave function which means the angles defining the probability amplitudes. This will correspond exactly to the problem for Nash equilibrium considered in section 3 of this paper. Some "quantum casino" can be organized following this rule.

5. Opportunism as a Quantum Phenomena

Social interactions are often accompanied by information asymmetry, in which case one of the participants fully controls the situation while the others do not at all. The simplest model of this kind is the well-known "Leader–Follower" model proposed by von Stackelberg [Moulin, 1981] which analyses the behavior of duopolists. One of them, the Leader, is fully informed on the intentions of the other one, while his partner, the Follower, is fully ignorant of the preferences of his competitor. Taking advantage of his information superiority, the Leader relies in his choice on all possible responses of the Follow. He openly proclaims the strategy, that he finds most profitable for himself. The Follower has to search the optimal strategy for himself within the scope of the opportunities left available for him. The outcome of such a game is known as Stackelberg equilibrium.

The Stackelberg model is a limit case of information asymmetry, in which each party is actually devoid of choice. Real interactions, in most cases, do not comply with such a rigid scheme. Information asymmetry, as a rule, does not exclude the initiative of the participants, thus leading to the opportunistic behavior. This results, typically, in "moral hard" and "adverse selection. G. Akerlof [Akerlof, 1970] was the first to investigate this sort of phenomena on the "lemon market" . However, his model is based entirely on the statistical characteristics of the ensemble of the players, without taking into account the reciprocal action (the "feedback", counter-reaction), when the control on the part of one of the players pushes the partner towards opportunistic behavior. The effect of reciprocal action can by illustrated by an example of the interaction of a police interrogator and a suspect. Being questioned by the former, the latter tries to do his best to hide undesirable information.

This phenomenon is well known in quantum mechanical measurements. In section 4, to imitate this effect we will consider the game of Alice and Bob with a ball located on a square. In this case the effect of adaptation is a consequence of the information asymmetry of the freedom of choice, which the Stackelberg model lacks: having received the answer of Alice Bob has an opportunity to move the ball to any of the adjacent vertices. Due to the fact that negative answers are not profitable for him he, in all possible cases, moves the ball to a convenient adjacent vertex. So being in vertices **2** or **4** and getting from Alice the question "Are you in the vertex **1**?" Bob quickly puts his ball in the asked vertex and honestly answers "yes". However, if the Bob's ball was initially in the vertex **3**, to whatever vertex he moves his ball, he cannot escape the negative answer and, consequently, fails. It should be noted that in this case Alice not only gets the profit but also obtains the exact information on the initial position of the ball: Bob's honest answer immediately reveals its initial position. This example illustrates the mechanisms of the creation of opportunism in the presence of information asymmetry. The payoff matrix of Alice (Tab. 1) shows that the model of the antagonistic game considered is based on a simplistic model of opportunism:

- In spite of the difference in outcomes (1 : 1), (1 : 2), (1 : 4), the payoff of Alice is the same in all these situations;
- outcomes (1 : 2) and (1 : 4) also differ, although, actually, they correspond to different opportunistic trajectories of Bob.

In real interactions different opportunistic trajectories may result in different payoff. A more realistic model is described by a bimatrix game

Table. 3: Payoff bymatrix

	BOB			
ALICE	**1**	**2**	**3**	**4**
1	$a_{11} : 0$	$a_{12} : 0$	$a_{13} : -b_3$	$a_{14} : 0$
2	$a_{21} : 0$	$a_{22} : 0$	$a_{23} : 0$	$a_{24} : 0$
3	$a_{31} : -b_1$	$a_{32} : 0$	$a_{33} : 0$	$a_{34} : -b_4$
4	$a_{41} : 0$	$a_{42} : -b_2$	$a_{43} : 0$	$a_{44} : 0$

in which Bob's payoff is non-positive. However, even a very simple antagonistic model leads to an unexpected problem. It can be easily shown that in this case (see Tab. 1) there is no Nash equilibrium in pure strategies. However, any attempt to seek the solution of the game in mixed strategies encounters an unexpected problem of calculation of the average payoff of a player in the framework of information asymmetry. According to the game theory, the average payoff of Alice is given by the following expression:

$$H = c_1p_1q_3 + c_3p_3q_1 + c_2p_2q_4 + c_4p_4q_2,$$

where p_j, q_k are corresponding probabilities of application of pure strategies. In this case the latter expression is not valid. The point is that the logic underlying the present game interaction is not Boolean and a conventional probability concept leads a contradiction.

In contrast with the Boolean logic, where "not **1**" is equivalent to "or **2** or **3** or **4**", in this case proposition "not **1**" is equivalent to "**3**". Similarly, "not **2**" is equivalent to "**4**". As a result, a set of true propositions

$$Pr(\mathbf{1} \vee \mathbf{3}) = Pr(\mathbf{1}) + Pr(\mathbf{3}), \quad Pr(\mathbf{2} \vee \mathbf{4}) = Pr(\mathbf{2}) + Pr(\mathbf{4})$$

leads to an absurd conclusion

$$Pr(\mathbf{1} \vee \mathbf{2} \vee \mathbf{3} \vee \mathbf{4}) = 2$$

It should be noted that although, contrary to classical logic, $\mathbf{2} \wedge \mathbf{4} = \mathbf{false}$, $\mathbf{2} \vee \mathbf{4} = \mathbf{true}$, "**2**" is not "not **4**". Thus, the principal axiom of the probability theory

$$a \wedge b = \varnothing \quad \Longrightarrow \quad Pr(a \vee b) = Pr(a) + Pr(b) \tag{4}$$

is no longer valid.

It is the mathematical formalism developed in quantum mechanics that provides a means to overcome this difficulty. This technique, elaborated long ago, allows to calculate the average in the case when random events are no longer governed by the Boolean logic. In quantum mechanics the notion of probability is transferred from the Boolean algebra to more general structures called ortholattices [Kalmbach, 1983]. An ortholattice is a set $\mathcal{L}$ with three operations

$$\vee : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}, \qquad \wedge : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}, \qquad \neg : \mathcal{L} \longrightarrow \mathcal{L}$$

such that:

a) $a \wedge b = b \wedge a,\ a \vee b = b \vee a$ – *commutativity*
b) $(a \wedge b) \wedge c = a \wedge (b \wedge c),\ (a \vee b) \vee c = a \vee (b \vee c)$ – *associativity*
c) $(a \vee b) \wedge a = a,\ (a \wedge b) \vee a = a$ – *absorption*
d) $\neg(a \vee b) = \neg a \wedge \neg b,\ \neg(a \wedge b) = \neg a \vee \neg b$ – *de Morgan laws*
e) $\exists\, \varnothing, \mathbb{I} : \quad a \vee \varnothing = a,\ a \wedge \mathbb{I} = a$ – *zero and unit exist*
f) $a \vee \neg a = 1,\ a \wedge \neg a = 0,\ \neg\neg a = a$ – *invertibility*

In the general case *distributivity*

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c), \quad (a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$$

is not observed for ortholattices. However, if distributivity laws are satisfied, an ortholattice amounts to a common Boolean algebra. The relation "$\sqsubset$" for ortholattices:

$$a \sqsubset b \quad \Longleftrightarrow \quad a \wedge b = a$$

is an analogue of the inclusion "$\subset$" in Boolean algebra. Elements a and b are called *compatible*, if $a \sqsubset b$ or $a \sqsubset b$. We say that a *commutes* with b if

$$a = (a \wedge b) \vee (a \wedge \neg b)$$

In ortholattices beside disjoint $(a \wedge b = \varnothing)$ a stronger relation, the *orthogonality*:

$$a \perp b \quad \Longleftrightarrow \quad a \sqsubset \neg b$$

is considered. It is instrumental in the formulation of the quantum version

$$a \perp b = \varnothing \quad \Longrightarrow \quad Pr(a \vee b) = Pr(a) + Pr(b).$$

of the Kolmogorov's axiom (4).

According to the Gleason's theorem [Gleason, 1957], quantum probability measures, "quantum states", are constructed on the basis of the representations of the elements of an ortholattice $a \in \mathcal{L}$ by means of projectors $E(a)$ in a Hilbert space $\mathcal{H}$ such that

$$E(a \wedge b) = E(a) \cdot E(b), \quad E(\neg a) = I - E(a), \quad E(\mathbb{I}) = I$$

for any elements $a, b \in \mathcal{L}$, that commute. In this case the probabilities of the elements $a \in L$ are calculated according to:

$$Pr_W(a) = \mathrm{tr}(W \cdot E(a))$$

where W is a density matrix. In present paper only pure states: $w \in \mathcal{H},\ \langle w|w\rangle = 1$ are considered. Their probabilities are calculated according to

$$Pr_w(a) = \langle E(a)w\,|w\rangle$$

where $w \in \mathcal{H}$ – “probability amplitudes”.

Let us consider a general game model of an interaction, when one participant acts in an opportunistic way. We suppose that each player has a finite number of strategies. Alice is given the right of control over the situation via finding out Bob’s strategy. However, her means of obtaining information are limited, which makes opportunistic behavior possible for Bob. On a set of strategies Ω of an opportunist there exists a structure of a non-oriented graph. While answering a corresponding question Bob can replace his initial strategy by an adjacent one and claim it to be his initial strategy without the risk that this will be discovered. However, if he chooses a non-adjacent strategy Bob’s opportunism may be uncovered and punished. This is reflected in the structure of his payoff matrix:

$$b_{ik} \begin{cases} < 0, & \text{if } i, k \text{ are not adjacent} \\ \geqslant 0, & \text{in the opposite case.} \end{cases}$$

As for the payoff matrix of Alice $||a_{ik}||$, no specific requirements are put forward. Thus, the model is described by the following characteristics:

- the payoff matrix of Alice;
- the payoff matrix of Bob;
- the vertex of adjacency of the graph.

The interaction between the players takes place according to their payoff matrices, while the logic of Bob’s behavior is formalized by the following mathematical structure. Each strategy $s \in \Omega$ of Bob is associated with an *opportunistic neighborhood* $O(s)$, i.e. a set of graph vertices comprised of the vertex s and adjacent vertices. Thus, an opportunist can avoid his previous obligation s by shifting to any point $O(s)$. The complement of the opportunistic neighborhood: $C(s) = \Omega \setminus O(s)$ is called *zone of control* of the vertex s. It is comprised of the vertices non-adjacent to s. The definition of the *opportunistic structure*, comprised of all possible intersections of the zones of control, is based on the set comprised of the zones of control:

$$\mathcal{L} = \{C(s_1) \cap \ldots \cap C(s_n) \mid s_i \in \Omega\}.$$

The natural order $A \subset B$ existing on the set $\mathcal{L}$ allows to introduce the following operations:

$$A \vee B = \min\{X \in L|\, X \supset A, B\} = \bigcap_{X \supset A, B} \{X \in \mathcal{L}\}$$

$$A \wedge B = \max\{X \in L|\, X \subset A, B\} = A \cap B, \qquad \neg A = \bigcap_{s \in A} C(s).$$

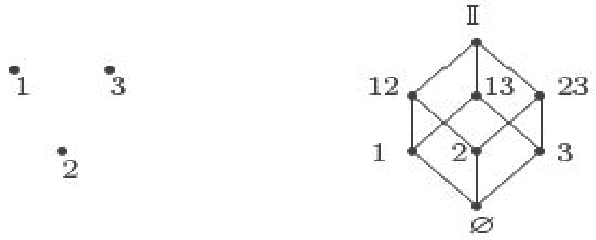

Fig. 7: Boolean algebra with 3 elements

It turns out that thus defined set $\mathcal{L}$ is an ortholattice. This ortholattice expresses the logic of the actions of an opportunist. For example, the following Boolean logic corresponds to a disjoint graph, consisting of three isolated points: While opportunistic translations along a four-link linear chain result in a ortholattice of 10 elements. The interconnection between vertex-connection of graph and the logic of the behavior of an opportunist is revealed in the following facts.

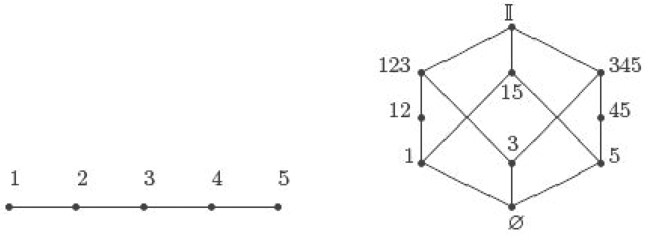

Fig. 8. Linear chain and their logic

Proposition 4. *To each ortholattice there corresponds a non-oriented graph, for which the former represents an opportunistic structure.* Thus, different logics of interaction correspond to different types of opportunism. Besides, the specificity of opportunism is entirely expressed in the logic of interaction.

Proposition 5. *An ortholattice is Boolean if the corresponding graph is without ribs.* Thus, the presence of opportunism inevitably leads to a failure of the traditional logic.

The analysis of the correspondence between the graphs and the logics engendered by these graphs (see Fig. 9, 10, 11) reveals an interesting feature: more numerous the possibilities for manifesting opportunism are simpler the logic of the interaction of the players is organized. The information asymmetry in the considered game manifests itself in the fact that Alice does not change her strategies, while her partner demonstrates opportunism. That is why in a repetitive game Alice uses classical mixed strategies, while Bob's strategies are described by state vectors. As a result, a quantum game amounts to a classical game with the payoff function

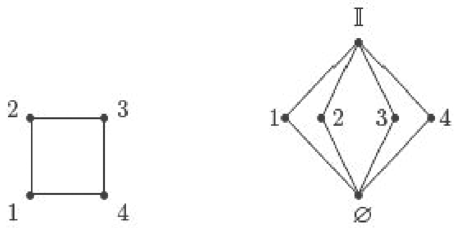

Fig. 9. Graph of opportunism type 1 and their lattice

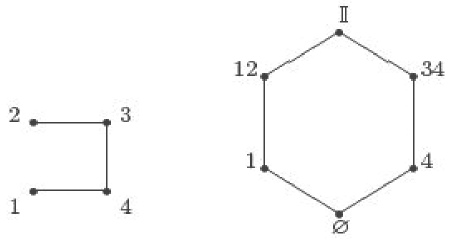

Fig. 10: Graph of opportunism type 2 and their lattice

$$A(\mathbf{p},\mathbf{x}) = \sum_{i,k=1}^{n} a_{ik} p_i \|S_k \mathbf{x}\|^2, \qquad B(\mathbf{p},\mathbf{x}) = -\sum_{i,k=1}^{n} b_{ik} p_i \|S_k \mathbf{x}\|^2,$$

where S_k are the projectors representing the ortholattice $\mathcal{L}$ and

$$\mathbf{p} = (p_1, \dots p_n), \quad p_i \geqslant 0, \quad \sum_{i=1}^{n} p_i = 1, \qquad \mathbf{x} = (x_1, \dots x_n), \quad \sum_{i=1}^{n} |x_i|^2 = 1$$

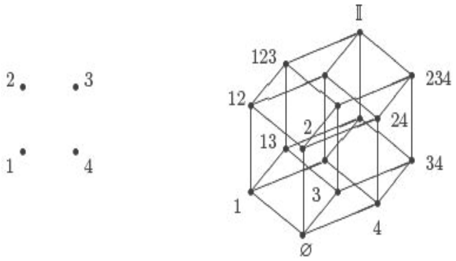

Fig. 11.: Graph without opportunism Boolean algebra

Acknowledgement

One of the authors (A.A.G.) is indebted to the Ministry of Science and Education of Russia for the financial support,grant RNP.2.1.1.6826,of this work.

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PGN-Value for Dynamic Games with Changing Partial Cooperation[1]

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Abstract. The game with partial cooperation with perfect information in extensive form is considered. The optimal solution PMS-vector in such a game has been proposed in [Petrosjan, 2000].

In our paper the characteristic functions are defined for each coalition $S\,(S \subset N)$ according to some unified principle (for example, the best response to Nash equilibrium), but they are not necessarily supper additive.

A new principle of optimal behavior in such a game is established, based on the nucleolus as optimality principle for the allocation of coalitional payoff. On the first part of this paper, we have made an assumption that once the player announced that he would take cooperative behavior and never change this announcement, namely, he could not leave the coalition.

Based on this assumption, we construct algorithm for the solution of the game. And in the second part in this paper, we try to eliminate this limitation and, so, we construct a new method to achieve the goal. Algorithm of PGN-value of this kind of a game is offered and the optimal trajectory is found. The existence and uniqueness of nucleolus leads to the existence and uniqueness of the new solution.

Keywords: Game with changing partial cooperation, nucleolus, Nash equilibrium, perfect information, PGN-value.

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1. Dynamic games with partial cooperation and the monotonous increase of player's coalition

1.1. Introduction

In recent years, the study on cooperative game has gone through the development from complete cooperation to partial cooperation. People have been already used to the so-called cooperative assumption in the research of the static or dynamic complete cooperative game, i.e. game rule agrees that players act to maximize payoff of the affiliated coalition. The player participates in the affiliated coalition throughout the game process, and it's decided that a coalition of players should not change after being formed in the whole game process. In the game defined in literature [Petrosjan, 1998] and [Ayoshin, 1998] any player may proceed with cooperative activity from a given stage instead of in the whole game process. To be specific, before the game starts each player must, independent of other players, point out a particular stage, to cooperate with other players and participate in the coalition of players who are ready to cooperate, but keep individual rational behavior and don't cooperate with any other player before this stage. Players are not permitted to alter the declared options in game process. Following the above rule, one player begins cooperation from the stage he has chosen in any case, no matter which concrete path is taken.

Literature [Petrosjan, 2000] weakens the conditions mentioned above. Different from the game defined in [Petrosjan, 1998] and [Ayoshin, 1998], the identical player might choose cooperation or non-cooperation respectively in different paths on the same stage according to [Petrosjan, 2000]. However, there's an obvious limitation to the definition of partial cooperation, i.e. player can not still change the declared choice in the game process, which is displayed in the demand of the monotonous increase of players' coalition. The existing optimal solution defined in the complete cooperative game is unable to get in the study of partial cooperative game based on non-cooperative games in finite extensive form. The limitation is obvious for [Petrosjan, 2000] to establish the optimal solution of game through defining *PMS*-vector, because the computation of *PMS*-vector bases the value of characteristic function of complete cooperative game on the value of 2-person zero sum game in special coalition, then assign payoff of the coalition with the distribution principle of Shapley value. The limitation of the above-mentioned *PMS*-vector can only be overcome through changing the structure principle of characteristic function in the coalition, but the change of structure principle of characteristic function will influence the choice of the optimal rule of behavior directly, namely, if the characteristic function established fails to satisfy super additivity, the application of Shapley vector will be nonsensical (because it is not individually reasonable at this moment), so *PMS*-vector loses the foundation of existence.

1.2. Notations and definition

Let Γ be a n-person non-cooperative game in finite extensive form with perfect information and without chance moves. Denote the set of players by $N = \{1, \ldots, n\}$.

Let $K(x_0)$ be the game tree with the origin x_0. According to the definition of a game in extensive form, on $K(x_0)$ there exists a partition $P_1, \dots, P_n, P_{n+1}$ of the set of game tree nodes, where $P_i (i \in N)$ is the set of decision points of player i, and P_{n+1} is the set of endpoints. The payoff of player i is specified by terminal real-valued functions $h_i : P_{n+1} \to R^1_+, i \in N$.

Definition 1. *$f_i : P_i \to \{0,1\}, i \in N$, is called a cooperative function of player i, if for a given path $\{x_0, \dots, x', x'', \dots, \bar{x}\}$, where $x' \in P_i$ and $\bar{x} \in P_{n+1}$, from $f_i(x'') = 1$it follows that $f_i(y) = 1$ for each $y \in P_i \cap \{x'', \dots, \bar{x}\}$. Player i keeps cooperative behavior when $f_i(x) = 1$ and plays individually when $f_i(x) = 0$.*

Definition 2. *$f = (f_1, \dots, f_n)$ is called a cooperative function of the game.*

Definition 3. *According to cooperative function $f = (f_1, \dots, f_n)$, players can cooperate or play individually in the switched game process, identical player may choose cooperation or non-cooperation in diffident paths of one stage of game process. The switched game is called a partial cooperative game $\Gamma_f(x_0)$. Function $f = (f_1, \dots, f_n)$ defines a special coalition structure on every node of the game tree $K(x_0)$.*

Definition 4. *Let $K_i = \{K(x^1), \dots, K(x^q)\}$ be a combination of non-intersecting subtrees of $K(x_0)$, with their origins $x^1, \dots, x^q$ being in P_i. The combination K_i is called the cooperative region of player i, i.e., player i pledges himself to proceed his cooperative behavior on the decision points in $K_i \cap P_i$. On the nodes in $P_i \backslash K_i$ player i plays individually.* Suppose that f has been defined and after several moves the game party came to a decision point x of player i. Assume that cooperative function of player i satisfies $f_i(x) = 1$. Consider the set

$$S^1_f(x) = \{j \in N \,|\exists y \in P_j \cap \{x_0, \dots, x\} : f_j(y) = 1\} \tag{1}$$

$S^1_f(x)$ consists of players who are ready to cooperate on x and the players who have made a move to cooperate before x. According to the definition of the cooperative function, players in $S^1_f(x)$ will continue to cooperate on every node of the subtree $K(x)$ with the initial node x.

Definition 5. *A subtree $K(x)$ is the trustiness region (TR) of player j if for every $y \in P_j \cap K(x), j \in N$, the cooperative function $f_j(y) = 1$.*

Hence,

$$S^2_f(x) = \left\{j \in N \backslash S^1_f \,| K(x) \text{ is TR of player } j \right\}. \tag{2}$$

$S^2_f(x)$ consists of the players who haven't taken cooperative action in the path $\{x_0, \dots, x\}$ but are going to cooperate on $K(x)$. Saying that player i proceeds the cooperative behavior on a node $x \in K(x_0)$, we mean that on x player i acts in the interests of the coalition

$$S_f(x) = S^1_f \cup S^2_f. \tag{3}$$

Players in $S_f(x)$ are not permitted to leave the coalition once the coalition is determined. The rest of the players in $N \backslash S_f(x)$ are considered as individual ones on x.

Since $S_f(x)$ is defined by the cooperative function f, the whole coalition structure

$$S_f(x), \{j_1\}, \{j_2\}, \ldots, \{j_{|N \setminus S_f(x)|}\} \tag{4}$$

is specified by f as well. The player set N_f in $\Gamma_f(x_0)$ consists of the subsets of the set N and are formed according to cooperative function f. The player set N_f is defined as follows. Take an arbitrary decision point x. Suppose that $x \in P_i$. Introduce

$$i_f(x) = \begin{cases} S_f(x), \text{if} f_i(x) = 1 \\ \{i\}, \text{if} f_i(x) = 0 \end{cases}. \tag{5}$$

Consider the set

$$I(S) = \{x \in K(x_0) \,|i_f(x) = S\}, \tag{6}$$

where $S \subset N$ is independent of x, i.e., for all $x \in I(S)$ the sets $i_f(x)$ coincide. The set (coalition) $S \subset N$ will be considered as *a player in game* $\Gamma_f(x_0)$ making decisions on the nodes $x \in I(S)$. The payoff of player S of $\Gamma_f(x_0)$ is defined as the sum of payoffs of player $i \in S$ on the endpoints of $K(x_0)$:

$$h_s(x) = \sum_{i \in S} h_i(x), x \in P_{n+1}, h_i(x) \geq 0, i \in N. \tag{7}$$

Obviously, player S in game $\Gamma_f(x_0)$ may only consist of one player in game Γ, i.e., we could find decision point $x \in P_i$, such that $i_f(x) = \{i\}$. It may also happen that the game $\Gamma_f(x_0)$ will be a one-player game (the set N_f consists of only one player N). This occurs in the case when $i_f(x) = N$ for all $x \in P_i, i \in N$. In the most complicated case the set N_f may consist of all subsets of the set N.

During the explanation we will often use the following notations. Assume that x is an arbitrary node. Let the set of immediate successors of x be $Z(x)$. Denote the decision making player on x, $x \in P_i, i \in N$ by $i(x) \in N$. The decision of player $i(x)$ on x leads to the node $\bar{x} \in Z(x)$. Finally, the rule c_f is determined by a cooperative function $f = (f_1, \ldots, f_n)$ if x is a decision point of player i, where

$$c_f(x) = \begin{cases} 1, \text{ if } f_i(x) = 1 \\ 0, \text{if} f_i(x) = 0 \end{cases}. \tag{8}$$

Suppose that the longest path of the tree $K(x_0)$ passes through T decision points. Introduce a partition of all nodes on $T+1$ sets $X_0, X_1, \ldots, X_t, \ldots, X_T$ and $X_T = \{x_0\}$, where X_t is composed of nodes which are reachable from x_0 after $T-t$ sequential moves. Denote decision points belonging to X_t by $x_t, t = 1, \ldots, T$.

1.3. The construction and algorithm of the optimal path

Consider the game $\Gamma_f(x_0)$. In this section we shall try to construct the solution concept for $\Gamma_f(x_0)$ which will lead to the construction of the corresponding optimal path. The optimal path is determined by means of backward induction, moving from the final nodes toward the initial one. The procedure is similar to the one used in the scheme of sub game–perfect Nash equilibrium construction, but the difference as follows is essential as well.

Let $K(x)$ belong to a cooperation region of player i. Then, on the endpoints of $K(x)$ we have to consider the payoffs of a coalition which includes player i instead of the payoffs of player i. By the Nash scheme the decisions of player i maximizing the payoff of coalition (in which he is included) can be easily determined with respect to $K(x)$. However, since the player i's payoff is not picked out from the coalition payoff, there occur difficulties on the decision points of player i between x and the root x_0, where player i plays individually. Therefore, the definition of players' payoffs corresponding to nodes where the individual behavior is replaced by the cooperative one is the main problem considered in the algorithm.

The initial stage. Consider the set P_{n+1} of endpoints. Since no player makes any move on P_{n+1}, the coalition structure on x and that on its immediate predecessor x_1,$x \in Z(x_1)$ are the same. On the node x_1 the given f specifies coalition structure $S_f(x_1), \{j_1\}, \ldots, \{j_{|N \backslash S_f(x_1)}\}$. The terminal payoffs $h_1(x), \ldots, h_n(x)$ in Γ specify the new payoff structure in $\Gamma_f(x_0)$ in correspondence with the coalition structure on x_1. Hence, the coalition $S_f(x_1)$ gets $\sum\limits_{i \in S_f(x_1)} h_i(x)$ on x and an individual player $j_k, k = 1, \ldots, |N \backslash S_f(x_1)|$, gets $h_{j_k}(x)$ on x.

Stage 1. Shift back from the endpoints x to their predecessors x_1. Consider an arbitrary taken x_1. If $c_f(x_1) = 1$, player $i(x_1)$ cooperates on x_1, from which it follows that $i(x_1)$ maximizes the payoff of the coalition $S_f(x_1)$ (thus he is playing in $\Gamma_f(x_0)$ as player $i_f(x) = S_f(x_1)$, $i_f(x) \in N_f$). We purpose him to select $\bar{x}_1 \in Z(x_1)$ from the condition

$$\max_{x \in z(x_1)} \sum_{i \in S_f(x_1)} h_i(x) = \sum_{i \in S_f(x_1)} h_i(\bar{x}_1). \tag{9}$$

In case $c_f(x_1) = 0$, player $i(x_1)$ maximizes the payoff of the coalition $i_f(x_1)$ consisting of himself only:

$$\max_{x \in Z(x_1)} h_{i(x_1)}(x) = h_{i(x_1)}(\bar{x}_1). \tag{10}$$

In the same way, we can construct trajectories starting from the arbitrary nodes on X_1. Therefore, instead of considering the terminal payoff function $h_i, i \in N$, on P_{n+1}, we may deal with payoff function $r_i^1 : X_1 \to R_+^1, i \in N$, on X_1 such that

$$r_i^1(x_1) = \begin{cases} h_i(\bar{x}_1), \text{ if } x_1 \notin P_{n+1}; \\ h_i(x_1), \text{if } x_1 \in P_{n+1}. \end{cases} \tag{11}$$

Stage t. Continue moving toward the tree root. Since the procedures on the further stages are the same, omitting explanation of every stage we deal with a stage t as an example of the general approach. Hence, suppose that we have reached a set of nodes X_t by continuing the moving on the game tree toward the origin x_0. Let

r_i^{t-1}-payoffs obtained on the stage $t-1$ for X_{t-1}. We don't deal with the endpoints belonging to $X_t \cap P_{n+1}$. Find the decisions of players on the set of non-terminal nodes $X_t \backslash P_{n+1}$. Let $Y(x_t) = Y_1(x_t) \cup Y_2(x_t)$, where

$$\begin{aligned} Y_1(x_t) &= \{x \in Z(x_t) | c_f(x_t) = 0 \text{ and } i(x_t) \in S_f(x)\} \\ Y_2(x_t) &= \{x \in Z(x_t) | c_f(x_t) = 1 \text{ and } S_f(x) \backslash S_f(x_t) \neq \emptyset\} \end{aligned} \tag{12}$$

If individually playing player $i(x_t)$ enters into multi-player coalition $S_f(y_{t-1})$ on $y_{t-1} \in Z(x_t)$, or coalition $S_f(x_t)$ gets a new member $i(x_t)$ on y_{t-1}, the coalition structure will be changed on y_{t-1}. For each node $x_t \in X_t$ we deal with two main cases.

1) Assume that $Y(x_t) = \emptyset$ for all $x_t \in X_t \backslash P_{n+1}$. In this case, the functions r_i^{t-1} specify the payoff obtained at the end of the game for each player $i(x_t)$, i.e., if the decision of player $i(x_t)$ leads to node $\bar{x}_t \in Z(x_t)$, then at the end of the game the coalition $S_f(x_t)$ will get $\sum\limits_{i \in S_f(x_t)} r_i^{t-1}(\bar{x}_t)$, and the payoff of individual player j_k will be $r_{j_k}^{t-1}(\bar{x}_t)$. Therefore, we can easily determine the nodes $\bar{x}_t$, where $\bar{x}_t \in Z(x_t)$ and $x_t \in X_t \backslash P_{n+1}$.

If $c_f(x_t) = 0$, then $\bar{x}_t$ has to satisfy

$$\max_{x \in Z(x_t)} r_{i(x_t)}^{t-1}(x) = r_{i(x_t)}^{t-1}(\bar{x}_t) \tag{13}$$

Now assume that $c_f(x_t) = 1$. By the definition of the cooperative function, the coalition $S_f(x_t)$ is included in coalition $S_f(x_{t-1})$ for each $x_{t-1} \in Z(x_t)$. Therefore, the coalitions $S_f(x_t)$ and $S_f(x_{t-1})$ coincide since $Y(x_t) = \emptyset$. Then since player $i(x_t)$ belongs to the coalition $S_f(x_t)$ on x_t, the node $\bar{x}_t$ has to satisfy

$$\max_{x \in Z(x_t)} \sum_{i \in S_f(x_t)} r_{i(x_t)}^{t-1}(x) = \sum_{i \in S_f(x_t)} r_{i(x_t)}^{t-1}(\bar{x}_t). \tag{14}$$

2) Now, suppose that there exists x_t such that the subset $Y(x_t)$ of nodes where the payoff of the coalition including player $i(x_t)$ is not defined by the functions r_i^{t-1}, is not empty. When $c_f(x_t) = 0$ our procedure did not define the payoff of the player $i(x_t)$. On the other hand, in the case of $c_f(x_t) = 1$, we have $S_f(x_{t-1}) \backslash S_f(x_t) \neq \emptyset$. Once $S_f(x_t) \subset S_f(x_{t-1})$, the payoff of the coalition $S_f(x_t)$ is included into the payoff of the coalition $S_f(x_{t-1})$ and, thus, is not defined either.

To construct a path on $K(x_t)$, it is necessary to define some imputation of payoff of coalition $S_f(y_{t-1})$ for each $y_{t-1} \in Y(x_t)$. We do it by considering an auxiliary cooperative game $G_f(y_{t-1}, S_f(y_{t-1}))$ on the subtree $K(y_{t-1})$ with the set of players $S_f(y_{t-1})$ and the characteristic function $v_f(y_{t-1}, R)$, $R \subset S_f(y_{t-1})$, for each $y_{t-1} \in Y(x_t)$. The payoff of the grand coalition in $G_f(y_{t-1}, S_f(y_{t-1}))$ is defined as

$$v_f(y_{t-1}, S_f(y_{t-1})) = \sum_{i \in S_f(y_{t-1})} r_i^{t-1}(y_{t-1}). \tag{15}$$

The explanation of the cooperative function construction will be provided in section 4.

Consider the nucleolus of the cooperative game

$$Nu^f(y_{t-1}) = (Nu^f_{k_1}(y_{t-1}), \dots, Nu^f_{k_{\left|S_f(y_{t-1})\right|}}(y_{t-1})), \tag{16}$$

where $\sum\limits_{j=1}^{|S_f(y_{t-1})|} Nu^f_{k_j}(y_{t-1}) = v_f(y_{t-1}, S_f(y_{t-1}))$ is taken as an optimal imputation of coalition $S_f(y_{t-1})$ payoff.

Note

$$PGN(y_{t-1}) = (PGN_1(y_{t-1}), \dots, PGN_n(y_{t-1})), \tag{17}$$

where $PGN_i(y_{t-1}) = Nu^f_i(y_{t-1})$, $i \in S_f(y_{t-1})$. Hence, the changed payoffs on X_{t-1} are specified by functions $\bar{r}^{t-1}_i : X_{t-1} \to R^1_+$, $i \in N$, such that for $x_{t-1} \in Z(x_t)$

$$\bar{r}^{t-1}_i(x_{t-1}) = \begin{cases} PGN_i(x_{t-1}), \text{ if } x_{t-1} \in Y(x_t) \text{ and } i \in S_f(x_{t-1}); \\ r^{t-1}_i(x_{t-1}), \text{otherwise.} \end{cases} \tag{18}$$

Suppose that $c_f(x_t) = 0$. Then player $i(x_t)$ chooses $\bar{x}_t \in Z(x_t)$ from the condition

$$\max_{x \in Z(x_t)} \bar{r}^{t-1}_{i(x_t)}(x) = \bar{r}^{t-1}_{i(x_t)}(\bar{x}_t). \tag{19}$$

If $c_f(x_t) = 1$, then player $i(x_t)$ cooperates on x_t with the coalition $S_f(x_t)$. Hence, $\bar{x}_t$ has to satisfy

$$\max_{x \in Z(x_t)} \sum_{i \in S_f(x_t)} \bar{r}^{t-1}_{i(x_t)}(x) = \sum_{i \in S_f(x_t)} \bar{r}^{t-1}_{i(x_t)}(\bar{x}_t). \tag{20}$$

Finally, we know the evolution of game on any subtree $K(x_t)$ since the decisions of players have been determined for every node $x_t \in X_t$. Hence, to construct the path on a subtree $K(x_{t+1})$ we have to consider just the decisions of player $i(x_{t+1})$. When $Y(x_t) \neq \emptyset$, the payoffs of players are different from those in the case of $Y(x_t) = \emptyset$. Define the payoffs on X_t by functions r^t_i: $X_t \to R^1_+$, $i \in N$, such that for $x_t \in X_t$ and $i \in N$

$$r^t_i(x_t) = \begin{cases} r^{t-1}_i(\bar{x}_t), \text{if } Y(x_t) = \emptyset; \\ \bar{r}^{t-1}_i(\bar{x}_t), \text{if } Y(x_t) \neq \emptyset; \\ h_i(x_t), \text{if } x_t \in P_{n+1}. \end{cases} \tag{21}$$

Definition 6. *We construct a path which is realized using the above algorithm if the cooperative function $f = (f_1, \dots, f_n)$ is given in Γ. Denote this path by $x(f)$ and call it optimal path of the partial cooperative game $\Gamma_f(x_0)$.*

Definition 7. *With the construction of the optimal path $x(f)$ we get the final payoffs $r^T(x_0)$ of players, we shall call it PGN-value of partial cooperative game $\Gamma_f(x_0)$.*

1.4. The concept of the best response to Nash equilibrium in cooperative subgames and construction of characteristic functions

In this section we establish the characteristic function $v_f(x,R)$, $R \subset S_f(x)$, of $G_f(x,S_f(x))$, $x \in Y(x_t)$. When constructing the optimal path in section 3 we define the behavior of players in each of its decision points. Such a fixed behavior regarded as a function of decision point is called a *strategy*. Denote n-tuple of strategies defined in section 3 by $\psi^*(\cdot) = (\psi_1^*(\cdot),\ldots,\psi_n^*(\cdot))$.

The cooperative game is constructed with the help of these strategies.

Consider the trace $\psi_x^*(\cdot) = (\psi_{1x}^*(\cdot),\ldots,\psi_{nx}^*(\cdot))$ of the n-tuple ψ^* in $K(x)$. For $i \notin S_f(x)$ fix the strategies $\psi_{ix}^*(\cdot)$ and consider the subgame $\bar{\Gamma}_f(x)$ of $\Gamma_f(x_0)$ in which the choice of player $i \notin S_f(x)$ is fixed according to $\psi_{ix}^*(\cdot)$ in his corresponding decision points. Thus, the subgame $\bar{\Gamma}_f(x)$ is a game among the players in the coalition $S_f(x)$.

Introduce the concept of the best response to Nash equilibrium. It is easy to know that deviation of the single player to Nash equilibrium will not cause the increase of individual payoff, however, the deviation of part of the coalition (including individual), or the whole coalition, will probably lead to the increase of coalition payoff.

Definition 7. *Suppose that the set of all players is M in subgame. The behavior that the deviation of part of the coalition R or the whole coalition R, maximizes the realized payoff of the coalition R when the players of $M\backslash R$ act on Nash equilibrium for any given coalition $R \subset M$, is called the best response to Nash equilibrium of coalition R.*

We will get the characteristic function of cooperative subgame $G(x,S_f(x))$ using the concept of the best response to Nash equilibrium. Firstly, we must find the Nash equilibrium of subgame $\bar{\Gamma}_f(x)$, then consider the best response to Nash equilibrium for every subcoalition $R \subset S_f(x)$, and take the realized maximum of payoff of coalition R which is obtained by the best response as the value $v(x,R)$ of characteristic function of R. We can decrease the number of possible equilibrium through assuming every player serves as a well-meaned restriction to the other players since the Nash equilibrium can be more than one. It can be shown that the payoff of every coalition R defined in such a way can not exceed $v(x,S_f(x)) = \sum\limits_{i\in S_f(x)} r_i^t(x)$. Using the above $v(x,R)$, one can construct the nucleolus $Nu^f(x)$ of cooperative subgame $G_f(x,S_f(x))$ and define PGN-value.

1.5 The example of PGN-value

Example 1. Consider a non-cooperative game Γ_1 with the tree $K(x_0)$ given in Figure 1 (broken line is excluded). The set of players is $N = \{1,2,3,4,5\}$. The player 1's decision points are x_0, x_9, x_{21}, player 2's -x_1, x_{12}, x_{23}, player 3's – x_4, x_{13}, x_{25}, player 4's-x_5, x_{16}, x_{26}, player 5's -x_8, x_{17}, x_{29}. Terminal payoffs are written vertically, with in every column the payoff of player 1 being the upper number, and so on.

Suppose that the cooperative function f^1 has the following form: $f_1^1(x_0) = f_1^1(x_{21}) = 0$, $f_1^1(x_9) = 1$, $f_2^1(x_1) = 0$, $f_2^1(x_{12}) = f_2^1(x_{23}) = 1$, $f_3^1(x_4) = f_3^1(x_{13}) =$

$f_3^1(x_{25}) = 0$, $f_4^1(x_5) = f_4^1(x_{16}) = f_4^1(x_{26}) = 0$, $f_5^1(x_{17}) = 1$, $f_5^1(x_8) = f_5^1(x_{29}) = 0$. Thus, the player set in the game $\Gamma_{f^1}(x_0)$ is $N_{f^1} = \{1,2,3,4,5,\{1,2\},\{1,2,5\}\}$. In $\Gamma_{f^1}(x_0)$ player 1 makes decision on x_0, x_{21}, player 2 – on x_1, player 3 – on x_4, x_{13}, x_{25}, player 4 – on x_5, x_{16}, x_{26}, player 5 – on x_8, x_{29}, player $\{1,2\}$ – on x_9, x_{23}, player $\{1,2,5\}$ – on x_{12}, x_{17}. The payoff of player $\{1,2\} \in N_f$ in $\Gamma_{f^1}(x_0)$ is defined as the sum of corresponding terminal payoffs of players 1 and 2 from N in Γ. The payoff of player $\{1,2,5\} \in N_f$ in $\Gamma_{f^1}(x_0)$ is defined as the sum of corresponding terminal payoffs of players 1, 2 and 5 from N in Γ.

Construct the optimal path of the partial cooperative game $\Gamma_{f^1}(x_0)$. The procedure of optimal path construction starts on the endpoints x_{19}, x_{20}, x_{30}, x_{31}. The coalition structure on x_{19}, x_{20} is the same as on x_{17}, i.e., $S_f(x_{17}) = \{1,2,5\}, \{3\}, \{4\}$. Hence, for players from N_{f^1}, the payoffs on x_{19} and x_{20} are given by triples $(8,4,5)$ and $(6,0,0)$ respectively, where the first component is the payoff of player $\{1,2,5\}$, the second one – player 3's and the third – player 4's. On x_{17}, player $\{1,2,5\}$ goes left to get 1+2+5=8. Hence, $r^1(x_{17}) = (1,2,4,5,5)^*$(notation* denotes transposition) and $r^1(x_{18}) = (1,1,1,4,1)^*$. Since $r_4^1(x_{17}) > r_4^1(x_{18})$, on x_{16} for player 4 it is optimal to go left to get 5. Hence, $r^2(x_{16}) = (1,2,4,5,5)^*$, $r^2(x_{15}) = (1,1,3,1,1)^*$.

Since $r_3^2(x_{16}) > r_3^2(x_{15})$, on x_{13} for player 3 it is optimal to go right to get 4. Hence, $r^3(x_{13}) = (1,2,4,5,5)^*$, $r^3(x_{14}) = (1,3,1,1,1)^*$. On x_{12} player 2 maximizes the payoff of the coalition $\{1,2,5\}$, and should go left to get $r_1^3(x_{13}) + r_2^3(x_{13}) + r_5^3(x_{13}) = 8$. Hence, $r^4(x_{12}) = (1,2,4,5,5)^*$. We get $r^4(x_{23}) = (2,1,3,4,5)^*$ similarly.

On x_9 player $\{1,2\}$ makes decision. But his share in the proposed payoff of $\{1,2,5\}$ is not known. Construct the cooperative subgame $G_{f^1}(x_{12}, S_{f^1}(x_{12}))$, $S_{f^1}(x_{12}) = \{1,2,5\}$, on the subtree $K(x_{12})$ with the initial node x_{12}. Fixing the above determined decisions of player 3 on x_{13} and 4 on x_{16}, we can define the characteristic function $v_{f^1}(x_{12}, R)$, $R \subset S_{f^1}(x_{12})$ with the concept of the best response to Nash equilibrium. The values of $v_{f^1}(x_{12}, R)$ are as the following: $v_{f^1}(x_{12}, \{1,2,5\}) = 8$, $v_{f^1}(x_{12}, \{1\}) = 1$, $v_{f^1}(x_{12}, \{2\}) = 3$, $v_{f^1}(x_{12}, \{5\}) = 1$,$v_{f^1}(x_{12}, \{1,2\}) = 4$, $v_{f^1}(x_{12}, \{1,5\}) = 2$, $v_{f^1}(x_{12}, \{2,5\}) = 7$. Thus, the nucleolus

$Nu^{f^1}(x_{12}) = (Nu_1^{f^1}(x_{12}), Nu_2^{f^1}(x_{12}), Nu_5^{f^1}(x_{12}))$ equals $(1, \frac{9}{2}, \frac{5}{2})^*$.

Hence, $PGN(x_{12}) = (1, \frac{9}{2}, \frac{5}{2})^*$, $\bar{r}^4(x_{12}) = (1, \frac{9}{2}, 4, 5, \frac{5}{2})^*$. It isn't necessary to construct cooperative subgame on subtree $K(x_{23})$ because node $x_{23} \notin Y(x_9) \neq \emptyset$. Thus, player $\{1,2\}$ should choose x_{12} to get payoff $\bar{r}_1^4(x_{12}) + \bar{r}_2^4(x_{12}) = 1 + \frac{9}{2} = \frac{11}{2}$ for maximizing his own payoff. Hence, $r^5(x_9) = (1, \frac{9}{2}, 4, 5, \frac{5}{2})^*$, $r^5(x_{22}) = (1,1,2,1,1)^*$. On x_{21} player 1 does not belong to coalition $\{1,2\}$ and plays individually. Since his share in the proposed payoff of $\{1,2\}$ is not known, construct the cooperative subgame $G_{f^1}(x_9, S_{f^1}(x_9))$, $S_{f^1}(x_9) = \{1,2\}$, on the subtree $K(x_9)$ with the initial node x_9. Fixing the above determined decisions of player 3 on x_{13}, x_{25}, 4 on x_{16}, x_{26}, and 5 on x_{17}, x_{29}, we can define the characteristic function $v_{f^1}(x_9, R), R \subset S_{f^1}(x_9)$. The values of $v_{f^1}(x_9, R)$ are as the following: $v_{f^1}(x_9, \{1,2\}) = \frac{11}{2}$, $v_{f^1}(x_9, \{1\}) = 2$, $v_{f^1}(x_9, \{2\}) = 1$. Thus, the nucleolus $Nu^{f^1}(x_9) = (Nu_1^{f^1}(x_9), Nu_2^{f^1}(x_9)) = (\frac{13}{4}, \frac{9}{4})^*$.

Hence, $PGN(x_9) = (\frac{13}{4}, \frac{9}{4})^*$, $\bar{r}^5(x_9) = (\frac{13}{4}, \frac{9}{4}, 4, 5, \frac{5}{2})^*$, $r^6(x_{21}) = (\frac{13}{4}, \frac{9}{4}, 4, 5, \frac{5}{2})^*$, $r^6(x_{10}) = (2,1,1,1,1)^*$. According to the given f^1, comparing $r_5^6(x_{21})$ and $r_5^6(x_{10})$, player 5 does not cooperate on x_8 and goes left to get $\frac{5}{2}$ for maximizing his own payoff. Hence, $r^7(x_8) = (\frac{13}{4}, \frac{9}{4}, 4, 5, \frac{5}{2})^*$, $r^7(x_7) = (1, 1, 1, 3, 1)^*$. Comparing $r_4^7(x_7)$ and $r_4^7(x_8)$, it is optimal for player 4 to go right to get 5. Hence, $r^8(x_5) = (\frac{13}{4}, \frac{9}{4}, 4, 5, \frac{5}{2})^*$, $r^8(x_6) = (1, 1, 3, 1, 1)^*$. On x_4 it is optimal for player 3 to go left to get 4. Hence, $r^9(x_4) = (\frac{13}{4}, \frac{9}{4}, 4, 5, \frac{5}{2})^*$, $r^9(x_3) = (1, 1, 1, 1, 1)^*$. According to the given f^1, player 2 does not cooperate on x_1 and go right to get $\frac{9}{4}$ for maximizing his own payoff. Hence, $r^{10}(x_1) = (\frac{13}{4}, \frac{9}{4}, 4, 5, \frac{5}{2})^*$, $r^{10}(x_2) = (2, 1, 1, 1, 1)^*$. Player 1 does not cooperate on x_0 and should go left to get $\frac{13}{4}$. Hence, $r^{11}(x_0) = (\frac{13}{4}, \frac{9}{4}, 4, 5, \frac{5}{2})^*$. Thus, for the game $\Gamma_{f^1}(x_0)$ the optimal path is

$$x(f^1) = \{x_0, x_1, x_4, x_5, x_8, x_{21}, x_9, x_{12}, x_{13}, x_{16}, x_{17}, x_{19}\}, \tag{22}$$

and PGN-value of $\Gamma_{f^1}(x_0)$ is

$$r^1(x_0) = (\frac{13}{4}, \frac{9}{4}, 4, 5, \frac{5}{2})^*. \tag{23}$$

Consider a complete cooperative game $G(x_0)$ constructed on the same tree $K(x_0)$ in Figure 1. The path $x^* = \{x_0, \ldots, x_{19}\}$ giving the maximal payoff to the grand coalition N in $G(x_0)$, coincides with the optimal path $x(f^1)$ of the game $\Gamma_{f^1}(x_0)$. The Shapley value of $G(x_0)$ is $Sh_(x_0) = \{\frac{13}{3}, \frac{10}{3}, \frac{10}{3}, 3, 3\}$. The optimal path using the algorithm of [Petrosjan, 2000] coincides with $x(f^1)$ of the game $\Gamma_{f^1}(x_0)$, but the *PMS*-value is $(\frac{8}{3}, \frac{8}{3}, 4, 5, \frac{8}{3})^*$.

Example 2. Consider game Γ_0 with the tree $K(x_0)$ given in Figure 1 composed of broken line part and bold part of Figure 1. The values of cooperative function f^0 are the same as those of f^1 of the game Γ_1. We can get the game tree $K(x_0)$ of example 2 in [Petrosjan, 2000] through changing the payoff on x_3 to $(1, 2, 1, 1, 1)^*$. We can get the same optimal path as $x(f^1)$ in $\Gamma_{f^0}(x_0)$ when the payoff on x_3 is $(1, 1, 1, 1, 1)^*$ in game Γ_0 according to [Petrosjan, 2000], the *PMS*-value is $(\frac{7}{2}, \frac{5}{2}, 4, 5, 2)^*$. Consider a complete cooperative game $G(x_0)$ in Γ_0. The path giving the maximal payoff to the grand coalition N in $G(x_0)$ still coincides with $x(f^1)$, and the Shapley value of $G(x_0)$ is

$Sh_{(x_0)} = \{\frac{13}{3}, \frac{10}{3}, \frac{10}{3}, 3, 3\}$. The optimal path of game $\Gamma_{f^0}(x_0)$ coincides with $x(f^1)$ when characteristic function is constructed using the concept of the best response to Nash equilibrium, PGN-value is $r^0(x_0) = (\frac{15}{4}, \frac{3}{2}, 4, 5, \frac{11}{4})^*$.

Example 3. Denote the new game which the value of cooperative function on x_1 for player 2 is changed into 1 in game Γ_0 by Γ_2, denote cooperative function by f^2. We can get the same optimal path as $x(f^1)$ in $\Gamma_{f^2}(x_0)$ according to [Petrosjan, 2000], the *PMS*-value is $(\frac{7}{2}, \frac{5}{2}, 4, 5, 2)^*$. Consider a complete cooperative game $G(x_0)$ in Γ_2. The path giving the maximal payoff to the grand coalition N in $G(x_0)$ still coincides with $x(f^1)$, and the Shapley value of $G(x_0)$ is $Sh(x_0) = \{\frac{13}{3}, \frac{10}{3}, \frac{10}{3}, 3, 3\}$. The optimal path of game$\Gamma_{f^2}(x_0)$coincides with $x(f^1)$when characteristic function is constructed using the concept of the best response to Nash equilibrium, PGN-value is $r^2(x_0) = (\frac{17}{8}, \frac{25}{8}, 4, 5, \frac{11}{4})^*$.

1.6. Conclusion

An algorithm of the optimal path and *PGN*-value for partial cooperative game in finite extensive form with perfect information are proposed to overcome the limitation of *PMS*-value in this paper. To be specific, the *PGN*-value proves that some players would make necessary sacrifice for the sake of constructing coalition under the supposition of giving up the extremely opposing behavior to other players.

Comparing the component of *PGN*-value, *PMS*-value and Shapley value for player 2 in the above three examples, we find that the payoff of player 2 according to *PGN*-value is less than those according to *PMS*-value and Shapley value, which shows that player 2 makes more sacrifice for the sake of urging other players to cooperate. At the same time we can avoid the limitation of the superadditivity which the characteristic function should satisfy using the nucleolus concept.

Besides, according to the above three examples and the solution, we know that all optimal paths are showing no difference although the *PGN*-value got by the algorithm in this paper is different from the *PMS*-value and Shapley value. We find that the corresponding components of *PGN*-value, *PMS*-value for player 3 and 4 outnumbering the Shapley value $Sh_(x_0)$ in complete cooperative game. In this way, *PGN*-value and *PMS*-value prove that the cooperation between player 3 and 4 is loose under the supposition of complete cooperation. The algorithm of the optimal path built in this article has introduced a more stable optimal solution in this paper, and the research will promote the construction of the system of partial cooperative game.

Changing the cooperative function f we get a class $\Gamma_F(x_0)$ of all partial cooperative game $\Gamma_f(x_0)$ which can be defined on $K(x_0)$. We can get more stable optimal solution if the optimal payoff of complete cooperative game is established with the help of the *PGN*-value of partial cooperative games $\Gamma_f(x_0) \in \Gamma_F(x_0)$. Summing up the above algorithm and basing on the existence and uniqueness of nucleolus of complete cooperative game we get:

Theorem. *The PGN-value of partial cooperative game in finite extensive form with perfect information exists and keeps unique.*

2. Dynamic games with partial cooperation and the free-changing structure of coalition

2.1. Introduction

In the game defined in [Petrosjan, 1998] and [Ayoshin, 1998], one player begins cooperation from the stage he has chosen in any case, no matter which concrete path is taken. [Petrosjan, 2000] weakens the conditions mentioned above. The identical player might choose cooperation or non-cooperation respectively in different paths on the same stage.

However, players are not permitted to alter the declared options in game process of [Petrosjan, 1998], [Ayoshin, 1998], [Petrosjan, 2000] which is displayed in the demand of the monotonous increase of players' coalition. The paper tries to cancel the limit through introduction of pivotal definition 10.

2.2. The basic model

Let Γ be a n-person non-cooperative game in finite extensive form with perfect information and without chance moves. Denote the set of players by $N = \{1, \ldots, n\}$. Let $K(x_0)$ be the game tree with the origin x_0. On $K(x_0)$ there exists a partition $P_1, \ldots, P_n, P_{n+1}$ of the set of game tree nodes, where $P_i (i \in N)$ is the set of decision points of player i, and P_{n+1} is the set of endpoints. The payoff of player i is specified by terminal real-valued functions $h_i : P_{n+1} \to R^1_+, i \in N$.

Definition 9. *According to cooperative function $f = (f_1, \ldots, f_n)$, players can cooperate or play individually in the switched game process, identical player may choose cooperation or non-cooperation in different paths of one stage of game process. The switched game is called a partial cooperative game $\Gamma_f(x_0)$. Function $f = (f_1, \ldots, f_n)$ defines a special coalition structure on every node of the game tree $K(x_0)$.*

Definition 10. *Suppose that f has been defined and after several moves the game party came to a decision point x of player i. Consider the set*

$$H_f(x) = \left\{ j \in N \;\middle|\; \begin{array}{l} f_j(y) = 1, y \text{ is player } j\text{'s nearest personal decision} \\ \text{point to } x \text{ on path } \{x_0, \ldots, x\}, y \in P_j \end{array} \right\}.$$

$H_f(x)$ consists of players who are ready to cooperate on x and the players who have made a move to cooperate before x and don't leave the coalition.

Definition 10 indicates that the players in coalition $H_f(x)$ still can leave the coalition after coalition $H_f(x)$ is formed on node x. The players from the set $N \backslash H_f(x)$ on x are considered as individual players. Since $H_f(x)$ is defined by the cooperative function f, the whole coalition structure

$$H_f(x), \{j_1\}, \{j_2\}, \ldots, \left\{j_{|N \backslash H_f(x)|}\right\}.$$

is specified by g as well on node x.

Now let's define a kind of partial-cooperative game $\Gamma_f(x_0)$ with perfect information in an extensive form. The game $\Gamma_f(x_0)$ is generated by the game Γ and the cooperative function f. The game tree of $\Gamma_f(x_0)$ coincides with the game tree of Γ. Take an arbitrary decision point x. Suppose that $x \in P_i$. Introduce

$$i_f(x) = \begin{cases} H_f(x), & \text{if} \quad f_i(x) = 1 \\ \{i\}, & \text{if} \quad f_i(x) = 0 \end{cases}.$$

The player set N_f in $\Gamma_f(x_0)$ consists of the subsets of the set N (player set in Γ). The player set N_f is denoted as

$$N_f = \{ i_f(x) \,| x \in K(x_0) \}.$$

The payoff of player S of $\Gamma_f(x_0)$ is defined as the sum of payoffs of player $i \in S$ on the endpoints of $K(x_0)$

$$h_s(x) = \sum_{i \in S} h_i(x), \quad x \in P_{n+1}, h_i(x) \geq 0, i \in N.$$

The rule c_f is determined by a cooperative function $f = (f_1, \ldots, f_n)$ if x is a decision point of player i, where

$$c_f(x) = \begin{cases} 1, & \text{if} \quad f_i(x) = 1 \\ 0, & \text{if} \quad f_i(x) = 0 \end{cases} .$$

The algorithm of construction of the optimal trajectory

The optimal path is determined by means of backward induction. The procedure is similar to the one used in the section 1.3.

The initial stage. The coalition structure on x and that on its immediate predecessor x_1, $x \in Z(x_1)$, are the same. On x_1 the given f specifies coalition structure $H_f(x), \{j_1\}, \{j_2\}, \ldots, \{j_{|N \backslash H_f(x)|}\}$. Hence, the coalition $H_f(x_1)$ gets $\sum\limits_{i \in H_f(x_1)} h_i(x)$ on x and an individual player $j_k, k = 1, \ldots, |N \backslash H_f(x_1)|$, gets $h_{j_k}(x)$ on x.

Stage 1. Shift back from the endpoints x to their predecessors. Consider an arbitrarily taken x_1.

If $c_f(x_1) = 1$, player $i(x_1)$ selects $\bar{x}_1 \in Z(x_1)$ from the condition

$$\max_{x \in z(x_1)} \sum_{i \in H_f(x_1)} h_i(x) = \sum_{i \in H_f(x_1)} h_i(\bar{x}_1).$$

In case $c_f(x_1) = 0$, player $i(x_1)$ maximizes the payoff of the coalition $i_f(x_1)$ consisting of himself only

$$\max_{x \in Z(x_1)} h_{i(x_1)}(x) = h_{i(x_1)}.(\bar{x}_1)$$

In the same way, we can construct trajectories starting from the arbitrary nodes on X_1. Therefore, instead of considering the terminal payoff function $h_i, i \in N$, on P_{n+1}, we may deal with payoff function $r_i^1 : X_1 \to R_+^1, i \in N$, on X_1 such that

$$r_i^1(x_1) = \begin{cases} h_i(\bar{x}_1), & \text{if} \qquad x_1 \notin P_{n+1} \\ h_i(x_1), & \text{if} x_1 \in P_{n+1} \end{cases} .$$

Stage t. Continue moving toward the tree root. For nodes $x_t \in X_t \backslash P_{n+1}$, consider set $Y(x_t) = \{x \in Z(x_t) | H_f(x) \neq H_f(x_t)\}$.

For each node $x_t \in X_t$ we deal with two main cases.

1) Assume that $Y(x_t) = \emptyset$ for all $x_t \in X_t \backslash P_{n+1}$. In this case, the functions r_i^{t-1} specify the payoff obtained at the end of the game for each player $i(x_t)$, i.e. if the decision of player $i(x_t)$ leads to node $\bar{x}_t \in Z(x_t)$, then at the end of the game the coalition $H_f(x_t)$ will get $\sum\limits_{i \in H_f(x_t)} r_i^{t-1}(\bar{x}_t)$, and the payoff of individual player j_k will be $r_{j_k}^{t-1}(\bar{x}_t)$. Therefore, we can easily determine the nodes $\bar{x}_t$, where $\bar{x}_t \in Z(x_t)$ and

$x_t \in X_t \backslash P_{n+1}$. If $c_f(x_t) = 0$, then $\bar{x}_t$ has to satisfy $\max\limits_{x \in Z(x_t)} r^{t-1}_{i(x_t)}(x) = r^{t-1}_{i(x_t)}(\bar{x}_t)$. Now assume that $c_f(x_t) = 1$. The coalitions $H_f(x_{t-1})$ and $H_f(x_t)$ coincide on $x_{t-1} \in Z(x_t)$ since $Y(x_t) = \emptyset$. Then since player $i(x_t)$ belongs to the coalition $H_f(x_t)$ on x_t, the node $\bar{x}_t$ has to satisfy $\max\limits_{x \in Z(x_t)} \sum\limits_{i \in H_f(x_t)} r^{t-1}_{i(x_t)}(x) = \sum\limits_{i \in H_f(x_t)} r^{t-1}_{i(x_t)}(\bar{x}_t)$.

2) Now, suppose $Y(x_t) \neq \emptyset$. To construct a path on $K(x_t)$, it is necessary to define some imputation of payoff of coalition $H_f(y_{t-1})$ for each $y_{t-1} \in Y(x_t)$. We do it by considering an auxiliary $|H_f(y_{t-1})|$-person cooperative game $G_f(y_{t-1}, H_f(y_{t-1}))$ on the subtree $K(y_{t-1})$ with the set of players $H_f(y_{t-1})$ and the characteristic function $v_f(y_{t-1}, R)$,$R \subset H_f(y_{t-1})$, for each $y_{t-1} \in Y(x_t)$. The explanation of the cooperative function $v_f(y_{t-1}, R)$ is provided in [Kuhn, 1953]. The payoff of the grand coalition $H_f(y_{t-1})$ in $G_f(y_{t-1}, H_f(y_{t-1}))$ is defined as $v_f(y_{t-1}, H_f(y_{t-1})) =$ $= \sum\limits_{i \in H_f(y_{t-1})} r^{t-1}_i(y_{t-1})$. Compute the nucleolus of game $G_f(y_{t-1}, H_f(y_{t-1}))$, $Nu^f(y_{t-1}) = (Nu^f_{k_1}(y_{t-1}), \ldots, Nu^f_{k_{|H_f(y_{t-1})|}}(y_{t-1}))$, where $\sum\limits_{j=1}^{|H_f(y_{t-1})|} Nu^f_{k_j}(y_{t-1}) =$ $= v_f(y_{t-1}, H_f(y_{t-1}))$ is taken as an optimal imputation of $G_f(y_{t-1}, H_f(y_{t-1}))$. Note $PGN(y_{t-1}) = (PGN_1\ (y_{t-1}), \ldots,\ PGN_n(y_{t-1}))$, where $PGN_i(y_{t-1}) = Nu^f_i(y_{t-1})$, $i \in H_f(y_{t-1})$. Hence, the changed payoffs on X_{t-1} are specified by functions $\bar{r}^{t-1}_i :$ $X_{t-1} \to R^1_+$, $i \in N$, such that for $x_{t-1} \in Z(x_t)$,

$$\bar{r}^{t-1}_i(x_{t-1}) = \begin{cases} PGN_i(x_{t-1}), \text{ if } \ x_{t-1} \in Y(x_t) \ \text{ and } i \in H_f(x_{t-1}) \\ r^{t-1}_i(x_{t-1}), \text{ otherwise} \end{cases},$$

if $c_f(x_t) = 0$, player $i(x_t)$ chooses $\bar{x}_t \in Z(x_t)$ from the condition

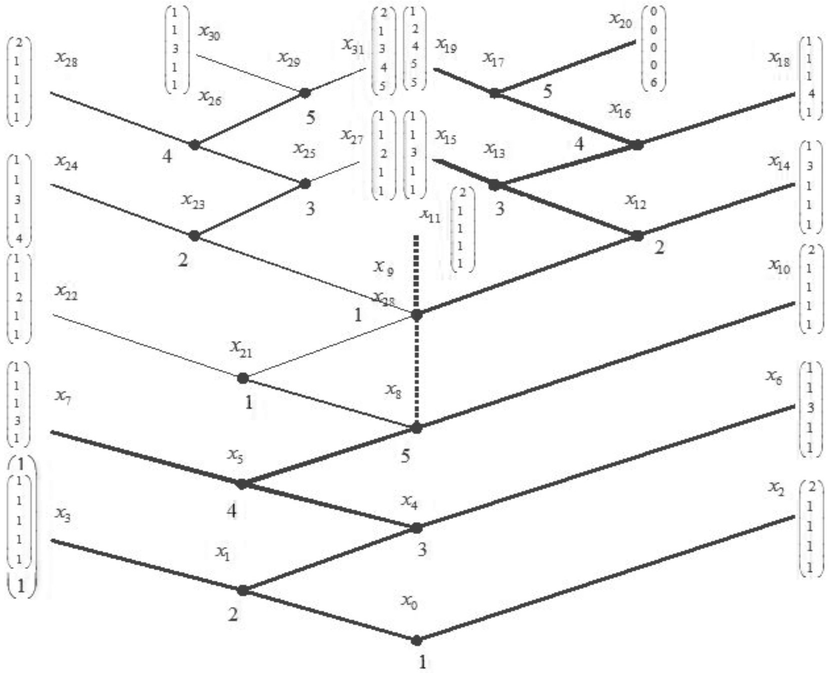
$$\max_{x \in Z(x_t)} \bar{r}^{t-1}_{i(x_t)}(x) = \bar{r}^{t-1}_{i(x_t)}(\bar{x}_t).$$

If $c_f(x_t) = 1$, $\bar{x}_t$ has to satisfy $\max\limits_{x \in Z(x_t)} \sum\limits_{i \in H_f(x_t)} \bar{r}^{t-1}_i(x) = \sum\limits_{i \in H_f(x_t)} \bar{r}^{t-1}_i(\bar{x}_t)$.

When $Y(x_t) \neq \emptyset$, the payoffs of players are different from those in the case of $Y(x_t) = \emptyset$. Define the payoffs on X_t by functions $r^t_i X_t \to R^1_+$, $i \in N$, such that for $x_t \in X_t$ and $i \in N$,

$$r^t_i(x_t) = \begin{cases} r^{t-1}_i(\bar{x}_t), & \text{if} \qquad Y(x_t) = \emptyset \\ \bar{r}^{t-1}_i(\bar{x}_t), & \text{if} \qquad Y(x_t) \neq \emptyset \\ h_i(x_t), & \text{if} x_t \in P_{n+1} \end{cases}.$$

By continuing moving backward on $K(x_0)$ toward the origin x_0, players' decisions on remaining sets X_τ,$\tau = t+1, \ldots, T$ are determined sequentially.



2.3 The example of PGN-value of the game with partial cooperation and the free-changing structure of coalition

Example. Consider a non-cooperative game Γ with the tree $K(x_0)$. We can get the game tree $K(x_0)$ of through changing the payoff on x_3 to $(1,2,1,1,1)^*$ and x_2 to $(1,1,1,1,1)^*$ in broken line part and bold part of picture. The set of players is $N = \{1,2,3,4,5\}$.

The player 1's decision points are x_0, x_9, player 2's – x_1, x_{12}, player 3's – x_4, x_{13}, player 4's – x_5, x_{16}, player 5's – x_8, x_{17}. Terminal payoffs are written vertically, with in every column the payoff of player 1 being the upper number, and so on.

Suppose that the cooperative function g has the following form: $f_1(x_0) = 1$, $f_1(x_9) = 0$, $f_2(x_1) = f_2(x_{12}) = 1$, $f_3(x_4) = f_3(x_{13}) = 0$,

$$f_4(x_5) = f_4(x_{16}) = 0, \quad f_5(x_8) = f_5(x_{17}) = 1.$$

Construct the optimal path of partial cooperative game $\Gamma_f(x_0)$ using above algorithm. Thus, we know $x(f) = \{x_0, x_1, x_4, x_5, x_8, x_9, x_{12}, x_{13}, x_{16}, x_{17}, x_{19}\}$, *PGN*-value of $\Gamma_f(x_0)$ is $r^{10}(x_0) = (\frac{7}{4}, \frac{15}{4}, 4, 5, \frac{5}{2})^*$.

2.4 Note

The changing process of coalition $H_f(\cdot)$ is as follows:$\{1\} \to \{1,2\} \to \{1,2,5\} \to \{2,5\}$. On x_1 players 1, 2 construct coalition $H_f(x_1) = \{1,2\}$ because player 2 obtains $\frac{15}{4}$ from coalition that is more than that when selecting x_3. On the other hand, player

1 can't select x_2 since he forecasts that he obtains $\frac{7}{4}$ after constructing coalition with player 2. Similarly we know coalition $\{1,2\}$ changes into $\{1,2,5\} = H_f(x_8)$ on x_8 because player 5 can obtain $\frac{5}{2}$ after joining coalition $\{1,2\}$, which is good for player 5. But on x_9 the coalition changes into $H_f(x_9) = \{2,5\}$ from $\{1,2,5\}$, because player 1 finds his share from coalition is less than that when leaving coalition. The case which coalition decreases occurs.

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Construction of Internal Time Consistent Optimality Principle

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Abstract. Many real conflicting situations occur on a given time interval and can be described by dynamic cooperative games. Most important properties in dynamic cooperation are time consistency and internal time consistency. This properties allow to preserve players cooperation during the whole time period.
In this paper we investigate possibilities of construction of new optimality principles with properties of time consistency and internal time consistency in case of dynamic hierarchical game.
As a basic model consider multistage cooperative game G with hierarchical $n+1$ player game Γ, played on each stage. We choose core as a solution concept in each stage game, with the use of characteristic function defined in multistage game as sum of stage characteristic functions. The corresponding optimality principle is defined and internal time consistent. Example of a game with such optimality principle is also considered.

Keywords: Dynamic cooperative games, optimality principle, time consistency, internal time consistency, hierarchical games, multistage games.

Introduction

Consider two-level hierarchical system. Structure of this system consists of the center A_0, which carries out control and has material and working resources in system and n subordinated "productive" centers $B_i,\ i=1,\dots,n$.

Suppose that the administrative center A_0 on the first level of hierarchy chooses $u=(u_1,\dots,u_n)$ from a set U, where U is the strategy set of A_0, and u_i is control, which influence on subordinate center $B_i,\ i=1,\dots,n$ and limits the possibilities of actions set of B_i. Center B_i is situated on the second level hierarchy. It chooses strategy $v_i \in V(u_i)$, where $V(u_i)$ – is strategy set of B_i, which is predetermined by a choice of A_0.

Thus the center A_0 has the right of the first move and can limit opportunities of the subordinated centers. The purpose of the center A_0 is to maximize the payoff $K_0(u, v_1(u) \ldots v_n(u))$ by choosing u, and centers B_i have their own purposes and aspire to get maximum of payoff $K_i(u_i, v_i)$ choosing v_i.

1. One-stage game

Consider $(n+1)$ person hierarchical game and denote it as Γ. The players of this game are administrative center A_0 and subordinated centers B_i.

Let player A_0 chooses value $u \in U$, where:

$$U = \{u = (u_1, \ldots, u_n), u_i \geq 0, u_i \in R^1, \sum_{i=1}^{n} u_i \leq b,\ b \geq 0,\ i = 1, \ldots, n\}, \tag{1}$$

here u_i are resources, which A_0 supplies to B_i.

Having the information of the choice $u \in U$ B_i chooses $x_i \in V_i(u_i)$, where:

$$V_i(u_i) = \{x_i \in R^1 : x_i a_i \leq u_i + \alpha_i,\ x_i \geq 0,\ i = 1, \ldots, n\}, \tag{2}$$

value x_i can be interpreted as production program of B_i, x_i is function of u_i, a_i are production coefficients, also they can be interpreted as prime cost, α_i are amounts of available resources of B_i, $V_i(u_i)$ is strategy set of the player B_i.

Define the payoff functions of the players. For the player A_0, the payoff function is equal to:

$$K_0(u, x_1(\cdot), \ldots, x_n(\cdot)) = \sum_{i=1}^{n} c_i x_i(u_i),\ i = 1, \ldots, n,$$

where c_i are utilities of the output of the player B_i for the administrative center A_0.

The payoff function of B_i is equal to:
$K_i(u, x_1(\cdot), \ldots, x_n(\cdot)) = d_i x_i(u_i),\ i = 1, \ldots, n$, where d_i are utilities of the output of B_i for B_i.

Assume that cooperation between centers is permitted. In this case for each coalition $S \subset N = \{A_0, B_1, ..B_n\}$ define its maximal guaranteed payoff $v(S)$ as follows:

$$v(S) = \begin{cases} \quad 0,\ if\ S = \{A_0\} \\ \min\limits_{u} \max\limits_{\substack{x_i \\ i \in S}} (\sum\limits_{i \in S} d_i x_i) = \sum\limits_{i \in S} \dfrac{d_i \alpha_i}{a_i},\ if\ A_0 \notin S \\ \max\limits_{u} \max\limits_{\substack{x_i \\ i \in S}} \min\limits_{\substack{x_j \\ j \in N \setminus S}} (\sum\limits_{i \in S} (c_i + d_i) x_i + \sum\limits_{j \in N \setminus S} c_j x_j) = \sum\limits_{i \in S} \dfrac{(c_i + d_i)(\alpha_i + \overline{u_i})}{a_i}, \\ \hfill if A_0 \in S \\ \max\limits_{u} \max\limits_{\substack{x_i \\ i \in S}} (\sum\limits_{i \in S} (c_i + d_i) x_i) = \sum\limits_{i=1}^{n} \dfrac{(c_i + d_i)(\alpha_i + \overline{u}_i)}{a_i},\ if\ S = \{N\}, \end{cases} \tag{3}$$

where $\overline{u}_i$ is defined in the following way. Let $\dfrac{c_i + d_i}{a_i} = K_i,\ i = 1, \ldots, n$, then

$$\overline{u}_i = \begin{cases} b, & if\ K_i = \max\limits_{j=1,\ldots,n} K_j \\ 0, & \forall i \neq j. \end{cases} \quad . \tag{4}$$

The necessity of introduction of this values can be confirmed with the following argumentation. From (2) we have $x_i \leq \dfrac{(\overline{u}_i + \alpha_i)}{a_i}$. Take expression for $v(N)$ and substitute $\dfrac{(\overline{u}_i + \alpha_i)}{a_i}$ instead of x_i and get $v(N) = \sum\limits_{i=1}^{n} \frac{(c_i+d_i)(\alpha_i+\overline{u}_i)}{a_i}$.

2. Multistage game

Describe now the multistage cooperative game G. Suppose that on the stage $k-1, k = \overline{1,m}$ a stage game Γ^{k-1} is realized with characteristic function $v^{k-1}(S)$ and optimality principle C^{k-1}. Let $\gamma^{k-1} = (\gamma_0^{k-1}, \ldots, \gamma_n^{k-1})$ is the imputation from the optimality principle C^{k-1}, and γ_0^{k-1} is component of imputation γ^{k-1} for center A_0, γ_i^{k-1} are the components for centers $B_i, i = \overline{1,n}$.

It is assumed, that administrative center A_0 can employ i's own component γ_0^{k-1} of imputation γ^{k-1} to purchase resources and to its own needs, and also on each stage any of centers B_i can dispose its own components γ_i^{k-1} of imputation γ^{k-1} to improve the production coefficients, purchase personal resources and to its own needs. Hence, on the next stage k parameters of our model are changed and we get new characteristic function $v^k(S)$ and optimality principle C^k.

Suppose, that on stage $k-1$ each player assigns some part of his payoff to general system evolution. Denote this part as p.

On each stage parameters of our model are changed in accordance with following formulas:

$$b^k = b^{k-1} + \frac{p\gamma_0^{k-1}}{s_0}, \qquad \alpha_i^k = \alpha_1^{k-1} + \frac{p\gamma_i^{k-1}}{s_i},\ i = 1, \ldots, n, \tag{5}$$

where $s_i,\ i = 1, \ldots, n$ are cost of resources for each center.

Using formulas (5) we can construct new characteristic function $v^k(S)$ for a stage game Γ^k:

$$beginarrayl\ S = \{A_0\}, \qquad v^k(S) = 0;$$

$$\begin{aligned} A_0 \notin S, &\qquad v^k(S) = v^{k-1}(S) + \sum_{i\in S} \frac{d_i\alpha_i^{k-1}}{a_i}\frac{p\gamma_i^{k-1}}{s_i}; \\ A_0 \in S, &\qquad v^k(S) = v^{k-1}(S) + \frac{(c_1+d_1)}{a_1}\left(\frac{p\gamma_1^{k-1}}{s_1} + \frac{p\gamma_0^{k-1}}{s_0}\right) + \sum_{i\in S}\frac{(c_i+d_i)}{a_i}\frac{p\gamma_i^{k-1}}{s_i}; \end{aligned}$$

$$S = \{N\}, v^k(S) = v^{k-1}(S) + \frac{(c_1+d_1)}{a_1}\left(\frac{p\gamma_1^{k-1}}{s_1} + \frac{p\gamma_0^{k-1}}{s_0}\right) + \sum_{i=2}^{n}\frac{(c_i+d_i)}{a_i}\frac{p\gamma_i^{k-1}}{s_i}. \tag{6}$$

Choose core as an optimality principle for a stage game and one can see that the imputation $\gamma^k = (\gamma_0^k, \gamma_1^k, \dots, \gamma_n^k)$ with following components:

$$\gamma_0^k = \frac{(c_1 + d_1)}{a_1} b^k = \frac{(c_1 + d_1)}{a_1} \left(b^{k-1} + \frac{p\gamma_0^{k-1}}{s_0} \right);$$

$$\gamma_i^k = \frac{(c_i + d_i)}{a_i} \alpha_i^{k-1} = \frac{(c_i + d_i)}{a_i} \left(\alpha_i^{k-1} + \frac{p\gamma_i^{k-1}}{s_i} \right),\ i = 1, \dots, n$$

will belong to the core C^k for a stage game Γ^k. Formulas (6) define the dynamics of the characteristic function evolution:

$$v^k(S) = F_S(v^{k-1}(S), \gamma^{k-1}), \gamma^{k-1} \in C^{k-1},\ k = 1, \dots, m. \tag{7}$$

Since v^k is characteristic function in the game Γ^k, then for F_S we should have:

$$F_{S_1 \cup S_2}(v^{k-1}(S_1 \cup S_2), \gamma^{k-1}) \geq F_{S_1}(v^{k-1}(S_1), \gamma^{k-1}) + F_{S_2}(v^{k-1}(S_2), \gamma^{k-1}),$$

for coalitions $S_1,\ S_2 \subset N,\ S_1 \cap S_2 = \emptyset,\ \ F_\emptyset(v^{k-1}(\emptyset), \gamma^{k-1}) = 0$. Define a new characteristic function $\overline{v}(S)$ for multistage game G as:

$$\overline{v}(S) = \sum_{k=0}^{m} v^k(S),\ \text{for all } S \subset N. \tag{8}$$

This function is also superadditive.

For the multistage game G, with hierarchical game Γ^k, played on each stage, using expression (8) and (6) construct characteristic function. Formulas for the characteristic function $\overline{v}(S)$ are following:

$$\begin{aligned}
&S = \{A_0\}, \qquad \overline{v}(S) = \textstyle\sum\limits_{k=1}^{m} v^k(S) = 0; A_0 \notin S,\\
&\overline{v}(S) = \sum_{k=1}^{m} [v^{k-1}(S) + \sum_{i \in S} \frac{d_i \alpha_i^{k-1}}{a_i} \frac{p\gamma_i^{k-1}}{s_i}] = \left[m(v^0(S)) + \sum_{k=1}^{m-1} \left[\sum_{i \in S} \frac{d_i^k}{a_i} \frac{p\gamma^k}{s_i^k} \right] \right];\\
&A_0 \in S,\\
&\overline{v}(S) = \textstyle\sum\limits_{k=1}^{m} \left[v^{k-1}(S) + \dfrac{(c_1 + d_1)}{a_1} \left(\dfrac{p\gamma_1^{k-1}}{s_1} + \dfrac{p\gamma_0^{k-1}}{s_0} \right) + \displaystyle\sum_{i \in S} \frac{(c_i + d_i)}{a_i} \frac{p\gamma_i^{k-1}}{s_i} \right] =\\
&\qquad = \left[m(v^0(S)) + \sum_{k=1}^{m-1} \left[\frac{(c_1 + d_1)}{a_1} p \left(\frac{\gamma_0^k}{s_0^k} + \frac{\gamma_i^k}{s_1^k} \right) + \sum_{i \in S} \frac{(c_i + d_i)}{a_i} \frac{p\gamma_i^k}{s_i^k} \right] \right];\\
&S = \{N\},\\
&\overline{v}(S) = \textstyle\sum\limits_{k=1}^{m} \left[v^{k-1}(S) + \dfrac{(c_1 + d_1)}{a_1} \left(\dfrac{p\gamma_1^{k-1}}{s_1} + \dfrac{p\gamma_0^{k-1}}{s_0} \right) + \displaystyle\sum_{i=2}^{n} \frac{(c_i + d_i)}{a_i} \frac{p\gamma_i^{k-1}}{s_i} \right] =\\
&= \left[m(v^0(S)) + \sum_{k=1}^{m-1} \left[\frac{(c_1 + d_1)}{a_1} p \left(\frac{\gamma_0^k}{s_0^k} + \frac{\gamma_1^k}{s_1^k} \right) + \sum_{i=2}^{n} \frac{(c_i + d_i)}{a_i} \frac{p\gamma_i^k}{s_i^k} \right] \right].
\end{aligned} \tag{10}$$

Construct the core for the multistage game G with characteristic function $\overline{v}(S)$ and denote this core as $\overline{C}(\gamma)$, since the core is generated by characteristic function $\overline{v}(S)$ which according to (10) is generated by γ. It can be easily seen that the imputation $\overline{\gamma} = (\overline{\gamma}_0, \overline{\gamma}_1, \ldots, \overline{\gamma}_n)$ with components

$$\overline{\gamma}_0 = \sum_{k=0}^{m} \frac{(c_1+d_1)}{a_1} b^k, \ \overline{\gamma}_i = \sum_{k=0}^{m} \frac{(c_i+d_i)}{a_i} \alpha^k, \ i = 1, \ldots, n, \text{ belongs to the core } \overline{C}.$$

Definition 1. *Let* $\overline{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_i, \ldots, \gamma_n) \in \overline{C}$. *Vector sequence* $\beta = \beta_0, \beta_1, \ldots, \beta_m$, $(\beta_l = \beta_{0l}, \ldots, \beta_{nl})$, *such that*

$$\gamma_i = \sum_{l=1}^{m} \beta_{il}, \ \beta_{il} \geq 0, \tag{11}$$

is called imputation distribution procedure (IDP) (see [Filar, 2000]).

Let $\hat{G}^k$ be the subgame of multistage game G, starting from the stage k and lasts to the end of the game G.

Definition 2. *Following vector sequence* $\beta = \beta_0, \ldots, \beta_k, \ldots$, $(\beta_l = \beta_{0l}, \ldots, \beta_{nl})$ *we will call time consistent imputation distribution procedure, if*

$$\gamma_i = \sum_{l=1}^{m} \beta_{il}, \quad i \in N, \tag{12}$$

$$\hat{\gamma}_i^k = \sum_{l=k}^{m} \beta_{il}, \quad i \in N, \hat{\gamma}^0 = \overline{\gamma}, \tag{13}$$

$\hat{\gamma}^k = (\hat{\gamma}_0^k, \hat{\gamma}_1^k, \ldots, \hat{\gamma}_n^k) \in \overline{C}^k(\gamma)$, *where* $\overline{C}^k(\gamma)$ *is optimality principle in subgame* $\hat{G}^k$, *generated by* γ *(see [Filar, 2000]).*

One can interpret imputation distribution procedure in the following way: β_{ik} is the payment to player i on stage k in the multistage game G, or it is payment on the first stage of the subgame $\hat{G}^k$. In the game G each player has payoff γ_i as i-th component of the optimal imputation $\overline{\gamma} \in \overline{C}$ $(i = 0, 1, \ldots, n)$.

Definition 3. *Optimality principle* $\overline{C}(\gamma)$ *is time consistent in game* G, *if for each* $\overline{\gamma} \in \overline{C}(\gamma)$ *there exist IDP* $\beta = (\beta_0, \beta_1, \ldots, \beta_m)$, $\beta_k \geq 0$, $k = 0, 1, 2, \ldots, m$, *such as*

$$\overline{\gamma} = \sum_{l=1}^{m} \beta_l, \quad \hat{\gamma}^k = \sum_{l=k}^{m} \beta_l \in \overline{C}^k(\gamma), \hat{\gamma}^0 = \overline{\gamma},$$

where $\overline{C}^k(\gamma)$ *is optimality principle generated by* γ *in the subgame* $\hat{G}^k$.

Definition 4. *Optimality principle* $\overline{C}(\gamma)$ *is called internal time consistent in the game* G*, if it is time consistent, and for each* $\overline{\gamma} \in \overline{C}(\gamma)$ *we have*

$$(\hat{\overline{\gamma}}^k = \sum_{l \geq k} \xi^l), \quad \hat{\overline{\gamma}}^k \in \overline{C}^k(\gamma),\ k = 0, 1, \ldots, m,$$

where $\overline{C}^k(\gamma)$ *is optimality principle in subgame* $\hat{G}^k$ *of the multistage game* G*, generated by* $\gamma = \{\gamma^k\}_{k=0}^{k=m}$ *(see [Filar, 2000]).*

Definition 5. *Imputation* $\overline{\gamma} \in \overline{C}(\gamma)$ *is called internal time consistent, if condition* $\hat{\overline{\gamma}}^k \in \overline{C}^k(\gamma),\ k = 0, \ldots, m$ *is realized.*

Remark. Not every imputation γ can be presented in the following way:

$$\gamma = \sum_{k=0}^{m} \gamma^k,\ \gamma^k \in C^k. \tag{14}$$

In our case imputation $\overline{\gamma} \in \overline{C}(\gamma)$ of the multistage game G can be presented in form (14):

$$\overline{\gamma}_0 = \sum_{k=0}^{m} \gamma_0^{'k}, \quad \overline{\gamma}_i = \sum_{k=0}^{m} \gamma_i^{'k},\ i = 1 \ldots, n, \quad \gamma^{'k} \in C^k,\ k = 0, 1, \ldots, m.$$

Denote as $\tilde{\overline{C}}$ optimality principle where all imputations can be represented in form (14), $\tilde{\overline{\gamma}}$ is imputation from the optimality principle $\tilde{\overline{C}}$, $\hat{\gamma}^k$ is imputation in optimality principle $\tilde{\overline{C}}^k$ of subgame $\hat{G}^k, k = l, \ldots, m$.

Each component of the imputation $\tilde{\overline{\gamma}} = (\widetilde{\gamma_0}, \widetilde{\gamma_1}, \ldots, \widetilde{\gamma_n})$ on stage k can be represented as

$$\tilde{\gamma}_0^k = \frac{(c_1 + d_1)}{a_1} b^k,\ \tilde{\gamma}_i^k = \frac{(c_1 + d_1)}{a_1} \alpha^k,\ i = 1, \ldots, n.$$

Hence, each imputation $\tilde{\gamma}^k$, which can be presented in form (14), belongs to optimality principle C^k of the game Γ^k. Let all players on each stage distribute their intermediate payoff in according with the optimality principle, chosen on the first stage.

In our case imputation distribution procedure is constructed in the following way: β_{ik} is equal to $\tilde{\gamma}_i^k$ and β_{ik} for the imputation $\tilde{\overline{\gamma}}$ is time consistent, because conditions of definition 2 are realized.

3. Construction of internal time consistent optimality principle in multistage cooperative game

Consider possibility of construction internal time consistent optimality principle. It is a difficult process, and one possible way is to construct the intersection of optimality principles for multistage games, generated by different imputations. Describe the process.

Consider multistage game G and suppose that on each stage hierarchical game Γ is played. This game Γ has the same structure as in introduction section. Let $v^1(S)$ is characteristic function and C^1 is core of the game Γ^1, played on the first stage. Core is multiple optimality principle, hence, on the second stage, as in the previous section, we can construct the set of characteristic functions and the set of corresponding optimality principles in the following way. Take some finite set of imputations from the core C^1 and denote this set as $\Delta = \{\xi_1^1, \xi_2^1, \ldots, \xi_p^1\}$. Use the set of imputations Δ on stage two to construct the set of characteristic functions $v_j^2(S;\ \xi_j^1)$, $j = 1, \ldots, p$ and the set of corresponding optimality principles C_j^2. Then on the second stage of the multistage game G we have the set of games G_j with two stages. On the third stage of the multistage game G we get the set of characteristic functions $v_j^3(S;\ \xi_j^3)$ and corresponding set of optimality principles C_j^3.Continue the process in the similar way, on stage k, for each game G_j^k, we constructed characteristic functions $v_j^k(S; \xi_j^k)$ and corresponding optimality principles $C_j^k,\ j = 1, \ldots, p$.

After m stages we have the set of multistage games G_j. Every game has its own characteristic function

$$\overline{v}_j(S; \xi_j) = \sum_{k=1}^{m} v_j^k(S;\ \xi_j^k),\ j = 1, \ldots, p, \quad where \quad \xi_j^k = (\xi_{j0}^k, \ldots, \xi_{jn}^k). \tag{15}$$

For each characteristic function $\overline{v}_j(S; \xi_j)$ construct corresponding optimality principle $\overline{C}_j$ for every multistage game $G_j,\ j = 1, \ldots, p$. Consider the intersection of optimality principles $\overline{C}_j$ and denote it as M:

$$M = \bigcap_{j=1}^{p} \overline{C}_j(\xi_j). \tag{16}$$

Suppose that $M \neq \emptyset$. Define as $\widehat{\xi}_j^k = \sum_{l \geq k} \xi_j^l$. The main question is, if subset Δ belongs to intersection M or not. If Δ, then we can say, that for every imputation $\xi_j \in \Delta,\ 1 \leq j \leq p$, condition $\widehat{\xi}_l^k \in \overline{C}^k(\xi_j), 1 \leq j \leq p$ is realized, and so subset Δ is internal time consistent optimality principle. But it is difficult to check condition $\Delta \subset M$, because we must check that imputation ξ_j belongs to every optimality principle from the intersection M.

Consider new function $w(S)$: $w(S) = \max_j \overline{v}_j(S, \xi_j)$.

Function $w(S, \xi_j)$ can be not superadditive. Construct Δ as a finite set of

$$\{\xi_j = (\xi_{j_0}, \xi_{j_1}, \ldots, \xi_{j_n}) :\ \sum_{i \in S} \xi_{j_i} \geq w(S),\ S \subset N, \sum_{i=1}^{n} \xi_{j_i} = w(N)\}. \tag{17}$$

Then $\Delta \subset M$ and condition $\widehat{\xi}_l^k \in \overline{C}^k(\xi_j),\ 1 \leq j \leq p$ is realized (if $\Delta \neq \emptyset$ and $M \neq \emptyset$).

4. Example

Consider cooperative game Γ with three players. Let the values of characteristic function be as following: $v(1) = v(2) = v(3) = 1, v(1,2) = v(1,3) = v(2,3) = 5, v(1,2,3)=7$.

As in the previous sections, construct multistage game G. In our example we consider four stages game. We use two types of imputations: proportional imputation $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ and Shapley value $\Phi = (\phi_1, \phi_2, \phi_3)$. Using that imputations we get two optimality principles for multistage game, denote as $\overline{C}(\Phi)$ optimality principle, generated by Shapley value and denote as $\overline{C}(\gamma)$ optimality principle, generated by proportional imputation γ.

Construct the set L; (see (17)) which is an intersection of optimality principles $\overline{C}(\gamma)$ and $\overline{C}(\Phi)$. Then we take the imputation $\xi \in L$, and construct for ξ imputation distribution procedure β and a new optimality principle, generated by the new imputations $\hat{\xi}$ such as $\hat{\xi}_i^k = \beta_{ik},\ i = 0, 1, \ldots, n, k = 0, \ldots, m$ in stages games.

As a result, we get that new optimal principle $\overline{C}(\hat{\xi})$, which is practically congruent with the intersection L. To construct the graph of the intersection L we use (0-1) reduced form for each multistage game. Coordinations of vertexes of the optimality principle $\overline{C}(\Phi)$ for multistage game G_1 generated by Shapley value are:

$$\begin{gathered}(0.5000000012, 0.2499999994, 0.2499999994),\\ (0.2499999994, 0.5000000012, 0.2499999994),\\ (0.2499999994, 0.2499999994, 0.5000000012).\end{gathered}$$

Coordinations of vertexes of the optimality principle $\overline{C}(\gamma)$ for multistage game G_2 generated by proportional imputation are:

$$\begin{gathered}(0.4999999924, 0.2500000038, 0.2500000038),\\ (0.2500000038, 0.4999999950, 0.2500000012),\\ (0.2500000012, 0.2500000038, 0.4999999950).\end{gathered}$$

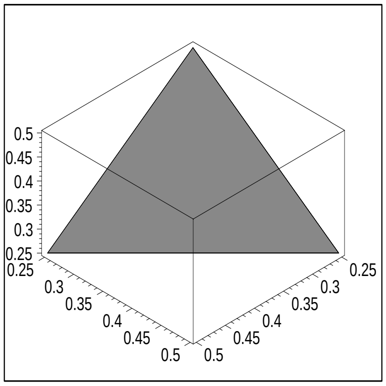
The vertexes of the intersection L have following coordinates:

$$\begin{gathered}(0.5000000012, 0.2499999994, 0.2499999994),\\ (0.2499999994, 0.5000000012, 0.2499999994),\\ (0.2499999994, 0.2499999994, 0.5000000012).\end{gathered}$$

Coordinates of the vertexes of the optimality principle $\overline{C}(\hat{\xi})$ for the new multistage game G_3 generated by imputation $\hat{\xi}$ are:

$$\begin{gathered}(0.5000000050, 0.2499999988, 0.2499999962),\\ (0.2499999962, 0.5000000076, 0.2499999962),\\ (0.2499999962, 0.2499999988, 0.5000000050).\end{gathered}$$

In our example we get that imputation $\xi \in L$ generates new optimality principle $\overline{C}(\hat{\xi})$ for the multistage game. For this optimality principle $\overline{C}(\hat{\xi})$ we have that imputation $\hat{\xi} \in L$ also belongs to optimality principle $\overline{C}(\hat{\xi})$, so conditions of definition 4 are realized and the intersection L is internal time consistent optimality principle.



Fig, 1: Intersection of optimality principles of multistage games.

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Abstract. Clean Development Mechanism (CDM) is a newly adopted scheme to give incentives to emission reduction projects in developing countries under Kyoto Protocol. We consider its implication under the demads for the products produced by firms engaging in CDM project are interrelated. In particular, we try to give examples where an adoption of a CDM project by one firm enhances the incentive of other firms to follow. What we found is that the condition for this to take place is rather stringent, indicating that the external push may be desirable for one to promote CDM activities in these situations.

Keywords: Kyoto Protocol, Clean Development Mechanism, related goods oligopoly, complements.

Introduction

CDM is a scheme introduced in Kyoto Protocol (1997) as the first attempt to convert emission reduction in developing country (and, hence, without an assigned limit) into the amount of emission toward fulfillment of assignment on the part of developed countries (more exactly, signatory of Kyoto Protocol). The scheme is more complicated than a mere subsidy scheme for emission reduction, and contains immense conceptual difficulty, which made some people dubious of the functioning of this mechanism (see [Bohm and Carlen, 2009] for example). Actually, after years

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[5] This work was supported by the French Foundation for Fundamental Researches under grants No.99-01-00146 and 96-15-96245.

of trial and errors, number of registered projects surpassed 400 (as of 2006) and now, inclusive of proposed projects (called ones in the pipeline), expected credits may reach 2 billion tones, (as of 2007) which may be already sufficient to fill the gap between demand and supply in the upcoming emission trade scheme under Kyoto Protocol (2008–2012) according to some speculation. (For a general overview of the current state of CDM, see [Capoor and Ambrosi, 2007] for instance.) Many developing countries were rather skeptical of the mechanism when it was proposed, but now they seem to find more interests in this mechanism and the category of eligible "projects" as CDM projects tend to be expanded (to include "program" CDM and further "policy" CDM or "sector" CDM is proposed). By contrast, some parties start criticizing complicated procedure of CDM as a burden, and propose to replace the mechanism by simpler schemes. In short, even though CDM seems to have launched successfully, but there remains a room for farther controversy.

One issue we have raised concerning CDM is the baseline methodologies [Imai and Akita, 2001]. Then we analyze the same issue for a private firm operating in an imperfectly competitive industry [Imai, Akita, and Niizawa, 2008]. In this paper, we again consider a private firm in an imperfectly competitive industry, but now our focus is on the incentive to undertake CDM projects for firms whose decisions are related through market demands. In particular, our interests are on if an early adoption of a CDM project by some firm affects the incentives of other firms to do the same, and if so, positively or negatively. The answer turns out to be very simple that under the most modes of oligopolistic competition the effect is negative. This itself has a significant meaning in terms of policy implication and we shall investigate this issue in depth in another opportunity.

Here, we shall pursue the possibility for this effect to be positive. Not surprisingly, this is the case if firms' demands are positively related, i.e. goods are complements. In this case, there is a positive externality across firms, and there one firm's enhancement of own demand or production level tends to raise demand for goods produced by other firms and so there would be an incentive to follow the suit.

Below, by means of an example, we show a case where adoption of CDM projects gives momentum for others to do so, which we call "CDM domino", although actual domino (CDM is adopted by firm, one after another) can occur under very stringent assumptions. We chose this way because such an example would exemplify the effect beautifully, and we can discuss welfare implication as well as some issues concerning mechanism design options in CDM.

1. Related Goods Oligopoly

We shall consider a group of firms 1 through n where relations among demands $q_i (i = 1, n)$ are characterized by the complement relations. Specifically, given prices $p_i (i = 1, n)$, the demand for good i is given by
$q_i = 1 - p_i + b \sum_{j \neq i} p_j (= 1 - (1 - b) p_i + bP$, where $P = \sum_j p_j)$ with $|b| < 1/n$.

Each firm has the unit costs $c_i(< 1)$ with $0 < c_1 \leq c_2 \leq ... \leq c_n$. For the sake of simplicity, we assume that there is no fixed costs (of production). (In the concluding remark we discuss the case where fixed cost may have some relevance.)

Firms decide their prices given the demand schedule which is determined by other firms' prices as well as its own, so as to maximize its profits: $p_i(1-p_i+b\sum\limits_{j\neq i} p_j)$ which yields the best response function:

$p_i^* = \frac{(1+b\Sigma p_j+c_i)}{2} = \frac{1+c_i+bP_{-i}}{2}$(where $P_{-i} = \sum\limits_{j\neq i} p_j$) which in turn yields $P = \frac{n}{2}+\frac{C}{2}+\frac{b(n-1)}{2}P$ (where $C = \Sigma c_j$), or

$$P = \frac{1}{2\left\{1-\frac{b(n-1)}{2}\right\}}[n+C] = \frac{n+C}{2-b\,(n-1)}$$

Thus,

$$P_{-i} = P - p_i = \frac{n+C}{2-b\,(n-1)} - p_i,$$

$$p_i = \frac{1+c_i+b\frac{n+C}{2-b(n-1)}}{2} - \frac{b}{2}p_i.$$

And, hence, the Nash equilibrium is

$$p_i = \frac{1+c_i+b\frac{n+C}{2-(n-1)b}}{2\left(1+\frac{b}{2}\right)} = \frac{1+c_i+\frac{b(n+C)}{2-b(n-1)}}{2+b}.$$

With equilibrium output level

$$q_i = 1-(1-b)\,\frac{1+C_i+\frac{b(n+C)}{2(n-1)b}}{2+b} + \frac{n+C}{2-b\,(n-1)} =$$
$$= 1-\frac{1-b}{2+b} + \frac{(n+C)}{2-(n-1)\,b} - \frac{(1-b)}{2+b}c_i,$$

and using notation

$$B = b\left(\frac{1+C}{2-(n-1)\,b}\right),$$

the equilibrium profits are

$$\pi_i = \left(\frac{1+c_i+b\left(\frac{1+C}{2-(n-1)b}\right)}{2+b} - c_i\right)\times$$
$$\times\left(1+b\frac{1+C}{2-(n-1)\,b} - (1+b)\,\frac{1+c_i+b\left(\frac{1+b}{2-(n-1)b}\right)}{2+b}\right) =$$

$$\begin{aligned}
&= \left(\frac{1-c_i+B}{2+b} - c_i\right)\left(1+B-(1+b)\,\frac{1+c_2+B}{2+b}\right) = \\
&= \left(\frac{1+c_i+B}{2+b} - c_i + \frac{B}{2+b}\right)\left(1 - \frac{1+b}{2+b}\,(1+c_i) + B\left(1-\frac{1+b}{2+b}\right)\right) = \\
&= \left(\frac{1}{2+b} - \frac{1+b}{2+b}c_i + \frac{B}{2+b}\right)^2 = \\
&= \frac{1}{(2+b)^2}\left(1-(1+b)\,c_i + \frac{b}{2-(n-1)\,b}c_i + \frac{b\,(2+C_{-i})}{(2-(n-1)\,b)}\right)^2 = \\
&= \frac{1}{(2+b)^2}\left(1-\left\{1+b-\frac{b}{2-(n-1)\,b}\right\}c_i + \frac{b\,(1+C_{-i})}{2-(n-1)\,b}\right),
\end{aligned}$$

where $C_{-i} = \sum\limits_{j\neq i} c_j$.

Next, consider a CDM project requiring investment costs for firm i, $I \geq 0$ which results in a reduction of unit costs to $c_i' \,(< c_i)$. A typical project which would produce this type of cost change would be energy switch projects. With an investment in equipments, energy source is switched to the one with less emission (with more or less costly fuel price). If it is less costly, then it must be the case that the increase in equilibrium profits due to the introduction of the CDM projects should be less than I, in order to meet the additionality requirement.

From above, we obtain that the change in profits of firm i when its unit cost shifts from c$_i$ to c$_i$' given the firms' costs $c_{-i} = (c_1, c_2, ..., c_{i-1}, c_{i+1}, ..., c_n)$ is given by

$$\begin{aligned}
&(\pi_i\,(c_i, C_{-i}) - \pi_i\,(c_i', C_{-i}))\,(2+b)^2 = \\
&= \frac{2-(n-2)\,b-(n-1)\,b^2}{2-(n-1)\,b}\,(c_i - c_i' + c_i') = \\
&= 2\,(c_i - c_i')\left\{1 + \frac{b\,(1+C_{-i})}{2-(n-1)\,b}\right\} = \\
&= (c_i - c_i')\left\{-2 - \frac{b + bC_i - \left(2-(n-2)\,b-(n-1)\,b^2\right)(c_i + c_i')}{2-(n-1)\,b}\right\},
\end{aligned}$$

and its sign can be confirmed by

$$-4 + 2\,(n-1)\,b - b - bC_i + \left(2-(n-2)\,b-(n-1)\,b^2\right)(c_i + c_i') < 0.$$

Now, suppose that conditions

$$\Delta\pi_i\,(c_i, \varepsilon, C_{-i}) > I > \Delta\pi_{i+1}\,(c_{i+1}, \varepsilon, C_{-i+1})$$

are met where

$$\begin{aligned}
&\varepsilon = c_i - c_i'; \quad \Delta\pi_i\,(c_1, \varepsilon, C_{-1}) = (\pi_i\,(c_i, C_{-i}) - \pi_i\,(c_i - \varepsilon, C_{-i}))\,; \\
&k = \frac{-\varepsilon^2 b}{(2+b)^2\,(2-(n-1)\,b)}; \quad c_{i+1} = c_i + k.
\end{aligned}$$

Further, assume that firms make decisions sequentially in the order of their suffixes. Then we obtain the sequential adoption of CDM projects by n firms. Obviously, this is a sort of a forced realization of CDM domino. Our question is if this could take place even without the condition on ordering.

One easy answer is that firms are myopic. That is, when making decision on adoption of a CDM project, firms take other firms' decisions as given. However, appealing to irrationality on the side of firms may not be a very attractive story. In fact, if firms are given a chance to make this decision simultaneously, rational firms would decide to adopt the project as it is one of the Nash equilibria. So, we can rephrase our question in the following form: if this could occur with rational firms even if firms have a chance to make decisions simultaneously (and sequentially). To answer this question affirmatively, we shall appeal to the possibility of incomplete information on the side of firms.

2. Incomplete Information

In order to produce a more plausible example of the CDM domino, we introduce a noise to the investment costs of each firm. This could be justified by the presence of the so, called capacity building costs, i.e. firms may lack employees equipped with sufficient knowledge, experiences, and capability to adapt to new machines or new methods of production, and this cost may not be observable to outsiders.

Let us write: $\Pi_i\left(c_i, c_{-i}\right) = \tilde{\Pi}_i\left(c_i, C_{-i}\right)$ and $c_i - c'_i = \varepsilon$.

We assume that each firm's investment cost could be high or low, and accordingly we call them type H and type L for each firm. In particular we assume that for each firm i type L:

$$\begin{aligned} &\tilde{\Pi}_i c'_i, (n-1)\, c_i - (i-1) - \tilde{\Pi}_i\left(c_i, (n-1)\, c_i - (i-1)\, \varepsilon\right) > I_i > \\ &\qquad > \tilde{\Pi}_i\left(c'_i, (n-1)\, c_i - (i-2)\, \varepsilon\right) - \tilde{\Pi}_i\left(c_i, (n-1)\, c_i - (i-2)\, \varepsilon\right); \end{aligned}$$

and for type H:

$$\tilde{\Pi}_i \left(c'_i, (n-1)\, c'_j - \tilde{\Pi}_i \left(c_i, (n-1)\, c'_j \right) < I'_j \right)$$

Each firm i suspects firm $j\, (\neq i)'\, s$ type to be L with a probability η_j, so that

$$\eta_j \left\{ \tilde{\Pi}_i\left(c'_i, (n-1)\, c_i - (i-1)\, \varepsilon\right) - \tilde{\Pi}_i\left(c_i, (n-1)\, c_i - (i-1)\, \varepsilon\right) \right\} + (1-\eta_j)$$

$$\left\{ \tilde{\Pi}_i\left(c'_i, (n-1)\, c_i - (i-2)\, \varepsilon\right) - \tilde{\Pi}_i\left(c_i, (n-1)\, c_i - (i-2)\, \varepsilon\right) \right\} < I'_j.$$

Under these suppositions, we consider a game played in n discrete periods. Each firm can make a (irreversible) decision to adopt a CDM project, at each period. As direct consequences of the above assumptions, we obtain the following properties:

1. Every firm of type L adopted the project simultaneously is not a Bayesian equilibrium.

2. For firm 1 of type L, adoption at the inception in the sequential decision problem does not result in a loss while for firm $j > 1$, adoption of the project yields expected loss.

3. Given firms 1 through $i - 1$ adopt the project, type L of firm I does not incur a loss by the adoption of the project, in the next period, while other firm $j > 1$, it suffers from an expected loss.

4. Given n periods, firm I of type L should adopt the project as soon as it finds the project profitable.

Theorem 1. *The following strategy profile with a belief system forms a perfect Bayesian equilibrium. If all the types are L, then CDM domino takes place. (i) Each type H never adopts the project. (ii) Type L of firm i adopts the project if and only if already $i - 1$ or more firms have adopted the project. (iii) Belief on firm j is given by $\mu\_\{j\}$ until period j, and after period j, the belief on firm j to be of type L is set at 0, while for all firms who have adopted the project, belief becomes 1.*

Proof.

The proof is almost straightforward due to the assumptions made. (i) is optimal by the assumption we made. (ii) is optimal given belief, i.e. given m firms adopting the project at period i with $m < i - 1$ and firm i having not doing so yet, then given the belief in (iii), for firm i (and firm j with $j >$ i) no adoption is optimal.(Note that type L of firm j $< m$ who has not adopted yet would do so (and is optimal to do so), but other firms believe that this firm is of type H. In some literature, the property of belief given in (iii) that the possibility of being type L vanished at one time ressurects by adoption is untenable. Here since adopted firm never acts again, so power of this criticism may be weak if valid.)

3. A simple dynamic model

As another alternative, we may provide a simple dynamic model incorporating learning and spillover effect with respect to capacity building. Now, suppose firm $i's$ costs of investment I_i decreases both with time and the number of firms already adopted the projects. Time is continuous and the rate of decline in investment costs given number of firms adopted the project is $\mu >$. Firm $i's$ investment cost at t when m firms have adopted the project $\delta^m e^{-rt}\left(I_{i0} + I_i e^{-\mu_m t}\right)$ where $I_{i0} > 0,\ 0 < \delta < 1$ and m is the number of firms adopted the project and $0 < \mu_0 < \mu_1 < \cdots < \mu_m$, we assum that $\delta^m I_{j0} < \int_0^\infty \Pi_m\left(c'_1, \cdots, c'_{m-1}, c'_m, c_{m+1}, \cdots, c_n\right) e^{-rt} dt < \delta^{m-1} I_{i0}$ with $i = m$ to make sure that the firms except for I may not have an incentive to adopt.

Consider the last firm's problem, provided that the last firm is firm n, and firm $1 \cdots\ n - 1$ adopted the project in that order (at t_i for firm i).

Firm n chooses t_n to maximize

$$\int_{t_{n-1}}^{t_n} e^{-rt}\pi_n\left(c_1', \cdots, c_{n-1}', c_n\right)dt + \int_{t_n}^{\infty} e^{-rt}\pi_n\left(c_1', \cdots, c_n'\right)dt$$
$$- e^{-rtn}\left(\delta^{n-1}I_{n0} + e^{-\sum\limits_{i=1}^{n}\mu_i(t_i - t_{i-1})}I_n\right)(with\ t_0 = 0).$$

The first order condition yields

$$e^{-rtn}\left(\Pi_n\left(c_1', \cdots, c_n'\right) - \Pi_n\left(c_1', \cdots, c_{n-1}', c_n\right)\right) =$$
$$= re^{-rtn}\left(\delta^{n-1}I_{n0} + e^{\overset{n}{-\Sigma}\mu_i(t_i - t_{i-1})}_{-i=1}I_n\right) + \mu_n e^{-rtn} \times e^{-\sum\limits_{i=1}^{n}\mu_i(t_i - t_{i-1})}I_n.$$

Rearranging, we have

$$\frac{\left(\Pi_n\left(c_1', \cdots, c_n'\right) - \Pi_n\left(c_1', \cdots, c_{n-1}', c_n\right)\right) - \delta^{n-1}I_{n0}}{(r + \mu_n)I_n} = e^{-\sum\limits_{i=1}^{n}\mu_i(t_i - t_{i-1})}$$

or

$$t_n^* = t_{n-1} - \sum_{i=1}^{n-1}\frac{\mu_i}{\mu_n}(t_i - t_{i-1}) -$$
$$- \log\left(\frac{\Pi_n\left(c_1', \cdots, c_n'\right) - \Pi_n\left(\left(c_1', \cdots, c_{n-1}', c_n\right)\right) - \delta^{n-1}I_{n0}}{(r + \mu_n)I_n}\right) =$$
$$= \sum_{i=1}^{n-1}\left(\frac{\mu_{i+1} - \mu_i}{\mu_n}\right)t_i -$$
$$- \log\left(\frac{\Pi_n\left(c_1', \cdots, c_n'\right) - \Pi_n\left(\left(c_1', \cdots, c_{n-1}', c_n\right)\right) - \delta^{n-1}I_{n0}}{(r + \mu_n)I_n}\right).$$

From this, we obtain that

$$\frac{dt_n^*}{dt_i} = \frac{\mu_{i+1} - \mu_i}{\mu_n} > 0\,(\text{but } < 1),\ \text{for } i \leqq n - 1.$$

Next, we consider firm $i's$ problem with $i < n$.

Firm i maximizes

$$\int_{t_{i-1}}^{t_i} e^{-rt}\Pi_i\left(c_1', \cdots, c_{n-1}', c_n\right)dt + \sum_{j=1}^{n}\int_{t_j}^{t_{j+1}} e^{-rt}\Pi\left(c_1', \cdots, c_j', c_{j+1}, \cdots, c_n\right)dt -$$
$$- e^{-rt_i}\left(\delta^{i-1}I_{i0} + e^{-\sum\limits_{j=1}^{i}\mu_i(t_i - t_{i-1})}I_i\right)$$

(with $t_j = t_j^*$ for $i < j \leqq n$ and $t_{n+1} = \infty$).

Usually this condition is not so tractable. When $n = 2$, the payoff for firm1 becomes:

$$\int_0^{t_i} e^{-rt}\pi_1(c,c)\,dt + \int_{t_1}^{t_2(t_1)} e^{-rt}\Pi_1(c',c)\,dt + \int_{t_2(t_1)}^{\infty} e^{-rt}\Pi_1(c',c')\,dt -$$

$$-e^{-rt_1}\left(I_{10} + e^{-\mu_1 t_1} I_1\right) = 0,$$

and the first order condition is

$$\begin{aligned} &e^{-rt_1}\left(\Pi_1(c',c) - \Pi_1(c,c)\right) + \frac{dt_2(t_1)}{dt_1} e^{-rt_2(t_1)}\left(\Pi_1(c',c') - \Pi_1(c',c)\right) = \\ &= e^{-rt_1}\left(rI_{10} + (r+\mu_1)\, e^{-\mu_1 t} I_1\right). \end{aligned}$$

Writing $t_2(t_1) = \left(\frac{\mu_2-\mu_1}{\mu_2}\right) t_1 - A$ and $\frac{dt_2(t_1)}{dt_2} = 1 - \frac{\mu_1}{\mu_2}$, we have

$$\frac{dt_2(t_1)}{dt_1} e^{-rt_2(t_1)} = \left(1 - \frac{\mu_1}{\mu_2}\right) e^{-r\left\{\left(1-\frac{\mu_1}{\mu_2}\right)t_1 - A\right\}},$$

and so

$$[\Pi,(c',c)] - rI_{10} - (r+\mu_1)\, I_1 e^{-\mu_1 t_1} + \frac{e^{rA}}{\mu_2}\left(\Pi_1(c',c') - \Pi_1(c',c)\right) e^{\frac{r\mu_1}{\mu_2} t_1} = 0$$

determines t_1^* (and, hence, $t_2^* = t_2 = t_2(t_1^*)$, the detail of which depends upon μ_1, μ_2 and r.) To obtain a tractable solution, we may adapt the assumption that the complementation is unilateral. I.e. adoption of the firm i affects the cost of firm $j > i$, but not vice versa. (In terms of demand functions, p_1 affects demand of firm j but p_j does not affect firm i's demand for $i < j$.)

Under this assumption, for $i > j$, t_j does not enter into the expression for profits of firm i. Thus, the first order condition becomes like that one for the n-th firm. I.e. writing

$$\begin{aligned} &c_1^i = \left\{\Pi_i(c_1', \cdots, c_i', c_{i+1}, \cdots, c_n) - \Pi_i\left(c_1', \cdots, c_{i-1}', c_i, \cdots, c_n\right)\right\}, \\ &c_2^i = r\delta^{i-1} I_{i0}, \end{aligned}$$

the first order condition for firm i becomes

$$\mu_i t_i = -\log\left(c_1^i - c_2^i\right) + \log(r+\mu_i)\, I_i + \sum_{j=1}^{i-1}(\mu_{j+1} - \mu_j)\, t_j.$$

In particular, $\frac{dt_i}{dt_{i-1}} = \frac{\mu_i - \mu_{i-1}}{\mu_i} > 0$.

4. Discussion

4.1. Baseline Methodology

Earlier we compared several baseline setting methods in their effects on incentives and overall performances. The CDM credits in this paper are computed through the method which we called the "ex post" baseline. That is, the credits are computed as the gap between the emission level with old and new technology of production provided that the output level is given by the actually observed output level.

One alternative method is to define credits as the gap between the actual emission level and the level obtained when the old technology is used with the output level forecast ex ante under that situation. In particular, under the stationary environment, forecast output level could be given by the actual output level observed before the project and so before the new technology is introduced. These two methods could give rise to quite distinctive incentives to the firm when output level is controlled by the form itself (see [Imai and Akita, 2003] and [Laurikka, 2002]). Also see [Fischer, 2005]. We adopted the ex post method in the main part because it is the chiefly utilized method in reality probablly both for its practicality and intuitive appealingness, in such CDM projects that emission level can be broken down into the output level and emission coefficients. However, there is a history that initially COP and the EB considered the ex ante method as one alternative, and possibly more legitimate method [cf. Marrakech Accord, 2001].

In the static version of the above model, we assumed that the "effective" marginal costs decrease as a result of an adoption of a CDM project. This is based upon several presumptions, but at least the ex post method contributes to the reduction of the effective marginal costs because for each unit produced, revenue from the sales of an additional reduction (compared to the old technology) brings a reduction of marginal costs (inclusive of the proceeds of credits). By contrast, under the ex ante baseline, credits decreases with the output (after CDM). Thus, an adoption of a CDM project works to increases the effective marginal costs.

Since a CDM project is adopted only if the firm expects a positive return from the engagement, it may be natural to think that the firm's effective marginal costs are still lower than the pre-CDM level even under the ex ante baseline. In that case, our argument above continues to hold although the range for which the assumption is valid may be narrower. However, in the extreme, one may imagine a case where this effective marginal costs go up as a whole as a consequence of the adoption of a CDM project (while the profitability of the project is assured by a decline in fixed costs, whose presence we had not assumed in our earlier model). In this case, the story is completely reversed. Under complementarity in demands CDM by one firm would induce a contraction of its output (due to ex ante baseline) which reduces the incentive for other firms to follow the suit (due to complementarity in demand). Rather the industry where firms' demands are characterized by the substitution relation becomes the suitable case for CDM domino.

4.2. Additionality

Another issue frequently raised related to CDM is the issue of additionality. Again consider the static model in the main text. The additionality constraint would be naturally given by that the profit difference without CDM credits is not sufficient to induce the firm to adopt the project. Letting ρ be the (expected) price of credits, this constraint can be expressed as $\pi_i(c'_1, ..., c'_i+\rho, c_{i+1}, ..., c_n) > I_i$ where the opportunity costs of investment are supposedly included in I_i. We have implicitly assumed that this constraint was met. As a matter of fact, this constraint only affects the decision over if this project is accepted or not, and does not affect the level of profits. In this sense, this constraint does not directly affect arguments for the CDM domino. However, once one takes into consideration the possibillity of the timing when the calculation of profits is made on which additionaly test is conducted, we get some insights on the importance of timing in CDM. For instance, suppose that the firm may adopt the project simultaneously or independently. If the additionality test compares the profits before the adoption and the profits the firm would earn when the firm adopts the project by itself, then there could be the case where the adoption by one firm alone is not profitable without credits, but if many firms adopt together, the project is profitable even without the credits. This provides an incentive for firms with sufficient foresight to adopt earlier expecting that other firms to adopt later, and to this effect all the firms may adopt at an earlier stage.

4.3. Double Counting

A related issue is that of double counting. Admittedly, externality among firm's demands does not raise a serious issue concerning double counting because this is not related to emission accounting or baseline itself. However, theoretically one could argue that the ensuing adoption of a CDM project upon an adoption by one firm may indicate that the firm adopting earlier may claim a part of subsequent emission reduction for itself as the fact of the adoption is attributable to the decision made by the first firm and, hence, may enlarge its project boundary. In some marginal situation, such enlargement would be crucial to induce the firm to employ the CDM project (provided that such enlargement does not affect the adoption decision of the firms adopting later). These issues can be oversome by a suitably packaged program CDM.

4.4. Contactual Form

We assumed that firms maximize their profits (inclusive of credits). The exact incentives of firms vary depending upon the details of the contract which the firm has with the contractual partners if any.

In the case of unilateral CDM, our profit maximization assumption holds without question. With a contractwith a buyer of the credits at the spot price with a fixed quantity without a penalty on non-delivery, or fixed price with a buyer's guarantee to purchase whole quantity delivered, our assumption is still valid. Otherwise, an obvious modification is necessary.

4.5. Welfare & Environment

It is well recognized that the Kyoto mechanism workes given the quota levied on each or any unit in terms of GHG emission.

And this implies that any reduction brought about by the CDM project would be used up by those units which emission level is bound by the quota. (There is some exceptional case where emission credits are bought by some entity which intentionally let the credit retire, or some entity banks (i.e. somes) for the usage for an indefinite future although this possibility is explicitly restricted by Marrakech Accord) to a certain extent.

Thus emission reduction under CDM project is merely a replacement of emission reduction somewherelse, where reduction costs are supposedly much higher. In this sense, the sheer effect of CDM would be the costs saved relative to other opportunities which would be given by the gap between the value of credits (evaluated at the emission price given by the emission trade market of Kyoto mechanism or EU-ETS, minus the actual costs, which in turn is the gross profits of CDM project.

Another effect of CDM often raised is the promotion of technology transfer. Generally speaking, a free transfer of profit creating technology is unlikely to take place. However, through several channels, like learning by watching or communications through employed workers, this may be realized.

The example developed in this paper is apparently extreme and making gevite convinient assumption, and, thus, not suited for empirical purpose, live as indicating a way to incorporate those factors like technology transfer and capacity building through CDM, we claim this is a really pioneering attempt.

Acknowlegments

The authors are grateful for the financial aid from the MEXT Grants-in-Aidfor Scientific Research 18310031.

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Monopoly, Diversification through Adjacent Technologies, and Market Structure

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Abstract. The theoretical literature on technological competition has been mostly concerned with various aspects of innovative activity in a single market. By contrast, this paper studies the adoption of a sequence of product innovations in two markets characterized by a common technology base, and illustrates the effects of technological rivalry and preemption. Under a perfect information scenario, it is shown in a two incumbent model that if the innovation is drastic (total replacement of the old product), under certain conditions the fear of being preempted by the entrant forces the firms to diversify their product lines by adopting the innovations across each otits's markets. On the ot hand, with non-drastic innovation (partial replacement of the old product), it is more likely for the firms to diversify in their own product lines.

Out of a class of equilibria characterized under non-drastic innovation, one is optimal in which innovations are adopted in the firms' own markets. In the Pareto inferior equilibria, the firms either adopt innovations in each other's market so that incumbency changes hands or jointly adopt both innovations in two separate product lines. Perfect Bayesian equilibria are characterized under an asymmetric information scenario where one of the firms is assumed to have complete information about the relevant costs of adopting an innovation in a separate product line. If the priors are based on pessimism, it is more often subject to exploitation by the informed firm leading to pooling equilibrium, while optimistism more often leads to diversification and to a competitive market structure in both product lines under a separating equilibrium.

In all the cases considered, both innovations are adopted, and in most cases they are adopted by the high cost entrant. The former is socially desirable, but the latter is not. More competitiveness necessarily implies wasteful expenditure by the high cost firm. Lack of competitiveness and technological rivalry, on the other hand, imply that maximum product diversity may not be achieved.

Keywords: Tehnological rivalry, preemption, adoption of innovations, upgrading.

Introduction

Almost all product innovations in recent times have been part of a sequence of innovations and upgrades in a certain product line rather than being a totally generic innovation that has never been materialized before. Once a product innovation has been first adopted in its most rudimentary form and has been successful in the market, it is almost certain that successive generations of upgraded versions with additional features will follow. As successive innovations generate rich product diversity in the marketplace, it is always those firms at the cutting edge of technology which persistently carry out R&D activity for upgrading and "perfecting" the product, to achieve or maintain a monopoly position that will skim the profits until a next generation product comes along. This, simply, is the Schumpeterian innovative process.

We also observe in the "sequences of innovations" process that the upgrading products will incorporate developments and/or findings of other product lines. While the underlying basic science required for a certain product innovation in a sequence of innovations could be very distinct from another innovation in a separate product line, it is becoming more likely that both product lines will share and contain the latest developments in some other technology and R&D base. Examples of these innovations and 'add ons' are all around us in cell phones, digital cameras, pocket pcs (PDAs), portable audio, video, digital imaging, communication and storage devices. We observe many other examples of technological convergence in recent product generations especially in consumer electronics industries that is fuelled by the shrinking size of semiconductors.

I call the process in which composite technologies are increasingly embodied in generations of successive product innovations "technological convergence". For example, cameras and camcorders use the same optics technology base. As they become more 'sophisticated' we observe that microprocessors of various sorts are being incorporated to increase their functions and capacity. VHS recorders followed by 8 mm camcorders which also function as players, are now being replaced with digital recording and digital cameras which all can be incorporated in a cell phone with still and movie camera features. This sequence illustrates the common practice of leading firms of technologically progressive industries diversifying their R&D base further from their initially established technological base, as the composite technology required to upgrade a product becomes increasingly diverse.

This aspect of increasing product diversification under technological convergence has been ignored in the theoretical R&D and innovation adoption literature. According to the prevailing theory firms may diversify into multiple product lines in response to excess capacity of productive factors [Montgomery and Wernerfelt, 1988]. Accordingly, technological scope economies embodied in the excess heterogeneous productive factors create incentives to diversify[1]. In this article I attempt to explain firm diversification based on strategic adoption of innovations and add on features. Specifically, I explore the effects of the strategic adoption of product innovations by separate monopolists on market structure. I conjecture that most innovations which are subject to strategic behavior on the part of the innovating or adopting firms are part of a sequence of innovations in a product line. By a "sequence of innovations" in a product line I mean a series of either qualitative (better functioning and/or more durable, etc.)[2] or quantitative (with additional functions/features incorporated)[3], or some combination of both, upgrading opportunities. Rosen calls this process 'add-on innovations' [Rosen, 1991] but emphasises only the case of process innovations. I follow Reinganum's [Reinganum, 1985] market driven definition of innovation, but focus only on the product innovations instead of process innovations. Reinganum calls an innovation "drastic" if the innovator becomes a monopolist in the post adoption market. I, instead, call an innovation drastic if the add-on technology that upgrades the product makes the previous version obsolete by completely replacing it. Hence, a drastic innovation that embodies an add-on technology can be adopted by more than one firm. Consider the audio cassette tapes that almost totally replaced the earlier betamax tapes, or the color TV that in most part replaced the black and white TV which in turn will possibly be replaced by HDTV in the very near future. These are new products that use the same basic technology with the old products. Similarly, a non-drastic innovation that partly replaces the old product – in the sense of a demand shift – can make the adopting firm a monopolist in the upgraded product market. Sony, for example, used to be a monopoly in the "recordable CD player" market in the early 1990's while the "CD player" market was competitive. Consequently, a single success does not mean that the successful firm reaps monopoly profits forever [Reinganum, 1985]. Rather, monopoly profits are earned only until the next, better innovation is developed and successfully adopted by the innovative firm and is accepted by the consumers[4]

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[1] In related literature Aron stresses the principal agent relationship between the owners and managers of firms and shows that diversification is an optimal response to the moral hazard problem facing firms' owners [Aron, 1988].

[2] Examples to the idea of qualitative upgrading could be found in the innovations in pharmaceuticals industry (in headache pills and cold medicine market it would imply rapid effectiveness and fewer side effects), synthetics industry (in cassette/video tapes and photograph films/digital imagemaking industry it would imply higher resolution), and high definition TV (better picture quality).

[3] Examples to the quantitative upgrading would be certain products in consumer electronics industry (calculators, computers, DVD players, cell phones, cameras, etc).

[4] Take the case of Home Video Revolution: Ampex in 1963 offered the first consumer version of a videotape recorder at an exorbitant price of $30,000; other iterations would follow, such as Sony's introduction of the videocassette recorder (VCR) in 1969, and the introduction of the U-Matic in 1972. In 1972, the AVCO CartriVision system was the first videocassette recorder to have prerecorded tapes of popular movies (from Columbia Pictures) for sale and rental – three years before Sony's Betamax system emerged into the market. However, the company went out of business a year

The idea of the models used in this paper is that each monopolist must choose either to preserve its own monopoly position by adopting an innovation in its product line or to challenge the incumbent monopolist in another market by adopting an innovation in that product line. Special features of the models arise from the differential costs in adopting innovations across separate product lines. Namely, if there is a product innovation to be adopted, any firm that is considering adoption is a potential entrant to the market which the adoption is expected to create. With the assumption that innovations have no patent protection and are common knowledge to all firms, the cost of entry to the potential market is the cost of adopting the innovation. I maintain that the cost of adoption would not be identical across existing firms considering undertaking R&D for this purpose. Unless the innovation is an original idea unrelated to any existing product market, the costs of adoption will be different across firms, depending on how suitable each firm's R&D program is to the requirements of adoption and how close its existing product line(s) is to the innovation under consideration. If the innovation isa part of a sequence of innovations in a product line, the existing firm(s) in that product line would have a cost advantage in adoption. This is due to the experience in production and learning by doing in R&D as well as the already established and technologically substitutable cospecialized assets which they might simply alter for the new product. This paper seeks to shed light on the importance of cost advantages in adoption and answers the question why firms sometimes give up these advantages and choose to enter a new market.

The theoretical literature on R&D and the adoption of new technologies has been concerned with different aspects of innovative activity in a single market. One

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later. The appearance of Sony's Betamax (the first home VCR, or videocassette recorder) in 1975 offered a cheaper sales price of $2,000 and recording time up to one hour; this led to a boom in sales – it was a technically-superior format when compared to the VHS system that was marketed by JVC and Matsushita beginning in 1976. In 1976, Paramount became the first to authorize the release of its film library onto Betamax videocassettes. In 1977, 20th Century Fox would follow suit, and begin releasing its films on videotape In 1977, RCA introduced the first VCRs in the United States based on JVC's system, capable of recording up to four hours on 1/2" videotape. By the late 70s, Sony's market share in sales of Betamax VCRs was below that of sales of VHS machines; consumers chose the VHS' longer tape time and larger tape size, over Sony's smaller and shorter tape time (of 1 hour). Video sales – the first films on videotape were released by the Magnetic Video Corporation (a company founded in 1968 by Andre Blay in Detroit, Michigan, the first video distribution company) - it licensed fifty films for release from 20th Century Fox for $300,000 in October, 1977; it began to license, market and distribute half-inch videotape cassettes (both Betamax and VHS) to consumers; it was the first company to sell pre-recorded videos; M*A*S*H (1970) was Magnetic's most popular title.Video rentals – in 1977, George Atkinson of Los Angeles began to advertise the rental of 50 Magnetic Video titles of its own collection in the Los Angeles Times, and launched the first video rental store, Video Station, on Wilshire Boulevard, renting videos for $10/day; within 5 years, it franchised more than 400 Video Station stores across the country.In 1978, Philips introduced the video laser disc (aka laserdisc and LD) – the first optical disc storage media for the consumer market; Pioneer began selling home LaserDisc players in 1980; eventually, the laserdisc systems would be replaced by the DVD ("digital versatile disc") format in the late 1990s.VHS video players, laser disc players and the release of films on videocassette tapes and discs multiplied as prices plummeted, creating a new industry and adding substantial revenue and profits for the movie studios. (15 April 2007 http://www.filmsite.org/70sintro.html) main body of research concentrates on the incentives and the process of bringing about inventions. To cite a few representative contributions among many, see: [Dasgupta and Stiglitz, 1980], [Dasgupta, 1986] on the industrial structure, uncertainty and the speed of R&D [Harris and Vickers, 1985], on patent races and persistence of monopoly, [Gilbert and Newberry, 1982]; and on licensing of innovations and network externalities [Katz and Shapiro, 1985].

Another line of research concentrates on the strategic aspects of the adoption of new technologies, again among the many, the following are few representative contributions: on the timing of innovation and the diffusion of new technology [Reinganum, 1981]; on rent dissipation [Fudenberg and Tirole, 1985, 1987]; on market entry dynamics [Smirnov and Wait, 2007]; on the sequence of innovations and industry evolution [Reinganum, 1985], [Vickers, 1986]; and on divisionalization [Schwartz and Thompson, 1986].

The main effects governing R&D and technological rivalry have been mostly analyzed using game theoretical tools. The results, with a few exceptions [Arrow, 1962], generally support Schumpeter's [Schumpeter, 1942] thesis that monopoly situations and innovativeness are intimately related. Nevertheless, the focus has been on a single market where an incumbent and an entrant engage in some sort of technological supremacy game mostly for process innovations ratits than product innovations.

The existing literature does not address the issues related to substitutability in basic science and technology when separate firms engage in strategic competition in R&D. That literature mostly treats an innovation as a generic idea unrelated to any existing product line. Furthermore, models which consider sequences of innovations [Reinganum, 1985], [Mclean and Riordan, 1989] do not establish the links between the sequence of innovations in separate product lines. These links can be quite important depending on the technology and R&D base which is common knowledge to the firms sharing it. The knowledge of the common technology and R&D base enables firms considering adoption of an innovation to recognize their potential challengers and their relative strengths and weaknesses. The empirical work of Cockburn and Henderson (1994) is an exception to this. The authors studied research activity by 10 major pharmaceutical companies in pursuit of the discovery of ethical drugs over 17 years and have found the presence of complementarities and spillovers between firms leading to multiple prizes out of a single R&D race. Hence, the authors show that the implication of the early theoretical "racing" models are inconsistent with the causal facts and their empirical results.

Of the product innovations mentioned earlier, I first focus on the drastic innovations using the perfect information framework. Following the general structure of the basic model, strategic competition for the new product markets are analyzed under three separate cases. Using the competitive payoffs as a benchmark for classifying the type of innovations, the Nash equilibrium points (NEP) are characterized. Under high cost drastic innovations the model is shown to represent a prisoner's dilemma situation where the firms only diversify, and, hence, switch their product lines. Under medium cost drastic innovations where pure strategy equilibrium does not exist, the mixed strategy equilibrium is characterized using a theorem. Following this, an example of mixed strategy equilibrium is presented which satisfies the criteria developed in the theorem.

Next, the basic model is modified to the case of non-drastic innovations. The assumption of symmetry in payoffs is relaxed by allowing differential market growth in separate product lines. The model is shown to yield equilibria where product upgrading is the more preferred best reply strategy. High cost non-drastic innovations are shown to exhibit multiplicity of equilibria. Under medium cost non-drastic innovations equilibrium it is proved that firms prefer upgrading, whereas under low cost non-drastic innovations the equilibrium is characterized where the preferred strategy is upgrading and diversifying into both markets.

1. Drastic Innovation Under Perfect Information

The following model assumes that there is no uncertainty related with post-adoption market conditions, and that a firm will successfully replace the old product if it adopts the innovation in that product line. Total replacement of the old product by making it obsolete is the 'drastic' nature of the innovation. Secondly, it is assumed that all agents involved in the innovative process are perfectly informed about all payoff relevant parameters of all agents, and that this perfect information is common knowledge to all agents.[5]

Consider a two period game with two identical firms sharing the closest technology base in two separate markets. In period one, assume that two separate products are exclusively and successfully produced by the two firms. Let firm 1 be the monopolist producing a_1, and firm 2 be the monopolist producing a_2. Both firms are earning monopoly profits equal to π^{m}. Firm 1, (2) can adopt a_1', (a_2') (the innovation in its own product line) with a cost $\boldsymbol{c}$, and it can adopt a_2', (a_1') (the innovation in the other firm's product line) with a cost $\boldsymbol{kc}$ where $k > 1$, or adopt both innovations with a cost $\mathrm{c}(1 + k)$, where $\mathrm{c}(1 + k) \leq \pi^{\mathrm{m}}$.

Firms can adopt four strategies in this non-cooperative, one shot game: adopt a_1', adopt a_2', adopt both a_1' & a_2', or adopt neither (stick with the existing product). If they both adopt either a_1' or a_2' they compete as Cournot duopolists in that product line. Both firms earn Cournot profits equal to π^{c} in this case. If only one firm adopts, it totally replaces the old product and becomes a monopolist earning monopoly profits equal to π^{m} in that product line.

Symmetry assumptions about period two profits and markets are restrictive but they focus the analysis on the role that incumbency plays in the adoption of new technologies. The assumption of equal profits in period one is also due to the same consideration. I introduce the following notation:

π_{ij}^{m}**:** Per period monopoly profit of firm i in market j, $\{i,j = 1,2\}$;

π_{ij}^{c}**:** Per period Cournot profit of firm i in market j, if it shares the market with the other firm;

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[5] See [Binmore, 1990].

c_{iu}**:** Cost to firm i of upgrading its product line by adopting the innovation in own market i;

c_{id}**:** Cost to firm i of diversifying into another market by adopting the innovation in market j.

Following restrictions are consistent with our discussion above.

$$\pi^{\mathrm{m}}_{\mathrm{ij}} > \pi^{\mathrm{c}}_{\mathrm{1j}} + \pi^{\mathrm{c}}_{\mathrm{2j}}; \quad \pi^{\mathrm{m}}_{\mathrm{ij}} > \pi^{\mathrm{c}}_{\mathrm{i1}} + \pi^{\mathrm{c}}_{\mathrm{i2}}; \quad \pi^{\mathrm{c}}_{\mathrm{1d}} = \pi^{\mathrm{c}}_{\mathrm{2d}} \tag{i}$$
$$\mathrm{c}_{\mathrm{id}} = k\mathrm{c}_{\mathrm{iu}} \quad ,k > 1 \tag{ii}$$
$$\pi^{\mathrm{m}}_{\mathrm{ij}} \geq \mathrm{c}_{\mathrm{id}} \quad \forall i, j = 1, 2 \tag{iii}$$

Inequalities in equation (i) are a general result of Cournot model [Tirole, 1988], [Shy, 2000], and symmetry assumptions imposed on the firms and markets.

Accordingly, monopoly profits in one market strictly exceed the total Cournot profits of both firms in that market. Alternately, the total of Cournot profits any firm can earn in two separate markets is strictly less than monopoly profits it can earn in one market.

Equation (ii) states that the cost of adopting the innovation in any firm's product line is strictly less than the cost of adopting the innovation in the other product line. In other words, diversifying is costlier than upgrading.

Equation (iii) imposes the restriction that payoffs to being a monopolist in any market are non-negative. If the inequality holds for the incumbent but not for the entrant, lacking any threat of entry, there would be no incentive for the incumbent to adopt the innovation in its own product line since it would simply be replacing itself as a monopolist. Thus, the restriction eliminates only a trivial case.

The following notation is introduced to simplify the characterization and manageability of the model. Accordingly the pure strategies for firm i are denoted by:

s_{i1} : no action (stick with the existing product);
s_{i2} : upgrade only (adopt the innovation in own product line);
s_{i3} : diversify only (adopt the innovation in the competitor's product line);
s_{i4} : upgrade and diversify (adopt both innovations).

Payoffs of the game are defined in terms of period 2 flow profits minus the cost of adoption. I use the following notation to denote the payoffs:

π^m_i : monopoly profits of firm i in own market i when no upgrading and entry has taken place;

V^m_{id} : monopoly profits in other market j , (requires i to adopt an innovation that allows it to diversify into market j);

V^m_{iu} : monopoly profits in own market i , (requires i to adopt an innovation that allows it to upgrade its product);

V^c_{iu} : profits in own market i when product is upgraded but competitor has diversified and entered market i;

V_{id}^{c} : profits in other market j when i has diversified but incumbent has upgraded;

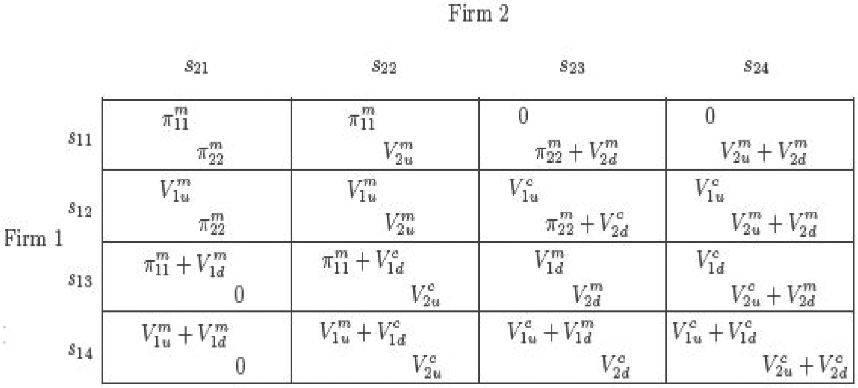
π_{ii}^{c} : profits of firm i in own market i when no upgrading has taken place and entrant has diversified in the upgraded product, (by definition, $\pi_{\mathrm{ii}}^{\mathrm{c}} \quad = \quad 0$ for drastic innovations).

For example, suppose in period two firm 1 decides to stick with its existing product a_1, and firm 2 diversifies by adopting a_1'. Since by definition of drastic innovations, a_1' totally replaces the market for a_1, firm 1 (the incumbent) is preyed upon by firm 2 (the entrant) in its own market. Firm 1 becomes a monopolist in both product lines producing a_1' and a_2. Thus, the payoffs to firm 1 and 2 respectively are $0, \pi_{22}^{\mathrm{m}} + \mathrm{V}_{2\mathrm{d}}^{\mathrm{m}}$.

Clearly the best scenario for each firm is that the incumbent monopolist sticks with its old product while the entrant adopts the innovation. Under this scenario the more aggressive firm preys upon the stagnant incumbent and becomes a monopoly in both markets.

In the following section, I characterize the equilibrium strategies of the players, and classify the results under different restrictions on the Cournot profits and the cost of adoption in the two product lines. Table 1. depicts the normal form representation of the general model under drastic innovations. The solution concept employed is payoff dominance. This method assumes economic agents behave rationally and it is common knowledge to the players that each player would play rationally. When players consider any two strategies , they would compare their payoffs in each cell of the corresponding strategies of the normal form.

Table 1: Strategic game with drastic innovations under perfect information



		Firm 2			
		s_{21}	s_{22}	s_{23}	s_{24}
Firm 1	s_{11}	π_{11}^{m} π_{22}^{m}	π_{11}^{m} V_{2u}^{m}	0 $\pi_{22}^{m} + V_{2d}^{m}$	0 $V_{2u}^{m} + V_{2d}^{m}$
	s_{12}	V_{1u}^{m} π_{22}^{m}	V_{1u}^{m} V_{2u}^{m}	V_{1u}^{c} $\pi_{22}^{m} + V_{2d}^{c}$	V_{1u}^{c} $V_{2u}^{m} + V_{2d}^{m}$
	s_{13}	$\pi_{11}^{m} + V_{1d}^{m}$ 0	$\pi_{11}^{m} + V_{1d}^{c}$ V_{2u}^{c}	V_{1d}^{m} V_{2d}^{m}	V_{1d}^{c} $V_{2u}^{c} + V_{2d}^{m}$
	s_{14}	$V_{1u}^{m} + V_{1d}^{m}$ 0	$V_{1u}^{m} + V_{1d}^{c}$ V_{2u}^{c}	$V_{1u}^{c} + V_{1d}^{m}$ V_{2d}^{c}	$V_{1u}^{c} + V_{1d}^{c}$ $V_{2u}^{c} + V_{2d}^{c}$

If in all pairwise comparisons one strategy yields payoffs that are strictly greater than the other, at least in one comparison and are equal in all the others, then it is said that the first strategy weakly dominates over the second one.

After players eliminate all dominated strategies, the game is then played on the remaining undominated strategies. Finally, Nash equilibrium point(s) (NEP) are searched.

Nash equilibrium points (NEP's) are strategy combinations s^*_{1i}, s^*_{2i}, that are the best replies to each other, such that $\mathrm{E}_1(s^*_1, s^*_2, = \max_{s_{11i}} E_1\,(s_1\,, s^*_2)$ and $E_2(s^*_1, s^*_2) = \max_{s_{2i}} E_2(s^*_1, s_2)$, where (s_i, s_j) is the combination of the ***i'***-th strategy of firm 1 and ***j'***-th strategy of firm 2.

Differences in the cost of adopting an innovation for the incumbent and the entrant result in three separate cases to consider.

Case 1: $\pi^{\mathrm{c}}_{\mathrm{ij}} \leq \mathrm{c_{iu}} < \mathrm{c_{id}}$, $\forall$ *i,j* = 1,2 $\{V^{\mathrm{c}}_{\mathrm{iu}} \leq 0\ ,\ V^{\mathrm{c}}_{\mathrm{id}} < 0\}$

Case 2: $\mathrm{c_{iu}} < \mathrm{c_{id}} \leq \pi^{\mathrm{c}}_{\mathrm{ij}}$, $\forall$ *i,j* = 1,2 $\{V^{\mathrm{c}}_{\mathrm{iu}} > 0\ ,\ V^{\mathrm{c}}_{\mathrm{id}} \geq 0\}$

Case 3: $\mathrm{c_{iu}} < \pi^{\mathrm{c}}_{\mathrm{ij}} < \mathrm{c_{id}}$, $\forall$ *i,j* = 1,2 $\{V^{\mathrm{c}}_{\mathrm{iu}} > 0\ ,\ V^{\mathrm{c}}_{\mathrm{id}} < 0\}$

Case 1 characterizes a situation where the net payoff to a firm when both firms undertake the same innovation in a given market is non-positive. I call this type of innovation as "high-cost" in this framework. Under Case 2 the net payoff to the firm, which is the Cournot profits net of adoption cost, is non-negative. I call this type of innovation "low-cost". Finally, Case 3 defines an intermediate scenario denoted as "medium cost". In the following sections the NEP's are characterized for each case.

1.1. High Cost Drastic Innovations

The conditions used in classifying an innovation as high-cost are (*i*) diversifying with an upgraded product is not profitable ($V^{\mathrm{c}}_{\mathrm{id}} < 0$), and (*ii*) competition in an upgraded product is not profitable either ($V^{\mathrm{c}}_{\mathrm{iu}} \leq 0$).

Next, the payoff dominant strategies -if they exist- are searched for, and after eliminating all possible dominated strategies the NEP's in pure strategies are found. If pure strategy NEP's do not exist, equilibrium in mixed strategies is characterized.

Lemma 1. *Suppose*
$\pi^{\mathrm{c}}_{\mathrm{ij}} \leq \mathrm{c_{iu}} < \mathrm{c_{id}}$, $\forall$ *i,j = 1,2* $\{V^{\mathrm{c}}_{\mathrm{iu}} \leq 0\ ;\ V^{\mathrm{c}}_{\mathrm{id}} < 0\}$.
Then,
(a) $\pi_1(s_1, s) \geq \pi_1(s_2, s)$ *and* $\pi_2(s, s_1) \geq \pi_2(s, s_2)\ \ \forall\ s$,
(b) $\pi_1(s_3, s) \geq \pi_1(s_4, s)$ *and* $\pi_2(s, s_3) \geq \pi_2(s, s_4)\ \ \forall\ s$,
where $\pi_{\mathrm{i}}(s_k, s_l)$ *is* ***i'*** *s payoff when firm 1 chooses* s_k *and firm 2 chooses* s_l.

Proof.
See the appendix for all proofs not given in the main body of the text.

Proposition 1. *Under drastic innovation where* $\pi^c_{ij}\ \leq\ _{iu}\ <\ c_{id}$*, the NEP is the strategy combination* $\{s_3, s_3\}$ *with payoffs* $(V^{\mathrm{m}}_{\mathrm{1d}}; V^{\mathrm{m}}_{\mathrm{2d}})$*. Thus, the monopolists cross over and diversify into the other firm's market.*

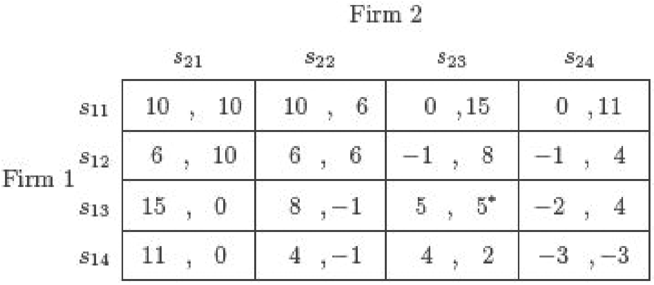
Remark. Since innovations are drastic, there is a one-hundred percent replacement of the incumbent's product line. From the firms' point of view (s_1, s_1) may be the "better" outcome; yet it is not self-enforcing and, therefore, not "stable". They are facing a classic 'prisoner's dilemma' situation.

Thus, each incumbent is preempted by the other monopolist, so that the monopolists switch markets. Although both innovations are adopted, the outcome is sub-optimal not only from the firms' viewpoint but also from a social standpoint since high cost entrants rather than the low cost incumbents are undertaking the adoption of innovations.

Example. $\pi_{\mathrm{i}}^{\mathrm{m}} = 10$, $c_{\mathrm{iu}} = 4$, $\mathrm{c}_{\mathrm{id}} = 5$, $V_{\mathrm{iu}}^{\mathrm{m}} = 6$, $V_{\mathrm{id}}^{\mathrm{m}} = 5$, $V_{\mathrm{iu}}^{\mathrm{c}} = -1$, $V_{\mathrm{id}}^{\mathrm{c}} = -2$. It is a simple exercise to plug in the values above into the generalized normal form given in Table 1 and see that $\{s_3, s_3\}$ is the NEP with corresponding payoffs of (5,5). High Cost Drastic Innovations:

π_{ij}^{c} i $c_{iu} < c_{id}$ $(V_{iu}^{c} < 0$; $V_{id}^{c} < 0)$

Example. π_{i}^{m}= 10 , c_{iu}= 4 , c_{id}= 5 , V_{iu}^{m} =6 , V_{id}^{m} =5 , V_{iu}^{c} =-1 , V_{iu}^{c} =-2.



		Firm 2			
		s_{21}	s_{22}	s_{23}	s_{24}
Firm 1	s_{11}	10 , 10	10 , 6	0 ,15	0 ,11
	s_{12}	6 , 10	6 , 6	−1 , 8	−1 , 4
	s_{13}	15 , 0	8 ,−1	5 , 5*	−2 , 4
	s_{14}	11 , 0	4 ,−1	4 , 2	−3 ,−3

Example

Fig. 1. High cost drastic innovations.

1.2. Low Cost Drastic Innovations

I say the innovations are low cost when both the diversification to compete with an upgraded product, and the competition in an upgraded product are profitable $(V_{\mathrm{iu}}^{\mathrm{c}} > 0;\ V_{\mathrm{id}}^{\mathrm{c}} \geq 0)$. In other words, no matter what the rival firm does, undertaking an innovation is profitable for both the entrant firm and the incumbent firm.

Lemma 2. *Suppose* $\mathrm{c}_{\mathrm{iu}} < \mathrm{c}_{\mathrm{id}} \leq \pi_{\mathrm{ij}}^{\mathrm{c}},\ \forall\, i, j = 1, 2,\ \{V_{\mathrm{iu}}^{\mathrm{c}} > 0,\ V_{\mathrm{id}}^{\mathrm{c}} \geq 0\}$. *Then*

(a) $\pi_1(s_3, s) \geq \pi_1(s_1, s)$, $\pi_2(s, s_3) \geq \pi_2(s, s_1)\ \ \forall\, s$

(b) $\pi_1(s_4, s) \geq \pi_1(s_2, s)$, $\pi_2(s, s_4) \geq \pi_2(s, s_2)\ \ \forall\, s$

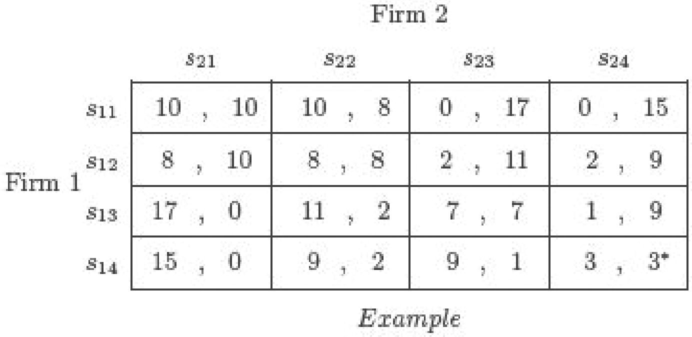
where $\pi_{\mathrm{i}}(s_k, s_l)$ *is* i'*s payoff when firm 1 chooses* s_k *and firm 2 chooses* s_l*.*

Proposition 2. *Under drastic innovation where* $\mathrm{c}_{\mathrm{iu}} < \mathrm{c}_{\mathrm{id}} \leq \pi_{\mathrm{ij}}^{\mathrm{c}}$, *the NEP is the strategy combination* $\{s_4, s_4\}$ *with payoffs* $(V_{1\mathrm{u}}^{\mathrm{c}} + V_{1\mathrm{d}}^{\mathrm{c}},\ V_{2\mathrm{u}}^{\mathrm{c}} + V_{2\mathrm{d}}^{\mathrm{c}})$. *Thus, the monopolists upgrade and diversify in both product lines by adopting both innovations.*

Remark Since competitive payoffs following a joint adoption of an innovation are greater than or equal to zero in each market, the monopolists upgrade and diversify in order to avoid being preempted by the entrant.

The outcome is clearly pro competitive even though the monopolists spend greater portion of their resources for a product innovation in which they have a comparative disadvantage. Their monopoly positions are replaced with Cournot Competition. Notice that in the reduced game the firms face a prisoner's dilemma situation as in the previous case of high cost innovations.

Example. $\pi_{\rm i}^{\rm m} = 10$, $c_{\rm iu} = 2$, $\rm c_{id} = 3$, $V_{\rm iu}^{\rm m} = 8$, $V_{\rm id}^{\rm m} = 7$, $V_{\rm iu}^{\rm c} = 2$, $V_{\rm id}^{\rm c} = 1$ It is a simple exercise to plug in the values above into the generalized normal form given in Table 1 and see that $\{s_4, s_4\}$ is the only NEP with corresponding payoffs of (3,3).



Firm 2

Firm 1

	s_{21}	s_{22}	s_{23}	s_{24}
s_{11}	10 , 10	10 , 8	0 , 17	0 , 15
s_{12}	8 , 10	8 , 8	2 , 11	2 , 9
s_{13}	17 , 0	11 , 2	7 , 7	1 , 9
s_{14}	15 , 0	9 , 2	9 , 1	3 , 3*

Example

Fig. 2. Low cost drastic innovations.

1.3. Medium Cost Drastic Innovations

Innovations are classified as medium cost under the following assumptions. The first is the condition that diversifying to compete with an upgraded product is not profitable, i.e. $V_{\rm id}^{\rm c} < 0$. The second is the condition that competition in an upgraded product is profitable, i.e. $V_{\rm iu}^{\rm c} > 0$. These conditions together with case 1 and case 2 exhaust the payoff spectrum under all possible strategy combinations chosen by the incumbent and the entrant. The idea of 'medium cost' embodies within it the concept of comparative advantage of being established in a product line. All else being equal, if the incumbent has to compete with an entrant in a product that it has upgraded, the incumbent will prevail and prey upon the entrant. Knowing its comparative disadvantage, the entrant will enter only if can secure a monopoly on the product line it is diversifying into. On the other hand, the incumbent monopolist is reasoning that if it does not adopt the innovation and upgrade its product, it will be preyed upon by a successful entrant–no matter how much a cost disadvantage exists. Because of the drastic nature of the innovation, no matter who adopts it, the existing product will be obsolete.

It is easy to see that there are not any payoff dominated strategies that can be eliminated to simplify the solution. It is also straightforward to verify that there are no NEP in pure strategies in this game either. Therefore, the NEP's has to be characterized using mixed strategies.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ denote firm 1's mixed strategy and $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ denote firm 2's mixed strategy. Then, write the expected payoffs for each firm as follows:

$$\begin{gathered}
\mathrm{E}_1(\alpha, \beta) = \alpha_1(\beta_1 + \beta_2)\pi_{11}^{\mathrm{m}} + \alpha_2\{(\beta_1 + \beta_2)V_{1\mathrm{u}}^{\mathrm{m}} + (\beta_3 + \beta_4)V_{1\mathrm{u}}^{\mathrm{c}}\}+ \\
+ \alpha_3\{(\beta_1 + \beta_2)\pi_{11}^{\mathrm{m}} + (\beta_1 + \beta_3)V_{1\mathrm{d}}^{\mathrm{m}} + (\beta_2 + \beta_4)V_{1\mathrm{d}}^{\mathrm{c}}\}+ \\
+ \alpha_4\{(\beta_1 + \beta_2)V_{1\mathrm{u}}^{\mathrm{m}} + (\beta_1 + \beta_3)V_{1\mathrm{d}}^{\mathrm{m}} + (\beta_2 + \beta_4)V_{1\mathrm{d}}^{\mathrm{c}} + (\beta_3 + \beta_4)V_{1\mathrm{u}}^{\mathrm{c}}\}; \\
\mathrm{E}_2(\alpha, \beta) = \beta_1(\alpha_1 + \alpha_2)\pi_{22}^{\mathrm{m}} + \beta_2\{(\alpha_1 + \alpha_2)V_{2\mathrm{u}}^{\mathrm{m}} + (\alpha_3 + \alpha_4)V_{2\mathrm{u}}^{\mathrm{c}}\}+ \\
+ \beta_3\{(\alpha_1 + \alpha_2)\pi_{22}^{\mathrm{m}} + (\alpha_1 + \alpha_3)V_{2\mathrm{d}}^{\mathrm{m}} + (\alpha_2 + \alpha_4)V_{2\mathrm{d}}^{\mathrm{c}}\}+ \\
+ \beta_4\{(\alpha_1 + \alpha_2)V_{2\mathrm{u}}^{\mathrm{m}} + (\alpha_1 + \alpha_3)V_{2\mathrm{d}}^{\mathrm{m}} + (\alpha_2 + \alpha_4)V_{2\mathrm{d}}^{\mathrm{c}} + (\alpha_3 + \alpha_4)V_{2\mathrm{u}}^{\mathrm{c}}\}.
\end{gathered}$$

Proposition 3. *Assume* $V_{\mathrm{iu}}^{\mathrm{m}} < \pi_{\mathrm{ii}}^{\mathrm{m}}$, $V_{\mathrm{iu}}^{\mathrm{c}} > 0$, $V_{\mathrm{id}}^{\mathrm{c}} < 0 \quad \forall \quad i = 1, 2$. *Then, any NEP in mixed strategies must satisfy the following:*

$$(\beta_1 + \beta_3)V_{1\mathrm{d}}^{\mathrm{m}} + (\beta_2 + \beta_4)V_{1\mathrm{d}}^{\mathrm{c}} = 0, \tag{1}$$

$$(\alpha_1 + \alpha_3)V_{2\mathrm{d}}^{\mathrm{m}} + (\alpha_2 + \alpha_4)V_{2\mathrm{d}}^{\mathrm{c}} = 0. \tag{2}$$

Proposition 4. *Under the assumptions of Proposition 3 any NEP in mixed strategies must satisfy the following:*

$$(\alpha_1 + \alpha_2)\pi_{22}^{\mathrm{m}} = (\alpha_1 + \alpha_2)V_{2\mathrm{u}}^{\mathrm{m}} + (\alpha_3 + \alpha_4)V_{2\mathrm{u}}^{\mathrm{c}}, \tag{3}$$

$$(\beta_1 + \beta_2)\pi_{11}^{\mathrm{m}} = (\beta_1 + \beta_2)V_{1\mathrm{u}}^{\mathrm{m}} + (\beta_3 + \beta_4)V_{1\mathrm{u}}^{\mathrm{c}}. \tag{4}$$

Theorem 1. $\forall\ \pi_{\mathrm{ij}}^{\mathrm{c}}$ *and* $\pi_{\mathrm{ii}}^{\mathrm{c}}$ *such that* $\mathrm{c}_{\mathrm{iu}} < \pi_{\mathrm{ij}}^{\mathrm{c}} < \mathrm{c}_{\mathrm{id}}$, $V_{\mathrm{iu}}^{\mathrm{m}} < \pi_{\mathrm{ii}}^{\mathrm{m}}$ *{i,j=1,2} holds, the set of all mixed strategy NEP's satisfies:*

$$\begin{gathered}
(\beta_1 + \beta_3)V_{1\mathrm{d}}^{\mathrm{m}} + (\beta_2 + \beta_4)V_{1\mathrm{d}}^{\mathrm{c}} = 0, \\
(\beta_1 + \beta_2)(\pi_{11}^{\mathrm{m}} - V_{1\mathrm{u}}^{\mathrm{m}}) - (\beta_3 + \beta_4)V_{1\mathrm{u}}^{\mathrm{c}} = 0, \\
\beta_1 + \beta_2 + \beta_3 + \beta_4 = 1, \quad \beta_{\mathrm{i}} \geq 0, \\
(\alpha_1 + \alpha_3)V_{2\mathrm{d}}^{\mathrm{m}} + (\alpha_2 + \alpha_4)V_{2\mathrm{d}}^{\mathrm{c}} = 0, \\
(\alpha_1 + \alpha_2)(\pi_{22}^{\mathrm{m}} - V_{2\mathrm{u}}^{\mathrm{m}}) - (\alpha_3 + \alpha_4)V_{2\mathrm{u}}^{\mathrm{c}} = 0,
\end{gathered}$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1, \quad \alpha_{\mathrm{i}} \geq 0,$$

where α and β are firm 1's and firm 2's mixed strategies respectively.

Proof of Theorem 1.

This theorem is a consequence of the definition of medium cost drastic innovations, and of Propositions 3 and 4.

The corresponding expected payoffs are:

$$\mathrm{E}_1(\alpha,\beta) = \alpha_1(\beta_1+\beta_2)\pi_{11}^{\mathrm{m}} + \alpha_2\{(\beta_1+\beta_2)V_{1\mathrm{u}}^{\mathrm{m}} + (\beta_3+\beta_4)V_{1\mathrm{u}}^{\mathrm{c}}\} + \alpha_3(\beta_1+\beta_2)\pi_{11}^{\mathrm{m}} +$$

$$+ \alpha_4\{(\beta_1+\beta_2)V_{1\mathrm{u}}^{\mathrm{m}} + (\beta_3+\beta_4)V_{1\mathrm{u}}^{\mathrm{c}}\};$$

$$\mathrm{E}_2(\alpha,\beta) = \beta_1(\alpha_1+\alpha_2)\pi_{22}^{\mathrm{m}} + \beta_2\{(\alpha_1+\alpha_2)V_{2\mathrm{u}}^{\mathrm{m}} + (\alpha_3+\alpha_4)V_{2\mathrm{u}}^{\mathrm{c}}\} + \beta_3(\alpha_1+\alpha_2)\pi_{22}^{\mathrm{m}} +$$

$$+ \beta_4\{(\alpha_1+\alpha_2)V_{2\mathrm{u}}^{\mathrm{m}} + (\alpha_3+\alpha_4)V_{2\mathrm{u}}^{\mathrm{c}}\}.$$

Simplifying the equations in *NEP** we obtain:

$$(\alpha_1+\alpha_2) = \frac{V_{2\mathrm{u}}^{\mathrm{c}}}{\pi_{22}^{\mathrm{m}} - V_{2\mathrm{u}}^{\mathrm{m}} + V_{2\mathrm{u}}^{\mathrm{c}}}, \tag{5}$$

$$(\alpha_3+\alpha_4) = \frac{\pi_{22}^{\mathrm{m}} - V_{2\mathrm{u}}^{\mathrm{m}}}{\pi_{22}^{\mathrm{m}} - V_{2\mathrm{u}}^{\mathrm{c}}}, \tag{6}$$

$$(\alpha_1+\alpha_3) = \frac{V_{2\mathrm{d}}^{\mathrm{c}}}{V_{2\mathrm{d}}^{\mathrm{m}} - V_{2\mathrm{d}}^{\mathrm{c}}}, \tag{7}$$

$$(\alpha_2+\alpha_4) = \frac{V_{2\mathrm{d}}^{\mathrm{m}}}{V_{2\mathrm{d}}^{\mathrm{m}} - V_{2\mathrm{d}}^{\mathrm{c}}}. \tag{8}$$

Remark. From (5) we note that the lower is the cost of upgrading the more likely is that the incumbent will stay in own product line and upgrade its product. Similarly, from (6) we note that the higher is the cost of upgrading, the more likely is that the entrant will either cross over to a separate product line or diversify into both product lines.

In the following numerical example I find the range of individual probabilities for mixed strategy NEP's.

Example. $\pi_{\mathrm{ij}}^{\mathrm{m}} = 10$, $\pi_{\mathrm{ij}}^{\mathrm{c}} = 4$, $\forall i, j = 1, 2$; $\mathrm{c_{iu}} = 3$, $\mathrm{c_{id}} = 5$, $V_{\mathrm{iu}}^{\mathrm{m}} = 7$, $V_{\mathrm{id}}^{\mathrm{m}} = 5$, $V_{\mathrm{iu}}^{\mathrm{c}} = 1$, $V_{\mathrm{id}}^{\mathrm{c}} = -1$

The following table depicts the normal form representation of this example. Let $A_1, \ldots, A_4$ be the expected payoffs firm 1 will receive if firm 2 plays the mixed strategies $(\beta_1, \ldots, \beta_4)$ and let $B_1, \ldots, B_4$ be the expected payoffs firm 2 will receive if firm 1 plays the mixed strategies $(\alpha_1, \ldots, \alpha_4)$. Then, using the payoffs in the

		Firm 2			
		s_{21}	s_{22}	s_{23}	s_{24}
Firm 1	s_{11}	10 , 10	10 , 7	0 , 15	0 , 12
	s_{12}	7 , 10	7 , 7	1 , 9	1 , 6
	s_{13}	15 , 0	9 , 1	5 , 5	−1 , 6
	s_{14}	12 , 0	6 , 1	6 , −1	0 , 0

Example

Fig. 3. Medium cost drastic innovations.

example, the following linear equation systems are set up for firm 1 and firm 2 respectively:

$$\begin{aligned}
\mathrm{A}_1 &= \pi_1(s_1,\beta) = 10\beta_1 + 10\beta_2,\\
\mathrm{A}_2 &= \pi_1(s_2,\beta) = 7\beta_1 + 7\beta_2 + \beta_3 + \beta_4,\\
\mathrm{A}_3 &= \pi_1(s_3,\beta) = 15\beta_1 + 9\beta_2 + 5\beta_3 - \beta_4,\\
\mathrm{A}_4 &= \pi_1(s_4,\beta) = 12\beta_1 + 6\beta_2 + 6\beta_3,\\
\mathrm{B}_1 &= \pi_2(\alpha,s_1) = 10\alpha_1 + 10\alpha_2,\\
\mathrm{B}_2 &= \pi_2(\alpha,s_2) = 7\alpha_1 + 7\alpha_2 + \alpha_3 + \alpha_4,\\
\mathrm{B}_3 &= \pi_2(\alpha,s_3) = 15\alpha_1 + 9\alpha_2 + 5\alpha_3 - \alpha_4,\\
\mathrm{B}_4 &= \pi_2(\alpha,s_4) = 12\alpha_1 + 6\alpha_2 + 6\alpha_3.
\end{aligned}$$

Equating $A_1 = A_4$, $A_2 = A_4$, $A_3 = A_4$ and using the constraint that the sum of the probabilities of random strategies is equal to one $\sum_{\mathrm{i}=1}^{4} \beta_{\mathrm{i}} = 1$, I proceed to find the interior solution to the following system for firm 1.

$$\begin{aligned}
2\beta_1 - 4\beta_2 + 6\beta_3 &= 0 \quad \text{(i)}\\
5\beta_1 - \beta_2 + 5\beta_3 - \beta_4 &= 0 \quad \text{(ii)}\\
-3\beta_1 - 3\beta_2 + \beta_3 + \beta_4 &= 0 \quad \text{(iii)}\\
\beta_1 + \beta_2 + \beta_3 + \beta_4 &= 0 \quad \text{(iv)}
\end{aligned}$$

We also want to place the restriction that $\beta_1, \beta_2, \beta_3, \beta_4 \geq 0$.

Note that (i)–(ii) = (iii). Therefore, we eliminate (**i**). From (**iii**) we have:

$$(\beta_1 + \beta_2) = \frac{(\beta_3 + \beta_4)}{3}.$$

Substituting this into (iv) we obtain,

$$\frac{(\beta_3 + \beta_4)}{3} + (\beta_3 + \beta_4) = 1$$

or,

$$(\beta_3 + \beta_4) = \frac{3}{4} \quad \left\{ = \frac{\pi_{11}^{\mathrm{m}} - V_{1\mathrm{u}}^{\mathrm{m}}}{\pi_{11}^{\mathrm{m}} - V_{1\mathrm{u}}^{\mathrm{m}} + V_{1\mathrm{u}}^{\mathrm{c}}} \right\} \tag{9}$$

It follows that,

$$(\beta_1 + \beta_2) = \frac{1}{4} \quad \left\{ = \frac{V_{1\mathrm{u}}^{\mathrm{c}}}{\pi_{11}^{\mathrm{m}} - V_{1\mathrm{u}}^{\mathrm{m}} + V_{1\mathrm{u}}^{\mathrm{c}}} \right\}. \tag{10}$$

From (ii) we have,

$$5(\beta_1 + \beta_3) = (\beta_2 + \beta_4).$$

Substituting this into (iv) we obtain $\beta_1 + \beta_3 + 5(\beta_1 + \beta_3) = 1$, or

$$(\beta_1 + \beta_3) = \frac{1}{6} \quad \left\{ = -\frac{V_{1\mathrm{d}}^{\mathrm{c}}}{V_{1\mathrm{d}}^{\mathrm{m}} - V_{1\mathrm{d}}^{\mathrm{c}}} \right\}. \tag{11}$$

It follows that,

$$(\beta_2 + \beta_4) = \frac{5}{6} \left\{ = \frac{V_{1\mathrm{d}}^{\mathrm{m}}}{V_{1\mathrm{d}}^{\mathrm{m}} - V_{1\mathrm{d}}^{\mathrm{c}}} \right\}. \tag{12}$$

We have shown that equations (9) through (12) satisfy equations (5) through (8) respectively. It is also immediate that, from equations (9) through (12) we can write the following conditions:

$$0 \leq \beta_1 \leq \frac{1}{6}, \quad 0 \leq \beta_2 \leq \frac{1}{4}, \quad 0 \leq \beta_3 \leq \frac{1}{6}, \quad 0 \leq \beta_4 \leq \frac{3}{4}. \tag{13}$$

Hence, it is seen that, in this example, any interior solution for the mixed strategy equilibria must satisfy (13). Simulations show that one solution that satisfies the non-negativity constraints are the following β_{i} values:

$$\beta_1 = 0, \quad \beta_2 = 0.25, \quad \beta_3 = 0.17, \quad \beta_4 = 0.58.$$

1.4. Summary of Results

If the innovation is drastic, i.e. that it would totally replace the existing product in that product line, then the firms would: (*i*) diversify into the incumbent's product line only and in the process switch markets as monopolists if and only if $\pi_{\mathrm{ij}}^{\mathrm{c}} \leq \mathrm{c_{iu}} < \mathrm{c_{id}}$; (*ii*) upgrade in own product line and diversify into the competitor's product line and compete as Cournot competitors by adopting the innovations in both product lines if and only if $\mathrm{c_{iu}} < \mathrm{c_{id}} < \pi_{\mathrm{ij}}^{\mathrm{c}}$; and (*iii*) use a mixed strategy equilibrium in deciding which innovation(s) they would adopt if and only if, $\mathrm{c_{iu}} < \pi_{\mathrm{ij}}^{\mathrm{c}} < \mathrm{c_{id}}$.

Under the first type of equilibrium with "drastic" innovation we find that firms diversify their product lines by crossing over markets and totally replace the incumbents, if payoffs from the competitive outcome are non-positive (i.e. if the innovations are high cost) for both the entrant and for the incumbent. This type of equilibrium where the monopolists switch markets develops as a dominant 'defensive' strategy because under drastic innovation firms do not undertake adoptions in their own product lines, since it would only mean replacing themselves as incumbents. It can be concluded that, because competition reduces profits, each firm's incentive to become a monopolist is greater than its incentive to become a duopolist by jointly adopting the high cost innovation.

Under the second type of equilibrium, firms upgrade products not only in their own product line but also in the incumbent's product line. This type of total diversification arises when competitive payoffs from diversifying into the competitor's product line is non-negative, i.e. when the innovations are low cost.

We also observe that under some boundary values of cost of adoption and Cournot profits firms may use mixed strategy equilibrium. We get mixed strategy equilibrium with drastic innovation if the profits from Cournot competition in any market strictly cover the cost of adoption for the incumbent, but are strictly less than the cost of adoption for the entrant, i.e. if the innovations are medium cost. Both innovations are adopted, however, either through switching of incumbency, or by the incumbent itself, or by joint adoption in both product lines, as demonstrated with a numerical example.

The main tendency is that if the firms are facing drastic innovations, then they would either diversify into the incumbent's product line only, or upgrade and diversify into both product lines. An interesting observation is that, lacking such a technological rivalry, monopolist firms would not undertake adoptions in their own product lines, since it would mean replacing themselves as incumbents. Thus, the outcome of this technological rivalry is socially desirable, since maximum product diversity is achieved through adoption of new innovations. On the other hand, the optimal "cooperative" strategy from the firms' standpoint would be not adopting the drastic innovations and sticking with the old product. Yet, this strategy can not be enforced as a credible commitment.

2. Non-drastic Innovation Under Perfect Information

With non-drastic innovation (or partial replacement of the old market) we mean that successful adoption of the innovation: (*i*) suppresses the demand for the old product but does not make it completely obsolete, and (*ii*) generates new demand so that the total demand in that product line is growing. In this model, up to 4 product markets (2 in each product line) can coexist in the second period. If both innovations are adopted exclusively, either by the incumbent or by the entrant, the payoff that can be earned from each new product market is $V_{\mathrm{i}}^{\mathrm{m}}$. The costs of adoption, suppressed in $V_{\mathrm{i}}^{\mathrm{m}}$, are $\mathrm{c_i}$ where $\mathrm{c_{id}} > \mathrm{c_{iu}}$, for all i=1,2.

The profits incumbent firms earn from their respective old markets, if the innovation is adopted, are denoted by a parameter $\mathrm{r_i}$, where $0 < \mathrm{r_i} \leq \pi_{\mathrm{i}}^{\mathrm{m}}$. $\mathrm{r_i} = 0$ would imply a drastic innovation where the old product market is totally replaced by the new one. On the other hand, $\mathrm{r_i} = \pi_{\mathrm{i}}^{\mathrm{m}}$ implies that adoption of the innovation has no effect on the old product market. While this is an extreme case of non-drastic innovation, no replacement indicates that the two products are possibly unrelated or not considered to be on the same product line on the demand side.

I modify the notation used for drastic innovations to simplify the characterization of the model.

The pure strategies for firm i are denoted by:

s_{i1} : no action (stick with the existing product);

s_{i2} : upgrade only (adopt the innovation in own product line, and continue producing the old product if $r > 0$;

s_{i3} : diversify only (adopt the innovation in the competitor's product line);

s_{i4} : upgrade and diversify (adopt both innovations, and continue producing the old product if $r > 0$.

Payoffs to firm i are denoted by:

π_i^m : pre-innovation monopoly profits in own market i (when firm i sticks with its old product and no upgrading has taken place);

π_{ij}^c : Cournot profits of firm i when both firms jointly adopt the innovation in market j;

r_i : post-innovation monopoly profits the incumbent earns from its old product market i (requires upgrading by the incumbent and/or diversification through adoption of an innovation by the competitor into market i);

V_{id}^m : monopoly profits in other market j (requires i to adopt an innovation that allows it to diversify into market j, but the incumbent does not upgrade);

V_{iu}^m : profits from having the monopoly of the new product in own market i (requires i to adopt an innovation that allows it to upgrade its product);

V_{iu}^c : profits from the new product in own market i when product is upgraded but competitor has diversified and entered market i;

V_{id}^c : profits in other market j when i has diversified by adopting an innovation but incumbent has upgraded.

Note that the following set of restrictions (i) through (iii), defined in section 2 before and repeated below for convenience is coupled with (iv). These are based on reasonable assumptions some of which follow directly from the economic theory.

$$\pi_{ij}^m > 2\pi_{ij}^c \quad \forall \quad i,j = 1,2, \qquad (i)$$

$$c_{id} > c_{iu} \quad \forall \quad i,j = 1,2, \qquad (ii)$$

$$V_{iu}^m,\ V_{id}^m > 0 \quad \forall \quad i,j = 1,2, \qquad (iii)$$

$$0 \leq r \leq \pi^m. \qquad (iv)$$

Recall that (i) states that the total of Cournot profits any firm can earn in two separate markets is strictly less than the monopoly profits it can earn in a single market. Hence, monopoly is always a preferred status by both firms.

Equation (ii) states that the cost of adopting the innovation in any firm's product line is strictly less than the cost of adopting the innovation in the competitor's product line capturing the idea of comparative cost advantage obtained by being an established firm in a product line. It implies[6] $V_{iu}^m > V_{id}^m$, and $V_{iu}^c > V_{id}^c \ \forall\, i = 1, 2$ meaning *'upgrading to keep monopoly is better than diversifying to obtain monopoly'* for the former, and *'divesifying to compete with an upgraded product is less profitable than upgrading to compete with an entrant that has diversified in the upgraded product'* for the latter.

Equation (iii) imposes the restriction that payoffs to being a monopolist in any market are non-negative. If not, it is either too costly for both the entrant and the incumbent to earn positive profits following an adoption, or it is too costly only for the entrant to earn positive profits even if it became a monopolist in the incumbent's product line. It implies that $\pi^m > c_{iu}, c_{id} \ \forall i = 1, 2$. If the inequality holds for the incumbent ($V_{iu}^m > 0$), but not for the entrant ($V_{id}^m < 0$) then, under drastic innovation ($r = 0$) there would be no reason for the incumbent to adopt the innovation in its own product line-since it would be replacing itself as a monopolist ($V_{iu}^m = \pi^m$). Thus, this restriction not only eliminates a trivial case but also captures the idea of technological closeness and rivalry. When *(iii)* holds, firms find themselves within the technological boundaries of one another; hence, they see themselves as potential challengers and entrants in the incumbent's product line.

Note that (ii) and (iii) together imply the following:

First, upgrading to obtain monopoly is better than diversifying into another market to obtain monopoly, $V_{iu}^m > V_{id}^m \ \forall\, i = 1, 2$.

Second, diversifying to compete with an upgraded product is less profitable than upgrading to compete with an entrant that has diversified in the upgraded product, $V_{iu}^c > V_{id}^c \ \forall\, i = 1, 2$.

Inequality (iv) captures the degree of replacement of the old product market by the adopted innovation. A drastic innovation where the old product market is totally replaced by the new one will be denoted by $r = 0$. On the other hand, $r = \pi^m$ implies that adoption of the innovation has no effect on the old product market, since the incumbent can earn the same amount of profits from its old product market. While this is an extreme case of drastic innovation, no replacement indicates that the two products are possibly unrelated.

Finally, note that using equation (iv) together with (ii) and (iii) we require: $V_{id}^m + r > \pi^m \ \forall\, i = 1, 2 \ \& \ 0 \leq r \leq \pi^m$.

Hence, for a drastic innovation where $r = 0$, we require that $V_{iu}^m, V_{id}^m > \pi^m$. This defines the lower bound for profits so that the firms would not be worse off as monopolists if they considered upgrading and/or diversifying their product lines.

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[6] However, it should be noted that Eq. (ii) does not imply either $c_{1u} > c_{2u}$, or $c_{1d} > c_{2d}$, or both necessarily hold.

I assume equal replacement in the two product lines, i.e., $r_1 = r_2$, without loss of generality, to simplify the notation, as this does not change the results.

Assume $V_{\mathrm{iu}}^{\mathrm{m}} + \mathrm{r_i} > \pi_{\mathrm{i}}^{\mathrm{m}}$ as a proxy that the total demand following adoption of an innovation in a product line is growing[7] Next assume $V_{\mathrm{iu}}^{\mathrm{m}} > V_{\mathrm{id}}^{\mathrm{m}}$ upgrading to obtain a monopoly is better than diversifying into another market to obtain monopoly. Finally, assume $V_{\mathrm{iu}}^{\mathrm{c}} > V_{\mathrm{id}}^{\mathrm{c}}$, diversifying to compete with an upgraded product is less profitable than upgrading to compete with an entrant that has diversified in the upgraded product.

Table 2 depicts the normal form representation of the general model. For example, if the firms exclusively adopt the innovations in their own product lines, the strategy combination for firm 1 and firm 2 would be denoted by (s_2, s_2), respectively. In this case, each firm maintains its monopoly position for both the old and the new markets in its product line. This strategy yields payoffs of $\pi_{\mathrm{i}}(s_2, s_2) = V_{\mathrm{iu}}^{\mathrm{m}} + \mathrm{r_i}$ and it is the maximum that can be earned as a monopolist in a single product line. The strategy combination (s_4, s_4) means that the firms both adopt the innovations in their product lines and across product lines while continuing to produce their original products. Thus, the firms become Cournot competitors in the new product markets and maintain their monopoly positions in the old product markets which is suppressed by the new products. In this case, payoffs to firm 1 and firm 2, in respective order, are as follows:

$$\pi_1(s_4, s_4) = V_{\mathrm{1u}}^{\mathrm{c}} + V_{\mathrm{1d}}^{\mathrm{c}} + \mathrm{r_1}; \quad \pi_2(s_4, s_4) = V_{\mathrm{2u}}^{\mathrm{c}} + V_{\mathrm{2d}}^{\mathrm{c}} + \mathrm{r_2}.$$

Similar to the case of drastic innovations, differences in the cost of upgrading and diversification result in three separate cases to consider under this scenario also:

Case 1: $\pi_{\mathrm{ij}}^{\mathrm{c}} < \mathrm{c_{iu}} < \mathrm{c_{id}}$, $\forall\ i,j = 1,2\ \{V_{\mathrm{iu}}^{\mathrm{c}} < 0;\ V_{\mathrm{id}}^{\mathrm{c}} < 0\}$;

Case 2: $\mathrm{c_{iu}} < \mathrm{c_{id}} < \pi_{\mathrm{ij}}^{\mathrm{c}}$, $\forall\ i,j = 1,2\ \{V_{\mathrm{iu}}^{\mathrm{c}} > 0;\ V_{\mathrm{id}}^{\mathrm{c}}A > 0\}$;

Case 3: $\mathrm{c_{iu}} \leq \pi_{\mathrm{ij}}^{\mathrm{c}} \leq \mathrm{c_{id}}$, $\forall\ i,j = 1,2\ \{V_{\mathrm{iu}}^{\mathrm{c}} \geq 0;\ V_{\mathrm{id}}^{\mathrm{c}} \leq 0\}$.

I label the above cases as high cost, low cost and medium cost respectively.

2.1. High Cost Non-drastic Innovations

If competitive payoffs to both the incumbent firm and the entrant firm following a joint adoption of an innovation in any market are strictly negative, I call them high cost innovations[8]: $(V_{\mathrm{iu}}^{\mathrm{c}} < 0\ ;\ V_{\mathrm{id}}^{\mathrm{c}} < 0)$.

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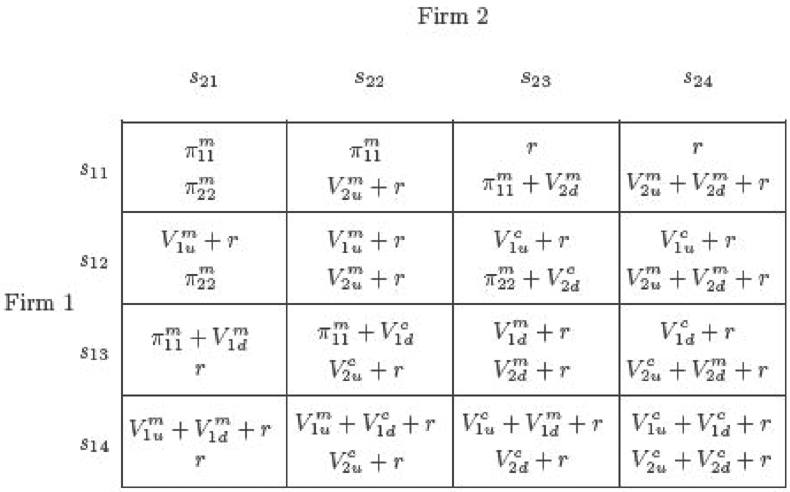
[7] $V_{iu}^{m} + \mathrm{r_i} = \pi_i^m$ implies the innovation is non-drastic, but it has simply generated new demand and revenues enough to compensate exactly for the cost of its adoption.

[8] Recall that high cost innovations were defined as $(V_{\mathrm{iu}}^{\mathrm{c}} \leq 0;\ V_{\mathrm{id}}^{\mathrm{c}} < 0)$ earlier. I ignore the discrepancy for the boundary values around zero and use the same terminology for both situations. This difference arises as a consequence of our concern for classifying types of innovations according to the outcomes they lead to in the solution of the game.

Table 2: Strategic game with non-drastic innovations under perfect information



Firm 2

		s_{21}	s_{22}	s_{23}	s_{24}
Firm 1	s_{11}	π_{11}^m π_{22}^m	π_{11}^m $V_{2u}^m + r$	r $\pi_{11}^m + V_{2d}^m$	r $V_{2u}^m + V_{2d}^m + r$
	s_{12}	$V_{1u}^m + r$ π_{22}^m	$V_{1u}^m + r$ $V_{2u}^m + r$	$V_{1u}^c + r$ $\pi_{22}^m + V_{2d}^c$	$V_{1u}^c + r$ $V_{2u}^m + V_{2d}^m + r$
	s_{13}	$\pi_{11}^m + V_{1d}^m$ r	$\pi_{11}^m + V_{1d}^c$ $V_{2u}^c + r$	$V_{1d}^m + r$ $V_{2d}^m + r$	$V_{1d}^c + r$ $V_{2u}^c + V_{2d}^m + r$
	s_{14}	$V_{1u}^m + V_{1d}^m + r$ r	$V_{1u}^m + V_{1d}^c + r$ $V_{2u}^c + r$	$V_{1u}^c + V_{1d}^m + r$ $V_{2d}^c + r$	$V_{1u}^c + V_{1d}^c + r$ $V_{2u}^c + V_{2d}^c + r$

Lemma 3. *Suppose* $\pi_{\mathrm{ij}}^{\mathrm{c}} < \mathrm{c_{iu}} < \mathrm{c_{id}}$, $(V_{\mathrm{iu}}^{\mathrm{c}},\ V_{\mathrm{id}}^{\mathrm{c}} < 0)$ $i = j$. *Then,*

(a) $\pi_1(s_1, s_4) > \pi_1(s, s_4)$ *and* $\pi_2(s_1, s_4) > \pi_2(s_1, s) \forall\ s$,

(b) $\pi_1(s_2, s_2) > \pi_1(s, s_2)$ *and* $\pi_2(s_2, s_2) > \pi_2(s_2, s) \forall\ s$,

(c) $\pi_1(s_3, s_3) > \pi_1(s, s_3)$ *and* $\pi_2(s_3, s_3) > \pi_2(s_3, s) \forall\ s$,

(d) $\pi_1(s_4, s_1) > \pi_1(s, s_1)$ *and* $\pi_2(s_4, s_1) > \pi_2(s_4, s) \forall\ s$,

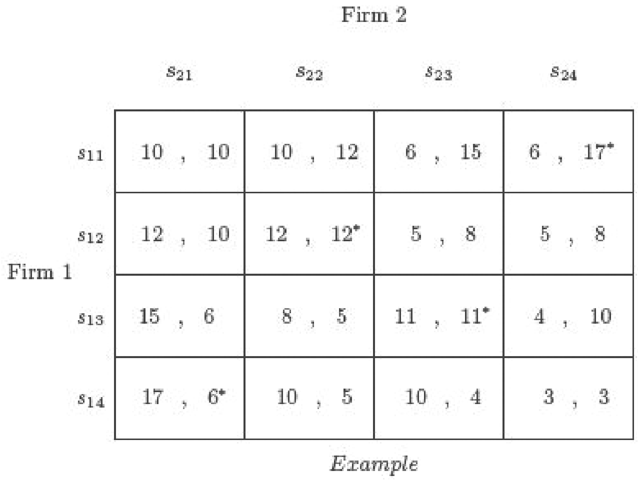
where $\pi_{\mathrm{i}}(s_k, s_l)$ *is i"s payoff when firm 1 chooses* s_k *and firm 2 chooses* s_l.

Proposition 5. *Under high-cost non-drastic innovation where* $\pi_{\mathrm{ij}}^{\mathrm{c}} < \mathrm{c_{iu}} < \mathrm{c_{id}}$, *the pure strategy NEP's are the strategy combinations* $\{s_1, s_4\}$, $\{s_1, s_2\}$, $\{s_3, s_3\}$, *and* $\{s_4, s_1\}$ *with payoffs* $(\mathrm{r_1};\ V_{\mathrm{2u}}^{\mathrm{m}} + V_{\mathrm{2d}}^{\mathrm{m}} + \mathrm{r_2})$, $(V_{\mathrm{1u}}^{\mathrm{m}} + \mathrm{r_1};\ V_{\mathrm{2u}}^{\mathrm{m}} + \mathrm{r_2})$, $(V_{\mathrm{1d}}^{\mathrm{m}} + \mathrm{r_1};\ V_{\mathrm{2d}}^{\mathrm{m}} + \mathrm{r_2})$, *and* $(V_{\mathrm{1u}}^{\mathrm{m}} + V_{\mathrm{1d}}^{\mathrm{m}} + \mathrm{r_1};\ \mathrm{r_2})$ *respectively.*

Remark Under two of the four equilibria we observe a passive incumbent and an aggressive entrant. The entrant monopolist diversifies across both its own product line and the entrant's product line, whereas the incumbent sticks with the old product and is partly preyed upon and replaced by the aggressive entrant. In the other two equilibria both firms are actively involved in diversification and specialization process. In the former equilibrium firms stay in their own market and upgrade in their own product lines. In the latter equilibrium, they diversify only in the incumbent's product line; and in this process of switching they partly replace the incumbent and are partly replaced by the entrant in their old markets.

Example. $\pi_{\mathrm{i}}^{\mathrm{m}} = 10$, $\mathrm{c_{iu}} = 4$, $\mathrm{c_{id}} = 5$, $V_{\mathrm{iu}}^{\mathrm{m}} = 6$, $V_{\mathrm{id}}^{\mathrm{m}} = 5$, $V_{\mathrm{iu}}^{\mathrm{c}} = -1$, $V_{\mathrm{id}}^{\mathrm{c}} = -2$, $\mathrm{r_1} = \mathrm{r_2} = 6$. It is a fairly easy exercise to plug in the values above into the generalized normal form given in Table 2 and see that $\{s_1, s_4\}$, $\{s_2, s_2\}$, $\{s_3, s_3,\}$ and $\{s_4, s_1\}$ are the NEP's with corresponding payoffs of (6,17), (12,12), (11,11), and

(17,6) respectively. r_1 and r_2, are given equal values to make the payoffs symmetric. Their equality is not necessary to drive the results.



Firm 2

		s_{21}	s_{22}	s_{23}	s_{24}
Firm 1	s_{11}	10 , 10	10 , 12	6 , 15	6 , 17*
	s_{12}	12 , 10	12 , 12*	5 , 8	5 , 8
	s_{13}	15 , 6	8 , 5	11 , 11*	4 , 10
	s_{14}	17 , 6*	10 , 5	10 , 4	3 , 3

Example

Fig. 4. High cost non-drastic innovations.

Lemma 4. *Suppose* $\mathrm{c_{iu}} \leq \pi^{\mathrm{c}}_{\mathrm{ij}}\ \forall\ i, j = 1, 2\ \ \{V^{\mathrm{c}}_{\mathrm{iu}} \geq 0\}$. *Then,*

(a) $\pi_1(s_2, s) \geq \pi_1(s_1, s)$ *and* $\pi_2(s, s_2) \geq \pi_2(s, s_1)\ \forall\ s$,

(b) $\pi_1(s_4, s) \geq \pi_1(s_3, s)$ *and* $\pi_2(s, s_4) \geq \pi_2(s, s_3)\ \forall\ s$,

where $\pi_{\mathrm{i}}(s_k, s_l)$ *is i's payoff when firm 1 chooses* s_k *and firm 2 chooses* s_l.

2.2. Low Cost Non-drastic Innovations

I classify the non-drastic innovations as low cost if the competitive payoffs the entrant and the incumbent would separately earn in any market following a joint adoption of an innovation are strictly positive ($V^c_{iu} > 0$, $V^c_{id} > 0$).

Proposition 6. *Under low-cost non-drastic innovations where* $\mathrm{c}_{iu} < \mathrm{c}_{id} < \pi^c_{ij}$, $\{\mathrm{V}^c_{iu} > 0, \mathrm{V}^c_{id} > 0\}$ *the NEP is the strategy combination* $\{s_4, s_4\}$ *with payoffs* $(\mathrm{V}^c_{1u} + \mathrm{V}^c_{1d} + r_1; V^c_{2u} + \mathrm{V}^c_{2d} + r_2)$. *Thus, each monopolist upgrades both in its own product line and diversifies into the competitor's product line while keeping the monopoly position in the old product market which is partly replaced by the innovation.*

Remark. Since innovations are non-drastic each firm stays as a monopolist in the old product market and also guarantees a non-negative payoff by upgrading its own product even if the competitor diversifies. On the other hand, diversifying also yields a non-negative competitive payoff. Thus, each monopolist successfully enters the incumbent's market. Although the incumbent can not prevent entry, it avoids being partly preyed upon by the entrant, through upgrading its own product. Monopoly is replaced by competition in both of the new product markets since each monopolist both upgrades its own product line and diversifies into the competitor's product line. However, each firm maintains the monopoly position in the old product market which, in part, is replaced by the innovation. The costly diversification may be justified for the presumably lower prices that would result under competition.

Example. $\pi_{\mathrm{i}}^{\mathrm{m}} = 10$, $\mathrm{c}_{\mathrm{iu}} = 2$, $\mathrm{c}_{\mathrm{id}} = 3$, $V_{\mathrm{iu}}^{\mathrm{m}} = 8$, $V_{\mathrm{id}}^{\mathrm{m}} = 7$, $V_{\mathrm{iu}}^{\mathrm{c}} = 2$, $V_{\mathrm{id}}^{\mathrm{c}} = 1$, $\mathrm{r}_1 = \mathrm{r}_2 = 6$.

Firm 2

Firm 1

	s_{21}	s_{22}	s_{23}	s_{24}
s_{11}	10 , 10	10 , 14	6 , 17	6 , 21
s_{12}	14 , 10	14 , 14	8 , 11	8 , 15
s_{13}	17 , 6	11 , 8	13 , 13	7 , 15
s_{14}	21 , 6	15 , 8	15 , 7	9 , 9*

Example

Fig. 5. Low cost non-drastic innovations.

When the above values are inserted into the generalized normal form given in table 2 we see that the NEP is $\{s_4, s_4\}$, and the corresponding payoffs are (9,9).

2.3. Medium Cost Non-drastic Innovations

Medium cost innovations are defined such that competition with an upgraded product is profitable ($V_{\mathrm{iu}}^{\mathrm{c}} \geq 0$) but diversifying to compete with an upgraded product is not ($V_{\mathrm{id}}^{\mathrm{c}} \leq 0$).

Proposition 7. *Under medium-cost non-drastic innovation where* $\mathrm{c}_{iu} \leq \pi_{ij}^{c} \leq \mathrm{c}_{id}$ $\{\mathrm{V}_{iu}^{c} > 0;\ \mathrm{V}_{id}^{c} > 0\}$, *the NEP is the strategy combination* $\{s_2, s_2\}$ *with payoffs* $(\mathrm{V}_{1u}^{m} + r_1;\ \mathrm{V}_{2u}^{m} + r_2)$. *Thus, the monopolists stay in their product lines and upgrade their own products.*

Remark. Since innovations are non-drastic each firm stays as a monopolist in the old product market and also guarantees a non-negative payoff by upgrading its own product even if the competitor diversifies. On the other hand, diversifying yields a non-positive competitive payoff. Thus, each incumbent effectively prevents entry by adopting only those innovations in its own product line and maintains its monopoly position. This is the optimal outcome from both the firms' and the society's standpoint. Both innovations are undertaken by the low cost firms established in those product lines. The threat of an incumbent prevents costly diversification which is desirable. Yet, both innovations are adopted through low cost upgrading by the incumbent firms which maintain their monopoly positions in their respective product lines. *Example:* $\pi_{\mathrm{i}}^{\mathrm{m}} = 10$, $\mathrm{c_{iu}} = 3$, $\mathrm{c_{id}} = 5$, $V_{\mathrm{iu}}^{\mathrm{m}} = 7$, $V_{\mathrm{id}}^{\mathrm{m}} = 5$, $V_{\mathrm{iu}}^{\mathrm{c}} = 1$, $V_{\mathrm{id}}^{\mathrm{c}} = -1$, $\mathrm{r}_1 = \mathrm{r}_2 = 6$. A simple inspection upon plugging the values above into table 2 shows that (s_2, s_2) is the NEP, with the corresponding payoffs of (13, 13).

Firm 2

		s_{21}	s_{22}	s_{23}	s_{24}
Firm 1	s_{11}	10 , 10	10 , 13	6 , 15	6 , 18
	s_{12}	13 , 10	13 , 13*	7 , 9	7 , 12
	s_{13}	15 , 6	9 , 7	11 , 11	5 , 12
	s_{14}	18 , 6	12 , 7	12 , 5	6 , 6

Example

Fig. 6. Medium cost non-drastic innovations.

2.4. Summary of Results

If the innovation is (*i*) non-drastic and high cost type, then the firms would find themselves in a multiple equilibria in pure strategies which ranges from sticking with their status quo to diversifying across both product lines; if the innovation is (*ii*) non-drastic and low cost type, they would upgrade and diversify across both product lines by adopting both of the innovations; and finally if the innovation is (*iii*) non-drastic and medium cost type, they would upgrade in their own product lines and maintain their monopoly positions.

Multiple equilibria arises with non-drastic innovation if competitive payoffs of both the entrant and the incumbent are strictly negative $(\pi_{ij}^{c} < \mathrm{c}_{iu} < \mathrm{c}_{id})$; in other words if the innovations are high cost type. Under this scenario a multiplicity of best reply strategies for each monopolist range from adopting both innovations, to not adopting any of the innovations. Both innovations are adopted, however, under any of the possible Nash equilibria. An interesting point is that, under two of the four possible equilibria, we observe a passive incumbent and an aggressive entrant. The entrant monopolist diversifies across both its own product line and the entrant's product line, whereas the incumbent sticks with the old product and is partly preyed upon and replaced by the aggressive entrant. In the other two multiple equilibria both firms are actively involved in diversification and specialization process. In one of the equilibrium firms stay in their own market and diversify in their own product lines. In the other equilibrium, they diversify only in the incumbent's product line; and in this process of switching they partly replace the incumbent and are partly replaced by the entrant in their old markets.

Under the second type of equilibrium with low cost innovations, firms upgrade products not only in their own product line but also in the incumbent's product line. This type of total diversification arises when competitive payoffs from diversifying into the competitor's product line is non-negative ($\mathrm{c}_{iu} < \mathrm{c}_{id} < \pi_{ij}^{c}$). This latter result is obtained under both the drastic and non-drastic low cost innovations.

Under the third type of Nash equilibrium, monopolists stay in their own markets and increase their specialization and upgrading of existing products. We get this result under Nash equilibrium with medium cost non-drastic innovations ($\mathrm{c}_{iu} \leq \pi_{ij}^{c} \leq$ $\leq \mathrm{c}_{id}$), i.e. when the cost of diversifying in another product line exceeds the flow profits of a possible competitive outcome.

The main tendency is that if the firms are facing non-drastic innovations, then they would either upgrade in their own product lines only, or upgrade and diversify into both product lines. Unlike the case of drastic innovation, firms do have the incentive to upgrade their own products even without the technological rivalry. The existence of a potential threat of entry into the incumbent's product line enhances the process of diversification and the firms might find themselves with excessive diversification across all possible product lines within their technological reach. The optimal outcome from both the society's and the firms' standpoints dictates that the firms adopt innovations in their own product lines since the same maximum product diversity could be achieved by the least cost monopolist. However, not only that this can not be enforced as a credible commitment, but it would also imply that the incumbents' monopoly positions would have to remain unchallenged. Clearly, this process of strategic inventiveness is in accord with the Schumpeterian concept of "creative destruction".

3. A Model of Innovation Adoption under Asymmetric Information

Next I turn to an asymmetric information scenario. Asymmetry means that one of the firms has access to private information which the other firm does not, and can be justified by historical leadership of a firm in R&D activity and other reputation effects. Hence, the informed firm is recognized and its leadership is accepted, giving it the first-mover advantage. Yet, the resulting signal about its private information would enable the uninformed firm to form conjectures, update its prior beliefs and make assessments.

In this section I work with the non-drastic innovation case and consider the drastic innovation as a special case. *(Refer to Table 2. for the generalized normal form game.)*

Payoffs are as defined in the base model of section 3. I shall, however, suppress the first subscript of the above notation when writing the payoffs under different strategy combinations. For example, the payoff to firm 2 under s_{12} and s_{22} strategy combinations will be denoted by $\pi_2(s_2, s_2) = V^m_{2u} + r$, and the strategy combination (s_4, s_4) now means that the firms adopt the innovations both in their product lines (upgrading) and across product lines (diversifying) while continuing to produce the old products.

Thus, the firms become Cournot competitors in the new product markets and maintain their monopoly position in the – now suppressed – old product markets. In this case, the payoffs to firm 1 and firm 2, in respective order, are as follows:

$$\pi_1(s_4, s_4) = V^c_{1u} + V^c_{2d} + r;$$
$$\pi_2(s_4, s_4) = V^c_{2u} + V^c_{2d} + r.$$

Next, recall that there are types of the players determined by 3 possible cost structures defined with respect to the competitive payoffs of each firm. I identify each cost structure with a possible firm type, denoted by N_{ij}(j^{th}type of firm i). These firm types are given as follows:

$$\begin{aligned} &(i) \quad N_{i1} : V^c_{iu} \leq 0, V^c_{id} < 0 \quad (\pi^c \leq c_{iu} < c_{id}),\\ &(ii) \quad N_{i2} : V^c_{iu} > 0, V^c_{id} < 0 \quad (c_{iu} < \pi^c < c_{id}),\\ &(iii) \quad N_{i3} : V^c_{iu} > 0, V^c_{id} \geq 0 \quad (c_{iu} < c_{id} \leq \pi^c).\end{aligned}$$

From the incumbent's point of view, the competitor's type is an indicator of whether it is a low cost, medium cost, or a high cost entrant. From the entrant's point of view, the incumbent's type is an indicator of whether it would fight back to block entry, accommodate the entrant and share the new product market, or yield the monopoly position in its product line.

Suppose, firm 1 has private information about its own type and that of firm 2, but firm 2 does not – throughout this paper we shall assume that firm 1 is the informed player, and first mover. Firm 1 observes both its own type (N_{1j}) and its competitor's type (N_{2j}).

Firm 2, on the other hand, knows only the adoption costs in its own product line c_{2u} which enables it to conclude whether $V^c_{2u} \leq 0$ or $V^c_{2u} > 0$. If, in fact, c_{2u} is too high such that $V^c_{2u} \leq 0$, then it follows that $V^c_{2u} < 0$ since by initial condition(ii) we have $c_{2u} < c_{2d}$. This enables firm 2 to conclude that its type is (N_{21}). Firm 2 does not need additional information to decide whether its type is (N_{21}) or not.

When firm 2 is of type (N_{21}), its best reply to the strategy played by firm 1 is based on only the information that it is a high cost firm. On the other hand, firm 2 has imperfect information about firm 1's type. Firm 2 knows that firm 1 is the informed player and has complete information about the types of both firms. This information advantage enables firm 1 to be the first mover regardless of its own type, and firm 2 accepts its leadership.

Lemma 5. *For the uninformed firm, its best reply to $s_{11}$is given as follows:*

$$s_{24} = b_2(s_{11}) \quad \forall \quad N_{2j}, j = 1, 2, 3.$$

Proof of lemma.

From Table 1 we immediately see that,

$$\begin{aligned} \pi_2(s_1, s_4 \mid N_{2j}) - \pi_2(s_1, s_3 \mid N_{2j}) &= ((V_{2u}^m + V_{2d}^m + r) - \pi^m + V_{2d}^m) = \\ &= V_{2u}^m \quad + \quad r - \pi^m \geq 0 \end{aligned}$$

$$\begin{aligned} \pi_2(s_1, s_4 \mid N_{2j}) - \pi_2(s_1, s_2 \mid N_{2j}) &= (V_{2u}^m + V_{2d}^m + r) - (V_{2u}^m + r) = \\ &= V_{2u}^m > 0 \end{aligned}$$

$$\begin{aligned} \pi_2(s_1, s_4 \mid N_{2j}) - \pi_2(s_1, s_1 \mid N_{2j}) &= (V_{2u}^m + V_{2d}^m + r) - (\pi^m) = \\ &= (V_{2u}^m \quad + \quad r - \pi^m) + V_{2d}^m > 0 \end{aligned}$$

We denote firm 2's best reply to the strategy s played by firm 1 by $b_2(s)$:

$$b_2(s) = \arg\max_{s'} \pi_2 \; \pi_2(s, s').$$

Then, a Stackelberg equilibrium is a pair of strategies $(s^*, b_2(s^*))$ such that

$$s^* = \arg\max_{s} \pi_1 \, (s^*, b_2(s^*)).$$

Next, the following lemma is used to construct normalized strategies of firm 2 as a function of its type, N_{2j}.

Lemma 6. *For the uninformed firm, its best replies given that it knows its type, are given as follows:*

$$\textbf{(a)} \qquad \begin{aligned} s_{22} &= b_2(s_{12}) \quad iff \quad N_2 = N_{21}, N_{22}, \\ s_{24} &= b_2(s_{12}) \quad iff \quad N_2 = N_{23}; \end{aligned}$$

$$\textbf{(b)} \qquad \begin{aligned} s_{23} &= b_2(s_{13}) \quad iff \quad N_2 = N_{21}, \\ s_{24} &= b_2(s_{13}) \quad iff \quad N_2 = N_{22}, N_{23}; \end{aligned}$$

Table 3: Stackelberg equilibria and payoffs under perfect information

Firm 1's Leader Strategy	*Firm2's Best Reply*	*Firm 1's Payoff*	*Firm 2's Payoff*
s_1	s_4	r	$V_{2u}^m + V_{2d}^m + r$
s_2	$s_2,\ if\ V_{2d}^c < 0$ $s_4,\ if\ V_{2d}^c > 0$	$V_{1u}^m + r$ $V_{1u}^c + r$	$V_{2u}^m + r$ $V_{2u}^m + V_{2d}^c + r$
s_3	$s_3,\ if\ V_{2u}^c < 0$ $s_4,\ if\ V_{2u}^c \geq 0$	$V_{1u}^m + r$ $V_{1d}^c + r$	$V_{2d}^m + r$ $V_{2u}^c + V_{2d}^m + r$
s_4	$s_1,\ if\ V_{2u}^c < 0,\quad V_{2d}^c < 0$ $s_2,\ if\ V_{2u}^c > 0,\quad V_{2d}^c < 0$ $s_4,\ if\ V_{2u}^c > 0,\quad V_{2d}^c > 0$	$V_{1u}^m + V_{1d}^m + r$ $V_{1u}^m + V_{1d}^c + r$ $V_{1u}^c + V_{1d}^c + r$	r $V_{2u}^c + r$ $V_{2u}^c + V_{2d}^c + r.$

$$\textbf{(c)} \qquad \begin{aligned} s_{21} &= b_2(s_{14}) \quad iff \quad N_2 = N_{21}, \\ s_{22} &= b_2(s_{14}) \quad iff \quad N_2 = N_{22}, \\ s_{24} &= b_2(s_{14}) \quad iff \quad N_2 = N_{23}. \end{aligned}$$

Table 3 summarizes firm 2's best reply strategies as a function of its type and the corresponding payoffs to both firms.

Lemma 7. *For the informed firm, the following payoff dominance relationships hold if the uninformed firm is of type* $N_{2J}\,\{J = 2, 3\,\}$*:*

$$\textbf{(a)} \qquad \begin{aligned} &\pi_1(s_2, s_2 \mid N_{11}) > \pi_1(s_k, s_2 \mid N_{11}) \quad \forall \quad k = 1, 3, 4, \\ &\pi_1(s_1, s_4 \mid N_{11}) > \pi_1(s_k, s_4 \mid N_{11}) \quad \forall \quad k = 2, 3, 4; \end{aligned}$$

$$\textbf{(b)} \qquad \begin{aligned} &\pi_1(s_2, s_2 \mid N_{12}) > \pi_1(s, s_2 \mid N_{12}), \\ &\pi_1(s_2, s_4 \mid N_{12}) > \pi_1(s, s_4 \mid N_{12}); \end{aligned}$$

$$\textbf{(c)} \qquad \begin{aligned} &\pi_1(s_4, s_2 \mid N_{13}) > \pi_1(s, s_2 \mid N_{13}), \\ &\pi_1(s_1, s_4 \mid N_{13}) > \pi_1(s, s_4 \mid N_{13}). \end{aligned}$$

3.1. Stackelberg Equilibria

It should be noted that only in the case where firm 1 leads by playing its s_{11} strategy, is firm 2's best reply not a function of its type; i.e. firm 2 does not need to know its type to play its best reply strategy, s_{24}. Consequently we can write the following proposition.

Proposition 8. *If firm 1 has perfect information, then* (s_4, s_1) *is a Stackelberg equilibrium if and only if firm 2's type is* N_{21}.

Proof of Proposition 8 can be easily proved by inspecting Table 2.

If (s_4, s_1) is a Stackelberg equilibrium, then from the last part of Table 3, $s_1 =$ $= b_2(s_4)$ if $V^c_{2u} > 0$, $V^c_{1d} < 0$, i.e. firm 2's type is N_{21}. Conversely, suppose firm 2's type is N_{21}. Then, consider each strategy of firm 1:

$$\begin{aligned}
&s_4 = b_2(s_1) \Rightarrow \pi_1(s_1, b_2(s_1)) = r,\\
&s_2 = b_2(s_2) \Rightarrow \pi_1(s_2, b_2(s_2)) = V^m_{1u} + r,\\
&s_3 = b_2(s_3) \Rightarrow \pi_1(s_3, b_2(s_3)) = V^m_{1u} + r,\\
&s_1 = b_2(s_4) \Rightarrow \pi_1(s_4, b_2(s_4)) = V^m_{1u} + V^m_{1d} + r.
\end{aligned}$$

From this we see that $s_4 = \arg\max_s \pi_1(s, b_2(s))$ and $s_1 = b_2(s_4)$. Thus, (s_4, s_1) is a Stackelberg equilibrium.

Remark. Under (N_{11}, N_{21}), (N_{12}, N_{21}) and (N_{13}, N_{21}) the uninformed player can deduce that it is a high cost firm without further information or signaling by firm 1, the informed player. Under the above states of the world, firm 2 need not know which type of a competitor (high cost/medium cost/low cost) it is facing. Firm 2's best response solely depends on its own type N_{21}. Firm 1 benefits from its information advantage only because it is the first mover. An interesting aspect of the NEP is that it is determined without any reference to the leader's type. Firm 1 might be a high cost or medium cost firm (N_{11} or N_{12}). But this is irrelevant for the particular NEP obtained, as long as firm 1 is the first mover. Firm 1 uses the first mover advantage; it upgrades and diversifies to obtain monopoly position in both the old and the new product markets. Firm 2, on the other hand, continues to produce the old product and exploits the residual demand as a declining monopolist.

Proposition 9. *The following strategy combinations obtain under the following states of the world*[9] *when firm 1 is the Stackelberg leader, and firm 2 is the follower.*
If (N_{11}, N_{22})*, then the Stackelberg equilibrium is* (s_2, s_2)*;*
If (N_{12}, N_{22})*, then the Stackelberg equilibrium is* (s_2, s_2)*;*
If (N_{13}, N_{22})*, then the Stackelberg equilibrium is* (s_4, s_2)*;*

---

[9] *A State of the World* is defined as the occurrence of a combination of the types of two firms such that $(N_{1i}, N_{2j}) \quad i, j = 1, 2, 3$.

If (N_{11}, N_{23})*, then the Stackelberg equilibrium is* (s_1, s_4)*;*
If (N_{12}, N_{23})*, then the Stackelberg equilibrium is* (s_2, s_4)*;*
If (N_{13}, N_{23})*, then the Stackelberg equilibrium is* (s_4, s_4)*.*

Proof. Analogous to the proof of Proposition 8 as it is seen from Table 3.

3.2. Perfect Bayesian Equilibrium with a medium/low cost follower

The Solution Concept:

To firm 2, firm 2 is either type N_{22} or N_{23}; firm 1 can be any of N_{11}, N_{12}, N_{13}. Being uninformed, firm 2 assigns prior probabilities $\Pr(N_{22}) = \phi$, $\Pr(N_{23}) = 1 - \phi$; $\Pr(N_{11}) = \theta_1$; $\Pr(N_{12}) = \theta_2$; and $\Pr(N_{13}) = \theta_3$, $\phi \geq 0$, $\theta_j \geq 0$, $\theta_1 + \theta_2 + \theta_3 = 1$.

These prior beliefs are common knowledge, i.e., they are known to both firms. Firm 2 knows that firm 1 knows its prior beliefs, and firm 1 knows that firm 2 knows that firm 1 knows firm 2's beliefs, and so on. Firm 2 has to update its beliefs after observing certain strategies played by firm 1. Suppose for example, that firm 2 observes s_{12} being played by firm 1. Her type might be either N_{22} or N_{23}. Seeing s_{12} should make firm 2 update the posterior belief, $\Pr(N_{22} \mid s_{12})$. The natural method is to use Bayes's rule, which shows how to revise the prior belief in the light of data. It uses two pieces of information that firm 2 knows: the likelihood of seeing s_{12} given that the state of the world is N_{22}, $\Pr(s_{12} \mid N_{22})$, and the likelihood of s_{12} given that the state of the world is, N_{23}, $\Pr(s_{12} \mid N_{23})$. Since there are only 2 alternatives to firm 2's type, the marginal likelihood of seeing s_{12} as a result of one or another possible types of firm 2 (N_{22} or N_{23}) is given by $\Pr(s_{12}) = \Pr(s_{12} \mid N_{22}) \Pr(N_{22}) + \Pr(s_{12} \mid N_{23}) \Pr(N_{23})$.

The probability that both the strategy s_{12} played and the state of the world N_{22} occurs is:

$$\begin{aligned}\Pr(s_{12}, N_{22}) = \Pr(s_{12} \mid N_{22}) \Pr(N_{22}) = \\ = \Pr(N_{22} \mid s_{12}) \Pr(s_{12})\end{aligned}$$

Firm 2's new belief – its posterior- is calculated using $\Pr(s_{12})$, which yields the following Bayes Rule:

$$\Pr(N_{22} \mid s_{12}) = \frac{\Pr(s_{12} \mid N_{22}) \Pr(N_{22})}{\Pr(s_{12})}.$$

The term Bayesian equilibrium is used to refer to Nash equilibrium when players update their beliefs according to Bayes's rule ([Rasmussen, 1989]). Perfect Bayesian equilibrium point (PBEP) is a Stackelberg equilibrium ($s*$, $\beta_2(s^*)$) where $\beta_2(s^*)$ is the Bayesian best reply to $s*$ such that

$$\beta_2(s) = \arg\max_{s'} \pi_2 E\pi_2(s, s'), \quad s^* = \arg\max_{s'} \pi_1\,(s, \beta_2(s)).$$

3.3. Characterization of Equilibrium

The main focus of this section is Theorem 2. Its proof shall be accomplished in a series of lemmas and observations in Appendix 2.

Table! 4: Reduced strategic game with non-drastic innovations under imperfect information

		Firm 2	
		s_{22}	s_{24}
Firm 1	s_{11}	π_{11}^m , $V_{2u}^m + r$	r , $V_{2u}^m + V_{2d}^m + r$
	s_{12}	$V_{1u}^m + r$, $V_{2u}^m + r$	$V_{1u}^c + r$, $V_{2u}^m + V_{2d}^m + r$
	s_{13}	$\pi_{11}^m + V_{1d}^c$, $V_{2u}^c + r$	$V_{1d}^c + r$, $V_{2u}^c + V_{2d}^m + r$
	s_{14}	$V_{1u}^m + V_{1d}^c + r$, $V_{2u}^c + r$	$V_{1u}^c + V_{1d}^c + r$, $V_{2u}^c + V_{2d}^c + r$

Theorem 2. *Under asymmetric, imperfect and incomplete information where firm 1 is the informed player and where the states of the world are* (N_{1j}, N_{2l}), $\forall = 1, 2, 3$, $l = 2, 3$, *the perfect Bayesian equilibrium points are the following strategy combinations:*

(a): (s_1, s_4) iff $(-)\dfrac{(V_{2d}^c \mid, N_{22})}{(V_{2d}^c \mid, N_{23})} \leq \dfrac{\theta_2(1-\phi)}{(\theta_1 + \theta_2)\phi}$ *and* $N_1 = N_{11}$,

(b): (s_2, s_4) iff $(-)\dfrac{(V_{2d}^c \mid, N_{22})}{(V_{2d}^c \mid, N_{23})} \leq \dfrac{\theta_2(1-\phi)}{(\theta_1 + \theta_2)\phi}$ *and* $N_1 = N_{12}$,

(c): (s_4, s_4) iff $(-)\dfrac{(V_{2d}^c \mid, N_{22})}{(V_{2d}^c \mid, N_{23})} \leq \dfrac{(1-\phi)}{\phi}$ *and* $N_1 = N_{13}$,

(d) : (s_2, s_2) iff $(-)\dfrac{(V_{2d}^c \mid, N_{22})}{(V_{2d}^c \mid, N_{23})} \geq \dfrac{\theta_2(1-\phi)}{(\theta_1 + \theta_2)\phi}$ *and* $N_1 = N_{11}, N_{12}$,

(e) : (s_4, s_2) iff $(-)\dfrac{(V_{2d}^c \mid, N_{22})}{(V_{2d}^c \mid, N_{23})} \leq \dfrac{(1-\phi)}{\phi}$ *and* $N_1 = N_{13}$.

First, recall from weak dominance firm 2's dominant strategies are s_{22} and $s_{24}$when it is either medium or low cost (see Table 4.2). Hence, firm 1 effectively faces a normal form game of dimension (4x2) if firm 2 is of either N_{22} or N_{23} type (see Table 4 for the reduced normal form).

I assume that firm 1, being an informed player, will always strive to reach the Stackelberg equilibrium that corresponds with its observed state. This assumption allows us to derive firm 2's conjectures of observing a strategy s_{1k}conditional on her type N_{2j}, i.e. $\Pr(s_{1k} \mid N_{2j})\ \forall\ J = 2,3 \quad \& \quad k = 1,\ldots,4$.

Lemma 8. *For the uninformed firm the following conjectures hold:*

$$\begin{aligned}
(a) : Pr(s_{11},N_{22}) &= \sum_{i=1}^{3}\Pr(s_{11} \mid N_{1i}, N_{22})\Pr(N_{1i}) = \Pr(s_{11} \mid N_{11}, N_{22})\Pr(N_{11}) + \\
&+\Pr(s_{11} \mid N_{12}, N_{22})\Pr(N_{12}) + \Pr(s_{11} \mid N_{13}, N_{22})\Pr(N_{13}) = 0; \\
\Pr(s_{11}, N_{23}) &= \sum_{i=1}^{3}\Pr(s_{11} \mid N_{1i}, N_{23})\Pr(N_{1i}) = \Pr(s_{11} \mid N_{11}, N_{23})\Pr(N_{11}) + \\
&+ \Pr(s_{11} \mid N_{12}, N_{23})\Pr(N_{12}) + \Pr(s_{11} \mid N_{13}, N_{23})\Pr(N_{13}) = \\
&= \Pr(N_{11}) = \theta_1; \\
(b) : \Pr(s_{12}, N_{22}) &= \sum_{i=1}^{3}\Pr(s_{12} \mid N_{1i}, N_{22})\Pr(N_{1i}) = \Pr(s_{12} \mid N_{11}, N_{22})\Pr(N_{11}) + \\
&+ \Pr(s_{12} \mid N_{12}, N_{22})\Pr(N_{12}) + \Pr(s_{12} \mid N_{13}, N_{22})\Pr(N_{13}) = \\
&= \Pr(N_{11}) + \Pr(N_{12}) = \theta_1 + \theta_2; \\
\Pr(s_{12}, N_{23}) &= \sum_{i=1}^{3}\Pr(s_{12} \mid N_{1i}, N_{23})\Pr(N_{1i}) = \Pr(s_{12} \mid N_{11}; N_{23})\Pr(N_{11}) + \\
&+ \Pr(s_{12} \mid N_{12}, N_{23})\Pr(N_{12}) + \Pr(s_{12} \mid N_{13}, N_{23})\Pr(N_{13}) = \\
&= \Pr(N_{12}) = \theta_2; \\
(c) : \Pr(s_{14}, N_{22}) &= \sum_{i=1}^{3}\Pr(s_{12} \mid N_{1i}, N_{22})\Pr(N_{1i}) = \Pr(s_{14} \mid N_{11}, N_{22})\Pr(N_{11}) + \\
&+ \Pr(s_{14} \mid N_{12}, N_{22})\Pr(N_{12}) + \Pr(s_{14} \mid N_{13}, N_{22})\Pr(N_{13}) = \\
&= \Pr(N_{13}) = \theta_3, \\
\Pr(s_{14}, N_{23}) &= \sum_{i=1}^{3}\Pr(s_{14} \mid N_{1i}, N_{23})\Pr(N_{1i}) = \Pr(s_{14} \mid N_{11}, N_{23})\Pr(N_{11}) + \\
&+ \Pr(s_{14} \mid N_{12}, N_{23})\Pr(N_{12}) + \Pr(s_{14} \mid N_{13}, N_{23})\Pr(N_{13}) = \\
&= \Pr(N_{13}) = \theta_3.
\end{aligned}$$

Remark on Separating and Pooling Equilibria:

Suppose part **(d)** of Theorem 2 holds, so that firm 2's best reply to s_{12} is s_{22}. We note that under **(d)** firm 1 can be either a high cost type or a medium cost type. Clearly, by deciding on a best reply of s_{22} under **(d)** firm 2 can not differentiate between the two types of competitors. From lemma 7(b), recall that a medium cost firm 1's best reply to either s_{22} or s_{24} is s_{12}. From lemma 7(a), also recall that a high cost type firm 1's best reply to s_{22} is also s_{12}. This indicates that under **(d)** an informed high cost firm successfully pretends that it is medium cost type. Under **(d)** we obtain a pooling equilibrium. Next, suppose **(a)** holds. Notice that **(a)** is a perfectly symmetric condition to **(d).** In this case, firm 2's best reply to s_{12} is s_{24}. But **(a)** holds only if firm 1 is high cost type so that his best reply to s_{24} is s_{11}. In this case, a high cost firm 1 is successfully differentiated from a medium cost one. Under **(a)** we obtain a separating equilibrium.

Rational Priors at a Boundary Payoff:

$V^c_{id} = 0$. Recall that a firm is defined as medium cost–type N_{i2} – if $V^c_{iu} > 0$ and $V^c_{id} < 0$ hold; and it is defined as low cost–type N_{i3}– if $V^c_{iu} > 0$ and $V^c_{id} \geq 0$ hold. Hence, note that the lower bounds of V^c_{id} under the two types are given by the following equations:

$$(V^c_{id} \mid N_{i2}) = -c_{id}, \quad (V^c_{id} \mid N_{i3}) = 0.$$

Corollary 1. *Suppose the uninformed firm's competitive payoff is given by,* $(V^c_{id} \mid N_{23}) = 0$. *Then, a separating equilibrium is obtained if and only if the following priors hold:* $1 - \phi = 1$, *and/or* $\theta_3 = 1$.

Proof.

Suppose firm 2 observes s_{12}. Then, from the proof of Theorem 2 part (b) we note that firm 2's best reply to s_{12} is s_{24} if and only if:

$$0 \leq \theta_2(1 - \phi)(V^c_{2d} \mid N_{23}) + (\theta_1 + \theta_2)\phi(V^c_{2d} \mid N_{22}).$$

Substituting $(V^c_{2d} \mid N_{23}) = 0$ into the above equation we obtain,

$$0 \leq (\theta_1 + \theta_2)\phi(V^c_{2d} \mid N_{11}, N_{22}). \tag{14}$$

But, since $(V^c_{2d} \mid N_{22}) < 0$ then we require either $\theta_1 + \theta_2 = 0$, and/or $\phi = 0$ for (14) to hold. Hence, it is easily seen that: $1 - \phi = 1$, and/or $\theta_3 = 1$.

Remark. If $(V^c_{2d} \mid N_{23})$ then from theorem 2**(b)** we require (14) to hold for a separating equilibrium (s_2, s_4). But, it is immediate that if there is a slight deviation in the priors so that the reverse inequality to (14) holds then we obtain the condition in part **(d)** which gives s_{22} as a Bayesian best reply to s_{12}. Therefore, a pooling equilibrium shall be obtained under part **(d)**. Obviously, firm 2 would prefer a separating equilibrium to a pooling equilibrium as opposed to firm 1 who would rather have a pooling equilibrium. This clearly requires (14) to hold, enabling firm 2 to derive the range of its rational priors as defined in the corollary.

3.4. A Proposed Equilibrium Using "Equally Likely" Assumption

A reasonable way to form priors for firm 2 is to conjecture that its type being N_{22} or N_{23} is equally likely (conditioned on not observing N_{21}), $\Pr(N_{22}) = \Pr(N_{23}) = 0.5$.

Further, suppose that it conjectures its facing a competitor of type N_{11}, N_{12} or N_{13} is also equally likely, $\Pr(N_{11}) = \Pr(N_{12}) = \Pr(N_{13}) = 0.333$.

Based on these common priors, firm 2 would update its beliefs. Accordingly, using the conjecture equations from lemma 8, firm 1 would play s_{12} 66.7 percent of the time, and play s_{14} 33.3 percent of the time, when firm 2's type is N_{22}. Hence, firm 2's conjecture upon observing s_{12} would be $\Pr(s_{11} \mid N_{22}) = 0.667$ and $\Pr(s_{14} \mid N_{22}) = 0.333$.

Similarly, when firm 2's type is N_{23}, firm 1 would play s_{11}, s_{12} and s_{14} 33.3 percent of the time, leading to conjectures $\Pr(s_{11}|N_{23}) = \Pr(s_{12}|N_{23}) = \Pr(s_{14}|N_{23}) = 0.333$

Suppose firm 1 plays its s_{12} strategy. Recall that this situation is characterized under parts **(b)** and **(d)** of theorem 2. Firm 2 would update its beliefs using her priors and Bayes's rule upon observing s_{12}. Then, we have:

$$\Pr(N_{22} \mid s_{12}) = \frac{\{\ \Pr(s_{12} \mid N_{22})\ \}\,\phi}{\{\ \Pr(s_{12} \mid N_{22})\ \}\,\phi + \{\Pr(s_{12} \mid N_{23})\ \}\,(1-\phi)}.$$

Substituting the conjectures,

$$= \frac{(\theta_1 + \theta_2)\phi}{(\theta_1 + \theta_2)\phi + \theta_2(1-\phi)},$$

and finally, substituting the values of priors,

$$\Pr(N_{22} \mid s_{12}) = \frac{(0.667)(0.5)}{(0.667)(0.5) + (0.333)(0.5)} = 0.667.$$

Also,

$$\Pr(N_{23} \mid s_{12}) = 1 - \Pr(N_{22} \mid s_{12}) = 1 - 0.667 = 0.333.$$

Thus, using the notation of theorem 2 we obtain,

$$\frac{\theta_2(1-\phi)}{(\theta_1 + \theta_2)\phi} = \frac{0.333}{0.667} = 0.5.$$

Hence, from the theorem 2, part **(d)**: if the condition $(-)\dfrac{(V_{2d}^c \mid N_{1i}, N_{22})}{(V_{2d}^c \mid N_{1i}, N_{23})} \geq 0.5$ holds, then firm 2's equilibrium strategy upon observing s_{22} is to play s_{22}.

On the other hand, if, instead, firm 2's conjectures satisfy part **(b)** of theorem 2, such that $(-)\dfrac{(V_{2d}^c| N_{1i}, N_{22})}{(V_{2d}^c |N_{1i}, N_{23})} \leq 0.5$ holds, then firm 2's best reply to s_{12} is s_{24}.

Next, suppose firm 1 plays its s_{14} strategy. This situation is characterized under parts **(c)** and **(e)** of theorem 2. Firm 2's updated beliefs would be

$$\Pr(N_{22} \mid s_{12}) = \frac{\{\ \Pr(s_{12} \mid N_{22})\ \}\,\phi}{\{\ \Pr(s_{12} \mid N_{22})\ \}\,\phi + \{\Pr(s_{12} \mid N_{23})\ \}\,(1-\phi)}.$$

substituting the conjectures we have,

$$= \frac{(\theta_1 + \theta_2)\phi}{(\theta_1 + \theta_2)\phi + \theta_2(1-\phi)}.$$

Finally, substituting the values of priors we obtain:

$$\Pr(N_{22} \mid s_{12}) = \frac{(0.667)(0.5)}{(0.667)(0.5) + (0.333)(0.5)} = 0.667,$$

and,

$$\Pr(N_{22} \mid s_{12}) = 1 - \Pr(N_{22} \mid s_{12}) = 1 - 0.667 = 0.333.$$

Thus, using the notation of theorem 2 we obtain,

$$= \frac{\theta_2(1-\phi)}{(\theta_1+\theta_2)\phi} = \frac{0.333}{0.667} = 0.5.$$

Hence, from the theorem 1, part **(d)** if the condition $(-)\dfrac{(V_{2d}^c \mid N_{1i}, N_{22})}{(V_{2d}^c \mid N_{1i}, N_{23})} \geq 0.5$ holds, firm 2's equilibrium strategy upon observing s_{22} is to play s_{22}. On the other hand, if, instead, firm 2's conjectures satisfy part **(b)** of theorem 2, such that $(-)\frac{(V_{2d}^c|N_{1i},N_{22})}{(V_{2d}^c|N_{1i},N_{23})} \leq 0.5$ holds, then firm 2's best reply to s_{12} is s_{24}.

Next, suppose firm 1 plays its s_{14}strategy. This situation is characterized under parts **(c)** and **(e)** of theorem 2. Firm 2's updated beliefs would be

$$\Pr(N_{22} \mid s_{14}) = \frac{\{\ \Pr(s_{14} \mid N_{22})\ \}\,\phi}{\{\ \Pr(s_{14} \mid N_{22})\ \}\,\phi + \{\Pr(s_{14} \mid N_{23})\ \}\,(1-\phi)}.$$

Substituting the conjectures,

$$= \frac{\theta_3\phi}{\theta_3\phi + \theta_3(1-\phi)}.$$

And finally, substituting the values of priors we get:

$$\Pr(N_{22} \mid s_{14}) = \frac{(0.333)(0.5)}{(0.333)(0.5) + (0.333)(0.5)} = 0.5.$$

It follows that, $\Pr(N_{23} \mid s_{14}) = 1 - \Pr(N_{22} \mid s_{14}) = 1 - 0.5 = 0.5$.
Substituting the notation of theorem 2 we obtain,

$$= \frac{(1-\phi)}{\phi} = \frac{0.5}{0.5} = 1.$$

Hence, from the theorem 2, part **(e)**: if the condition $(-)\dfrac{(V_{2d}^c \mid N_{1i}, N_{22})}{(V_{2d}^c \mid N_{1i}, N_{23})} \geq 1$ holds, upon observing s_{14}, firm 2's equilibrium strategy is to play s_{22}. Also, from theorem 1, part **(c)**: if the condition $(-)\dfrac{(V_{2d}^c \mid N_{1i}, N_{22})}{(V_{2d}^c \mid N_{1i}, N_{23})} \leq 1$ holds, then upon observing s_{14} firm 2's equilibrium strategy is to play s_{24}. On the other hand firm 1's best reply does not depend on firm 2's strategy since under type N_{13} its best reply to both s_{22} and

s_{24} is to play s_{14}. Nevertheless, the equilibrium (s_4, s_2) under **(e)** is preferred to the equilibrium (s_4, s_4) under (c) by firm 1, since it yields higher payoffs (see Table 2).

Last, suppose firm 1 plays s_{11} then from lemma 5 we know that firm 2's best reply is to play s_{24}. But this does not suffice for (s_1, s_4) to be a perfect Bayesian equilibrium, as I have argued in the proof of Theorem 2. We check for the binding condition which causes s_{11} to be firm 1's equilibrium strategy

$$\Pr(N_{22} \mid s_{12}) = \frac{\{\Pr(s_{12}| \, N_{22})\ \} \, \phi}{\{\ \Pr(s_{12} \mid N_{22})\ \} \, \phi + \{\Pr(s_{12} \mid N_{23})\ \} \, (1-\phi)}.$$

Substituting for the values of the priors and the "passive conjectures" we get

$$\Pr(N_{22} \mid s_{12}) = \frac{(0.667)(0.5)}{(0.667)(0.5) + (0.333)(0.5)} = 0.667,$$

also

$$\Pr(N_{22} \mid s_{12}) = 1 - \Pr(N_{22} \mid s_{12}) = 1 - 0.667 = 0.333.$$

Substituting the notation of Theorem 2 we obtain,

$$= \frac{\theta_2(1-\phi)}{(\theta_1+\theta_2)\phi} = \frac{0.333}{0.667} = 0.5.$$

Hence, we get the condition **(a)** $(-)\dfrac{(V_{2d}^c \mid N_{1i}, N_{22})}{(V_{2d}^c \mid N_{1i}, N_{23})} < 0.5$ for (s_1, s_4) to be an equilibrium strategy combination. That is firm 1 decides to play its best reply strategy of s_{11}, only after being certain that firm 2's Bayesian best reply to s_{12} is s_{24}. In return, firm 2's best reply to s_{11} is s_{24}. Firm 2 does not require to update its beliefs about its own type in order to decide on a best reply strategy, since from lemma 5 we know that s_{24} is its best reply under both N_{22} and N_{23}. In this case, firm 1 successfully recognizes a high cost competitor and we obtain a separating equilibrium (s_1, s_4) under **(a)**.

Thus, proposed priors and the "passive conjectures" support the set of perfect Bayesian equilibria characterized in theorem 2.

3.5. Concluding Remarks

It is most likely and plausible to define the initial conditions for an interior solution, i.e. that for a low-cost firm competitive payoff from diversification would be strictly positive, whereas for a high-cost firm it would be strictly negative. Hence, the uninformed firm will have to form prior beliefs unconstrained by any rationality criteria which is discussed in the corollary. Theorem 2 allows us to make comparative static conjectures about the likelihood of certain equilibrium outcomes with respect to a change in the uninformed firm's prior beliefs.

Suppose firm 1 is considering to play s_1 strategy, knowing from Theorem 2. That firm 2's best reply is s_4. Then, it is more likely to play s_1 the higher firm 2's prior belief is that it is a low-cost firm, ceteris paribus. Also, it is more likely to play s_1 the higher firm 1's prior belief is that it is facing a medium-cost competitor, ceteris paribus, thereby attaining the (s_1, s_4) equilibrium strategy combination. Hence, the higher the values of $1 - \phi$ and/or θ_2 are, the more likely it is to obtain a separating equilibrium. If the likelihood of firm 2's prior beliefs is such that it strongly believes it is a high-cost firm and that it is facing either a high-cost and/or medium-cost competitor, then the more likely it is that the equilibrium outcome would be (s_1, s_4) supporting a pooling equilibrium, ceteris paribus.

An interesting observation is that it is less likely for a high-cost firm to exploit its information advantage against an optimistic firm 2 which strongly believes that it is a low-cost type, ceteris paribus. With an optimistic firm 2 and a high cost firm 1 it is more likely to observe (s_1, s_4) than (s_2, s_2). On the other hand, a pessimistic firm 2 can be more easily forced to stay in its product line by a high-cost firm 1 which would stay in its product line itself. In this case we observe that firm 1 would be forced to maintain it monopoly position by upgrading its product line even though it would have been less costly for firm 2 to do so if it had chosen to diversify.

Another important observation on this point is that the efficiency choice from the society's standpoint is not one between having monopoly or competition, but it is one between a high-cost monopolist (firm 1) and a low-cost monopolist (firm 2). Hence, from an efficiency standpoint the society is losing from firm 2's ignorance, and more so if it does not have an optimistic outlook. This, possibly, is a supportive argument for having government research labs (or joint research projects coordinated by government institutions, like the MITI in Japan) provide and/or coordinate the flow of scientific information across companies which will enable them assess their potential strengths and weaknesses.

Next, suppose firm 1 plays its s_4 strategy. We know from theorem 2 that firm 1 would do so only when it is a low-cost firm. In other words, firm 1 would stick to s_4 if it is low-cost, no matter what firm 2's prior beliefs are. Firm 2's response, in this case, is a function of its prior beliefs about its own type only. It will tend to play s_4 the higher prior beliefs are that it is a low-cost firm, enabling its to upgrade and diversify into both markets following its competitor, ceteris paribus. This requires firm 2 to be optimistic about its own type, whereas firm 1 would prefer its to be pessimistic and stick to its product line.

Hence, it is more likely to get a competitive outcome (s_4, s_4) in both markets the more optimistic firm 2 is about its own type. On the other hand, it is more likely that a pessimistic firm 2 would lose its monopoly position in its product line by sharing it with firm 1, while firm 1 continues to be a monopolist in its product line.

From an efficiency standpoint, the choice is between having monopoly or competition in firm 1's product line. Since with a more optimistic firm 2 we tend to have competition in both markets, then policies that encourage firm 2 to increase its optimism about diversification should be devised. Such policies would possibly provide additional incentives that would enable downgrading of the uninformed firm's prior beliefs about the costs of diversification.

4. Conclusion

This paper has investigated strategic behavior of technologically progressive monopolist firms facing a choice of upgrading or diversifying their product lines by adopting product innovations which embody increasing composition of technology. The analysis starts with formalizing a set of plausible assumptions of the basic game theoretical model at the beginning of each chapter. First, the case of perfect information is taken up under drastic innovations. The assumptions of symmetry in payoffs and no market growth enabled us to focus on the role of pure strategic behavior on the part of the incumbent and the entrant firms considering adoption of innovations. Using the competitive payoffs as a benchmark for classifying the type of innovations, the Nash equilibrium points (NEPs) are characterized under three separate cases.

Under the first type of equilibrium with drastic innovation we find the interesting possibility that firms diversify their product lines by crossing over markets and totally replace the incumbents, if the innovations are high-cost. This type of equilibrium where the monopolists switch markets develops as a dominant "defensive" strategy because under drastic innovation firms do not undertake adoptions in their own product lines, since it would mean replacing themselves as incumbents. It can be concluded that, because competition reduces profits, each firm's incentive to become a monopolist is greater than its incentive to become a duopolist by jointly adopting the high-cost innovation.

We also observe that under some boundary values of cost of adoption and Cournot profits, firms may use mixed strategy equilibrium. We get this type of equilibrium with medium cost drastic innovations. We characterize the range of mixed strategy equilibria using a theorem. Both innovations are adopted, however, either through switching of incumbency, or by the incumbent itself, or by joint adoption in both product lines, as we have demonstrated with a numerical example.

Under the third type of equilibrium, firms upgrade products not only in their own product line but also in the incumbent's product line. This type of total diversification arises when the innovations are low-cost.

The main tendency is that if the firms are facing drastic innovations, then they would either diversify into the incumbent's product line only, or upgrade and diversify into both product lines. An interesting observation is that, lacking such a technological rivalry, monopolist firms would not undertake adoptions in their own product lines, since it would mean replacing themselves as incumbents. Thus, the outcome of this technological rivalry is socially desirable, since maximum product diversity is achieved through new innovations. On the other hand, the optimal "cooperative" strategy from the firms' standpoint would be not adopting the drastic innovations and sticking with the old product. Yet, this strategy can not be enforced as a credible commitment.

The basic model is later modified to the case of non-drastic innovations. The assumption of symmetry in payoffs is relaxed by allowing differential market growth in separate product lines. The model is shown to yield equilibria where product upgrading is the more often preferred best reply strategy. High cost non-drastic innovations are shown to exhibit multiple equilibria. Under this scenario a multiplicity of best reply strategies for each monopolist range from adopting both innovations to not adopting any of the innovations. Both innovations are adopted, however, under any of the possible Nash equilibria. An interesting point is that, under two of the four possible equilibria, we observe a passive incumbent and an aggressive entrant. The entrant monopolist diversifies across both its own product line and the entrant's product line, whereas the incumbent sticks with the old product and is partly preyed upon and replaced by the aggressive entrant. In the other two multiple equilibria both firms are actively involved in diversification and specialization process. In one of the equilibrium, firms stay in their own market and diversify in their own product lines. In the other equilibrium they diversify only in the incumbent's product line; and in this process of switching they partly replace the incumbent and are partly replaced by the entrant in their old markets.

Under the second type of Nash equilibrium monopolists stay in their own markets and increase their specialization and upgrading of existing products. We get this result under Nash equilibrium with medium cost non-drastic innovations, i.e. when the cost of diversifying in another product line exceeds the flow profits of a possible competitive outcome.

Under the third type of equilibrium with low cost innovations firms upgrade products not only in their own product line but also in the incumbent's product line. This latter result is obtained under both the drastic and non-drastic low cost innovations.

The main tendency under non-drastic innovations is found to be either upgrading in own product line only, or upgrading and diversifying into both product lines. Unlike the case of drastic innovation, firms do have the incentive to upgrade their own products even without the technological rivalry. The existence of a potential threat of entry into the incumbent's product line enhances the process of diversification, and the firms might find themselves with excessive diversification across all attainable product lines. The optimal outcome from both the society's and the firms' standpoints dictates that the firms adopt innovations in their own product lines since the same maximum product diversity could be achieved by the least cost monopolist. However, not only that this can not be enforced as a credible commitment, but it would also imply that the incumbents' monopoly positions would have to remain unchallenged. When put together, these models imply a process of strategic inventiveness that is in accord with the Schumpeterian concept of “creative destruction”.

Finally, the perfect information framework is extended to the case of asymmetric information. Three separate types are defined for each player. It is shown that if the uninformed firm is a high-cost type, then, Stackelberg equilibrium is the appropriate equilibrium concept. Under a low or a medium cost type follower, perfect Bayesian equilibria are characterized using a theorem. It is shown that equilibrium requires the uninformed firm should derive assessments which are best replies to the strategy chosen by the informed firm in response.

Following this, it is argued that the prior beliefs of the uninformed firm can support two different types of equilibrium. Under pessimistic prior beliefs, the outcome will be a pooling equilibrium in which the uninformed firm can not differentiate between a high cost competitor from a medium cost one. This allows a high cost informed firm to successfully pretend that it is a medium cost type. Alternatively, it is less likely for a high cost firm to exploit its information advantage against an optimistic firm which has strong prior beliefs that it is a low cost type, leading to a separating equilibrium. Exploiting the information advantage under a pooling equilibrium implies that a low cost uninformed firm would be forced to stay in its product line by a high cost informed firm who would stay in its product line itself. In this case we also observed that a high cost informed firm would be forced to maintain its monopoly position by upgrading its product line even though it would have been less costly for the uninformed firm to do so if it had chosen to diversify.

An important observation on this point is that the efficiency choice from the society's standpoint is not one between having monopoly or competition, but it is one between a high-cost monopolist (the informed firm) and a low-cost monopolist (the uninformed firm). Hence, from an efficiency standpoint the society is losing from not only the uninformed firm's ignorance, but also from its pessimistic outlook. This, possibly, is a supportive argument for a centrally coordinated industrial policy that is augmented by government research labs (or joint research projects coordinated by government institutions) providing and/or coordinating the flow of scientific information across companies which will enable them assess their potential strengths and weaknesses.

Hence, it is argued that a competitive outcome in both markets is more likely the more optimistic the uninformed firm is about its own type. On the other hand, it is more likely that a pessimistic uninformed firm would lose its monopoly position in its product line by sharing it with the informed firm, while it continues to be a monopolist in its product line.

From an efficiency standpoint, since with a more optimistic ignorance we tend to have competition in both markets, then, policies that encourage the uninformed firm to increase its optimism about diversification should be devised. Such policies would possibly provide additional incentives that would enable downgrading of the uninformed firm's prior beliefs about the costs of diversification.

The range of rational prior beliefs for a boundary payoff value of a low cost follower is characterized under a corollary. It is shown that these priors are the necessary conditions that satisfy the criteria for a separating equilibrium. Finally, an example of sensible equilibrium is presented using passive conjectures under "equally likely" assumption.

An obviously restrictive assumption of the models in this paper is that only 2 incumbent firms seeking diversification were allowed to be challengers to each other and choose between only two innovations to adopt under certainty. While introducing more firms and more innovations would complicate the analysis, it is our belief that the qualitative results would not change. However, an intriguing extension for future research would be to explore the equilibrium under post adoption market uncertainty. An adoption will be either a success or a failure[10] with a two point probability distribution. Uncertainty of a successful adoption might be another factor why firms want to diversify across markets. Finally, although the simple model in this paper provides some insights about the relationship of technological closeness and market structure, it is but a small step towards understanding the more complex dynamic process where market structure evolves with increasing product diversity, and where incumbent firms identify their potential competitors and try to enhance their comparative advantages in the basic research of their product lines.

Acknowledgements

The author expresses gratitude to two anonymous referees for helpful comments.

Appendix

First Appendix

Proof of Lemma 1. From Table 1 we immediately see that, $\pi_1(s_1, s_1) > \pi_1(s_2, s_1)$, $\pi_1(s_1, s_2) > \pi_1(s_2, s_2)$, $\pi_1(s_1, s_3) \geq \pi_1(s_2, s_3)$ and $\pi_1(s_1, s_4) \geq \pi_1(s_2, s_4)$. Therefore, $\pi_1(s_3, s) \geq \pi_1(s_4, s) \forall s$. Similarly, $\pi_2(s, s_1) \geq \pi_2(s, s_2) \forall s$. From Table 1 we see that, $\pi_1(s_3, s_1) - \pi_1(s_4, s_1) = c_{1u} > 0$, $\pi_1(s_3, s_2) - \pi_1(s_4, s_2) = c_{1u} > 0$, $\pi_1(s_3, s_3) - \pi_1(s_4, s_3) = -V^c_{1u} \geq 0$ and $\pi_1(s_3, s_4) - \pi_1(s_4, s_4) = -V^{\mathrm{c}}_{1\mathrm{u}} \geq 0$. Therefore, $\pi_1(s_3, s) \geq$ $\geq \pi_1(s_4, s) \; \forall \; s$. Similarly, by symmetry we obtain $\pi_2(s, s_3) \geq \pi_2(s, s_4) \; \forall \; s$.

Proof of Proposition 1.

From Lemma 1. we can replace the original game by the (2×2) game with the dominant strategies s_1 and s_3 for each firm. Notice that the resulting reduced game represents a "prisoner's dilemma" situation. Firms play their maximin strategies in order to avoid the worst outcome of being totally preyed upon by their competitors. Hence, the NEP is the strategy combination (s_3, s_3).

Proof of Lemma 2.

From Table 1 we see that, $\pi_1(s_3, s_1) - \pi_1(s_1, s_1 = V^m_{1d} > 0$; $\pi_1(s_3, s_1) - \pi_1(s_1, s_2) = V^c_{1d} \geq 0$; $\pi_1(s_3, s_3) - \pi_1(s_1, s_3) = V^m_{1d} > 0$, and $\pi_1(s_3, s_4) - \pi_1(s_1, s_4) = V^c_{1d} \geq 0$. Therefore, $\pi_1(s_3, s) \geq \pi_1(s_1, s) \; \forall \; s$. Similarly, $\pi_2(s, s_3) \geq \pi_2(s, s_1) \; \forall \; s$. From Table 1 we see that, $\pi_1(s_4, s_1) - \pi_1(s_2, s_1) = V^m_{1d} > 0$, $\pi_1(s_4, s_2) - \pi_1(s_2, s_2) = V^c_{1d} \geq 0$, $\pi_1(s_4, s_3) - \pi_1(s_2, s_3) = V^m_{1d} > 0$, and $\pi_1(s_4, s_4) - \pi_1(s_2, s_4) = V^c_{1d} \geq 0$. Therefore, $\pi_1(s_4, s) \geq \pi_1(s_2, s) \; \forall \; s$. Similarly, by symmetry we obtain $\pi_2(s, s_3) \geq \pi_2(s, s_1) \; \forall \; s$.

Proof of Proposition 2.

From Lemma 2 we can replace the original game by the (2×2) reduced game with the dominant strategies s_2 and s_4 for each firm. Notice that this reduced game represents a "prisoner's dilemma" situation. Hence, the only self-enforcing and stable strategy combination is (s_4, s_4) which is the only NEP.

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[10] Glazer (1985) considers possibility of entry and failure in this sense and gives a brief summary of the empirical literature on product failures.

Proof of Proposition 3.

To prove (1) it is sufficient to show that $(\beta_1+\beta_3)V^{\mathrm{m}}_{\mathrm{1d}}+(\beta_2+\beta_4)V^{\mathrm{c}}_{\mathrm{1d}}$ can not be either strictly positive or strictly negative in a Nash equilibrium.

Case 1. Suppose $(\beta_1+\beta_3)V^{\mathrm{m}}_{\mathrm{1d}}+(\beta_2+\beta_4)V^{\mathrm{c}}_{\mathrm{1d}}>0$ (i).

Then, upon inspection of the expression for $\mathrm{E}_1(\alpha,\beta)$ above, it is immediate that firm 1's best reply to β such that (i) holds must satisfy $\alpha_1=0$, $\alpha_2=0$. Then $\mathrm{E}_2(\alpha,\beta)$ becomes:
$\mathrm{E}_2(\alpha,\beta)=\beta_2(\alpha_3+\alpha_4)V^{\mathrm{c}}_{\mathrm{2u}}+\beta_3\{\alpha_3V^{\mathrm{m}}_{\mathrm{2d}}+\alpha_4V^{\mathrm{c}}_{\mathrm{2d}}\}!\beta_4\{\alpha_3V^{\mathrm{m}}_{\mathrm{2d}}+\alpha_4V^{\mathrm{c}}_{\mathrm{2d}}+(\alpha_3+\alpha_4)V^{\mathrm{c}}_{\mathrm{2u}}\}$ (ii).

If β is firm 2's best reply, it will maximize (ii) given α. If $\alpha_3V^{\mathrm{m}}_{\mathrm{2d}}+\alpha_4V^{\mathrm{c}}_{\mathrm{2d}}>0$, then the best reply β will consist of $\beta_1=0$, $\beta_2>0$, $\beta_3=0$, $\beta_4>0$. But these values of β contradict (i). If $\alpha_3V^{\mathrm{m}}_{\mathrm{2d}}+\alpha_4V^{\mathrm{c}}_{\mathrm{2d}}<0$, then the best reply β will consist of $\beta_1=0$, $\beta_2=0$, $\beta_3=0$, $\beta_4=0$. But these values of β also contradict (i). Finally, if $\alpha_3V^{\mathrm{m}}_{\mathrm{2d}}+\alpha_4V^{\mathrm{c}}_{\mathrm{2d}}=0$, then the best reply β will consist of $\beta_1=0$, $\beta_2>0$, $\beta_3=0$, $\beta_4>0$, which again contradicts (i). Therefore, no NEP can exist which satisfies (i).

Case 2. Suppose, $(\beta_1+\beta_3)V^{\mathrm{m}}_{\mathrm{1d}}+(\beta_2+\beta_4)V^{\mathrm{c}}_{\mathrm{1d}}<0$ (iii).

Upon inspection of $\mathrm{E}_1(\alpha,\beta)$ above, the inequality (iii) implies that firm 1's best reply must satisfy $\alpha_3=0$, $\alpha_4=0$. This implies that $\mathrm{E}_2(\alpha,\beta)$ becomes:
$\mathrm{E}_2(\alpha,\beta)=\beta_1(\alpha_1+\alpha_2)\pi^{\mathrm{m}}_{22}+\beta_2(\alpha_1+\alpha_2)V^{\mathrm{m}}_{\mathrm{2u}}+\beta_3\{(\alpha_1+\alpha_2)\pi^{\mathrm{m}}_{22}+\alpha_1V^{\mathrm{m}}_{\mathrm{2d}}+\alpha_2V^{\mathrm{c}}_{\mathrm{2d}}\}+$
$+\beta_4(\alpha_1+\alpha_2)V^{\mathrm{m}}_{\mathrm{2u}}+\alpha_1V^{\mathrm{m}}_{\mathrm{2d}}+\alpha_2V^{\mathrm{c}}_{\mathrm{2d}}\}$ (iv).

The best reply for firm 2 will maximize (iv) given α. Since $\pi^{\mathrm{m}}_{22}>V^{\mathrm{m}}_{\mathrm{2u}}$, then $\beta_2=0$ and $\beta_4=0$. But then (iii) can never hold when $\beta_2=0$, $\beta_4=0$. Therefore, it follows that (iii) cannot hold under a NEP.

This proves that under a NEP (1) must be true. Clearly, by symmetry, it follows that under a NEP (2) must also be true.

Proof of Proposition 4.

We first prove (4). Substituting 1) from Proposition 3 into $\mathrm{E}_1(\alpha,\beta)$ we get:

$$\begin{aligned}\mathrm{E}_1(\alpha,\beta)=&\alpha_1(\beta_1+\beta_2)\pi^{\mathrm{m}}_{11}+\alpha_2\{(\beta_1+\beta_2)V^{\mathrm{m}}_{\mathrm{1u}}+(\beta_3+\beta_4)V^{\mathrm{c}}_{\mathrm{1u}}\}\\&+\ \alpha_3(\beta_1+\beta_2)\pi^{\mathrm{m}}_{11}+\alpha_4\{(\beta_1+\beta_2)V^{\mathrm{m}}_{\mathrm{1u}}+(\beta_3+\beta_4)V^{\mathrm{c}}_{\mathrm{1u}}\}.\end{aligned}\tag{i}$$

We will prove that $(\beta_1+\beta_2)\pi^{\mathrm{m}}_{11}=(\beta_1+\beta_2)V^{\mathrm{m}}_{\mathrm{1u}}+(\beta_3+\beta_4)V^{\mathrm{c}}_{\mathrm{1u}}$ is impossible. First, consider the inequality,

$$(\beta_1+\beta_2)\pi^{\mathrm{m}}_{11}>(\beta_1+\beta_2)V^{\mathrm{m}}_{\mathrm{1u}}+(\beta_3+\beta_4)V^{\mathrm{c}}_{\mathrm{1u}}.\tag{ii}$$

But, maximizing (i) under (ii) implies that firm 1's best reply must satisfy $\alpha_1>0$, $\alpha_3>0$, $\alpha_2=0$, $\alpha_4=0$. But this strategy violates Proposition 3-2. Hence, (ii) can not hold in a Nash equilibrium. Next, consider the inequality,

$$(\beta_1+\beta_2)\pi^{\mathrm{m}}_{11}<(\beta_1+\beta_2)V^{\mathrm{m}}_{\mathrm{1u}}+(\beta_3+\beta_4)V^{\mathrm{c}}_{\mathrm{1u}}.\tag{iii}$$

Maximizing (i) under (iii) implies that firm 1's best reply must satisfy $\alpha_1=\alpha_2>0$, $\alpha_3=0$, $\alpha_4>0$. This also contradicts Proposition 3-2. Thus, we conclude that (4) must hold under a Nash Equilibrium. By symmetry we can write the expected payoff of firm 2 using Proposition 3-2 as follows:

$$\begin{aligned}\mathrm{E}_2(\alpha,\beta) =&\beta_1(\alpha_1+\alpha_2)\pi^{\mathrm{m}}_{22} + \beta_2\{(\alpha_1+\alpha_2)V^{\mathrm{m}}_{2\mathrm{u}} + (\alpha_3+\alpha_4)V^{\mathrm{c}}_{2\mathrm{u}}\}+\\ &+\beta_3(\alpha_1+\alpha_2)\pi^{\mathrm{m}}_{22} + \beta_4\{(\alpha_1+\alpha_2)V^{\mathrm{m}}_{2\mathrm{u}} + (\alpha_3+\alpha_4)V^{\mathrm{c}}_{2\mathrm{u}}. \qquad \text{(iv)}\end{aligned}$$

In a symmetric manner it is easy to see that a NEP that maximizes (iv) requires that (3) hold.

Proof of Lemma 3.

From Table 2 we immediately see that

$$\begin{aligned}&\pi_2(s_1,s_4) - \pi_2(s_1,s_3) = (V^{\mathrm{m}}_{2\mathrm{u}} + V^{\mathrm{m}}_{2\mathrm{d}} + r_2) - (\pi^{\mathrm{m}}_2 + V^{\mathrm{m}}_{2\mathrm{d}}) = (V^{\mathrm{m}}_{2\mathrm{u}} + r_2 - \pi^{\mathrm{m}}_2) > 0,\\ &\pi_2(s_1,s_4) - \pi_2(s_1,s_2) = (V^{\mathrm{m}}_{2\mathrm{u}} + V^{\mathrm{m}}_{2\mathrm{d}} + r_2) - (V^{\mathrm{m}}_{2\mathrm{u}} + r_2) = V^{\mathrm{m}}_{2\mathrm{d}} > 0,\\ &\pi_2(s_1,s_4) - \pi_2(s_1,s_1) = (V^{\mathrm{m}}_{2\mathrm{u}} + V^{\mathrm{m}}_{2\mathrm{d}} + r_2) - (\pi^{\mathrm{m}}_2) = (V^{\mathrm{m}}_{2\mathrm{u}} + r_2 - \pi^{\mathrm{m}}_2) + V^{\mathrm{m}}_{2\mathrm{d}} > 0.\end{aligned}$$

From Table 2 we see that

$$\begin{aligned}&\pi_1(s_1,s_4) - \pi_1(s_2,s_4) = r_1 - (V^{\mathrm{c}}_{1\mathrm{u}} + r_1) = -V^{\mathrm{c}}_{1\mathrm{u}} > 0,\\ &\pi_1(s_1,s_4) - \pi_1(s_3,s_4) = r_1 - (V^{\mathrm{c}}_{1\mathrm{d}} + r_1) = -V^{\mathrm{c}}_{1\mathrm{d}} > 0,\\ &\pi_1(s_1,s_4) - \pi_1(s_4,s_4) = r_1 - (V^{\mathrm{c}}_{1\mathrm{u}} + V^{\mathrm{c}}_{1\mathrm{d}} + r) = -(V^{\mathrm{c}}_{1\mathrm{u}} + V^{\mathrm{c}}_{1\mathrm{d}}) > 0.\end{aligned}$$

This proves **(a)**.

From Table 2 we see that

$$\begin{aligned}&\pi_2(s_2,s_2) - \pi_2(s_2,s_1) = (V^{\mathrm{m}}_{2\mathrm{u}} + r_2) - \pi^{\mathrm{m}}_2 > 0,\\ &\pi_2(s_2,s_2) - \pi_2(s_2,s_3) = (V^{\mathrm{m}}_{2\mathrm{u}} + r_2) - (V^{\mathrm{c}}_{2\mathrm{d}} + \pi^{\mathrm{m}}_2) = (V^{\mathrm{m}}_{2\mathrm{u}} + r_2 - \pi^{\mathrm{m}}_2) - V^{\mathrm{c}}_{2\mathrm{d}} > 0,\\ &\pi_2(s_2,s_2) - \pi_2(s_2,s_4) = (V^{\mathrm{m}}_{2\mathrm{u}} + r_2) - (V^{\mathrm{m}}_{2\mathrm{u}} + V^{\mathrm{c}}_{2\mathrm{d}} + r_2) = -V^{\mathrm{c}}_{2\mathrm{d}} > 0.\end{aligned}$$

Similarly, from symmetry we see that, $\pi_1(s_2,s_2) - \pi_1(s,s_2) > 0 \ \forall \ s$.

This proves **(b)**.

From Table 2 we see that

$$\begin{aligned}&\pi_2(s_3,s_3) - \pi_2(s_3,s_1) = (V^{\mathrm{m}}_{2\mathrm{d}} + r_2) - r_2 = V^{\mathrm{m}}_{2\mathrm{d}} > 0\\ &\pi_2(s_3,s_3) - \pi_2(s_3,s_2) = (V^{\mathrm{m}}_{2\mathrm{d}} + r_2) - (V^{\mathrm{c}}_{2\mathrm{u}} + r_2) = V^{\mathrm{m}}_{2\mathrm{d}} - V^{\mathrm{c}}_{2\mathrm{d}} > 0\\ &\pi_2(s_3,s_3) - \pi_2(s_3,s_4) = (V^{\mathrm{m}}_{2\mathrm{d}} + r_2) - (V^{\mathrm{c}}_{2\mathrm{u}} + V^{\mathrm{m}}_{2\mathrm{d}} + r_2) = -V^{\mathrm{c}}_{2\mathrm{u}} > 0\end{aligned}$$

Similarly, from symmetry it is immediate that, $\pi_1(s_3,s_3) - \pi_1(s,s_3) > 0 \ \forall \ s$.

This proves **(c)**.

Also note from Table 2 that the strategy combination (s_4,s_1) implied by **(d)** is symmetrically opposite to the strategy combination (s_1,s_4) implied by **(a)**. To prove **(d)** it is sufficient to see that the payoffs are also distributed symmetrically. Hence, using **(a)** we get, $\pi_1(s_4,s_1) - \pi_1(s,s_1) > 0 \ \forall \ s$ and $\pi_2(s_4,s_1) - \pi_2(s_4,s) > 0 \ \forall \ s$. This proves **(d)**.

Proof of Proposition 5.

Suppose firm 1 plays its s_1 strategy. Then, firm 2's best reply strategy is s_4. Since from Lemma 3 **(a)** we have $\pi_2(s_1, s_4) > \pi_2(s_1, s) \;\; \forall \;\; s$. Alternately, suppose firm 2 plays its s_4 strategy. Then, firm 1's best reply strategy is s_1. Since from Lemma 3 **(a)** we have $\pi_1(s_1, s_4) > \pi_1(s, s_4) \; \forall \; s$. Hence, $\{s_1, s_4\}$ is a NEP.

Suppose firm 1 plays its s_2 strategy. Then, firm 2's best reply strategy is s_2. Since from Lemma 3 **(b)** we have, $\pi_2(s_2, s_2) > \pi_2(s_2, s) \; \forall \; s$. Alternately, suppose firm 2 plays its s_2 strategy. Then, firm 1's best reply strategy is s_2. Since from Lemma 3 **(b)** we have $\pi_1(s_2, s_2) > \pi_1(s, s_2) \forall s$. Hence, $\{s_2, s_2\}$ is another NEP.

Suppose firm 1 plays its s_3 strategy. Then, firm 2's best reply strategy is s_3. Since from Lemma 3 **(c)** we have $\pi_2(s_3, s_3) > \pi_2(s_3, s) \; \forall \; s$. Alternately, suppose firm 2 plays its s_3 strategy. Then, firm 1's best reply strategy is s_3. Since from Lemma 3 **(c)** we have $\pi_1(s_3, s_3) > \pi_1(s, s_3) \forall s$. Hence, $\{s_3, s_3\}$ is another NEP.

Suppose firm 1 plays its s_4 strategy. Then, firm 2's best reply strategy is s_1. Since from Lemma 3 **(d)** we have $\pi_2(s_4, s_1) > \pi_2(s_4, s) \; \forall \; s$. Alternately, suppose firm 2 plays its s_1 strategy. Then, firm 1's best reply strategy is s_4. Since from Lemma 3 **(d)** we have $\pi_1(s_4, s_1) > \pi_1(s, s_1) \forall s$. Hence, $\{s_4, s_1\}$ is another NEP.

Proof of Lemma 4.

From Table 2 we immediately see that $\pi_1(s_2,s_1) > \pi_1(s_1,s_1)$, $\pi_1(s_2,s_2) > \pi_1(s_1,s_2)$, $\pi_1(s_2, s_3) \geq \pi_1(s_1, s_3)$ and $\pi_1(s_2, s_4) \geq \pi_1(s_1, s_4)$. Therefore, $\pi_1(s_2, s) \geq \pi_1(s_1, s) \forall s$. Similarly, $\pi_2(s, s_2) \geq \pi_2(s, s_1) \forall s$.

From Table 3 we see that $\pi_1(s_4,s_1) > \pi_1(s_3,s_1)$, $\pi_1(s_4,s_2) > \pi_1(s_3,s_2)$, $\pi_1(s_4,s_3) \geq$ $\geq \pi_1(s_3, s_3)$ and $\pi_1(s_4, s_4) \geq \pi_1(s_3, s_4)$. Therefore, $\pi_1(s_4, s) \geq \pi_1(s_3, s) \forall s$. Similarly, by symmetry we obtain $\pi_2(s, s_4) \geq \pi_2(s, s_3) \forall s$.

Proof of Proposition 6.

From Lemma 4 we can replace the original game by the reduced form (2×2) game with the dominant strategies s_2 and s_4 for each firm. Notice that the resulting reduced game can be further iterated for payoff dominance.

From Table 2 it is immediate that $\pi_1(s_4, s_2) - \pi_1(s_2, s_2) = V_{1d}^c > 0$ and $\pi_1(s_4, s_4) - \pi_1(s_2, s_4) = V_{1d}^c > 0$. Therefore, $\pi_1(s_4, s) > \pi_1(s_2, s) \; \forall s$. Similarly, $\pi_2(s, s_4) > \pi_2(s, s_2) \forall s$. Hence, NEP is the strategy combination (s_4, s_4).

Proof of Proposition 7.

From Lemma 4 we can replace the original game 3. by the (2×2) game with the dominant strategies s_2 and s_4 for each firm. Notice that the resulting reduced game can be further iterated for payoff dominance. To see this note from Table 2. that, $\pi_1(s_2, s_2) - \pi_1(s_4, s_2) = V_{\mathrm{1d}}^{\mathrm{c}} \leq 0$, $\pi_1(s_2, s_4) - \pi_1(s_4, s_4) = V_{\mathrm{1d}}^{\mathrm{c}} \leq 0$. Therefore, $\pi_1(s_2, s) \geq \pi_1(s_4, s) \; \forall \; s$. Similarly, $\pi_2(s, s_2) \geq \pi_2(s, s_4) \; \forall \; s$. Hence, the NEP is the strategy combination $\{s_2, s_2\}$.

Proof of Lemma 6.
From Table 1 we immediately see that

$$\pi_2(s_2, s_2 \mid N_{2j}) - \pi_2(s_2, s_1 \mid N_{2j}) = V_{2u}^m + r + \pi^m > 0;$$
$$\pi_2(s_2, s_2 \mid N_{2j}) - \pi_2(s_2, s_3 \mid N_{2j}) = (V_{2u}^m + r + \pi^m) - V_{2c}^d > 0;$$
$$\pi_2(s_2, s_2 \mid N_{2j}) - \pi_2(s_2, s_4 \mid N_{2j}) = -V_{2d}^c > 0 \quad \forall \quad j = 1, 2.$$

And,

$$\pi_2(s_2, s_4 \mid N_{23}) - \pi_2(s_2, s_1 \mid N_{23}) = V_{2u}^m + r - \pi^m + V_{2d}^c > 0;$$
$$\pi_2(s_2, s_4 \mid N_{23}) - \pi_2(s_2, s_2 \mid N_{23}) = V_{2d}^c \geq 0;$$
$$\pi_2(s_2, s_4 \mid N_{23}) - \pi_2(s_2, s_3 \mid N_{23}) = V_{2u}^m + r - \pi^m > 0.$$

This proves **(a)**.
From table 1 we see that

$$\pi_2(s_3, s_3 \mid N_{21}) - \pi_2(s_3, s_1 \mid N_{21}) = V_{2d}^m > 0;$$
$$\pi_2(s_3, s_3 \mid N_{21}) - \pi_2(s_3, s_2 \mid N_{21}) = V_{2d}^m - V_{2d}^c > 0;$$
$$\pi_2(s_3, s_3 \mid N_{21}) - \pi_2(s_3, s_4 \mid N_{21}) = -V_{2u}^C \geq 0;$$
$$\pi_2(s_3, s_4 \mid N_{2j}) - \pi_2(s_3, s_1 \mid N_{2j}) = V_{2u}^C + V_{2d}^m > 0;$$
$$\pi_2(s_3, s_4 \mid N_{2j}) - \pi_2(s_3, s_2 \mid N_{2j}) = V_{2d}^m > 0;$$
$$\pi_2(s_3, s_4 \mid N_{2j}) - \pi_2(s_3, s_3 \mid N_{2j}) = V_{2u}^C \geq 0 \quad \forall \quad j = 2, 3.$$

This proves **(b)**.
From Table 1 we immediately see that

$$\pi_2(s_4, s_1 \mid N_{21}) - \pi_2(s_4, s_2 \mid N_{21}) = -V_{2u}^C \geq 0;$$
$$\pi_2(s_4, s_1 \mid N_{21}) - \pi_2(s_4, s_3 \mid N_{21}) = -V_{2d}^C > 0;$$
$$\pi_2(s_4, s_1 \mid N_{21}) - \pi_2(s_4, s_4 \mid N_{21}) = -(V_{2u}^c + V_{2d}^C) > 0.$$

We also see that

$$\pi_2(s_4, s_2 \mid N_{22}) - \pi_2(s_4, s_1 \mid N_{22}) = -V_{2u}^C > 0;$$
$$\pi_2(s_4, s_2 \mid N_{22}) - \pi_2(s_4, s_3 \mid N_{22}) = V_{2u}^C - V_d^c > 0;$$
$$\pi_2(s_4, s_2 \mid N_{22}) - \pi_2(s_4, s_4 \mid N_{22}) = -V_{2d}^C > 0.$$

And

$$\pi_2(s_4, s_4 \mid N_{23}) - \pi_2(s_4, s_1 \mid N_{23}) = V_{2u}^c + V_{2d}^C > 0;$$
$$\pi_2(s_4, s_4 \mid N_{23}) - \pi_2(s_4, s_2 \mid N_{23}) = V_{2d}^C \geq 0;$$
$$\pi_2(s_4, s_4 \mid N_{23}) - \pi_2(s_4, s_3 \mid N_{23}) = V_{2u}^c > 0.$$

This proves **(c).**

Proof of Lemma 7.
From Table 3 we note that,

$$\begin{aligned}
\pi_1(s_2, s_2 \mid N_{11}) - \pi_1(s_1, s_2 \mid N_{11}) &= (V_{1u}^m + r) - \pi^m > 0;\\
\pi_1(s_2, s_2 \mid N_{11}) - \pi_1(s_3, s_2 \mid N_{11}) &= (V_{1u}^m + r) - (\pi + V_{1d}^c) =\\
&= (V_{1u}^m + r - \pi^m) - V_{1d}^c > 0;\\
\pi_1(s_2, s_2 \mid N_{11}) - \pi_1(s_4, s_2 \mid N_{11}) &= (V_{1u}^m + r) - (V_{1u}^m + V_{1d}^c + r) = -V_{1d}^c > 0.
\end{aligned}$$

We also note that

$$\begin{aligned}
\pi_1(s_1, s_4 \mid N_{11}) - \pi_1(s_2, s_4 \mid N_{11}) &= r - (V_{1u}^c + r) = -V_{1u}^c > 0;\\
\pi_1(s_1, s_4 \mid N_{11}) - \pi_1(s_3, s_4 \mid N_{11}) &= r - (V_{1d}^c + r) = -V_{1d}^c > 0;\\
\pi_1(s_1, s_4 \mid N_{11}) - \pi_1(s_4, s_4 \mid N_{11}) &= r - (V_{1u}^c + V_{1d}^c + r) = -(V_{1u}^c + V_{1d}^c) > 0.
\end{aligned}$$

This proves **(a)**.
From Table 3 we see that

$$\begin{aligned}
\pi_1(s_2, s_2 \mid N_{12}) - \pi_1(s_1, s_2 \mid N_{12}) &= (V_{1u}^m + r) - \pi^m > 0;\\
\pi_1(s_2, s_2 \mid N_{12}) - \pi_1(s_1, s_2 \mid N_{12}) &= (V_{1u}^m + r) - (\pi^m + V_{1d}^c) =\\
&= (V_{1u}^m + r - \pi^m) - V_{1d}^c) > 0;\\
\pi_1(s_2, s_2 \mid N_{12}) - \pi_1(s_4, s_2 \mid N_{12}) &= (V_{1u}^m + r) - (V_{1u}^m V_{1d}^c + r) = -V_{1d}^c > 0.
\end{aligned}$$

We also note that

$$\begin{aligned}
\pi_1(s_2, s_4 \mid N_{12}) - \pi_1(s_1, s_4 \mid N_{12}) &= (V_{1u}^c + r) - r > 0;\\
\pi_1(s_2, s_4 \mid N_{12}) - \pi_1(s_3, s_4 \mid N_{12}) &= (V_{1u}^c + r) - (V_{1d}^c + r) = V_{1u}^c - V_{1d}^c > 0;\\
\pi_1(s_2, s_4 \mid N_{12}) - \pi_1(s_4, s_4 \mid N_{12}) &= (V_{1u}^c + r) - (V_{1u}^c V_{1d}^c + r) = -V_{1d}^c > 0.
\end{aligned}$$

This proves **(b)**.
From Table 3 we see that

$$\begin{aligned}
\pi_1(s_4, s_2 \mid N_{13}) - \pi_1(s_1, s_2 \mid N_{13}) &= (V_{1u}^m + V_{1d}^c + r) - \pi^m > 0;\\
\pi_1(s_4, s_2 \mid N_{13}) - \pi_1(s_2, s_2 \mid N_{13}) &= (V_{1u}^m + V_{1d}^c + r) - (V_{1u}^m + r) = V_{1d}^c > 0 =\\
&= (V_{1u}^m + r - \pi^m) > 0.
\end{aligned}$$

We also note that

$$\begin{aligned}
\pi_1(s_4, s_4 \mid N_{13}) - \pi_1(s_1, s_4 \mid N_{13}) &= (V_{1u}^c + V_{1d}^c + r) - r > 0,\\
\pi_1(s_4, s_4 \mid N_{13}) - \pi_1(s_2, s_4 \mid N_{13}) &= (V_{1u}^c + V_{1d}^c + r) - (V_{1u}^c + r) = V_{1d}^c > 0;\\
\pi_1(s_4, s_4 \mid N_{13}) - \pi_1(s_3, s_4 \mid N_{13}) &= (V_{1u}^c + V_{1d}^c + r) - (V_{1d}^c + r) = V_{1d}^c > 0.
\end{aligned}$$

This proves **(c)**.

Proof of Lemma 8.

First, we prove lemma 8 **(c)**. Recall from lemma 7 **(c)** that s_{14} is the dominant leading strategy if firm 1 is of type N_{13}. Hence, firm 2 uses the following joint distribution of the states of the world and firm 1's dominant strategies to derive its conjectures, $\Pr(s_{1k} \mid N_{2j}) \forall j = 2, 3 \& k = 1, ..., 4$. In particular, when strategy s_{14} observed, then the following equations must hold:

$$\Pr(s_{14} \mid N_{11}, N_{22}) = 0 \, \Pr(s_{14} \mid N_{11}, N_{23}) = 0;$$
$$\Pr(s_{14} \mid N_{12}, N_{22}) = 0 \, \Pr(s_{14} \mid N_{12}, N_{23}) = 0;$$
$$\Pr(s_{14} \mid N_{13}, N_{22}) = 1 \, \Pr(s_{14} \mid N_{13}, N_{23}) = 1.$$

Therefore, it follows that

$$\begin{aligned}\Pr(s_{14}, N_{22}) = \sum_{i=1}^{3} \Pr(s_{14} \mid N_{1i}, N_{22}) \Pr(N_{1i}) = \\ = \Pr(s_{14} \mid N_{13}; N_{22}) \Pr(N_{13}) = \Pr(N_{13}) = \theta_3,\end{aligned}$$

and,

$$\begin{aligned}\Pr(s_{14}, N_{23}) = \sum_{i=1}^{3} \Pr(s_{14} \mid N_{1i}, N_{23}) \Pr(N_{1i}) = \\ = \Pr(s_{14} \mid N_{13}; N_{23}) \Pr(N_{13}) = \Pr(N_{13}) = \theta_3\end{aligned}$$

This proves part **(c)** of lemma 8. Also, note that under (N_{11}, N_{12}) the Stackelberg equilibrium is (s_1, s_2). This allows firm 2 to use the following joint distribution toderive its conjectures:

$$\Pr(s_{12} \mid N_{11}, N_{22}) = 1 \, \Pr(s_{12} \mid N_{11}, N_{23}) = 0;$$
$$\Pr(s_{12} \mid N_{12}, N_{22}) = 1 \, \Pr(s_{12} \mid N_{12}, N_{23}) = 1;$$
$$\Pr(s_{12} \mid N_{13}, N_{22}) = 0 \, \Pr(s_{12} \mid N_{13}, N_{23}) = 0;$$

It follows that

$$\begin{aligned}\Pr(s_{12}, N_{22}) &= \sum_{i=1}^{3} \Pr(s_{12} \mid N_{1i}, N_{22}) \Pr(N_{1i}) = \\ &= \Pr(s_{12} \mid N_{11}; N_{22}) \Pr(N_{11}) + \Pr(s_{12} \mid N_{12}, N_{22}) \Pr(N_{12}); \\ &= \Pr(N_{11}) + \Pr(N_{12}) = \theta_1 + \theta_2,\end{aligned}$$

and

$$\begin{aligned}\Pr(s_{12}, N_{23}) &= \sum_{i=1}^{3} \Pr(s_{12} \mid N_{1i}, N_{23}) \Pr(N_{1i}) = \\ &= \Pr(s_{12} \mid N_{11}; N_{23}) \Pr(N_{12}) + \Pr(s_{12} \mid N_{13}, N_{23}) \Pr(N_{13}) = \\ &= \Pr(N_{12}) = \theta_2.\end{aligned}$$

This proves part **(b)** of Lemma 8.

Last, we note that (s_1, s_4) is a Stackelberg equilibrium only if the state of the world is (N_{11}, N_{23}). In other words, s_{11} can only be observed if (i) firm 1 is of type N_{11} (recall Lemma 7 (a)), and (ii) firm 2 is of type N_{23} (recall Lemma 6). Hence, we conclude that if and when s_{11} is observed the following distribution should hold:

$$\begin{aligned}
&\Pr(s_{11} \mid N_{11}, N_{22}) = 0 \Pr(s_{11} \mid N_{11}, N_{23}) = 1\\
&\Pr(s_{11} \mid N_{12}, N_{22}) = 0 \Pr(s_{11} \mid N_{12}, N_{23}) = 0\\
&\Pr(s_{11} \mid N_{13}, N_{22}) = 0 \Pr(s_{11} \mid N_{13}, N_{23}) = 0
\end{aligned}$$

which are used by firm 2 to calculate its conjectures of its type upon observing s_{11}.

$$\begin{aligned}
\Pr(s_{11}, N_{22}) &= \sum_{i=1}^{3} \Pr(s_{11} \mid N_{1i}, N_{22}) \Pr(N_{1i})\\
&= \Pr(s_{11} \mid N_{11}, N_{22}) \Pr(N_{11}) + \Pr(s_{11} \mid N_{12}, N_{22}) \Pr(N_{12})\\
&\quad + \Pr(s_{11} \mid N_{13}, N_{22}) \Pr(N_{13}) = 0
\end{aligned}$$

and,

$$\begin{aligned}
\Pr(s_{11}, N_{23}) &= \sum_{i=1}^{3} \Pr(s_{11} \mid N_{1i}, N_{23}) \Pr(N_{1i})\\
&= \Pr(s_{11} \mid N_{11}, N_{23}) \Pr(N_{11}) + \Pr(s_{11} \mid N_{12}, N_{23}) \Pr(N_{12})\\
&\quad + \Pr(s_{11} \mid N_{13}, N_{23}) \Pr(N_{13})\\
&= \Pr(s_{11} \mid N_{11}; N_{23}) \Pr(N_{11}) = \Pr(N_{11}) = \theta_1
\end{aligned}$$

This proves part **(a)** of Lemma 8.

Second Appendix

Proof of Theorem 2.

First, we prove **(e)**.

We will prove that s_{22} is the best reply to s_{14} iff

$$(-)\frac{(V_{2d}^{c} \mid N_{22})|}{(V_{2d}^{c} \mid N_{23})} \geq \frac{(1-\phi)}{\phi}. \tag{15}$$

Recall from weak dominance that firm 2's dominant strategies ares_{22} and s_{24} under either N_{22} or N_{23}. Therefore, it follows that s_{22} is the best reply tos_{14} if and only if

$$E\pi_2(s_4 s_2) \geq E\pi_2(s_4, s_4), \tag{16}$$

where

$$E\pi_2(s_4 s_2) = \Pr(N_{22} \mid s_{14})\pi_2(s_4, s_2 \mid N_{22}) + \Pr(N_{23} \mid s_{14})\pi(s_4, s_2 \mid N_{23}), \tag{17}$$

$$E\pi_2(s_4 s_4) = \Pr(N_{22} \mid s_{14})\pi_2(s_4, s_4 \mid N_{22}) + \Pr(N_{23} \mid s_{14})\pi(s_4, s_4 \mid N_{23}). \tag{18}$$

We use Bayes's rule to calculate $\Pr(N_{22} \mid s_{14})$ and $\Pr(N_{23} \mid s_{14})$:

$$\Pr(N_{22} \mid s_{14}) = \frac{\Pr(s_{14} \mid N_{22}) \Pr(N_{22})}{\Pr(s_{14})} =$$

$$= \frac{\{ \Pr(s_{14} \mid N_{22}) \}\, \phi}{\{ \Pr(s_{14} \mid N_{22}) \}\, \phi + \{\Pr(s_{14} \mid N_{23}) \}\, (1-\phi)}. \tag{19}$$

where ϕ and $1-\phi$ are the priors for $\Pr(N_{22})$ and $\Pr(N_{23})$, respectively.

But from Lemma 8 **(c)** we have

$$\Pr(s_{14} \mid N_{22}) = \Pr(N_{13}) = \theta_3, \quad \Pr(s_{14} \mid N_{23}) = \Pr(N_{13}) = \theta_3,$$

which upon substitution into (5) yields:

$$\Pr(N_{22} \mid s_{14}) = \phi.$$

It follows that

$$\Pr(N_{23} \mid s_{14}) = 1-\phi.$$

From Table 2 and the definitions of N_{22}, N_{23} we observe that:

$$\pi_2(s_4, s_2 \mid N_{22}) = \pi_2(s_4, s_2 \mid N_{23}) = V^c_{2u} + r > 0,$$

and

$$\pi_2(s_4, s_4 \mid N_{2j}) = (V^c_{2u} \mid N_{2j}) + (V^c_{2d} \mid N_{2j}) + r,$$

where $(V^c_{2u} \mid N_{2j}) = V^c_{2u} > 0$ for $j = 2, 3$ and

$$(V^c_{2d} \mid N_{2j}) = \left\{ \begin{array}{c} V^c_{2d} < 0 \;\; for \;\; j = 2 \\ V^c_{2d} \geq 0 \;\; for \;\; j = 3 \end{array} \right\}.$$

Therefore, substituting the expressions for $\Pr(N_{2j} \mid s_{14}), \pi_2(s_4, s_2 \mid N_{2j})$ and $P\pi_2(s_4, s_2 \mid N_{2j})$, $j = 2, 3$ into (3) and (4), we can write the inequality (2) as

$$V^c_{2u} + r \geq V^c_{2u} + r + (V^c_{2d} \mid N_{22})\phi + V^c_{2d} \mid N_{23}(1-\phi) \qquad \text{or,}$$
$$(-)\frac{(V^c_{2d} \mid N_{22})}{(V^c_{2d} \mid N_{23})} \geq \frac{(1-\phi)}{\phi}.$$

What we have shown so far is that s_{22} is the best reply to s_{14} if and only if the inequality (1) holds independently of player 1's type. The arguments also show that s_{24} is the best reply to s_{14} if the reverse inequality to (1) holds independently of player 1's type.

Next, we will prove that s_{24} is the best reply to s_{12} iff,

$$(-)\frac{(V^c_{2d} \mid N_{22})}{(V^c_{2d} \mid N_{23})} \leq \frac{\theta_2(1-\phi)}{(\theta_1+\theta_2)\phi}. \tag{20}$$

We know from weak dominance that firm 2's dominant strategies are s_{22} and s_{24} under either N_{22} or N_{23}. Therefore, it follows that s_{24} is a best reply to s_{12} if and only if

$$E\pi_2(s_4 s_4) \geq E\pi_2(s_2, s_2) \tag{21}$$

where

$$E\pi_2(s_2 s_4) = \Pr(N_{22} \mid s_{12})\pi_2(s_2, s_4 \mid N_{22}) + \Pr(N_{23} \mid s_{12})\pi_2(s_2, s_4 \mid N_{23}), \tag{22}$$

$$E\pi_2(s_2 s_2) = \Pr(N_{22} \mid s_{12})\pi_2(s_2, s_2 \mid N_{22}) + \Pr(N_{23} \mid s_{12})\pi_2(s_2, s_2 \mid N_{23}). \tag{23}$$

We use Bayes's rule to calculate $\Pr(N_{22} \mid s_{12})$ and $\Pr(N_{23} \mid s_{12})$:

$$\Pr(N_{22} \mid s_{12}) = \frac{\Pr(s_{12} \mid N_{22}) \Pr(N_{22})}{\Pr(s_{12})} =$$

$$= \frac{\{\Pr(s_{12} \mid N_{22})\}\phi}{\{\Pr(s_{12} \mid N_{22})\}\phi + \{\Pr(s_{12} \mid N_{23})\}(1-\phi)}, \tag{24}$$

where ϕ and $1-\phi$ are the priors for $\Pr(N_{22})$ and $\Pr(N_{23})$ respectively. But from Lemma 8 **(b)** we have

$$\Pr(s_{12} \mid N_{22}) = \Pr(N_{11}) + \Pr(N_{12}) = \theta_1 + \theta_2,$$

$$\Pr(s_{12} \mid N_{23}) = \Pr(N_{12}) = \theta_2,$$

which upon substitution into (10) yields:

$$\Pr(N_{22} \mid s_{12}) = \frac{(\theta_1 + \theta_2)\phi}{\theta_1\phi + \theta_2}.$$

Then, it follows that

$$\begin{aligned} \Pr(N_{23} \mid s_{12}) &= 1 - \Pr(N_{22} \mid s_{12}) = \\ &= \frac{\theta_2(1-\phi)}{\theta_1\phi + \theta_2}. \end{aligned}$$

From Table 2 and the definitions of N_{22}, N_{23} we observe that:

$$\pi_2(s_2, s_4 \mid N_{2j}) = V_{2u}^m + (V_{2d}^c \mid N_{2j}) + r > 0,$$

and

$$\pi_2(s_2, s_2 \mid N_{22}) = \pi_2(s_2, s_2 \mid N_{23}) = V_{2u}^m + r,$$

where

$$(V_{2d}^c \mid N_{2j}) = \left\{ \begin{array}{l} V_{2d}^c < 0 \;\; for \;\; j = 2 \\ V_{2d}^c \geq 0 \;\; for \;\; j = 3 \end{array} \right\}.$$

Therefore, substituting the expressions for $\Pr(N_{2j} \mid s_{12})$,$\pi_2(s_2, s_4 \mid N_{2j})$ and $\pi_2(s_2, s_2 \mid N_{2j}), j = 2, 3$ into (8) and (9), we can write the inequality (7) as

$$0 \leq \frac{(\theta_1 + \theta_2)\phi}{\theta_1\phi + \theta_2}(V^c_{2d} \mid N_{22}) + \frac{\theta_2(1-\phi)}{\theta_1\phi + \theta_2}(V^c_{2d} \mid N_{23}),$$

or

$$(-)\frac{(V^c_{2d} \mid N_{22})}{(V^c_{2d} \mid N_{23})} \leq \frac{\theta_2(1-\phi)}{(\theta_1 + \theta_2)\phi}.$$

Hence, we have shown that s_{24} is the best reply to s_{12} if and only if the inequality (6) holds independently of player 1's type. The arguments also show that s_{22} is the best reply to s_{12} if and only if the reverse inequality to (6) holds independently of player 1's type.

What we have shown so far is $s_{22} = b_2(s_{24})$ if and only if (1) holds; and $s_{24} = b_2(s_{12})$ if and only if (6) holds. Conversely, suppose (1) holds. Then we know that

$$s_{24} \neq b_2(s_{14}).$$

Similarly, suppose (6) holds. Then we know that

$$s_{22} \neq b_2(s_{14}).$$

To see this we reproduce (1) and (6) below for convenience:

$$(-)\frac{(V^c_{2d} \mid N_{22})}{(V^c_{2d} \mid N_{23})} \geq \frac{(1-\phi)}{\phi}, \quad (-)\frac{(V^c_{2d} \mid N_{22})}{(V^c_{2d} \mid N_{23})} \leq \frac{\theta_2(1-\phi)}{(\theta_1 + \theta_2)\phi}.$$

It is easily seen that

$$\frac{\theta_2(1-\phi)}{(\theta_1 + \theta_2)\phi} \leq \frac{(1-\phi)}{\phi}\, \forall\, 0 \leq \theta_i \leq 1. \tag{25}$$

From (11) it is obvious that (1) and (6) can not hold together.

Now, if (s_4, s_2) is a perfect Bayesian equilibrium point (PBEP), then we know s_{22} is the best reply to s_{14} so that inequality (1) holds. Conversely, suppose (1) holds. Then we know s_{22} is the best reply to s_{14}. However, in order for (s_4, s_2) to be a PBEP we must have

$$\pi_1(s_4, s_2) \geq \pi_1(s_i, b_2(s_i)) \quad \forall \quad i = 1, 2, 3, 4 \qquad \text{where} \qquad s_{22} = b_2(s_{14}).$$

Recall that under Lemma 7 s_{11}, s_{12} and s_{14} are identified as best replies to s_{22} and s_{24} under N_{1j} for all $j = 1, 2, 3$. Therefore it follows that s_{14} is a Bayesian best reply to s_{22} if and only if,

$$\begin{array}{llll} (a) & \pi_1(s_4, s_2) & \geq & \pi_1(s_1, s_4); \\ (b) & \pi_1(s_4, s_2) & \geq & \pi_1(s_2, s_2), \\ (c) & \pi_1(s_4, s_2) & \geq & \pi_1(s_2, s_4). \end{array} \tag{26}$$

But we have shown that $s_{24} \neq b_2(s_{12})$ when (1) holds. Therefore (12.(c)) drops out.

From Table 2 and the definitions of N_{11}, N_{12} and N_{13} we observe that

$$\pi_1(s_4, s_2) = V_{1u}^m \mid V_{1d}^c + r \geq = \left\{ \begin{array}{ll} \pi_1(s_1, s_4) = r & always \quad holds \\ \pi_1(s_2, s_2) = V_{1u}^m + r & holds \ if \quad N_1 = N_{13} \end{array} \right\} \quad (27)$$

(s_2, s_4) is a PBEP iff,

$$(-)\frac{(V_{2d}^c \mid N_{22})}{(V_{2d}^c \mid N_{23})} \geq \frac{(1-\phi)}{\phi} \ and \quad N_1 = N_{13}.$$

This proves part **(e)** of Theorem 2.

If (s_2, s_4) is a PBEP then we know s_{24} is the best reply to s_{12} so that inequality (6) holds. Conversely, suppose (6) holds. Then, we know s_{24} is a best reply to s_{12}. However, in order for (s_2, s_4) to be a PBEP we must have $\pi_1(s_2, s_4) \geq \geq \pi_1(s_i, b_2(s_i)) \ \forall \ i = 1, 2, 3, 4$ where $s_{24} = b_2(s_{12})$.

Since from Lemma 7 s_{11}, s_{12}, s_{14} are identified as best replies to s_{22} and s_{24} under N_{1j} for all $j = 1, 2, 3$, it follows that s_{12} is a Bayesian best reply to s_{24} if and only if

$$\begin{array}{llll} (a) & \pi_1(s_2, s_4) & \geq & \pi_1(s_1, s_4); \\ (b) & \pi_1(s_2, s_4) & \geq & \pi_1(s_4, s_4); \\ (c) & \pi_1(s_2, s_4) & \geq & \pi_1(s_4, s_2). \end{array} \quad (28)$$

But we have shown that $s_{22} \neq b_2(s_{14})$ when (6) holds. Therefore, (14 **(c)**) drops out.

From Table 4 and the definitions of N_{11}, N_{12} and N_{13} we observe that

$$\pi_1(s_2, s_4) = V_{1u}^c + r \geq$$

$$\geq \left\{ \begin{array}{ll} \pi_1(s_1, s_4) = r & holds \ if \ N_1 = N_{12}, N_{13} \\ \pi_1(s_4, s_4) = V_{1u}^c + V_{1d}^c + r & holds \ if \quad N_1 = N_{11}, N_{12} \end{array} \right\} \quad (29)$$

(s_2, s_4) is a PBEP iff

$$(-)\frac{(V_{2d}^c \mid N_{22})}{(V_{2d}^c \mid N_{23})} \geq \frac{\theta_2(1-\phi)}{(\theta_1 + \theta_2)\phi} \quad and \quad N_1 = N_{12} \ .$$

This proves part **(b)** of Theorem 2.

We have already proved that s_{22} is the best reply to s_{12} if and only if

$$(-)\frac{(V_{2d}^c \mid N_{22})}{(V_{2d}^c \mid N_{23})} \geq \frac{\theta_2(1-\phi)}{(\theta_1 + \theta_2)\phi} \quad (30)$$

Thus, (s_2, s_2) is a PBEP we know s_{22} is the best reply to s_{12} so that (16) must hold.

Conversely, suppose (16) holds. Then we know s_{22} is the best reply to s_{12}. But in order for (s_2, s_2) to be a PBEP, we must have $\pi_1(s_2, s_2) \geq \pi_1(s_i, b_2(s_i)) \ \forall \ i = 1, 2, 3, 4$

where $s_{22} = b_2(s_{12})$. Since from Lemma 7 s_{11}, s_{12} and s_{14} are best replies to s_{22} and s_{22} under N_{1j} for all $j = 1, 2, 3$, it follows that s_{12} is a Bayesian best reply to s_{24} if and only if

$$\begin{array}{llll} (a) & \pi_1(s_2, s_2) & \geq & \pi_1(s_1, s_4), \\ (b) & \pi_1(s_2, s_2) & \geq & \pi_1(s_4, s_2), \\ (c) & \pi_1(s_2, s_2) & \geq & \pi_1(s_4, s_4). \end{array}$$

From Table 2 and the definitions of N_{11}, N_{12} and N_{13} we observe that

$$\pi_1(s_2, s_2) = V_{1u}^m + r \geq$$

$$\geq \left\{ \begin{array}{ll} \pi_1(s_1, s_4) = r & always \quad holds \\ \pi_1(s_4, s_2) = V_{1u}^m + V_{1d}^c + r & holds \ if \quad N_1 = N_{11}, N_{12} \\ \pi_1(s_4, s_4) = V_{1u}^c + V_{1d}^c + r & always \ \ holds \end{array} \right\} \tag{31}$$

(s_2, s_2) is a PBEP iff

$$(-)\frac{(V_{2d}^c \mid N_{22})}{(V_{2d}^c \mid N_{23})} \geq \frac{\theta_2(1-\phi)}{(\theta_1 + \theta_2)\phi} \qquad and \qquad N_1 = N_{11}, N_{12}.$$

This proves part **(d)** of Theorem 2.

We now consider the necessary and sufficient conditions for $(s_4,\ s_4)$ to be a PBEP.

We have already proved that s_{24} is the best reply to s_{14} iff

$$(-)\frac{(V_{2d}^c \mid N_{22})}{(V_{2d}^c \mid N_{23})} \leq \frac{(1-\phi)}{\phi}. \tag{32}$$

Thus, if $(s_4,\ s_4)$ is a PBEP we know s_{24} is the best reply to s_{14} so that inequality (18) must hold. Conversely, suppose (18) holds. Then we know s_{24} is the best reply to s_{14}. But in order for $(s_4,\ s_4)$ to be a PBEP we must have

$$\pi_1(s_4, s_4) \geq \pi_1(s_i, b_2(s_i)) \qquad \forall \qquad i = 1, 2, 3, 4,$$

where

$$s_{24} = b_2(s_{14}).$$

Since from Lemma 7 s_{11}, s_{12} and s_{14} are identified as the best replies to s_{22} and s_{24} under N_{1j} for all $j = 1, 2, 3$, it follows that s_{14} is a Bayesian best reply to s_{24} if and only if

$$\begin{array}{llll} (a) & \pi_1(s_4, s_4) & \geq & \pi_1(s_1, s_4); \\ (b) & \pi_1(s_4, s_4) & \geq & \pi_1(s_2, s_4); \\ (c) & \pi_1(s_4, s_4) & \geq & \pi_1(s_2, s_2). \end{array} \tag{33}$$

But we have shown under part (d) that (s_2, s_2) cannot be a PBEB if $N_1 = N_{13}$. Hence, $s_{22} \neq b_2(s_{14})$ when (18) holds. Therefore, (19 **(c)**) drops out.

From Table 2 and the definitions of N_{1j} for all $j = 1, 2, 3$ we observe that

$$\pi_1(s_4, s_4) = V^c_{1u} + V^c_{1d} + r \geq$$

$$\geq \left\{ \begin{array}{ll} \pi_1(s_1, s_4) = r & holds\ if\ \ V^c_{1u} + V^c_{1d} + \geq 0 \\ & \quad and\ \ N_1 = N_{13} \\ \pi_1(s_2, s_4) = V^c_{1u} + r & \quad holds\ if\ \ N_1 = N_{13} \end{array} \right\} \tag{34}$$

(s_4, s_4) is a PBEP iff

$$(-)\frac{(V^c_{2d} \mid N_{22})}{(V^c_{2d} \mid N_{23})} \leq \frac{(1-\phi)}{\phi} \qquad and \qquad N_1 = N_{13}.$$

This proves part **(c)** of Theorem 2.

Finally, we consider the necessary and sufficient conditions for (s_1, s_4) to be a PBEP. First recall from Lemma 5 that $s_{24} = b_2(s_{11})$ under N_{2j} for all $j = 1, 2, 3$ independent of player 1's type. But this is an ex-post condition, i.e. s_{24} is the best reply to s_{11} for all N_{2j} $j = 1, 2, 3$ once player 1 chooses s_{11} strategy. However, player 1 chooses s_{11} as a Bayesian best reply to s_{24} if and only if $E\pi_2(s_1, s_4) \geq E\pi_2(s_1, s_2)$ and $N_1 = N_{11}$.

Recall that firm 2's dominant strategies are s_{22} and s_{24} under either N_{22} or N_{23}. Hence, we will first prove that, ex-ante, s_{24} is the best reply s_{11} if and only if

$$E\pi_2(s_1, s_4) \geq E\pi_2(s_1, s_2), \tag{35}$$

where,

$$E\pi_2(s_1 s_4) = \Pr(N_{22} \mid s_{11})\pi_2(s_1, s_4 \mid N_{22}) + \Pr(N_{23} \mid s_{11})\pi_2(s_1, s_4 \mid N_{23}), \tag{36}$$

$$E\pi_2(s_1 s_2) = \Pr(N_{22} \mid s_{11})\pi_2(s_1, s_2 \mid N_{22}) + \Pr(N_{23} \mid s_{11})\pi_2(s_1, s_2 \mid N_{23}). \tag{37}$$

We use Bayes's rule to calculate $\Pr(N_{22}|s_{11})$ and $\Pr(N_{23}|s_{11})$:

$$\Pr(N_{22} \mid s_{11}) = \frac{\Pr(s_{11} \mid N_{22}) \Pr(N_{22})}{\Pr(s_{11})} =$$

$$= \frac{\{\ \Pr(s_{11} \mid N_{22})\ \}\ \phi}{\{\ \Pr(s_{11} \mid N_{22})\ \}\ \phi + \{\Pr(s_{11} \mid N_{23})\ \}\ (1-\phi)} \tag{38}$$

where ϕ and $1 - \phi$ are the priors for $\Pr(N_{22})$ and $\Pr(N_{23})$ respectively.

But from Lemma **8 (a)** we have,

$$\Pr(s_{11} \mid N_{22}) = 0, \qquad \Pr(s_{11} \mid N_{22}) = \Pr(N_{11}) = \theta_1,$$

which upon substitution into (24) yields:

$$\Pr(N_{22} \mid s_{11}) = 0.$$

It follows that

$$\Pr(N_{23} \mid s_{11}) = 1.$$

From Table 2 and the definitions of N_{22}, N_{23} we observe that:

$$\pi_2(s_1, s_4 \mid N_{22}) = \pi_2(s_1, s_2 \mid N_{23}) = V^m_{2u} + V^m_{2d} + r > 0,$$

and

$$\pi_2(s_1, s_2 \mid N_{22}) = \pi_2(s_1, s_2 \mid N_{23}) = V^m_{2u} + r > 0.$$

Therefore, substituting the expressions for $\Pr(N_{2j}|s_{11})$, $\pi_2(s_1, s_4|N_{2j})$ and $\pi_2(s_1, s_4|N_{2j})$, $j = 2, 3$ into (22) and (23), we can write the inequality (21) as

$$V^m_{2u} + V^m_{2d} + r \geq V^m_{2u} + r \quad \Rightarrow \quad V^m_{2d} \geq 0.$$

But, $V^m_{2d} \geq 0$ is always true by the definition of payoffs. Therefore, (21) always holds. Hence, we have shown that s_{14} is always the best reply to s_{11} independent of player 1's and player 2's types.

Now, if (s_1, s_4) is a PBEP then we know s_{24} is a best reply to s_{11} so that (21) holds. Conversely, suppose (21) holds. Then we know s_{24} is a best reply to s_{11}. However, in order for (s_1, s_4) to be a PBEP we must have

$$\pi_1(s_1, s_4) \geq \pi_1(s_i, b_2(s_i)) \, \forall \; i = 1, 2, 3, 4,$$

where $s_{24} = b_2(s_{11})$. But from Table 3 this can only occur if

$$\begin{array}{ll} (a) & \pi_1(s_1, s_4) \geq \pi_1(s_2, s_4), \\ (b) & \pi_1(s_1, s_4) \geq \pi_1(s_2, s_2), \\ (c) & \pi_1(s_1, s_4) \geq \pi_1(s_4, s_2), \\ (d) & \pi_1(s_1, s_4) \geq \pi_1(s_4, s_4). \end{array} \tag{39}$$

Recalling from part **(b)** that we have shown $s_{24} = b_2(s_{12})$ iff (6) holds.

Also recall from part **(e)** that we have shown $s_{22} = b_2(s_{12})$ iff (1) holds.

But, we have proved that (1) and (6) can not hold together. Therefore, (25(c)) drops out since $s_{22} \neq b_2(s_{12})$ when (6) holds. We also noted in proving part **(b)** that $s_{22} \neq b_2(s_{14})$ when (6) holds. Therefore, (25(b)) also drops out.

This argument shows that in order fo rs_{24} to be the best reply to either s_{12} or s_{14} must hold as the binding condition. From Table 2 and the definitions N_{11}, N_{12} and N_{13} we observe that:

$$\pi_1(s_1, s_4) = r \geq \left\{ \begin{array}{lll} \pi_1(s_2, s_4) = V^c_{1u} + r & holds & if \;\; N_1 = N_{11} \\ \pi_1(s_4, s_4) = V^c_{1u} + V^c_{1d} + r & holds & if \;\; N_1 = N_{11} \end{array} \right\} \tag{40}$$

Therefore, we conclude that (s_1, s_4) is a PBEP iff

$$(-)\frac{(V^c_{2d} \mid N_{22})}{(V^c_{2d} \mid N_{23})} \leq \frac{\theta_2(1-\phi)}{(\theta_1 + \theta_2)\phi}$$

and $N_1 = N_{11}$.

This proves part **(a)** of Theorem 2.

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Stochastic Differential Games and Queueing Models To Innovation and Patenting

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Abstract. Dynamic differential games have been widely applied to the timing of product and device innovations. Uncertainty is also inherent in the process of technological innovation: R&D expenditures will be engaged in an unforeseeable environment and possibly lead to innovations after a random time interval. Reinganum [Reinganum, 1982] enumerates such uncertainties and risks: feasibility, delays in the process, imitation by rivals. Uncertainties generally affect the fundamentals of the standard differential game problem: discounted profit functional, differential state equations of the system, initial states. Two ways of resolution may be taken [Dockner, 2000]: firstly, stochastic differential games with Wiener process and secondly differential games with deterministic stages between random jumps (Poisson driven probabilities) of the modes. The player will then maximize the expected flows of his discounted profits subject to the stochastic state constraints of the system. In this context, the state evolution is described by a stochastic differential equation SDE (the Ito equation or the Kolmogorov forward equation KFE). According to the Dasguspta and Stiglitz's model [Dasguspta, 1980], R&D efforts exert direct and induced influences (through accumulated knowledge) about the chances of success of innovations. The incentive to innovate and the R&D competition can be supplemented by a competition around a patent. This presentation is focused on such essential economic and managerial problems (R&D investments by firms, innovation process , and patent protection) with uncertainties using stochastic differential games [Friedman, 2004], [Yeung, 2006], [Kythe, 2003], modeling with It SDEs [Allen, 2007] and queueing models [Gross, 1998]. The computations are carried out using the software *Mathematica* 5.1 and other specialized packages [Wolfram, 2003], [Kythe, 2003].

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Keywords: Stochastic differential game, uncertainty, Markov-Nash equilibrium, technological innovation

JEL Classification: C73, O31, Q55 – *Mathematics Subject Classification* . Primary 60K25, 91A23; Secondary 60H15, 90B22, 91A80.

1. Uncertainties and basic stochastic processes

Definition 1. *(stochastic process). Given a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with filtration* [2] $\{\mathcal{T}_t, t \geq 0\}$ *satisfying the conditions of right continuity and completion, a stochastic process is a collection of random variables* $\{x_t\}_{t \in T}$ *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with values in* $\mathbb{R}^n$*. The process may be represented by the function [Øksendal, 2003]*

$$(t, \omega) \mapsto x(t, \omega) : T \times \Omega \mapsto \mathbb{R}^n.$$

We will thus have the number of random events that occur in [0,t].

Definition 2. *(random variable, path). For each* $t \in T$ *fixed, the state of the process is given. We then have a random variable (RV)defined by*

$$\omega \mapsto x_t(\omega), \omega \in \Omega.$$

For each experiment $\omega \in \Omega$*, we have a path of* x_t *defined by*

$$t \mapsto x_t(\omega), t \in T.$$

Definition 3. *(stationary process). A process* x_t *is stationary if* $x_{t_1}, \ldots, x_{t_n}$ *and* $x_{t_{1+s}}, \ldots, x_{t_{n+s}}$ *have the same joint distribution for all n and s.*[3]

1.1. Brownian motion

Definition 4 *(one-dimensional Brownian motion, or Wiener process [Friedman, 2004]. A stochastic process* $\{z_t\}_{t \geq 0}$*, is a Brownian motion satisfying the conditions:* (i) $z_0 = 0$*,* (ii) *the process has stationary independent increments* $z_{t_k} - z_{t_{k-1}} (1 \leq k \leq n)$*, and* (iii) *if* $0 \leq s < t$*,* $z_t - z_s$ *is normally distributed with* $\mathbf{E}[z_t - z_s] = (t - s)\mu$ *and* $\mathbf{E}[\{z_t - z_s\}^2] = (t - s)\sigma^2$*, where* μ *is the drift and* σ^2 *– the variance.*

According to the first condition any z_t that starts at z_0 can be redefined as $z_t - z_0$. The second condition tells that the random increment $z_{t_{n+1}} - z_{t_n}$ is independent of the previous one $z_{t_{n-1}} - z_{t_{n-2}}$, for all n. Increments are stationary when $z_t - z_{t-s}$ has the same distribution for any t and s constants. With the third condition, the RV has

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[2] In algebra, a filtration of a group is ordinarily a sequence $G_n (n \in \mathbb{N})$ of subgroups such that $G_{n+1} \subseteq G_n$. A filtration is often used to represent the change of the sets of measurable events in terms of information quantity. A filtered σ-algebra is an increasing sequence of Borel σ-algebras $\{\mathcal{F}_t\}_{t \geq 0}$ with $\mathcal{F}_t \subseteq \mathcal{F}$ for each t and $t_1 \leq t_2 \Rightarrow \mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$.

[3] A weaker concept states that the first two moments are the same for all t and that the covariance between x_{t_1} and x_{t_2} depends on the time interval $t_1 - t_2$ [Ross, 1996].

the following probability density function (PDF) $f(z) = (\sqrt{2\pi\sigma^2 t})^{-1}\exp[-\frac{1}{2}(\frac{z-\mu}{\sigma})^2]$. Since the variance linearly increases in time, the Wiener process is non stationary. Stochastic differential equations (SDEs) frequently introduce uncertainty through a simple Brownian motion and are defined by

$$dx_t = \mu dt + \sigma dz_t,$$

where the constant μ is the drift rate, σ^2 the variance rate of x_t (σ denotes the diffusion rate), dt a short time interval, and dz_t the increment of the Brownian motion. The Figure 1 (a) shows three different realizations $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3$ of the Brownian process. The deterministic part is clearly governed by the ordinary differential equation (ODE) $\dot{y}_t = \mu$ which solution is linear in time.[4]

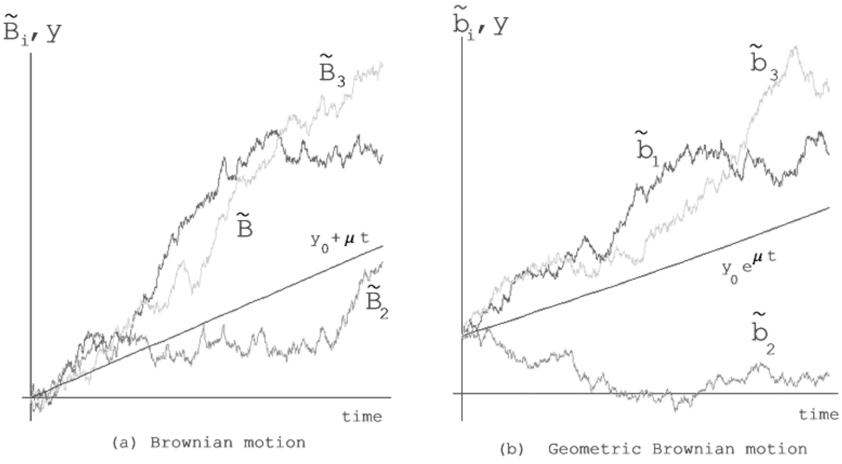

Fig. 1: Standard Brownian motions

Example (total factor productivity). Let us consider a differential representation for the production technology [Wälde, 2006]. With an AK technology, we have $Y_t = F(A_t, K) = A_t K$, where Y_t and A_t denote continuous functions of time t (a more convenient notation), where A states for the total factor productivity (TFP), Y, the

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[4] The calculations use the packages Statistics‘ ContinuousDistributions, StochasticEquations‘ EulerSimulate of *Mathematica* 5.1 and Itovns3 [Kendall, 1993].The primitive of EulerSimulate is EulerSimulate [drift, diffusion, {x, x0}, {duration, nsteps}]. It returns a list of simulated values for the corresponding Ito process. More generally , a system of Ito processes can also be simulated by specifying the drift vector and the matrix of diffusion. The Mathematica package ItosLemma implements Ito's lemma for stochastic multidimensional calculus, computing stochastic derivatives and Ito–Taylor series. The primitive Itomake[x[t], μ, σ], where μ is the drift and σ the diffusion, stores the rule $x[t+dt] = x[t] + \mu dt + \sigma dB_1$.

output and K, a fixed amount of global productive factors. Suppose TFP grows at a deterministic rate g with $\dot{A}_t = gA_t \Leftrightarrow dA_t = gA_t dt$. We have the differential $dF(A_t, K) = F_A dA_t + F_K dK$.

We easily deduce the growth of Y_t as $\dot{Y}_t = gKA_t$. The evolution in time is

$$Y_t = gK \int_1^t A_s ds.$$

A more realistic situation consists in the introduction of the uncertainties that may affect the TFP. Let suppose that A_t will be driven by a Brownian motion with drift such as $dA_t = gdt + \sigma dz_t$, where g and σ are constants. Solving the SDE, we have $A_t = A_0 + gt + \sigma z_t$. The time evolution of Y_t is given by

$$Y_t = A_0 K + gKt + \sigma K z_t.$$

In this example[5], the evolution of the output consists of two parts: a deterministic trend and a stochastic deviation component from the trend. However, since Y_t may be negative, we have to look for another specification.

A RV may also evolve according to a geometric Brownian, such as

$$\frac{dx_t}{x_t} = \mu dt + \sigma dz_t \Leftrightarrow dx_t = \mu x_t dt + \sigma x_t dz_t.$$

The Figure 1 (b) shows three different realizations $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3$ of such a geometric Brownian process. The deterministic part is governed by the ODE $\dot{Y}_t = aY_t$, which solution is clearly exponential in time.

Definition 5. *(one-dimensional Itô processes [Øksendal, 2003]). Let* B_t *be a one-dimensional Brownian motion on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *A stochastic Itô integral is a stochastic process* x_t *of the form:*

$$x_t = x_0 + \int_0^t u_s(\omega) ds + \int_0^t v_s(\omega) dB_s,$$

so that

$$\mathbb{P}\{\textstyle\int_0^t v_s(\omega)^2 ds < \infty \textit{ for all } t \geq 0\} = 1$$

$$\textit{and } \mathbb{P}\{\int_0^t |u_s(\omega)| ds < \infty \textit{ for all } t \geq 0\} = 1.$$

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[5] Wälde [Wälde, 2006] also gives another specification where the drift rate is AK and the diffusion rate σK. The solution takes the form

$$Y_t = Y_0 + AKt + \sigma K z_t.$$

Theorem 1 *(the one-dimensional Itô formula [Øksendal, 2003]). Let x_t be an Itô process given by $dx_t = udt + vdB_t$. Let a twice continuously differentiable function $g(t,x) \in \mathcal{C}^2([0,\infty) \times \mathbb{R})$ on $[0,\infty) \times \mathbb{R}$. Then $y_t = g(t,x_t)$ is again an Itô process, and*

$$dy_t = \frac{\partial g}{\partial t}(t,x_t)dt + \frac{\partial g}{\partial x}(t,x_t)dx_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x_t)(dx_t)^2,$$

where $(dx_t)^2 = (dx_t).(dx_t)$ is calculated according to the multiplication rules

$$dt.dt = dt.dB_t = dB_t dt = 0,\ dB_t.dB_t = dt.$$

Proof.

See Øksendal [Øksendal, 2003], p.46 (with slightly different notations).

Example (Total factor productivity (continued) [Wӓlde, 2006]). Let TFP follow a geometric [6] Brownian motion

$$\frac{dA_t}{A_t} = gdt + \sigma dz_t,$$

where g and σ are constants. Hence, we have

$$\int_0^t \frac{dA_s}{A_s} = gt + \sigma z_t,\ z_0 = 0. \tag{1}$$

To evaluate the integral on the LHS, the Itô formula is used for the logarithmic function $g(t,x) = \ln x,\ x > 0$. We have

$$dg(t,x) = g_t dt + g_x dx + \frac{1}{2} g_{xx}(dx)^2,$$

where $dx = gxdt + \sigma xdz$. Since $(dx)^2 = g^2x^2(dt)^2 + \sigma^2x^2(dz)^2 + 2g\sigma dtdz = \sigma^2x^2dt$, we deduce

$$dg(t,x) = g_t dt + g_x dx + \frac{1}{2} g_{xx}\sigma^2x^2dt.$$

We then obtain

$$\begin{aligned} d\ln A_t &= \frac{1}{A_t}dA_t + \frac{1}{2}(-\frac{1}{A_t^2})(dA_t)^2 = \\ &= \frac{dA_t}{A_t} - \frac{1}{2A(t)^2}\sigma^2A_t^2dt = \frac{dA_t}{A_t} - \frac{1}{2}\sigma^2dt. \end{aligned}$$

Hence, $\frac{dA_t}{A_t} = d\ln A_t + \frac{1}{2}\sigma^2dt$. The expression of the integral is then

$$\int_0^t \frac{dA_s}{A_s} = \ln\frac{A_t}{A_0} + \frac{1}{2}\sigma^2t.$$

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[6] The process rather describes the rate of change of a RV, than the random variable itself.

From (1), we then deduce the time evolution of the TFP: $A_t = A_0 \exp[(g - \frac{1}{2}\sigma^2)t+$ $+\sigma z_t]$.

1.2. Poisson Process

The occurrence of discrete events at times $t_0, t_1, t_2, \ldots$ (e.g. innovations) are often modeled as a Poisson process. For a Poisson process, the time intervals $\Delta t_1 = t_1 - t_0, \Delta t_2 = t_2 - t_1, \ldots$ between successive events are independent variables drawn from an exponential distributed population. The parameterized PDF is given by $f(x; \lambda) = \lambda \mathrm{e}^{-\lambda x}$ for some positive constant λ. Suppose a system that starts in state 0 at initial time t_0. It will change to state 1 at some time $t = T$, where T is drawn from an exponential distribution. The probability that the system will be in state 1 at time t_1 is given by the integral

$$P_1(t_1) = \int_0^{t_1} \lambda \mathrm{e}^{-\lambda t} \mathrm{dt} = 1 - \mathrm{e}^{-\lambda t_1}.$$

The probability of the system still being in state 0 is the complement $P_0(t_1) = \mathrm{e}^{-\lambda t_1}$. The absolute rate of change of being in state 1 is $\frac{dP_1}{dt} = \lambda \mathrm{e}^{-\lambda t}$. We deduce an exponential transition with rate λ such as

$$\frac{dP_1}{dt} = \lambda P_0. \tag{2}$$

More generally, for any number of states, a system of differential equations such as (2) will describe the probabilities of being in each state. Since the transition time from the state P_n to P_{n+1} is exponential for all n, a Poisson process will be deduced. Schematically, it is illustrated by the chain of the Figure 2, where P_j is the probability of the jth state when j events have occurred. The initial conditions are such that $P_0(0) = 1, P_j(0) = 0$ for all $j > 0$. We have to determine $P_n(t)$. Since the transitions are exponentially distributed, we have the system

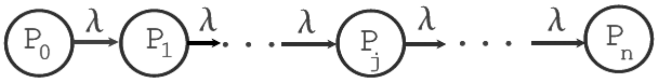


Fig. 2: Poisson process at constant rate

$$\frac{dP_0}{dt} = -\lambda P_0,\ \frac{dP_1}{dt} = \lambda P_0 - \lambda P_1, \ldots,\ \frac{dP_n}{dt} = \lambda P_{n-1} - \lambda P_n.$$

Given the initial condition $P_0(0) = 1$, the solution of the first equation is $P_0(t) = \mathrm{e}^{-\lambda t}$. Substituting this result into the second equation, we have the ODE

$$\frac{dP_1}{dt} + \lambda P_1 = \lambda \mathrm{e}^{-\lambda t}.$$

Solving the ODE, we obtain [7]

$$P_1(t) = (\lambda t)\mathrm{e}^{-\lambda t}.$$

Continuing by substitution, we have

$$P_2(t) = \frac{(\lambda t)^2}{2!}\mathrm{e}^{-\lambda t}, \ldots, P_n(t) = \frac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t}.$$

In this simple counting Poisson process, the probability $P_n(t)$ then expresses that exactly n events have occurred at time t. The expected number of occurrences by time t is

$$\mathbf{E}[n,t] = \sum_{n=0}^{\infty} nP_n(t) = \mathrm{e}^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!} = \mathrm{e}^{-\lambda t}(\lambda t)\sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!} = \lambda t.$$

Definition 6 *(Poisson process). A stochastic process q_t is a Poisson process with arrival rate* [8] *λ if:* (i) $q_0 = 0$, (ii) *the process has independent increments, and* (iii) *the increments $q_\tau - q_t$ (or jumps) in any time interval $\tau - t$ is Poisson distributed with mean $\lambda(\tau - t)$, say $q_\tau - q_t \rightsquigarrow \mathcal{P}oi(\lambda(\tau - t))$.*

The probability that the process increases n times between t and $\tau > t$ is given by

$$\mathbb{P}\{q_\tau - q_t = n\} = \mathrm{e}^{-\lambda(\tau-t)}\frac{(\lambda(\tau-t))^n}{n!},\ n = 0, 1, \ldots.$$

The SDE of a standard Poisson process is

$$dx_t = adt + bdq_t,$$

where the increment dq_t is driven by

$$dq_t = \begin{cases} 0 & \text{w.p. } 1 - \lambda dt, \\ 1 & \text{w.p. } \lambda dt, \end{cases}$$

The Figure 3 illustrates two situations: in figure (a) jumps have the same amplitude, in the second (b) jump amplitudes are random. If no jump occurs ($dq = 0$), the variable will follow a linear growth $x_t = x_0 + at$.When a jump occurs, x_t increases by b. The Figure 3 (b) shows an extension of the Poisson process where the amplitude of the jumps b_t are governed by some distribution, such as $b_t \rightsquigarrow \mathcal{N}(\mu, \sigma^2)$.

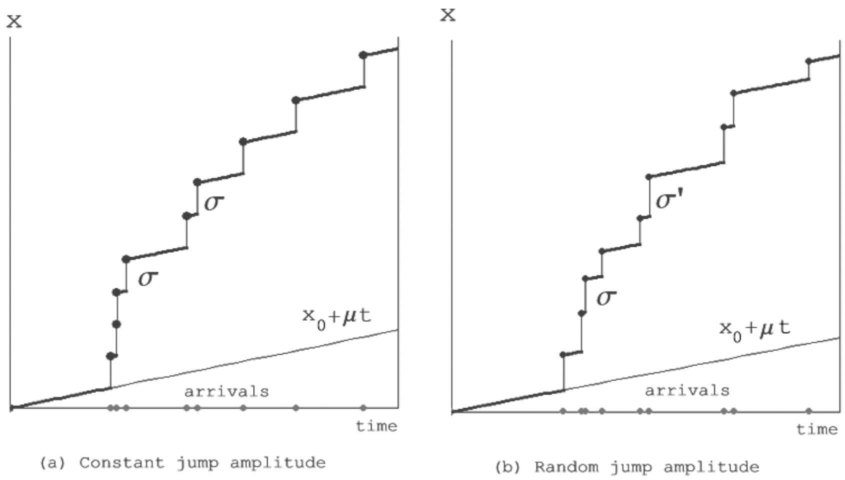
---

[7] The solution of the generalized ODE (in usual notations) with variable parameters $\dot{x} + a(t) = b(t)$ is given by

$$x(t) = \mathrm{e}^{-\int_1^t a(s)ds}\{C + \int_1^t \mathrm{e}^{\int_1^s a(u)du} b(s)ds\},$$

where C is a constant of integration. The solution for the process is achieved by setting $x \equiv P_1$, $a(t) \equiv \lambda$ and $b(t) = \lambda\mathrm{e}^{-\lambda t}$.

[8] A high arrival rate means that the process jumps more often.



(a) Constant jump amplitude (b) Random jump amplitude

Fig. 3: Poisson processes with constant and random jump amplitude

Lemma 1 *(Change of Variable Formula CVF). Let x_t be a Poisson stochastic process given by*

$$dx_t = a(t, x_t, q_t)dt + b(t, x_t, q_t)dq_t.$$

Let a twice continuously differentiable function $F(t, x) \in \mathcal{C}^2([0,\infty)\times\mathbb{R})$ on $[0,\infty)\times\mathbb{R}$, the differential is

$$dF(t, x_t) = (F_t + F_x a(.))dt + \{F(t, x_{t^-} + b(.)) - F(t^-, x_t)\}dq_t,$$

where t^- denotes a date that precedes a jump at time t.

Example (Total factor productivity (continued) [Wälde, 2006]). Let TFP follow a geometric Poisson process

$$\frac{dA_t}{A_t} = gdt + \sigma dq_t,$$

where g and σ are constants. Hence, applying the Itô's lemma for Poisson processes, we obtain the solution [9]

$$A_t = A_0 \exp[gt + (q_t - q_0)\ln(1+\sigma)].$$

The TFP will then follow a deterministic exponential trend and have a stochastic deviation, given by $(q_t - q_0)\ln(1+\sigma), \sigma \geq 0$.

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[9] The proof of lemma 1.9 is shown in appendix A. The detailed calculations for this example are given in the same appendix.

1.3. Queueing models

Queueing models are a typical application of exponential transitions and Poisson processes. Some events occur at some constant rate λ and are treated at a constant rate μ. Let us consider a M/M/1 queue, where the first M states for memoryless arrivals (i.e. inter-arrivals times of occurring events are often modeled as exponentially distributed variables), the second M is the same for departures and the 1 states a single server. The chain may by represented schematically by the Figure 4.

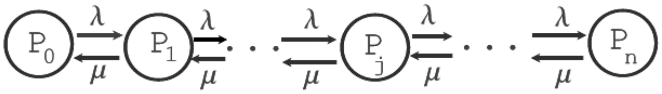

Fig. 4: Queues with a single service M/M/1

The system of dynamic equations is:

$$\frac{dP_0}{dt} = -\lambda P_0 + \mu P_1,\ \frac{dP_1}{dt} = \lambda P_0 - \lambda P_1 - \mu P_1 + \mu P_2, \ldots,$$
$$\frac{dP_n}{dt} = \lambda P_{n-1} - \lambda P_n - \mu P_n + \mu P_{n+1}.$$

The steady state probabilities, once the system has stabilized at the equilibrium, are characterized by the probability that exactly n events occur and by the expected number of presently waiting events[10]

$$P_n = (1 - \frac{\lambda}{\mu})(\frac{\lambda}{\mu})^n \text{ and } \mathbf{E}[n] = \sum_{n=0}^{\infty} nP_n = \frac{\lambda/\mu}{1-\lambda/\mu}.$$

2. Stochastic control problem and differential games

There are two particular ways to introduce uncertainty in the differential games. The first way is based on piecewise deterministic processes, where the system switches

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[10] In the steady state, the derivatives vanish and we deduce the geometric series

$$P_1 = \frac{\lambda}{\mu}P_0, P_1 = (\frac{\lambda}{\mu})^2 P_0, \ldots, P_n = (\frac{\lambda}{\mu})^n P_0, \ldots$$

which converges only if the rate of arrivals λ is less than the rate μ of processing. Under this condition we have

$$P_0\{1 + (\frac{\lambda}{\mu}) + (\frac{\lambda}{\mu})^2 + \ldots + (\frac{\lambda}{\mu})^n + \ldots\} = P_0(1 - \frac{\lambda}{\mu})^{-1} = 1.$$

The expression of P_n will then follow from $P_0 = 1 - (\lambda/\mu)$.

from one deterministic mode to another, at random jump times. The second way consists of introducing continuous stochastic noise processes[11].

2.1. Optimal control under uncertainty

The uncertainty may take the form of a Brownian motion (also Wiener process). This process will generally influence the evolution of the state variable. This evolution will be described by an SDE of the form [Dockner, 2000]:

$$dx_t = f(t, x_t, u_t)dt + \sigma(t, x_t, u_t)dw_t, x_0 \text{ given}, \tag{3}$$

where x_t denotes the n-dimensional vector of the states and u_t – an m-dimensional vector of controls. A k-dimensional Wiener process w_t is a continuous-time stochastic process, such as $w : [0, T) \times \Xi \mapsto \mathbb{R}^k$, where Ξ denotes the set of points ξ of possible realizations of the RV. The functions are such that $f : \Omega = \{(t, x, u)|t \in [0, T), x \in X, u \in U(t, x)\} \mapsto \mathbb{R}^n$ and $\sigma : \Omega \mapsto \mathbb{R}^{n\times k 12}$. A solution x_t of the SDE (3) must satisfy the following integral equation

$$x_t = \int_0^t f(s, x_s, u_s)ds + \int_0^t \sigma(s, x_s, u_s)dw_s$$

for all ξ of $w(s, \xi)$ in a set of probability 1^{13} footnote Any solution to the SDE is a stochastic process depending on the realizations of $\xi \in \Xi$. The correct notation is rather $x(t, \xi)$ than x(t) or x_t, showing that the value of x_t cannot be known, without knowing the realization of ξ (see [Dockner, 2000], p.228)..

Lemma 2 *(Ito's lemma). Suppose that x_t solves the SDE (3). Let $G : [0, T) \times X \mapsto \mathbb{R}$ be a function with continuous partial derivatives G_t, G_x, G_{xx}. The function $g(t) = G(t, x_t)$ will satisfy the SDE:*

$$\begin{aligned} dg(t) = \{G_t(t, x_t) + G_x(t, x_t)f(t, x_t, u_t) + \\ + \frac{1}{2}tr[G_{xx}(t, x_t)\sigma(t, x_t, u_t).\sigma(t, x_t, u_t)']\}dt + \\ + G_x(t, x_t)\sigma(t, x_t, u_t)dw_t. \end{aligned}$$

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[11] Memoryless Poisson models of patent race are associated with Dasgupta and Stiglitz [Dasguspta, 1980], Lee and Wilde [Lee, 1980], Loury [Loury, 1979], Reinganum [Reinganum, 2004], [**?**]. The probability to innovate and to obtain a patent depends on the current R&D investment. Reinganum [Reinganum, 1982] and Yeung and Petrosyan [Yeung, 2006] consider non cooperative and cooperative games.

[12] An entry σ_{ij} of this $n \times k$ matrix evaluates the direct impact of the jth component of the k-dimensional Wiener process on the evolution of the ith component of the n-dimensional state vector.

[13] The second integral is such as $\lim_{\delta\to 0} \sum_{l=1}^{L-1} \sigma(t_l, x_{t_l}, u_{t_l})\{w_{t_{l+1}} - w_{(t_l}\}$, where $0 = t_1 < t_2 < \cdots < t_L = t$ and $\delta = \max\{|t_{l+1} - t_l|, 1 \leq l \leq L - 1\}$

The stochastic control problem is given by

$$\begin{aligned} \max \mathbf{E}_{u(.)}[\int_0^T F(t,x_t,u_t))\mathrm{e}^{-rt}\, dt + \mathrm{e}^{-rT}S(x_T)], \\ \text{s.t.} \\ dx_t = f(t,x_t,u_t)dt + \sigma(t,x_t,u_t)dw_t, \\ x_0 \text{ given}, u_t \in U(t,x_t). \end{aligned} \tag{4}$$

The following optimality conditions are based on the Bellman equation (HJB).

Theorem 2 *(Optimality conditions). Let a function be defined as* $V : [0,T) \times X \mapsto \mathbb{R}$ *with continuous partial derivatives* V_t, V_x *and* V_{xx}*. Assume that* V *satisfies the HJB equation:*

$$\begin{aligned} rV(t,x_t) - V_t(t,x_t) = max\ \{F(t,x_t,u_t) + V_x(t,x_t)f(t,x_t,u_t) \\ + \frac{1}{2}tr[V_{xx}(t,x_t)\sigma(t,x_t,u_t).\sigma(t,x_t,u_t)']|u_t \in U(t,x_t)\}, \end{aligned} \tag{5}$$

for all $(t,x_t) \in [0,T) \times X$*. Let* $\Phi(t,x_t)$ *be the set of controls maximizing the RHS of (5) and* u_t *be a feasible control path with state trajectory* x_t *s.t.* $u_t \in \Phi(t,x_t)$ *holds a.s. for a.a.* $t \in [0,T)$*:*
(i) *if* $T < \infty$ *and if the boundary condition* $V(T,x) = S(x)$ *holds for all* $x \in X$ *then* $u_{(.)}$ *is an optimal control path;*
(ii) *if T=*∞ *and if either* V *is bounded and* $r > 0$*, or* V *is bounded below with* $\lim_{t\to\infty} e^{-rt}\boldsymbol{E}_{u_{(.)}}V(t,x_t) \leq 0$ *holds, then u(.) is a catching up optimal control path.*

Proof. See Dockner et al. [Dockner, 2000], p.229-30, Yeung and Petrosyan [Yeung, 2006], p.16 with different notations.

2.2. Differential games with random process

Let us consider a N-players game. The control variable by the ith player is denoted by u_t^i at time t for $i \in \{1,2,\ldots,N\}$. The vector of controls by the opponents of player i will be $u_t^{-i} = \{u_t^1, u_t^2, \ldots, u_t^{i-1}, u_t^{i+1}, \ldots, u_t^N\}$. The controls are subject to the constraints $u_t^i \in U^i(t,x_t,u_t^{-i}) \subseteq \mathbb{R}^{m_i}$, where the $x_t \in X$ are the state variables of the system. The state equation for the game will then be given (by omitting the time index of arguments)

$$dx_t = f(t,x,u^1,\ldots,u^N)dt + \sigma(t,x,u^1,\ldots,u^N)dw_t,$$

where w_t is a k-dimensional Wiener process. The functions f and σ are both defined on $\Omega = \{(t,x,u^i,u^{-i})|t \in [0,T), x \in X,\ u^i \in U^i(t,x,u^{-i})\}$ with values in $\mathbb{R}^n$ and $\mathbb{R}^{n\times k}$ respectively. The objective of each player is to maximize the expectation of his discounted flow of payoffs

$$J^i(u^i_{(.)}) = \mathbf{E}_{u_{(.)}}\{\int_0^T F^i(t,x_t,u^i,u^{-i})\mathrm{e}^{-r_it}dt + \mathrm{e}^{-r_it}S^i(x_T)|x_0 \text{ given}\},$$

where F^i is a real-valued utility function defined on Ω, S^i a real-valued scrap value function defined on X, and r_i the discount rate.

2.3. Piecewise deterministic control problem

Let us consider an autonomous problem defined over an unbounded time interval $[0, \infty)$. The evolution of the system may be deterministic, except at certain jump times given by the finite set $\{T_1, T_2, \ldots, T_M\}$. At each of these dates, the system switches from one mode to another. The following description is inspired from Dockner et al. [Dockner, 2000]. Let $X \subseteq \mathbb{R}^n$ denote the state space and $x_t \in X$ the state at time t. The set of controls, when the current mode is $h \in M$, is given by $U(h, x_t) \subseteq \mathbb{R}^m$.

The motion is described by the differential equation $\dot{x}_t = f(h, x_t, u_t)$ where $f(h, ., .)$ maps $\Omega(h) = \{(x, u) | x \in X, u \in U(h, x)\}$ into $\mathbb{R}^n$. The instantaneous payoffs of the player consist of $F(h, x_t, u_t)$, a real-valued function defined on $\Omega(h)$, and the lump sum payoff $S_{hk}(x_t)$, when a jump occurs from mode h to mode $k, (k \neq h)$. The payoffs are discounted at the constant rate $r > 0$. The motion of the system mode is a continuous-time stochastic process $h : [0, \infty) \times \Xi \mapsto M$, where the set Ξ of points ξ represents realizations of some random variable. Thus, the event, that the mode is h at time t, is $\{\xi \in \Xi | h_t(\xi) = h\}$ and its probability is denoted by $\mathbb{P}\{h_t(\xi) = h\}$. The probability that the system switches from mode h to mode k during the time interval $(t, t + \Delta t]$ is proportional to the length of Δ. We have

$$\lim_{\Delta \to 0} \frac{1}{\Delta} \mathbb{P}\{h_{t+\Delta} = k | h_t = h\} = q_{hk}(x_t, u_t)\}, \ k \neq h,$$

where $q_{hk} : \Omega(h) \mapsto \mathbb{R}_+$. The risk neutral player seeks to maximize the expectation of the discounted payoff flow, conditional on the initial state and mode. Initially, we state

$$J(u_{(.)}) = \mathbf{E}_{u_{(.)}}[\int_0^\infty F(h_t, x_t, u_t) e^{-rt} dt + \\ + \sum_{l \in \mathbb{N}} S_{h_{T_l -} h_{T_l}}(x_{T_l}) e^{-rT_l} \ | x_0, h_0 \text{ given}],$$

where T_l denotes the lth jump and $h_{T_l -}$, the mode immediately before the switch. We also have

$$J(u_{(.)}) = \mathbf{E}_{u_{(.)}}[\int_0^\infty \{F(h_t, x_t, u_t) + \\ + \sum_{k \neq h_t} q_{h_t k}(x_t, u_t) S_{h_t k}(x_t)\} e^{-rt} dt \mid x_0, h_0 \text{ given}]. \quad (6)$$

Definition 7 *(feasible path). Given* Ξ *a set of points* $\xi \in \Xi$ *representing possible realizations of a random variable, the control path* $u : [0, \infty) \times \Xi \mapsto \mathbb{R}^m$ *is feasible* *for the stochastic control problem if the following conditions are satisfied:* (i) *it is non-anticipating,* (ii) *the constraints* $x_t \in X$ *and* $u(t) \in U(h_t, x_t)$ *are a.s. verified,* (iii) *the process* $(h_{(.)}, x_{(.)})$ *and the integral in (4) are well defined. The control path is optimal, if it is feasible and if* $J(\bar{u}_{(.)}) \geq J(u_{(.)})$ *for all feasible paths.*

Dockner et al. [Dockner, 2000] p.206–7, deduce the fundamental theorem.

Theorem 3 *(optimal control path). Let us consider the autonomous problem* [14] *and suppose the existence of a bounded function* $V : M \times X \mapsto \mathbb{R}$ *which have the following properties. The function* $V(h, x)$ *is continuously differentiable in* x *for all* $h \in M$ *and is such that (omitting the time argument) the HJB equation*

$$rV(h,x) = max\,\{F(h,x,u) + V_x(h,x)f(h,x,u) + \\ + \sum_{k \neq h} q_{hk}(x,u)[S_{hk}(x) + V(k,x) - V(h,x)] | u \in U(h,x)\}, \quad (7)$$

is verified for all $(h, x) \in M \times X$*. Let* $\Phi(h, x)$ *be the set of controls maximizing the RHS of (7). Then the control path* u_t *is optimal if the following conditions are satisfied: the control* $u_{(.)}$ *is feasible and* $u_{(.)} \in \Phi(h_{(.)}, x_{(.)})$ *holds a.s.*

Proof. See Dockner et al. [Dockner, 2000], p. 207.

2.4. Piecewise deterministic differential games

Let a N-players autonomous game be a piecewise deterministic differential game over an infinite time horizon. Denote by u_t^i the control value by player i and

$$u_t^{-i} = (u_t^1, \ldots, u_t^{i-1}, u_t^{i+1}, \ldots, u_t^N)$$

the vector of controls of the opponents of player i. The player's i set of feasible controls is

$$U^i(h_t, x_t, u_t^{-i}) \subseteq \mathbb{R}^{m_i},$$

when the system is in mode $h_t \in M$ and state $x_t \in X$. The objective functional of the ith player is given by

$$J^i(u^i_{(.)}) = \mathbf{E}_{u_{(.)}}[\int_0^\infty F^i(h_t, x_t, u_t)\mathrm{e}^{-r_i t} dt \\ + \sum_{l \in \mathbb{N}} S^i_{h_{\theta_{l-}} h_{T_l}} x_{T_l} \mathrm{e}^{-r_i T_l} \mid x_0, h_0 \text{ given}],$$

where $S^i_{h_{(T_{l-}} h_{T_l}}(x_{T_l})$ denotes the payoff received if a jump occurs at time T_l from $h_{T_{l-}}$ to h_{T_l}. Suppose as in [Dockner, 2000], that all players use a stationary Markov

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[14] Dockner et al. [Dockner, 2000] also consider the non-autonomous problem.

strategy of the form $u_t^i = \phi^i(h_t, x_t)$ then the player i's optimal control problem is of the form

$$\max_{u^i_{(.)}} J^i_{\phi^{-i}}(u^i_{(.)}) = \mathbf{E}_{u_{(.)}}[\int_0^\infty F^i_{\phi^{-i}}(h_t, x_t, u_t^i)e^{-r_i t}dt \\ + \sum_{l \in \mathbb{N}} S^i_{h_{T_{l^-}}, h_{T_l}}(x_{T_l})e^{-r_i T_l} \mid x_0, h_0 \text{ given}],$$

$$\text{s.t.}$$

$$\dot{x}_t = f^i_{\phi^{-i}}(h_t, x_t, u_t^i),$$

$$x_0 \text{ given}, \; u_t^i \in U^i_{\phi^{-i}}(h_t, x_t),$$

where the piecewise deterministic process $h_{(.)}$ is determined by the initial condition h_0 and the switching rates $q^i_{\phi^{-i},hk}(x_t, u_t^i)$. The functions $F^i_{\phi^{-i}}$ and $f^i_{\phi^{-i}}$ have the same pattern as

$$F^i_{\phi^{-i}}(h, x, u^i) = F^i(h, x, \phi^1(h, x), \ldots, \phi^{i-1}(h, x), u^i, \phi^{i+1}(h, x), \ldots, \phi^N(h, x)).$$

The functions $U^i_{\phi^{-i}}$ and $q^i_{\phi^{-i},hk}$ are defined by

$$U^i_{\phi^{-i}}(h, x) = U^i(h, x, \phi^1(h, x), \ldots, \phi^{i-1}(h, x), \phi^{i+1}(h, x), \ldots, \phi^N(h, x)),$$
$$q^i_{\phi^{-i},hk}(x, u^i) = q^i(x, \phi^1(h, x), \ldots, \phi^{i-1}(h, x), u^i, \phi^{i+1}(h, x), \ldots, \phi^N(h, x)).$$

Ddefinition 8 *(stationary Markov–Nash equilibrium). A N-tuple of functions $\phi^i : M \times X \mapsto \mathbb{R}^{m_i}, i = 1, \ldots, N$ is a stationary Markov-Nash equilibrium of the game $\Gamma(h_0, x_0)$ if an optimal control path $u^i_{(.)}$ exists for each player i. If $(\phi^1, \ldots, \phi^N)$ is a stationary Markov–Nash for all games $\Gamma(h, x)$, then it is sub-game perfect.*

Theorem 4 *(stationary Markov–Nash equilibrium). Let us consider a given N-tuple of functions $\phi^i : M \times X \mapsto \mathbb{R}^{m_i}, i = 1, \ldots, N$ and assume that the piecewise deterministic process defined by state motion*

$$\dot{x}_t = f(h_t, x_t, \phi^1(h_t, x_t), \ldots, \phi^N(h_t, x_t))$$

and the switching rates

$$q_{hk}(x_t, \phi^1(h_t, x_t), \ldots, \phi^N(h_t, x_t))$$

is well defined for all initial conditions $(h_0, x_0) = (h, x) \in M \times X$. Suppose the existence of N bounded functions $V^i : M \times X \mapsto \mathbb{R}, i = 1, \ldots, N$ such that $V^i(h, x)$ is continuously differentiable in x and such that the HJB equations

$$r_i V^i(h, x) = max \; \{F^i_{\phi^{-i}}(h, x, u^i) + V^i_x(h, x) f^i_{\phi^{-i}}(h, x, u^i) \\ + \sum_{k \neq h} q^i_{\phi^{-i},hk}(h, u^i)(S^i_{hk}(x) + V^i(k, x) - V^i(h, x)) | \; u^i \in U^i_{\phi^{-i}}(h, x)\}, \quad (8)$$

are satisfied for all $i = 1, N$ *and all* $(h, x) \in M \times X$. *Denote by* $\Phi^i(h, x)$ *the set of all* $u^i \in U^i_{\phi-i}(h, x)$ *which maximize the RHS of (8). If* $\phi^i(h, x) \in \Phi^i(h, x)$ *and all* $(h, x) \in M \times X$ *then* $(\phi^1, \ldots, \phi^N)$ *is a stationary Markov-Nash equilibrium. The equilibrium is sub-game perfect.*

Proof. See Dockner et al.[Dockner, 2000], p.212.

3. A stochastic game of R&D competition

The following game is initially due to Reinganum [Reinganum, 1982] and has been detailed by Dockner et al. [Dockner, 2000]. In this game N firms have competing R&D projects. The dynamic game supposes that: *(i)* no firm knows in advance the amount of R&D that must be invested, *(ii)* R&D activities are costly but contribute to higher accumulation of common know-how and then have positive externalities *(iii)* one successful innovation may be achieved by using different paths. Resources in R&D positively influence the probability of successful innovations. Once a firm has won the competition, it acquires a monopolistic position. The problem is illustrated in the figure 5.

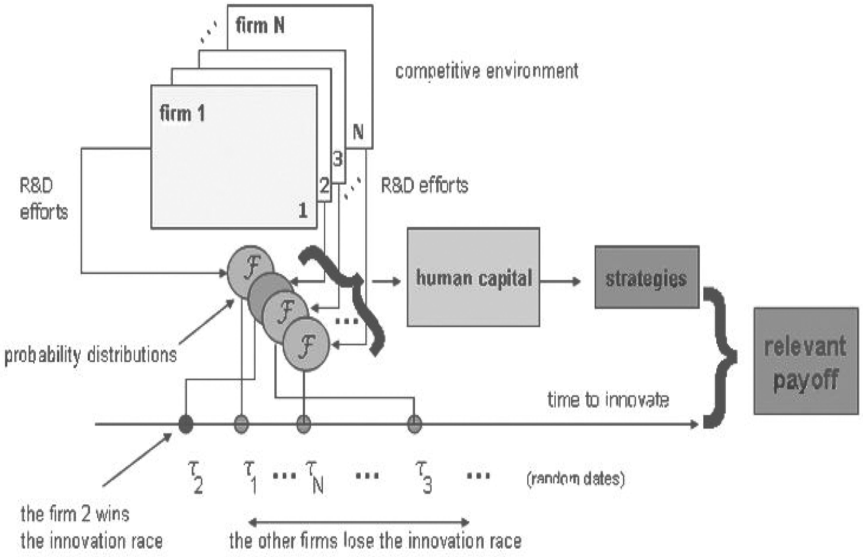


Fig. 5: A stochastic innovation game

Game presentation

The time τ_i to complete a project is a RV, whose probability distribution is $F_{it} = \mathbb{P}\{\tau_i \leq t\}$, where for convenience F_{it} is taken similar to $F_i(t)$. The RVs τ_i are stochastically independent, since knowledge is supposed to have no spillover between firms. Denote by $\tau = \min\{\tau_i, i = 1, 2, \ldots, N\}$ the date of an innovation. According to independency, we may read

$$\mathbb{P}\{\tau \leq t\} = 1 - \prod_{i=1}^{N}(1 - F_{it}).$$

Let $u_{it} \geq 0$ be the rate of R&D effort. The rate of the distribution F_i is assumed to be proportional to the R&D efforts

$$\dot{F}_{it} = \lambda u_{it}(1 - F_{it}),\ F_{i0} = 0, \lambda > 0, \tag{9}$$

where $1 - F_i$ is the survival probability and $\dot{F}_{it}(1 - F_{it})^{-1}$ is the hazard rate. Assume that the present value of the innovator's net benefits P_I is constant and greater than the ones of the other competitors P_F. The costs of R&D efforts are quadratic in the investment rate. The game is played over a finite horizon T. This stochastic game belongs to the class of piecewise deterministic games. The system is in mode 0 before the innovation is happens. Ones an innovation by firm i has occurred, the system switches from one mode to another mode $i, i = 1, 2, \ldots, N$ [Dockner, 2000].

Game analysis

The expected discounted profit to be maximized by the ith player is

$$\int_0^T \{P_I \dot{F}_{it} \prod_{j \neq i}(1 - F_{jt}) + P_F \sum_{j \neq i} \dot{F}_{jt} \prod_{k \neq j}(1 - F_{kt}) - \\ - \frac{\mathrm{e}^{-rt}}{2} u_{it}^2 \prod_{j=1}^{N}(1 - F_{jt})\} dt. \tag{10}$$

This expression consists of three terms, which weights are the probabilities: firstly the firm i's value of net payoff P_I if the firm becomes the first to innovate, secondly the firm i's value of payoff P_F if this firm looses the competition and, thirdly, the discounted value of the cost of R&D efforts. Substituting the RHS of (9) into (10) the payoff expression is simplified as follows

$$\int_0^T \{\lambda P_I u_{it} + \lambda P_F \sum_{j \neq i} u_{jt} - \frac{\mathrm{e}^{-rt}}{2} u_{it}^2\} \prod_{j=1}^{N}(1 - F_{jt}) dt. \tag{11}$$

Let introduce the state transformation

$$-\ln(1 - F_{it}) = \lambda z_{it} \Leftrightarrow 1 - F_{it} = \mathrm{e}^{-\lambda z_{it}}.$$

The state variable z_{it} denotes the firm i's accumulated know-how via the R&D efforts. A differentiation w.r.t. time yields $\dot{F}_i(1-F_i)^{-1} = \lambda\dot{z}_i$. Hence, we have $\dot{z}_i = u_{it},\ z_{i0} = 0$. The corresponding payoff is deduced from (11)

$$J^i = \int_0^T \{\lambda P_I u_{it} + \lambda P_F \sum_{j\neq i} u_{jt} - \frac{\mathrm{e}^{-rt}}{2} u_{it}^2\} \times \exp[-\lambda \sum_{j=1}^{N} z_{jt}]dt.$$

Let y_t be equal to $\exp[-\lambda \sum_{j=1}^{N} z_{jt}]$. The differentiation w.r.t. time yields

$$\dot{y}_t = -\lambda y_t \sum_{j=1}^{N} u_{jt},\ y_0 = 1. \tag{12}$$

The game is then transformed to the following exponential game

$$J^i = \int_0^T \{\lambda P_I u_{it} + \lambda P_F \sum_{j\neq i} u_{jt} - \frac{\mathrm{e}^{-rt}}{2} u_{it}^2\} y_t dt,$$

$$s.t.$$

$$\dot{y}_t = -\lambda y_t \sum_{i=1}^{N} u_{it},\ y_0 = 1.$$

Markov–Nash equilibrium

When omitting the time argument, the current-value Hamiltonians ($i = 1, 2, \ldots, N$) are

$$\mathcal{H}^i(t, y, u_i\mu_i) = y\{\lambda P_I u_i + \lambda P_F \sum_{j\neq i} u_j - \frac{\mathrm{e}^{-rt}}{2} u_i^2\} - \mu_i \lambda y(u_i + \sum_{j\neq i}^{N} u_j),$$

where $\mu_i, i = 1, 2, \ldots, N$ are the current costate variables. The first order conditions (FOCs) are

$$\begin{aligned} \frac{\partial \mathcal{H}(t, y, u_i, \mu_i)}{\partial u_i} &= 0 \Rightarrow u_i = \lambda \mathrm{e}^{rt}(P_I - \mu_{it}), \\ \dot{\mu}_i &= -\frac{\partial \mathcal{H}^i(t, y, u_i, \mu_i)}{\partial y},\ \mu_{iT} = 0; \end{aligned} \tag{13}$$

$$u_{it} = -b_t \lambda \mathrm{e}^{rt}. \tag{14}$$

We have $N+1$ boundary conditions[15] from the state equation (12) and FOCs (13). To solve the boundary value problem let us conjecture a solution.

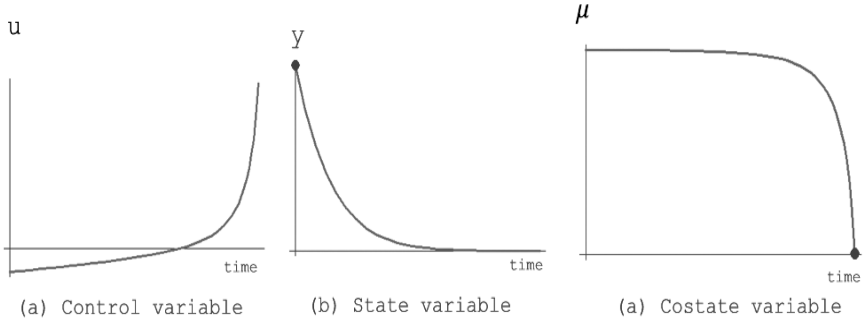
---

[15] There are $2N+1$ variables namely $y, \mu_i, u_i, i = 1, 2, \ldots, N$ and $2N+1$ equations.

Using the expression of u_i in (13) and substituting into (12) we have $\dot{y}_t = \lambda^2 N b_t y_t \mathrm{e}^{rt}$, $y_0 = 1$. To hold the conjectured solution, b_t must satisfy a Ricatti differential equation (RDE):

$$\dot{b}_t = -\frac{\lambda^2 \mathrm{e}^{rt}}{2}\{(2N-1)b_t^2 + 2b_t(1-N)(P_F - P_I)\}, \qquad b_T = -P_I. \tag{15}$$

The solution of the RDE (15) can be found by an CVF method letting $g(t) = -1/b_t$.



(a) Control variable (b) State variable (a) Costate variable

Fig. 6: Control, state, costate in an innovation game with two firms

Substituting the solution of b_t into (14) yields the Markovian identical strategies $u_t =$

$$= \frac{2\lambda P_I(P_I - P_F)(N-1)\mathrm{e}^{rt}}{(2N-1)P_I + \{P_I + 2(N-1)P_F\} \times \exp[\frac{1}{r}(P_I - P_F)(N-1)\lambda^2(\mathrm{e}^{rt} - \mathrm{e}^{rT})]}.$$

The Figure 6 shows the control variable, the state and the costate variables, when the game is reduced to two competitors ($N = 2$).

4. A stochastic game of patent race

In some market models N incumbents and one entrant aim to invent a new process or a new product, and patent their innovations. In the model of [Dasguspta, 1980], [Reinganum, 1982], [Tirole, 1990]the patent race is to develop a cost reducing production process for an existing product, in the model of Gayle [Gayle, 2001] the firm are patenting a new product. This model is in line with the one of Loury [Loury, 1979] and its developments by Lee and Wilde [Lee, 1980] where a stochastic relationship is assumed between R&D investments and the time at which an innovation will occur.

Description of the game

An output market is composed of $N+1$ firms, which are attempting to produce a new product and taking simultaneously a patent[16]. N of these firms are incumbents and one firm is a potential entrant. The rate of investment in R&D is $x_i, i = 1, \ldots, N$ for an incumbent, and z for the entrant. The date of success for an innovation is supposed to depend only on the R&D investment rate[17] such as $\tau_i(x_i)$. The probability that the firm i succeeds at or before the date t is

$$\mathbb{P}\{\tau_i(x_i) \leq t\} = 1 - \exp[-h(x_i)t],\ t \in [0, \infty),$$

where $h(.)$ denotes the hazard function. Let this function be twice differentiable, strictly increasing, concave and satisfying the following conditions: $h'(.) > 0, h''(.) < 0$ for all $x_i, z \in [0, \infty),$ and $\lim_{x_i, z \to \infty} h'(.) = 0$. The conditional probability that the firm i will succeed in the instant, given that it has not already succeeded is

$$\mathbb{P}\{\tau_i(x_i) \in (t, t+dt] | \tau_i > t\} = h(x_i)dt,\ t \in [0, \infty).$$

This result is due to the memoryless property of the exponential distribution[18]. Let $\bar{\tau}_i$ represents the date at which the first firm introduces an innovation. Then $\bar{\tau}_i = \min_{j \neq i}\{\bar{\tau}(x_j)\}$, and

$$\begin{aligned}\mathbb{P}\{\bar{\tau}_i \leq t\} &= 1 - \mathbb{P}\{\tau_j > t\},\ \text{for all } j \neq i\\ &= 1 - \exp[\sum_{j \neq i} h(x_j)].\end{aligned}$$

Finally, let us suppose the following constant non-discounted profits: P_i^A the incumbent i's profit before any innovation occurs, P_i^W the incumbent i's profit if he wins the patent race, P_i^{LE} the incumbent i's profit if he loses the patent race to the entrant, P_i^{LI} the incumbent i's profit if he loses the patent race to another incumbent, and P_E the profit of the entrant when he wins the patent race.

The expected profits of incumbents i, $i = 1, 2, \ldots, N$ are given by:

$$\begin{aligned}V^i = \int_0^\infty &\exp[-\{h(x_i) + \sum_{k \neq i}^N h(x_k) + h(z)\}t] \times\\ &\times \{P_i^A - x_i + h(x_i)\frac{P_i^W}{r} + h(z)\frac{P_i^{LE}}{r} + \sum_{k \neq i}^N h(x_k)\frac{P_i^{LI}}{r}\} \times \mathrm{e}^{-rt}dt. \quad (16)\end{aligned}$$

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[16] The following game of patent race is inspired from the presentation of Gayle [Gayle, 2001].

[17] Reinganum [Reinganum, 1982] [**?**] assumes that the probability of introducing an innovation is a function of both the current of investment in R&D and the accumulated stock of technology.

[18] Let the PDF of the R.V. T be $f(t)dt = \lambda \mathrm{e}^{-\lambda t}dt$. We have the moment $\mathbf{E}[T^k] = \int_0^\infty t^k f(t)dt$. Then $\mathbf{E}[T^k] = \Gamma(k+1)\lambda^{-k} = \lambda^{-k}k!$, since $\Gamma(k) = \int_0^\infty \mathrm{e}^{-u}u^{k-1}du = (k-1)!$. Hence, $\mathbf{E}[\tau] = \lambda^{-1}$.

The expected profit of the entrant is

$$V^E = \int_0^\infty \exp[-\{h(x_i) + \sum_{k\neq i}^{N} h(x_k) + h(z)\}t] \times \{h(z)\frac{P_E}{r} - z\} \times \mathrm{e}^{-rt} dt. \quad (17)$$

Integrating (16) and (17), we have the system for the R&D subgame

$$V^i = \frac{P_i^A - x_i + h(x_i)\frac{P_i^W}{r} + h(z)\frac{P_i^{LE}}{r} + \sum_{k\neq i}^{N} h(x_k)\frac{P_i^{LI}}{r}}{r + h(x_i) + \sum_{k\neq i}^{N} h(x_k) + h(z)},$$

$$V^E = \frac{h(z)\frac{P_E}{r} - z}{r + h(x_i) + \sum_{k\neq i}^{N} h(x_k) + h(z)}.$$

Best response functions

The reaction functions (BRFs) are then deduced from the following $N+1$ FOCs

$$\frac{\partial V^i}{\partial x_i} = 0,\ i = 1, \dots, N \text{ and } \frac{\partial V^E}{\partial z} = 0.$$

A Nash equilibrium of R&D spending must then satisfy the following conditions (omitting the nonzero denominator of the LHS):

$$\{r + h(x_i) + \sum_{k\neq i}^{N} h(x_k) + h(z)\} \times (h'(x_i)\frac{P_i^W}{r} - 1) -$$

$$- \{P_i^A - x_i + h(x_i)\frac{P_i^W}{r} + h(z)\frac{P_i^{LE}}{r} + \sum_{k\neq i}^{N} h(x_k)\frac{P_i^{LI}}{r}\}h'(x_i) = 0, \quad (18)$$

$$\{r + h(x_i) + \sum_{k\neq i}^{N} h(x_k) + h(z)\}(h'(z)\frac{P_E}{r} - 1) - (h(z)\frac{P_E}{r} - z)h'(z) = 0. \quad (19)$$

The equation (18) is the incumbent i's BRF with upward sloping[19], given the R&D investments of his opponents. Similarly, (19) is the entrant's BRF with upward sloping, given the R&D investments of his opponents.

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[19] In the case of symmetry between the BRFs of the incumbent and entrant, when $N = 1, PLE = PA = 0$, we have the incumbent's BRF

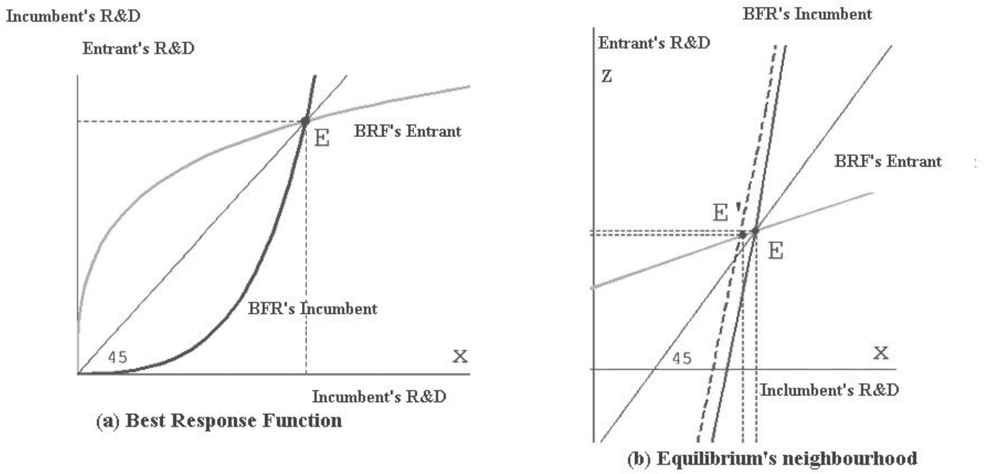
$$F(x, z) = \{h'(x) + \frac{1}{r}h'(x)h(z)\}P^W - h'(x)P^A - r - \{h(x) + h(z)\} + xh'(x).$$

By the implicit function theorem, we deduce

$$\frac{dx}{dz} = -\frac{F_z}{F_x} = -\frac{h'(z)\{h'(x)\frac{P^W}{r} - 1\}}{h''(x)\{1 + \frac{h(z)}{r}\}P^W + xh''(x)}.$$

The derivative dx/dz is positive because of a non-negative numerator with non-negative expected profits and due to a negative denominator with the concavity of h(.).

Symmetric Nash equilibrium



(a) Best Response Function

(b) Equilibrium's neighbourhood

Fig. 7: Symmetric Nash equilibria

A symmetry of the BRFs is achieved when $N = 1, PLE = PA = 0$. The BRFs are then defined by

$$\{h'(x) + \frac{1}{r}h'(x)h(z)\}P_i^W - h'(x)P_i^N - r - \{h(x) + h(z)\} + xh'(x) = 0,$$

$$\{h'(z) + \frac{1}{r}h'(z)h(x)\}P_E - r - \{h(x) + h(z)\} + zh'(z) = 0.$$

The Figure 7 (a) shows the reaction functions and the Nash equilibrium. Due to the stability condition, the incumbent's reaction function is steeper than the entrant's one[20]Both reactions functions intersect on the 45° line. The expected pre-innovation profit P^A only occurs in the incumbent's BRF. A positive P^A will then shift the incumbent's BRF to the left, as it is shown in figure Fig.(b). As a consequence to the shift, we observe with [Gayle, 2001] that the R&D spending equilibrium of the incumbent will be less than that the one of the entrant.

5. Concluding remarks

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[20] The stability condition is expressed by

$$|V^i_{xx}| > |V^i_{xz}| \text{ and } |V^E_{zz}| > |V^E_{zx}|$$

in the neighborhood of the equilibrium (see Figure 7 (b)).

The differential games in R&D economics give some indications and results about debates and questions, like the relation between the concentration of an industry and the intensity of R&D investments. The consequences of the market structures (socially managed market, pure monopolist with barriers to entry, competitive economy) on R&D have been studied notably by Dasguspta and Stiglitz [Dasguspta, 1980]. For example, the correlation between concentration and R&D efforts depends upon the existing degree of concentration and upon free entrance. For Loury [Loury, 1979], given a market structure, firms will rather more invest in R&D than is socially optimal. Reinganum [Reinganum, 1982] considers the optimal resource allocation in R&D, the social optimality of the game and implications of innovation policies (taxes and subsidies, patents). Noncooperation or cooperation aspects also play a great role. If costs are not too high, a firm may choose to cooperate in R&D investments with one or some other firms. Dasguspta and Stiglitz [Dasguspta, 1980], Reinganum [Reinganum, 1982], Amir, Evstigneev, Wooders [Amir, 2003] have studied such cooperations. Yeung and Petrosyan [Yeung, 2006] consider the theory of the cooperative stochastic differential games and analyze a cooperation R&D game under uncertainty over a finite planning period. This paper has introduced to the approach of stochastic games in theory and application. Two simple examples of noncooperative games in R&D have been developed. The first example of stochastic game belongs to the class of piecewise deterministic games. Then, when an innovation occurs, the system switches from one mode to another. Indeed, the winning firm is supposed to acquire immediately a monopolistic position. The second example considers a stochastic game between several incumbents of an industry and a potential entrant. In this model, the R&D spending equilibrium of the representative incumbent (the game is symmetric) is rather less than the one of the entrant. Recent papers relax or improve some strong assumptions. Sennewald [Sennewald, 2007] relaxes the strong assumption of boundedness conditions when applying the Bellman equation (bounded utility function, bounded coefficients in the differential equation). The HJB can then still be used with linearly boundedness. Let us also indicate the less recent model of endogenous growth by Aghion and Howitt [Aghion, 1992]. In this model, according to a forward-looking difference equation, the research expenses in any period depend upon the expected research investment next period.

Appendix

Ito's lemma for Poisson Processes

Ito's lemma for Poisson processes is not frequently presented in textbooks. This appendix shows the corresponding rule, its proof and one application to TFP.

Lemma 3 *(Ito's lemma for a Poisson process). Given a Poisson SDE*

$$dx_t = adt + bdq_t, \text{ where a and b are constant.} \tag{20}$$

Let $F(t, x_t)$ be a continuously differentiable equation of t and x. Then, we have

$$dF(t, x_t) = (F_t + aF_x)dt + \{F(t, x_t + b) - F(t, x_t)\}dq_t. \tag{21}$$

Proof. By differentiating the function $F(t, x_t)$ and using equation (20), we have

$$\begin{aligned} dF(t, x_t) &= F(t + dt, x_{t+dt}) - F(t, x_t) = \\ &= F(t + dt, x_t + adt + bdq_t) - F(t, x_t). \end{aligned} \tag{22}$$

Adding and subtracting $F(t + dt, x_t + adt)$ in equation (22) we have

$$\begin{aligned} dF(t, x_t) = F(t + dt, x_t + adt + bdq_t) - F(t + dt, x_t + adt) - \\ -F(t, x_t) + F(t + dt, x_t + adt). \end{aligned} \tag{23}$$

The last two terms of equation (23) correspond to a situation where we have no jump. Hence, with $x_t = at$ and $dx_t = adt$, we have

$$\begin{aligned} F(t + dt, x_t + dx_t) - F(t, x_t) &= F_t dt + F_x dx_t = \\ &= (F_t + aF_x)dt. \end{aligned} \tag{24}$$

According to equations (23) and (24), we deduce

$$\begin{aligned} dF(t, x_t) = F(t + dt, x_t + adt + bdq_t) - F(t + dt, x_t + adt) + \\ +(F_t + aF_x)dt. \end{aligned} \tag{25}$$

The first two terms of equation (25) have a different expression according that a jump may or not occur. We have

$$F(t + dt, x_t + adt + bdq_t) - F(t + dt, x_t + adt) =$$

$$\begin{aligned} &= \begin{cases} F(t, x_t + b) - F(t, x_t) & \text{w.p. } \lambda dt \text{, with jump} \\ 0 & \text{w.p. } 1 - \lambda dt \text{, without jump} \end{cases} \\ &= \{F(t, x_t + b) - F(t, x_t)\}dq_t. \end{aligned}$$

Then, we find the Ito's formula for a Poisson stochastic differential equation

$$dF(t, x_t) = (F_t + aF_x)dt + \{F(t, x_t + b) - F(t, x_t)\}dq_t.$$

Example (Total factor productivity). The stochastic differential equation is given by [Wälde, 2006])

$$\frac{dA_t}{A_t} = gdt + \sigma dq_t, \tag{26}$$

where g and σ are constants. The equation (26) is equivalent to

$$dA_t = gA_t dt + \sigma A_t dq_t.$$

In this example, the parameters of the stochastic differential equation (20) are $a \equiv gA_t = a_t$ and $b \equiv \sigma A_t = b_t$. Let us apply the Ito's formula (21). Taking $F(t, A_t) = \ln A_t$, we have $F_t = 0$ and $F_A = 1/A_t$. We also determine

$$\ln(A_t + \sigma A_t) - \ln A_t = \ln(1 + \sigma).$$

Then according to Ito's formula, we have

$$d \ln A_t = g + \ln(1 + \sigma) dq_t. \tag{27}$$

By integrating both sides of equation (27), we obtain

$$\int_0^t \ln A_s = gt + \ln(1 + \sigma) \int_0^t q_s.$$

Hence,

$$[\ln A_s]_0^t = gt + [q_s]_0^t,$$

and

$$\ln A_t = \ln A_0 + gt + \ln(1 + \sigma)(q_t - q_0).$$

The path of the total factor productivity is then given by

$$A_t = A_0 \exp[gt + \ln(1 + \sigma)(q_t - q_0)].$$

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Subgame Perfect Nash Equilibrium in a Quality–Price Competition Model with Vertical and Horizontal Differentiation

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Abstract. The paper describes how to construct the subgame perfect equilibrium by using a backwards induction procedure in a model of product differentiation that takes into account both vertical and horizontal differentiation features.

Keywords: Industrial organization, product differentiation, subgame perfect Nash equilibrium.

1. Presuppositions and Formalization of the Game-Theoretical Model with Vertical and Horizontal Differentiation Features

We propose a game-theoretical duopoly model of product differentiation that takes into account both vertical and horizontal differentiation features. The research is based on the following assumptions:

- Each consumer estimates the quality of the product and this estimation is important for determining the upper level of the price acceptable for the regarded consumer.
- Each consumer is characterized by a parameter $t : t \in [\underline{t}, \overline{t}] \subset [0, +\infty)$ which shows his willingness to pay for quality increasing, and all consumers range the substitute goods in the same way when their prices are equal.

- If a firm wants its product to be higher estimated by consumers it has to invest in R&D and generally it leads to the growth of both fixed and variable costs.
- Two competitive firms are supposed to be located on the different sides of a "linear city" ([Hotelling, 1929]; [Tirole, 1997]). All consumers are uniformly distributed on the segment between these points.
- Travel costs of the consumer are in direct proportion to the distance between the consumer and the firm.
- Firms play a two-staged game. In the first stage they choose simultaneously their respective quality level. In the second stage they choose simultaneously their price. Quality decisions are observable by both rivals before price decisions are made.

Now, let's regard the game more properly.

In the 1^{st} step (at the stage of their quality decisions) both firms choose their qualities simultaneously: $q_i \in [\underline{q}, \overline{q}]$ and, thus, they face some quality development costs $FC(q_i)$. For each one of the consumers (we suppose S consumers in all) the products of the firms are substitute goods. Each consumer has to choose between purchasing one unit of product from one of the firms or making no purchase.

The 1^{st} stage decisions $(q_1, q_2), q_1 \leq q_2$ become observable before the second stage starts. At the second stage (the stage of price-competition) the rivals choose their prices p_1 and p_2 respectively. Here and further the subscript 2 denotes the firm that has entered the market with a quality that exceeds the quality of the other firm. This firm is labeled "the high-quality seller". The other firm is labeled "the low-quality seller" and is denoted by the subscript 1.

Each consumer's strategy is maximizing his utility function of the following form:

$$U_t = max\{tq_2 - p_2 - k(\overline{\rho} - s), tq_1 - p_1 - k(s - \underline{\rho}), 0\}, \tag{1}$$

where $t : t \in [\underline{t}, \overline{t}] \subset [0, +\infty)$ – is a scalar parameter which shows consumer's willingness to pay for quality increasing; $s : s \in [\underline{\rho}, \overline{\rho}] \subset [0, +\infty)$ – is a scalar parameter that characterizes consumer's spatial position on the segment $[\underline{\rho}, \overline{\rho}]$; k – is a scalar parameter that can be interpreted as transport costs. Parameters t and s together can be regarded as a 2-dimensional variate uniformly distributed on a rectangle $[\underline{t}, \overline{t}] \times [\underline{\rho}, \overline{\rho}]$.

Consumer's reaction described above leads to self-segmentation of the consumers and unambiguously determines market shares D_1 and D_2. The consumer t will buy the hight quality product if and only if his surplus from buying the high-quality product will be higher than from buying the low-quality product or

$$tq_2 - p_2 - k(\overline{\rho} - s) > tq_1 - p_1 - k(s - \underline{\rho}).$$

And the position of marginal consumer $t_2(s)$ who is indifferent between purchasing low quality or high quality product can be defined by solving:

$$tq_2 - p_2 - k(\overline{\rho} - s) = tq_1 - p_1 - k(s - \underline{\rho}) \;\Rightarrow\; t_2(s) = \frac{k(\overline{\rho} + \underline{\rho} - 2s) - p_1 + p_2}{q_2 - q_1}. \tag{2}$$

The area above the line $t_2(s)$ is the high-quality seller's market share: D_2. Similarly

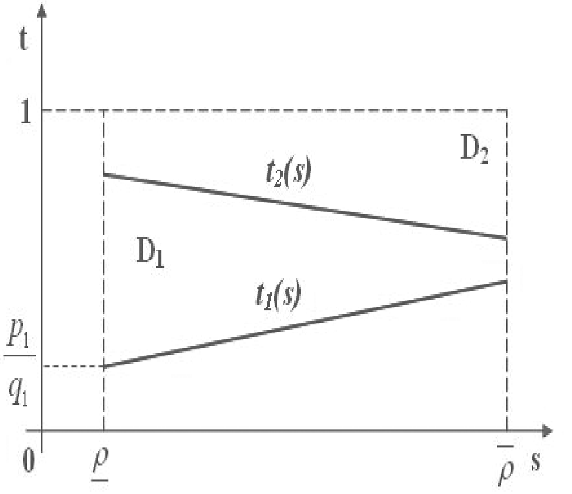

Fig.1: Self-segmentation of the consumers

we determine the low-quality seller's market share: D_1. In this case the position of marginal consumer who is indifferent between buying the low-quality product or making no purchase is a line $t_1(s)$. Its equation can be analytically found by solving:

$$tq_1 - p_1 - k(s - \underline{\rho}) = 0 \;\Rightarrow\; t_1(s) = \frac{p_1 + k(s - \underline{\rho})}{q_1}. \tag{3}$$

The consumers under the line $t_1(s)$ make no purchase. The market shares D_1 and D_2 are proportional to the areas of the corresponding figures (see Figure 1). The purpose of each firm is maximizing its profit. The solution concept traditionally used in similar models [Gabszewics, 1979], [Sutton, 1982], [Ronnen, 1991], [Motta, 1993], [Petrosjan, 1983], [Zenkevich, 2006] is a subgame perfect equilibrium SPE [Selten, 1975]. The equilibrium is solved by the backwards induction method. Here and further we suppose nontrivial situation when $q_2 > q_1$.

2. The Construction of Second-stage (Price) Equilibrium

Now let us consider some private case of the model with elements of vertical and horizontal differentiation described above:

$$\circ\; q \in (0, +\infty); [\underline{t}, \overline{t}] = [0, 1]; S = 1.$$

$$\circ\ FC(q) = \frac{1}{2}q^2.$$

$$\circ\ t_2(\underline{\rho}) \leq 1; t_2(\overline{\rho}) \geq t_1(\overline{\rho}). \tag{4}$$

Denote $d \doteq \overline{\rho} - \underline{\rho}$. The last conditions in the list restricts the research by the case when the lines $t_1(s)$ and $t_2(s)$ do not intersect within the area regarded and $t_2(\underline{\rho}) \leq 1$ (like in Figure 1).

According to the backwards induction procedure we begin from the second (and the last) stage of the game – the stage of price competition.

Let q_1 and q_2 be the first stage firm's quality decisions.

Theorem 1. *In the game-theoretical model of duopoly with vertical and horizontal differentiation features there exists second-stage price equilibrium:*

$$\begin{cases} p_1^*(q_1, q_2) = \frac{(q_2-q_1)(q_1-kd)}{4q_2-q_1}, \\ \\ p_2^*(q_1, q_2) = \frac{(q_2-q_1)(4q_2-kd)}{4(4q_2-q_1)}. \end{cases} \tag{5}$$

Proof.

In order to find the second-stage price equilibrium we construct the reaction functions of the firms.

Firstly, we construct the reaction function of the high-quality seller. Notice that a consumer (t, s) will buy a product of this firm if the point (t, s) is located above the curve $t_2(s)$(see Figure 1). Thus, we can conclude that the high quality seller's market share is

$$D_2 = \frac{d(q_2 - q_1 - p_2 + p_1)}{q_2 - q_1} \tag{6}$$

And the high-quality seller's profit will be:

$$\Pi_2(q_1, q_2, p_1, p_2) = p_2 D_2 - \frac{q_2^2}{2} = p_2 \frac{d(q_2 - q_1 - p_2 + p_1)}{q_2 - q_1} - \frac{q_2^2}{2}. \tag{7}$$

Thus, the reaction function of the high quality seller will look like

$$p_2(p_1) = \frac{q_2 - q_1 + p_1}{2}, \tag{8}$$

where p_1 meets the following conditions (according to the conditions (4)):

$$p_1 < \frac{q_2 - q_1 - 2kdr}{2r - 1}, \tag{9}$$

$$p_1 > 2kd - q_2 + q_1. \tag{10}$$

The low-quality seller's market share is

$$D_1 = \frac{2d(p_2 q_1 - p_1 q_2) - kd^2(q_2 - q_1)}{2q_1(q_2 - q_1)}. \tag{11}$$

Its profit will be

$$\Pi_1(q_1,q_2,p_1,p_2) = p_1 D_1 - \frac{q_1^2}{2} = p_1 \frac{2d(p_2 q_1 - p_1 q_2) - kd^2(q_2 - q_1)}{2q_1(q_2 - q_1)} - \frac{q_1^2}{2}. \qquad (12)$$

And the reaction function of the low-quality seller will be

$$p_2(p_1) = \frac{2q_1 p_2 - kd(q_2 - q_1)}{4q_2}, \qquad (13)$$

where p_2 meets the following conditions (according to the conditions (4)):

$$p_2 > \frac{krd(q_2 - q_1 + 1)}{2(2q_2 - q_1 r)}, \qquad (14)$$

$$p_2 < \frac{4q_2(q_2 - q_1 - kd) - kd(q_2 - q_1)}{2q_1 - 1}. \qquad (15)$$

The two reaction curves intersect uniquely and give the vector of equilibrium prices (5).

3. The Equilibrium at the Stage of Quality Competition

Let us note that when prices p_1^* and p_2^* are selected the profits of the rivals are equal to

$$\Pi_1(q_1,q_2) = \frac{q_2 d(q_2 - q_1)(q_1 - kd)^2}{q_1(4q_2 - q_1)^2} - \frac{1}{2}q_1^2; \qquad (16)$$

$$\Pi_2(q_1,q_2) = \frac{d(q_2 - q_1)(4q_2 - kd)^2}{4(4q_2 - q_1)^2} - \frac{1}{2}q_2^2. \qquad (17)$$

In accordance to the subgame perfect equilibrium concept now we regard the first stage of the game – the stage of quality competition. If the vector of selected qualities (q_1^*, q_2^*) is the equilibrium, the following conditions should be held:

$$\begin{cases} \frac{\partial \Pi_1}{\partial q_1}(q_1^*, q_2^*) = 0, \\ \frac{\partial \Pi_2}{\partial q_2}(q_1^*, q_2^*) = 0. \end{cases} \qquad (18)$$

The system (18) have been numerically solved for the private case:
$\overline{\rho} = 1, \underline{\rho} = 0, k = 0,01$.
The following decisions have been found:

A. $q_{1A}^* \approx 0,0063, q_{2A}^* \approx 0,0041;$

B. $q_{1B}^* \approx 0,0009, q_{2B}^* \approx 0,0025;$

C. $q_{1C}^* \approx 0,0111, q_{2C}^* \approx 0,2502;$

D. $q^*_{1D} \approx 0,0485, q^*_{2D} \approx 0,2528$.

We had supposed $q_1 < q_2$, thus, the decision A mismatches. The decisions B and C do not match either because $\Pi_1(q^*_{1C}, q^*_{2C}) < 0$ and $\Pi_2(q^*_{1B}, q^*_{2B}) < 0$.

Now let's check whether the vector (q^*_{1D}, q^*_{2D}) meeting the conditions (18) is a Nash equilibrium.

Let the firm 1 deviates from q^*_{1D} and chooses some $q_1 < q^*_{2D}$. In this case the profit of the low-quality firm will be:

$$\Pi_1(q_1, q^*_{2D}) = \frac{0,2528(0,2528 - q_1)(q_1 - 0,01)^2}{q_1(1,0112 - q_1)^2} - \frac{q_1^2}{2}. \tag{19}$$

Investigating the derivative $\frac{\partial \Pi_1}{\partial q_1}(q_1, q^*_{2D})$, we conclude that the low-quality seller's profit is a decreasing function in the segment (-0,009; 0,0115) and (0,048;1,7501). Considering conditions: $0,01 \leq q_1 \leq q^*_{2D}$, and

$$\Pi_1(0,01, q^*_{2D}) = -\frac{0,01^2}{2} < 0, \tag{20}$$

we receive

$$\arg \max_{0,01 \leq q_1 \leq q^*_{2D}} \Pi_1(q_1, q^*_{2D}) = q^*_{1D} \approx 0,0485. \tag{21}$$

We have, therefore, proved that the low-quality seller (firm 1) has no incentive to deviate from the strategy q^*_{1D}.

In a similar manner it is possible to prove that firm 2 won't deviate from its strategy q^*_{2D} on the segment $(q^*_{1D}, +\infty)$ either.

Besides, it is also necessary to check that the 1^{st} firm will not deviate from q^*_{1D} on the segment $(q^*_{2D}, +\infty)$, i.e. that the 1^{st} firm has no incentive to "jump over" the quality q^*_{2D} and to become the high-quality seller.

Let the 1^{st} firm choose some quality $q_1 > q^*_{2D}$. In this case it's profit will be

$$\Pi_1(q_1, q^*_{2D}) = \frac{(q_1 - 0,2528)(4q_1 - 0,01)^2}{4(4q_1 - 0,2528)^2} - \frac{q_1^2}{2}. \tag{22}$$

Investigating the derivative $\frac{\partial \Pi_1}{\partial q_1}(q_1, q^*_{2D})$ we conclude that the 1^{st} firm's profit is a decreasing function in the segment $(0,3337; +\infty)$. Considering $q_1 > q^*_{2D}$ we receive:

$$\arg \max_{q^*_{2D} \leq q_1 \leq +\infty} \Pi_1(q_1, q^*_{2D}) \approx 0,3337. \tag{23}$$

The value of profit $\Pi_1(0,3337, q^*_{2D}) = -0,0253 < 0$, i.e. it isn't advantageous for the 1^{st} firm to deviate from the value q^*_{1D} in the segment $(q^*_{2D}, +\infty)$.
In a similar manner it is possible to prove that firm 2 won't "jump over" the quality q^*_{1D} and become the low-quality seller. Therefore, the pair of the rival's strategies

$(q^*_{1D}, p^*_1(q^*_{1D}, q^*_{2D}))$ and $(q^*_{2D}, p^*_2(q^*_{1D}, q^*_{2D}))$ make a subgame perfect equilibrium in the game. In this equilibrium qualities, prices and profits are equal accordingly

$$\begin{aligned} & q^*_{1D} \approx 0,0485, q^*_{2D} \approx 0,2528; \\ & p^*_1 \approx 0,0082,\ p^*_2 \approx 0,1062; \\ & \Pi^*_1 \approx 0,0005,\ \Pi^*_2 \approx 0,0233. \end{aligned} \tag{24}$$

Thus, we managed to construct the subgame perfect equilibrium in a proposed 2-space (vertical and horizontal) product differentiation model using the backwards induction procedure.

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Three-Sided Matchings and Separable Preferences

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Abstract. In this paper we provide sufficient conditions for the existence stable matchings for three-sided systems.

Introduction

The two-sided matching model of Gale and Shapley [Shapley, 1962] can be interpreted as one where a non-empty finite set of firms need to employ a non-empty finite set of workers. Further, each firm can employ at most one worker and each worker can be employed by at most one firm. Each worker has preferences over the set of firms, and each firm has preferences over the set of workers. An assignment of workers to firms is said to be stable if there does not exist a firm and a worker who prefer each other to the ones they are associated with in the assignment. Gale and Shapley [Shapley, 1962] proved that every two-sided matching problem admits at least one stable matching.

In this paper we extend the above model by including a non-empty finite set of techniques. A technique can be likened to a machine that is owned by a technologist who is neither a firm nor a worker, and which the firm and worker together use for production. Further each technologist owns exactly one technique. Each firm has preferences over the set of ordered pairs of workers and techniques, each worker has preferences over the set of ordered pairs of firms and techniques, and each technologist has preferences over ordered pairs of firms and workers. Such models (see [Alkan, 1988]) are called three-sided systems. A matching in a three-sided system consists of disjoint triplets, each triplet comprising a firm, a worker and a technologist. A stable matching for a three-sided system is a matching which does not admit a triplet whose members are better off together than at their current designations. Alkan [Alkan, 1988] provided an example of a three-sided system that does not admit a stable matching. Danilov [Danilov, 2003] established the existence of a stable matching for lexicographic three-sided systems.

The preference of a firm is separable if its preference over workers is independent of the technique and its preference over techniques is independent of the worker. The preference of a worker is separable if its preference over firms is independent of the technique, and its preference over techniques is independent of the firm. A three-sided system is said to be separable if preferences of all firms and workers are separable. Through, out the paper, we assume that the preferences of the workers are separable between firms and techniques. A special case of such preferences is lexicographic preferences, with firms enjoying priority over techniques. If, in addition, the preferences of the firms are lexicographic, with workers enjoying priority over techniques, then the system is called lexicographic. Lexicographic systems are clearly separable.

In this paper we show that if a three-sided system is lexicographic for workers and satisfies a property called *Technical Specialization* then there exists a stable matching. Technical Specialization says: given two distinct firm–worker pairs, the technique that is best for the firm in one pair is different from the technique that is best for the firm in the other one. Note that the discrimination property is strictly stronger than the weak discrimination property that we discussed earlier. We also provide an example of a three-sided system with preferences of workers being both lexicographic as well as separable, that does not admit a stable matching. In this example the preferences of the firms are neither lexicographic nor separable.

Neither technical specialization nor the proof of theorem that establishes the existence of a stable matching when technical specialization is satisfied by a three-sided system, takes cognizance of the preferences of the technologists. In a way, the stable matching that is obtained, may have resulted by "coercing" the technologists. While this may make the technical specialization an unpalatable assumption, it is worth remembering, that a stable matching for a three-sided system, does not require that every side of the system play an active role in determining its viability. Alternatively one may assume that the three-sided system is strongly separable, i.e. lexicographic for workers and separable for firms. In such a scenario we need to assume that the preferences of firms and technologists over workers are in "agreement" (i.e. given a firm, a technologists ranks the workers in the same way that the firm does) to show that a three-sided system admits a stable matching. Agreement over workers in a strongly separable environment implies some kind of a hierarchy where the worker cares only about the firm and forms the bottom layer, whereas the technologist's preferences over the workers "echoes" the preferences of the firm it is engaged with.

Following the tradition of Gale and Shapley, we model our analysis in terms of a firm employing at most one worker. By present day reckoning, a firm employing at most one worker, is usually a small road-side shop, rather than an industrial unit. Hence, it might appear as if our analysis has little if no relevance to the more common real world situations. However, it may well be a reasonable starting point for the cooperative theory of multi-sided systems. [Roth and Sotomayor, 1988] contain an elaborate discussion of matching models, where firms may employ more than one worker. It turns out in their analysis, that the cooperative theory for such firms is almost identical to the cooperative theory arising out of [Gale and Shapley, 1962] framework. This occurs, since each firm can be replicated as often as the number of workers it can employ, with each replica having the same preferences over workers as the original firm. Further, the preferences of the workers between replicas of two different firms should be exactly the same as the preferences between the originals. On the other hand, the non-cooperative theory where each firm employs more than one worker is considerably different from the non-cooperative theory where firms may employ at most one. It is noteworthy that the cooperative theory for many-to-many two-sided matching models does not permit the same replication argument. This has been shown in [Lahiri, 2006].

The analysis reported in this paper, attempts at extending results pertaining to the existence of stable matchings in a labor market, by introducing technology as an essential determinant of the results that we obtain. Since our paper, is concerned with the cooperative theory of three-sided systems, the model that we use of a firm employing at most one worker, continues to provide valuable insights concerning the existence of stable matchings in labor markets.

1. The model

Let W be a non-empty finite set denoting the set of workers, F a non-empty finite set denoting the set of firms and T a non-empty finite set denoting the set of techniques. We assume for the sake of simplicity that the $|T|$ cardinality of T is equal to the number of firms ($|F|$) which in turn is equal to the number of workers ($|W|$).

Each $w \in W$ has preference over $F \times T$ defined by a linear order (i.e. anti-symmetric, reflexive, complete and transitive binary relation) $\geq_w$ which asymmetric part is denoted $>_w$. Each $f \in F$ has preference over $W \times T$ defined by a linear order $\geq_f$ whose asymmetric part is denoted $>_f$. Each $t \in T$ has preference over $F \times W$ defined by a linear order $\geq_t$ whose asymmetric part is denoted $>_t$. A three-sided system is given by the array

$$[\{\geq_f\colon f \in F\},\ \{\geq_w\colon w \in W\},\ \{\geq_t\colon t \in T\}].$$

A job-matching is a one-to-one function m from F to W. A technique matching is a one-to-one function n from F to T.

Since F, T and W all have the same cardinality, every job-matching and every technique matching is of necessity a bijection.

A pair (m, n) where m is a job-matching and n is a technique matching is called a matching (for the three-sided system).

A matching (m, n) is said to be stable if there does not exist $f \in F$, $w \in W$ and $t \in T$ such that: $(w, t) >_f (m(f), n(f))$,

$$(f, t) >_w (m^{-1}(w), n(m^{-1}(w))),$$

and

$$(f, w) >_t (m(n^{-1}(t), n^{-1}(t))).$$

A three-sided system is said to be separable for workers if for all $w \in W$ there exist linear orders P_w on F and Q_w on T such that for all (f, t)

$$(f', t') \in F \times T : (f, t) \geq_w (f', t)$$

if and only if fP_wf' and $(f, t) \geq_w (f, t')$ if and only if tQ_wt'.

A three-sided system is said to be separable for firms if for all $f \in F$ there exist linear orders P_f on W and Q_f on T such that for all (w, t)

$$(w', t') \in W \times T : (w, t) \geq_f (w', t)$$

if and only if wP_fw' and $(w, t) \geq_f (w, t')$ if and only if tQ_ft'.

A three-sided system is said to be separable if it is separable for both firms and workers.

A separable three-sided system is said to be lexicographic for workers if for all $w \in W$ there exists linear orders P_w on F and Q_w on T such that:

(a) for all $f, f' \in F$ with $f \neq f'$ and $t, t' \in T$: fP_wf' implies $(f, t) >_w (f', t')$;

(b) for all $f \in F$ and $t, t' \in T$ with $t \neq t'$: tQ_wt' implies $(f, t) >_w (f, t')$.

A three-sided system is said to be lexicographic for firms if for all $f \in F$ there exists linear orders P_f on W and Q_f on T such that:

(a) for all $w, w' \in W$ with $w \neq w'$ and $t, t' \in T$: wP_fw' implies $(w, t) >_f (w', t')$;

(b) for all $w > F$ and $t, t' \in T$ with $t \neq t'$: tQ_ft' implies $(w, t) >_f (w, t')$.

A three-sided system is said to be lexicographic if it is both lexicographic for workers as well as for firms.

Since every lexicographic binary relation over a finite set admits a numerical representation, which is additively separable, every lexicographic preference relation must of necessity be separable. Hence, if a three-sided system is lexicographic for workers (firms), then it must be separable for workers (firms). A lexicographic three-sided system is, thus, separable.

A three-sided system is said to be strongly separable if it is separable for firms and lexicographic for workers.

Danilov (2003) proved that if a three-sided system is lexicographic, then it admits a stable matching.

2. Existence of Stable Matchings

A three-sided system is said to satisfy **Technical Specialization** (TS) if there exists a function $\beta : F \times W \longrightarrow T$ such that:

(a) for all $w, w_1 \in W$ and $f, f_1 \in F$ with $w \neq w_1$ and $f \neq f_1$: $\beta(f, w) \neq \beta(f_1, w_1)$;

(b) for all $w \in W$, $f \in F$ and $t \in T$: $(w, \beta(f, w)) \geq_f (w, t)$.

Theorem 1. *Suppose a three-sided system that is lexicographic for workers satisfies TS. Then there exists a stable matching.*

Proof.

Suppose preferences are lexicographic for workers and the system satisfies TC.

Hence, for all $w \in W$ there exist linear orders P_w on F and Q_w on T such that:

(a) for all $f, f' \in F$ with $f \neq f'$ and $t, t' \in T$: fP_wf' implies $(f,t) >_w (f',t')$;

(b) for all $f \in F$ and $t, t' \in T$ with $t \neq t'$: tQ_wt' implies $(f,t) >_w (f,t')$.

For $f \in F$ let P_f be the linear order on W such that for all w, $w' \in W$: wP_fw' if and only if $(w, \beta(f,w)) \geq_f (w', \beta(f,w'))$.

Consider the two-sided matching problem where the preference of a firm f is given by P_f, and the preference of a worker w is given by P_w.

As in Gale and Shapley (1962) we get a stable job-matching m, i.e. for all $w \in W$ and $f \in F$: either $m(f)P_fw$ or $m^{-1}(w)P_wf$.

The technique-matching n is defined as follows:

For all $f \in F$: $n(f) = \beta(f, m(f))$.

By TS, n is well defined.

Suppose the matching (m,n) is not stable. Thus, there exists $w \in W$, $f \in F$ and $t \in T$ such that:

$$(f,t) >_w (m^{-1}(w), n(m^{-1}(w))), (w,t) >_f (m(f), n(f))$$

and

$$(f,w) >_t ((n^{-1}(t), m(n^{-1}(t)).$$

Let $m^{-1}(w) = f_0$, and $n^{-1}(t) = f_1$.

Since the preferences of workers are lexicographic (with firms receiving priority over techniques),

$$(f,t) >_w (m^{-1}(w), n(m^{-1}(w))$$

implies fP_wf_0.

However, since m is stable, fP_wf_0 implies $m(f)P_fw$.

Thus,

$$(m(f), n(f)) = (m(f), \beta(f, m(f)))P_f(w, \beta(f,w)).$$

Clearly $(w, \beta(f,w)) \geq_f (w,t)$. Hence, $(m(f), \beta(f, m(f)))P_f(w,t)$, contrary to our assumption.

Thus, (m,n) is stable. Q.E.D.

Note: The above proof is not valid if instead of assuming that preferences are lexicographic for workers, we merely assume that they are separable for them. The conflict arises since TC defines a best technique according to the preferences of the firms and not that of the workers.

The following example shows that if a three-sided system is merely lexicographic for workers then the existence of a stable matching is not guaranteed.

Example 1. Let $W = \{w_1, w_2\}$, $F = \{f_1, f_2\}$, $T = \{t_1, t_2\}$.

Assume that the system is lexicographic for workers with both w_1 and w_2 preferring t_1 to t_2 for any given firm f. Suppose that both w_1 and w_2 prefer f_1 to f_2.

Suppose, f_1 prefers (w_2, t_1) to (w_1, t_1) to (w_1, t_2) to (w_2, t_2) and f_2 prefers (w_1, t_1) to (w_2, t_1) to (w_2, t_2) to (w_1, t_2).

Suppose that t_1 prefers (f_2, w_1) to (f_1, w_2) to (f_2, w_2) to (f_1, w_1) and t_2 prefers (f_1, w_1) to (f_1, w_2).

Let us consider the following four matchings:

(1) $\{(f_1, w_1, t_1), (f_2, w_2, t_2)\}$;
(2) $\{(f_1, w_1, t_2), (f_2, w_2, t_1)\}$;
(3) $\{(f_2, w_1, t_1), (f_1, w_2, t_2)\}$;
(4) $\{(f_2, w_1, t_2), (f_1, w_2, t_1)\}$.

Matching (1) is blocked by (f_2, w_2, t_1) since w_2 prefers (f_2, t_1) to (f_2, t_2), f_2 prefers (w_2, t_1) to (w_2, t_2) and t_1 prefers (f_2, w_2) to (f_1, w_1).

Matching (2) is blocked by (f_1, w_2, t_1) since w_2 prefers (f_1, t_1) to (f_2, t_1), f_1 prefers (w_2, t_1) to (w_1, t_2) and t_1 prefers (f_1, w_2) to (f_2, w_2).

Matching (3) is blocked by (f_1, w_1, t_2) since w_1 prefers (f_1, t_2) to (f_2, t_2), f_1 prefers (w_1, t_2) to (w_2, t_2) and t_2 prefers (f_1, w_1) to (f_1, w_2).

Matching (4) is blocked by (f_2, w_1, t_1) since w_1 prefers (f_2, t_1) to (f_2, t_2), f_2 prefers (w_1, t_1) to (w_1, t_2) and t_1 prefers (f_2, w_1) to (f_1, w_2).

Hence, none of the four matchings are stable. Further,

$$\beta(f_1, w_2) = \beta(f_2, w_1) = t_1.$$

This contradicts TS.

Note: In the above example, the preferences are separable for workers as well. Hence, above is an example of a three-sided system that is separable for workers, and yet does not admit a stable matching.

It is worth noting that TS is not necessary for the existence of a stable matching for a three-sided system, as the following example reveals.

Example 2. Let $W = \{w_1, w_2, w_3\}$, $F = \{f_1, f_2, f_3\}$ and $T = \{t_1, t_2, t_3\}$.

Suppose that for each $w \in W$ there exists a linear order P_w on F satisfying $f_1 P_w f_2 P_w f_3$ and for each $f \in F$ there exists a linear order P_f on W satisfying $w_1 P_f w_2 P_f w_3$. Suppose for each $w \in W$ there exists a linear order Q_w on T and for each $f \in F$ there exists a linear order Q_f on T.

Suppose, $t_1 Q_a t_2 Q_a t_3$ for $a \in \{f_1, w_1, w_2\}$ and $t_3 Q_a t_2 Q_a t_1$ for $a \in \{w_3, f_2, f_3\}$. Further suppose that for all $w, w' \in W$, $f, f' \in F$ and $t, t' \in T$ with $w \neq w'$, $f \neq f'$ and $t \neq t'$:

(a) $(w, t) >_f (w', t')$ if and only if $w P_f w_1$;

(b) $(w, t) >_f (w, t')$ if and only if $t Q_f t'$;

(c) $(f, t) >_w (f', t')$ if and only if $f P_w f'$;

(d) $(f, t) >_w (f, t')$ if and only if $t Q_w t'$.

In addition, suppose that for all $t \in T$, $f' \in F$, $w' \in W$ and $i \in \{1, 2, 3\}$: $(f_i, w_i) \geq_t (f', w')$ if and only if $t = t_i$. Towards a contradiction suppose that this system satisfies TS.

Then there exists a function $\beta : F \times W \longrightarrow T$ such that:

(a) for all $w, w' \in W$ and $f, f' \in F$ with $w \neq w'$ and $f \neq f'$: $\beta(f, w) \neq \beta(f', w')$;

(b) for all $w \in W$ and $f \in F$: $[(w, \beta(f, w)) \geq_f (w, t)$ for all $t \in T]$.

Thus,

$$\beta(f_1, w_1) = t_1, \quad \text{and } \beta(f_3, w_3) = t_3.$$

Since $\beta(f_2, w_2) \in \{t_1, t_3\}$, the requirements of TS are violated. Thus, this system does not satisfy TS. However, the matching with the associated triplets being (w_i, f_i, t_i) for $i = 1, 2, 3$ is indeed a stable matching.
A three-sided system

$$[\{\geq_f : f \in F\}, \{\geq_w : w \in W\}, \{\geq_t : t \in T\}]$$

is said to satisfy agreement over workers if for $f \in F$, $t \in T$ and $w' \in W$: $(w, t) >_f (w', t)$ implies $(f, w) >_t (f, w')$.

Theorem 2. *Suppose a three-sided system is strongly separable (i.e. separable for firms and lexicographic for workers) and satisfies Agreement over Workers. Then there exists a stable matching.*

Proof.

Suppose that for all $w \in W$ there exists linear orders P_w on F and Q_w on T such that:

(a) for all $f, f' \in F$ with $f \neq f'$ and $t, t' \in T$: fP_wf' implies $(f, t) >_w (f', t')$;

(b) for all $f \in F$ and $t, t' \in T$ with $t \neq t'$: tQ_wt' implies $(f, t) >_w (f, t')$.

Suppose, in addition that for all $f \in F$, there exists a linear order P_f on W and Q_f on T such that for all $w, w' \in F$ with $w \neq w'$ and $t, t' \in T$ with $t \neq t'$: wP_fw' implies $(w, t) >_f (w', t)$ and tP_ft' implies $(w, t) >_f (w, t')$.

Consider the two-sided matching model based on F and W where for each $f \in F$ and $w \in W$ preferences are given by P_f and P_w respectively. As in Gale and Shapley (1962), we get a job-matching m that is stable, i.e. for all $w \in W$ and $f \in F$: either $m(f)P_fw$ or $m^{-1}(w)P_wf$.

For $t \in T$ let P_t be a linear order on F such that for all $f, f' \in F$ with $f \neq f'$: fP_tf' if and only if

$$(f, m(f)) >_t (f', m(f')).$$

Consider the two-sided matching model based on F and T where for each $f \in F$ and $t \in T$, preferences are given by Q_f and P_t respectively. As in Gale and Shapley (1962), we get a technique-matching n such that for all $t \in T$ and $f \in F$: either $n(f)Q_ft$ or $n^{-1}(t)P_tf$.

Towards a contradiction suppose that the matching (m, n) is not stable. Thus, there exists $w \in W$, $f \in F$ and $t \in T$ such that:

$$(f, t) >_w (m^{-1}(w), n(m^{-1}(w))), (w, t) >_f (m(f), n(f))$$

and

$$(f, w) >_t (n^{-1}(t), m(n^{-1}(t))).$$

Since the preferences of workers are lexicographic with firms receiving priority over techniques, it must be case either that

(a) $f = m^{-1}(w)$ and $tQ_w n(m^{-1}(w))$ or **(b)** $fP_w m^{-1}(w)$.

(b) Suppose $f = m^{-1}(w)$. Thus, $tQ_w n(m^{-1}(w))$. Since preferences of firms are separable we must have $tQ_f n(f)$. $tQ_f n(f)$ and the stability of the matching n imply:

$$(n^{-1}(t), m(n^{-1}(t))) >_t (f, m(f)).$$

Thus, $(f, w) >_t (f, m(f))$. This contradicts $w = m(f)$. Hence, suppose $fP_w m^{-1}(w)$. Since preferences of firms are separable we must have $tQ_f n(f)$. $tQ_f n(f)$ and the stability of the matching n implies $(n^{-1}(t), m(n^{-1}(t))) >_t (f, m(f))$.

Thus, $(f, w) >_t (f, m(f))$. Since the three-sided system is assumed to satisfy agreement over workers and the preferences of firms are separable, $(f, w) >_t (f, m(f))$ implies $wP_f m(f)$.

By the stability of the matching m we get $m^{-1}(w)P_w f$ contrary to our assumption.

Thus, (m, n) is stable. Q.E.D.

In example 1 the preferences of the workers are lexicographic (with firms getting priority over technologists), but the preferences of the firms are not separable. Thus, although the three-sided system satisfies agreement over workers, it does not admit a stable matching. In the following example preferences are strongly separable but the system does not satisfy agreement over workers and does not admit a stable matching.

Example 3. Let $W = \{w_1, w_2\}$, $F = \{f_1, f_2\}$, $T = \{t_1, t_2\}$.

Assume that the system is lexicographic for workers (with firms receiving priority over technologists). Suppose that for any given firm both workers prefer t_2 to t_1 and both workers prefer f_1 to f_2. Suppose, the preferences of the firms are also lexicographic (although not in the sense that we have defined in this paper), with technologists receiving priority over workers. Hence the preferences of the firms are separable. Suppose, both firms prefer t_2 to t_1 and for any given technique prefer w_2 to w_1.

Suppose, t_1 prefers (f_1, w_2) to (f_1, w_1).

Suppose, t_2 prefers (f_2, w_2) to (f_1, w_1) and (f_1, w_1) to (f_2, w_1) to $f_1, w_2)$.

Let us consider the following four matchings:

(1) $\{(f_1, w_1, t_1), (f_2, w_2, t_2)\}$;

(2) $\{(f_1, w_1, t_2), (f_2, w_2, t_1)\}$;

(3) $\{(f_2, w_1, t_1), (f_1, w_2, t_2)\}$;

(4) $\{(f_2, w_1, t_2), (f_1, w_2, t_1)\}$.

Matching (1) is blocked by (f_1, w_2, t_1) since w_2 prefers (f_1, t_1) to (f_2, t_2), f_1 prefers (w_2, t_1) to (w_1, t_1) and t_1 prefers (f_1, w_2) to (f_1, w_1).

Matching (2) is blocked by (f_2, w_2, t_2) since w_2 prefers (f_2, t_2) to (f_2, t_1), f_2 prefers (w_2, t_2) to (w_2, t_1) and t_2 prefers (f_2, w_2) to (f_1, w_1).

Matching (3) is blocked by (f_2, w_1, t_2) since w_1 prefers (f_2, t_2) to (f_2, t_1), f_2 prefers (w_1, t_2) to (w_2, t_1) and t_2 prefers (f_2, w_1) to (f_1, w_2).

Matching (4) is blocked by (f_1, w_1, t_2) since w_1 prefers (f_1, t_2) to (f_2, t_2), f_1 prefers (w_1, t_2) to (w_2, t_1) and t_2 prefers (f_1, w_1) to (f_2, w_1).

Hence none of the four matchings are stable.

Note that given t_2, f_1 prefers w_2 to w_1, whereas given f_1, t_2 prefers w_1 to w_2. Hence, the system does not satisfy agreement over workers.

It is instructive to note that in example 3, although the preferences of the workers and firms are both lexicographic (although not in the sense in which it is defined here) with workers giving priority to firms over techniques, a stable matching does not exist, since firms accord priority to techniques over workers. The reciprocation of priority between firms and workers that was assumed by Danilov (2003) is absent in example 3.

Acknowledgment

This paper is a revised extension of Stable Matchings for Three-Sided Systems: A Comment (Grand Coalition Working Paper). *Excepts from this paper and related themes were presented at the Second World Congress of the Game Theory Society held at Marseilles from July 5 to 9, 2004, Seventh International Meetings of the Society for Social Choice and Welfare held at Osaka from July 21–25, 2004 and at the FourthConference on Models and Methods in Economics held at Indian Statistical Institute Kolkata from December 8 to 10, 2004. I would like to put on record a very deep acknowledgment to Arunava Sen for pointing out to me that lexicographic preferences are separable. Thanks also to the participants of the Kolkakta Conference who made observations during my presentation. Further, I would like to thank Ahmet Alkan, Francis Bloch, V.I. Danilov, Ashok Srinivasan and William Zwicker for comments.*

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Multiple Membership and Federal Structures

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Abstract. We consider a model of the "world" with several regions that may create a unified entity or be partitioned into several unions (countries). The regions have distinct preferences over policies chosen in the country to which they belong and equally share the cost of public policies. It is known that stable "political maps" or country partitions, that do not admit a threat of secession by any group of regions, may fail to exist. To rectify this problem, in line with the recent trend for an increased autonomy and various regional arrangements, we consider *federal structures*, where a region can simultaneously be a part of several unions. We show that, under very general conditions, there always exists a stable federal structure.

Keywords: Partitions, Federal Structures, Stability, Cooperative games.
JEL Classification Numbers: C71, D71, H41.

Introduction

In this note we consider a model of country formation with a "world" consisting of multiple regions that may either form a unified entity or to be partitioned into several countries. Each country chooses a (possibly multidimensional) public policy which cost is shared among country's regions. However, since regions have heterogenous preferences over public policies, some of them may find centrally chosen policies sufficiently distant from their ideal choices and may pose a threat of secession from the country to which they belong. A natural question is whether there are *stable partitions* of the world that do not admit a group of regions each benefiting by breaking away from the status quo. It turns out that, in general, the stability cannot be guaranteed. In particular, in the case where all regions within a country equally share the cost of public policies, even uni-dimensionality of the policy space and single-peakedness of regions' preferences do not guarantee the existence of a stable world structure [Bogomolnaia et al., 2007]. The paradigm of the centralized decision-making, however, has been recently revised on both theoretical and empirical grounds. As Alesina et al. (2005, p.602) point out: "Historically, the nation state concentrated most of the authority in every policy domain. In recent decades, however, a more complex structure has begun to emerge, characterized by a demand for more autonomy (if not secession) at a sub-national level of country unions which assume certain policy prerogatives". Indeed, in the context of the federal structures and multiple public goods, the local governments assume an increasing responsibility for providing local public goods while macroeconomic and redistribution policies with the federal government.

Regional projects that tackle the trade, environmental and migration issues play an ever increasing role across the globe. These observations indicate that from the theoretical point of view, the players (regions) may belong to several unions, each assigned to a certain aspect of the public policy and we therefore permit an option of the multiple union membership for every region. Note that a membership in several unions is a wide-spread phenomena as some European countries may sustain a simultaneous membership in the European Union, European Monetary Union, NATO, United Nations or WTO.

Thus, the regions could be members in several unions, and we even allow for an opt-out option [Makarov, 2003] where some regions forego their participation in the provision of some public goods. Every formed union of regions S is assigned a certain public project and a participation weight $\Lambda(S)$, where for every region r the sum of the participation weights over the unions r belongs to, must be equal to one. A set of unions with the corresponding participation weights will be called a *federal structure*. Given a federal structure, some of the regions may reject the proposed arrangement and pose a threat of secession. Our secession requirement is very mild: a group of regions S poses a threat of secession if it can guarantee to every region in S a higher payoff than at least in one of the unions r belongs to. The main result of this paper yields the existence of stable federal structures. In order to prove our result we use the framework of cooperative game without transferable utility and rely on the Danilov's (1999) variant of the celebrated Scarf (1967) theorem on non-emptiness of the core.

1. The Model

Consider a model with a finite set $N = \{1, \ldots, n\}$ of regions, which can either constitute one (unified) country or be partitioned into several countries. Each country chooses a public policy, and we assume that the set of feasible policies is given by a multi-dimensional Euclidean space $P = \mathcal{R}^k$, where $k \geq 1$. If a country $S \subset N$ forms it must choose a policy p in P. The policy implementation incurs monetary costs, denoted by $g(S)$. We naturally assume that the costs are positive for every S and are weakly increasing with respect to inclusion:

Definition 1 [A.1 – Cost Monotonicity]. *If the set of regions S is contained in a larger set S' then $g(S') \geq g(S)$.*

Every region $r \in N$ has an ideal point $p^r \in P$ and the choice of any other policy $p \in P$ would generate a disutility for r, represented by the Euclidean distance $||p^r - p||$ between its ideal policy p^r and the policy p. We assume that every region $r \in N$ has an initial endowment $y_r > 0$, a part of which is spent on the implementation of the public policy chosen in the country of region r. That is, if country S chooses a policy $p \in P$ then every region $r \in S$ is assigned the monetary contribution t_r. We assume the total contributions of the regions cover the cost of public projects they participate in:

Definition 2 [A.2 – Budget-balancedness]. $\sum_{r \in S} t_r = g(S)$.

Definition 3 [A.3 – Utilities]. *The utility of region r assigned to a monetary contribution* t_r *in the country with a public policy p, is given by* $u(y_r - t_r, ||p^r - p||) = y_r - t_r - ||p^r - p||$.

In order to proceed with our results we make further assumptions on policy choices and allocation of their costs across regions. We assume that the policy choice in the country is determined through the majority voting mechanism. If the policy set P is unidimensional, for every country $S \subset N$ consider the set of its *median locations.* It is easy to see that every median location minimizes the aggregate cost of regions in S, and is, in fact, a solution to the following minimization problem:

$$\min_{p \in P} \sum_{r \in S} ||p^r - p||. \tag{1}$$

We denote the set of solutions to (1) by $M(S)$. If the policy set P is multidimensional and the set of ideal points p^r is not located along a straight line, a solution to (1), denoted by $m(S)$, is unique.[1] In the unidimensional case, $M(S)$ could be an interval, and in this case $m(S)$ will stand for the middle point of $M(S)$. We impose the efficiency requirement:

Definition 4 [A.4 – Efficiency]. *Every country S chooses its public policy at* $m(S)$.

Following [Alesina and Spolaore, 1997], [Casella, 2001], [Jéhiel and Scotchmer, 2001], [Haimanko et al., 2004], [Bogomolnaia et al. 2007] we assume, for simplicity, that all regions of the same country make an equal contribution towards the policy cost. The regions are hold responsible for their preferences, and are not compensated for their disutility of location of the chosen policy in the policy space P:[2]

Description 5 [A.5 – Equal Share]. *If the country S is created, every region in S makes the same monetary contribution:* $t_r = \frac{g(S)}{|S|}$ *for every* $r \in S$.

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[1] In the mathematical programming literature, the value of the problem (1) is called Minimal Aggregate Transportation Cost of the set S.

[2] We could have assumed that each region contributes proportionally to its population, without altering the paper's main result.

The assumptions we impose allow us to reduce a country formation problem described above to determination of countries' composition; once formed, their policy actions and cost contributions are prescribed as above. We will denote by $v(r, S)$ the indirect utility (or payoff) of the region $r \in S$ when country S forms: $v(r, S) = y_r - \frac{g(S)}{|S|} - ||p^r - m(S)||$.

We will examine partitions which are stable under secession threats. In other words, no group of regions could reduce their costs by forming a new country. Formally,

Definition 6. *A collection* $\pi = \{S_1, \ldots, S_K\}$ *of pairwise disjoint subsets of* N *is called a* partition *if* $\bigcup_{k=1}^{K} S_k = N$. *The set of all partitions of* N *is denoted by* Π.
Consider a partition π of N and denote by $S^r(\pi) \in \pi$ the country in π that contains r. Then, the utility of the region r is given by $v(r, \pi) = v(r, S^r(\pi))$.
We now offer the standard definition of (core) stability:

Definition 7. *A partition* $\pi = \{S_1, \ldots, S_K\}$ *of* N *is called* stable *if there is no group of regions* $S \subset N$ *such that* $v(r, S) > v(r, \pi)$ *for every* $r \in S$.

Proposition 1. *There exists a set of regions* N*, satisfying assumptions A1–A5, which does not admit a stable partition.*

2. Federal Structures

Note that the definition of partitions introduced in the previous section rules out situations where different facets of public policy are carried out under different group structures. For example, defense or foreign policy can be implemented by the grand coalition of regions, while education or health fall into jurisdiction of local authorities. A natural framework to incorporate this possibility would be allowing the regions to enter several unions, each responsible for a certain facet of public policy. A union does not necessarily include all regions, as one could easily imagine the case where only a group of regions is keen on developing of a public policy on, say, environment or migration, while other (possibly distant) regions may have a limited interest in those issues. In line with this comment we allow unions of regions to pursue different aspects of public policy. Degrees of participation intensity may vary across unions but are the same for all members of the same "rigid" (in terminology of Alesina et al., 2005) union. Some "disinterested" regions may even decide to forego their participation in certain parts of the public project, and, following [Makarov, 2003], we allow for an opt-out option. Formally, we consider a notion of *federal structure*, which consists of unions of regions formed to pursue different facets of public projects. Each union is assigned a participation weight, and we only require that the sum of the participation weights over the unions a region belongs to, is equal to one for all regions:

Definition 8. *A federal structure is a function* $\Lambda : 2^N \setminus \emptyset \to [0,1]$ *that assigns every non-empty subset* S *of* N *a non-negative value such that the equality* $\sum_{S \in \mathcal{S}^r} \Lambda(S) = 1$ *holds for all* $r \in N$*, where* $\mathcal{S}^r$ *is the collection of subsets of* N *that contain* r[3].

Note that every partition $\pi \in \Pi$ induces a federal structure Λ_π by assigning the value of one to every S from π and zero to all other subsets.

We will now introduce a secession requirement. First, for every federal structure Λ define the set of *essential* unions S with a positive degree of participation: $\mathcal{S}_\Lambda = \{S \subset N | \Lambda(S) > 0\}$.

Then the utility level derived by region r, given Λ would be defined as follows:

$$v_m(r, \Lambda) = \min_{S \in \mathcal{S}_\Lambda \cap \mathcal{S}^r} v(r, S). \tag{2}$$

Now a *secession threat* by a group of regions S would simply require that S can guarantee to every region $r \in S$ a payoff which is higher than at least in one of effective unions r belongs to. Formally,

Definition 9. *A group of regions* S *poses a threat of secession to the federal structure* Λ *if* $v(r,S) > v_m(r,\Lambda)$ *for every* $r \in S$*. A federal structure* Λ *is called stable if no group of regions poses a threat of secession to* Λ*.*

Now, we state our main result.

Proposition 2. *Under Assumptions A.1–A.5, there exists a stable federal structure* Λ*.*

Proof of Proposition 2.

To prove this result we will use Danilov (1999). For every non-empty $S \subset N$ denote by $\Re^S$ the projection of the set $\Re^n$ on coordinates in S. For every vector $y = (y_1, \ldots, y_n) \in \Re^n$ let $y^S \in \Re^S$ be a natural projection of y, that is $y^S_r = y_r$ for every $r \in S$. A *non-cooperative NTU-game* V is a correspondence that assigns to each S a subset $V(S)$ of $\Re^S$. Danilov (1999) proves the following variant of the Scarf's result:

D-Theorem. *Consider a non-cooperative NTU-game* V*, where for every* $S \subset N$ *the set* $V(S)$ *is closed, bounded from above and satisfies the free disposal condition. Then there exists a vector* $y \in \Re^n$*, and a balanced collection (federal structure)* Λ*, satisfying two requirements: there is no* $S \subset N$ *such that* $y^S \in int\{V(S)\}$*, where* $int\{V(S)\}$ *stands for the interior of* $V(S)$*; (ii) If* $\Lambda_S > 0$ *then* $y^S \in V(S)$*.*

Now, for every non-empty $S \subset N$ define the set $\hat{V}(S) \subset \Re^S$ as follows:

$$\hat{V}(S) = \{y^S \in \Re^S : \ v(r,S) \geq y^S_r \ \forall r \in S\}. \tag{3}$$

---

[3] Note that in the cooperative game theory a collection $\{\Lambda_S\}_{S \subset N}$ satisfying this equality is called *balanced.*

Clearly, the game $\hat{V}$ satisfies the conditions of the D-theorem. Thus, there exist vector $y \in \Re^n$ and a balanced collection Λ which satisfy the requirements of D-Theorem. We claim that Λ is a stable mixed federal structure.

Indeed, suppose a group of regions S poses a threat of secession to Λ. That is for each region $r \in S$ there exists $T(r) \in \mathcal{S}_\Lambda \cap \mathcal{S}^r$ such that $v(r, S) > v(r, T(r))$.

Since $\Lambda_{T(r)} > 0$, the assertion (ii) of the D-theorem implies that $y^{T(r)} \in \hat{V}(T(r))$, and, therefore, $v(r, T(r)) \geq y_r^{T(r)} = y_r$ for every $r \in S$. The last two inequalities guarantee that $v(r, S) > y_r$ for all $r \in S$. But then $y^S \in int\{\hat{V}(S)\}$, a contradiction to assertion (i) of the D-theorem.

3. Participation Distributions

We introduce *participation distributions* over the set of partitions Π and allow every region to be a part of different partitions each assigned to a certain aspect of the public policy.

Let μ be a *participation distribution* on Π, where for every $\pi \in \Pi$, a non-negative value of $\mu(\pi)$ represents the relative weight assigned to the dimension of the public project carried out by partition π, where $\sum_{\pi \in \Pi} \mu(\pi) = 1$.

Note that the set of participation distributions includes all partitions π. Indeed, every partition π is, in fact, a dichotomous participation contribution, which assigns the value one to π and zero to all other partitions.

Denote by $\Pi(\mu)$ the set of μ-essential partitions, i.e. those partitions π for which the value of $\mu(\pi)$ is positive. Naturally,

$$\sum_{\pi \in \Pi(\mu)} \mu(\pi) = 1. \tag{4}$$

In order to introduce a notion of stability in this modified framework, one has to define a utility level derived by a region r from the distribution μ. To remain in line with the discussion of the previous section we simply assume that for every region r the value $v(r, \mu)$ is the weighted average of the utilities derived from essential partitions:

$$v(r, \mu) = \sum_{\pi \in \Pi(\mu)} \mu(\pi) v(r, S^r(\pi)). \tag{5}$$

Given a participation distribution μ a group of countries S, may consider a withdrawal from the proposed arrangement if it can make every region r in S better off:

Definition 10. *A participation distribution μ is called* stable *if there is no group of regions $S \subset N$ such that $v(r, S) > v(r, \mu)$ for every $r \in S$.*

Proposition 3 indicates that a stable participation distribution that generates a single partition may fail to exist. The conclusion is reversed if we consider the entire set of stable participation distributions:

Proposition 3. *For every set of regions N, satisfying assumptions A1-A5, there is a stable participation distribution.*

Moreover, every participation distribution μ induces a federal structure as well. Indeed, for any participation distribution μ and every $S \subset N$ define the participation weights Λ_μ by:

$$\Lambda_\mu(S) = \sum_{\{\pi \in \Pi(\mu): S \in \Pi\}} \mu(\pi) \tag{6}$$

That is for every union S we calculate the sum of the weights over all partitions that contain S as its element. Obviously, Λ_μ is a federal structure as for every $r \in N$ we have

$$\sum_{S \in \mathcal{S}^r} \Lambda_\mu(S) = \sum_{\pi \in \Pi(\mu)} \mu(\pi) = 1. \tag{7}$$

It is worth pointing out that the converse is not necessarily true, and there are federal structures, whose weights can not be supported by a participation distribution. Consider an example with three regions $N = \{1, 2, 3\}$ and the following weights:

$$\Lambda(S) = \begin{cases} \frac{1}{2} & \text{if} \quad |S| = 2 \\ 0 & \text{if} \quad \text{otherwise} \end{cases} \tag{8}$$

Obviously, Λ is a federal structure. Note that the partition $\{\{1, 2\}, \{3\}\}$ is the only one that contains $\{1, 2\}$. Thus, for (6) to hold we must have $\mu(\{1, 2\}) = 1/2$. Similarly, $\mu(\{1, 3\}) = \mu(\{2, 3\}) = 1/2$, which rules out the existence of a participation distribution that would support the weights Λ. These observations can be summarized by the following corollary:

Corollary. *The set of partitions Π is a proper subset of the family of all participation distributions, whereas the latter is a proper subset of the set of all federal structures.*

4. Conclusions

In this paper we consider a model of the "world" with multiple regions. The world is partitioned into countries, each consisting of one or several regions. The regions have distinct preferences over public policies chosen in their country and finance the cost of public projects through the equal share mechanism. [Bogomolnaia et al., 2007] have shown that stable partitions, that are immune against threats of secession by groups of regions, do not necessarily exist. In order to rectify this problem and to examine a more flexible distribution of power and responsibility within countries, we allow every regions to belong to several unions, assigned to different facets of the public good project. This so-called *federal structure* consists of unions of regions, where each union is assigned a participation weight so that for every region the sum of the participation weights over the unions a region belongs to is equal to one. By using the Danilov variant (1999) of the Scarf theorem (1967) on non-emptiness of the core of a balanced game without side payments, we show that under our assumptions there always exists a stable federal structure. There are two natural questions related to the result of this paper that remain open and are left for future research. What are conditions on the distribution of regions' preferences that guarantee the existence of stable partitions? Could one characterize the set of stable federal structures, especially in the environments that do not admit stable partitions?

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A Game-theoretic Approach to Multicriteria Problems

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Abstract. A new concept of the single-valued solution for the multicriteria problems using the principles of the consistency and equilibrium from the game theory is introduced. For this solution new equations are constructed and conditions of its Pareto optimality are established. Examples are considered.

Keywords: Multicriteria optimisation, Pareto optimality, consistency, equilibrium, game theory.

Introduction

It is known that any "solution" of the multicriteria problem must be Pareto optimal (effective). It means that the improvement of such a solution in any criterion leads to its deterioration in another criterion. The systematical study of the Pareto optimal solutions of the multicriteria problems and of the methods of finding such solution are given in the monograph [Podinovsky and Noghin, 1982]. One of such methods of finding a solution is the method of reducing to one criterion, in particular to the non-negative linear combination of the criteria. This approach contains also in [Kostreva et al., 2004]; [Leskinen et al., 2004]. Another approach to the multicriteria problems consists of an ordering of the criteria by their relative importance. To this approach monograph [Noghin, 2002] and papers [Angilella et al., 2004]; [Doumpos and Zopounidis, 2004] are devoted.

In the papers [Liapounov, 2005a; 2005b; 2005c; 2007] a new concept of the solution the multicriteria problems based on the axiomatic approach with using principles of consistency and equilibrium from the game theory is proposed. The principle of consistency consists of the following: a problem is considered as an element of a class of problems depending on a parameter and the form of dependence of the solution from this parameter is postulated. In our case (see subsection 2.1) the principle of consistency is reduced to the continuous and monotone dependence of the solution for the segment on the angle of rotation. In our case (see subsection 3.1) the principle of equilibrium consists of the following: the principle of consistency must fulfil in every variable.

On the other hand, the proposed approach is connected with the bargaining problem [Abhinay, 1999]; [Nash, 1950]; [Rubinstein and Osborne, 1990]; [Thomson, 1994].

This paper contains the systematic description of this solution and its properties. Here the equations for this solution are given, its existence and the conditions of its Pareto optimality are proved. Examples are considered.

The section 1 contains definitions. In section 2 the problem with one variable is considered. In subsection 2.1 from the principle (the axiom) of consistency the equations for the solution of the problem with the linear criteria (for the segment) are established. In subsection 2.2 this solution with using of the axiom of the additivity is generalized on the nonlinear problem with one variable (or for the curve). In subsection 2.3 the conditions of the Pareto optimality are established. In subsection 2.4 Liapunov's function is given.

In section 3 the general problem is considered. In subsection 3.1 from the solution for the problem with one variable with using of the equilibrium axiom the basic equations are established. In subsection 3.2 the existence theorem is proved. In subsection 3.3 the properties of the solution are given, in subsection 3.4 the problem with the linear criteria is considered, in the subsection 3.5 the problem with the concave criteria is considered, in the subsection 3.6 the examples are given and in subsection 3.7 the non-cooperative game connected with the multicriteria problem is considered. In section 4 (conclusion) some features of the solution are discussed.

1. Preliminaries

For $x, y \in \mathbb{R}^n$ we shall write $x \geq y$ if $x_j \geq y_j$, $j = 1, \ldots, n$. Let $X \subset \mathbb{R}^n$ be a set.

Definition 1. *A point $x \in X$ is called* Pareto optimal (or effective) in the set X *if do not exists a point $y \in X$ such that $y \geq x$, $y \neq x$. The set of Pareto optimal points of the set X we shall denote $\mathcal{E}(X)$. If $\mathcal{E}(X) = X$, then we say that the set X is* Pareto optimal.

Let $X \subset \mathbb{R}^n$ be a set, $f : X \to \mathbb{R}^m$ be a map.

Definition 2. *The pair $\mathcal{P} = (X, f)$ is called* multicriteria problem*, or merely* problem*. The components f_i, $i = 1, \ldots, m$, of the map f are called* criteria *of the problem $\mathcal{P}$.*

Definition 3. *A point $x \in X$ is called* Pareto optimal (or effective) in the problem *$\mathcal{P}$ if do not exists a point $y \in X$ such that $f(y) \geq f(x)$, $f(y) \neq f(x)$. The set of the Pareto optimal points of the problem $\mathcal{P}$ will be denoted by $\mathcal{E}(\mathcal{P}) = \mathcal{E}(X, f)$. If $\mathcal{E}(\mathcal{P}) = X$, then we say that the problem $\mathcal{P}$ is Pareto optimal.*

Obviously, $\mathcal{E}(X) = \mathcal{E}(X, f)$ where $f(x) = x$.

Definition 4. *A map s such that $s(\mathcal{P}) \in X$ is called a* solution of the problem $\mathcal{P}$. *If $s = s(\mathcal{P})$ is a solution of the problem $\mathcal{P}$ then the vector $v = f(s)$ is called* the vector value of the problem $\mathcal{P}$ *or merely* the value of the problem $\mathcal{P}$*.*

The solution must be Pareto optimal, i.e. the following axiom must be fulfil:

A1. Axiom of Pareto optimality. $s(\mathcal{P}) \in \mathcal{E}(\mathcal{P})$.

It may be required to fulfil a stronger axiom: the solution must be determined only by the Pareto optimal set of the problem. More precise, let $\mathcal{P} = (X, f)$ be a problem. Consider the problem $\widehat{\mathcal{P}} = (\mathcal{E}(\mathcal{P}), \widehat{f})$, where $\widehat{f}$ is the restriction f to $\mathcal{E}(\mathcal{P})$. Obviously, $\mathcal{E}(\widehat{\mathcal{P}}) = \mathcal{E}(\mathcal{P})$. Then the axiom consists of the following:

A2. Strengthened axiom of Pareto optimality. $s(\mathcal{P}) = s(\widehat{\mathcal{P}})$. In the paper the conditions under which the axiom A1 and A2 are satisfied are given.

2. Case of One Variable (Solution for a Curve)

2.1. Solution for a Segment

Let $x \in \mathbb{R}^n$. Put $x^+ = (x_1^+, \ldots, x_n^+)$, where $x_j^+ = \max(x_j, 0)$. Consider the following function:

$$\varphi(x) = \frac{\|x^+\|}{\|x\|}\ , \tag{1}$$

where $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^n$.

In l_2 norm the function (1) is the cosinus of the angle between the vector x and the positive ortant $\mathbb{R}^n_+$.

Function (1) has the following properties:

F1. The function φ is continuous for all $x \in \mathbb{R}^n$, $x \neq 0$.
F2. $0 \leq \varphi(x) \leq 1$ for all $x \in \mathbb{R}^n$, $x \neq 0$, $\varphi(x) = 1$ for $x \in \mathbb{R}^{\ltimes}_+$, $\varphi(x) = 0$ for $x \in \mathbb{R}^n_-$.
F3. $\varphi(x) + \varphi(-x) \geq 1$, and in l_1-norm $\varphi(x) + \varphi(-x) = 1$.
F4. For $\lambda > 0$ $\varphi(\lambda x) = \varphi(x)$.

Lemma 1. *φ is the nondecreasing function and for $x \notin \mathbb{R}^n_+$ and $x \notin \mathbb{R}^n_-$ it is the strictly increasing function.*

Proof.

In the l_p-norm, where $p > 1$, we have from (1)

$$\left(\sum_{j=1}^{n} |x_j|^p\right)^2 \frac{\partial \varphi^p(x)}{\partial x_j} = p(x_j^+)^{p-1} \sum_{j=1}^{n} |x_j|^p - p|x_j|^{p-1}\mathrm{sign} x_j \sum_{j=1}^{n} (x_j^+)^p,$$

$$j = 1, \ldots, n.$$

Since

$$(x_j^+)^{p-1} \geq |x_j|^{p-1}\mathrm{sign}x_j\varphi^p(x), \qquad (2)$$

the following inequality holds:

$$\frac{\partial\varphi(x)}{\partial x_j} \geq 0.$$

Moreover, if $0 < \varphi(x) < 1$ and $x_j \neq 0$, from (2) it follows that

$$\frac{\partial\varphi(x)}{\partial x_j} > 0.$$

The proof for $p = 1$ we get by going to limit as $p \to 1$. □

Consider the problem $\mathcal{P} = (X, f)$, where

$$X = [0,1], \quad f(x) = a(1-x) + bx, \quad a, b \in \mathbb{R}^m \ . \qquad (3)$$

Definition 5. The point $s(a,b)$ is called *the solution* of the problem (3) or *the solution* to the segment $[a,b]$ if it fulfils to the following axiom:

A3. Axiom of consistency

$$s(a,b) = a + \varphi(b-a)(b-a) \ , \qquad (4)$$

where φ is defined by (1) .

Remark 1. Equality (4) expresses *the property of the consistency:* the solution s continuously and monotonously depends on the rotation angle of the vector $b - a$ around the point a.

Solution (4) has the following properties:

S1. The function s is continuous on $\mathbb{R}^m \times \mathbb{R}^m$.
S2. $s(a,b) \in [a,b]$.
S3. $\|s(a,b) - a\| = \|(b-a)^+\|$.
S4. $s(a,b)$ is Pareto optimal for all $a, b \in \mathbb{R}^m$.

Remark 2. The solution defined in (4) is not symmetric: $s(a,b) \neq s(b,a)$. Instead (4) we may consider the symmetric solution:

$$\widehat{s}(a,b) = \frac{1}{2}(s(a,b) + s(b,a)) = a + \frac{b-a}{2}(1 + \varphi(b-a) - \varphi(a-b)) \ . \qquad (5)$$

From (5) it follows that for the symmetric solution function (1) must be replaced by the following function:

$$\varphi^s(x) = \frac{1}{2}\left(1 + \varphi(x) - \varphi(-x)\right) \ , \qquad (6)$$

and $\|x^+\|$ must be replaced by

$$n(x) = \frac{1}{2}\left(\|x\| + \|x^+\| - \|x^-\|\right) \ . \tag{7}$$

Note that in the l_1-norm

$$\varphi^s(x) = \varphi(x), \quad n(x) = \|x^+\|.$$

For the sequel we shall use function (4).

Example 1. Let in problem (3)

$$m = 2, \quad a = 0, \quad \|b\| = \sqrt{b_1^2 + b_2^2} = 1,$$

and the vector b rotates around origin. Then the solution $s = s(0, b)$ describes the following curve:

$$s_1^2 + s_2^2 = 1, \quad \text{if} \quad s_1 \geq 0, \quad s_2 \geq 0,$$

$$s_1^2 + \left(s_2 - \frac{1}{2}\right)^2 = \frac{1}{4}, \quad \text{if} \quad s_1 \leq 0, \quad s_2 \geq 0,$$

$$\left(s_1 - \frac{1}{2}\right)^2 + s_2^2 = \frac{1}{4}, \quad \text{if} \quad s_1 \geq 0, \quad s_2 \leq 0.$$

2.2. Solution for a Curve

Let $X = [\alpha, \beta] \subset \mathbb{R}$ be a segment, $f : [\alpha, \beta] \to \mathbb{R}^m$ be a set of criteria. Consider the problem

$$\mathcal{P} = ([\alpha, \beta], f). \tag{8}$$

Together with problem (8) consider the following curve

$$L = \{y \in \mathbb{R}^m \mid y = f(x), \quad \alpha \leq x \leq \beta\}. \tag{9}$$

Let $[\alpha', \beta'] \subset [\alpha, \beta]$. Put

$$L(\alpha', \beta') = \{y \in \mathbb{R}^m \mid y = f(x), \ \alpha' \leq x \leq \beta'\}. \tag{10}$$

By $l(\alpha', \beta')$ denote the length of the curve $L(\alpha', \beta')$. It is known that

$$l(\alpha', \beta') = \int_{\alpha'}^{\beta'} \|f'(x)\| \, dx. \tag{11}$$

Let $s(\alpha', \beta') = x^*$, $\alpha' \leq x^* \leq \beta'$, be a solution for curve (10). Let us require that the solution satisfies the following axiom:

A4. Axiom of additivity. For all $\alpha \leq \alpha' \leq \gamma \leq \beta' \leq \beta$ the equality must hold:

$$l(\alpha', s(\alpha', \beta')) = l(\alpha', s(\alpha', \gamma)) + l(\gamma, s(\gamma, \beta')) \ . \tag{12}$$

Note that equality (8) for the segment holds in view of (4), property F4 of the function φ and property S3 of the solution.

Let $x_0 = \alpha \leq x_1 \leq \ldots \leq x_k = \beta$ be a partition of the segment $[\alpha, \beta]$ and let L^k be a piecewise line corresponding to this partition, i.e.

$$L^k = [f(x_0), f(x_1)] \cup \ldots \cup [f(x_{k-1}), f(x_k)] \ .$$

From (8) and property S3 of the solution for the segment it follows that if s^k is the solution for L^k and $l^k(s^k)$ is the length of the part of L^k from $f(\alpha)$ to s^k then the following equality fulfils:

$$l^k(s^k) = \sum_{i=1}^{k} \|(f(x_i) - f(x_{i-1}))^+\|. \tag{13}$$

Let $s = x^*$ be a solution for curve (9), $l(s) = l(\alpha, x^*)$ be length (11). Going to the limit in (13) in view of (11) we get the following equation for x^*:

$$\int_{\alpha}^{x^*} \|f'(x)\| \, dx = \int_{\alpha}^{\beta} \|(f'(x))^+\| \, dx. \tag{14}$$

Equation (14) is the extension of the property S3 of the solution to the segment. Equation (14) has the following interpretation.

The norm of the vector determines the length of the curve L by the formula:

$$\|x\| \rightsquigarrow l(L) = \int_{\alpha}^{\beta} \|f'(x)\| \, dx.$$

The norm of the positive part of the vector determines the "quasilength" of the curve by the formula:

$$\|x^+\| \rightsquigarrow \widehat{l}(L) = \int_{\alpha}^{\beta} \|(f'(x))^+\| \, dx.$$

Equation (14) implies the equality

$$l(\alpha, x^*) = \widehat{l}(\alpha, \beta).$$

2.3. Conditions of the Pareto Optimality

Let in problem (8) the functions f_i, $i = 1, \ldots, m$ be strictly concave on $[\alpha, \beta]$ and points x_i, $i = 1, \ldots, m$ be determined by the following conditions:

$$f_i(x_i) = \max_{x \in [\alpha, \beta]} f_i(x), \quad i = 1, \ldots, m \ . \tag{15}$$

Without loss of generality it can be assumed that

$$\alpha \leq x_1 \leq \ldots \leq x_m \leq \beta \ . \tag{16}$$

Lemma 2. *If condition* (16) *holds then* $\mathcal{E}(\mathcal{P}) = [x_1, x_m]$.

Proof.

Let $x^* \in [x_1, x_m]$. If $x < x^*$ then $f_m(x) < f_m(x^*)$ and if $x > x^*$ then $f_1(x) < f_1(x^*)$. Hence, there does not exist $x \in [\alpha, \beta]$ such that $f(x) \geq f(x^*)$, $f(x) \neq f(x^*)$. If $x^* \in [\alpha, x_1)$ then $f(x_1) > f(x^*)$, and if $x^* \in (x_m, \beta]$ then $f(x_m) > f(x^*)$. □

Theorem 1. *If the functions* f_i*,* $i = 1, \ldots, m$ *are strictly concaved then the solution* x^* *of equation* (14) *satisfies the axiom* A2.

Proof.

From the conditions of the theorem and (15), (16) it follows that $f'(x) > 0$ for $x \in [\alpha, x_1)$ and $f'(x) < 0$ for $x \in (x_m, \beta]$. Hence equation (14) is reduced to

$$\int_{x_1}^{x^*} \|f'(x)\| \, dx = \int_{x_1}^{x_m} \|(f'(x))^+\| \, dx.$$

The theorem follows from lemma 1. □

Corollary 1. *If* $m = 1$ *and* f *is strictly concave then the solution* x^* *of equation* (14) *is the maximum of the function* f.

Example 2. Consider problem (8), where $m = 2$, $\alpha = 0$, $f(x) = (f_1(x), f_2(x))$, $f(0) = (0, c)$ and assume that $f_1'(x) > 0$, $f_2'(x) < 0$. Obviously, the all points of this curve are Pareto-optimal. Equation (14) in l_1-norm is reduced to the equation

$$f_1(x^*) - f_2(x^*) = f_1(\beta) - c, \tag{17}$$

and in l_2-norm to the equation

$$\int_0^{x^*} \sqrt{(f_1'(x))^2 + (f_2'(x))^2} \, dx = f_1(\beta). \tag{18}$$

Let $f_1(x) = x$, $f_2(x) = c - \frac{1}{2}x^2$. From equation (17) we get $x^* = \sqrt{2\beta + 1} - 1$, and from equation (18) we get the following equation for x^*:

$$x^* \sqrt{1 + x^{*2}} + \ln\bigl(x^* + \sqrt{1 + x^{*2}}\,\bigr) = 2\beta.$$

Let us compare this solution with Nash's solution for bargaining problem (Nash, 1950). For this aim we must solve the following problem:

$$f_1(x) f_2(x) \to \max, \quad 0 \leq x \leq \beta.$$

Taking into account that $c = \beta^2/2$ we get $x^* = \beta/\sqrt{3}$.

Example 3. Let in problem (8) $X = [0, \pi/2]$, $f(x) = (\cos x, \sin x)$. We have $f'(x) = (-\sin x, \cos x)$. Equation (14) in the l_1-norm is reduced to

$$\int_0^x (\sin x' + \cos x')\, dx' = \int_0^{\pi/2} \cos x\, dx.$$

Hence, $x^* = \pi/4$, $f(x^*) = (1/\sqrt{2}, 1/\sqrt{2})$.

In the l_2-norm equation (14) is reduced to

$$\int_0^x dx = \int_0^{\pi/2} \cos x\, dx.$$

Hence, $x^* = 1$, $f(x^*) = (\cos 1, \sin 1)$.

For symmetric solution in the l_2-norm (see (5), (6), (7)) we have $x^* = \pi/4$, $f(x^*) = (1/\sqrt{2}, 1/\sqrt{2})$.

For Nash's solution we have the problem:

$$\sin x \cos x \to \max, \quad 0 \leq x \leq \frac{\pi}{2}.$$

Solving this problem we get $x^* = \pi/4$.

2.4. Lyapunov's function

Consider the following function:

$$F(x) = \int_\alpha^x (x - t)\|f'(t)\|\, dt - (x - \alpha) \int_\alpha^\beta \|(f'(t))^+\|\, dt. \tag{19}$$

The equation $F'(x^*) = 0$ is equation (14). Moreover,

$$F''(x) = \|f'(x)\| > 0.$$

Therefore, function (19) is Lyapunov's function for problem (3) (Lyapunov, 1907).

3. The general case

3.1. The basic equation

Let $X \subset \mathbb{R}^n$ be a convex compact set such that $\dim X = n$, $f : X \to \mathbb{R}^m$ is a set of criteria. Consider the problem

$$\mathcal{P} = (X, f). \tag{20}$$

For $x \in \mathbb{R}^n$ we shall write $x = (x_j, x_{-j})$, where x_j is the j-th component of x and x_{-j} is the set of other components. For $x \in X$ put

$$a_j(x_{-j}) = \min\{x_j \mid (x_j, x_{-j}) \in X\}, \quad b_j(x_{-j}) = \max\{x_j \mid (x_j, x_{-j}) \in X\}. \tag{21}$$

Note that in view of the made assumptions about the set X functions (21) are well defined, continuous and $a_j(x_{-j}) < b_j(x_{-j})$ a.e. $x \in X$.

Take an arbitrary point $\widehat{x} \in X$, choose j and fixe $\widehat{x}_{-j}$. Consider the problem with one variable

$$\mathcal{P}_j(\widehat{x}_{-j}) = ([a_j(\widehat{x}_{-j}),\, b_j(\widehat{x}_{-j})],\, f(\cdot, \widehat{x}_{-j})). \tag{22}$$

Let us require that the solution of problem (20) satisfies the following axiom:

A5. Axiom of equilibrium. The point $x^* = s(\mathcal{P}) \in X$ is *a solution* of the problem $\mathcal{P}$ if the following equalities hold:

$$x_j^* = s(\mathcal{P}_j(x_{-j}^*)), \quad j = 1, \dots, n, \tag{23}$$

i.e. x_j is a solution of problem (22), $j = 1, \dots, n$.

Remark 3. This definition expresses *the principle of the equilibrium:* equations (23) are analogous to Nash's equilibrium points in the non-coopertive games.

Applying equation (14) to each variable x_j, we obtain the following system for solution x^*:

$$\int_{a_j(x_{-j}^*)}^{x_j^*} \left\| \frac{\partial f}{\partial x_j}(x_j, x_{-j}^*) \right\| dx_j = \int_{a_j(x_{-j}^*)}^{b_j(x_{-j}^*)} \left\| \left(\frac{\partial f}{\partial x_j}(x_j, x_{-j}^*) \right)^+ \right\| dx_j,\ j = 1, \dots, n. \tag{24}$$

3.2. The existence theorem

Theorem 2. *If* X *is a convex compact set such that* $\dim X = n$ *and the functions* f *are continuously differentiable, then equations* (24) *have the solution* $x^* \in X$.

The proof is based on the following lemma. For $x \in X$ put

$$Q(x) = \{y \in \mathbb{R}^n \mid a_j(x_{-j}) \le y_j \le b_j(x_{-j}), \quad j = 1, \dots, n\}, \tag{25}$$

where $a_j(x_{-j})$ and $b_j(x_{-j})$, are determined by (21) and let

$$Q = \{x \in \mathbb{R}^n \mid a \le x \le b\} \tag{26}$$

be a minimal parallelepiped containing the set X.

Lemma 3. *Let* $\Phi : X \to 2^{\mathbb{R}^n}$ *be a multivalued map such that the following conditions hold:*

1. *The map* Φ *is upper semicontinuous.*
2. $\Phi(x)$ *is a convex set for all* $x \in X$.
3. $\Phi(x) \subset Q(x)$ *for all* $x \in X$.

Then the map Φ *has in* X *a fixed point* x^**:* $x^* \in \Phi(x^*)$.

Proof.

Let π be the projection operator on the set X. Determine the map $\Psi : Q \to 2^Q$, where Q is the parallelepiped (26), by the equality $\Psi = \Phi\pi$. Obviously, the map Ψ

satisfies the conditions 1 and 2 of the lemma and, hence, has a fixed point $x^* \in Q$: $x^* \in \Psi(x^*)$. Prove that $x^* \in X$.

Assume the converse: $x^* \notin X$. Then $c = \pi(x^*) - x^* \neq 0$. Consider the hyperplane

$$cx = c\pi(x^*).$$

We have $cx^* < c\pi(x^*)$ and for all $x \in X$

$$cx \geq c\pi(x^*). \tag{27}$$

Show that inequalities (27) hold for all $x \in Q(\pi(x^*))$. Since this contradicts the condition 3, the lemma will be proved.

Since $\pi(x^*)$ is the solution of the problem

$$\|x - x^*\| \to \min, \quad x \in X,$$

for all $x \in X$ the following inequalities hold

$$c(x - x^*) \geq \|c\|^2.$$

From this inequalities and also from (25) it follows that for $j = 1, \ldots, n$

$$\pi_j(x^*) = \begin{cases} a_j(\pi_{-j}(x^*)), & \text{if} \quad c_j > 0, \\ b_j(\pi_{-j}(x^*)), & \text{if} \quad c_j < 0, \\ a_j(\pi_{-j}(x^*)) \leq \pi_j(x^*) \leq b_j(\pi_{-j}(x^*)), & \text{if} \quad c_j = 0. \end{cases}$$

From this conditions the required proposition follows.

Proof of theorem 2. Consider the map $\Phi : X \to \mathbb{R}^n$, where $y = \Phi(x)$ is determined from the system

$$\int_{a_j(x_{-j})}^{y_j} \left\| \frac{\partial f}{\partial x_j}(x_j, x_{-j}) \right\| dx_j = \int_{a_j(x_{-j})}^{b_j(x_{-j})} \left\| \left(\frac{\partial f}{\partial x_j}(x_j, x_{-j}) \right)^+ \right\| dx_j, j = 1, \ldots, n.$$

Obviously, the map Φ satisfies the conditions of lemma 2 and, hence, has a fixed point x^* which is a solution of system (24).

3.3. The properties of the solution

Property 1. *If in problem* (20)

$$f(x) = \sum_{j=1}^{n} h_j(x_j), \tag{28}$$

then from equations (24) *it follows that the solution satisfies the following equations:*

$$\int_{a_j(x^*_{-j})}^{x^*_j} \left\| h'_j(x_j) \right\| dx_j = \int_{a_j(x^*_{-j})}^{b_j(x^*_{-j})} \left\| \left(h'_j(x_j) \right)^+ \right\| dx_j, \quad j = 1, \ldots, n. \tag{29}$$

Property 2. *If in problem* (20) *for some* $j = 1, \dots, n$

$$\frac{\partial f}{\partial x_j}(x) \geq 0, \tag{30}$$

then from equations (24) *it follows that*

$$x_j^* = b_j(x_{-j}^*). \tag{31}$$

Corollary 2. *If inequalities* (30) *are satisfied for all* $j = 1, \dots, n$, *then from* (31) *it follows that*

$$x \in \mathcal{E}(X),$$

and every point of $\mathcal{E}(X)$ *is the solution of equations* (24).

3.4. The problem with linear criteria

Let in problem (20)

$$f(x) = Cx = \sum_{j=1}^{n} c^j x_j, \tag{32}$$

where C is a $m \times n$ matrix and $c^j = (c_{1j,\dots,c_{mj}})$, $j = 1, \dots, n$ are its columns. Since f has form (28) equations (29) are reduced to the following ones:

$$x_j = a_j(x_{-j})(1 - \varphi(c^j)) + b_j(x_{-j})\varphi(c^j), \quad j = 1, \dots, n. \tag{33}$$

If X is parallelepiped (26) then

$$x_j = a_j(1 - \varphi(c^j)) + b_j\varphi(c^j), \quad j = 1, \dots, n. \tag{34}$$

If X is a unique cube, i.e. in (26) $a_j = 0$, $b_j = 1$, $j = 1, \dots, n$, then $x_j = \varphi(c^j)$, $j = 1, \dots, n$, and

$$v = f(x) = \sum_{j=1}^{n} c^j x_j = \sum_{j=1}^{n} s(c^j),$$

i.e. value of the problem is the sum of the values for the consisting vectors.

Example 4. Let $X \subset \mathbb{R}^2$ be a rectangular with the vertex (1,0), (0,1), (2,3), (3,2),

$$f_1(x) = x_1 + x_2, \quad f_2(x) = -x_1 - x_2.$$

Find the solution in the l_1-norm. Since $c^1 = c^2 = (1, -1)$, from (1) it follows that $\varphi(c^1) = \varphi(c^2) = 1/2$, and by (33) we have

$$x_1 = \frac{1}{2}(a_1(x_2) + b_1(x_2)), \quad x_2 = \frac{1}{2}(a_2(x_1) + b_2(x_1)).$$

Obviously, the solutions of this equations are the points of the segment L with the ends (1,1) and (2,2). Put $y = x_1 + x_2$, for $x = (x_1, x_2) \in L$. Then $2 \leq y \leq 4$.

Put $\widehat{f}_1(y) = y$, $\widehat{f}_2(y) = -y$ and solve the problem $([2,4], \widehat{f})$ by (34) (or merely by (4) for the segment [(2,-2),(4,-4)]). We get $y = 3$, $x_1 = x_2 = 3/2$, $v = (3, -3)$.

Example 5. Let in problem (20)

$$X = \{x \in \mathbb{R}^n \mid \sum_{j=1}^{n} \frac{x_j}{d_j} \leq 1\},$$

where $d_j > 0$, $j = 1, \ldots, n$, and f are criteia (32). By (21) we have

$$a_j(x_{-j}) = 0, \quad b_j(x_{-j}) = d_j\Big(1 - \sum_{k \neq j} \frac{x_k}{d_k}\Big), \quad j = 1, \ldots, n.$$

From equations (33) we get

$$x_j = \varphi(c^j) d_j \Big(1 - \sum_{k \neq j} \frac{x_k}{d_k}\Big), \quad j = 1, \ldots, n. \tag{35}$$

Put

$$A = 1 - \sum_{k=1}^{n} \frac{x_k}{d_k}.$$

Then equations (35) may be written in the following form

$$(1 - \varphi(c^j)) \frac{x_j}{d_j} = \varphi(c^j) A, \quad j = 1, \ldots, n. \tag{36}$$

Case 1. There exists such j, that $\varphi(c^j) = 1$. In this case, obviously, $A = 0$, e.g.

$$\sum_{j=1}^{n} \frac{x_j}{d_j} = 1.$$

Let $\varphi(c^j) = 1$ for $j = 1, \ldots, k$ and $\varphi(c^j) < 1$ for $j = k+1, \ldots, n$, where $k \geq 1$. Then from equations (36) it follows that

$$x_j = 0, \quad \text{for} \quad j = k+1, \ldots, n \tag{37}$$

and

$$\sum_{j=1}^{k} \frac{x_j}{d_j} = 1. \tag{38}$$

Let $k > 1$. We get a new problem $\widehat{\mathcal{P}} = (\hat{X}, \hat{f})$, where in view of (37), (38)

$$\hat{X} = \left\{ x \in \mathbb{R}^k \left| \sum_{j=1}^{k} \frac{x_j}{d_j} = 1 \right. \right\}, \quad \hat{f}(x) = \sum_{j=1}^{k} c^j x_j. \tag{39}$$

1.1. Lagrange's method. Introduce Lagrange's function for problem (39):

$$l(x,\lambda)=\sum_{j=1}^{k}\Big(c^j-e\frac{\lambda}{d_j}\Big)x_j+\lambda e,$$

where $e\in\mathbb{R}^k$, $e=(1,\dots,1)$. Put

$$K=\{x\in\mathbb{R}^k \mid 0\le x_j\le d_j,\quad j=1,\dots,k\}$$

and consider the problem $(K,l(\cdot,\lambda))$, where λ is the parameter. Solving this problem by (34), we get

$$x_j=d_j\varphi\Big(c^j-e\frac{\lambda}{d_j}\Big). \tag{40}$$

By property F4 of the function φ equation (40) may be written in the following form:

$$x_j=d_j\varphi(c^jd_j-\lambda e), \tag{41}$$

where λ is determined from condition (38):

$$\sum_{j=1}^{k}\varphi(c^jd_j-\lambda e)=1. \tag{42}$$

From lemma 1 it follows that equation (42) has the unique solution.

1.2. The method of excluding of the variable. In the problem $\widehat{\mathcal{P}}$ (39) exclude the variable x_k. We have

$$x_k=d_k\left(1-\sum_{j=1}^{k-1}\frac{x_j}{d_j}\right). \tag{43}$$

Substituting in (39) for x_k (43) we get the problem $\widetilde{\mathcal{P}}=\widetilde{X},\widetilde{f}$, where

$$\widetilde{X}=\left\{x\in\mathbb{R}^{k-1}\,\middle|\,\sum_{j=1}^{k-1}\frac{x_j}{d_j}\le 1\right\}, \tag{44}$$

$$\widetilde{f}=\sum_{j=1}^{k-1}\widetilde{c^j}x_j+d_kc^k, \tag{45}$$

and

$$\widetilde{c^j}=c^j-\frac{d_k}{d_j}c^k,\quad j=1,\dots,k-1. \tag{46}$$

From (21) we have

$$a_j(x_{-j})=0,\quad b_j(x_{-j})=d_j\left(1-\sum_{l\ne j}\frac{x_l}{d_l}\right),\quad j=1,\dots,k-1. \tag{47}$$

If $\varphi(\widetilde{c}^j) < 1$ for all $j = 1, \ldots, k-1$, then we get case 2 for the problem $\widetilde{\mathcal{P}}$ ((44), (45), (46) (47)) (with replacing $k-1$ by n and $\widetilde{c}^j$ by c^j).

If there exists j such that $\varphi(\widetilde{c}^j) = 1$ then the process of excluding of the variable is repeated.

Case 2. For all $j = 1, \ldots, n$ $\varphi(c^j) < 1$. From (36) we get

$$x_j = \frac{\varphi(c^j)d_j}{1-\varphi(c^j)}A, \quad j = 1, \ldots, n,$$

where A is determined from the equation

$$A\Big(1 + \sum_{j=1}^{n} \frac{\varphi(c^j)}{1-\varphi(c^j)}\Big) = 1.$$

Note that the vector $(\frac{x_1}{d_1}, \ldots, \frac{x_n}{d_n}, A) \in \mathbb{R}^{n+1}$ is the optimal strategy in the diagonal matrix game

$$\text{diag}\left\{\frac{1-\varphi(c^1)}{\varphi(c^1)}, \ldots, \frac{1-\varphi(c^n)}{\varphi(c^n)}, 1\right\},$$

and A is its value [Vorob'ev, 1985].

3.5. The problem with the concave criteria

Let in problem (20) $X = \mathbb{R}^n$ and the criteria satisfy following condition:

Condition C. The functions f_i, $i = 1, \ldots, m$, are strictly concave and have their maxima on $\mathbb{R}^n$.

Lemma 4. *If the functions f_i, $i = 1, \ldots, m$ satisfy condition C, then for $p \in \mathbb{R}^m_+$, $p \neq 0$, the function $g = pf = \sum_{i=1}^m p_i f_i$ also satisfies condition C.*

Proof.

It is sufficient to prove the lemma for $g = f_1 + f_2$. Obviously g is strictly concave. Show that g has the maximum. Without loss of generality it can be assumed that $f_1(x) \leq 0$. Let $S^i_\beta = \{x \in \mathbb{R}^n \mid f_i(x) \geq \beta\}$, $i = 1, 2$, and $S_\beta = \{x \in \mathbb{R}^n \mid g(x) \geq \beta\}$ be Lebesgue's sets, accordingly, of the functions f_i, $1 = 1, 2$ and g. From condition C it follows that the sets S^i_β, $i = 1, 2$, are bounded and non-empty for $\beta \leq \min\{\max f_1, \max f_2\}$. Since $f_1(x) \leq 0$, $g(x) \leq f_2(x)$ and $S_\beta \subset S^2_\beta$. Hence, S_β is bounded and non-empty and g has the maximum. □

With

$$\Delta_m = \{p \in \mathbb{R}^m_+ \mid \sum_{i=1}^{m} p_i = 1\}$$

denote a simplex in $\mathbb{R}^m$.

Let $\mathcal{P} = (\mathbb{R}^n, f)$ be a problem and condition C holds. It is known [Podinovsky and Noghin, 1982] that for $x^0 \in \mathcal{E}(\mathcal{P})$ it is necessary and sufficient, that for some $p \in \Delta_m$ x^0 is the solution of the following problem:

$$pf(x) \;\rightarrow\; \max, \quad x \in \mathbb{R}^n. \tag{48}$$

From condition C and lemma 4 it follows that problem (48) has the unique solution for every $p \in \Delta_m$. Denote this solution with $x(p)$. Then $\mathcal{E}(\mathcal{P}) = x(\Delta_m)$. Put $g(p) = f(x(p))$ and consider the problem

$$\widehat{\mathcal{P}} = (\Delta_m, g). \tag{49}$$

Introduce Lagrange's function for problem (49):

$$l(p, \lambda) = g(p) - e\lambda \left(\sum_{i=1}^{m} p_i - 1 \right), \tag{50}$$

where $e \in \mathbb{R}^m$, $e = (1, \dots, 1)$.

Consider, finally, the third problem:

$$\mathcal{P}^l = (K, l), \tag{51}$$

where K is the unit cube and l is function (50). In problem (51) λ is the parameter. Solving problem (51) by (24) we get the solution $p(\lambda)$ as the function of λ. The λ is found from condition $p(\lambda) \in \Delta_m$, that is

$$\sum_{i=1}^{m} p_i(\lambda) = 1. \tag{52}$$

Remark 4. The solution of problem (49), (51) at the same time gives us the weights p_i, $i = 1, \dots, m$, of the criteria.

Remark 5. Consider two problems: (49) and $\mathcal{P}^0 = (\mathcal{E}(\mathcal{P}), f)$. This problems are not equivalent: if p^* is the solution of the problem (49) and x^* is the solution of the problem $\mathcal{P}^0$ then $x^* \neq x(p^*)$.

Example 6. Let

$$f_i(x) = -\sum_{j=1}^{n} (x_j - d_j^i)^2, \quad i = 1, \dots, m.$$

Solving problem (48) we get

$$x(p) = \sum_{i=1}^{m} p_i d^i,$$

where $d^i = (d^i_1, \ldots, d^i_n)$, $i = 1, \ldots, m$. This means that

$$\mathcal{E}(\mathcal{P}) = \mathrm{conv}\{d^1, \ldots, d^m\}.$$

Put $m = n + 1$,

$$f_0(x) = -\sum_{j=1}^{n} x_j^2, \quad f_i(x) = -\sum_{j \neq i} x_j^2 - (x_i - a_i)^2, \quad a_i > 0, \quad i = 1, \ldots, n.$$

We have

$$\mathcal{E}(\mathcal{P}) = \left\{ x \in \mathbb{R}^n_+ \,\middle|\, \sum_{j=1}^{n} \frac{x_j}{a_j} \leq 1 \right\}.$$

Equalities (21) are reduced to

$$a_j(x_{-j}) = 0, \quad b_j(x_{-j}) = a_j \left(1 - \sum_{k \neq j} \frac{x_k}{b_k} \right), \quad j = 1, \ldots, n.$$

Put

$$A = 1 - \sum_{j=1}^{n} \frac{x_j}{a_j.}.$$

Then

$$b_j(x_{-j}) = a_j A + x_j, \quad j = 1, \ldots, n.$$

Since

$$\frac{1}{2} \frac{\partial f}{\partial x_j} = (-x_j, \ldots, \overbrace{-x_j + a_j}^{j}, \ldots, -x_j), \quad j = 1, \ldots, n,$$

equations (24) are reduced in l_1-norm to the following equations:

$$\int_0^{x_j} (a_j + (n-1)x'_j)\, dx'_j = \int_0^{a_j A + x_j} (a_j - x'_j)\, dx'_j, \quad j = 1, \ldots, n.$$

From this equations we find

$$x_j = \frac{\sqrt{n^2 - n + 1} - 1}{n(n-1)} a_j, \quad j = 1, \ldots, n.$$

3.6. The examples

Example 7. Let us find a solution for the Pareto optimal part of the sphere. Consider problem (20), where

$$X = \{x \in \mathbb{R}^n_+ \mid \sum_{j=1}^{n} x_j^2 = 1\}, \quad f_i(x) = x_i, \quad i = 1, \ldots, n. \tag{53}$$

1. The method of the excluding of the variable. We have

$$x_n = \sqrt{1 - \sum_{j=1}^{n-1} x_j^2}.$$

Consider the problem $\widehat{\mathcal{P}} = (\widehat{X}, \widehat{f})$, where

$$\widehat{X} = \{x \in \mathbb{R}_+^n \mid \sum_{j=1}^{n-1} x_j^2 \leq 1\}, \quad \widehat{f}_i(x) = x_i, \quad i = 1, \ldots, n-1, \quad \widehat{f}_n = \sqrt{1 - \sum_{j=1}^{n-1} x_j^2}.$$

From (21) we have

$$a_j(x_{-j}) = 0, \quad b_j(x_{-j}) = \sqrt{1 - \sum_{k \neq j} x_k^2} = \sqrt{\widehat{f}_n^2(x) + x_j^2}, \quad j = 1, \ldots, n-1. \tag{54}$$

Hence, we get

$$\frac{\partial \widehat{f}_i}{\partial x_j} = \begin{cases} 0, & \text{if} \quad i \neq j, n, \\ 1, & \text{if} \quad i = j, \\ -\dfrac{x_j}{\widehat{f}_n(x)}, & \text{if} \quad i = n, \end{cases} \qquad i = 1, \ldots, n, \qquad j = 1, \ldots, n-1.$$

Note that

$$\widehat{f}_n(0, x_{-j}) = b_j(x_{-j}). \tag{55}$$

In the l_1-norm equations (24) are reduced to the following equations

$$\int_0^{x_j} \left(1 + \left| \frac{\partial \widehat{f}_n}{\partial x_j}(x_j', x_{-j}) \right| \right) dx_j' = \int_0^{b_j(x_{-j})} dx_j,$$

or

$$x_j - \widehat{f}_n(x_j, x_{-j}) + \widehat{f}_n(0, x_{-j}) = b_j(x_j), \quad j = 1, \ldots, n-1.$$

In view of (55)

$$x_j = \widehat{f}_n(x), \quad j = 1, \ldots, n-1,$$

and finally

$$x_1 = \ldots = x_n = \frac{1}{\sqrt{n}}. \tag{56}$$

In the l_2-norm equations (24) have the following form

$$\int_0^{x_j} \sqrt{1 + \frac{x_j'^2}{\widehat{f}_n^2(x_j, x_{-j})}}\, dx_j' = \int_0^{b_j(x_{-j})} dx_j.$$

In view of (54) we have

$$\int_0^{x_j} \sqrt{1+\frac{x_j'^2}{\widehat{f}_n^2(x_j', x_{-j})}}\, dx_j' = \int_0^{x_j} \frac{b_j(x_{-j})}{\sqrt{b_j^2(x_{-j}) - x_j'^2}}\, dx_j' = b_j(x_{-j}) \arcsin \frac{x_j}{b_j(x_{-j})}.$$

Hence,

$$\arcsin \frac{x_j}{b_j(x_{-j})} = 1, \quad j = 1, \ldots, n-1.$$

From this equations we find

$$x_j = \frac{\mathrm{tg} 1}{\sqrt{1+(n-1)\mathrm{tg}^2 1}}, \quad j = 1, \ldots, n-1, \quad x_n = \frac{1}{\sqrt{1+(n-1)\mathrm{tg}^2 1}}.$$

Let us find the symmetric solution in l_2-norm (see remark 2 eq. (5), (6), (7)). We have

$$\int_0^{x_j} \sqrt{1+\frac{x_j'^2}{\widehat{f}_n^2(x_j, x_{-j})}}\, dx_j' =$$

$$= \frac{1}{2} \int_0^{b_j(x_{-j})} \left(\sqrt{1+\frac{x_j^2}{\widehat{f}_n^2(x_j, x_{-j})}} + 1 - \left| \frac{\partial \widehat{f}_n}{\partial x_j}(x_j, x_{-j}) \right| \right) dx_j.$$

Integrating we find

$$x_j = b_j(x_{-j}) \sin \frac{\pi}{8}, \quad j = 1, \ldots, n-1,$$

or

$$x_j = \widehat{f}_n(x) \mathrm{tg} \frac{\pi}{8} = \widehat{f_n}(x)(\sqrt{2}-1), \quad j = 1, \ldots, n-1.$$

Finally,

$$x_j = \frac{\sqrt{2}-1}{\sqrt{(n-1)(3-2\sqrt{2})+1}}, \quad j = 1, \ldots, n-1, \quad x_n = \frac{1}{\sqrt{(n-1)(3-2\sqrt{2})+1}}.$$

2. Lagrange's method. Introduce Lagrange's function for problem (53):

$$l(x, \lambda) = f(x) - \frac{1}{2} e\lambda (\sum_{j=1}^{n} x_j^2 - 1),$$

and consider the problem $\widehat{\mathcal{P}} = (K, l)$. Obviously, the problem $\widehat{\mathcal{P}}$ satisfies property 1 (subsection 3.3) and by the symmetry of equations (29) the solution is given by (56).

Example 8. Let us find the solution for the Pareto optimal part of the sphere determined paramertically. Consider problem (20) where

$$X = \{x = (x_1, x_2) \in \mathbb{R}_+^2 \mid 0 \le x_1, x_2 \le \frac{\pi}{2}\},$$

$$f_1(x) = \cos x_1 \cos x_2, \quad f_2(x) = \cos x_1 \sin x_2, \quad f_3(x) = \sin x_1.$$

In the l_1-norm equations (24) have the following form:

$$\int_0^{x_1} (\sin x_1' (\cos x_2 + \sin x_2) + \cos x_1')\, dx_1' = \int_0^{\pi/2} \cos x_1\, dx_1,$$

$$\int_0^{x_2} \cos x_1 (\sin x_2' + \cos x_2')\, dx_2' = \int_0^{\pi/2} \cos x_1 \cos x_2\, dx_2.$$

From the second equation it follows that $\sin x_2 = \cos x_2 = 1/\sqrt{2}$ and $f_1(x) = f_2(x) = \dfrac{\cos x_1}{\sqrt{2}}$. From the first equation we get

$$\sin x_1 - \sqrt{2} \cos x_1 = 1 - \sqrt{2}.$$

Obviously, this equation has a unique solution in $[0, \pi/2]$.

In the l_2-norm equations (24) take the following form:

$$\int_0^{x_1} dx_1' = \int_0^{\pi/2} \cos x_1\, dx_1, \qquad \int_0^{x_2} \cos x_1\, dx_2' = \int_0^{\pi/2} \cos x_1 \cos x_2\, dx_2.$$

From this equations we have

$$x_1 = x_2 = 1, \quad f_1(x) = \cos^2 1, \quad f_2(x) = \sin 1 \cos 1, \quad f_3(x) = \sin 1.$$

For symmetric solution (see remark 2 eq. (6), (7), (8)) in the l_2-norm equations (24) are reduced to the following equations:

$$\int_0^{x_1} dx_1' = \frac{1}{2} \int_0^{\pi/2} (1 + \cos x_1 - \sin x_1)\, dx_1,$$

$$\int_0^{x_2} \cos x_1\, dx_2' = \frac{1}{2} \int_0^{\pi/2} (\cos x_1 + \cos x_1 \cos x_2 - \cos x_1 \sin x_2)\, dx_2.$$

From this equations we have

$$x_1 = x_2 = \frac{\pi}{4}, \quad f_1(x) = f_2(x) = \frac{1}{2}, \quad f_3(x) = \frac{1}{\sqrt{2}}.$$

3.7. The non-cooperative game

In the subsection 2.4 Liapunov's function for the problem with one variable was been constructed. In the general case there may be constructed the non-cooperative game (see [Vorob'ev, 1985]) with n players that is equivalent to the multicriteria problem.

By analogy with function (19) introduce the following functions

$$F_j(x) = \int_{a_j(x_{-j})}^{x_j} (x_j - t) \left\| \frac{\partial f}{\partial x_j}(t, x_{-j}) \right\| dt- \tag{57}$$

$$-(x_j - a_j(x_{-j})) \int_{a_j(x_{-j})}^{b_j(x_{-j})} \left\| \left(\frac{\partial f}{\partial x_j}(t, x_{-j}) \right)^+ \right\| dt, \quad j = 1, \dots, n$$

and consider the following non-cooperative game:

$$\Gamma = \langle J,\ X,\ \{F_j \mid j \in J\} \rangle, \tag{58}$$

where $J = \{1, \dots, n\}$ is the set of the players, X is the set of the feasible vectors of the strategies, and F_j is the payoff function of the player $j \in J$ determined by (57).

Note that in game (58) the players want to minimize their payoffs.

Theorem 3. 1. *Game* (58) *with payoff function* (57) *has an equilibrium point in* X.

2. *The set of the equilibrium points in game* (58) *coincides with the set of the solutions of system* (24).

Proof. 1. Since

$$\frac{\partial^2 F_j}{\partial x_j^2}(x) = \left\| \frac{\partial f}{\partial x_j}(x) \right\| \geq 0, \quad j = 1, \dots, n,$$

the functions F_j are convex in x_j for $j = 1, \dots, n$. Therefore,

$$\psi_j(x_{-j}) = \operatorname{argmin}_{x_j \in [a_j(x_{-j}), b_j(x_{-j})]} F_j(x_j, x_{-j}), \quad j = 1, \dots, n$$

are convex sets. Determine the map $\Phi : X \to 2^{\mathbb{R}^n}$ by the equality:

$$\Phi(x) = \prod_{j=1}^{n} \psi_j(x_{-j}).$$

Obviously, the map Φ satisfies the conditions of lemma 3 and, hence, has in X a fixed point $x^* = \Phi(x^*)$. By the definition Φ we have

$$F_j(x_j^*, x_{-j}^*) = \min_{x_j : (x_j, x_{-j}^*) \in X} F_j(x_j, x_{-j}^*), \quad j = 1, \dots, n.$$

Hence, x^* is the equilibrium point in game (58).

2. Since F_j, $j = 1, \dots, n$ are convex in x_j, the equilibrium point x^* satisfies the following equations:

$$\frac{\partial F_j}{\partial x_j}(x^*) = 0, \quad j = 1, \dots, n.$$

This system coincides with system (24).

4. Conclusion

Let us note some features of the proposed solution.

1. The solution of equations (24) may be not Pareto optimal, as it is seen from examples 4, 5 (subsection 3.4). Here two approaches are possible: a) the determining of the conditions for the Pareto optimality of the solution; b) the restriction of the consideration only the Pareto optimal problems.

2. The solution of equation (24) may be not unique, as it is seen from corollary 1 (property 2, subsection 3.3) and examples 4, 5 (subsection 3.4). In this case it is possible to iterate equations (24), as in example 4, 5.

3. The solution of equations (24) depends on the selected norm and also on the method of the solution: excluding the variable or Lagrange's method.

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Cooperative Incentive Equilibrium for a Bioresource Management Problem

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Keywords: dynamic games, bioresource management problem, cooperative equilibrium, incentive equilibrium

Introduction

We consider here a dynamic game model related with the bioresource management problem (fish catching). The center (referee) shares a reservoir between the competitors. The players (countries) which harvest the fish stock are the participants of this game. Each player is independent decision-maker, being guided by maximization of the profit of fish sale. In traditional statement [Basar, 1982], [Clark, 1993], [Ehtamo, 1993], [Hamalainen, 1984] the center's objective is catch regulation by introduction quotas on fishing. In the series of papers [Mazalov, 2004], [Mazalov, 2005], [Mazalov, 2006] it was developed a new approach where the policy of the center is to determine the optimal share of the aquatic environment where fishing is prohibited. In the papers we considered different dynamic game models with reserved territory, that take into account a population's distribution in the reservoir, age distribution of the population, migration, different types of players' profits division. In this paper we use this developed approach for bioresource sharing problem for two players. We investigate a new type of equilibrium – cooperative incentive equilibrium. For finite horizon the Pontryagin's maximal principle is applied to determine the equilibrium. The incentive equilibriums are constructed in the case when the players punish each other for a deviation from the cooperative equilibrium, and in the case when the center punishes them for a deviation. The numerical modelling and the results comparison are given for two types of the punishment.

1. The model of bioresource sharing

The bioresource management problem (fish catching) is considered. Let us divide the water area into two parts: s and $1-s$, where two countries exploit the fish stock. The center (referee) shares the reservoir. The players (countries) which exploit the fish stock during T time periods on their territory are the participants of this game.

The dynamics of the fishery is described by the equation

$$\begin{aligned} x'(t) &= F(x(t)) - q_1E_1(t)(1-s)x(t) - q_2E_2(t)sx(t), \\ & \quad 0 \le t \le T,\ x(0) = x_0\,, \end{aligned} \tag{1}$$

where $x(t) \ge 0$ – size of the population at a time t; F – natural growth function of the population; $E_1(t), E_2(t) \ge 0$ – countries' fishing efforts measured as the number of vessels involved in fishing at time t and $q_1, q_2 > 0$ – catchability coefficients related to the unit fishing effort of the country.

We assume that E_1, E_2 belong to decision sets D_1, D_2. Let $D_1 = D_2 = [0, \infty)$.

Assume that population evolves in accordance with Ferhulst model of the form:

$$F(x) = rx(1 - \frac{x}{K})\,,$$

where r – the intrinsic growth rate, and K – maximal natural object capacity.

The players' net revenues over a fixed time period $[0, T]$ are

$$\begin{aligned} J_1 &= g_1(x(T)) + \int_0^T [\tfrac{1}{2}a_1E_1^2(t)(1-s)^2x^2(t) + b_1E_1(t)(1-s)x(t) + \\ &+ c_1E_1(t)E_2(t)(1-s)sx^2(t) + d_1E_2(t)sx(t) + \tfrac{1}{2}l_1E_2^2(t)s^2x^2(t)]dt, \\ J_2 &= g_2(x(T)) + \int_0^T [\tfrac{1}{2}a_2E_2^2(t)s^2x^2(t) + b_2E_2(t)sx(t) + \\ &+ c_2E_1(t)E_2(t)(1-s)sx^2(t) + d_2E_1(t)(1-s)x(t) + \\ &+ \tfrac{1}{2}l_2E_1^2(t)(1-s)^2x^2(t)]dt, \end{aligned} \tag{2}$$

where the coefficients a_i, b_i, c_i, d_i, l_i correspond to the concrete problem and include discount rate as $e^{-\rho_i t}$ ($a_i < 0$, $l_i < 0$, $i = 1, 2$).

Functions $g_i(x)$ describe the salvage value of the stock at time T and $g_i'(x) \ge 0$, $g_i''(x) \le 0$, $i = 1$, 2.

We investigate different types of the equilibrium for this model.

1.1. Cooperative equilibrium

We have to solve the following optimization problem to define the cooperative equilibrium.

$$\begin{cases} \max(\mu_1J_1(E_1(t), E_2(t)) + \mu_2J_2(E_1(t), E_2(t))\,, \\ \qquad \text{where } x(t) \text{ is defined in (1)}\,. \end{cases} \tag{3}$$

The solution of this problem is given in the next theorem.

Theorem 1. *The cooperative equilibrium of the problem (1)–(2) is*

$$E_1^d = \frac{u_1^d}{(1-s)x}; \quad E_2^d = \frac{u_2^d}{sx},$$

where

$$u_1^d = \frac{(\mu_1 c_1 + \mu_2 c_2)(\mu_2 b_2 + \mu_1 d_1 - \lambda q_2) - (\mu_2 a_2 + \mu_1 l_1)(\mu_1 b_1 + \mu_2 d_2 - \lambda q_1)}{((\mu_1 a_1 + \mu_2 l_2)(\mu_2 a_2 + \mu_1 l_1) - (\mu_1 c_1 + \mu_2 c_2)^2)},$$

$$u_2^d = \frac{(\mu_1 c_1 + \mu_2 c_2)(\mu_1 b_1 + \mu_2 d_2 - \lambda q_1) - (\mu_1 a_1 + \mu_2 l_2)(\mu_2 b_2 + \mu_1 d_1 - \lambda q_2)}{((\mu_1 a_1 + \mu_2 l_2)(\mu_2 a_2 + \mu_1 l_1) - (\mu_1 c_1 + \mu_2 c_2)^2)}.$$

Proof. Let's use the maximal principle [Pontryagin, 1976]. The Hamilton function is

$$\begin{aligned} &H(E_1, E_2, s, x) = \mu_1(\tfrac{1}{2} a_1 E_1^2 (1-s)^2 x^2 + b_1 E_1 (1-s) x + c_1 E_1 E_2 (1-s) s x^2 + \\ &+ d_1 E_2 s x + \tfrac{1}{2} l_1 E_2^2 s^2 x^2) + \mu_2(\tfrac{1}{2} a_2 E_2^2 s^2 x^2 + b_2 E_2 s x + c_2 E_1 E_2 (1-s) s x^2 + \\ &+ d_2 E_1 (1-s) x + \tfrac{1}{2} l_2 E_1^2 (1-s)^2 x^2) + \lambda(r x - \frac{r x^2}{K} - q_1 E_1 (1-s) x - q_2 E_2 s x). \end{aligned}$$

Solving the system

$$\begin{cases} \dfrac{\partial H}{\partial E_1} = 0, \\ \dfrac{\partial H}{\partial E_2} = 0 \end{cases}$$

we receive that the optimal controls are of the form

$$E_1^d = \frac{(\mu_1 c_1 + \mu_2 c_2)(\mu_2 b_2 + \mu_1 d_1 - \lambda q_2) - (\mu_2 a_2 + \mu_1 l_1)(\mu_1 b_1 + \mu_2 d_2 - \lambda q_1)}{((\mu_1 a_1 + \mu_2 l_2)(\mu_2 a_2 + \mu_1 l_1) - (\mu_1 c_1 + \mu_2 c_2)^2)(1-s)x},$$

$$E_2^d = \frac{(\mu_1 c_1 + \mu_2 c_2)(\mu_1 b_1 + \mu_2 d_2 - \lambda q_1) - (\mu_1 a_1 + \mu_2 l_2)(\mu_2 b_2 + \mu_1 d_1 - \lambda q_2)}{((\mu_1 a_1 + \mu_2 l_2)(\mu_2 a_2 + \mu_1 l_1) - (\mu_1 c_1 + \mu_2 c_2)^2)sx},$$

where $\lambda(t)$ – conjugate variable satisfying the equation

$$\lambda'(t) = -\frac{\partial H}{\partial x}, \;\; \lambda(T) = \mu_1 g_1'(x(T)) + \mu_2 g_2'(x(T)).$$

We define

$$u_1^d = \frac{(\mu_1 c_1 + \mu_2 c_2)(\mu_2 b_2 + \mu_1 d_1 - \lambda q_2) - (\mu_2 a_2 + \mu_1 l_1)(\mu_1 b_1 + \mu_2 d_2 - \lambda q_1)}{((\mu_1 a_1 + \mu_2 l_2)(\mu_2 a_2 + \mu_1 l_1) - (\mu_1 c_1 + \mu_2 c_2)^2)},$$

$$u_2^d = \frac{(\mu_1 c_1 + \mu_2 c_2)(\mu_1 b_1 + \mu_2 d_2 - \lambda q_1) - (\mu_1 a_1 + \mu_2 l_2)(\mu_2 b_2 + \mu_1 d_1 - \lambda q_2)}{((\mu_1 a_1 + \mu_2 l_2)(\mu_2 a_2 + \mu_1 l_1) - (\mu_1 c_1 + \mu_2 c_2)^2)}.$$

So, we receive the optimal controls. Let us prove their optimality.

Substituting the expressions $E_1^d(t)$ and $E_2^d(t)$ in $\frac{\partial H}{\partial x}$, we obtain that the equation for the conjugate variable takes the form

$$\lambda'(t) = -\lambda(t)(r - \frac{2rx(t)}{K}) , \ \ \lambda(T) = \mu_1 g_1'(x(T)) + \mu_2 g_2'(x(T)).$$

Then

$$\lambda(t) = (\mu_1 g_1'(x(T)) + \mu_2 g_2'(x(T)))e^{\int\limits_t^T r - 2rx(\tau)/K \, d\tau} > 0.$$

Let us denote

$$H^0(x, \lambda, t) = \max_{E_1, E_2 \in R} H(x, E_1, E_2, \lambda, t).$$

Substituting $E_1^d(t)$ and $E_2^d(t)$ and simplifying we obtain

$$H^0(x, \lambda, t) = \frac{(\mu_2 b_2 - \lambda q_2)^2}{2a_2\mu_2} + \frac{(\mu_1 b_1 - \lambda q_1)^2}{2a_1\mu_1} + \lambda F(x); \quad H_{xx}^0(x, \lambda, t) = -\frac{2r}{K}\lambda.$$

With $\lambda(t) > 0$ it yields that H^0 – concave. Using concavity and definition of H^0 we obtain

$$H(x, \lambda, t) \le H^0(x^d, \lambda, t) - \lambda'(t)(x(t) - x^d(t)).$$

Integrating and simplifying it is not difficult to show that

$$\mu_1 J_1(E_1(t), E_2(t)) + \mu_2 J_2(E_1(t), E_2(t)) \le \mu_1 J_1(E_1^d(t), E_2^d(t)) + \mu_2 J_2(E_1^d(t), E_2^d(t)).$$

So we proved that $E_1^d(t)$ and $E_2^d(t)$ – solution of the problem (3).

1.2. Incentive equilibrium

Following [Ehtamo, 1993] we assume that the strategy of the player i is a causal mapping $\gamma_i : D_j \to D_i$ ($E_i \in D_i$). Let give the definition of the incentive equilibrium.

Definition 1. *A strategy pair* (γ_1, γ_2) *is called the incentive equilibrium at* (E_1^d, E_2^d)*if*

$$\begin{aligned} &E_1^d = \gamma_1(E_2^d) , \ \ E_2^d = \gamma_2(E_1^d), \\ &J_1(E_1^d, E_2^d) \ge J_1(E_1, \gamma_2(E_1)) , \ \ \forall E_1 \in D_1, \\ &J_2(E_1^d, E_2^d) \ge J_2(\gamma_1(E_2), E_2) , \ \ \forall E_2 \in D_2. \end{aligned}$$

We change the scheme of the punishment for a deviation from the cooperative equilibrium. We assume that the center punishes the players for a deviation from the equilibrium point, but not themselves, as it was in [Ehtamo, 1993]. Assume then, if the first player deviates the center increases s, but if the second player deviates – decreases s.

Let consider the second player's deviation $E_2 = E_2^d + \Delta$. We find the center's strategy in the following form $s^* = s^d - \eta(E_2 - E_2^d)$. Then the first player's punishment strategy is

$$\gamma_1(E_2) = \frac{u_1^d}{(1 - s^d + \eta(E_2 - E_2^d))x}\,.$$

Next we should determine the coefficient η. For that we solve the next problem

$$\begin{cases} & & \max(J_2(\gamma_1(E_2), E_2)), \\ x'(t) & = & rx(t)(1 - x(t)/K) - q_1\gamma_1(E_2(t)) \\ & & (1 - s^d + \eta(E_2 - E_2^d))x - q_2E_2(t)(s^d - \eta(E_2 - E_2^d))x. \end{cases} \tag{4}$$

Using the Pontryagin's maximal principle we write the Hamilton function:

$$\begin{aligned} &H_2(E_2, s^*, x) = \tfrac{1}{2}a_2E_2^2(s^d - \eta(E_2 - E_2^d))^2x^2 + b_2E_2(s^d - \eta(E_2 - E_2^d))x + \\ &c_2E_2\tfrac{u_1^d}{(1-s^d+\eta(E_2-E_2^d))x}(1 - s^d + \eta(E_2 - E_2^d))(s^d - \eta(E_2 - E_2^d))x^2 + \\ &+ d_2\tfrac{u_1^d}{(1-s^d+\eta(E_2-E_2^d))x}(1 - s^d + \eta(E_2 - E_2^d))x + \tfrac{1}{2}l_2\tfrac{(u_1^d)^2}{(1-s^d+\eta(E_2-E_2^d))^2x^2}\times \\ &\times(1 - s^d + \eta(E_2 - E_2^d))^2x^2 + \lambda_2(rx - \tfrac{rx^2}{K} - q_1\tfrac{u_1^d}{(1-s^d+\eta(E_2-E_2^d))x}(1 - s^d + \\ &+ \eta(E_2 - E_2^d))x - q_2E_2(s^d - \eta(E_2 - E_2^d))x) = \tfrac{1}{2}a_2E_2^d(s^d - \eta(E_2 - E_2^d))^2\times \\ &\times x^2 + (b_2 - q_2\lambda_2)E_2(s^d - \eta(E_2 - E_2^d))x + c_2E_2u_1^d(s^d - \eta(E_2 - E_2^d))x + \\ &+ d_2u_1^d + \tfrac{1}{2}l_2(u_1^d)^2 + \lambda(rx - rx^2/K - q_1u_1^d). \end{aligned}$$

The maximum of this function is achieved at

$$\frac{\partial H_2}{\partial E_2} = x(a_2E_2(s^d - \eta(E_2 - E_2^2))x + b_2 - \lambda_2q_2 + c_2u_1^d)(s^d - \eta(E_2 - E_2^2) - \eta E_2) = 0. \tag{5}$$

γ_1 is the incentive equilibrium if the cooperative equilibrium is the solution of the problem (4). Consequently, $E_2 = E_2^d$ should satisfy the equation (5). Here the expression in the first brackets is non-zero, and the expression in the second brackets is equal to zero when

$$\eta = \frac{s^d}{E_2^d}\,.$$

Analogously, when the first player deviates we define the center's strategy $s^* = = s^d + \theta(E_1 - E_1^d)$, and the second player's strategy

$$\gamma_2(E_1) = \frac{u_2^d}{(s^d + \theta(E_1 - E_1^d))x}.$$

It yields

$$\theta = \frac{1 - s^d}{E_1^d}\,.$$

So, we proved the next theorem.

Theorem 2. *The incentive equilibrium of the problem (1)–(2) is*

$$\gamma_1(E_2) = \frac{u_1^d}{(1-s_2^*)x}, \qquad \gamma_2(E_1) = \frac{u_2^d}{s_1^* x},$$

where

$$s_2^* = s^d - \frac{s^d}{E_2^d(t)}(E_2(t) - E_2^d(t)), \quad s_1^* = s^d + \frac{1-s^d}{E_1^d(t)}(E_1(t) - E_1^d(t)).$$

2. The model with functionals of the form $(\mathbf{p} - \mathbf{cu})\mathbf{u}$.

We consider the model with quadratic players' profits. As before, the dynamics of the fishery is described by the equation

$$\begin{aligned} x'(t) \quad &= \quad rx(t)(1 - x(t)/K) - q_1E_1(t)(1-s)x(t) - q_2E_2(t)sx(t), \\ & \quad\quad 0 \le t \le T, \; x(0) = x_0, \end{aligned} \tag{6}$$

where $x(t) \ge 0$ – size of the population at a time t; F – natural growth function of the population; $E_1(t), E_2(t) \ge 0$ – countries' fishing efforts measured as the number of vessels involved in fishing at time t and $q_1, q_2 > 0$ – catchability coefficients related to the unit fishing effort of the country.

The players' net revenues over a fixed time period $[0, T]$ are

$$\begin{aligned} J_1 \quad &= \quad g_1(x(T)) + \int\limits_0^T e^{-\rho_i t}[(p_1 - k_1q_1E_1(t)(1-s)x(t))q_1E_1(t)(1-s)x(t)]dt, \\ J_2 \quad &= \quad g_2(x(T)) + \int\limits_0^T e^{-\rho_i t}[(p_2 - k_2q_2E_2(t)sx(t))q_2E_2(t)sx(t)]dt, \end{aligned} \tag{7}$$

or substituting $a_i = 2k_iq_i^2e^{-\rho_i t}$, $b_i = p_iq_ie^{-\rho_i t}$, we receive

$$\begin{aligned} J_1 \quad &= \quad g_1(x(T)) + \int\limits_0^T e^{-\rho_i t}[-\tfrac{1}{2}a_1E_1^2(t)(1-s)^2x^2(t) + b_1E_1(t)(1-s)x(t)]dt, \\ J_2 \quad &= \quad g_2(x(T)) + \int\limits_0^T e^{-\rho_i t}[-\tfrac{1}{2}a_2E_2^2(t)s^2x^2(t) + b_2E_2(t)sx(t)]dt. \end{aligned} \tag{7'}$$

At first we define the cooperative equilibrium. Using Theorem 1, we receive the next proposition.

Proposition 1. *The cooperative equilibrium of the problem (6)-(7) is*

$$E_1^d = \frac{b_1 - \mu_1^{-1}q_1\lambda}{a_1(1-s)x}, \quad E_2^d = \frac{b_2 - \mu_2^{-1}q_2\lambda}{a_2sx},$$

with $\lambda(t)$ satisfying the differential equation

$$\lambda'(t) = -\lambda(t)(r - \frac{2rx(t)}{K}), \quad \lambda(T) = \mu_1 g_1'(x(T)) + \mu_2 g_2'(x(T)).$$

Next we define the incentive equilibrium. Using Theorem 2, we receive the next proposition.

Proposition 2. *The incentive equilibrium of the problem (6)-(7) is*

$$\gamma_1(E_2) = \frac{b_1 - \mu_1^{-1} q_1 \lambda}{a_1(1 - s_2^*)x}, \qquad \gamma_2(E_1) = \frac{b_2 - \mu_2^{-1} q_2 \lambda}{a_2 s_1^* x},$$

where

$$s_2^* = s^d - \frac{s^d}{E_2^d(t)}(E_2(t) - E_2^d(t)), \quad s_1^* = s^d + \frac{1 - s^d}{E_1^d(t)}(E_1(t) - E_1^d(t)).$$

Next we give the results of computer modelling for the next set of parameters

$$\begin{array}{llll} \rho = 0.02\,, & p_1 = p_2 = 6000\,, & q_1 = q_2 = 0.02 & s^d = 0.5 \\ K = 300000\,, & r = 0.06\,, & \mu_1 = 0.505\,, & \mu_2 = 0.495\,. \end{array}$$

We give two examples. In the first case a player who deviates from the cooperative equilibrium continues this deviation until the end of the planning period. In the second case just after a deviation he returns back to the cooperation.

1. We consider the case when after the second player's deviation there is no return to cooperative behavior. Let $k_1 = k_2 = 2$. The initial size of the population $x(0) = 100000$. The time when the second player deviates is $t_0 = 20$, and the size of a deviation is $\Delta = 0.5$.

   In the Figure 1.1–1.6 you can see the difference in parameters in the case of cooperative behavior and in the case of the deviation (dotted line). Figure 1.1 shows the population's dynamics. Figigure 1.2 and 1.3 present the first and the second player's controls, respectively We notice that the second player increases his fishing efforts and the first player decreases it. Figure 1.4 shows water area sharing (s). You can see that s decreases from 0.5 to 0.1. Figure 1.5 and 1.6 present the first and the second player's catch, respectively ($v_1(t) = q_1 E_1(t)(1 - s(t))x(t)$, $v_2(t) = q_2 E_2(t) s(t) x(t)$). We notice that the first player's catch increases slightly, while the second player's catch decreases quickly (from 1420 to 900 individuals per time moment).

   The players' profits under the cooperative behavior are

   $$J_1^d = 248136465, \qquad J_2^d = 248098365.$$

   If the second player deviates the players' profits are

   $$J_1^{otk} = 249513396, \qquad J_2^{otk} = 234237606.$$

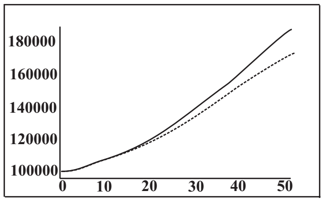

Fig.1.1: $x(t)$

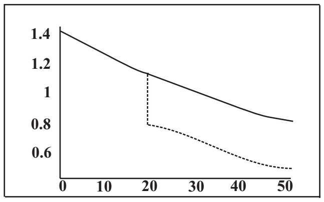

Fig.1.2: $E_1(t)$

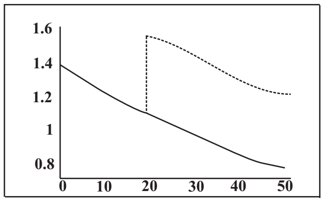

Fig.1.3: $E_2(t)$

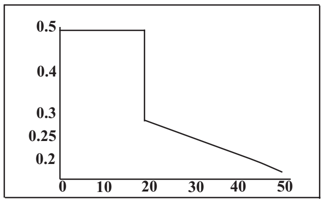

Fig.1.4: $s(t)$

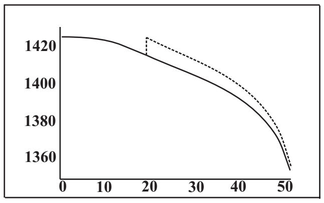

Fig.1.5: $v_1(t)$

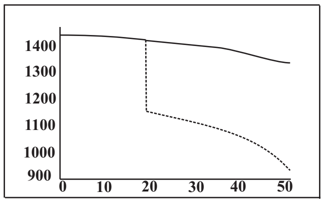

Fig.1.6: $v_2(t)$

2. All parameters are the same. The difference is the second player deviates at time $t_0 = 20$ but at time $t_0 + 1$ he returns to the initial cooperative behavior.

   In the Figures 2.1–2.6 you can see the difference in parameters in the case of cooperative behavior and in the case of the deviation at the interval $[t_0, t_0 + 1]$ (dotted line). Figure 2.1 shows the population's dynamics. Figures 2.2 and 2.3 present the first and the second players' controls, respectively. We notice that the second player increases his fishing efforts and the first player decreases it at the time interval $[t_0, t_0 + 1]$. Figure 2.4 shows water area sharing (s). You can see that s decreases from 0.5 to 0.3 at the interval $[t_0, t_0 + 1]$. Figure 2.5 and 2.6 present the first and the second player's catch, respectively. We notice that the first player's catch almost doesn't change, while the second players' catch decreases quickly at the time interval $[t_0, t_0 + 1]$ (from 1420 to 1150 individuals per the moment).

   The players' profits under the cooperative behavior are as before

   $$J_1^d = 248136465, \qquad J_2^d = 248098365.$$

   If the second player deviates the players' profits are

   $$J_1^{otk} = 246975859, \qquad J_2^{otk} = 246443739.$$

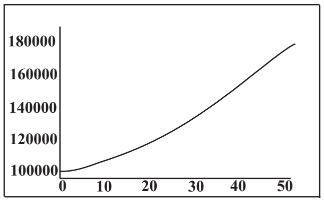

Fig.2.1: $x(t)$

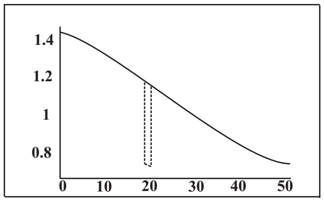

Fig.2.2: $E_1(t)$

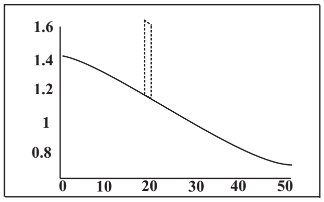

Fig.2.3: $E_2(t)$

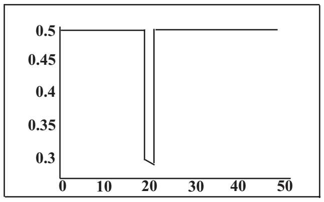

Fig.2.4: $s(t)$

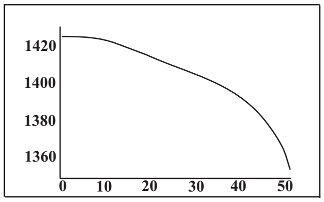

Fig.2.5: $v_1(t)$

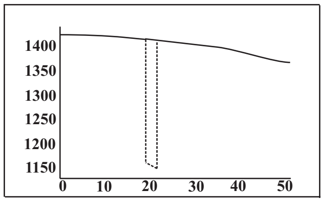

Fig.2.6: $v_2(t)$

3. Conclusion

So, we can see from the results of numerical modelling the center's participation in optimal resource using regulation has several interesting features.

In the scheme of short-time deviation from the cooperative equilibrium the center's punishment is rather strong with respect to both players. If the center's strategy is to punish defaulter player until the end of the planning period then the honest player has visible advantages even in comparison with cooperative equilibrium, and his opponent incurs remarkable losses.

The center's strategy here is the territory sharing. The player who breaks the agreement achieved at the beginning of the game is punished by gradually decreasing the harvesting territory. This scheme can be easily realized in practice.

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Dynamic Oligopoly Competition with Public Environmental Information Disclosure[1]

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Abstract. The main purpose of this paper is to study the impact of traditional and emergent environmental regulations on firms' strategies and outcomes. The former corresponds to, e.g., emissions taxing, and the latter to emissions reporting. To do so, we consider a differential game between two polluting firms.
The market potential of each firm varies with its goodwill, which evolution depends on its advertising effort and on its emissions, as well as those of its competitor. We characterize the open-loop Nash equilibrium and contrast the results obtained under different regulatory regimes (laisser-faire, traditional regulation, emergent regulation and dual regulation). We also carry out a sensitivity analysis to assess the impact of some key model parameters on strategies, steady-state goodwill stocks and payoffs.

Keywords: Traditional Environmental Regulation; Public Disclosure Program; Pricing; Advertising; Goodwill; Differential games.

Introduction

Environmental issues, ranging from smog in cities to global warming, undoubtedly rank very high on the political and scientific agendas. Almost all decision-makers are

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[1] Research supported by NSERC, Canada, and FQRSC, Quebec. Research completed when the first author was a visiting professor at GERAD, HEC Montréal. The first author's research was partially supported by MEC under project SEJ2005-03858/ECON and by JCYL under project VA045A06, co-financed by FEDER funds.

calling for at least some environmental regulation to limit the negative externalities of human activities, and to face the problem of market failure. Environmental regulations can be schematically classified into two categories. The first one, to which one usually refers as traditional regulation, consists in monitoring firms and enforcing reductions in their pollutant emissions. Instruments used by the regulator include, e.g., emissions quotas, taxes and subsidies. The second approach, emergent regulation, consists of forcing the firm to provide information on its environmental record, through what is known as a Public Disclosure Program (PDP). The rationale here is that the consumer and financial markets will react and make decisions that are consistent with the environmental performance of the firm. The increasing popularity of the PDP is due to its supposedly lower cost and higher effectiveness, as compared to traditional means.

When it comes to the impact of environmental regulation on firms' profit and competitiveness, two opposite opinions are found in the literature:

- Supporters of the "win-win" paradigm, or what is referred to as the Porter hypothesis [Porter, 1991], suggest that a severe environmental regulation may have a beneficial effect on firms by stimulating innovation. Porter and van der Linde (1995) suggest that environmental regulation can create a double dividend by improving social welfare as well as firm profit. Taking stock of several case studies of firms, they argue that "properly designed environmental standards can trigger innovation that may partially or more than fully offset the costs of complying with them". Mohr (2002) shows that endogenous technical change makes the Porter hypothesis feasible, but concludes that when the result is consistent with the hypothesis, the policy adopted is not necessarily optimal. Through the use of a principal-agent model, Ambec and Barla (2002) show that, by reducing agency costs, an environmental regulation may enhance firm benefits as well as innovation.

- Supporters of the traditional economic paradigm consider that environmental regulation, since it is a cost, harms profit and spoils competitiveness. For instance, Smith and Walsh (2000) report an experimental test of the Porter Hypothesis. They provide one explanation that the empirical difficulties in rejecting the Porter hypothesis are due to limitations in the economic methods of decomposing productivity. Gray (1987) shows that OSHA[2] and EPA[3] regulations may be responsible for about 30% of the reduction in productivity growth in the American manufacturing sector during the seventies. Dufour et al. (1998) state that the decline in productivity growth in the Quebec manufacturing sector during the eighties is at least partially attributed to environmental regulation. Stewart (1993) argues that imposing stringent environmental regulation and liability rules on firms may harm their international competitiveness.

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[2] Occupational Safety and Health Administration.

[3] Environmental Protection Agency.

To conclude, empirical findings are clearly not unanimous regarding the impact of environmental regulation on firms' profits and welfare. It also seems that, to date, there is no theoretical framework that fully accounts for the complex relationship between environmental regulations and competitiveness (see, e.g., Jaffe et al. (1995) for a discussion of this topic[4]). An alternative explanation of the "win-win" possibility may lie in the firm image (goodwill or reputation). By being greener, whether compelled by regulation or through voluntary actions, a firm may enhance its competitiveness, at least in market segments where consumers value the environment attribute. The main purpose of this paper is precisely to study the impact of a PDP and of traditional regulation on the long-term goodwill of competing firms, and on their equilibrium policies. For this, we adopt a differential game model in which two firms compete in prices and advertising. The latter, as well as the environmental records of both firms, feed the goodwill stock of each, which in turn determines its market potential[5]. Assuming that consumers prefer the products of greener firms, and by linking public disclosure actions to the firm's profit via its goodwill, we extend to a competitive framework the monopoly setting studied in [Sokri and Zaccour, 2007]. As in that paper, we consider here that the regulator can use a tax/subsidy regulation and a PDP to curb pollution. We analyze and contrast different scenarios, namely a *laisser-faire* scenario where there is no regulation; a scenario with one regulation (either a traditional or emergent regulation); and finally, a dual-regulation scenario. We employ numerical experiments to illustrate the kind of insight that one can get from the model. The remainder of the paper is organized as follows. In Section 2 we set up the conceptual model. In Section 3 we characterize the open-loop Nash equilibrium strategies. In Section 4 the numerical results under the different regulatory regimes are interpreted and a sensitivity analysis on key model parameters is conducted. Finally, Section 5 makes some brief concluding remarks.

1. The Model

We consider a duopoly where two competing firms produce differentiated goods over an infinite time horizon $[0, \infty)$. At each moment $t \in [0, \infty)$ the demand for

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[4] The Porter hypothesis was controversial and heavily criticized by certain economists. Bradford and Simpson (1996), for instance, assert that it is unlikely that environmental regulations enhance industrial advantage. Palmer et al. (1995) harshly criticize the theoretical foundation of this hypothesis and develop a simple static model of innovation in technology to prove that, on the contrary, an increase in the stringency of environmental regulation makes the polluting firm worse off. For Jaffe and Palmer (1997), the Porter hypothesis is hindered by ambiguity. The authors distinguish at least two different hypotheses:

- The weak version, which says only that regulation will stimulate certain kinds of innovation, and that the additional innovation comes at an opportunity cost that exceeds its benefits;
- The strong version, which suggests that firms do not necessarily find or pursue all profitable opportunities for new products or processes. The regulation induces innovation whose benefits outweigh its cost. For these authors, this strong form of Porter hypothesis is anecdotal and presents environmental regulation as a free lunch.

[5] Surveys of goodwill models are found in [Jorgensen and Zaccour, 2004] for the competitive setting, and in [Feichtinger, 1994] for the monopolistic case.

brand i, $q_i(t)$, depends on this firm's goodwill, $G_i(t)$, the price of its product, $p_i(t)$, and the price of its competitor, $p_j(t)$. The demand function for brand i is assumed to be linear and given by

$$q_i(t) = a + b_i G_i(t) - p_i(t) + \mu p_j(t), \quad i, j = 1, 2, \ \ i \neq j, \tag{1}$$

where $a, b_i > 0$ and $\mu \in [0, 1]$ measures the degree of substitutability between the two varieties. If $\mu = 1$, then the products are perfect substitutes; if $\mu = 0$, then the two brands do not compete against one other, and the firms behave as monopolists.

As an inevitable by-product, the production of brand i yields $e_i(t)$ as pollutant emissions. We assume that these emissions are proportional to production: $e_i(t) = = \alpha q_i(t)$, with $0 < \alpha < 1$.

In order to improve its goodwill, $G_i(t)$, and, hence, its market potential measured by $a + b_i G_i(t)$, firm i can count either on its advertising effort, $A_i(t)$, or on its capacity to reduce its emissions, $e_i(t)$. The latter impact will be endogenous in our model. The firm's image is also affected by the emission behavior of its competitor, $e_j(t)$. We assume that the dynamics of firm i's goodwill stock obeys the following law:

$$\dot{G}_i(t) = A_i(t) - \varphi_i e_i(t) + \gamma_i e_j(t) - \delta G_i(t), \ G_i(0) = G_{i0}, i, j = 1, 2, i \neq j, \tag{2}$$

in which $\varphi_i, \gamma_i, \delta > 0$. This differential equation extends the standard Nerlove and Arrow (1962) setting by adding the term $(\gamma_i e_j - \varphi_i e_i)$ which expresses the effect of both firms' environmental records on firm i's goodwill. This term captures the idea of environmental reporting, that is the PDP, on the image of the firm. Parameter δ is the decay rate of the goodwill stock, which is assumed equal for both firms.

Parameter φ_i measures the marginal effect of the firm's own emissions on goodwill. The more the firm emits, the more it hurts its image and, consequently, its market potential. Except for its take on the role of innovation, this assumption is firmly within the win-win point of view (see [Porter and van der Linde, 1995], [Blend and Ravenswaay, 1999], [Kristrom and Lundgren, 2003]). On the other hand, each firm benefits, image-wise, from its competitor's emissions. The inclusion of the differential environmental records term, i.e. $\gamma_i e_j - \varphi_i e_i$, emphasizes that, when judging companies, consumers acknowledge the fact that emissions are unavoidable, and that their grading is comparative. We assume that the marginal effect of a firm's own emissions on its goodwill is greater than or equal to the marginal effect of the competitor's emissions, that is, $\varphi_i \geq \gamma_i$.[6]

Each firm has a constant cost of production, c, and pays a tax τ per unit of emissions. The advertising cost of firm i is assumed to be quadratic and convex, i.e. $C(A_i) = 1/2 A_i^2$. Denoting by r the constant positive discount rate, the optimization problem of player i is given by

$$\max_{p_i, A_i} \Pi_i = \int_0^\infty e^{-rt} \left\{ (p_i(t) - c)\, q_i(t) - \frac{1}{2} A_i^2(t) - \tau e_i(t) \right\} dt, \tag{3}$$

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[6] This assumption is not essential and can be relaxed.

subject to the dynamics in (2).

To recapitulate, by (3) and (2) we have described a two-player differential game with two state variables, the firms' goodwill stocks $G_i(t)$ and $G_j(t)$, and two control variables for each player, the price and the advertising effort, $p_k(t), A_k(t)$, $k = i, j$. We assume that the game is played à la Nash and that both firms noncooperatively and simultaneously decide their prices and their advertising investments. We consider that the firms select open-loop strategies, which means that pricing and advertising strategies are functions of time.

From now on we assume that $h = a - (1-\mu)(c + \alpha\tau) \geq 0$, which guarantees that the firms produce a positive quantity when their goodwill stocks are zero. We eliminate the time argument when no confusion may arise.

2. Pricing and Advertising Strategies

Each firm seeks two control paths $p_k(\cdot)$, $A_k(\cdot)$ so as to maximize its objective functional while considering the dynamics of both goodwill firms, $\dot{G}_k, k = i, j$.

The following proposition characterizes the firms' pricing and advertising strategies and outcomes at equilibrium.

Proposition 1. *Assuming interior solutions, the firms' pricing and advertising strategies at equilibrium satisfy:*

$$\begin{aligned} p_k &= \frac{(a+c+\alpha\tau)(2+\mu)+2b_kG_k+b_l\mu G_l+2\alpha[(\gamma_k\mu+\varphi_k)\psi_{kk}-(\gamma_l+\varphi_l\mu)\psi_{kl}]}{4-\mu^2} - \\ &- \frac{\alpha\mu[(\gamma_k+\varphi_k\mu)\psi_{lk}-(\gamma_l\mu+\varphi_l)\psi_{ll}]}{4-\mu^2}, \quad k, l = i, j, k \neq l, \qquad (4) \\ A_k &= \psi_{kk}, \quad k = i, j. \qquad (5) \end{aligned}$$

An open-loop Nash equilibrium is fully characterized by the following system of linear first-order differential equations:

$$\dot{y}(t) = My(t) + n, \qquad (6)$$

where

$$y(t) = (G_i(t), G_j(t), \psi_{ii}(t), \psi_{ij}(t), \psi_{ji}(t), \psi_{jj}(t))^T,$$

ψ_{kl} denotes the costate variable that firm k associates with the goodwill of firm l, and matrix M and vector n are given in the Appendix.

The initial conditions $G_k(0) = G_{k0}, k = i, j$, and the transversality conditions

$$\lim_{t\to\infty} e^{-rt}\psi_{lk}(t)G_k(t) = 0, \; k, l = i, j,$$

have to be satisfied.

Proof.

See the Appendix.

The advertising strategy in (5) follows the familiar rule of marginal cost (A_k) equals marginal revenue (ψ_{kk}), here, the shadow price of its own goodwill stock. The pricing strategy in (4) shows that each firm fixes its price depending on both firms' goodwill stocks and all costate variables. The price increases with both firms' goodwill stocks as well as with the shadow price that each firm assigns to its own goodwill stock. However, the price decreases with the shadow price associated with the competitor's goodwill stock.

An open-loop Nash equilibrium of the differential game at hand is fully characterized by a solution to (6), which satisfies the initial and transversality conditions, (4) and (5). To obtain a solution to (6), first, we derive the fundamental solution of (6) and, second, we compute a particular solution that satisfies the initial and transversality conditions. As is usually done in infinite-horizon dynamic games to ensure that the transversality conditions are satisfied, we look for a solution to system (6) which converges to a steady state.

Unfortunately, we cannot obtain a closed-form expression of the six eigenvalues of the system matrix M (see the Appendix). However, it is well known (see, e.g. [Hirch and Smale, 1974]) that the general, asymptotically stable solution of the homogeneous system, $\dot{y}(t) = My(t)$, can be written

$$y_{gh}(t) = \sum_{i \in I} K_i e^{\xi_i t} v_i,$$

where K_i is constant, I denotes the set of indices such that ξ_i is a real negative eigenvalue of matrix M with $\xi_i \neq \xi_j, i, j \in I, i \neq j$ and v_i denotes the corresponding eigenvector.

The particular solution of the non-homogeneous system (6) is given by

$$y_p(t) = -M^{-1} n,$$

which corresponds to the steady state of the dynamical system, which is denoted, from now on, by

$$y_{ss} = (G_i^{ss}, G_j^{ss}, \psi_{ii}^{ss}, \psi_{ij}^{ss}, \psi_{ji}^{ss}, \psi_{jj}^{ss})^T.$$

Therefore, the general solution of system (6) can be expressed as

$$y(t) = y_{gh}(t) + y_p(t) = \sum_{i \in I} K_i e^{\xi_i t} v_i + y_{ss}.$$

The expression of the steady state is set out in the Appendix.

The following remark collects the steady-state values when the game is symmetric in all its aspects.

Remark. In the symmetric scenario, i.e. $b_i = b_j = b$, $\gamma_i = \gamma_j = \gamma$, $\varphi_i = \varphi_j = \varphi$, the steady state reduces to:

$$
\begin{aligned}
G_i^{ss} &= G_j^{ss} = \frac{h[b(r+\delta)(\alpha^2(\gamma-\varphi)(\gamma\mu-\varphi)-1)-\alpha((r+\delta)^2(\gamma-\varphi)+b^2\varphi)]}{Den},\\
\psi_{ii}^{ss} &= \psi_{jj}^{ss} = -\frac{hb\delta(r+\delta+b\alpha\varphi)}{Den},\\
\psi_{ij}^{ss} &= \psi_{ji}^{ss} = -\frac{hb^2\alpha\gamma\delta}{Den},
\end{aligned}
$$

where

$$
\begin{aligned}
Den &= (r+\delta)\{(r+\delta)[\delta(\mu-2)+b\alpha(\gamma-\varphi)]+b\alpha\delta(\gamma\mu+(2\mu-3)\varphi)\}\\
&+ b^2[b\alpha\varphi+(r+\delta)(1-\alpha^2(\gamma-\varphi)(\gamma\mu-\varphi))+\alpha^2\delta(\gamma^2-\varphi^2)(1-\mu)].
\end{aligned}
$$

The steady-state level for the demands of both firms, $q_i^{ss} = q_j^{ss}$, is given by

$$q_i^{ss} = q_j^{ss} = -\frac{h\delta(r+\delta)}{Den}.$$

Since h is assumed to be positive, in order to ensure a positive demand in the long run, expression Den has to be negative. Under this condition, the steady-state levels for the costate variables $\psi_{kk}^{ss}, \psi_{kl}^{ss}, k, l = i, j$ are also positive. Therefore, the advertising investments are also positive in the long run. The steady-state levels for the goodwill stocks are positive if and only if $b(r+\delta)(\alpha^2(\gamma-\varphi)(\gamma\mu-\varphi)-1)-\alpha((r+\delta)^2(\gamma-\varphi)+b^2\varphi)>0$.

The symmetric steady-state prices of the products in the long run are

$$p_i^{ss} = p_j^{ss} = c+\alpha\tau-\frac{h\delta[(r+\delta+b\alpha\varphi)^2-b^2\alpha^2\gamma^2]}{Den}.$$

Since we are assuming $Den < 0$ and $\varphi \geq \gamma$, the steady-state prices are greater than the total cost, $c+\alpha\tau$.

3. Numerical Results

It is well known that, in infinite-horizon linear-quadratic optimization models, the time paths of the state (goodwill) and of the controls are exponential functions of time that converge towards the steady state, either from above or below depending on whether the initial condition is higher or lower than the steady state. The initial conditions for the goodwill stocks, and the relationship between the initial values for the state and costate variables, that are required to have a trajectory that converges towards the steady state, give us the initial conditions for the firms' control variables: advertising investments and prices.

As previously stated, we cannot analytically derive the time paths for the goodwill stocks, and, hence, for the advertising efforts and prices, since the eigenvalues of the

6×6 system matrix M in (6) cannot be explicitly obtained. Therefore, in order to illustrate the behavior of the strategies and the outcomes, we provide some numerical examples. Being interested in the long-run behavior of the decision variables, the goodwill stocks, and the firms' demands and profits, their values are shown at the steady state.

For all reported numerical simulations, the matrix associated with the state dynamics has two negative and four positive real eigenvalues. Therefore, the steady state has a two-dimensional stable manifold. In the tables $q_k^{ss}, \Pi_k^{ss}, k = i, j$ denote, respectively, the demand and the instantaneous profit at the steady state for firm k.

We retain the following parameter values for the "base" case:

$$\begin{aligned}
&r = 0.1,\ \alpha = 0.5,\ c = 3,\ a = 7,\ \mu = 0.25,\ \tau = 0.2,\ \delta = 0.1,\\
&b_k = 0.1,\ \varphi_k = 0.5,\ \gamma_k = 0.25, \qquad k = i, j.
\end{aligned}$$

The analysis is done in two steps. First, we assume that the game is symmetric in all aspects. Second, we study some scenarios in which the game is symmetric in all its aspects but one[7].

3.1. Reference Cases

Our aim is to assess the impact on the long-term firm policies of the different regulatory regimes. To start, we wish to contrast the following reference scenarios:

1. *Laisser-faire* policy: $\varphi_k = \gamma_k = \tau = 0, k = i, j$;
2. Traditional regulation: $\varphi_k = \gamma_k = 0, k = i, j$ and $\tau = 0.2$;
3. Emergent regulation: $\tau = 0$, $\varphi_k = 0.5, \gamma_k = 0.25, k = i, j$;
4. Dual regulation: $\tau = 0.2$, $\varphi_k = 0.5, \gamma_k = 0.25, k = i, j$.

In the above scenarios, the regulation parameters (i.e. tax and PDP parameters) assume either their "base" case values or zero when the parameter(s) is (are) not a part of the considered scenario. In all four scenarios, the other parameters are kept at their base case levels. The steady-state values are collected in Table 1 and allow for the following comments:

- Comparing the results of the traditional regulation scenario to their *laisser-faire* policy counterparts shows a reduction in emissions and demand, an increase in price to consumer, lower advertising and goodwill levels, and finally, less profit. Therefore, the regulation achieves its environmental goal of forcing the firms to pollute less. Being an additional cost to the firms, the regulation increases their prices. Consumers are paying more and the firms are making less profit.

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[7] Note that the same qualitative effects as those collected in the tables below have been obtained for numerical simulations carried out for different base values of the model's parameters.

Table 1: Steady-state results for the three special regimes

	Laisser-faire	Traditional R.	Emergent R.	Dual R.
Variables				
G_k^{ss}	19.00	18.70	12.34	12.14
p_k^{ss}	6.80	6.84	6.67	6.71
A_k^{ss}	1.90	1.87	1.64	1.61
e_k^{ss}	1.90	1.87	1.61	1.59
q_k^{ss}	3.80	3.74	3.23	3.18
Π_k^{ss}	12.64	12.24	10.52	10.19

- Comparing the results of the emergent regulation scenario to the *laisser-faire* one leads to the same observations as above, regarding emissions, advertising, goodwill and profit. Surprisingly, the price under a PDP is lower than under a *laisser-faire* policy. One possible explanation is that, by damaging the reputation of the firm, the PDP is shrinking the market size of the (symmetric) firms, and the firm's response is to reduce prices. This leads to lower revenues, and hence, less funds are available to invest in advertising to counter the negative effect of emissions on goodwill and profits.

- Comparing the two regulatory regimes shows that the PDP is more efficient in reducing emissions than a tax (15.26% versus 1.58%). Also, the long-term goodwill value is lower under a PDP than under a tax mechanism. Given that the results are obtained with some particular parameter values, one should not consider them as definitive. Nevertheless, we note that in all the numerical simulations carried out for different values of τ and φ, the impact of a PDP is more pronounced than that of the traditional regime. Actually, contrasting the results of a dual regulation to the results with one regulation shows that adding a PDP to a tax regulation has a much higher impact than adding a tax to a PDP. One possible explanation for this greater impact is that, while a tax hurts "only" the cost side, a PDP has a dual effect. Indeed, it directly hurts the goodwill dynamics (and, hence, the cost of advertising), and, indirectly, the revenues of the firm.

To summarize, both regulations achieve lower emissions, and a dual regulation is more effective than any single one. This result was also obtained in the monopoly case in [Sokri and Zaccour, 2007]. Both of the regulations lower the firms' profits, its goodwill and demand. Interestingly, the two regulations differ in terms of their impact on price to consumer. Whereas the traditional regulation leads to a higher price, the emergent one decreases the price to consumer. Comparing the regulatory regimes, it seems that a PDP has a more pronounced effect than taxing emissions do. Finally, note that in all simulations the goodwill steady-state values happen to be positive, which is not a requirement in our model.

3.2. Sensitivity Analysis

We turn now to a sensitivity analysis, to assess the impact of key parameters on the equilibrium results in the dual regulatory regime. Recall that the model has 13 parameters, namely:

$$\begin{aligned} \text{Demand and cost parameters:} &\quad a, b_i, b_j, \mu, \alpha, c, \\ \text{Regulation parameters:} &\quad \tau, \varphi_i, \varphi_j, \gamma_i, \gamma_j, \\ \text{Other parameters:} &\quad r, \delta. \end{aligned}$$

All parameters that are common to the two players have a rather intuitive and straightforward impact on the equilibrium's control and state values. We, therefore, do not include the results[8]. For instance, the parameter a has a positive scale effect, i.e. any increment in its value leads to an increase in any steady-state value. Conversely, a greater unit cost c (or a higher decay rate δ) implies a decrease in the steady-state values of the goodwill stocks, advertising, emission levels, and demand and profit levels. Further, increasing α amounts to an additional cost, and the impact is the same as the production cost increasing. Finally, varying the discount rate r has a pure translation effect on the results.

Since the focus of this paper is environmental regulation, we make an exception for the common tax rate. Table 2 collects the results of varying τ. Actually, it can be shown analytically that an increase in τ leads to lower steady-state values of the goodwill stocks, the advertising efforts, the emission levels, and the firms' demand and profit levels. In order to compensate for this loss of profits, the firm reduces its advertising effort. This in turn negatively affects the firm's goodwill. A lower goodwill leads to lower demand, and, therefore, lower emission levels. Further, the higher the tax, the higher the price the consumer pays.

Hence, the firms are simply asking the consumer to pay at least part of the regulation bill. A relevant result to point out is that the impact of the tax is rather low. Indeed, multiplying the tax rates by four (from 0.1 to 0.4) leads to a variation of less than 10% in the steady-state values of the strategies, the goodwill, the demand and profit. We now turn to assessing the impact on the strategies and outcomes of varying the parameters that affect goodwill stocks, i.e. b_k, γ_k and φ_k. Tables 3, 5, 7 collect the results when these changes preserve the symmetric structure of the game; that is, each modification affects both firms identically. In all cases the other parameters are fixed as in the base case. Afterwards, three asymmetric scenarios are considered:

1. Firms are asymmetric with respect to parameter b_k ($b_i = 0.1, b_j \in \{0.08, 0.09, 0.1, 0.11, 0.12\}$);

2. Firms are asymmetric with respect to parameter φ_k ($\varphi_i = 0.5, \varphi_j \in \{0.25, 0.5, 0.75, 1, 1.25, 1.5\}$);

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[8] The results are available from the authors upon request.

Table 2: Sensitivity of steady-state results for changes in τ

	$\tau = 0$	$\tau = 0.1$	$\tau = 0.2$	$\tau = 0.3$	$\tau = 0.4$
Variables					
G_k^{ss}	12.34	12.24	12.14	12.04	11.95
p_k^{ss}	6.67	6.69	6.71	6.74	6.76
A_k^{ss}	1.64	1.62	1.61	1.60	1.59
e_k^{ss}	1.61	1.60	1.59	1.57	1.56
q_k^{ss}	3.23	3.20	3.18	3.15	3.13
Π_k^{ss}	10.52	10.35	10.19	10.03	9.87

3. Firms are asymmetric with respect to parameter γ_k
($\gamma_i = 0.25, \gamma_j \in \{0, 0.15, 0.25, 0.35, 0.5\}$);

Tables 4, 6 and 8 collect the results of these numerical simulations. Recall that in all tables the base case corresponds to a symmetric scenario, and, therefore, symmetric steady-state values are attained.

3.3. Impact of b_k

Table 3 shows that an increment in parameter b_k moves up the steady-state values of all model variables. In our framework, increasing b_k amounts to an expansion of the firm's market potential and an increase in its prestige, which reduces the impact of price on demand. Therefore, it is not surprising that the firms increase their prices with b_k. The fact that consumers are becoming less sensitive to price results in higher demand, and consequently, more pollution. The firms increase their advertising levels to compensate for, and even offset, the damage caused by the PDP to their goodwill. The ultimate consequence is a higher profit in the long run. In Table 4 the two columns to the left of the base case correspond to scenarios in which parameter b_j is lower than parameter $b_i = 0.1$, whereas the two columns to the right of the base case represent the opposite situation ($b_j > b_i$). A quick inspection of the results leads to the following comments:

Table 3: Sensitivity of steady-state results for changes in b_k

	$b_k = 0.08$	$b_k = 0.09$	Base case	$b_k = 0.11$	$b_k = 0.12$
Variables					
G_k^{ss}	8.12	10.01	12.14	14.62	17.56
p_k^{ss}	6.33	6.50	6.71	6.98	7.31
A_k^{ss}	1.18	1.38	1.61	1.88	2.21
e_k^{ss}	1.45	1.51	1.59	1.69	1.81
q_k^{ss}	2.91	3.03	3.18	3.37	3.62
Π_k^{ss}	8.68	9.33	10.19	11.31	12.82

- The firm enjoying a higher impact of goodwill on demand (higher b_k) achieves better long-term results in terms of goodwill stock, demand and profits. It invests more in advertising, charges a higher price to consumers and pollutes the environment more. This qualitative behavior is the same as the one seen in the symmetric case.
- The increase in the emission level of firm j pushes up the goodwill stock of its competitor, G_i. In turn, this shifts upward the demand function, which results in a greater emissions level and a higher price. Both effects trigger a rise in the firm's profits.

Table 4: Sensitivity of steady-state results for changes in b_j

	$b_j = 0.08$	$b_j = 0.09$	Base case	$b_j = 0.11$	$b_j = 0.12$
Variables					
G_i^{ss}	11.72	11.91	12.14	12.43	12.78
G_j^{ss}	8.47	10.22	12.14	14.29	16.75
p_i^{ss}	6.65	6.68	6.71	6.76	6.81
p_j^{ss}	6.38	6.53	6.71	6.93	7.20
A_i^{ss}	1.58	1.60	1.61	1.63	1.66
A_j^{ss}	1.20	1.39	1.61	1.86	2.15
e_i^{ss}	1.56	1.57	1.59	1.61	1.63
e_j^{ss}	1.48	1.53	1.59	1.66	1.76
q_i^{ss}	3.12	3.15	3.18	3.22	3.27
q_j^{ss}	2.96	3.06	3.18	3.33	3.52
Π_i^{ss}	9.82	9.98	10.19	10.44	10.75
Π_j^{ss}	8.99	9.52	10.19	11.03	12.10

To wrap up, in both the symmetric and asymmetric cases, a higher b_k results in good news for the firm, and bad news for consumers (at least price-wise) and the environment (more emissions). The moral here is crystal clear: if the consumer values the firm's reputation more, by increasing his willingness-to-pay and buying more then the firm takes note of this and satisfies the higher demand, with the unavoidable consequence of polluting more.

3.4. Impact of φ_k and γ_k

We now turn to an assessment of the impact of the PDP parameters, φ_k and γ_k, on the strategies, goodwill stocks and payoffs. These parameters represent how consumers are reacting to the firms' environmental records, which are made available thanks to the public disclosure program. The simulations presented here are conducted under the assumption that the firm's goodwill is more sensitive to its environmental record than to its competitor's. This translates in setting φ_k higher than γ_k. (Clearly, the opposite can be also considered, which would lead to results that mirror those discussed below). Recalling that the parameter φ_k is fixed at 0.5 in the base case, we, therefore, allow γ_k to vary from zero (i.e. the dynamics of the

Table 5: Sensitivity of steady-state results for changes in γ_k

	$\gamma_k = 0$	$\gamma_k = 0.15$	Base case	$\gamma_k = 0.35$	$\gamma_k = 0.5$
Variables					
G_k^{ss}	7.33	10.12	12.14	14.31	17.90
p_k^{ss}	6.40	6.58	6.71	6.85	7.08
A_k^{ss}	1.47	1.55	1.61	1.68	1.79
e_k^{ss}	1.47	1.54	1.59	1.65	1.74
q_k^{ss}	2.93	3.07	3.18	3.29	3.48
Π_k^{ss}	8.60	9.51	10.19	10.94	12.25

Table 6: Sensitivity of steady-state results for changes in γ_j

	$\gamma_j = 0$	$\gamma_j = 0.15$	Base case	$\gamma_j = 0.35$	$\gamma_j = 0.5$
Variables					
G_i^{ss}	11.73	11.98	12.14	12.31	12.56
G_j^{ss}	7.42	10.23	12.14	14.09	17.07
p_i^{ss}	6.66	6.69	6.71	6.74	6.77
p_j^{ss}	6.44	6.60	6.71	6.83	7.00
A_i^{ss}	1.58	1.60	1.61	1.62	1.64
A_j^{ss}	1.48	1.56	1.61	1.66	1.75
e_i^{ss}	1.56	1.58	1.59	1.60	1.62
e_j^{ss}	1.48	1.55	1.59	1.63	1.70
q_i^{ss}	3.12	3.16	3.18	3.20	3.24
q_j^{ss}	2.97	3.09	3.18	3.26	3.39
Π_i^{ss}	9.87	10.06	10.19	10.33	10.53
Π_j^{ss}	8.81	9.62	10.19	10.79	11.73

goodwill stock is not affected by the competitor's emissions) to 0.5 (i.e. both players' emissions have the same effect on the evolution of the goodwill stock). Table 5 presents the results in the symmetric scenario and Table 6, those resulting from the asymmetric case. In Table 6 the middle column corresponds to the base case, where $\gamma_j = \gamma_i = 0.25$. The first two columns represent scenarios in which the marginal impact of firm i's emissions on firm j's goodwill stock is lower than the marginal impact of firm j's emissions on firm i's goodwill stock (that is $\gamma_j < \gamma_i = 0.25$).

The last two columns consider opposite scenarios ($\gamma_j > \gamma_i$). Tables 7 and 8 report the results of varying φ_k in symmetric and asymmetric ways, respectively.

Tables 5–8 allow for the following remarks:

- As can easily be noticed from Tables 5–8 (and should actually be expected), increasing φ_k has the same qualitative impact as decreasing γ_k. A higher value of φ_k, meaning that consumers are more sensitive to the environmental record of the firm, leads to a lower goodwill stock, and consequently, to lower demand and prices. This means less revenues, and, therefore, less available funds for advertising.

- The lowest emissions levels are registered when $\varphi_k > 0$ and $\gamma_k = 0$, i.e. when consumers judge a firm's environmental record on its own merit and not on a comparative basis (see Table 5).
- The symmetric variations in γ have a pronounced, but less than proportional, impact on goodwill stocks, and a modest impact on all other variables. For instance, from Table 5 we see that moving from $\gamma_k = 0.15$ to $\gamma_k = 0.5$ (i.e. more than tripling the value of this parameter) leads to a 75% increase in the goodwill and a variation of less than 50% on emissions, prices, advertising and profits. The symmetric variations in φ have a much greater effect (see Table 7). Note that the goodwill steady-state value can even be negative, meaning that the market potential is lower than what is normally expected (a case where the market potential is given by a).

Table 7: Sensitivity of steady-state results for changes in φ_k

	$\varphi_k = 0.25$	Base case	$\varphi_k = 0.75$	$\varphi_k = 1$	$\varphi_k = 1.25$	$\varphi_k = 1.5$
Variables						
G_k^{ss}	18.27	12.14	7.30	3.38	0.14	-2.58
p_k^{ss}	6.97	6.71	6.51	6.25	6.22	6.10
A_k^{ss}	1.83	1.61	1.44	1.30	1.19	1.09
e_k^{ss}	1.80	1.59	1.42	1.28	1.18	1.08
ψ_{kl}^{ss}	0.11	0.09	0.08	0.07	0.06	0.05
q_k^{ss}	3.60	3.18	2.84	2.57	2.35	2.16
Π_k^{ss}	12.26	10.19	8.67	7.52	6.62	5.90

To summarize the findings on φ and γ, if the values assumed in this exercise were empirical, then the message to each firm would be twofold: first, a PDP can be quite damaging in terms of reputation; and, second, what really matters is each firm's environmental record rather than a comparative one.

4. Conclusion

In this paper we explored the relationships between environmental regulations and the goodwill of competing firms. One result is that regulation, regardless of the type, seems to correspond to bad news for the firms. Therefore, green goodwill seems not to be sufficient to achieve the "win-win" outcome that was hoped for. Our numerical results allow us to note two differences between traditional and emergent regulations. First, whereas a higher tax rate induces a higher price, an increase in the impact of information reporting leads to a lower price. In the tax regulation, the firms shift to the consumer any cost increase that results from the higher tax rate, which violates the "polluter pays" principle on which the tax mechanism was founded. In the PDP scenario the firms lower their prices in an attempt to compensate for the decrease in demand resulting from a loss of brand image that would be associated with a more stringent impact of a PDP, as measured by φ_k. Second, our simulations tend to show

Table 8 Sensitivity of steady-state results for changes in φ_j

	$\varphi_j = 0.25$	Base case	$\varphi_j = 0.75$	$\varphi_j = 1$	$\varphi_j = 1.25$	$\varphi_j = 1.5$
Variables						
G_i^{ss}	12.74	12.14	11.66	11.25	10.91	10.62
G_j^{ss}	17.58	12.14	7.73	4.07	0.98	-1.65
p_i^{ss}	6.77	6.71	6.67	6.63	6.59	6.57
p_j^{ss}	6.91	6.71	6.56	6.43	6.32	6.23
A_i^{ss}	1.64	1.61	1.59	1.57	1.56	1.54
A_j^{ss}	1.80	1.61	1.46	1.34	1.23	1.14
e_i^{ss}	1.16	1.59	1.57	1.55	1.54	1.53
e_j^{ss}	1.77	1.59	1.44	1.32	1.21	1.12
q_i^{ss}	3.23	3.18	3.14	3.11	3.08	3.05
q_j^{ss}	3.54	3.18	2.88	2.63	2.43	2.25
Π_i^{ss}	10.51	10.19	9.93	9.72	9.54	9.39
Π_j^{ss}	11.88	10.19	8.90	7.88	7.06	6.38

that a PDP seems to be more efficient in curbing emissions than taxation is. Not surprisingly, two regulations achieve better environmental results than any one taken separately.

This study is the first attempt to study in a dynamic and competitive framework the relationships mentioned above. Therefore, we made two simplifying assumptions that are worth relaxing. First, it would be interesting to adopt the more conceptually appealing feedback-information structure and contrast the resulting Nash equilibrium with the open-loop equilibrium obtained in this paper. Clearly, there is no hope of analytically fully characterizing the feedback equilibrium, and, therefore, we would need to resort to numerical methods. Second, it would also be interesting to introduce abatement capital, and, hence, fully participate in the debate surrounding the Porter hypothesis.

Appendix

Proof of Proposition 1.

We have to prove that the following system of linear first order differential equations completely characterizes an open-loop Nash equilibrium:

$$\dot{y}(t) = My(t) + n, \tag{7}$$

where

$$\begin{aligned} y(t) &= (G_i(t), G_j(t), \psi_{ii}(t), \psi_{ij}(t), \psi_{ji}(t), \psi_{jj}(t))^T, \\ n &= \left(\frac{\alpha h(\gamma_i - \varphi_i)}{2-\mu}, \frac{\alpha h(\gamma_j - \varphi_j)}{2-\mu}, -\frac{b_i h}{2-\mu}, 0, 0, -\frac{b_j h}{2-\mu} \right)^T, \end{aligned}$$

with

$$h = a - (1-\mu)(c + \alpha\tau),$$

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & m_{36} \\ 0 & 0 & m_{43} & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{55} & m_{56} \\ m_{32} & m_{62} & m_{63} & m_{64} & m_{65} & m_{66} \end{pmatrix},$$

and $m_{rs}, r, s = 1, \ldots, 6$ are the following constants:

$$m_{11} = \frac{\alpha b_i(\gamma_i \mu - 2\varphi_i)}{4-\mu^2} - \delta, \quad m_{12} = \frac{\alpha b_j(2\gamma_i - \mu\varphi_i)}{4-\mu^2},$$

$$m_{13} = \frac{4 - \mu^2 + \alpha^2(\gamma_i\mu + \varphi_i)(\gamma_i\mu + (2-\mu^2)\varphi_i)}{4-\mu^2},$$

$$m_{14} = -\frac{\alpha^2(\gamma_j + \mu\varphi_j)(\gamma_i\mu + (2-\mu^2)\varphi_i)}{4-\mu^2},$$

$$m_{15} = \frac{\alpha^2(\gamma_i + \mu\varphi_i)(\gamma_i(2-\mu^2) + \mu\varphi_i)}{4-\mu^2},$$

$$m_{16} = -\frac{\alpha^2(\gamma_j\mu + \varphi_j)(\gamma_i(2-\mu^2) + \mu\varphi_i)}{4-\mu^2}, \quad m_{31} = -\frac{2b_i^2}{4-\mu^2},$$

$$m_{32} = -\frac{b_i b_j \mu}{4-\mu^2}, \quad m_{33} = r + \delta + \frac{\alpha b_i((2-\mu^2)\varphi_i - 2\gamma_i\mu)}{4-\mu^2},$$

$$m_{34} = \frac{\alpha b_i(2\mu\varphi_j - \gamma_j(2-\mu^2))}{4-\mu^2}, \quad m_{35} = \frac{\alpha b_i \mu(\gamma_i + \mu\varphi_i)}{4-\mu^2},$$

$$m_{36} = -\frac{\alpha b_i \mu(\gamma_j\mu + \varphi_j)}{4-\mu^2}, \quad m_{43} = -\alpha b_j \gamma_i, \quad m_{44} = r + \delta + \alpha b_j \varphi_j.$$

The other constants can be obtained following next rule:

$$\begin{aligned} & m_{11} \hookrightarrow m_{22},\ m_{12} \hookrightarrow m_{21},\ m_{13} \hookrightarrow m_{26},\ m_{14} \hookrightarrow m_{25},\ m_{15} \hookrightarrow m_{24},\ m_{16} \hookrightarrow m_{23}, \\ & m_{31} \hookrightarrow m_{62},\ m_{33} \hookrightarrow m_{66},\ m_{34} \hookrightarrow m_{65},\ m_{35} \hookrightarrow m_{64},\ m_{36} \hookrightarrow m_{63}, \\ & m_{43} \hookrightarrow m_{56},\ m_{44} \hookrightarrow m_{55}, \end{aligned}$$

where the arrow indicates that in each case the subscripts i and j have been interchanged. The current Hamiltonian of firm k can be written as:

$$\begin{aligned} H_k(G_k, G_l, A_k, p_k, \psi_{kk}, \psi_{kl}) &= (p_k - c)q_k - \frac{1}{2}A_k^2 - \tau e_k + \\ &+ \psi_{kk}(A_k - \varphi_k e_k + \gamma_k e_l - \delta G_k) + \psi_{kl}(A_l - \varphi_l e_l + \gamma_l e_k - \delta G_l), \end{aligned}$$

where $\psi_{kl}, k, l = i, j$, denote the firm's k costate variables associated with the state variables $G_l, l = i, j$. Replacing the expression of the emission levels in terms of the production and rearranging terms, the current Hamiltonian of firm k can be rewritten as:

$$\begin{aligned} H_k(G_k, G_l, A_k, p_k, \psi_{kk}, \psi_{kl}) &= (p_k - c - \tau\alpha)q_k - \frac{1}{2}A_k^2 + \\ &\psi_{kk}(A_k - \alpha(\varphi_k q_k + \gamma_k q_l) - \delta G_k) + \psi_{kl}(A_l - \alpha(\varphi_l q_l + \gamma_l q_k) - \delta G_l). \end{aligned}$$

Substituting the demand functions of both firms given by (1) the Hamiltonian of firm k reads:

$$\begin{aligned} &H_k(G_k, G_l, A_k, p_k, \psi_{kk}, \psi_{kl}) = (p_k - c - \tau\alpha)(a + b_k G_k - p_k + \mu p_l) - \frac{1}{2}A_k^2 + \\ &+\psi_{kk}[A_k + \alpha(a(\gamma_k - \varphi_k) + (\varphi_k + \mu\gamma_k)p_k - (\varphi_k\mu + \gamma_k)p_l + \gamma_k b_l G_l) - (\alpha\varphi_k b_k + \delta)G_k] + \\ &+\psi_{kl}[A_l + \alpha(a(\gamma_l - \varphi_l) + (\varphi_l + \mu\gamma_l)p_l - (\varphi_l\mu + \gamma_l)p_k + \gamma_l b_k G_k) - (\alpha\varphi_l b_l + \delta)G_l]. \end{aligned}$$

Assuming interior solution, the first order necessary optimality conditions for manufacturer k are given by:

$$\begin{aligned} &\frac{\partial H_k}{\partial A_k}(\cdot) = \psi_{kk} - A_k = 0, \qquad (8) \\ &\frac{\partial H_k}{\partial p_k}(\cdot) = a + c + \alpha\tau + b_k G_k - 2p_k + \mu_k p_l + \alpha[(\gamma_k\mu_l + \varphi_k)\psi_{kk} - (\gamma_l + \mu_l\varphi_l)\psi_{kl}] = 0 \\ &\dot{G}_k = A_k + \alpha[a(\gamma_k - \varphi_k) + (\varphi_k + \mu\gamma_k)p_k - (\varphi_k\mu + \gamma_k)p_l + \gamma_k b_l G_l] - (\alpha\varphi_k b_k + \delta)G_k, \\ &G_k\,(0) = G_{k0}, \\ &\dot{\psi}_{kk} = r\psi_{kk} - \frac{\partial H_k}{\partial G_k}(\cdot) = (r + \delta + b_k\alpha\varphi_k)\psi_{kk} + b_k(c + \alpha\tau - p_k - \alpha\gamma_l\psi_{kl}), \\ &\dot{\psi}_{kl} = r\psi_{kl} - \frac{\partial H_k}{\partial G_l}(\cdot) = (r + \delta + b_l\alpha\varphi_l)\psi_{kl} - b_l\alpha\gamma_k\psi_{kk}, \\ &\lim_{t\to\infty} e^{-rt}\psi_{kk}(t)G_k(t) = 0,\ \lim_{t\to\infty} e^{-rt}\psi_{kl}(t)G_l(t) = 0,\ k \neq l. \end{aligned}$$

From equation (8) we get the expression of the optimal advertising investment given in (5). The advertising effort of each firm coincides with the costate variable that this firm associates to its own goodwill stock. To obtain the optimal expression of prices p_i and p_j in (4), we have to solve simultaneously the two equations system given by (8) for $k, l = i, j, i \neq j$. Once the optimal advertising investments and prices in (5) and (4), respectively, have been replaced in the dynamics of the goodwill stocks and of the costate variables, the differential equations appearing in the first order optimality conditions can be rewritten equivalently in matrix notation.

Steady state

The steady state of the dynamical system (6) has been computed using Mathematica 5.2. as follows. From the differential equations for the costate variables ψ_{ij}

and ψ_{ji}, the following relationships linking the values of the costate variables at the steady state are obtained:

$$\psi_{kl}(\psi_{kk}) = \frac{b_l \alpha \gamma_k \psi_{kk}}{r + \delta + b_l \alpha \varphi_l}, \quad k, l = i, j, k \neq l. \tag{9}$$

Replacing these expressions into the differential equation for the costate variables ψ_{ii} and ψ_{jj} and equating to zero, the steady-state values of G_i and G_j in terms of ψ_{ii} and ψ_{jj} are given by: $G_k(\psi_{kk}, \psi_{ll}) =$

$$\frac{2(r+\delta)^2 + \alpha(r+\delta)(2b_l\varphi_l + b_k(\varphi_k - \gamma_k\mu)) + \alpha^2 b_k b_l(\varphi_k\varphi_l - \gamma_k\gamma_l)}{b_k^2(r + \delta + b_l\alpha\varphi_l)}\psi_{kk} -$$

$$-\frac{(r+\delta)^2 + \alpha(r+\delta)(b_l\varphi_l + b_k\varphi_k) + \alpha^2 b_k b_l(\varphi_k\varphi_l - \gamma_k\gamma_l)}{b_k b_l(r + \delta + b_k\alpha\varphi_k)}\mu\psi_{ll} -$$

$$-\frac{(a - (1-\mu))(c + \alpha\tau)}{b_k}, \quad k, l = i, j, k \neq l.$$

Substituting G_i and G_j given in (11) into the differential equations for these variables in the dynamical system (6), equating to zero and solving for ψ_{ii} and ψ_{jj} the steady-state values ψ_{ii}^{ss} and ψ_{jj}^{ss} are obtained:

$$\psi_{kk}^{ss} = \frac{Num(\psi_{kk}^{ss})}{Den(\psi_{kk}^{ss})}, \quad k = i, j, \tag{10}$$

where

$$Num(\psi_{kk}^{ss}) = -b_k\delta h(r + \delta + b_l\alpha\varphi_l)\{(r+\delta)(b_k\alpha\gamma_k + \delta(2+\mu))(r + \delta + b_k\alpha\varphi_k) -$$

$$-b_l^2((r+\delta)(1 + \alpha^2\gamma_l\mu\varphi_l) + b_k\alpha\varphi_k) + b_l\alpha[(r+\delta)((r + \delta(2+\mu))\varphi_l - \gamma_l\delta\mu) +$$

$$+b_k\alpha((r + \delta(2+\mu))\varphi_k\varphi_l) - (r\mu + \delta(1+2\mu))\gamma_k\gamma_l]\},$$

$$Den(\psi_{kk}^{ss}) = \alpha^2\varphi_i\varphi_j b_i^3 b_j^3 + B_i b_i^3 b_j^2 + B_j b_i^2 b_j^3 + C_i b_i^3 b_j + C_j b_j^3 b_i + D_i b_i^3 + D_j b_j^3 +$$

$$+E b_i^2 b_j^2 + F_i b_i^2 + F_j b_j^2 + H_i b_i^2 b_j + H_j b_j^2 b_i + I b_i b_j + J_i b_i + J_j b_j + \delta^2(r+\delta)^4(4 - \mu^2),$$

$$B_i = \alpha[(r+\delta)\varphi_i(1 + \alpha^2(\gamma_i\mu\varphi_i - \varphi_j^2)) + \alpha^2\delta\varphi_j(\gamma_i\gamma_j - \varphi_i\varphi_j)],$$

$$C_i = \alpha^2(r+\delta)[\alpha^2(r+3\delta)\mu\varphi_i\gamma_i(\gamma_i\gamma_j - \varphi_i\varphi_j) - (r+4\delta)\varphi_i\varphi_j + \gamma_i\gamma_j\delta],$$
$$D_i = -\alpha\delta(r+\delta)^2\varphi_i(2 + 3\alpha^2\gamma_i\mu\varphi_i),$$

$$E = (r+\delta)^2 - \alpha^4(\gamma_i\gamma_j - \varphi_i\varphi_j)[(r+2\delta)^2(\gamma_i\gamma_j\mu^2 + \varphi_i\varphi_j) - \delta^2(\gamma_i\gamma_j + \varphi_i\varphi_j\mu^2)] +$$
$$+ \alpha^2(r^2 + 3r\delta + 2\delta^2)((\gamma_i\mu - \varphi_i)\varphi_i + (\gamma_j\mu - \varphi_j)\varphi_j),$$

$$F_i = -\delta(r+\delta)^2[2(r+\delta) + \alpha^2\varphi_i(2\gamma_i\mu(2r+3\delta) - \varphi_i(2r+\delta(4-\mu^2)))],$$

$$H_i = -\alpha(r+\delta)\{(r+\delta)^2(\varphi_j - \alpha^2(\gamma_i\mu - \varphi_i)(\gamma_i\gamma_j - \varphi_i\varphi_j)) - \delta[r(\gamma_j(\mu(1+ \\ + \alpha^2\gamma_i^2) - \alpha^2\gamma_i\varphi_i(1-3\mu^2)) + (4\alpha^2\varphi_i(\varphi_i - \gamma_i\mu) - 3)\varphi_j) + \delta(\gamma_j(\mu(1+2\alpha^2\gamma_i^2) - \\ - \alpha^2\gamma_i\varphi_i(4-5\mu^2)) - (3+\alpha^2\varphi_i(5\gamma_i\mu - 7\varphi_i + 2\mu^2\varphi_i))\varphi_j)]\},$$

$$I = -\alpha^2(r+\delta)^2[(r+\delta)^2(\gamma_i\gamma_j - \varphi_i\varphi_j) - \delta(\gamma_i(\gamma_j(2r\mu^2 + 5\delta\mu^2 - 4\delta) - (2r+ \\ + 3\delta)\mu\varphi_j) + \varphi_i(-(2r+3\delta)\mu\gamma_j + \varphi_j(6r+15\delta-4\delta\mu^2)))],$$

$$J_i = -\alpha(r+\delta)^3(\gamma_i\mu(r+3\delta) - 2\varphi_i(r+\delta(4-\mu^2))).$$

Constants B_j, C_j, D_j, F_j, H_j and J_j can be obtained replacing the subscript i by j, and vice versa.

The final expressions of the steady-state values $G_i^{ss}, G_j^{ss}, \psi_{ij}^{ss}$ and ψ_{ji}^{ss} can be easily obtained replacing (12) in (11) and (10).

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Generalized Kernels and Bargaining Sets for Families of Coalitions [1]

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Abstract. For a fixed collection of subsets of the player set, two generalizations of Aumann–Maschler theory of the bargaining set for cooperative TU-games, where objections and counter-objections are permitted only between elements of this collection, and corresponding generalizations of the kernel are considered. We describe conditions on the fixed collection of coalitions that ensure existence of corresponding sets of imputations for all n-person games.
All sufficient conditions are based on a generalization of [Peleg]. Here relations are defined not on the player set, but on the set of coalitions, and acyclicity is not assumed. Obtained sufficient conditions are also necessary for both generalized bargaining sets if the number of players is no more than five and for one of generalized kernels.

Keywords: Cooperative games, kernel, bargaining set.

Introduction

The theory of the bargaining set, the kernel, and the nucleolus for cooperative TU-games develops more than forty years (see [Maschler, 1992]). The existence theorems for the kernel and the bargaining set $\mathcal{M}_1^i$ are well known. See [Davis, 1963], [Davis, 1967], [Peleg, 1967], [Maschler, 1966].

In [Naumova, 1976] the bargaining set $\mathcal{M}_1^i$ is generalized. For each family of coalitions $\mathcal{A}$, she permits objections and counter-objections only between the members of this family. The resulting $\mathcal{A}$-bargaining set is denoted $\mathcal{M}_{\mathcal{A}}^i$. In this spirit a

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[1] This research was supported by NWO (The Netherlands Organization for Scientific Research), grant NL-RF 047.017.017 and RFBR (Russian Foundation for Basic Research), grant 05-01-89005-HBO-a.

$\mathcal{A}$-kernel, $\mathcal{K}_{\mathcal{A}}$, which is contained in $\mathcal{M}^i_{\mathcal{A}}$, are also defined. If $\mathcal{A}$ is the set of all singletons, then $\mathcal{M}^i_{\mathcal{A}} = \mathcal{M}^i_1$ and $\mathcal{K}_{\mathcal{A}}$ is the kernel. Here another generalization of $\mathcal{M}^i_1$, denoted by $\bar{\mathcal{M}}^i_{\mathcal{A}}$ and the corresponding generalization of the kernel $\bar{\mathcal{K}}_{\mathcal{A}}$ are defined. $\bar{\mathcal{K}}_{\mathcal{A}} \subset \bar{\mathcal{M}}^i_{\mathcal{A}} \subset \mathcal{M}^i_{\mathcal{A}}$ but neither $\bar{\mathcal{K}}_{\mathcal{A}} \subset \mathcal{K}_{\mathcal{A}}$ nor $\mathcal{K}_{\mathcal{A}} \subset \bar{\mathcal{K}}_{\mathcal{A}}$.

Papers [Naumova, 1976] and [Naumova, 1978] contain sufficient conditions on $\mathcal{A}$ which guarantee that each game would have a non-empty $\mathcal{A}$-kernel for every coalition structure.

Sufficient conditions on $\mathcal{A}$ for existence of $\bar{\mathcal{M}}^i_{\mathcal{A}}$ and new sufficient conditions on $\mathcal{A}$ for existence of the generalized bargaining set $\mathcal{M}^i_{\mathcal{A}}$ are obtained.

All these sufficient conditions are formulated in terms of special directed $\mathcal{A}$-admissible (or weakly $\mathcal{A}$-admissible) graphs, where $\mathcal{A}$ is the set of vertices. For fixed $\mathcal{A}$, they can be verified in a finite number of steps.

If the number of players is no more than 5, for both generalizations of bargaining sets their sufficient conditions on $\mathcal{A}$ for existence result are also necessary.

Sufficient condition on $\mathcal{A}$ for existence of $\mathcal{K}_{\mathcal{A}}$ which was obtained in [Naumova, 1978], is also necessary.

The new existence theorems are based on a generalization of Peleg's theorem in [Peleg]. Peleg proved the existence of equilibrium imputation for open acyclic relations on the set of players. This generalization guarantees the existence of equilibrium imputations for open relations on the considered family of coalitions under some additional conditions without acyclicity assumption.

1. Definitions

A cooperative TU-game is a pair (N, v), where $N = \{1, \dots, n\}$ is a set of players, v is a map that takes each $S \subset N$ to a number $v(S)$, $v(\emptyset) = 0$.

For cooperative TU-game (N, v), an *imputation* is a vector $x = \{x_i\}_{i \in N}$ such that $\sum_{i \in N} x_i = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in N$.

Let Γ^0 be the set of games (N, v) such that $v(\{i\}) = 0$ for all $i \in N$ and $v(S) \geq 0$ for all $S \subset N$. (Such games are 0-normalizations of games (N, v) with $\sum_{i \in S} v(\{i\}) \leq v(S)$ for all $S \subset N$.)

In what follows, we consider only $(N, v) \in \Gamma^0$.

Let (N, v) be a cooperative TU-game, $K, L \subset N$, x be an imputation for (N, v). An *objection of* K *against* L *at* x is a pair (C, y_C), such that $K \subset C \subset N$, $L \cap C = \emptyset$, $y_C = \{y_i\}_{i \in C}$, $\sum_{i \in C} y_i = v(C)$, $y_i > x_i$ for all $i \in K$, and $y_i \geq x_i$ for all $i \in C$.

A *counter-objection of* L *against* K *to this objection* is a pair (D, z_D) such that $L \subset D \subset N$, $K \not\subset D$, $\sum_{i \in D} z_i = v(D)$, $z_i \geq x_i$ for all $i \in D$, $z_i \geq y_i$ for all $i \in C \cap D$.

A *strong counter-objection of* L *against* K *to this objection* is a pair (D, z_D) such that $L \subset D \subset N$, $K \cap D = \emptyset$, $\sum_{i \in D} z_i = v(D)$, $z_i \geq x_i$ for all $i \in D$, $z_i \geq y_i$ for all $i \in C \cap D$.

An objection is *justified* if there is no counter-objection to it. An objection is *weakly justified* if there is no strong counter-objection to it.

Let $\mathcal{A}$ be a set of subsets of N. An imputation x of (N, v) belongs to the *bargaining set* $\mathcal{M}^i_{\mathcal{A}}(N, v)$ if for all $K, L \in \mathcal{A}$ there are no justified objections of K against L at x.

An imputation x of (N,v) belongs to the *strong bargaining set* $\bar{\mathcal{M}}^i_{\mathcal{A}}(N,v)$ if for all $K,L\in\mathcal{A}$ there are no weakly justified objections of K against L at x.

If $\mathcal{A}$ is the set of all singletons, then $\bar{\mathcal{M}}^i_{\mathcal{A}}(N,v)=\mathcal{M}^i_{\mathcal{A}}(N,v)=\mathcal{M}^i_1(N,v)$.

Note that $\bar{\mathcal{M}}^i_{\mathcal{A}}(N,v)\subset\mathcal{M}^i_{\mathcal{A}}(N,v)$.

For a set $\mathcal{A}$ of subsets of N consider the following two generalizations of the kernel.

Let $K,L\subset N$ and x be an imputation of (N,v). *K overweights L at x* if $K\cap L=\emptyset$, $\sum_{i\in L}x_i>v(L)$, and $s_{K,L}(x)>s_{L,K}(x)$, where

$$s_{P,Q}(x)=\max\{v(S)-\sum_{i\in S}x_i : S\subset N, P\subset S, Q\not\subset S\}.$$

The set $\mathcal{K}_{\mathcal{A}}(N,v)$ is the set of all imputations x of (N,v) such that no $K\in\mathcal{A}$ overweights any $L\in\mathcal{A}$.

K prevails L at x if $K\cap L=\emptyset$, $\sum_{i\in L}x_i>v(L)$, and $t_{K,L}(x)>t_{L,K}(x)$, where

$$t_{P,Q}(x)=\max\{v(S)-\sum_{i\in S}x_i : S\subset N, P\subset S, Q\cap S=\emptyset\}.$$

The set $\bar{\mathcal{K}}_{\mathcal{A}}(N,v)$ is the set of all imputations x of (N,v) such that no $K\in\mathcal{A}$ prevails any $L\in\mathcal{A}$.

If $\mathcal{A}$ is the set of all singletons, then $\mathcal{K}_{\mathcal{A}}$ and $\bar{\mathcal{K}}_{\mathcal{A}}$ coincide with the kernel. It was proved in [Naumova, 1976] that $\mathcal{K}_{\mathcal{A}}(N,v)\subset\mathcal{M}^i_{\mathcal{A}}(N,v)$. It will be proved below that $\bar{\mathcal{K}}_{\mathcal{A}}(N,v)\subset\bar{\mathcal{M}}^i_{\mathcal{A}}(N,v)$. It will be shown below (Examples 4 and 14) that neither $\bar{\mathcal{K}}_{\mathcal{A}}(N,v)\subset\mathcal{K}_{\mathcal{A}}(N,v)$ nor $\bar{\mathcal{K}}_{\mathcal{A}}(N,v)\supset\mathcal{K}_{\mathcal{A}}(N,v)$. In what follows for each $x\in R^n$, we denote $x(S)=\sum\limits_{i\in S}x_i$.

2. Existence results

2.1. Equilibrium points for general relations on $\mathcal{A}$

Let $N=\{1,\dots,n\}$, $X\subset R^n$, $\mathcal{A}$ be a collection of subsets of N, $\{\succ_x\}_{x\in X}$ be a collection of binary relations $\succ_x$ defined on $\mathcal{A}$. Then $x^0\in X$ is an *equilibrium vector on $\mathcal{A}$* if $K\not\succ_{x^0}L$ for all $K,L\in\mathcal{A}$.

For $b>0$, $K\in\mathcal{A}$ denote

$$X(b)=\{x\in R^n : x_i\geq 0,\ \ x(N)=b\},$$

$$F^K(b)=\{x\in X(b): L\not\succ_x K \quad \text{for all} \quad L\in\mathcal{A}\}.$$

In this subsection sufficient conditions for existence of equilibrium vectors in $X(b)$ are described. The following result of [Peleg, 1963] will be used in the proof. Let $\mathcal{P}$ be a partition of the set N, w be a non-negative function defined on $\mathcal{P}$,

$$X(\mathcal{P},w)=\{x\in R^n :\ \ x(S)=w(S)\ \forall S\in\mathcal{P},\ x_i\geq 0\ \forall i\in N\}.$$

Peleg's Lemma. *Let $c_1, \dots, c_n$ be nonnegative continuous functions defined on $X(\mathcal{P}, w)$. If for each $x \in X(\mathcal{P}, w)$, $S \in \mathcal{P}$, there exists $i_0 = i_0(x) \in S$ such that $c_{i_0}(x) \geq x_{i_0}$, then there exists $x^0 \in X(\mathcal{P}, w)$ such that $c_i(x^0) \geq x_i^0$ for all $i \in N$.*

Theorem 1. *Let a family of binary relations $\{\succ_x\}_{x \in X(b)}$ on $\mathcal{A}$ satisfy the conditions:*
1) for all $K \in \mathcal{A}$, the set $F^K(b)$ is closed;
2) if $x_i = 0$ for all $i \in K$, then $x \in F^K(b)$;
3) for each $x \in X(b)$, the set of coalitions $\{L \in \mathcal{A} : K \succ_x L$ for some $K \in \mathcal{A}\}$ does not cover N.

Then there exists an equilibrium vector $x^0 \in X(b)$ on $\mathcal{A}$.

Proof.

Denote $\mathcal{A}_i = \{K \in \mathcal{A} : i \in K\}$, $N_0 = \{i \in N : \cap_{K \in \mathcal{A}_i} F^K(b) \neq \emptyset\}$.

If $x \in X(b)$, then by condition 3, there exists $i_0 = i_0(x) \in N$ such that the set

$$\{L \in \mathcal{A} : K \succ_x L \quad \text{for some} \quad K \in \mathcal{A}\}$$

does not cover i_0, i.e. $x \in F^K(b)$ for all $K \in \mathcal{A}_{i_0}$. This implies $x \in \cap_{K \in \mathcal{A}_{i_0}} F^K(b)$ and $i_0 \in N_0$, hence $N_0 \neq \emptyset$.

Let $\mathcal{P}$ be the partition of N that consists of N_0 and $\{j\}$ for all $j \in N \setminus N_0$, $w(N_0) = b$, $w(\{j\}) = 0$ for all $j \in N \setminus N_0$. Then

$$X(\mathcal{P}, w) = \{x \in X(b) : x_j = 0 \quad \text{for all} \quad j \in N \setminus N_0\}.$$

Define functions c_i $(i \in N)$ on the set $X(\mathcal{P}, w)$ by

$$c_i(x) = x_i - x_i \rho(x, \cap_{K \in \mathcal{A}_i} F^K(b))/b \qquad \text{for } i \in N_0,$$

$$c_i(x) = x_i = 0 \quad \text{otherwise},$$

where $\rho(x, y) = \max_{i \in N} |x_i - y_i|$, $\rho(x, Y) = \inf_{y \in Y} \rho(x, y)$ for all $Y \subset R^n$.

Then c_i are nonnegative continuous functions. Let $x \in X(\mathcal{P}, w)$, then $x \in X(b)$ and by the proved above, there exists $i_0 = i_0(x) \in N_0$ such that $x \in \cap_{K \in \mathcal{A}_{i_0}} F^K(b)$. Then $c_{i_0}(x) = x_{i_0}$. By Peleg's Lemma, there exists $x^0 \in X(\mathcal{P}, w)$ such that $c_i(x^0) \geq x_i^0$ for all $i \in N$.

We prove that $x^0 \in F^M(b)$ for all $M \in \mathcal{A}$. If $x_i^0 = 0$ for all $i \in M$, then $x^0 \in F^M(b)$ by condition 2. If $x_j^0 > 0$ for some $j \in M$, then $j \in N_0$ and $\rho(x^0, \cap_{K \in \mathcal{A}_j} F^K(b)) = 0$. Since the sets $F^K(b)$ are closed by condition 1, $x^0 \in F^K(b)$ for all $K \in \mathcal{A}_j$ and, in particular, $x^0 \in F^M(b)$.

Corollary. *Let G be an undirected graph with the set of vertices $\mathcal{A}$ and a family of binary relations $\{\succ_x\}_{x \in X(b)}$ on $\mathcal{A}$ satisfy the conditions:*
1) for all $K \in \mathcal{A}$ the set $F^K(b)$ is closed;
2) if $x_i = 0$ for all $i \in K$, then $x \in F^K(b)$;

3) for each $x \in X(b)$ the relation $\succ_x$ is acyclic;
4) if a single vertex is taken in each connected component of G, then the remaining elements of $\mathcal{A}$ do not cover N.

Then there exists an equilibrium vector $x^0 \in X(b)$ on $\mathcal{A}$.

This corollary coincides with Theorem 2 in [Naumova, 1978], which provides sufficient condition for existence of the generalized kernel $\mathcal{K}_{\mathcal{A}}$ and the bargaining set $\mathcal{M}^i_{\mathcal{A}}$.

2.2. Necessary and sufficient condition for existence of generalized kernels

A set of coalitions $\mathcal{A}$ generates the undirected graph $G = G(\mathcal{A})$, where $\mathcal{A}$ is the set of vertices and $K, L \in \mathcal{A}$ are adjacent iff $K \cap L = \emptyset$.

Theorem 2. *Let $\mathcal{A}$ be a set of subsets of N. Then $\mathcal{K}_{\mathcal{A}}(N, v) \neq \emptyset$ for all (N, v) if and only if $\mathcal{A}$ satisfies the following condition .*

C0) If a single vertex is taken in each connected component of $G(\mathcal{A})$, then the remaining elements of $\mathcal{A}$ do not cover N.

Proof.

It was proved in [Naumova, 1978] that if $\mathcal{A}$ satisfies C0, then $\mathcal{K}_{\mathcal{A}}(N, v) \neq \emptyset$ for all (N, v).

Now suppose that $\mathcal{A}$ does not satisfy C0. We can assume without loss of generality that each connected component of $G(\mathcal{A})$ contains at least two vertices. Indeed, let $\bar{\mathcal{A}}$ be obtained from $\mathcal{A}$ by deleting all isolated vertices of $G(\mathcal{A})$. Then $\bar{\mathcal{A}}$ does not satisfy C0 and $\mathcal{K}_{\mathcal{A}}(N, v) = \mathcal{K}_{\bar{\mathcal{A}}}(N, v)$.

Let $G(\mathcal{A})$ have k connected components. Let $S_1, \ldots, S_k$ be taken in all different connected components of $G(\mathcal{A})$ and the union of the remaining elements of these components cover N. We construct a game (N, w) such that $\mathcal{K}_{\mathcal{A}}(N, w) = \emptyset$. We consider two cases.

Case 1. The family $\mathcal{A}$ does not contain singletons. We can assume that each S_i does not include the other elements of $\mathcal{A}$. Indeed, let $Q \in \mathcal{A}$, $Q \subset S_i$. There exists $P \in \mathcal{A}$ such that $P \cap S_i = \emptyset$. Then $P \cap Q = \emptyset$, hence, S_i and Q belong to the same connected component and it is possible to take Q instead of S_i. Define (N, w) as follows:

$$w(N) = w(S_i) = 1 \quad \text{for } i = 1, \ldots, k,$$

$$w(Q) = 0 \quad \text{otherwise.}$$

Suppose that there exists $x \in \mathcal{K}_{\mathcal{A}}(N, w)$. It will be proved that for each $i = 1, \ldots, k$, if Q and S_i belong to the same connected component, $Q \neq S_i$, then $x(Q) = 0$. This implies $x(N) = 0$ and contradicts $x(N) = w(N) = 1$.

Fix $i \leq k$. Since $x(S_i) \leq 1 = w(S_i)$, $v(S_i) - x(S_i) \geq 0$. Let $L \in \mathcal{A}$, $L \cap S_i = \emptyset$. Then $S_j \not\supset L$ for all $j = 1, ..., k$, so $v(P) - x(P) \leq 0$ for all $P \supset L$, $P \neq N$. Thus, $s_{S_i,L}(x) \geq 0$, $s_{L,S_i}(x) \leq 0$ and $x \in \mathcal{K}_{\mathcal{A}}(N, w)$ implies $x(L) = 0$. Let $T \in \mathcal{A}$,

$T \neq S_i$, $T \cap L = \emptyset$. Since $x(L) = 0$, $s_{L,T}(x) \geq 0$. Since $S_j \not\supset T$ for all $j = 1, \ldots, k$, $s_{T,L}(x) \leq 0$, therefore, $x \in \mathcal{K}_{\mathcal{A}}(N, w)$ implies $x(T) = 0$. Thus, by induction on the distance from S_i in $G(\mathcal{A})$ we prove that $x(P) = 0$ for all $P \in \mathcal{A} \setminus \{S_1, \ldots, S_k\}$.

Case 2. The family $\mathcal{A}$ contains a singleton. Denote it by $\{1\}$. Then all connected components of $G(\mathcal{A})$ that do not contain $\{1\}$, contain only one element. Therefore, we can assume that $G(\mathcal{A})$ is a connected graph.

Let S be a coalition in $\mathcal{A}$ such that the remaining coalitions cover N and S is minimal in $\mathcal{A}$. If $|S| > 1$, then take the same game as in case 1.

Let $|S| = 1$. Assume that $S = \{1\}$. We can not take $w(S) = 1$ because $w(\{i\}) = 0$ for all $i \in N$. There exist $S_1, Q_1 \in \mathcal{A}$ such that $1 \in S_1$, $|S_1| > 1$, and $S_1 \cap Q_1 = \emptyset$. Define (N, w) as follows:
$w(N) = w(Q_1 \cup \{1\}) = 1$, $w(P) = 0$ otherwise.

Let $x \in \mathcal{K}_{\mathcal{A}}(N, w)$. For each imputation x of (N, w),

$$s_{Q_1,S_1}(x) \geq w(Q_1 \cup \{1\}) - x_1 - x(Q_1) \geq 0, \quad s_{S_1,Q_1}(x) \leq 0.$$

Therefore, $x \in \mathcal{K}_{\mathcal{A}}(N, w)$ implies $x(S_1) = 0$, hence, $x_1 = 0$. Let $L \in \mathcal{A}$, $1 \notin L$, then $s_{\{1\},L}(x) \geq v(\{1\}) - x_1 = 0$ and $s_{L,\{1\}}(x) \leq 0$, hence, $x \in \mathcal{K}_{\mathcal{A}}(N, w)$ implies $x(L) = 0$. In particular, $x(Q_1) = 0$.

Let $T \in \mathcal{A}$, $1 \in T$. If $T \subset (Q_1 \cup \{1\})$, then $x(T) = 0$ by the proved above. Let $T \not\subset (Q_1 \cup \{1\})$. There exists $M \in \mathcal{A}$ such that $M \cap T = \emptyset$. Since $x(M) = 0$, $s_{M,T}(x) \geq v(M) - x(M) = 0$. Since $T \not\subset (Q_1 \cup \{1\})$, $s_{T,M}(x) \leq 0$. Therefore, $x \in \mathcal{K}_{\mathcal{A}}(N, w)$ implies $x(T) = 0$. Thus, $x(P) = 0$ for all P in the main connected component, hence, $x(N) = 0$ and this contradicts $x(N) = w(N) = 1$.

Let us consider the collections of subsets $\mathcal{A}$ that satisfy the condition C0 of Theorem 2. If N is not covered by the elements of $\mathcal{A}$, then this condition is obviously fulfilled. Let $\mathcal{A}^0$ be the set of isolated vertices of graph $G(\mathcal{A})$, $\bar{\mathcal{A}} = \mathcal{A} \setminus \mathcal{A}^0$. Then the collections of coalitions $\mathcal{A}$ and $\bar{\mathcal{A}}$ satisfy the condition C0 simultaneously.

A player i is called a *fanatic for* $\mathcal{A}$ if it belongs to precisely one element of $\mathcal{A}$.

Let us describe for some n all collections of coalitions $\bar{\mathcal{A}}$ that cover N and satisfy C0.

If $G(\bar{\mathcal{A}})$ has only one connected component, then each element of $\bar{\mathcal{A}}$ has a fanatic.

Example 1. For $n = 3$, if $\mathcal{A}$ satisfies C0, then either $\bar{\mathcal{A}} = \{\{1\}, \{2\}, \{3\}\}$ or $\bar{\mathcal{A}} = \{\{i, j\}, \{k\}\}$.

Example 2. For $n = 4$, if $\mathcal{A}$ satisfies C0, then either each element of $\bar{\mathcal{A}}$ contains a fanatic of $\bar{\mathcal{A}}$ or $\bar{\mathcal{A}} = \{\{i, j\}, \{k, l\}, \{i, k\}, \{j, l\}\}$. Indeed, if $\bar{\mathcal{A}}$ contains 1-person or 3-person coalition, then $G(\mathcal{A})$ has only one connected component. In the remaining case, if $G(\bar{\mathcal{A}})$ has 3 connected components, then C0 is not fulfilled, and only the case when $G(\bar{\mathcal{A}})$ has 2 connected components, is non-trivial.

Example 3. For $n = 5$, there are two non-trivial cases of $\mathcal{A}$ that satisfy C0.
1) $\bar{\mathcal{A}} = \{\{i, j\}, \{k, l, m\}, \{i, k\}, \{j, l\}\}$;
2) $\bar{\mathcal{A}} = \{\{i, j\}, \{k, l, m\}, \{i, k\}, \{j, l, m\}\}$.

Example 4. $\bar{\mathcal{K}}_{\mathcal{A}} \not\subset \mathcal{K}_{\mathcal{A}}$. Indeed, let $N = \{1, 2, 3\}$, $\mathcal{A} = \{\{1\}, \ \{2\}, \ \{1, 3\}\}$. By Theorem 2, there exists a game (N, v) such that $v(\{i\}) = 0$ for $i = 1, 2, 3$, $0 \leq v(S) \leq v(N) = 1$ for $S \subset N$, and $\mathcal{K}_{\mathcal{A}}(N, v) = \emptyset$. We prove that $\bar{\mathcal{K}}_{\mathcal{A}}(N, v) \neq \emptyset$.

Let $x_1 = (1 - v(\{2, 3\}))/1$, $x_2 = (1 - v(\{1, 3\}))/2$, $x_3 = 1 - x_1 - x_2$. Then x is an imputation for (N, v) and

$$v(\{1\}) - x_1 = (v(\{2, 3\}) - 1)/2, \ v(\{2\}) - x_2 = (v(\{1, 3\}) - 1)/2,$$

$$v(\{1, 3\}) - x_1 - x_3 = (v(\{1, 3\}) - 1)/2, \ v(\{2, 3\}) - x_2 - x_3 = (v(\{2, 3\}) - 1)/2.$$

Therefore,

$$t_{\{1\},\{2\}}(x) = \max\{(v(\{2, 3\}) - 1)/2, \ (v(\{1, 3\}) - 1)/2\} = t_{\{2\},\{1\}}(x),$$

$$t_{\{2\},\{1,3\}}(x) = (v(\{1, 3\}) - 1)/2 = t_{\{1,3\},\{2\}}(x),$$

so $x \in \bar{\mathcal{K}}_{\mathcal{A}}(N, v)$.

2.3. Existence conditions for generalized bargaining sets

The condition C0 of Theorem 2 is sufficient for non-emptyness of $\mathcal{M}^i_{\mathcal{A}}$ but is not necessary.

Example 5. Let $\mathcal{A}$ consists of all 1-person and $(n - 1)$-person coalitions. Then $\mathcal{M}^i_{\mathcal{A}} = \mathcal{M}^i_1$.

Indeed, the objections of 1-person coalition against $(n - 1)$-person coalition are impossible and if $(n - 1)$-person coalition has a justified objection against 1-person coalition, then each singleton, contained in $(n - 1)$-person coalition, has the same justified objection against 1-person coalition.

Example 6. Let $\mathcal{A} = \{K, L, M\}$, where $K \subset L$, $K \neq L$, $M \cap L = \emptyset$, $M \cup L = N$. Then $\mathcal{A}$ does not satisfy the condition C0, but $\mathcal{M}^i_{\mathcal{A}}(N, v) \neq \emptyset$ for each $(N, v) \in \Gamma^0$.

Indeed, for (N, v), let x^0 be an imputation such that

$$x^0(M) = \min\{v(N), v(M)\}, x^0_i = 0 \quad \text{for all } i \in K.$$

Then $x^0 \in \mathcal{M}^i_{\mathcal{A}}(N, v)$. Indeed, if there exists a justified objection (Q, y_Q) at x^0, then, since $x^0(K) \leq v(K)$ and $x^0(M) \leq v(M)$, (Q, y_Q) is an objection of M against L and $Q = M$. If $v(M) \leq v(N)$, then $x^0(M) \geq v(M)$ and such objection is impossible. If $v(M) > v(N)$, then $x^0(L) = 0 \leq v(L)$ and there exists a counter-objection (L, z_L).

Now we describe more weak conditions on $\mathcal{A}$ that ensure the non-emptyness of $\mathcal{M}^i_{\mathcal{A}}$ for all games in Γ^0. The following lemmas generalize well known results on the bargaining set $\mathcal{M}^i_1$.

Lemma 1. *Let (Q, y_Q) be an objection of K against L at x. Let $S \subset N$, $L \subset S$, $K \not\subset S$. Then there exists a counter-objection (S, z_S) to this objection if $(y-x)(Q \cap S) \leq v(S) - x(S)$.*

Proof.

Let $(y-x)(Q \cap S) > v(S) - x(S)$. Suppose that there exists a counter-objection (S, z_S) to (Q, y_Q). Then

$$z(S) = z(Q \cap S) + z(S \setminus Q) \geq y(Q \cap S) + x(S \setminus Q) = (y-x)(Q \cap S) + x(S),$$

this contradicts $z(S) = v(S)$.

Let $(y-x)(Q \cap S) \leq v(S) - x(S)$. Then we can take z_S such that

$z_{Q \cap S} = y_{Q \cap S}$,

$z_{S \setminus Q} \geq x_{S \setminus Q}$,

$(z-x)(S \setminus Q) = v(S) - x(S) - (y-x)(Q \cap S)$.

Let us verify that $z(S) = v(S)$.

$z(S) = z(S \setminus Q) + z(S \cap Q) = z(S \setminus Q) + y(S \cap Q) = (z-x)(S \setminus Q) + (y-x)(S \cap Q) + x(S)$
$= v(S) - x(S) - (y-x)(Q \cap S) + (y-x)(S \cap Q) + x(S) = v(S)$.

Thus (S, z_S) is a counter-objection to (Q, y_Q).

Lemma 2. *Let x be an imputation for (N, v), $K, L \subset N$, $K \subset Q \subset N$, $L \cap Q = \emptyset$. Let $S \subset N$, $L \subset S$, $K \not\subset S$, $x(Q) < v(Q)$.*

If either $Q \cap S = \emptyset$ and $x(S) \leq v(S)$, or $Q \cap S \neq \emptyset$ and $v(Q) - x(Q) \leq v(S) - x(S)$, then for each objection (Q, y_Q) of K against L at x there exists a counter-objection (S, z_S) of L against K.

Proof.

Let y_Q be an objection of K against L at x.

If $Q \cap S = \emptyset$ and $x(S) \leq v(S)$, then take z_S such that $z_S(S) = v(S)$ and $z_S \geq x_S$.

Suppose that $Q \cap S \neq \emptyset$ and $v(Q) - x(Q) \leq v(S) - x(S)$. Note that

$$v(Q) - x(Q) = y(Q) - x(Q) = (y-x)(Q \cap S) + (y-x)(Q \setminus S),$$

so $(y-x)(Q \setminus S) \geq 0$ implies

$$(y-x)(Q \cap S) \leq v(Q) - x(Q) \leq v(S) - x(S).$$

By Lemma 1, there exists a counter-objection (S, z_S) to (Q, y_Q).

Let $\mathcal{A}$ be a family of subsets of N. A directed graph Gr is called $\mathcal{A}$-*admissible* if $\mathcal{A}$ is the set of its vertices and there exists a map f defined on the set of the edges of Gr, that takes each oriented edge (K, L) to a pair $f(K, L) = (Q, r)$ ($Q \subset N$, $r \in R^1$) and satisfies the following 3 conditions.

C1. If $f(K, L) = (Q, r)$, then $K \subset Q$, $Q \cap L = \emptyset$, $|Q| > 1$.

C2. If $f(K, L) = (Q, r)$, $f(R, P) = (S, t)$, $L \subset S$, $K \not\subset S$, then $Q \cap S \neq \emptyset$.

C3. If $f(K, L) = (Q, r)$, $f(R, P) = (S, t)$, $L \subset S$, $K \not\subset S$, then $r > t$.

Example 7. Let $\mathcal{A}_1 = \{K, L, M\}$, where $K \subset L$, $K \neq L$, $M \cap L = \emptyset$, $M \cup L = N$. Let Gr_1 be a digraph, where $\mathcal{A}_1$ is the set of vertices and $\{(K, M), (M, L)\}$ is the set of edges.

Then Gr_1 is not $\mathcal{A}_1$-admissible. Indeed, if Gr_1 is $\mathcal{A}_1$-admissible and f is the corresponding map, then, by C1, $f(M, L) = (M, r)$, $f(K, M) = (Q, t)$, $Q \cap M = \emptyset$, but this contradicts C2.

Example 8. Let $N = \{1, 2, 3, 4\}$, $\mathcal{A}_2 = \{\{1\},\ \{2\},\ \{3\},\ \{3, 4\}\}$. Let Gr_2 be a digraph with the set of vertices $\mathcal{A}_2$ and the set of edges
$\{(\{3\}, \{2\}),\ (\{2\}, \{3, 4\}),\ (\{3, 4\}, \{1\})\}$.

Then Gr_2 is $\mathcal{A}_2$-admissible. Indeed, take
$f(\{3\}, \{2\}) = (\{1, 3\}, 2)$,
$f(\{2\}, \{3, 4\}) = (\{1, 2\}, 1)$,
$f(\{3, 4\}, \{1\}) = (\{2, 3, 4\}, 3)$.

Theorem 3. *Let $\mathcal{A}$ be a set of subsets of N. If for each $\mathcal{A}$-admissible graph Gr the set of the ends of its edges does not cover N, then $\mathcal{M}^i_{\mathcal{A}}(N, v) \neq \emptyset$ for all $(N, v) \in \Gamma^0$.*

Proof.

Let $(N, v) \in \Gamma^0$, x be an imputation for (N, v). For $K, L \in \mathcal{A}$, denote $K \succ_x L$ if there exists a justified objection of K against L at x. Let $b = v(N)$, then $X(b)$ is the set of all imputations for (N, v). We prove that the family of binary relations $\{\succ_x\}_{x \in X(b)}$ satisfies all conditions of Theorem 1.

It follows from Lemma 1 that the set $F^K(b)$ is closed for all $K \in \mathcal{A}$. If $x_i = 0$ for all $i \in K$, then $x(K) \leq v(K)$ and, by Lemma 2, there exists a counter-objection (K, z_K) to each objection against K at x.

Let us verify that for each imputation x the set of coalitions $\{L \in \mathcal{A} : K \succ_x L \text{ for some } K \in \mathcal{A}\}$ does not cover N. Let $Gr(x)$ be the digraph, where $\mathcal{A}$ is the set of vertices and (K, L) is the edge of $Gr(x)$ iff $K \succ_x L$. Let us prove that $Gr(x)$ is $\mathcal{A}$-admissible.

Define the map f as follows. $f(K, L) = (C, r)$ if there exists a justified objection (C, y_C) of K against L at x. (If there exist several justified objections, then one of them is fixed arbitrary.) The numbers r must satisfy the following: if $f(K, L) = (C, r)$, $f(M, P) = (Q, t)$, then $r \leq t$ iff $e(C, x) \leq e(Q, x)$, where $e(S, x) = v(S) - x(S)$.

Let us check that f satisfies the conditions C1–C3. C1 follows from the definition of objection; $|f_1(K, L)| > 1$ because $x_i \geq v(i)$ for all $i \in N$.

C2. Suppose that there exist the edges (K, L) and (M, P) of $Gr(x)$ such that $f(K, L) = (Q, r)$, $f(M, P) = (S, t)$, $L \subset S$, $K \not\subset S$, and $Q \cap S = \emptyset$. Then $x(S) < v(S)$ by the definition of objection, hence, by Lemma 2, there exists z_S such that (S, z_S) is the counter-objection of L against K to any objection (Q, y_Q) of K against L at x, this contradicts the definition of f.

C3. Suppose that there exist the edges (K, L) and (M, P) of $Gr(x)$ such that $f(K, L) = (Q, r)$, $f(M, P) = (S, t)$, $L \subset S$, $K \not\subset S$, and $r \leq t$. Then $v(Q) - x(Q) \leq$

$v(S) - x(S)$. By Lemma 2, there is a counter-objection to each objection (Q, y_Q) of K against L at x, this contradicts the definition of f.

Thus, $Gr(x)$ is $\mathcal{A}$-admissible, hence the ends of its edges do not cover the set N. By Theorem 1, there exists an equilibrium imputation x^0 on $\mathcal{A}$, i.e. $x_0 \in \mathcal{M}^i_{\mathcal{A}}(N, v)$.

The following propositions help to use Theorem 3.

Proposition 1. *Each $\mathcal{A}$-admissible digraph Gr generates an acyclic binary relation on $\mathcal{A}$.*

Proof.

Suppose that for some N, $\mathcal{A}$, and $\mathcal{A}$-admissible digraph Gr there exist $K^1, \dots, K^t \in \mathcal{A}$ such that (K^i, K^{i+1}) for $i = 1, \dots, t-1$ and (K^t, K^1) are edges of Gr.

For each integer j, let $\overline{j} \in \{1, \dots, t\}$, $\overline{j} \equiv j$ (mod. t).

Let $f(K^i, K^{\overline{i+1}}) = (Q_i, r_i)$, $r_{i_0} = \max_{i=1,\dots,t} r_i$. If $Q_{i_0} \not\supset K^{\overline{i_0-1}}$, then as $Q_{i_0} \supset K^{i_0}$, the condition C3 implies $r_{i_0} < r_{\overline{i_0-1}}$, thus, $Q_{i_0} \supset K^{\overline{i_0-1}}$. Similarly, $Q_{i_0} \supset K^{\overline{i_0-2}}$,..., and $Q_{i_0} \supset K^{\overline{i_0+1}}$, but this contradicts C1.

Proposition 2. *Let $\mathcal{A}$, $\mathcal{B}$ be families of subsets of N, $\mathcal{B} \subset \mathcal{A}$ and $\mathcal{A} \setminus \mathcal{B}$ contains only $(n-1)$-person coalitions. Then $\mathcal{M}^i_{\mathcal{B}}(N, v) \neq \emptyset$ for all $(N, v) \in \Gamma^0$ implies $\mathcal{M}^i_{\mathcal{A}}(N, v) \neq \emptyset$ for all $(N, v) \in \Gamma^0$.*

Proof.

Suppose that there exists $i_0 \in N$ such that $i_0 \in S$ for all $S \in \mathcal{B}$. Then $G(\mathcal{A})$ may contain only one non-trivial connected component $\{\{i_0\}, N \setminus \{i_0\}\}$. By Proposition 1, for each $\mathcal{A}$-admissible digraph Gr the ends of its edges do not cover N and, by Theorem 3, $\mathcal{M}^i_{\mathcal{A}}(N, v) \neq \emptyset$ for all $(N, v) \in \Gamma^0$. Otherwise, $\mathcal{M}^i_{\mathcal{B}}(N, v) = \mathcal{M}^i_{\mathcal{A}}(N, v)$. Indeed, let $x \in \mathcal{M}^i_{\mathcal{B}}(N, v)$. Suppose that there exists a justified objection (Q, y_Q) of K against L at x. Then $|Q| > 1$, hence, $|L| < n-1$ and $|K| = n-1$, $L = \{i_0\}$, and $Q = K$. In considered case, there exists $T \in \mathcal{B}$, $T \subset K$. Then (Q, y_Q) is an objection of T against L at x, hence, there exists a counter-objection (D, z_D) to this objection. By the definition of counter-objection, (D, z_D) is also a counter-objection of L against K to (Q, y_Q), and the objection is not justified.

Example 9. Let $\mathcal{A}_1 = \{K, L, M\}$, where $K \subset L$, $K \neq L$, $M \cap L = \emptyset$, $M \cup L = N$. If the ends of the edges of digraph Gr cover N, then, by Proposition 1, $Gr = Gr_1$ (see Example 7), where $\{(K, M), (M, L)\}$ is the set of edges of Gr_1. Since Gr_1 is not $\mathcal{A}_1$-admissible, $\mathcal{M}^i_{\mathcal{A}_1}(N, v) \neq \emptyset$ for all $(N, v) \in \Gamma^0$.

The next theorem states that for $|N| \leq 5$ the sufficient condition on $\mathcal{A}$ for existence of $\mathcal{M}^i_{\mathcal{A}}(N, v)$ is also necessary.

Theorem 4. *Let $\mathcal{A}$ be a set of subsets of N and $|N| \leq 5$. Then $\mathcal{M}^i_{\mathcal{A}}(N, v) \neq \emptyset$ for all $(N, v) \in \Gamma^0$ if and only if for each $\mathcal{A}$-admissible graph Gr the set of the ends of its edges does not cover N.*

Proof.

Let $|N| \leq 5$. Suppose that there exists a set of subsets of N $\mathcal{A}$ and an $\mathcal{A}$-admissible digraph Gr such that the ends of its edges cover N. We construct a cooperative game

(N,v) such that $\mathcal{M}^i_{\mathcal{A}}(N,v)=\emptyset$. Let $End(Gr)$ be the set of the ends of the edges of Gr.

We reduce the graph Gr. For each $L\in End(Gr)$, we delete all edges (K,L) except one. The reduced graph is also $\mathcal{A}$-admissible, the set $End(Gr)$ is preserved, and each edge of new Gr is defined by its end.

Let f be a map defined on the set of the edges of Gr that takes each edge (K,L) to a pair $f(K,L)=(Q(L),r(L))$, satisfies the conditions C1–C3 and the following condition. If $\bar{f}$ is a map defined on the set of the edges of Gr that takes each edge (K,L) to a pair $\bar{f}(K,L)=(\bar{Q}(L),r(L))$, satisfies the conditions C1–C3 and $Q(L)\subset\bar{Q}(L)$ for each $L\in End(Gr)$, then $\bar{f}=f$. (This means that it is impossible to extend $Q(L)$ with preserving $r(L)$ and C1–C3.)

Define (N,v) as follows:

$$v(N)=1, v(Q(L))=1 \quad \text{for each } L\in End(Gr),$$

$$v(T)=0 \quad \text{otherwise.}$$

Note that (N,v) is well defined since $|Q(L)|>1$ by C1. We shall prove that $M^i_{\mathcal{A}}(N,v)=\emptyset$.

Step 1. Denote

$$\mathcal{B}(i)=\cup_{S\in End(Gr):r(S)\leq i}S.$$

We prove that for all $L\in End(Gr)$,

$$N\setminus Q(L)\subset\mathcal{B}(r(L)). \tag{1}$$

Suppose that there exists $L\in End(Gr)$ such that $N\setminus Q(L)\not\subset\mathcal{B}(r(L))$. Let $q\in N\setminus(Q(L)\cup\mathcal{B}(r(L)))$. Let $\bar{Q}=Q(L)\cup\{q\}$.

Let us replace $Q(L)$ by $\bar{Q}$. Since it is impossible to extend $Q(L)$ with preserving C1–C3, one of these conditions violates.

Since $L\in\mathcal{B}(r(L))$, $q\notin L$, hence, the condition C1 preserves. If C2 or C3 violate, then there exists an edge (R,P) such that $P\subset\bar{Q}$, $P\not\subset Q(L)$, $R\not\subset\bar{Q}$ and
either C2 violates, i.e. $\bar{Q}\cap Q(P)=\emptyset$,
or C3 violates, i.e. $r(L)\geq r(P)$.

Let $\bar{Q}\cap Q(P)=\emptyset$, then $K\cap Q(P)=\emptyset$. If $L\subset Q(P)$, then, by C2, $Q(L)\cap Q(P)\neq\emptyset$, hence, $\bar{Q}\cap Q(P)\neq\emptyset$. Therefore, $L\not\subset Q(P)$, $|L\setminus Q(P)|\geq 1$. By C1, $|Q(P)|\geq 2$, $|\bar{Q}|\geq 3$. Since the sets $L\setminus Q(P)$, $Q(P)$, $\bar{Q}$ are mutually disjoint, $|N|\geq 6$, this contradicts $|N|\leq 5$.

Let $r(L)\geq r(P)$, then $P\subset\mathcal{B}(r(L))$. Since $P\not\subset Q(L)$, $P\subset\bar{Q}$, we have $q\in P$, hence, $q\in\mathcal{B}(r(L))$, and this contradicts the definition of q. Thus, (1) is proved.

Step 2. Suppose that there exists $x^0\in M^i_{\mathcal{A}}(N,v)$. We shall prove by induction on $r(L)$ that $x^0(L)=0$ for all $L\in End(Gr)$. Since the set $End(Gr)$ covers N, this would imply $x^0(N)=0$, but $x^0(N)=v(N)=1$.

Let $r(L)=r_0$ be minimal, (K,L) be an edge of Gr. Suppose that $x^0(L)>0$. Then $x^0(Q(L))<1$ and there exists an objection of K against L at x^0. There exists a counter-objection (D, z_D) to this objection. Since $v(D) = z_D(D) \geq x^0(L) > 0$, $v(D) = 1$ and $D = Q(P)$ for some $P \in End(Gr)$. By the definition of counter-objection, $L \subset D$, $K \not\subset D$ and C3 implies $r(P) < r(L) = r_0$, but r_0 is minimal. Thus, $x^0(L) = 0$ for all L with $r(L) = r_0$ and $x^0(\mathcal{B}(r_0)) = 0$. In view of (1), this implies

$$x^0(Q(L)) = 1 \quad \text{for all} \quad L \in End(Gr) \quad \text{with} \quad r(L) = r_0.$$

Suppose that $x^0(L) = 0$ and $x^0(Q(L)) = 1$ for $r(L) < i$. Let $r(L) = i$, (K, L) be an edge of Gr. Assume that $x^0(L) > 0$, then $x^0(Q(L)) < 1$, $v(Q(L)) - x^0(Q(L)) > 0$ and there is an objection $(Q(L), y_{Q(L)}$ of K against L at x^0. There exists a counter-objection (S, z_S) to this objection. Since $v(S) = z_S(S) \geq x^0(L) > 0$, $v(S) = 1$ and $S = Q(T)$ for some $T \in End(Gr)$. By the definition of counter-objection, $L \subset S$, $K \not\subset S$ and C3 implies $r(T) < r(L)$. By induction assumption, $x^0(S) = 1$, hence, $v(S) - x^0(S) = 0$. By Lemma 1,

$$v(S) - x^0(S) \geq y_{Q(L)}(S \cap Q(L)) - x^0(S \cap Q(L)).$$

By C2, $S \cap Q(L) \neq \emptyset$, and by the definition of objection,

$$y_{Q(L)}(S \cap Q(L)) - x^0(S \cap Q(L)) > 0.$$

This contradicts $v(S) - x^0(S) = 0$, hence, $x^0(L) = 0$. Thus,

$$x^0(L) = 0 \quad \text{for all} \quad L \in End(Gr) \quad \text{with} \quad r(L) = i.$$

In view of (1), this implies

$$x^0(Q(L)) = 1 \quad \text{for all} \quad L \in End(Gr) \quad \text{with} \quad r(L) = i.$$

Example 10. Let $N = \{1, 2, 3, 4\}$, $\mathcal{A}_2 = \{\{1\},\ \{2\},\ \{3\},\ \{3, 4\}\}$.
Let Gr_2 be a digraph with the set of vertices $\mathcal{A}_2$ and the set of edges

$$\{(\{3\}, \{2\}),\ (\{2\}, \{3, 4\}),\ (\{3, 4\}, \{1\})\}.$$

The set of the edges of Gr_2 covers N, and Gr_2 is $\mathcal{A}_2$-admissible (see Example 8), hence, $\mathcal{M}^i_{\mathcal{A}_2}(N, v) = \emptyset$ for some $(N, v) \in \Gamma^0$.

The conditions on $\mathcal{A}$ that ensure the non-emptyness of $\bar{\mathcal{M}}^i_{\mathcal{A}}(N, v)$ for all $(N, v) \in \Gamma^0$ are similar.

Let $\mathcal{A}$ be a family of subsets of N. A directed graph Gr is called *weakly $\mathcal{A}$-admissible* if $\mathcal{A}$ is the set of its vertices and there exists a map g defined on the set of the edges of Gr, that takes each oriented edge (K, L) to a pair $g(K, L) = (Q, r)$ ($Q \subset N$, $r \in R^1$) and satisfies the following 3 conditions.

C1. If $g(K, L) = (Q, r)$, then $K \subset Q$, $Q \cap L = \emptyset$, $|Q| > 1$.

C2'. If $g(K,L)=(Q,r)$, $g(R,P)=(S,t)$, $L\subset S$, $K\cap S=\emptyset$, then $Q\cap S\neq\emptyset$.
C3'. If $g(K,L)=(Q,r)$, $g(R,P)=(S,t)$, $L\subset S$, $K\cap S=\emptyset$, then $r>t$.

Example 11. Let $N=\{1,2,3,4\}$, $\mathcal{A}_3=\{\{1\},\ \{2\},\ \{1,3\},\ \{1,4\}\}$.
Then digraph Gr_3 with the set of edges $\{\alpha,\beta,\gamma\}$, where

$$\alpha=(\{1\},\{2\}),\quad \beta=(\{2\},\{1,3\}),\quad \gamma=(\{2\},\{1,4\})$$

is not weakly $\mathcal{A}_3$-admissible.

Indeed, suppose that Gr is weakly $\mathcal{A}_3$-admissible. Then, by C1, $g(\beta)=(\{2,4\},i)$, $g(\gamma)=(\{2,3\},j)$ and, by C1 and C2', $g(\alpha)=(\{1,3,4\},k)$. Since $\{2\}\subset\{2,4\}$ and $\{1\}\cap\{2,4\}=\emptyset$, C3' implies $i<k$. Since $\{1,3\}\subset\{1,3,4\}$ and $\{2\}\cap\{1,3,4\}=\emptyset$, C3' implies $k<i$, this contradicts $i<k$.

Note that weakly $\mathcal{A}$-admissible digraph Gr does not generate an acyclic binary relation on $\mathcal{A}$.

Example 12. Let $N=\{1,2,3,4\}$, $\mathcal{A}_4=\{\{1,2\},\{3\},\{4\}\}$. Then digraph with the set of edges $\{\alpha,\beta,\gamma\}$, where $\alpha=(\{1,2\},\{3\})$,$\beta=(\{3\},\{4\})$, $\gamma=(\{4\},\{1,2\})$, is weakly $\mathcal{A}_4$–admissible because we can take

$$\begin{aligned} g(\alpha) &= (\{1,2,4\},2),\\ g(\beta) &= (\{1,3\},3),\\ g(\gamma) &= (\{3,4\},1). \end{aligned}$$

Lemma 3. *Let $(Q,y)Q)$ be an objection of K against L at x. Let $S\subset N$, $L\subset S$, $K\cap S=\emptyset$. Then there exists a strong counter-objection (S,z_S) to this objection iff*

$$(y-x)(Q\cap S)\le v(S)-x(S).$$

Proof.
The proof is similar to the proof of Lemma 1.

Lemma 4. *Let x be an imputation for (N,v), $K,L\subset N$, $K\subset Q\subset N$, $L\cap Q=\emptyset$. Let $S\subset N$, $L\subset S$, $K\cap S=\emptyset$, $x(Q)<v(Q)$.*

If either $Q\cap S=\emptyset$ and $x(S)\le v(S)$ or $Q\cap S\neq\emptyset$ and $v(Q)-x(Q)\le v(S)-x(S)$, then for each objection (Q,y_Q) of K against L at x there exists a strong counter-objection (S,z_S) of L against K.

Proof.
The proof is similar to the proof of Lemma 2.

Corollary. $\bar{\mathcal{K}}_{\mathcal{A}}(N,v)\subset\bar{\mathcal{M}}^i_{\mathcal{A}}(N,v)$.

Proof.
Let $x\in\bar{\mathcal{K}}_{\mathcal{A}}(N,v)$. Suppose that there exists an objection (C,y_C) of K against L at x for some $K,L\in\mathcal{A}$. We have either $x(L)\le v(L)$ (and then there exists a strong counter-objection (L,z_L)) or $t_{K,L}(x)\le t_{L,K}(x)$ and then there exists $D\subset N$ such

that $L \subset D$, $K \cap D = \emptyset$, $v(D) - x(D) \geq v(C) - x(C)$. Since $v(C) - x(C) > 0$, by Lemma 4, there exists a strong counter-objection to (C, y_C).

Theorem 5. *Let $\mathcal{A}$ be a set of subsets of N. If for each weakly $\mathcal{A}$-admissible graph Gr the set of the ends of its edges does not cover N, then $\bar{\mathcal{M}}^i_{\mathcal{A}}(N, v) \neq \emptyset$ for all $(N, v) \in \Gamma^0$.*

Proof.

The proof is similar to the proof of Theorem 3, Lemmas 1 and 2 are changed by Lemmas 3, 4.

Example 13. Let $N = \{1, 2, 3, 4\}$, $\mathcal{A} = \{\{1\}, \{2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}\}$. Then $\bar{\mathcal{M}}^i_{\mathcal{A}}(N, v) \neq \emptyset$ for all $(N, v) \in \Gamma^0$.

Let us check the condition of Theorem 5. Suppose that there exists a weakly admissible digraph Gr such that the ends of its edges cover N. Then $\{2, 3, 4\}$ is not the end of an edge by C1 and Gr contains the edges $\{\alpha, \beta, \gamma\}$, where

$$\alpha = (\{1\}, \{2\}), \quad \beta = (\{2\}, \{1, 3\}), \quad \gamma = (\{2\}, \{1, 4\}).$$

It was proved (see Example 11) that such Gr is not weakly $\mathcal{A}$-admissible.

Theorem 6. *Let $\mathcal{A}$ be a set of subsets of N and $|N| \leq 5$. Then $\bar{\mathcal{M}}^i_{\mathcal{A}}(N, v) \neq \emptyset$ for all $(N, v) \in \Gamma^0$ if and only if for each weakly $\mathcal{A}$-admissible graph Gr the set of the ends of its edges does not cover N.*

Proof.

The proof is the same as the proof of Theorem 4.

Example 14. $\mathcal{K}_{\mathcal{A}} \not\subset \bar{\mathcal{K}}_{\mathcal{A}}$.

Let $N = \{1, \ldots, 4\}$, $\mathcal{A} = \mathcal{A}_4 = \{\{1, 2\}, \{3\}, \{4\}\}$. There exists $\mathcal{A}_4$-admissible digraph such that the ends of its edges cover N (see Example 12), hence, by Theorem 6, there exists a game (N, v) with $\bar{\mathcal{M}}^i_{\mathcal{A}_4}(N, v) = \emptyset$ and then by Corollary to Lemma 4, $\bar{\mathcal{K}}_{\mathcal{A}_4}(N, v) = \emptyset$. But by Theorem 2, $\mathcal{K}_{\mathcal{A}_4}(N, v) \neq \emptyset$.

3. Conclusion

The paper considers conditions on the families of coalitions that ensure the existence of two types of generalized bargaining sets with respect to these families. Sufficient conditions are valid for each finite number of players and, for a fixed family, can be verified in finite number of steps. It is proved that these conditions are also necessary if the number of players is no more than 5.

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Some Cases of Cooperation in Differential Pursuit Games

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Abstract. In this paper we study a time-optimal model of pursuit in which players move on a plane with bounded velocities. The game is supposed to be a nonzero-sum simple pursuit game between an evader and m pursuers acting irrespective of each other. The key point of the work is to construct some cooperative solutions of the game and compare them with non-cooperative solutions such as Nash equilibria. It is important to give a reasonable answer to the question if cooperation is profitable in differential pursuit games or not. We consider all possible coalitions of the players in the game. For example, the pursuers promise some amount of the total payoff to the evader for cooperation with him. In that way, a cooperative game in characteristic function form is constructed, and its various cooperative solutions are found. We prove that in the game $\Gamma_v(x_1^0, \ldots, z_m^0, z^0)$ there exists the nonempty core for any initial positions of the players. In a dynamic game existence of the core at the initial moment of time is not sufficient for being accepted as a solution in it. We prove that the core in this game is time-consistent.

Keywords: Group pursuit game, cooperative game, Nash equilibrium, core.

Introduction

The process of pursuit represents a typical conflict situation. When only two players are involved in the process of pursuit we deal with a classical zero-sum differential pursuit game. These games grew out of the problem of setting and solving military pursuit games and were developed by [Isaaks, 1965]. In the case when more than two players participate in the game and the players' objectives are not strictly opposite it would be rather reasonable to consider such a game as a non zero-sum one. Although even in this case some game theorists used zero-sum models dividing all the players into two groups with opposite interests (see [Chikrii, 1992] and [Melikjan, 1981]).

In contrast of this approach to the problem of pursuit we consider a group pursuit game as nonzero-sum (see [Petrosjan, 1983], [Tarashnina, 1998]). Earlier differential games were used to model military problems. Now we try to apply them to some economic situations. It is obvious that players' goals are not always strictly opposed. We want to illustrate how differential games can be used for solving economic problems. In such kind of games under "capture" we understand just meeting of players and delivering some goods or information. In other words, players are not aimed to destruct each other. These are the socalled nonzero-sum pursuit games. In order to investigate such a nonzero-sum game we construct both a corresponding game in normal form and its TU-cooperative version. Then we consider two different games (cooperative and noncoopertive) and find their solutions. The key moment of this paper is comparing and analyzing of these solutions.

In the framework of classical cooperative game theory with transferable utilities many solution concepts have been known, and there is a famous concept of the *core*, among them [Scarf, 1967]. Basic notations and results are described in the monograph of [Pecherskii and Yanovskaya, 2004].

Dynamic aspects of solving of classical cooperative games are considered in [Chistyakov, 1993] and [Petrosjan and Kuzyutin, 2008].

In dynamic games the property that provides for a solution to be feasible throughout the game is very important. This requirement is called *time-consistency* of a solution of a game. This property and the connected with it imputation distribution procedure (IDP) were introduced by Petrosjan, 1989.

Moving along cooperative trajectory, the players on some sense travel over the subgames, which differ from each other with initial states and duration. It is obvious that when time is passing either the players' opportunities or the players' interests may change. Therefore, at some instant t, being in the corresponding current subgame, the originally adopted optimal solution may either not exist or not satisfy the players' interests any more. In other words, time-consistency of a solution of a dynamic game means that at each time instant within the game the players do not have any reasons to deviate from the originally adopted "optimal" behaviour.

1. Differential game of pursuit with one evader and m pursuers

The game under study is a time-optimal model of pursuit in which n players – m pursuers P_1,. . ., P_m and one evader E – move on a plane with bounded velocities α_1, . . ., α_m and β, respectively. Moreover,

$$\beta < \min\{\alpha_1, \alpha_2, \dots, \alpha_m\}, \quad \alpha_1, \alpha_2, \dots, \alpha_m, \ \beta = const.$$

The players $P_1, \dots, P_m$ and E start their motion at the positions x_1^0, . . ., x_m^0 and z_0, respectively, and have the possibility of making decisions continuously in time. At each instant they may choose directions of their motion (velocity vectors) and velocities within prescribed limits. Thus, sets of control variables of players have the following forms

$$\begin{aligned} U_{P_j} &= \{\mathbf{u}_{P_j} = (u^1_{P_j}, u^2_{P_j}) : (u^1_{P_j})^2 + (u^2_{P_j})^2 \le \alpha_j^2\}, \quad j = \overline{1, m}, \\ U_E &= \{\mathbf{u}_E = (u^1_E, u^2_E) : (u^1_E)^2 + (u^2_E)^2 \le \beta^2\}. \end{aligned}$$

In this case, the motion of the players is described by the following system of differential equations

$$\begin{aligned} &\dot{x}_j = \mathbf{u}_{P_j}, \qquad \mathbf{u}_{P_j} \in U_{P_j}, \qquad j = \overline{1, m}, \\ &\dot{z} = \mathbf{u}_E, \qquad \mathbf{u}_E \in U_E \end{aligned} \tag{1}$$

with initial conditions

$$x_1(0) = x_1^0, \quad \ldots, \quad x_m(0) = x_m^0, \quad z(0) = z^0. \tag{2}$$

We describe the case of perfect information. This means that each player, choosing control variables $\mathbf{u}_{P_j}(x_1^t, \ldots, x_m^t, z^t)$ and $\mathbf{u}_E(x_1^t, \ldots, x_m^t, z^t)$ at each time instant $t \ge 0$ knows the time t and his own as well as all other players positions.

We assume that the pursuers use strategies with discrimination against the evader. This means that at each instant t players P_1, P_2, … , P_m have additional information about the value of the vector-parameter $\mathbf{u}_E$ chosen by evader E at the same instant t. In such a situation, the evader is said to be discriminated. The evader uses piecewise open-loop strategies.

Denote as $\mathcal{U}_{P_j}$ and $\mathcal{U}_E$ the admissible strategy sets of the players P_j and E, respectively.

The functions $x_j(t)$ $j = 1, \ldots, m$ and $z(t)$, $t \in [0, \overline{t}]$, which satisfy equations (1) and initial conditions (2), where $\overline{t}$ – the end of the game, are called *trajectories* for the players P_j and E. Here by capture we mean coincidence of players' positions.

Denote by $P_j^t = x_j^t$ and by $E^t = z^t$ the current positions of pursuer P_j and evader E at the time instant t and

$$t_{P_j}(x_j^0, z^0, u_{P_j}(\cdot), u_E(\cdot)) = \min\{t : x_j^t = z^t\}, \quad j = \overline{1, m}.$$

If there is no such t then $t_{P_j} = +\infty$.

Let

$$t_E(x_1^0, \ldots, x_m^0, z^0, u_{P_1}(\cdot), \ldots, u_{P_m}(\cdot), u_E(\cdot)) = \min\{t_{P_1}, \ldots, t_{P_m}\}.$$

Denote by K_E the payoff function of the evader. It is equal to its meeting time with the first of the pursuers multiplied by some number $\gamma > 0$. Here γ is a price of a time unit. So,

$$\begin{aligned} &K_E(x_1^0, \ldots, x_m^0, z^0, u_{P_1}(\cdot), \ldots, u_{P_m}(\cdot), u_E(\cdot)) = \\ &\quad = \gamma \times t_E(x_1^0, \ldots, x_m^0, z^0, u_{P_1}(\cdot), \ldots, u_{P_m}(\cdot), u_E(\cdot)). \end{aligned}$$

The payoff function to player P_j $(j = 1, 2, \ldots, m)$ is given as follows

$$K_{P_j}(x_1^0, \ldots, x_m^0, z^0, u_{P_1}(\cdot), \ldots, u_{P_m}(\cdot), u_E(\cdot)) =$$
$$= -\gamma \times t_{P_j}(x_1^0, \ldots, x_m^0, z^0, u_{P_1}(\cdot), \ldots, u_{P_m}(\cdot), u_E(\cdot)).$$

The evader gets the total time of the pursuit. Each pursuer gets a negative value of his time of pursuit. The game ends as soon as at least one of the pursuers catches the evader.

The objective of each player in the game is to maximize his own payoff function. In other words, all this means that each pursuer has a reason to meet the evader before the other players do, and the evader wants to be caught as late as possible.

So, we define the nonzero-sum pursuit game as a normal form game as follows

$$\Gamma(x_1^0, \ldots, x_m^0, z^0) = \langle N, \{\mathcal{U}_j\}_{j \in N}, \{K_j\}_{j \in N} \rangle,$$

where $N = \{P_1, \ldots, P_m, E\}$ is the set of players, $\mathcal{U}_j$ is the set of admissible strategies of player j and K_j is a payoff function of the j-th player $(j \in N)$.

As a solution of this game we consider a concept of Nash equilibrium.

Here the discrimination of player E has an important sense as the pursuers choose directions of their movement depending on a choice of the evader.

If $\mathbf{u}_E = (u_E^1, u_E^2)$ is a control variable of the evader, then movement with the vector velocities $\mathbf{u}_{P_j} = \left(u_E^1, \sqrt{\alpha_j^2 - (u_E^1)^2}\right)$ is called the parallel pursuit strategy (Π-strategy) of the pursuer P_j, $j = 1, \ldots, m$.

It is known that if pursuer P uses Π-strategy and evader E uses a piecewise open-loop strategy, then their meeting happens within the Apollonius circle.

The *Apollonius circle* is the set of points M such that

$$\frac{|E^0M|}{\beta} = \frac{|P^0M|}{\alpha}.$$

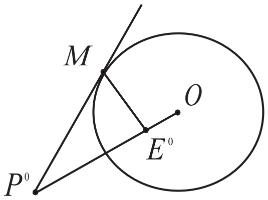

Fig. 1: Apollonius circle

The *Apollonius point* is the point on the Apollonius circle which is the most far from the evader's position.

Construct the Apollonius circles for a case of two pursuers. The Apollonious circle $A_i = A(x_i^0, z^0)$ for the pursuit game $\Gamma_{P_i \setminus E}(x_i^0, z^0)$ between P_i and E is defined in the following way

$$A_i = A(x_i^0, z^0) = S(O_i^0, R_i^0),$$

where

$$\left|E^0 O_i^0\right| = a_i = \beta^2 \times \frac{\left\|x_i^0 - z^0\right\|}{\alpha_i^2 - \beta^2}, \quad R_i = \alpha_i \times \beta \times \frac{\left\|x_i^0 - z^0\right\|}{\alpha_i^2 - \beta^2}.$$

Similarly, $A(x_j^0, z^0) = S(O_j^0, R_j^0)$ is the Apollonius circle for the pursuit game $\Gamma_{P_j \setminus E}(x_j^0, z^0)$ between P_j and E, where

$$\left|E^0 O_j^0\right| = a_j = \beta^2 \times \frac{\left\|x_j^0 - z^0\right\|}{\alpha_j^2 - \beta^2}, \quad R_j = \alpha_j \times \beta \times \frac{\left\|x_j^0 - z^0\right\|}{\alpha_j^2 - \beta^2}.$$

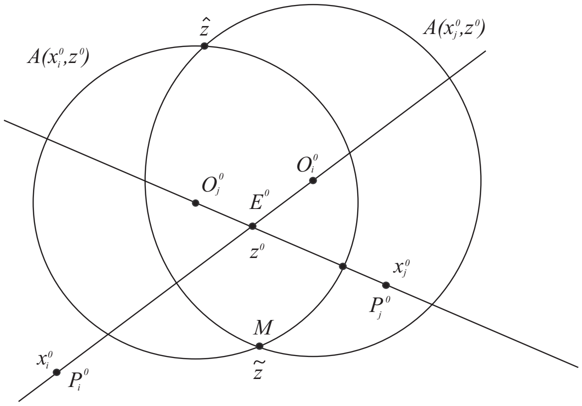

Fig. 2: Apollonius circles

1.1. A cooperative form of the game $\Gamma(x_1^0, \ldots, x_m^0, z^0)$

Let us assume that utilities of players are transferable, i.e. the players in the game are in such conditions that the total payoff, which is earned by a coalition $S \subseteq N$, can be arbitrarily divided between members of the coalition. It is interesting to consider all possible variants of players cooperation in this game. Suppose that each pursuer tries to agree with evader and promises to divide their payoff between two of them. It is important to find out, whether such behavior is favorable for the players, and namely, whether each player increases his own payoff by cooperation with the other players.

This game can be interpreted in the following way: imagine that the evader has something that each pursuer needs to have. It can be a kind of good or information.

Moreover, it is supposed the information to disappear once any of the pursuers reaches the evader. Thus, each pursuer wants to get it before the other does. It seems quite interesting to consider all possible cooperation between the players in this game, assuming the payoffs to be transferable. It would be rather helpful to know what the best way for the pursuers to "share" the evader is: whether to cooperate with each other, or to try to win over the evader to his side, or to form the grand coalition of n players. With this purpose with every game $\Gamma(x_1^0, \ldots, x_m^0, z^0)$ we associate the corresponding game in characteristic function form $\Gamma_v(x_1^0, \ldots, x_m^0, z^0)$.

Now we introduce a cooperative form of the game $\Gamma(x_1^0, \ldots, x_m^0, z^0)$.

Let 2^N be the set of all subsets of N. The function $v\ :\ 2^N \to R^1$ with the following two properties

1. $v(\emptyset) = 0$,
2. $v(S \cup R) \geq v(S) + v(R)$, $S, R \subset N$, $S \cap R = \emptyset$,

is called *the characteristic function* (c.f.). Condition 2 is the superadditivity property. For any coalition $S \subset N$ we define the characteristic function as follows

$$v(S) = \max_{u_S} \min_{u_{N\setminus S}} \sum_{i \in S} K_i\big(u_S, u_{N\setminus S}\big),$$

where u_S and $u_{N\setminus S}$ are vectors of admissible strategies of coalitions S and $N \setminus S$, respectively.

In paper [Tarashnina, 2002] the characteristic function of the game $\Gamma(x_i^0, x_j^0, z^0)$ is defined as

$$v(\{P_i\}, x_i^0, z^0) = -\gamma \times \frac{\|x_i^0 - z^0\|}{\alpha_i - \beta} = -g_i^0,$$

$$v(\{P_j\}, x_j^0, z^0) = -\gamma \times \frac{\|x_j^0 - z^0\|}{\alpha_j - \beta} = -g_j^0,$$

$$v(\{P_i, P_j\}, x_i^0, x_j^0, z^0) = -2\gamma \times \frac{\|z^0 - \hat{z}\|}{\beta} = -2g^0,$$

$$v(\{E\}, x_i^0, x_j^0, z^0) = \gamma \times \frac{\|z^0 - \hat{z}\|}{\beta} = g^0,$$

$$v(\{P_i, E\}, x_i^0, z^0) = 0,$$

$$v(\{P_j, E\}, x_j^0, z^0) = 0,$$

$$v(\{P_i, P_j, E\}, x_i^0, x_j^0, z^0) = -\gamma \times \overline{t} = -\gamma \times \frac{\|z^0 - \tilde{z}\|}{\beta} = -g^*,$$

where $\overline{t}$ is the minimal total pursuit time.

In this case the goal of the players is to choose such strategies that maximize the total pursuit time t_E. These strategies make all the players move to the point $\tilde{z}$ which is the intersection point of the Apollonius circles, the nearest to the initial position of E.

Definition 1. *The trajectory* $\big(\bar{x}_1(\cdot), \ldots, \bar{x}_m(\cdot), \bar{z}(\cdot)\big)$ *of system (1)–(2) such that* K_N $\big(\bar{x}_1(\cdot), \ldots, \bar{x}_m(\cdot), \bar{z}(\cdot)\big) =$

$$= \sum_{i \in N} K_i\big(\bar{x}_1(\cdot), \ldots, \bar{x}_m(\cdot), \bar{z}(\cdot)\big) = v(N; x_1^0, \ldots, x_m^0, z^0)$$

is called a cooperative trajectory *in the game* $\Gamma(x_1^0, \ldots, x_m^0, z^0)$.

All in all, the non-cooperative solution, which is a Nash equilibrium, proposes all the players to move to the point $\hat{z}$, which is the farest intersection point of the Apollonius circle, whereas according to the cooperative solution all the players should move to the point $\tilde{z}$.

In the case of the pursuit game considered above with two pursuers we construct the characteristic function using the values of corresponding zero-sum games. However, already for a game with three pursuers to construct the characteristic function in such a way is not possible, as the value of zero-sum game of pursuit with three pursuers is not known (to get a solution of such a game is a really complicated and still not solved problem).

Consider a game with m pursuers and one evader. We assume that all Apollonius circles have not empty intersection. If it is not so the i-th circle contains the j-th one that we count the player i to be "dummy". The "dummy" player cannot influence on the process of pursuit, therefore, this game is reduced to a game with $m-1$ pursuers.

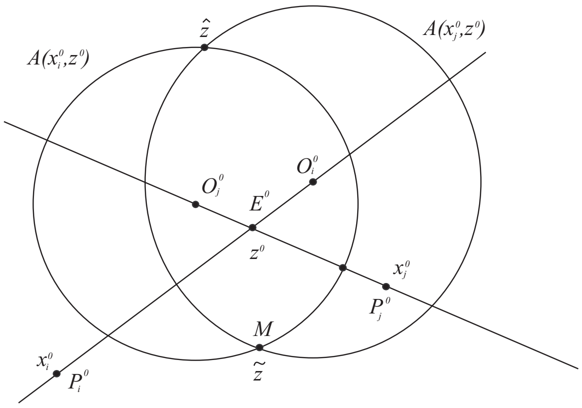

Fig. 3: The game $\Gamma(x_1^0, \ldots, x_m^0, z^0)$

In the game $\Gamma(x_1^0,\dots,x_m^0,z^0)$ as the worth of a coalitions S we take the guaranteed payoff of S against $N\backslash S$.

Denote $g_i^0 = \gamma \times \dfrac{\|x_i^0 - z^0\|}{\alpha_i - \beta}$, $g^0 = \gamma \times \dfrac{\|z^0 - \hat{z}\|}{\beta}$, $\hat{g}_{i,j}^0 = \gamma \times \dfrac{\|z^0 - \hat{z}_{i,j}\|}{\beta}$, $\hat{g}_{i,j,k}^0 = 3\gamma \times \dfrac{\|z^0 - \hat{z}_{i,j,k}\|}{\beta}$, $\tilde{g}_{i,j}^0 = \gamma \times \dfrac{\|z^0 - \tilde{z}_{i,j}\|}{\beta}$, $g^* = \gamma \times \dfrac{\|z^0 - \tilde{z}\|}{\sigma}$ and, using this approach, we construct the characteristic function for the game $\Gamma(x_1^0,\dots,x_m^0,z^0)$ in the following form:

$$
\begin{aligned}
&v(\{P_i\},x_i^0,z^0) = -g_i^0, \quad i=\overline{1,m},\\
&v(\{E\},x_i^0,z^0) = g^0,\\
&v(\{P_i,P_j\},x_i^0,x_j^0,z^0) = -2\hat{g}_{i,j}^0,\\
&v(\{P_i,E\},x_i^0,z^0) = 0, \ i=\overline{1,m},\\
&v(\{P_i,P_j,P_k\},x_i^0,x_j^0,x_k^0,z^0) = -3\hat{g}_{i,j,k}^0,\\
&v(\{P_i,P_j,E\},x_i^0,x_j^0,z^0) = -\tilde{g}_{i,j}^0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots..\\
&v(\{P_1,\dots,P_m\},x_1^0,\dots,x_m^0,z^0) = -m\times g^0,\\
&v(\{P_1,\dots,P_m,E\},x_1^0,\dots,x_m^0,z^0) = -g^*,
\end{aligned}
\tag{3}
$$

where $\hat{z}$ and $\tilde{z}$ are defined by formulas

$$
\begin{aligned}
&\hat{z} = \mathop{\arg\max}_{z\in A_1\cap\ldots\cap A_m} ||z^0 - z||,\\
&\hat{z}_{i,j} = \mathop{\arg\max}_{z\in A_i\cap A_j} ||z^0 - z||, \quad i,j=\overline{1,m},\ i\neq j,\\
&\hat{z}_{i,j,k} = \mathop{\arg\max}_{z\in A_i\cap A_j\cap A_k} ||z^0 - z||, \quad i,j,k=\overline{1,m},\ i\neq j\neq k,\\
&\tilde{z}_{i,j} = \mathop{\arg\min}_{z\in A_i\cap A_j} ||z^0 - z||, \quad i,j=\overline{1,m},\ i\neq j,\\
&\tilde{z} = \mathop{\arg\min}_{z\in A_1\cap\ldots\cap A_m} ||z^0 - z||.
\end{aligned}
\tag{4}
$$

Definition 2. *The pair* $\langle N, v(S;x_1^0,\dots,x_m^0,z^0)\rangle$*, where* $N=\{P_1,\dots,P_m,E\}$ *is the set of players, and* v *is the characteristic function defined by (3) and (4), is called a* cooperative differential *game in characteristic function form and denoted by* $\Gamma_v(x_1^0,\dots,x_m^0,z^0)$.

Proposition 1. *In the differential TU game of pursuit $\Gamma_v(x_1^0, \ldots, x_m^0, z^0)$ the constructed characteristic function (3) is superadditive for any initial positions and velocities of the players E, $P_1, \ldots, P_m$.*

By checking of corresponding inequalities it is possible to be convinced that the constructed characteristic function possesses the superadditivity property.

Definition 3. *The cooperative n-person game $\langle N, v\rangle$ is* equivalent *to the game $\langle N, v'\rangle$ if there exists a positive number k and n arbitrary real numbers c_i $(i \in N)$ such that for any coalition $S \subset N$*

$$v'(S) = kv(S) + \sum_{i \in S} c_i.$$

In fact, by setting $k = 1$, $c_{P_i} = g_i(0) = g_i^0$, $c_E = 0$, we construct the game $\Gamma_{v'}(x_1^0, \ldots, x_m^0, z^0)$, which is equivalent to the game $\Gamma_v(x_1^0, \ldots, x_m^0, z^0)$. In such a case the characteristic function v' has the form

$$\begin{aligned}
&v(\{P_i\}, x_i^0, z^0) = 0, \quad i = \overline{1, m},\\
&v(\{E\}, x_i^0, z^0) = g^0,\\
&v(\{P_i, P_j\}, x_i^0, x_j^0, z^0) = g_i^0 + g_j^0 - 2\hat{g}_{i,j}^0,\ i, j = \overline{1, m},\ i \neq j,\\
&v(\{P_i, E\}, x_i^0, z^0) = g_i^0,\ i = \overline{1, m},\\
&v(\{P_i, P_j, P_k\}, x_i^0, x_j^0, x_k^0, z^0) = g_i^0 + g_j^0 + g_k^0 - 3\hat{g}_{i,j,k}^0, \qquad (5)\\
&v(\{P_i, P_j, E\}, x_i^0, x_j^0, z^0) = g_i^0 + g_j^0 - \tilde{g}_{i,j}^0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&v(\{P_1, \ldots, P_m\}, x_1^0, \ldots, x_m^0, z^0) = g_i^0 + \ldots + g_m^0 - m \times g^0,\\
&v(\{P_1, \ldots, P_m, E\}, x_1^0, \ldots, x_m, z^0) = -g^*.
\end{aligned}$$

We shall examine this game by using dominance relation. Recall that the imputation ξ dominates the imputation η with respect to the coalition S $(\xi \succ_S \eta)$ if the following conditions hold

$$\xi_i > \eta_i, \quad i \in S,$$
$$\xi(S) = \sum_{i \in S} \xi_i \leq v(S).$$

The following theorem is needed for the sequel.

Theorem 1. *Suppose $\langle N, v\rangle$ and $\langle N, v'\rangle$ are two equivalent games, then the map $\xi \mapsto \xi'$, where*

$$\xi_i' = k\xi_i + c_i, \quad i \in N,$$

establishes one-to-one mapping between the imputation set of the game $\langle N, v\rangle$ *and the imputation set of the game* $\langle N, v'\rangle$ *such that* $\xi \succ_S \eta$ *implies* $\xi' \succ_S \eta'$*.*

1.3. Existence of the Core in the cooperative pursuit game $\Gamma_v(x_1^0, \ldots, x_m^0, z^0)$

It follows from the superadditivity condition that it is advantageous for the players to form the maximal coalition N and obtain the maximal total payoff $v(N; x_1^0, \ldots, x_m^0, z^0)$ that is possible in the game $\Gamma_v(x_1^0, \ldots, x_m^0, z^0)$. Various methods for "equitable" distribution of the total profit between players are considered as optimality principles. The set of distributions that satisfies an optimality principle is called a solution in a cooperative game (in the sense of this optimality principle).

Let us describe the imputation set in the game $\Gamma_v(x_1^0, \ldots, x_m^0, z^0)$. Denote by $\xi = (\xi_{P_1}, \ldots, \xi_{P_m}, \xi_E)$ an imputation. The imputation set is defined as follows

$$E_v(x_1^0, \ldots, x_m^0, z^0) = \left\{\xi : \xi_{P_i} \geq v(P_i),\ i = \overline{1, m},\ \xi_E \geq v(E); \sum_{i \in N} \xi_i = v(N)\right\},$$

or

$$E_v(x_1^0, \ldots, x_m^0, z^0) = \left\{\xi :\ \xi_{P_i} \geq -g_i^0,\ i = \overline{1, m},\ \xi_E \geq g^0;\ \sum_{i \in N} \xi_i = -g^*\right\}. \tag{6}$$

As an optimality principle in $\Gamma_v(x_1^0, \ldots, x_m^0, z^0)$ we take the core. An analytical description of the core is provided by the following theorem, which was independently proved by [Bondareva, 1989] and [Shapley, 1969].

Theorem 2. *For the imputation* $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ *to belong to the core it is necessary and sufficient that the inequality*

$$\xi(S) = \sum_{i \in S} \xi_i \geq v(S)$$

holds for all $S \subset N$*.*

By Theorem 2, for the imputation ξ to belong to the core it is necessary and sufficient that the following system of inequalities holds

$$\begin{cases} \xi_{P_i} + \xi_{P_j} \geq -2 \cdot \hat{g}_{i,j}^0, & i, j = \overline{1, m},\ i \neq j, \\ \xi_{P_k} + \xi_E \geq 0, & k = \overline{1, m}, \\ \xi_{P_i} + \xi_{P_j} + \xi_{P_k} \geq -3 \cdot \hat{g}_{i,j,k}^0, & i, j, k = \overline{1, m},\ i \neq j \neq k, \\ \xi_{P_i} + \xi_{P_j} + \xi_E \geq -\tilde{g}_{i,j}^0, & i, j = \overline{1, m},\ i \neq j, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \xi_{P_1} + \ldots + \xi_{P_m} \geq -m \cdot g^0, & \\ \underbrace{\xi_{P_i} + \ldots + \xi_{P_k}, \xi_E}_{m-1} \geq -\tilde{g}_{i,\ldots,k}^0, & i, \ldots, k = \overline{1, m},\ i \neq \ldots \neq k. \end{cases} \tag{7}$$

In paper [Tarashnina, 2002] it is proved that in the cooperative game with two pursuers and one evader there is a nonempty C-core for any initial positions of the players. In this paper we show that this is true for a general case of n-persons of game. The following proposition holds.

Proposition 2. *In differential TU game of pursuit* $\Gamma(x_1^0, \ldots, x_m^0, z^0)$ *there exists the non-empty core for any initial positions of the players.*

Proof.

Summing the inequalities (7) and multiplying result by (-1), we obtain

$$\begin{aligned}
&-(C_m^1 + \ldots + C_m^{m-1}) \cdot (\xi_{P_1} + ... + \xi_{P_m} + \xi_E) \leq 2\,\underbrace{(\hat{g}^0_{1,2} + \hat{g}^0_{1,3} + \ldots)}_{C_m^2} + \\
&+ 3\,\underbrace{(\hat{g}^0_{1,2,3} + \hat{g}^0_{1,2,4} + \ldots)}_{C_m^3} + \underbrace{(\tilde{g}^0_{1,2} + \tilde{g}^0_{1,3} + \ldots)}_{C_m^2} + \\
&+ 4\,\underbrace{(\hat{g}^0_{1,2,3,4} + \hat{g}^0_{1,2,3,5} + \ldots)}_{C_m^4} + 2\,\underbrace{(\tilde{g}^0_{1,2,3} + \tilde{g}^0_{1,2,4} + \ldots)}_{C_m^3} + \\
&+ \ldots + m \times \underbrace{g^0}_{C_m^m} + (m-2) \times \underbrace{(\tilde{g}^0_{1,2,\ldots,m-1} + \tilde{g}^+_{1,2,\ldots,m-2,m} \cdots)}_{C_m^{m-1}}.
\end{aligned} \tag{8}$$

Let us simplify the left part of an inequality.

It is known that

$$C_m^0 + C_m^1 + C_m^2 + \ldots + C_m^{m-1} + C_m^m = 2^m,$$

then

$$C_m^1 + C_m^2 + \ldots + C_m^{m-1} = 2^m - 2.$$

On construction, we have $v(N) = \xi_{P_1} + \ldots + \xi_{P_m} + \xi_E = -g^*$. Compare $v(N)$ and the worths of the other coalitions. It is obvious that the following relations hold

$$\begin{aligned}
g^* &= \min_{z \in A_1 \cap \ldots \cap A_m} ||z^0 - z|| \leq \ldots \leq \min_{z \in A_i \cap A_j} ||z^0 - z||, \\
g^0 &= \max_{z \in A_1 \cap \ldots \cap A_m} ||z^0 - z|| \leq \ldots \leq \max_{z \in A_i \cap A_j} ||z^0 - z||.
\end{aligned}$$

Therefore, $g^* \leq g^0$. This means, it is possible to replace the right part of inequality (8) by the following estimation:

$$\begin{gathered}
(2C_m^2 + 3C_m^3 + C_m^2 + 4C_m^4 + 2C_m^3 + \ldots + \\
+ m \times C_m^m + (m-2) \times C_m^{m-1}) \times g^*.
\end{gathered}$$

Hence,

$$(2^m-2)\cdot g^* \leq (2C_m^2+3C_m^3+C_m^2+4C_m^4+2C_m^3+\ ...\ +m\times C_m^m+(m-2)\times C_m^{m-1})\times g^*,$$

and

$$(2^m-2) \leq 2C_m^2+3C_m^3+C_m^2+4C_m^4+2C_m^3+\ldots+m\times C_m^m+(m-2)\times C_m^{m-1}. \qquad (9)$$

Now it is left to prove that inequality (9) holds. It is possible to confirm, that

$$C_m^1+2C_m^2+3C_m^3+\ldots+m\times C_m^m = m\times 2^{m-1}.$$

Then

$$2C_m^2+3C_m^3+\ldots+m\times C_m^m = m\times (2^{m-1}-1).$$

It is obvious that

$$2\times (2^{m-1}-1) \leq m\times (2^{m-1}-1).$$

Consequently, the inequality (9) is fulfilled. This implies that the inequality (8) is satisfied for any initial positions of the players. Hence, system (7), which describes the core, is combined. This means that there exists the non-empty core in the described above cooperative game.

It can be easily checked that the vector $\eta^0 = (-g^0, -g^0, \ldots, -g^0, -g^*, (m-1)\times g^0)$ is an imputation from the core. This completes the proof.

2. Time-consistency of the core

We focus our attention on time-consistency of the core $C_v(x_1^0,\ldots,x_m^0,z^0)$ in the game

$$\Gamma_v(x_1^0,\ldots,x_m^0,z^0).$$

Let an optimality principle be chosen in the game $\Gamma_v(x_1^0,\ldots,x_m^0,z^0)$. Let it be the core. The solution of this game constructed at the initial moment $t=0$ is $C_v(x_1^0,\ldots,x_m^0,z^0)$. It follows from Proposition 1 that $C_v(x_1^0,\ldots,x_m^0,z^0)\neq\emptyset$. Remind that here $(\bar{x}_1(\cdot),\ldots,\bar{x}_m(\cdot),\bar{z}(\cdot))$ is the cooperative trajectory in the game Γ_v $(x_1^0,\ldots,x_m^0,z^0)$.

We study behavior of the set $C_v(x_1^0,\ldots,x_m^0,z^0)$ along the cooperative optimal trajectory

$$(\bar{x}_1(\cdot),\ldots,\bar{x}_m(\cdot),\bar{z}(\cdot)).$$

With this end in view we enter the notion of a current subgame. At each current state $(\bar{x}_1(t),\ldots,\bar{x}_m(t),\bar{z}(t))$ a current subgame $\Gamma_v(\bar{x}_1(t),\ldots,\bar{x}_m(t),\bar{z}(t))$ is defined like the game $\Gamma_v(x_1^0,\ldots,x_m^0,z^0)$ with the only difference: it starts at the current state lying on the cooperative trajectory and has duration $(\bar{t}-t)$. In the subgame

$\Gamma_v(\bar{x}_1(t),\ldots,\bar{x}_m(t),\bar{z}(t))$ we define the characteristic function as it was done for the original game:

$$v(\{P_i\},\bar{x}_i(t),\bar{z}(t)) = -\gamma \times \frac{\|\bar{x}_i(t)-\bar{z}(t)\|}{\alpha_i-\beta}, \quad i=\overline{1,m},$$

$$v(\{E\},\bar{x}_i(t),\bar{z}(t)) = \gamma \times \frac{\|\bar{z}(t)-\hat{z}(t)\|}{\beta},$$

$$v(\{P_i,P_j\},\bar{x}_i(t),\bar{x}_j(t),\bar{z}(t)) = -2\gamma \times \frac{\|\bar{z}(t)-\hat{z}_{i,j}(t)\|}{\beta},$$

$$v(\{P_i,E\},\bar{x}_i(t),\bar{z}(t)) = 0,\ i=\overline{1,m},$$

$$v(\{P_i,P_j,P_k\},\bar{x}_i(t),\bar{x}_j(t),\bar{x}_k(t),\bar{z}(t)) = -3\gamma \times \frac{\|\bar{z}(t)-\hat{z}_{i,j,k}(t)\|}{\beta},$$

$$v(\{P_i,P_j,E\},\bar{x}_i(t),\bar{x}_j(t),\bar{z}(t)) = -\gamma \times \frac{\|\bar{z}(t)-\widetilde{z}_{i,j}(t)\|}{\beta},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$$

$$v(\{P_1,\ldots,P_m\},\bar{x}_1(t),\ldots,\bar{x}_m(t),\bar{z}(t)) = -m \times \gamma \times \frac{\|\bar{z}(t)-\hat{z}(t),\|}{\beta},$$

$$v(\{P_1,\ldots,P_m,E\},\bar{x}_1(t),\ldots,\bar{x}_m(t),\bar{z}(t)) = -\gamma \times (\bar{t}-t),$$

where

$$\hat{z}(t) = \underset{z(t)\in A_1(t)\cap\ldots\cap A_m(t)}{\arg\max} ||\bar{z}(t)-z(t)||,$$

$$\hat{z}_{i,j}(t) = \underset{z(t)\in A_i(t)\cap A_j(t)}{\arg\max} ||\bar{z}(t)-z(t)||, \quad i,j=\overline{1,m},\ i\neq j,$$

$$\hat{z}_{i,j,k}(t) = \underset{z\in A_i(t)\cap A_j(t)\cap A_k(t)}{\arg\max} ||\bar{z}(t)-z(t)||, \quad i,j,k=\overline{1,m},\ i\neq j\neq k,$$

$$\widetilde{z}_{i,j}(t) = \underset{z\in A_i(t)\cap A_j(t)}{\arg\min} ||\bar{z}(t)-z(t)||, \quad i,j=\overline{1,m},\ i\neq j,$$

$$\widetilde{z} = \underset{z\in A_1(t)\cap\ldots\cap A_m(t)}{\arg\min} ||\bar{z}(t)-z(t)||,$$

and $A_i(t)$ are Apollonius circle in the games $\Gamma_{P_i\setminus E}(\bar{x}_i(t),\bar{z}(t))$.

Let us consider the functions

$$g_i(t) = \gamma\frac{\|\bar{x}_i(t)-\bar{z}(t)\|}{\alpha_1-\beta}, \quad i=1,\ldots,m, \quad g(t) = \gamma\frac{\|\bar{z}(t)-\hat{z}(t)\|}{\beta}.$$

These functions are continuous monotonically decreasing functions in t on the interval $[0,\bar{t}]$.

Remark 1. From the definition of Apollonius circle it follows that

$$g_i(t) = \gamma \left(1 - \frac{t}{\bar{t}}\right) \frac{\|x_i^0 - z_0\|}{\alpha_1 - \beta}, \quad i = 1, \ldots, m.$$

Proposition 3.[1] *The function $g^0(t)$ is linear function with respect to t, i.e.*

$$g^0(t) = \gamma \left(1 - \frac{t}{\bar{t}}\right) \frac{\|z_0 - \hat{z}\|}{\beta}.$$

The characteristic function of the game $\Gamma_v(\bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t))$ has the form

$$\begin{aligned}
&v(\{P_i\}, \bar{x}_i(t), \bar{z}(t)) = -g_i(t), \quad i = \overline{1, m},\\
&v(\{E\}, \bar{x}_i(t), \bar{z}(t)) = g(t),\\
&v(\{P_i, P_j\}, \bar{x}_i(t), \bar{x}_j(t), \bar{z}(t)) = -2\hat{g}_{i,j}(t),\\
&v(\{P_i, E\}, \bar{x}_i(t), \bar{z}(t)) = 0,\ i = \overline{1, m},\\
&v(\{P_i, P_j, P_k\}, \bar{x}_i(t), \bar{x}_j(t), \bar{x}_k(t), \bar{z}(t)) = -3\hat{g}_{i,j,k}(t),\\
&v(\{P_i, P_j, E\}, \bar{x}_i(t), \bar{x}_j(t), \bar{z}(t)) = -\tilde{g}_{i,j}(t),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\\
&v(\{P_1, \ldots, P_m\}, \bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t)) = -m \times g(t),\\
&v(\{P_1, \ldots, P_m, E\}, \bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t)) = -g^*(t).
\end{aligned}$$

The imputation set in the game $\Gamma_v(\bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t))$ is of the form

$$E_v\left(\bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t)\right) = \Big\{\xi^t \ :\ \xi^t_{P_i} \geq -g_i(t),\ i = \overline{1, m},$$

$$\xi^t_E \geq g(t);\ \sum_{i \in N} \xi^t_i = -g^*(t)\Big\}.$$

The core of the current game is defined as follows

$$C_v\big(\bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t)\big) = \Big\{\xi^t \ :\ \xi^t \in E_v(\bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t)),$$

$$\xi^t \text{ satisfies system (10)}\Big\}.$$

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[1] It was proved by V. Reshetilova in her diploma thesis.

$$\begin{cases} \xi_{P_i} + \xi_{P_j} \geq -2 \times \hat{g}_{i,j}(t), & i,j = \overline{1,m},\ i \neq j, \\ \xi_{P_k} + \xi_E \geq 0, & k = \overline{1,m}, \\ \xi_{P_i} + \xi_{P_j} + \xi_{P_k} \geq -3 \times \hat{g}_{i,j,k}(t), & i,j,k = \overline{1,m},\ i \neq j \neq k, \\ \xi_{P_i} + \xi_{P_j} + \xi_E \geq -\tilde{g}_{i,j}(t), & i,j = \overline{1,m},\ i \neq j, \\ \dots\dots\dots\dots\dots\dots & \\ \xi_{P_1} + \xi_{P_2} + ... + \xi_{P_m} \geq -\, m \times g(t), & \\ \underbrace{\xi_{P_i} + \xi_{P_j} + ... + \xi_{P_k} + \xi_E}_{m-1} \geq -\tilde{g}_{i,j,\dots,k}(t), & i,j,\dots,k = \overline{1,m},\ i \neq j \neq \dots \neq k. \end{cases} \tag{10}$$

Suppose $C_v\big(\bar{x}_1(t),\dots,\bar{x}_m(t),\bar{z}(t)\big) \neq \emptyset$ along the cooperative trajectory for any t, $t \in [0,\bar{t}]$. If this condition is not satisfied, then it is impossible for players to adhere to the chosen optimality principle, since at the very first instant t, when $C_v\big(\bar{x}_1(t),\dots,\bar{x}_m(t),\bar{z}(t)\big) = \emptyset$, the players have no possibility to follow this principle. And as we assumed above at the initial state $(x_1^0,\dots,x_m^0,z_0)$ the players agree upon the imputation $\xi^0 \in C_v\big(\bar{x}_1^0,\dots,\bar{x}_m^0,\bar{z}^0\big)$ such that the share of the i-th player is equal to ξ_i^0. Let the payoff of the player i (his share) on the time interval $[0,t]$ be $\xi_i\big(\bar{x}_1(t),\dots,\bar{x}_m(t),\bar{z}(t)\big)$. Then on the remaining time interval $[t,\bar{t}]$, according to the imputation ξ^0, he is to receive the gain

$$\xi_i^t = \xi_i^0 - \xi_i\big(\bar{x}_1(t),\dots,\bar{x}_m(t),\bar{z}(t)\big).$$

For the original agreement (the imputation ξ^0) to remain in force at the instant t it is essential that the vector $\xi^t = (\xi_{P_1}^t,\dots,\xi_{P_m}^t,\xi_E^t)$ belongs to the set $C_v\big(\bar{x}_1^0,\dots,\bar{x}_m^0,\bar{z}^0\big)$. If this condition is satisfied at each instant $t \in [0,\bar{t}]$, then the imputation ξ^0 is realized. This is conceptual meaning of time-consistency. In a view of Petrosjan, the definition of time-consistency of an imputation in the game $\Gamma_v\big(x_1^0,\dots,x_m^0,z_0\big)$ has the form.

Definition 4. *The imputation $\xi^0 \in C_v\big(x_1^0,\dots,x_m^0,z_0\big)$ is called time-consistent in the time-optimal game $\Gamma_v\big(x_1^0,\dots,x_m^0,z_0\big)$ if the following conditions are satisfied:*

1. *$C_v\big(\bar{x}_1(t),\dots,\bar{x}_m(t),\bar{z}(t)\big) \neq \emptyset$ along the cooperative optimal trajectory $\big(\bar{x}_1(t),\dots,\bar{x}_m(t),\bar{z}(t)\big)$ at each instant t, $0 \leq t \leq \bar{t}$;*
2. *there exists an integrable function $\tau(t) = \big(\tau_{P_1}(t),\dots,\tau_{P_m}(t),\tau_E(t)\big)$ on $[0,\bar{t}]$ such that $\tau_i(t) \geq 0$ ($i \in N$) for each $t \in [0,\bar{t}]$ and*

$$\xi^0 - \xi(\bar{x}_1(t),\dots,\bar{x}_m(t),\bar{z}(t)) \in C_v\big(\bar{x}_1(t),\dots,\bar{x}_m(t),\bar{z}(t)\big), \tag{11}$$

where

$$\xi\left(\bar{x}_1(t),\dots,\bar{x}_m(t),\bar{z}(t)\right) = \Big(\xi_{P_1}\left(\bar{x}_1(t),\dots,\bar{x}_m(t),\bar{z}(t)\right),\dots$$

$$\ldots, \xi_{P_m}\left(\bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t)\right),\ \xi_E\left(\bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t)\right)\Big)$$

and $\quad \xi_i(\bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t)) = \int\limits_0^t \tau_i(y)dy,\ i \in N.$

Remark 2. The cooperative differential game $\Gamma_v\left(x_1^0, \ldots, x_m^0, z_0\right)$ has a time consistent solution if each imputation $\xi^0 \in C_v\left(x_1^0, \ldots, x_m^0, z_0\right)$ is time-consistent.

The following theorem holds.

Theorem 3. *In the cooperative differential time-optimal pursuit game* $\Gamma_v\left(x_1^0, \ldots, x_m^0, z_0\right)$ *there exists the non-empty core* $C_v\left(x_1^0, \ldots, x_m^0, z_0\right)$ *that is time-consistent.*

Proof. Consider the family of the current subgames

$$\Big\{\Gamma_v(\bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t)),\ 0 \leq t \leq \bar{t}\Big\}.$$

Now our aim is to show that $C_v\left(\bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t)\right) \neq \emptyset$ for each $t \in [0, \bar{t}]$. Summing inequality (10) and multiplying by (-1), we obtain

$$\begin{aligned}
-(C_m^1 + \ldots + C_m^{m-1}) \times (\xi_{P_1} + \xi_{P_2} + ... + \xi_{P_m} + \xi_E) \leq 2\,\underbrace{(\hat{g}_{1,2}(t) + \hat{g}_{1,3}(t) + \ldots)}_{C_m^2} + \\
+ 3\,\underbrace{(\hat{g}_{1,2,3}(t) + \hat{g}_{1,2,4}(t) + \ldots)}_{C_m^3} + \underbrace{(\tilde{g}_{1,2}(t) + \tilde{g}_{1,3}(t) + \ldots)}_{C_m^2} + \\
+ 4\,\underbrace{(\hat{g}_{1,2,3,4}(t) + \hat{g}_{1,2,3,5}(t) + \ldots)}_{C_m^4} + 2\,\underbrace{(\tilde{g}_{1,2,3}(t) + \tilde{g}_{1,2,4}(t) + \ldots)}_{C_m^3} + \\
+ \ldots + m \times \underbrace{g(t)}_{C_m^m} + (m-2) \times \underbrace{(\tilde{g}_{1,2,\ldots,m-1}(t) + \tilde{g}_{1,2,\ldots,m-2,m}(t) + \ldots)}_{C_m^{m-1}}.
\end{aligned} \tag{12}$$

Note that

$$g(t) = \frac{\gamma}{\beta}\|\bar{z}(t) - \hat{z}(t)\| = \max_{z(t) \in A_1(t) \cap \ldots \cap A_m(t)} \|\bar{z}(t) - z(t)\| \leq \ldots \leq \hat{g}_{i,j,k}(t) =$$

$$= \max_{z(t) \in A_i(t) \cap A_j(t) \cap A_k(t)} \|\bar{z}(t) - z(t)\| \leq \hat{g}_{i,j}(t) \max_{z(t) \in A_i(t) \cap A_j(t)} \|\bar{z}(t) - z(t)\|$$

and

$$g^*(t) = \frac{\gamma}{\beta}\|\bar{z}(t) - \tilde{z}(t)\| = \min_{z(t) \in A_1(t) \cap \ldots \cap A_m(t)} \|\bar{z}(t) - z(t)\| \leq \ldots \leq \tilde{g}_{i,j,k}(t) =$$

$$= \min_{z(t) \in A_i(t) \cap A_j(t) \cap A_k(t)} \|\bar{z}(t) - z(t)\| \leq \tilde{g}_{i,j}(t) \min_{z(t) \in A_i(t) \cap A_j(t)} \|\bar{z}(t) - z(t)\|,$$

where $A_i(t) \subset A_i$ is Apollonius circle in the game $\Gamma_{P_i \setminus E}(\bar{x}_i(t), \bar{z}(t))$, $i = 1, \dots, m$. Obviously, $g^*(t) \leq g(t)$ for all $t \in [0, \bar{t}]$. Therefore, inequality (12) we can write in the following form

$$(2^m - 2) \times g^* \leq (2C_m^2 + 3C_m^3 + C_m^2 + 4C_m^4 + 2C_m^3 + \dots + m \times C_m^m + (m-2) \times C_m^{m-1}) \times g^*.$$

Then inequality (12) holds for all $t \in [0, \bar{t}]$. (The proof of this fact is like the proof of Proposition 2.) It is clear that there exists an imputation

$$\xi^t = (-g(t), -g(t), \dots, -g^*, (m-1) \times g(t)),$$

which belongs to $E_v\big(\bar{x}_1(t), \dots, \bar{x}_m(t), \bar{z}(t)\big)$ and satisfies system (10). So, $C_v\big(\bar{x}_1(t), \dots, \bar{x}_m(t), \bar{z}(t)\big) \neq \emptyset$ for all $t \in [0, \bar{t}]$.

Now it remains to check condition 2 in definition 4. According to Remark 2, we must prove that condition (11) holds for all imputations from the core. For convenience we need to get an equivalent game $\Gamma_{v'}(x_1^0, \dots, x_m^0, z^0)$ (see system (5)). The core in this game can have various forms corresponding to initial positions of the players. In order to prove time-consistency of the core it is sufficient to consider a case of the widest core. This core $C_{v'}\big(x_1^0, \dots, x_m^0, z_0\big)$ represents a convex hull of the imputations

$$\begin{array}{rcl}
\eta^1 & = & (g_1^0 - g^0, g_2^0 - g^0, \dots, g_{m-1}^0 - g^0, g_m^0 - g^*, (m-1)g^0), \\
\eta^2 & = & (g_1^0 - g^0, g_2^0 - g^0, \dots, g_{m-1}^0 - g^*, g_m^0 - g^0, (m-1)g^0), \\
 & \dots & \\
\eta^{m-1} & = & (g_1^0 - g^0, g_2^0 - g^*, \dots, g_{m-1}^0 - g^0, g_m^0 - g^0, (m-1)g^0), \\
\eta^m & = & (g_1^0 - g^*, g_2^0 - g^0, \dots, g_{m-1}^0 - g^0, g_m^0 - g^0, (m-1)g^0), \\
\eta^{m+1} & = & (0, g_2^0 - g^*, \dots, g_{m-1}^0, g_m^0, g_1^0), \\
\eta^{m+2} & = & (g_1^0 - g^*, 0, \dots, g_{m-1}^0, g_m^0, g_2^0), \\
 & \dots & \\
\eta^{2m-1} & = & (g_1^0, g_2^0, \dots, 0, g_m^0 - g^*, g_{m-1}^0), \\
\eta^{2m} & = & (g_1^0, g_2^0, \dots, g_{m-1}^0 - g^*, 0, g_m^0), \\
\eta^{2m+1} & = & (0, g_1^0 + g_2^0 - 2g^0, \dots, g_{m-1}^0 - g^0, g_m^0 - g^0, mg^0 - g^*), \\
\eta^{2m+2} & = & (0, g_2^0 - g^0, \dots, g_{m-1}^0 - g^0, g_m^0 - g^0, mg^0 - g^*), \\
 & \dots & \\
\eta^{3m-1} & = & (0, g_2^0 - g^0, \dots, g_1^0 + g_{m-1}^0 - 2g^0, g_m^0 - g^0, mg^0 - g^*), \\
\eta^{3m} & = & (0, g_2^0 - g^0, \dots, g_{m-1}^0 - g^0, g_1^0 + g_m^0 - 2g^0, mg^0 - g^*), \\
\eta^{3m+1} & = & (g_1^0 + g_2^0 - 2g^0, 0, \dots, g_{m-1}^0 - g^0, g_m^0 -^0, mg^0 - g^*), \\
\eta^{3m+2} & = & (g_1^0 - g^0, 0, \dots, g_{m-1}^0 - g^0, g_m^0 -^0, mg^0 - g^*), \\
 & \dots & \\
\eta^{p-1} & = & (g_1^0 - g^0, g_2^0 - g^0, \dots, +0, g_{m-1}^0 + g_m^0 - 2g^0, mg^0 - g^*), \\
\eta^p & = & (g_1^0 - g^0, g_2^0 - g^0, \dots, g_{m-1}^0 + g_m^0 - 2g^0, 0, mg^0 - g^*),
\end{array} \tag{13}$$

where $p = 2m + 2 \times C_m^2$.

Hence, any imputation $\eta^0 \in C_{v'}\big(x_1^0, \dots, x_m^0, z_0\big)$ can be represented as

$$\eta^0 = \sum_{j=1}^{p} \lambda_j \eta^j, \quad \sum_{j=1}^{p} \lambda_j = 1, \quad \lambda_j \geq 0 \ \text{ for } \ 1 \leq j \leq 6. \tag{14}$$

By substituting (13) into (14) and denoting

$$s_1 = \begin{pmatrix} \sum_{j=1}^{m} \lambda_j + \sum_{j=m+2}^{2m} \lambda_j + \sum_{j=3m+1}^{p} \lambda_j \\ \lambda_{2m+1} \\ \ldots \\ \lambda_{m+1} \end{pmatrix},$$

$$s_2 = \begin{pmatrix} \lambda_{2m+1} \\ \sum_{j=1}^{m+1} \lambda_j + \sum_{j=m+3}^{3m} \lambda_j + \sum_{j=4m+1}^{p} \lambda_j \\ \ldots \\ \lambda_{m+2} \end{pmatrix},$$

$$\ldots$$

$$s_{m+1} = \begin{pmatrix} -\sum_{j=1}^{m-1} \lambda_j - 2\lambda_{3m+1} - \sum_{j=3m+2}^{p} \lambda_j \\ -\sum_{j=1}^{m-2} \lambda_j - \lambda_m - 2\lambda_{2m+1} - \sum_{j=2m+1}^{3m} \lambda_j - \sum_{j=4m}^{p} \lambda_j \\ \ldots \\ (m-1)\sum_{j=1}^{m} \lambda_j + m \sum_{j=2m+1}^{p} \lambda_j \end{pmatrix},$$

$$s_{m+2} = \begin{pmatrix} \lambda_m + \lambda_{m+2} \\ \lambda_{m-1} + \lambda_{m+2} \\ \ldots \\ \sum_{j=2m+1}^{p} \lambda_j \end{pmatrix},$$

we have $\eta^0 = g_1^0 s_1 + g_2^0 s_2 + \ldots + g_m^0 s_m + g^0 s_{m+1} - g^* s_{m+2}$. The main idea is to prove that η^0 is time-consistent and, namely, to find an integrable vector function $\tau(t)$ on $[0, \bar{t}]$ such that $\tau_i(t) \geq 0$, $i \in N$, and condition (11) holds.

Indeed, at the last moment $t = \bar{t}$ the core $C_{v'}(\bar{x}_1(\bar{t}), \ldots \bar{x}_m(\bar{t}), \bar{z}(\bar{t})) = \emptyset$ as a solution of the current game $\Gamma_{v'}(\bar{x}_(\bar{t}), \ldots \bar{x}_m(\bar{t})$ with integral payoffs and zero-duration. Thus, from condition (11) it follows that

$$\eta(\bar{x}(\bar{t}), \ldots \bar{x}_m(\bar{t}) = \int_0^{\bar{t}} \tau(y)dy = \eta^0. \tag{15}$$

On account of (15), we can put $\tau(y) = \frac{g_1^0}{\bar{t}} s_1 + \frac{g_2^0}{\bar{t}} s_2 + \ldots + \frac{g^0}{\bar{t}} s_{m+1} - \gamma s_{m+2}$. Finally, according to (15), we have

$$\eta^0 = \int_0^{\bar{t}} \tau(y)dy = \int_0^{\bar{t}} \Big[\frac{g_1^0}{\bar{t}} s_1 + \frac{g_2^0}{\bar{t}} s_2 + \ldots + \frac{g^0}{\bar{t}} s_{m+1} - \gamma s_4\Big] dy =$$

$$= g_1^0 s_1 + g_2^0 s_2 + \ldots + g^0 s_{m+2} - g^* s_{m+2}.$$

Now the aim is to show that the imputation $\eta^0 - \eta(\bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t))$ belongs to the core of the current game $\Gamma_{v'}(\bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t))$ for any $t \in [0, \bar{t}]$. Substituting the vector-function $\tau(t)$ into (11) and taking into account Proposition 3 and Remark 1, we obtain

$$\eta^0 - \eta(\bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t)) = \eta^0 - \int_0^t \tau(y)dy =$$

$$= \left[g_1^0 s_1 + g_2^0 s_2 + \ldots + g^0 s_{m+1} - g^* s_{m+2}\right]\left(1 - \frac{t}{\bar{t}}\right) = \eta^t$$

for all $t \in [0, \bar{t}]$.

It is not hard to prove that η^t belongs to the core $C_{v'}\big(\bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t)\big)$ of the game $\Gamma_{v'}(\bar{x}_1(t), \ldots, \bar{x}_m(t), \bar{z}(t))$.

So, condition 2 of definition 4 holds for all $t \in [0, \bar{t}]$ and for all $\eta^0 \in C_{v'}$ $\big(x_1^0, \ldots, x_m^0, z_0\big)$. Since $\Gamma_v(x_1^0, \ldots, x_m^0, z^0)$ and $\Gamma_{v'}(x_1^0, \ldots, x_m^0, z^0)$ are equivalent we have $\eta_i^0 = \xi_i^0 + g_i^0$ $i = 1, m+1$. Therefore, all our conclusions are true for ξ^0. This completes the prove of the theorem.

Acknowlegments

The author expresses the gratitude to D. Kuzyutin and S. Tarashnina for useful discussions on the subjects.

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Subgame Consistency of Shapley Value in Cooperative Data Transmission Game in Wireless Network

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Introduction

This paper examines an example of cooperative stochastic game application and considers the problem of data transmission in the simple wireless network. The simple network of data transmission consisting of three nodes is taken as a basis of network topology. Two of the nodes generate data packages in each time slot with the corresponding probabilities. The third node is the destination one. The first two nodes are connected by a channel, the connection is one-way, i.e. the first node (first player) can transmit a package directly to node 3 or to node 2. For the transmission of a package to node 2 node 1 receives a non-negative reward. The system of rewards and costs makes it possible to support cooperation between nodes 1 and 2 which are players 1 and 2 in the game, respectively. The described situation can be solved as a cooperative stochastic game.

In the papers [Michiardi, 2003], [Buttyan, 2003] theoretical-game models of behaviour in ad hoc wireless networks with emphasis on the development of cooperation mechanisms to stimulate package forwarding are considered. Paper [Sagduyu, 2006] examines a simple network topology, and the players' payoffs are the players' expected average payoffs.

In this article the payoffs of the players are the mathematical expectations of players' payoffs in the whole stochastic game. The theoretical results were taken from work [Shapley, 2006]. We calculate the maximal expected total payoff of the players and the values of the characteristic function. We can take any known allocation of cooperative game theory as an allocation of the maximal expected total payoff. All theoretical results used in this work are in [Shapley, 1953], [Petrosjan, 2006].

A cooperative stochastic game is a game which takes place in dynamics. In any dynamic cooperative game the condition of keeping the cooperation is important. The condition of subgame consistency of a chosen optimality principle allows players in each time slot of the game to expect the receiving of the allocation belonging to the same optimality principle. The condition of subgame consistency was suggested by L.A. Petrosjan for cooperative differential games in [Petrosjan, 1977]. Nowadays many works (see [Petrosjan, 2004], [Herings, 2004]) are devoted to the examination of the condition of the subgame consistency of optimality principles.

1. Stochastic Game of Data Transmission in Wireless Network

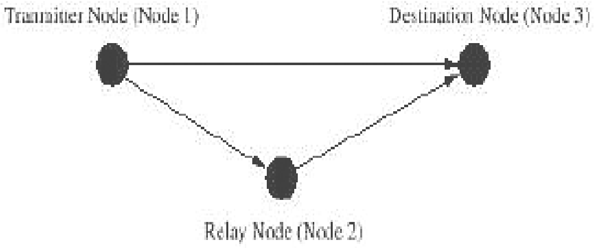

Fig. 1: Simple channel topology of wireless network

We consider a slotted synchronous system in which nodes 1 and 2 independently generate packages in each time slot with probability a_1 and a_2, respectively, provided that their individual queues were empty at the end of the previous time slot. Some assumptions about this system are as follows:

1. Nodes 1 and 2 (players 1 and 2, respectively) are going to send their packages to a common destination (node 3).

2. The maximum buffer capacity of any node is equal to one. The destination node can accept only one transmitted package in one time slot. We do not assume multiple package transmissions or simultaneous transmissions and reception by any node in any time slot.

3. If players simultaneously transmit packages to the destination node the last one rejects these packages and they return to their initial nodes, i.e. at the next time slot no new packages can be generated in nodes 1 and 2.

4. All transmitted packages have the same length, and it requires one time slot to transmit a package from one node to the other which has the direct channel with the first one.

5. Player 1 chooses between sending a package directly to node 3 or relying on node 2 to forward the package to the final destination (node 3).
6. If player 1 (node 1) transmits a package to player 2 (node 2) which has already had a package in its queue, player 2 rejects this package. Otherwise, player 2 decides on whether to accept or reject the package from player 1.

We suggest the following system of rewards and costs:

- $f \geqslant 0$ is a reward of player 1 or player 2 for each successful transmission to the common destination node;
- player 1 receives a reward $c \geqslant 0$ from player 2 for delivering a package to player 2 which can obtain the value f only after successful transmission that particular package to the final destination in a subsequent time slot;
- each time slot of package delay results in an additional cost $d \geqslant 0$ for the node that has that particular package in its queue (regardless of the source of that package);
- D_{ij} is an energy cost of one package transmission from node i to node j.

We suppose that the game ends in any time slot with the probability $0 < q < 1$. The probability q can be interpreted as a discount factor. The transmission problem in a wireless network can be solved as a Markov game (or a stochastic game in stationary strategies).

Denote the pair (Q_1, Q_2) as the state of the stochastic game where Q_i is a queue content of node i, $i = 1, 2$. The queue content Q_i can be equal to 0 or 1 if no or one package is present at the queue of node i, respectively.

The four states of Markov game are as follows:

$$K = \{(0,0); (0,1); (1,0); (1,1)\}.$$

We assume players have information not only on their own queues but on the other player's queue. This suggestion is important for this work because we are going to find the cooperative decision of this stochastic game.

Form the simultaneous game for each state:

$$\Gamma(0,0)$$

Player 1 has one strategy W (waiting), player 2 has the same strategy W (waiting).

The payoffs of the players will be $(0, 0)$.

$$\Gamma(0,1)$$

Player 1 has one strategy W (waiting), player 2 also has one strategy $\xrightarrow{3}$ (transmission to node 3).

The payoffs of the players will be $(0, f - D_{23})$.

$$\Gamma(1,0)$$

Player 1 has two strategies: 1) $\xrightarrow{3}$ (transmision to node 3), 2) $\xrightarrow{2}$ (transmission to node 2), player 2 has two strategies: 1) Ac (accepting a package from node 1), 2) Rej (rejecting a package from node 1).

The payoffs of the players will be

$$\begin{pmatrix} (f - D_{13}, 0) & (f - D_{13}, 0) \\ (c - D_{12}, -c) & (-d - D_{12}, 0) \end{pmatrix}.$$

$$\Gamma(1,1)$$

Player 1 has two strategies: 1) $\xrightarrow{3}$ (transmission to node 3), 2) W (waiting), player 2 has two strategies: 1) $\xrightarrow{3}$ (transmission to node 3), 2) W (waiting).

The payoffs of the players will be as follows:

$$\begin{pmatrix} (-d - D_{13}, -d - D_{23}) & (f - D_{13}, -d) \\ (-d, f - D_{23}) & (-d, -d) \end{pmatrix}.$$

Without loss of generality, we add $z = -\min\{0, f - D_{13}, f - D_{23}, -d, -d - D_{13}, -d - D_{23}, -d - D_{12}, c - D_{12}, -c\}$ to all payoffs of the players in all simultaneous games to get the nonnegative payoffs.

2. Matrix of Transition Probabilities

Consider the players' set of stationary strategies. In the game defined in stationary strategies the players' choice of a strategy in simultaneous games in the states depends neither on the history, nor on the time slot, in which the game is at present, but depends only on the simultaneous game of the state. For applications of stochastic games it's important to use a simple set of strategies for simplicity of the calculations of the expected payoffs of the players. Denote the set of mixed stationary strategies of player i as X_i, $i = 1, 2$.

According to the game structure the player 1's mixed stationary strategy assigns him to chose strategy W with probability one in the states $(0,0)$, $(0,1)$, strategy $\xrightarrow{3}$ with probability p_{11} in the state $(1,0)$, and strategy $\xrightarrow{3}$ with probability p_{12} in the state $(1,1)$. The player 2's mixed stationary strategy assigns him to chose strategy W with probability one in the state $(0,0)$, strategy $\xrightarrow{3}$ in the state $(0,1)$, strategy Ac with probability p_{21} in the state $(1,0)$, and strategy $\xrightarrow{3}$ with probability p_{22} in the state $(1,1)$.

Denote a player i's mixed stationary strategy as $x_i = (p_{i1}, p_{i2})$ and the set of player i's mixed stationary strategies as X_i, $i = 1, 2$. A stationary strategy-profile is $x = (x_1, x_2) = (p_{11}, p_{12}, p_{21}, p_{22})$. The matrix of transition probabilities of the stationary strategy-profile x is as follows:

$$T(x) = \begin{pmatrix} (1-a_1)(1-a_2) & (1-a_1)a_2 & a_1(1-a_2) & a_1a_2 \\ (1-a_1)(1-a_2) & (1-a_1)a_2 & a_1(1-a_2) & a_1a_2 \\ p_{11}(1-a_1)(1-a_2) & \begin{array}{r} p_{11}(1-a_1)a_2+ \\ (1-p_{11})p_{21}(1-a_1) \end{array} & \begin{array}{l} p_{11}a_1(1-a_2)+ \\ +(1-p_{11})(1-p_{21})(1-a_2) \end{array} & \begin{array}{r} p_{11}a_1a_2+(1-p_{11})p_{21}a_1+ \\ +(1-p_{11})(1-p_{21})a_2 \end{array} \\ 0 & p_{12}(1-p_{22})(1-a_1) & (1-p_{12})p_{22}(1-a_2) & \begin{array}{r} p_{12}p_{22}+(1-p_{12})(1-p_{22})+ \\ +p_{12}(1-p_{22})a_1+(1-p_{12})p_{22}a_2 \end{array} \end{pmatrix}.$$

If the stationary strategy-profile x is realized player 1's payoff in the stochastic game is as follows:

$$K_1(x) = \begin{pmatrix} z \\ z \\ p_{11}(z+f-D_{13})+(1-p_{11})p_{21}(z+c-D_{12})+ \\ +(1-p_{11})(1-p_{21})(z-d-D_{12}) \\ p_{12}p_{22}(z-d-D_{13})+ \\ +p_{12}(1-p_{22})(z+f-D_{13})+(1-p_{12})(z-d) \end{pmatrix}.$$

If the stationary strategy-profile x is realized player 2's payoff in the stochastic game is as follows:

$$K_2(x) = \begin{pmatrix} z \\ z+f-D_{23} \\ (1-p_{11})p_{21}(z-c) \\ p_{12}p_{22}(z-d-D_{23})+ \\ +(1-p_{12})p_{22}(z+f-D_{23})+(1-p_{22})(z-d) \end{pmatrix}.$$

Consider the set of player i's pure stationary strategies which is denoted as Ξ_i, $i = 1, 2$. For example, player 1's pure stationary strategy $\eta_1 = (1, 0)$ assigns player 1 to choose strategy $\xrightarrow{3}$ in the state $(1, 0)$ and strategy W in the state $(1, 1)$. Each player has 4 pure stationary strategies in the stochastic game, so we have 16 strategy-profile in pure stationary strategies. For each strategy-profile in pure stationary strategies $\eta = (\eta_1, \eta_2)$ we can simplify the matrix of transition probabilities $T(\eta)$.

For example, for $\eta^1 = (1,1,1,1)$ the matrix of transition probabilities will be as follows:

$$T(\eta^1) = \begin{pmatrix} (1-a_1)(1-a_2) & (1-a_1)a_2 & a_1(1-a_2) & a_1a_2 \\ (1-a_1)(1-a_2) & (1-a_1)a_2 & a_1(1-a_2) & a_1a_2 \\ (1-a_1)(1-a_2) & (1-a_1)a_2 & a_1(1-a_2) & a_1a_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For each strategy-profile $\eta \in \Xi = \prod_{i=1}^{2} \Xi_i$ we can calculate the mathematical expectation of players' payoffs for each subgame beginning from the fixed state. The mathematical expectation of player i' payoffs for subgames is denoted as: $E_i = (E_i^{(0,0)}, E_i^{(0,1)}, E_i^{(1,0)}, E_i^{(1,1)})$,

$$E_i(\eta) = (E - (1-q)T(\eta))^{-1} K_i(\eta), \tag{1}$$

where $K_i(\eta)$, $T(\eta))$ are determined above.

The mathematical expectation of player i's payoff in the whole game taking into account the choice of the case of the initial state is as follows:

$$\overline{E}_i(\eta) = \pi E_i(\eta), \tag{2}$$

where $\pi = (\pi_{(0,0)}, \pi_{(0,1)}, \pi_{(1,0)}, \pi_{(1,1)})$ is a vector of the initial probabilities, and π_k is the probability that the first state in the stochastic game will be $k \in K$.

3. Cooperative stochastic transmission game in wireless network

If we consider the cooperative transmission game we will find *the cooperative decision* $\overline{\eta} = (\overline{\eta}_1, \overline{\eta}_2)$ of the game which is a strategy-profile in pure stationary strategies such as

$$\sum_{i \in \{1,2\}} \overline{E}_i(\overline{\eta}) = \max_{\eta \in \Xi} \sum_{i \in \{1,2\}} \overline{E}_i(\eta).$$

The characteristic functions for the subgames

$$V(S) = (V^{(0,0)}(S), V^{(0,1)}(S), V^{(1,0)}(S), V^{(1,1)}(S))$$

can be calculated using the following formula:

$$V^k(S) = \max_{\eta_S} \min_{\eta_{N \setminus S}} \sum_{i \in S} E_i^k(\eta_S, \eta_{N \setminus S}). \tag{3}$$

The value of characteristic function of the whole cooperative transmission game for coalition $S \neq \emptyset, \{1,2\}$ is as follows:

$$\overline{V}(S) = \pi V(S). \tag{4}$$

Definition 1. *A cooperative stochastic transmission game in wireless network is pair* $\left\langle \{1,2\}, \overline{V}(S) \right\rangle$, *where* $\overline{V}(S)$ *is a characteristic function defined by* (4), $\overline{V}(\emptyset) = 0$ *and* $\overline{V}(\{1,2\}) = \sum\limits_{i \in \{1,2\}} \overline{E}_i(\overline{\eta})$.

We can find any allocation of the mathematical expectation of players' total payoff using the characteristic function $V(S)$ for cooperative subgames and $\overline{V}(S)$ for the whole cooperative transmission game. When the number of players in the game is equal to two, many imputations are equal to each other, i.e. the Shapley Value and nucleolus coincide.

For simplicity, suppose that optimality principle is Shapley Value. Denote Shapley Value calculating for subgames as $Sh = (Sh_1, Sh_2)$, where $Sh_i = (Sh_i^{(0,0)}, Sh_i^{(0,1)}, Sh_i^{(1,0)}, Sh_i^{(1,1)})$, and Shapley Value calculating for the game as $\overline{Sh} = (\overline{Sh}_1, \overline{Sh}_2)$.

4. Cooperative Payoff Distribution Procedure

Before the game players (node 1 and 2) come to the agreement about the choice of the strategies which guarantee the maximal total payoff $\overline{V}(\{1,2\})$ and expect to receive the components of the allocation (we use Shapley Value) $\overline{Sh}_1$ and $\overline{Sh}_2$ to player 1 and player 2, respectively.

The situation will be "natural" if the payments to the players in the simultaneous games in the states are equal to their payoffs in these simultaneous games. It is equivalent to the recurrence equation:

$$Sh_i = K_i(\overline{\eta}) + (1-q)T(\overline{\eta})Sh_i, \tag{5}$$

where $K_i(\overline{\eta})$ is a payoff of player i in the simultaneous game on the condition of using the cooperative decision $\overline{\eta}$, and $(1-q)T(\overline{\eta})Sh_i$ is the expected value of the i-th component of Shapley Value on the condition that the transmission data game hasn't ended.

Definition 2. *We will call Shapley Value* $\overline{Sh}$ *where* $\overline{Sh}_i = \pi Sh_i$ *natural consistent if* Sh_i *satisfies the equation* (5) *for any* $i = 1, 2$.

For the protection of cooperation in any time slot players should expect getting payoffs in accordance with Shapley Value. Unfortunately, realizing the payments in each time slot of the transmission game in accordance with the payoffs in simultaneous games on these states, it is impossible to obtain that the rest payments to the players will coincide with the components of Shapley Value. It can frustrate plans of the players and the cooperation can break. We suggest to redistribute payoffs of players in the simultaneous games to overcome this problem.

Let $\beta_i = (\beta_i^{(0,0)}, \beta_i^{(0,1)}, \beta_i^{(1,0)}, \beta_i^{(1,1)})$ be a vector of real payments to player i in the states $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, respectively, such that

$$\sum_{i=1}^{2} \beta_i^k = \sum_{i \in N} K_i^k(\overline{\eta})$$

for any $k \in \{(0,0),(0,1),(1,0),(1,1)\}$.

Definition 3. *We will call $\beta^k = (\beta_1^k, \ldots, \beta_2^k)$, $k \in K$, a cooperative payoff distribution procedure* (*see* [8], [9]) *in the state k, where β_i^k is a real payment to player i in the state k.*

We can always determine these payments from the equation

$$Sh_i = \beta_i + (1-q)T(\overline{\eta})Sh_i, \tag{6}$$

or

$$\beta_i = (E - (1-q)T(\overline{\eta}))Sh_i. \tag{7}$$

We can calculate the cooperative payoff distribution procedure in the game using (7).

Denote the mathematical expectation of the payments to player i in the cooperative stochastic game $\overline{B}_i = \pi B_i$, where $B_i = (B_i^1, \ldots, B_i^t)$ and B_i^k is the mathematical expectation of the payments to the player i in the cooperative stochastic subgame beginning from the state k.

Lemma 1. *The equality $\overline{B}_i = \overline{Sh}_i$ holds for any $i \in N$.*

Lemma 1 says that the mathematical expectation of sums β_i (defined by (7)), which are the payments to players realized along the cooperative decision, equals to the mathematical expectation of player i's payoff in this game (i.e. to the ith component of Shapley Value $\overline{Sh}_i$).

5. Subgame Consistency of Shapley Value

Someone can require that β_i^k should be non-negative for any state $k \in K$ and for each $i \in N$. It is equal to the condition when the system of equations on the variable $\beta_i = (\beta_i^{(0,0)}, \beta_i^{(0,1)}, \beta_i^{(1,0)}, \beta_i^{(1,1)})$

$$Sh_i = (E - (1-q)\Pi(\overline{\eta}(\cdot)))^{-1}\beta_i \tag{8}$$

has got a non-negative solution. But in a general case we can't guarantee the non-negativeness of the components of vector $\beta_i = (\beta_i^{(0,0)}, \beta_i^{(0,1)}, \beta_i^{(1,0)}, \beta_i^{(1,1)})$.

Definition 4. *The Shapley Value $\overline{Sh} = (\overline{Sh}_1, \ldots, \overline{Sh}_n)$, $\overline{Sh}_i = \pi Sh_i$ is called subgame consistent in the cooperative stochastic game* (see [Baranova, Petrosjan, 2006], [Petrosjan, 2006]) *if for any $i \in N$ β_i is a non-negative solution of the system* (8).

6. Numerical Example of Cooperative Data Transmission Game in Wireless Network

We have to turn to the numerical example to introduce the results of the calculation of the cooperative decision because in the general case $(E-(1-q)T(\overline{\eta}))^{-1}$ isn't representable in the paper.

$$\begin{aligned} a_1 &= 0.5, & q &= 0.01,\\ a_2 &= 0.1, & f &= 1,\\ D_{12} &= 0.1, & d &= 0.1,\\ D_{13} &= 0.6, & c &= 0.3,\\ D_{23} &= 0.2, & \pi &= (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}). \end{aligned}$$

The table shows the following values for each strategy-profile in pure stationary strategies η:

- $E_1(\eta) = (E_1^{(0,0)}(\eta), E_1^{(0,1)}(\eta), E_1^{(1,0)}(\eta), E_1^{(1,1)}(\eta))$ is a vector of mathematical expectations of player 1's payoffs in subgames,
- $E_2(\eta) = (E_2^{(0,0)}(\eta), E_2^{(0,1)}(\eta), E_2^{(1,0)}(\eta), E_2^{(1,1)}(\eta))$ is a vector of mathematical expectations of player 2's payoffs in subgames,
- $\sum\limits_{i\in\{1,2\}} \overline{E}_i(\eta)$ is the mathematical expectation of players' total payoff.

η	$E_1(\eta)$	$E_2(\eta)$	$E_1(\eta)+E_2(\eta)$
$\eta^1=(1,1,1,1)$	14.75966387 14.75966387 15.15966387 0	45.70756302 46.50756302 45.70756302 40.	60.46722689 61.26722689 60.86722689 40.
$\eta^2=(1,1,1,0)$	89.80000000 89.80000000 90.20000000 90.20000000	76.24887286 77.04887286 76.24887286 76.71127141	166.0488729 166.8488729 166.4488729 166.9112714
$\eta^3=(1,0,1,1)$	88.22724883 88.22724883 88.62724883 88.30952131	77.92000000 78.72000000 77.92000000 78.72000000	166.1472488 166.9472488 166.5472488 167.0295213
$\eta^4=(1,0,1,0)$	64.67563026 64.67563026 65.07563026 60.	62.34621849 63.14621849 62.34621849 60.	127.0218488 127.8218488 127.4218488 120.

$\eta^6 = (1,1,0,0)$	89.80000000	76.24887286	166.0488729
	89.80000000	77.04887286	166.8488729
	90.20000000	76.24887286	166.4488729
	90.20000000	76.71127141	166.9112714
$\eta^7 = (1,0,0,1)$	88.22724883	77.92000000	166.1472488
	88.22724883	78.72000000	166.9472488
	88.62724883	77.92000000	166.5472488
	88.30952131	78.72000000	167.0295213
$\eta^8 = (1,0,0,0)$	64.67563026	62.34621849	127.0218488
	64.67563026	63.14621849	127.8218488
	65.07563026	62.34621849	127.4218488
	60.	60.	120.
$\eta^9 = (0,1,1,1)$	3.870077599	41.81391498	45.68399258
	3.870077599	42.61391498	46.48399258
	2.815688411	41.29388792	44.10957633
	0.	40.	40.
$\eta^{10} = (0,1,1,0)$	85.30045000	82.15225000	167.4527000
	85.30045000	82.95225000	168.2527000
	85.58955000	82.29775000	167.8873000
	85.78955000	82.49775000	168.2873000
$\eta^{11} = (0,0,1,1)$	75.28276807	93.56491025	168.8476783
	75.28276807	94.36491025	169.6476783
	75.40659007	93.89870345	169.3052935
	75.23559576	94.52135936	169.7569551
$\eta^{12} = (0,0,1,0)$	60.82133034	60.79766590	121.6189962
	60.82133034	61.59766590	122.4189962
	60.70655852	60.59084462	121.2974031
	60.	60.	120.
$\eta^{13} = (0,1,0,1)$	5.432827686	43.10048870	48.53331639
	5.432827686	43.90048870	49.33331639
	4.587155964	42.75229358	47.33944954
	0.	40.	40.
$\eta^{14} = (0,1,0,0)$	62.35393639	75.10793102	137.4618674
	62.35393639	75.90793102	138.2618674
	62.07747575	75.07981035	137.1572861
	63.29742280	75.59292249	138.8903453
$\eta^{15} = (0,0,0,1)$	51.37623762	77.92000001	129.2962376
	51.37623762	78.72000001	130.0962376
	50.99000000	77.92000000	128.9100000
	51.09000000	78.72000000	129.8100000

For our numerical example the cooperative decision is

$$\eta_{11} = ((W, W, \xrightarrow{2}, W), (W, \xrightarrow{3}, Ac, \xrightarrow{3})).$$

The maximum of the mathematical expectation of total payoff of the players in the game is as follows:

$$\max_{\eta \in \Xi} \sum_{i \in N} \overline{E}_i(\eta) = 169.39,$$

$$V(\{1\}) = (64.68, 64.68, 65.08, 60),$$

$$V(\{2\}) = (61.09, 61.89, 60.92, 60),$$

$$V(\{1,2\}) = (168.85, 169.65, 169.31, 169.76).$$

The characteristic function of the whole cooperative data transmission game is as follows:

$$\overline{V}(\{S\}) = \pi V(\{S\}), \tag{9}$$

or for our numerical example:

$$\overline{V}(\{1\}) = 63.61,$$

$$\overline{V}(\{2\}) = 60.97,$$

$$\overline{V}(\{1,2\}) = 169.39,$$

Shapley Value is as follows:

- for subgames:
  - $Sh_1 = (86.22, 86.22, 86.73, 84.88)$,
  - $Sh_2 = (82.63, 83.43, 82.57, 84.88)$,
- for the whole game:
  - $\overline{Sh}_1 = 86.01$,
  - $\overline{Sh}_2 = 83.38$.

The cooperative payoff distribution procedure β_1 for player 1 and β_2 for player 2 will be as follows:

- $\beta_1 = (0.7, 0.7, 2.04, -0.8)$,
- $\beta_2 = (0.7, 1.5, -0.74, 2.9)$,

where β_i^k is a real payment to player i in the state k.

Remind that the payoffs of the players in the states defined in the matrix forms are as follows:

- $K_1 = (0.7, 0.7, 0.9, 0.6)$,

- $K_2 = (0.7, 1.5, 0.4, 1.5)$,

If players always want to receive components of Shapley Value in the rest of the data transmission game the payments to the players should be as follows:

- in the state $(1, 0)$
  - 2.04 to player 1 instead of 0.9,
  - -0.74 to player 2 instead of 0.4,
- in the state $(1, 1)$
  - -0.8 to player 1 instead of 0.6,
  - 2.9 to player 2 instead of 1.5.

In our numerical example Shapley Value $\overline{Sh} = (86.01, 83.38)$ is not a subgame consistent optimality principle because some of the real payments to the players are negative.

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Proportionality in NTU Games: Proportional Excess, Nucleolus and Prenucleolus

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Abstract. An axiomatic approach is developed to define the proportional excess function on the space of positively generated NTU games. This excess generalizes to NTU games the proportional TU excess $v(S)/x(S)$. Five axioms are proposed, and it is shown that the proportional excess is the unique excess function satisfying the axioms. The properties of proportional excess and corresponding nucleolus, prenucleolus and, in particular, *status quo*-proportional solution for bargaining games are studied.

Keywords: NTU games, excess function, proportional excess, Minkowski gauge function, nucleolus, prenucleolus, bargaining solution.

Introduction

In this paper we discuss the problem of definition of proportional solutions for NTU games. Of course, proportional allocation is not a new idea. It goes back to Aristotle (see, for example, [Moulin, 2002]). Young [8] writes that "proportionality is deeply rooted in law and custom as a norm of distributive justice". Thompson [Thomson, 1998] puts proportionality "at the heart of equity theory". It is also the commonly recognized standard of business practice. The relevant results on proportionality are found in the accounting and social choice literature. For two person positive TU games the proportional solution is defined in an obvious way:

$$x_i = v(\{1,2\})\frac{v(\{i\})}{v(\{1\}) + v(\{2\})},\ i = 1,2.$$

Unfortunately, for TU games with more than two players the straightforward definition of proportional solution is not possible, since it is not clear what the proportionality should mean when different coalitions are possible. Hence, the definition of proportional solution for positive TU games can be realized as some extensions of proportional solution for two person games.

The first such solution, the proportional nucleolus, was defined by Lemaire [Lemaire, 1991]. He defined it in a usual manner, but he used the relative excess $e_r(S, x, v) = = \frac{v(S)-x(S)}{v(S)}$ instead of standard excess $e(S, x, v) = v(S) - x(S)$. Note that the relative excess just mentioned is ordinally equivalent to the proportional excess $\frac{v(S)}{x(S)}$. Another proportional solution defined independently by Feldman [Feldman, 1999] and Ortmann [Ortmann, 2000] can be considered as a consistent (in Hart-Mas-Colell sense) extension of the proportional solution for two-person positive games on n-person positive games.

Yanovskaya (see [Pechersky, 2004]) used the proportional excess to define the proportional solution, proportionality of solution being defined as follows.

A solution Ψ on the set of positive TU games G_{N+} with the set N of players is called *proportional*, if for any two games (N, v), $(N, w) \in G_{N+}$ and any two payoff vectors $x \in X(N, v)$, $y \in X(N, w)$

$$\frac{v(S)}{x(S)} = \frac{w(S)}{y(S)} \text{ for all } S \subset N$$

implies

$$x = \Psi(N, v) \iff y = \Psi(N, w). \tag{1}$$

For example, Ψ defined by

$$\Psi(N, v) = \arg \max_{x \in X(N,v)} \prod_{S \subset N} x(S)^{w(s)v(S)} \tag{2}$$

for some non-negative numbers $w(s), s \leq n - 1, n = |N|$, where $X(N, v)$ is the set of preimputations for v, is proportional solution. Clearly, (1) is an analogues of shift covariant solution for standard excess: let Φ be an arbitrary shift covariant solution on G_N, and $(N, v), (N, w) \in G_N$. Then

$$v(S) - x(S) = w(S) - y(S) \text{ for all } S \subset N \tag{3}$$

implies

$$x \in \Phi(N, v) \iff y \in \Phi(N, w). \tag{4}$$

It is well-known that the generalization of solutions based on cardinal notion of excess to NTU games creates difficulties. In particular, the problem is that there is no natural analogues of excesses in NTU case. In his survey Maschler [Maschler, 1992] noted that "research concerning the extension of the kernel and the nucleolus to games without side payments is still scarce, ... the main issue is to decide what the analogue of the excess function should be". Although, there have been several suggestions (see, for example, [Kalai, 1973], [Kalai, 1975], [Billera, 1972], [Bondareva, 1989], [McLean, 1989], [Nakayama, 1983],...) that these proposals have not yet achieved the status of a general theory similar to the one that exists for the side payment case.

Kalai [Kalai, 1975] defined a family of excess functions for cooperative NTU games. Using these excess functions he defined the ε-core, the kernel and the nucleolus of a NTU game in a way that preserves a significant portion of the structure that these concepts exhibit in the TU case. These excess functions satisfy some natural conditions, which seem to be required for such functions.

In the frameworks of Kalai's approach in [Pechersky, 2000, 2001] the *gauge excess* which generalizes the relative excess $e_r(S,x,v) = \frac{v(S)-x(S)}{v(S)}$ to NTU games was defined and axiomatically characterized.

Our aim in this paper is to define, axiomatically characterize the proportional excess for NTU games, and consider its property. In particular, we study the corresponding nucleolus and prenucleolus. For bargaining games the nucleolus defines the *status quo*-proportional solution. The axiomatic characterization of this solution is given.

The paper is organized as follows. Section 1 provides definitions and notations. In Section 2 we describe axioms, state the existence and uniqueness theorem, and study the properties of the proportional excess function and the properties of corresponding nucleolus and prenucleolus. In Section 3 we give an axiomatic characterization of the *status quo*-proportional solution for bargaining games. The proofs are given in Appendix 1. The geometric characterization of the nucleolus and prenucleolus is given in Appendix 2.

1. Definitions and Notations

Let $N = \{1, 2, \ldots, n\}$ be a non-empty finite set of players. A *coalition* is a non-empty subset S of N. For a subset $S \subset N$ let $\mathbb{R}^S$ denote $|S|$-dimensional Euclidean space with axes indexed by elements of S. A *payoff vector* for S is a vector $x \in \mathbb{R}^S$. For $z \in \mathbb{R}^N$ and $S \subset N$, z^S will denote the projection of z on the subspace

$$\mathbb{R}^{[S]} = \{x \in \mathbb{R}^N \,:\, x_i = 0 \text{ for } i \notin S\},$$

and z_S – the restriction of z on $\mathbb{R}^S$. To simplify notations, if $|S| = 1$ or $|S| = 2$, i. e. $S = \{i\}$ or $S = \{i, j\}$ for some $i, j \in N$, we write $\mathbb{R}^{[i]}$ and $\mathbb{R}^{[i,j]}$ instead of $\mathbb{R}^{[\{i\}]}$ and $\mathbb{R}^{[\{i,j\}]}$, respectively.

Let $x, y \in \mathbb{R}^N$. We will write $x \geq y$, if $x_i \geq y_i$ for all $i \in N$; $x > y$, if $x_i > y_i$ for all $i \in N$. Denote

$$\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \,:\, x \geq \mathbf{0}\},$$
$$\mathbb{R}^N_{++} = \{x \in \mathbb{R}^N \,:\, x > \mathbf{0}\},$$

where $\mathbf{0} = (0, 0, ..., 0)$. We denote the coordinate-wise product by $x * y$, i. e. $x * y = = (x_1 y_1, \ldots, x_n y_n)$, and coordinate-wise division by $x : y = (x_1/y_1, \ldots, x_n/y_n)$ for $y > \mathbf{0}$.

Let $A \subset \mathbb{R}^N$. If $x \in \mathbb{R}^N$, then $x + A = \{x + a \,:\, a \in A\}$ and $\lambda A = \{\lambda a \,:\, a \in A\}$. A is *comprehensive*, if $x \in A$ and $x \geq y$ imply $y \in A$. A is *bounded above*, if $A \cap (x + \mathbb{R}^N_+)$

is bounded for every $x \in \mathbb{R}^N$. The boundary of A is denoted by ∂A. The interior of a set A will be denoted by *int* A, and the relative interior by *rel int* A. The closed convex hull of a set A we denote with *co* A.

A *nontransferable utility game* (or shortly *NTU game*) is a pair (N, V), where N is the set of players, and V is the set-valued map that assigns to each coalition $S \subset N$ a set $V(S)$, that satisfies:

(1) $V(S) \subset \mathbb{R}^{[S]} = \{x \subset \mathbb{R}^N \ : \ x_i = 0 \text{ for } i \notin S\}$;

(2) $V(S)$ is closed, non-empty, comprehensive and bounded above.

(Usually $V(\emptyset) = \emptyset$). The following particular cases should be mentioned.

TU game. A TU game v can be considered as a NTU game of the following form:

$$V(S) = \{x \in \mathbb{R}^{[S]} : x(S) \leq v(S)\},$$

where $x(S) = \sum_{i \in S} x_i$. The boundary of $V(S)$ is a hyperplane in $\mathbb{R}^{[S]}$ with normal e^S, where $e = (1, 1, \ldots, 1)$.

Hyperplane game. A hyperplane game V is defined as follows: for every $S \subset N$

$$V(S) = \left\{ x \in \mathbb{R}^{[S]} \ : \ \sum_{i \in S} p_i^{(S)} x_i \leq r^{(S)} \right\},$$

where $p_i^{(S)} > 0$ for every $i \in S$, $S \subset N$ ($p_i^{(S)} = 0$, $i \notin S$). The boundary of $V(S)$ is a hyperplane in $\mathbb{R}^{[S]}$ with a normal $p^{(S)}$. Clearly TU game is a hyperplane game with $p_i^{(S)} = 1$ for all $i \in S$, $S \subset N$.

Bargaining game. A n-person bargaining game is a pair (q, Q), where $q \in \mathbb{R}^N$ is the *status quo* point, $Q \subset \mathbb{R}^N$ and $N = \{1, 2, \ldots, n\}$. When interpreting this pair one can think as follows: if the players act separately the only possible outcome for the players is q giving utility q_i to player $i = 1, 2, \ldots, n$. If all players cooperate they can potentially agree on an arbitrary outcome $x \in Q$. The corresponding NTU game can be defined as follows:

$$V(N) = \{x \in \mathbb{R}^N \ : \ \text{there is } y \in Q \text{ such that } x \leq y\},$$

$$V(S) = \{x \in \mathbb{R}^{[S]} \ : \ x_i \leq q_i \text{ for every } i \in S\} \text{ for } S \neq N.$$

2. Proportional Excess for NTU games

2.1. The Problem and the Space $\mathcal{G}_{N+}$

As we have mentioned above, at the last time there is a growing interest to proportional excess, defined for every positive TU game u (i. e. $u(S) > 0$ for every coalition S) by formula

$$h_S(u, x) = \frac{u(S)}{x(S)}.$$

Our goal is to generalize this excess to NTU games. We restrict our attention to the space $\mathcal{G}_{N+}$ of all normally generated NTU games. Roughly speaking (the formal definition will be given further) a game V belongs to $\mathcal{G}_{N+}$, if every set $V(S)$ is compactly generated, contains $\mathbf{0} = (0, \ldots, 0)$ as its relatively interior point and coincides with the comprehensive hull of its positive part $V_+(S) = V(S) \cap \mathbb{R}^{[S]}_+$.

To define the corresponding NTU excess we impose five axioms (continuity axiom, scale invariance axiom, MIN and MAX axioms and TU game axiom) which describe desirable properties of an excess function. Continuity axiom asserts that the excess (of a coalition) should be continuous jointly in x and V. Scale invariance asserts that excess does not depend on linear transformations of the game and payoff vector. MIN and MAX axioms state that the excess in the "intersection game" ($V = V_1 \cap V_2$) and in the "union game" ($V = V_1 \cup V_2$) should be equal to the minimum and to the maximum of two component games excesses, respectively. TU game axiom asserts that in the TU case the excess should coincide with the proportional excess.

These five axioms uniquely define the *proportional excess* $h_S(V, x)$, defined by formula

$$h_S(V, x) = 1/\gamma(V(S), x^S),$$

where $\gamma(W, .)$ is the *gauge* (or the *Minkowski gauge function*) of a set W (see, for example, [Rockafellar, 1997]):

$$\gamma(W, x) = \inf\{\lambda > 0 : x \in \lambda W\}.$$

We study the properties of the proportional excess and corresponding nucleolus and prenucleolus.

Let us define the space $\mathcal{G}_{N+}$. A game $V \in \mathcal{G}_{N+}$ iff for every S:

(a) $V(S)$ is positively generated (i.e. $V(S) = (V(S) \cap \mathbb{R}^{[S]}_+) - \mathbb{R}^{[S]}_+$, and $V_+(S) = V(S) \cap \mathbb{R}^{[S]}_+$ is compact), and every ray $L_x = \{\lambda x : \lambda \geq 0\}$, $x \neq \mathbf{0}$ does not intersect the boundary of $V(S)$ more than once;

(b) $\mathbf{0}$ is an interior point of the set $V^\wedge(S) = V(S) + \mathbb{R}^{[N \setminus S]}$.

For $S \subset N$ a set $V(S) \subset \mathbb{R}^{[S]}$ will be called a *game subset*, if it satisfies (a) and (b). The space consisting of all game subsets satisfying (a) and (b) will be denoted by $\mathcal{G}^S_{N+}$. Clearly every $V \in \mathcal{G}_{N+}$ is a game in Kalai's sense (cf. [Kalai, 1975]).

It is often convenient to consider some modification of a game (N, V) defined by the set-valued map $\overline{V}$ with $\overline{V}(S) = V_S(S)$, where $V_S(S)$ is the restriction of $V(S)$ on $\mathbb{R}^S$. Clearly $V(S) = \overline{V}(S) \times \mathbf{0}_{N \setminus S}$.

Obviously, every set $V_+(S)$ is *normal* (or **0**-*comprehensive*), i.e. if $x \in V_+(S)$, $y \in \mathbb{R}^{[S]}$ and $\mathbf{0} \leq y \leq x$, then $y \in V_+(S)$. In the sense a game $V \in \mathcal{G}_{N+}$ is *normally generated*. It is clear also that every $U \in \mathcal{G}^S_{N+}$ is *star-shaped* (cf. [Rubinov, 1986]), i.e. U is closed, it contains $\mathbf{0}$ as a relatively interior point (in $\mathbb{R}^{[S]}$), and every ray L_x does not intersect the boundary ∂U of U more than once. (This definition is stronger than the usual one: a star-shaped subset of a real vector space contains a distinguished member, the center, which can be connected with every other element by a line segment which is completely contained in the set.) Note also that for every $V(S) \in \mathcal{G}^S_{N+}$ and every $x \in \mathbb{R}^{[S]}_+$, $x \neq \mathbf{0}$ there is a unique $\lambda > 0$ such that $\lambda x \in \partial V(S)$.

It is clear that if $V_1, V_2 \in \mathcal{G}_{N+}$, then the games $V_1 \cap V_2$ and $V_1 \cup V_2$, defined by

$$(V_1 \cap V_2)(S) = V_1(S) \cap V_2(S)\ ,\ (V_1 \cup V_2)(S) = V_1(S) \cup V_2(S),$$

also belong to $\mathcal{G}_{N+}$.

Next, if $V(S) \in \mathcal{G}^S_{N+}$, $A \in \mathbb{R}^{[S]}_{++}$ and $A * V(S) = \{A * y\ :\ y \in V(S)\}$, then $A * V(S) \in \mathcal{G}^S_{N+}$.

Remark 1. It is not difficult to prove that if a game V is such that for every S the set $V(S)$ is normally generated, possesses (b) and satisfies traditional non-levelness condition:

$$x, y \in \partial V(S) \cap \mathbb{R}^{[S]}_+, \quad x \geq y \ \Rightarrow\ x = y,$$

then $V \in \mathcal{G}_{N+}$.

Indeed, let $V(S)$ satisfies these properties. It is sufficient to check that non-levelness condition implies the second part of (a). Suppose the contrary, i. e. for some $z \in \mathbb{R}^{[S]}_+$, $z \neq \mathbf{0}$ the ray $L_z = \{\lambda z\ :\ \lambda \geq 0\}$ intersects the boundary of $V(S)$ more than once. Hence, there are $\lambda_1 > \lambda_2 > 0$, such that $u = \lambda_1 z \in \partial V(S)$, $v = \lambda_2 z \in \partial V(S)$ and $u^S \geq v^S$, $u^S \neq v^S$, which contradicts the non-levelness condition.

Two particular cases should be mentioned: NTU games corresponding the TU games and hyperplane games. Of course, they are not exactly TU games and hyperplane games (in the standard sense), but we leave the usual name.

TU game. Let v be a positive TU game, i. e. $v(S) > 0$ for every S. Then the corresponding NTU game $V \in \mathcal{G}_{N+}$ can be defined by

$$V(S) = \{x \in \mathbb{R}^{[S]}_+\ :\ x(S) \leq v(S)\} - \mathbb{R}^{[S]}_+.$$

Hyperplane game. Let V be a hyperplane game, i. e. for every $S \subset N$,

$$V(S) = \left\{x \in \mathbb{R}^{[S]}\ :\ \sum_{i \in S} p_i^{(S)} x_i \leq r_S\right\},$$

where $p_i^{(S)} > 0$ for every $i \in S$, and $r_S > 0$. Then corresponding NTU game $V_1 \in \mathcal{G}_{N+}$ can be defined by:

$$V_1(S) = \left\{x \in \mathbb{R}^{[S]}_+\ :\ \sum_{i \in S} p_i^{(S)} x_i \leq r_S\right\} - \mathbb{R}^{[S]}_+.$$

Let us recall some definitions from Kalai's paper ([Kalai, 1975]) adopting them to the case under consideration and taking into account our goal to generalize the proportional excess. In particular, in property (C) we require that excess should be equal to 1 on the boundary, and not 0.

Define *excess function* for a coalition $S \neq \emptyset$ as such function $E_S\ :\ \mathcal{G}_{N+} \times \mathbb{R}^N \to \mathbb{R}$, that

(A) If $x, y \in \mathbb{R}^N$ and $x_i = y_i$ for every $i \in S$, then for every V, $E_S(V, x) = E_S(V, y)$.

(B) If $x, y \in \mathbb{R}^N$ are such that $x_i < y_i$ for every $i \in S$, then for every V, $E_S(V, x) >$ $> E_S(V, y)$.

(C) For every game V, $x \in \partial V(S) \Rightarrow E_S(V, x) = 1$.

(D) $E_S(V, x)$ is continuous jointly in x and V.

The metric on $\mathcal{G}_{N+}$ is the Hausdorff metric: for $V, W \in \mathcal{G}_{N+}$

$$\rho(V, W) = \max_S H_S(V(S), W(S)).$$

(Recall that the Hausdorff metric is defined as follows. Let A and B be the subsets of $\mathbb{R}^S$, then $H_S(A, B) = \max(l(A, B), l(B, A))$, where

$$l(A, B) = \sup\{d(x, B) \, : \, x \in A\}$$

and d is Euclidean metric.)

E_S will be said to be independent of other coalitions if for every two games V and W such that $V(S) = W(S)$, and for every $x \in \mathbb{R}^N$, $E_S(V, x) = E_S(W, x)$.

We restrict our attention to nonnegative vectors x only since any reasonable solution of a game $V \in \mathcal{G}_{N+}$, should be, clearly, non-negative. We consider only those excesses which are independent of other coalitions. Taking into account property (A) and, of course, (B), we will consider in what follows excess functions as those functions on $\mathcal{G}^S_{N+} \times \mathbb{R}^{[S]}_+$ (or $\mathcal{G}^S_{N+} \times \mathbb{R}^N_+$).

The following notations will be used throughout the paper ([Kalai, 1975]): $IR(V) = \{x \in V(N) : \forall i \in N \; x_i \geq y_i$ for every $y \in V(i)\}$ – the set of individually rational points of a game V;
$GR(V) = \{x \in V(N)$: there is no $y \in V(N)$, such that $y > x\}$ – the set of (weakly) Pareto optimal points of a game V;
$C(V) = \{x \in V(N)$: there is no S, $y \in V(S)$, such that $y_i > x_i \; \forall i \in S\}$ – the core of a game V.

Finally, we recall the definitions of the nucleolus and the prenucleolus of a game (cf., for example, [Kalai, 1975]).

Let $\{E_S\}_S$ be a fixed family of excess functions, and let X be a closed subset of $\mathbb{R}^N$. For an arbitrary $x \in X$ and game V define a vector $\theta(x)$ to be:

$$\theta(x) = \theta(V, x) = (E_{S_1}(V, x), \ldots, E_{S_{2^n}}(V, x)),$$

where various excesses of all coalitions are arranged in decreasing (nonincreasing) order. The components of $\theta(x)$ are well defined and vary continuously for "good" excess functions. We say that $\theta(x)$ is lexicographically smaller than $\theta(y)$, $\theta(x) \prec_{lex}$ $\theta(y)$, if there is such a positive integer q that $\theta_i(x) = \theta_i(y)$ for all $i < q$ and $\theta_q(x) <$ $< \theta_q(y)$.

The *nucleolus* of V (with respect to X and given family of excess functions $\{E_S\}_S$) – we denote it by $N(X, V)$ – is the set of vectors in X which θ's are lexicographically least, i. e.

$$N(X, V) = \{x \in X : \theta(x) \preceq_{lex} \theta(y) \text{ for all } y \in X\}.$$

If $X = IR(V) \cap GR(V)$, then $N(X, V) := N(V)$ is called the *nucleolus of* V. If $X = GR(V)$, then $N(X, V) := PN(V)$ is called the *prenucleolus of* V.

2.2. Proportional Excess: Axioms, Existence and Solutions

Let H_S be an excess function, i. e. $H_S : \mathcal{G}^S_{N+} \times \mathbb{R}^{[S]}_+ \to \mathbb{R}$. Let us impose the following axioms (we write V instead of $V(S)$).

Continuity. $H_S(V, x)$ is continuous jointly in V and $x \neq \mathbf{0}$.

Scale invariance. If $V \in \mathcal{G}^S_{N+}$, $A \in \mathbb{R}^{[S]}_{++}$ and $A * V = \{A * y : y \in V\}$, then

$$H_S(A * V, A * x) = H_S(V, x).$$

MIN. Let $V_1, V_2 \in \mathcal{G}^S_{N+}$, then

$$H_S(V_1 \cap V_2, x) = \min\{H_S(V_1, x), H_S(V_2, x)\}.$$

MAX. Let $V_1, V_2 \in \mathcal{G}^S_{N+}$, then

$$H_S(V_1 \cup V_2, x) = \max\{H_S(V_1, x), H_S(V_2, x)\}.$$

Proportionality for TU games. If $V \in \mathcal{G}_{N+}$ corresponds to TU game v, i. e.

$$V(S) = \{x \in \mathbb{R}^{[S]}_+ : x(S) \leq v(S)\} - \mathbb{R}^{[S]}_+,$$

then $H_S(V, x) = v(S)/x(S)$.

Continuity axiom is self-explanatory, and there is no need to comment on it. Scale independence seems also to be clear, but the absence of translation invariance should be stressed. Note that in the TU case the translation invariance can be justified by the lack of “income effect”. However, in the NTU case, where the “income effect” can be of great importance, translation invariance seems to be not justified.

Note also that the proportional, or its ordinal equivalent relative excesses make good sense in economic environment, when the players are e.g. companies. If an excess at x is measure of dissatisfaction of a coalition of companies from x (satisfaction, if negative) it is reasonable to assume that a rich coalition will “tolerate” a large loss and a poor coalition will not “tolerate” a much smaller loss. Thus, a nucleolus based on the above and similar excess functions, is a reasonable solution concept. If a poor person looses, say, \$ 1000 he will protest strongly, whereas a large conglomerate may not bother to even rise the issue. In such cases the fact that the excess is not invariant under translation is even a merit.[2] MIN and MAX axioms seem to be natural conditions. The interpretation of games $V_1 \cap V_2$ and $V_1 \cup V_2$ is

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[2] I am grateful to Michael Maschler for this observation.

evident: corresponding set $V_1(S) \cap V_2(S)$ represents the payoffs vectors feasible for coalition S in both games V_1 and V_2, and $V_1(S) \cup V_2(S)$ is the set of payoffs vectors feasible for coalition S for at least in one of two given games V_1 and V_2. (Note that these two properties are trivially fulfilled in TU case for all excesses: the standard $v(S) - x(S)$, the relative $(v(S) - x(S))/v(S)$ and the proportional $v(S)/x(S)$).

Finally, proportionality axiom is the reformulation of our aim to generalize the TU proportional excess to NTU games.

Let $V \in \mathcal{G}_{N+}$ be an arbitrary game. Define a function $h_S : \mathcal{G}^S_{N+} \times \mathbb{R}^N_+ \to \mathbb{R}$ as follows:

$$h_S(V, x) = 1/\gamma(V(S), x^S), \tag{5}$$

where $\gamma(W, y) = \inf\{\lambda > 0 : y \in \lambda W\}$ is the gauge (or Minkowski gauge) function ([Rockafellar, 1997]).

Theorem 1. *There is a unique function $H_S : \mathcal{G}^S_{N+} \times \mathbb{R}^N_+ \to \mathbb{R}$, satisfying continuity, scale invariance, MIN, MAX, and proportionality for TU games axioms. Moreover, $H_S = h_S$, where h_S is defined by (5).*

For the proof of the theorem we use four lemmas which we formulate for $S = N$ (the proof for an arbitrary S is the same). We also omit index N to simplify notations. The proofs of lemmas are given in Appendix.

Let $z \in \mathbb{R}^N_{++}$ and $P_z = \{y \in \mathbb{R}^N : y \le z\}$, i.e. $P_z = z - \mathbb{R}^N_+$. Clearly P_z is star-shaped. Also star-shaped is every finite union of such sets, i.e. for every natural number M and arbitrary vectors $z^m \in \mathbb{R}^N_{++}$, $m = 1, 2, \ldots, M$ the set $P(z^1, \ldots, z^M)$, defined by

$$P(z^1, \ldots, z^M) = \bigcup_{m=1}^{M} P_{z^m},$$

is star-shaped. Obviously, $P(z^1, \ldots, z^M) \in \mathcal{G}^N_{N+}$, since $z^m > \mathbf{0}$ for every m.

Lemma 1. *The proportional excess defined by (5) satisfies all axioms mentioned.*

Let us denote the metric on $\mathcal{G}^N_{N+}$ by ρ.

Lemma 2. *Let $V \in \mathcal{G}^N_{N+}$ be an arbitrary game subset. Then for every $\varepsilon > 0$ there are such natural number M and points $z^m \in \mathbb{R}^N_{++}$, $m = 1, 2, \ldots, M$ that*

1) *$P(z^1, \ldots, z^M) \supset V$;*
2) *$\rho(P(z^1, \ldots, z^M), V) \le \varepsilon n^{1/2}$.*

Lemma 3. *Let $V \in \mathcal{G}^N_{N+}$ be a hyperplane game subset, i. e.*

$$V = \left\{ z \in \mathbb{R}^N_+ : \sum_{i \in N} p_i z_i \le r \right\} - \mathbb{R}^N_+$$

for some $p_i > 0$, $i = 1, \ldots, n$, and $r > 0$. Then $H(V, x) = r / \sum_{i \in N} p_i x_i = h(V, x)$.

Lemma 4. *Let* $z^1, \dots, z^M \in \mathbb{R}^N_{++}$, *then*

$$H(P(z^1, \dots, z^M), x) = \max_{m=1,\dots,M} \min_{i=1,\dots,n} \left(\frac{z_i^m}{x_i} \right).$$

Moreover, $H(P(z^1, \dots, z^M), x) = 1/\gamma(P(z^1, \dots, z^M), x)$.

The proof of the Theorem follows from these lemmas and continuity of H and h.

We call this function the proportional excess though it is not an excess function in Kalai sense (recall that we replaced the equality $E_S(V, x) = 0$ on the boundary by $E_S(V, x) = 1$).

2.3. Proportional Nucleolus and Prenucleolus

As we have just mentioned the proportional excess is not an excess in Kalai sense, nevertheless the assertions of Kalai's Theorem [Kalai, 1975] hold. (It should be noted that the ε-core in our case (proportional excess equals 1 on the boundary) can be defined in different ways. For example, $h_S(V, x) \leq 1/(1 - \varepsilon)$ instead of $h_S(V, x) \leq 1 + \varepsilon$ can be required).

In the following proposition we state only the part of theorem mentioned concerning proportional nucleolus and prenucleolus.

Proposition 1. *Let* $\{h_S\}_S$ *be a family of proportional excesses, and* $V \in \mathcal{G}_{N+}$. *Then if* $IR(V) \cap GR(V) \neq \emptyset$, *then* $N(V) \neq \emptyset$ *and consists of a finite number of points. For every* V $PN(V)$ *is non-empty and consists of a finite number of points.*

Proof.

Since $h_S - 1$ is an excess in Kalai sense his proof can be applied directly. Though the non-emptiness of the prenucleolus is not discussed in Kalai's theorem it follows in a similar manner to the non-emptiness of the nucleolus.

Corollary 1. *If* $V \in \mathcal{G}_{N+}$ *is a hyperplane game then the nucleolus (if not empty) consist of precisely one point. The prenucleolus consist of precisely one point.*

Proposition 2. *Let* $V, V' \in \mathcal{G}_{N+}$ *be such games that* $V(N) = V'(N)$ *and* $V'(S) = = aV(S), \forall S \neq N$ *for some* $a > 0$. *Then* $x \in PN(V) \Leftrightarrow x \in PN(V')$, *where* $PN(V)$ *is the prenucleolus of* V.

Proof.

Let us consider an arbitrary coalition $S \subset N$ and the corresponding set $V(S)$. It is clear that for every $x \in \mathbb{R}^{[S]}$

$$\gamma(V'(S), x) = \gamma(aV(S), x) = \inf\{\lambda > 0 \, : \, x \in \lambda a V(S)\} =$$

$$= \inf\{\mu/a > 0 \, : \, x \in \mu V(S)\} = (1/a)\gamma(V(S), x).$$

Then $h_S(V'(S), x) = 1/\gamma(V'(S), x) = a/\gamma(V(S), x)$. Therefore,

$$h_S(V', x) = a h_S(V, x), \quad \theta(V', x) = a\theta(V, x).$$

Note that this property is analogous to one of the characteristic property of the prenucleolus in sidepayments case for classical excess function (see, for example, [Pechersky, 1995]): let v and v' be two such side payment games that $v(N) = v'(N)$ and for an arbitrary real number a, $v'(S) = v(S) + a$ for every $S \neq N$. Then $PN(v') = PN(v)$.

Remark 2. In the proof of the Proposition we have used the property

$$\gamma(aV(S), x) = (1/a)\gamma(V(S), x).$$

It is important, and it will be used to modify Maschler–Peleg–Shapley's geometric characterization of the nucleolus in side payment case ([Maschler, 1979]) for NTU games. Of course, it is more cumbersome. We consider it in Appendix 2.

It is clear that Proposition 2 can be reformulated in the following manner.

Proposition 3. *Let* $V, V^{'} \in \mathcal{G}_{N+}$ *be such games that* $V^{'}(N) = aV(N)$ *and* $V^{'}(S) = = V(S)$, $\forall S \neq N$ *for some* $a > 0$. *Then* $x \in PN(V) \Leftrightarrow ax \in PN(V^{'})$.

Example [Bargaining game]. Let $V \in \mathcal{G}_{N+}$ be a bargaining game, i.e. for some $q \in \mathbb{R}^N_{++}$

$$V(S) = \{x \in \mathbb{R}^S : x_i \leq q_i \text{ for every } i \in S\},$$

for every $S \neq N$ and $q \in \text{int}V(N)$. Let $V(N) \bigcap (q + \mathbb{R}^N_+)$ be nonlevelled. Then $N(V) = PN(V) = \lambda q$, where λ is such that $\lambda q \in \partial V(N)$.

The following Proposition is a simple corollary of the definition of the proportional excess and equality $\gamma(V, x) = \gamma(A * V, A * x)$ for every $A \in \mathbb{R}^S_{++}$.

Proposition 4. *The nucleolus and the prenucleolus are scale covariant, i. e. if* $A \in \mathbb{R}^N_{++}$, *then for every* $V \in \mathcal{G}_{N+}$, $N(AV) = A*N(V)$, *and* $PN(AV) = A*PN(V)$, *where* AV *is defined by*

$$AV(S) = A * V(S) \text{ for every } S.$$

The next Proposition is an analogue of the corresponding property of prenucleolus in TU case (cf., for example, [Pechersky, 1995]).

Proposition 5. *Let* $U, V \in \mathcal{G}_{N+}$ *be such games that* $U(N) = V(N)$, *and* $x \in PN(U) \cap PN(V)$. *Let* $W = U \cup V$, *then* $x \in PN(W)$.

Proof.

Suppose the contrary, i.e. $x \notin PN(W)$. Then there is an $y \in N(W)$ such that $\theta(W, y) \prec_{lex} \theta(W, x)$. However, since $h_S(W, z) = \max\{h_S(U, z), h_S(V, z)\}$ for any $z \in \mathbb{R}^N_+$ and every S, we have $h_S(U, z) \leq h_S(W, z)$ and $h_S(V, z) \leq h_S(W, z)$.

Therefore, $\theta(U, y) \prec_{lex} \theta(U, x)$ and $\theta(V, y) \prec_{lex} \theta(V, x)$. Hence, $x \notin PN(U) \cap PN(V)$, and this contradiction proves the proposition.

3. Status quo Proportional Solution for Bargaining Games

In example we noted that the nucleolus (and prenucleolus) of a nonlevelled bargaining game is a Pareto optimal point proportional to the *status quo* point. Let us consider this solution in more details.

We consider bargaining games (q, Q) with positive *status quo* points q. Moreover, we suppose that the sets Q possess not only properties (a)–(b) characterizing the sets $V_+(S)$ and $\mathcal{G}_N+$, but also that they are nonlevelled and normal, and for every bargaining game (q, Q) there is such $x \in Q$ that $x \geq q$.

For a bargaining game (q, Q) define solution R as follows: let

$$\mu(q, Q) = \max\{t \in \mathbb{R}_+ : tq \in Q\},$$

and $R(q, Q) = \mu(q, Q)q$. This solution will be called *sq*-proportional (*status quo* proportional).

It can be shown easily that the solution defined in such a manner is proportional in the sense of (1). (Recall that we consider bargaining games as NTU games of special structure.) Of course, the proportional excess for NTU games defined in previous section should be used instead of $v(S)/x(S)$.

Proposition 6. *SQ-proportional solution is proportional in the sense of (1).*

Proof.

Let (q, Q) and (q^1, Q^1) be two bargaining games, V and V^1 be corresponding NTU games (note that $V_S = P_{q^S}$). Let $x \in \partial Q$, $y \in \partial Q^1$, $x = \mu(q, Q)q$ and $h_S(V, x) = h_S(V^1, y)$ for every $S \subset N$. Clearly $h_S(V, x) = 1/\mu(q, Q)$. Then $h_S(V^1, y) = 1/\mu(q, Q)$ for every $S \subset N$. In particular, $y_1 = \mu(q, Q)q_1^1, \ldots, y_n = = \mu(q, Q)q_n^1$. Since $y \in \partial Q^1$, then $\mu(q, Q) = \mu(q^1, Q^1)$, and y is *sq*-proportional solution of bargaining game (q^1, Q^1).

Let F be a bargaining solution. Consider following axioms.

Pareto optimality. $F(q, Q) \in \pi Q$, where πQ denotes the set of Pareto optimal points of Q.

Scale covariance. Let $\lambda \in \mathbb{R}^N_{++}$. Then for every bargaining game (q, Q)

$$F(\lambda * q, \lambda * Q) = \lambda * F(q, Q),$$

where $*$ denotes the coordinate-wise product.

Anonymity. If τ is an arbitrary permutation of N, then

$$F(\tau^* q, \tau^* Q) = \tau^* F(q, Q),$$

where τ^* is the transformation of $\mathbb{R}^N$, induced by τ, i.e. $\tau^*(x) = (x_{\tau(1)}, ..., x_{\tau(n)})$.

Strong monotonicity. If $Q^{'} \supset Q$, then $F(q, Q^{'}) \geq F(q, Q)$.

The following proposition follows immediately from the definition.

Proposition 7. *SQ-proportional solution satisfies Pareto optimality, scale covariance, anonymity and strong monotonicity axioms.*

Theorem 2. *SQ-proportional solution is the unique solution satisfying Pareto optimality, scale covariance, anonymity and strong monotonicity axioms.*

Proof.

Let F be a bargaining solution satisfying axioms mentioned. Consider an arbitrary bargaining game (e, Q), where $e = (1, 1, \ldots, 1)$. Let $x = \mu e$, where $\mu = \max\{t \in \mathbb{R}_+ : te \in Q\}$, and let $F(e, Q) \neq x$.

Consider the bargaining game (e, Q^{Π}), where

$$Q^{\Pi} = \bigcup_{\tau \in \Pi} \tau^* Q,$$

and Π denotes the set of all permutations of N. Clearly the set Q^{Π} is invariant relative to any permutation of N, hence, $F(e, Q^{\Pi}) = \lambda e$ for some $\lambda > 0$. Therefore, $\lambda e = x$. (On the one hand, $x \in Q^{\Pi}$, and on the other hand, it cannot be $F(e, Q^{\Pi}) > x$, since $F(e, Q^{\Pi}) \in \tau^* Q$ for every permutation τ, but $\mu e \in \tau^* Q$ for every τ, and, hence, $\mu \neq \max\{t \in \mathbb{R}_+ : te \in Q\}$.)

Since $Q^{\Pi} \supset Q$, then $x = F(e, Q^{\Pi}) \geq F(e, Q)$. However (by non-levelness condition) this is possible if $F(e, Q) = x$ only. Since Q was an arbitrary set the scale covariance proves the statement.

SQ-proportional solution can be easily characterized in another manner. Namely, let us formulate Proposition 2 as an axiom.

Invariance under common change of status quo *point.* If the bargaining games (q, Q) and (q', Q) are such that $q' = aq$ for some $a > 0$, then $F(q', Q) = F(q, Q)$.

Proposition 8. *SQ-proportional solution is the unique bargaining solution satisfying individual rationality, Pareto optimality and invariance under common change of* status quo *point axioms.*

Proof.

Consider an arbitrary bargaining game (q, Q). If $q \in \pi Q$, then by non-levelness condition $F(q, Q) = q$. Let now $q \notin \pi Q$, then (since Q is normal and $\exists\, y \in Q : y \geq q$) there is such $a > 0$, that $aq \in \pi Q$. Then $F(aq, Q) = aq$, and by invariance under common change of *status quo* point axiom $F(q, Q) = aq$.

The last proposition can be easily reformulated in the spirit of Proposition 3. Let us consider the following axiom.

Positive homogeneity with respect to changes of feasible set. If (q, Q) and (q, Q') be such that $Q' = aQ$ for some $a > 0$, then $F(q, Q') = aF(q, Q)$.

Proposition 9. *SQ-proportional solution is the unique bargaining solution satisfying individual rationality, Pareto optimality and positive homogeneity with respect to changes of feasible set axioms.*

The proof is obvious.

Acknowledgments

The author expresses his gratitude to Elena Yanovskaya for useful discussions on the subject, and to Michael Maschler for useful comments on the early versions of the paper.

Appendix

First Appendix

Here we give the proofs of the Lemmas.

Proof of Lemma 1.

Continuity, MIN and MAX follow immediately from the continuity of gauge function γ and its well-known properties:

$$\gamma(V_1 \cap V_2, x) = \max\{\gamma(V_1, x), \gamma(V_2, x)\},$$

and

$$\gamma(V_1 \cup V_2, x) = \min\{\gamma(V_1, x), \gamma(V_2, x)\}.$$

Now check the scale invariance. Let $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^N_{++}$. Since

$$\gamma(a * V, a * x) = \inf\{\lambda > 0 \, : \, a * x \in a * V\} =$$

$$= \inf\{\lambda > 0 \, : \, \mathrm{x} \in \mathrm{V}\} = \gamma(\mathrm{V}, \mathrm{x}),$$

it is clear that h is scale invariant.

Let now $V \in \mathcal{G}_{N+}$ corresponds to a TU game v, i. e.

$$V(S) = \{x \in \mathbb{R}^{[S]}_+ \, : \, x(S) \leq v(S)\} - \mathbb{R}^{[S]}_+.$$

Then

$$\gamma(V, x) = \inf\{\lambda > 0 \, : \, x \in \lambda V\} = \inf\{\lambda > 0 \, : \, x(S) \leq \lambda v(S)\} =$$

$$= \inf\{\lambda > 0 \, : \, x(S)/v(S) \leq \lambda\} = x(S)/v(S).$$

Hence, $h(V, x) = 1/\gamma(V, x) = v(S)/x(S)$.

Proof of Lemma 2.

Since V is normally generated it is sufficient to consider only the positive parts of V and $P(z^1, \dots, z^M)$. Consider the covering of V_+ by the cubes with the edge equal to ε and vertices in the nodes of the ε-lattice in $\mathbb{R}^N$. Since V_+ is compact, it is contained in a finite union of such cubes, each having non-empty intersection with V_+. Denote the number of such cubes with M, and let $z^1, \dots, z^M$ be the maximum vertices of these cubes (i.e. vertices with coordinate-wise maximum).

Consider $P(z^1, \ldots, z^M) = \bigcup_{m=1}^M P_{z^m}$. By construction, $P(z^1, \ldots, z^M) \supset V$. Moreover, the following inclusion holds also by construction:

$$V + (\varepsilon n^{1/2})B \supset P(z^1, \ldots, z^M),$$

where B is the unit ball in $\mathbb{R}^N$. Hence, $\rho(P(z^1, \ldots, z^M), V) \leq \varepsilon n^{1/2}$.

Proof of Lemma 3.

Let $p = (p_1, \ldots, p_n) \in \mathbb{R}^N_{++}$. Then p transforms the game subset V into the TU game subset $p * V = \{w \in \mathbb{R}^N_+ : \sum_{i \in N} w_i \leq r\} - \mathbb{R}^N_+$, where $w_i = p_i z_i$. By TU game axiom

$$H(p * V, p * x) = \frac{r}{\sum w_i},$$

and by scale invariance

$$H(V, x) = H(p * V, p * x) = r / \sum p_i x_i.$$

It is easy to check that $\gamma(V, x) = \sum p_i x_i / r$. Indeed,

$$\inf\{\lambda > 0 : x \in \lambda V\} = \inf\{\lambda > 0 : \sum p_i x_i \leq \lambda r\} =$$

$$= \inf\{\lambda > 0 : \sum p_i x_i / r \leq \lambda\} = \sum p_i x_i / r.$$

Therefore, $H(V, x) = 1/\gamma(V, x) = h(V, x)$.

Proof of Lemma 4.

Prove firstly that if $z \in \mathbb{R}^N_{++}$, then $H(P_z, x) = \min_i \left\{ \frac{z_i}{x_i} \right\} = h(P_z, x)$.

Clearly, the faces of P_z are the sets

$$Q_i = \{y \in \mathbb{R}^N : ye^i = ze^i, y \leq z\},\ i = 1, 2, \ldots, n,$$

where

$$e^i_j = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases}$$

Consider for some natural k the hyperplane game subset

$$V^i = \{y \in R^N_+ : y(e^i + \frac{1}{k} e^{N \setminus \{i\}}) \leq z(e^i + \frac{1}{k} e^{N \setminus \{i\}})\} - \mathbb{R}^N_+.$$

Then by lemma 3,

$$H(V^i, x) = \frac{z_i + \frac{1}{k} z(N \setminus \{i\})}{x_i + \frac{1}{k} x(N \setminus \{i\})}.$$

Now consider a game V_k with $V_k = \bigcap_{i=1}^n V^i$. Clearly $V_k \in \mathcal{G}^N_{N+}$, and we have by MIN axiom

$$H(V_k, x) = \min \left\{ \frac{z_i + \frac{1}{k} z(N \setminus \{i\})}{x_i + \frac{1}{k} x(N \setminus \{i\})} \right\}.$$

Besides, $H(V_k, x) = 1/\gamma(V_k, x)$. Letting $k \to +\infty$, we get by continuity

$$H(P_z, x) = \min_i \left\{ \frac{z_i}{x_i} \right\} = 1/\gamma(P_z, x) = h(P_z, x).$$

By MAX axiom

$$H(P(z^1, \ldots, z^M), x) = \max_{m=1,\ldots,M} H(P_{z^m}, x) = \max_{m=1,\ldots,M} \min_i \left(\frac{z_i^m}{x_i} \right).$$

$$\gamma(V \cap V^{'}, x) = \max(\gamma(V, x), \gamma(V^{'}, x)); \quad \gamma(V \cup V^{'}, x) = \min(\gamma(V, x), \gamma(V^{'}, x))$$

for every $V, V^{'} \in \mathcal{G}_{N+}^N$, we have

$$H(P(z^1, \ldots, z^M), x) = 1/\gamma(P(z^1, \ldots, z^M), x).$$

Second Appendix

In the Appendix we consider the geometric characterization of the proportional nucleolus in Maschler–Peleg–Shapley spirit (see [Maschler, 1979]).

To define ε-core it is convenient to require $h_S(V, x) \leq\ 1/(1-\varepsilon)$. In this case it coincides with the ε-core defined for NTU games by the gauge excess $g_S(V, x) = 1 - \gamma(V(S), x^S)$ (cf., [Pechersky, 2000]), which is ordinally equivalent to the proportional excess and is an excess in Kalai sense. We start with the definition of lexicographic centers of a NTU game. Of course, it is more cumbersome, but the intuition behind it is transparent and very close to that by Maschler–Peleg–Shapley in TU case.

Let V be a game in $\mathcal{G}_{N+}$. It is convenient to introduce the function $g(S, x) \equiv g_S(V, x) = 1 - 1/h_S(V, x)$, since $h_S(V, x) \leq\ 1/(1-\varepsilon)$ is equivalent to $g(S, x) \leq \varepsilon$.

Define $X^0 = IR(V) \cap GR(V)$ (we suppose it is not empty) and $\Sigma^0 = \{S \subset N : S \neq N, \emptyset\}$. Let

$$\varepsilon^1 = \min_{x \in X^0} \max_{S \in \Sigma^0} g(S, x),\ X^1 = \{x \in X^0 : \max_{S \in \Sigma^0} g(S, x) = \varepsilon^1\}. \tag{6}$$

Both ε^1 and X^1 are well-defined, and X^1 is a compact set. Let now $x \in X^1$. Define

$$\Sigma^1(x) = \{S \in \Sigma^0 : g(S, x) = \varepsilon^1\}.$$

Since Σ^0 is finite the set X^1 is partitioned into finite family of compact sets $X_1^1, X_2^1, \ldots, X_{r_1}^1$ such that $\Sigma^1(x') = \Sigma^1(x)$ for any $x,\ x' \in X_l^1$, $l = 1, \ldots, r_1$. Denote such $\Sigma^1(x)$ by Σ_{1l}, i. e.

$$\Sigma_{1l} = \{S \in \Sigma^0 : g(S, x) = \varepsilon^1 \text{ for all } x \in X_l^1\}.$$

Let $\sigma_1 = \min_{l=1,\ldots,r_1} |\Sigma_{1l}|$ and $\tilde{\Sigma}_1 = \{\Sigma_{1l} : |\Sigma_{1l}| = \sigma_1\} \equiv \{\Sigma_{11}, \ldots, \Sigma_{1M'}\}$, $M' > 0$. Consider $\Sigma_m^1 = \Sigma^0 \setminus \Sigma_{1m} \neq \emptyset$, $m = 1, \ldots, M$. If all $\Sigma_m^1 = \emptyset$ we stop the process. If $M \geq 1$, we continue the process for each m.

Let $\varepsilon_m^2 = \min_{X_m^1} \max_{S \in \sum_m^1} g(S, x)$, and take only such ε_m^2, that

$$\varepsilon_m^2 = \min\{\varepsilon_1^2, \ldots, \varepsilon_M^2\}. \tag{7}$$

Denote such ε_m^2 by ε^2. Further we can define the set

$$X_m^2 = \left\{ x \in X_m^1 \,:\, \max_{S \in \Sigma_m^1} g(S, x) = \varepsilon^2 \right\}$$

and corresponding sets $X_{m1}^2, X_{m2}^2, \ldots, X_{mr_2}^2$ and $\Sigma_{m21}, \ldots, \Sigma_{m2r_2}$ for m, satisfying (7).

Let $\sigma_2 = \min_m \max_{l=1,\ldots,r_2} |\Sigma_{m2l}|$ and $\tilde{\Sigma}_2 = \{\Sigma_{m2l} \,:\, |\Sigma_{m2l}| = \sigma_2\}$, and so on. Clearly, this process will stop after finite number of steps. It defines a finite set of points in $IR(V) \cap GR(V)$, called lexicographic centers, and this set coincides with the nucleolus N(V) of a game V. The geometry of the process is akin to that in sidepayment case and can be described in the following way. We start with an ε large enough so that the ε-core is non-empty:

$$C_\varepsilon(V) = \{x \in GR(V) \cap IR(V) \,:\, \forall S \neq N,\ g_S(V, x) \leq \varepsilon\} \neq \emptyset.$$

If $g_S(V, x) \leq \varepsilon$, then $1 - \gamma(V(S), x) \leq \varepsilon$ and $\gamma(V(S), x) \geq 1 - \varepsilon$. Consider $aV(S)$ for $a > 0$. Since $\gamma(aV(S), x) = (1/a)\gamma(V(S), x)$, we have

$$\gamma(aV(S), x) \geq 1 \Leftrightarrow \gamma(V(S), x) \geq a,$$

and we can take $a = 1 - \varepsilon$. Therefore, if ε is decreasing then a is increasing.

The process can be characterized by expanding ("blowing up") the sets $V(S)$: we start with sufficiently small $a > 0$ such that

$$X(a) = \{x \in GR(V) \cap IR(V) \,:\, \gamma(V(S), x) \geq a, \forall S \neq N\} =$$

$$= \{x \in GR(V) \cap IR(V) \,:\, x^S \notin \ \mathrm{rel\,int}\ aV(S), \forall S \neq N\} \neq \emptyset.$$

Then we begin to expand ("blow up") all sets $aV(S)$ (by increasing a) "pushing up" some x from $GR(V) \cap IR(V)$. This expansion (blowing up) is performed at equal speed and is stopped either when the set $X(a)$ becomes empty or would become disjoint from $GR(V) \cap IR(V)$ (or $C(V)$ if it is non-empty). This bring us to the sets X_m^1. Any further increase in a will render the sets X_m^1 empty. We, therefore, continue to expand only those sets $V(S)$, where $S \in \Sigma_m^1$ (there are M such Σ_m^1). These we expand so long as the corresponding modification of the set $X(a)$ is neither empty nor disjoint from $GR(V) \cap IR(V)$ (or $C(V)$ if it is non-empty). This bring us to the sets X_{mr}^2, and so on. The process continues in the same manner until all the sets $aV(S)$ have been expanded to their respective limits (i.e. corresponding to appropriate $1 - a = \varepsilon^k$).

Remark 3. The same procedure can be used for the prenucleolus. We have to replace $GR(V) \cap IR(V)$ by $GR(V)$, and to consider only whether set $X(a)$ is empty.

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Time Consistency of Cooperative Solutions

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Introduction

Advances in technology, communications, industrial organization, regulation methodology, international trade, economic integration and political reform have created rapidly expanding social and economic networks incorporating cross-personal and cross-country activities and interactions. From a decision- and policy-maker's perspective, it has become increasing important to recognize and accommodate the interdependencies and interactions of human decisions under such circumstances. The strategic aspects of decision making are often crucial in areas as diverse as trade negotiation, foreign and domestic investment, multinational pollution planning, market development and integration, technological R&D, resource extraction, competitive marketing, regional cooperation, military policies, and arms control.

Game theory has greatly enhanced our understanding of decision making. As socioeconomic and political problems increase in complexity, further advances in the theory's analytical content, methodology, techniques and applications as well as case studies and empirical investigations are urgently required. In the social sciences, economics and finance are the fields which most vividly display the characteristics of games. Not only would research be directed towards more realistic and relevant analysis of economic and social decision-making, but the game-theoretic approach is likely to reveal new and interesting questions and problems, especially in management science.

The origin of differential games traces back to the late 1940s. Rufus Isaacs modeled missile versus enemy aircraft pursuit schemes in terms of descriptive and navigation variables (state and control), and formulated a fundamental principle called the tenet of transition. For various reasons, Isaacs's work did not appear in print until 1965. In the meantime, control theory reached its maturity in the Optimal

Control Theory of Pontryagin et al. (1962) and Bellman's Dynamic Programming (1957). Research in differential games focused in the first place on extending control theory to incorporate strategic behavior. In particular, applications of dynamic programming improved Isaacs' results. Berkovitz (1964) developed a variation approach to differential games, and Leitmann and Mon (1967) investigated the geometry of differential games. Pontryagin (1966) solved differential games in open-loop solution in terms of the maximum principle.

First paper about differential games in Soviet Union appeared in 1965 [Krasovsky, 1966], [Petrosjan, 1965], [Pontryagin, 1967].

Research in differential game theory continues to appear over a large number of fields and areas. Applications in economics and management science are surveyed in Dockner et al. (2000). In the general literature, derivation of open-loop equilibria in nonzero-sum deterministic differential games first appeared in [Petrosjan, Murzov, 1967], [Case, 1967, 1969], and [Starr and Ho, 1969a, 1969b] were the first to study open-loop and feedback Nash equilibria in nonzero-sum deterministic differential games. While open-loop solutions are relatively tractable and easy-to-apply, feedback solutions avoid time inconsistency at the expense of reduced intractability. In following research, differential games solved in feedback Nash format were presented by [Clemhout and Wan, 1974], [Fershtman, 1987], [Jorgensen, 1985], [Jorgensen and Sorger, 1990], [Leitmann and Schmitendorf, 1978], [Lukes, 1971a, 1971b], [Sorger, 1989], and [Yeung, 1987, 1989, 1992, 1994].

Cooperative games suggest the possibility of socially optimal and group efficient solutions to decision problems involving strategic action. Formulation of optimal behavior for players is a fundamental element in this theory. In dynamic cooperative games, a stringent condition on cooperation and agreement is required. In the solution, the optimality principle must remain optimal throughout the game, at any instant of time along the optimal state trajectory determined at the outset. This condition is known as dynamic stability or time consistency. In other words, dynamic stability of solutions to any cooperative differential game involved the property that, as the game proceeds along an optimal trajectory, players are guided by the same optimality principle at each instant of time, and, hence, do not possess incentives to deviate from the previously adopted optimal behavior throughout the game.

The question of dynamic stability in differential games has been rigorously explored in the past three decades. Haurie (1976) raised the problem of instability when the Nash bargaining solution is extended to differential games. Petrosjan (1977) formalized the notion of dynamic stability (time consistency) in solutions of differential games. Kydland and Prescott (1977) found time inconsistency of optimal plans (Nobel Prize, 2005). Petrosjan and Danilov (1982) introduced the notion of "imputation distribution procedure" for cooperative solution. Tolwinski et al. (1986) investigated cooperative equilibria in differential games in which memory-dependent strategies and threats are introduced to maintain the agreed-upon control path. Petrosjan (1993) and Petrosjan and Zenkevich (1996) presented a detailed analysis of dynamic stability in cooperative differential games, in which the method of regularization was introduced to construct time-consistent solutions. Yeung and Petrosjan (2001) designed time-consistent solutions in differential games and characterized the conditions that the allocation-distribution procedure must satisfy. Petrosjan (2003) employed the regularization method to construct time-consistent bargaining procedures. Petrosjan and Zaccour (2003) presented time-consistent Shapley value allocation in a differential game of pollution cost reduction.

In the field of cooperative stochastic differential games, little research has been published to date, mainly because of difficulties in deriving tractable time-consistent solutions. Haurie et al. (1994) derived cooperative equilibria in a stochastic differential game of fishery with the use of monitoring and memory strategies. In the presence of stochastic elements, a more stringent condition – that of *subgame consistency* – is required for a credible cooperative solution. In particular, a cooperative solution is subgame-consistent if an extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behavior would remain optimal.

As pointed out by [Jorgensen and Zaccour, 2002] conditions ensuring time consistency of cooperative solutions are generally stringent and intractable. A significant breakthrough in the study of cooperative stochastic differential games can be found in the recent work of [Yeung and Petrosjan, 2004]. In particular, these authors developed a generalized theorem for the derivation of an analytically tractable "payoff distribution procedure" which would lead to subgame-consistent solutions. In offering analytical tractable solutions, Yeung and Petrosjan's work is not only theoretically interesting in itself, but would enable hitherto insurmountable problems in cooperative stochastic differential games to be fruitfully explored. When payoffs are nontransferable in cooperative games, the solution mechanism becomes extremely complicated and intractable. Recently, a subgame-consistent solution was constructed by [Yeung and Petrosjan, 2005] for a class of cooperative stochastic differential games with nontransferable payoffs. The problem of obtaining subgame-consistent cooperative solutions has been rendered tractable for the first time.

Stochastic dynamic cooperation represents perhaps decision-making in its most complex form. Interactions between strategic behavior, dynamic evolution and stochastic elements have to be considered simultaneously in the process, thereby leading to enormous difficulties in the way of satisfactory analysis. Despite urgent calls for cooperation in the politics, environmental control, the global economy and arms control, the absence of formal solutions has precluded rigorous analysis of this problem.

1. Cooperative solution

It is essential to begin with basic definitions. Since the main subject of the paper is game theory applications in management studies, corresponding models and solutions will be considered.

In *general* we treat *cooperative solution* as solution of participants (players) joined by will to make decision about actual problem. Suppose that such decision requires players' behavior coordination guaranted by an agreement. Thus, cooperation means any coordinated agreement of parties involved. Consideration of time consistency problems is directly connected with cooperative solutions in such general context.

Cooperative decision problems appear in various fields of management and management science. Note the problem of signing of contract as a result of a given agreement. In strategic management, this could be merger and takeover, strategic alliance agreements and other type's interfirm cooperation. In financial management – long term investment decisions. On a firm level this is a longterm agreement between owners and managers about profit distribution. There are many other examples. At the same time cooperative solutions are possible in legal contracts or agreement forms with legal or not, obvious or secret aims. More complicated cooperative agreement forms are possible also.

In analyzing cooperative decision making some important aspects are usually considered. Firstly, what are participants' motivations to make cooperative decision? If such motivations exist, are they sufficient? Often categories of utility and equity of coordinated agreement serve as such motivation. Secondly, what coordinated agreement is to be chosen as optimal (what optimality principle is to be chosen)? How to choose optimal solution (what is algorithm of decision making)? Thirdly, how to realize the cooperation solutions? In this paper we will be interested in behavior of cooperative solutions in time, so the third question of decision making will be a key problem.

Cooperative solutions in general are divided to static and dynamic. In static case solution is made once, instantaneously realized and players get the outcomes right away. In spite of seeming simplicity of such an approach, classic game theory deals with static models. However, management and management science deals with control, and therefore – with processes evolving in time (with conflict processes in our case). To understand cooperative solution concept, it is necessary to begin with consideration of static game.

Game in normal form Γ is defined as:

$$\Gamma = \left\langle N, \left\{U^i\right\}_{i=1}^{n}, \left\{K_i\right\}_{i=1}^{n} \right\rangle,$$

where $N = \{1, \ldots, n\}$ is the set of players, U^i – the set of strategies ($u_i \in U^i$), $K_i(u_1, \ldots, u_n)$ – the payoff function of player $i \in N$.

What is the solution of the game Γ? The answer to the question is given by concepts (principles) of optimality, formulated in the definitions below. In general the solution is the set of n-tuples of strategies $(u_1, \ldots, u_n)$, satisfying required optimality conditions. Nash equilibrium is most widespread optimality concept in nonzero sum game theory.

Definition 1. *[Nash, 1951] The n-tuple of strategies $(\bar{u}_1, \ldots, \bar{u}_n)$ is called Nash equilibrium if for all $u_i \in U^i$ and $i \in N$ the following inequalities hold*

$$K_i(\bar{u}_1, \ldots, \bar{u}_{i-1}, \bar{u}_i, \bar{u}_{i+1}, \ldots, \bar{u}_n) \geq K_i(\bar{u}_1, \ldots, \bar{u}_{i-1}, u_i, \bar{u}_{i+1}, \ldots, \bar{u}_n).$$

Nash Equilibrium (NE-solution) is cooperative solution in general, because the choice of such solution requires coordinated players' behavior. If there is more than one NE-solution, the following notice is especially important. In such case players also have to agree what NE-solution they would realize, since the payoffs in different NE-solutions are different in general.

Definition 2. *The n-tuple of strategies $(\bar{u}_1, \dots, \bar{u}_n)$ is called Pareto optimal if there is no such n-tuple of strategies $(u_1, \dots, u_n)$ that the following inequalities hold for all $i \in N$:*

$$K_i(u_1, \dots, u_i, \dots, u_n) \geq K_i(\bar{u}_1, \dots, \bar{u}_i, \dots, \bar{u}_n)$$

and for at least one $j \in N$:

$$K_j(u_1, \dots, u_i, \dots, u_n) > K_j(\bar{u}_1, \dots, \bar{u}_i, \dots, \bar{u}_n).$$

There may be many Pareto optimal solutions with different payoffs for players. This is the reason why Pareto optimal solution (PO-solution) is also a cooperative solution, because choosing such solution requires coordinated players' behavior and contains the property of group rationality.

Typical representative of Pareto optimal solution is *Nash bargaining solution.* Nash bargaining solution $(u_1^{'}, \dots, u_n^{'})$ is the solution of the optimization problem [Nash, 1950]:

$$\max_{u_1, \dots, u_n} \prod_{i=1}^{n} [K_i(u_1, \dots, u_n) - K_i^0] = \prod_{i=1}^{n} [K_i(u_1^{'}, \dots, u_n^{'}) - K_i^0]$$

subject to

$$K_i(u_1, \dots, u_n) \geq K_i^0, \qquad i \in N.$$

Here $(u_1^0, \dots, u_n^0)$ is given "reference solution", defining the "status quo" point

$$K^0 = (K_1^0, \dots, K_i^0, \dots, K_n^0), \qquad K_i^0 = K_i(u_i^0, \dots, u_n^0), \qquad i \in N.$$

Nash bargaining solution (*NB-solution*) is cooperative solution which selects a special Pareto optimal solution.

Another representative of Pareto optimal solution is *Kalai–Smorodinskiy bargaining solution.*

Kalai–Smorodinskiy bargaining solution $(u_1^{''}, \dots, u_n^{''})$ is the solution of the following optimization problem [Kalai, Smorodinskiy, 1975]:

$$\max_{\lambda \in [0,1]} \lambda$$

subject to

$$K_i = K_i^0 + \lambda(\hat{K}_i - K_i^0), \qquad i \in N,$$

$$K_i = K_i(u_1, \dots, u_n), u_i \in U^i, i \in N,$$

where $(u_i^0, \dots, u_n^0)$ is given "reference solution", defined by the "status quo" point

$$K^0 = (K_1^0, \dots, K_i^0, \dots, K_n^0), \qquad K_i^0 = K_i(u_i^0, \dots, u_n^0)$$

and by "ideal" point

$$\hat{K} = (\hat{K}_1, \dots, \hat{K}_i, \dots, \hat{K}_n), \qquad \hat{K}_i = \max_{u_i, \dots, u_n} K_i(u_i, \dots, u_n), \qquad i \in N.$$

Usually it is not possible to obtain ideal point $K^{'}$ in any solution (else, this point would be optimal solution), i.e. it doesn't belong to the set of feasible estimates. Geometrically Kalai–Smorodinsky solution is defined intersection point of the line segment connecting "status quo" and "ideal" points with the set of feasible estimates. Note that Kalai–Smorodinsky bargaining solution (*KS-solution*) is cooperative solution in general as special case of Pareto optimal solutions.

All mentioned above optimality principles are strategic in sense that they are constructed on base of coordinated or joint strategy choice.

Consider now a special type of cooperative solution. Such cooperative solution concept assumes two-stage cooperation: selection of n-tuple of strategies, which maximize the sum of players' payoffs, and allocation of the aggregate maximal cooperation payoff. Recall that the *cooperative game* in characteristic function form is defined as a system:

$$\Gamma = \langle N, \nu \rangle,$$

where $N = \{1, \dots, n\}$ is the set of players, $\nu(S) \geq 0$, $S \subset N$, $\nu(\emptyset) = 0$ is characteristic function, possessing *the superadditivity property:*

$$\nu(S \cap T) \geq \nu(S) + \nu(T) \qquad S \cap T = \emptyset.$$

The characteristic function value $\nu(S)$ is often interpreted as *maximal guaranted payoff of coalition* S, $S \in N$. From the superadditivity property of characteristic function we have $\nu(S) \geq \nu(S^{'})$, when $S^{'} \subset S \subseteq N$. Therefore it is advantageous to create maximal coalition N to obtain maximum possible aggregate payoff $\nu(N)$ during game evolution.

Let Γ_V be a cooperative game, constructed on the game Γ structure (with transferable payoffs), where players play according to some accepted in advance optimality principle [Petrosjan, Zenkevich, 1996]. Then, as mentioned above, the value $\nu(S)$ is interpreted as *maximal guaranted payoff of coalition* S, i.e. maximum payoff of coalition S in the worst case, when other players create coalition $N \setminus S$ to play against the coalition S.

The agreement about how exactly realize cooperation and share the gain of joined cooperative payoff is *optimality principle of cooperative game solution.* In particular, a solution of cooperative game is:

- Agreement about the cooperative n-tuple of strategies, oriented at receiving maximal cooperative payoff.

- Method for share of aggregate maximal payoff between participants.

The set of all allocations of maximal aggregate payoff is called *imputation set.* Denote ξ_i player's $i \in N$ payoff under cooperation, when aggregate cooperation payoff is $\nu(N)$.

Vector (aggregate payoff allocation)

$$\xi = (\xi_1, \xi_2, \ldots, \xi_n),$$

is called *imputation* in game Γ_ν, if the following conditions are satisfied:

(i) $\xi_i \geq \nu(\{i\}), i \in N$,

(ii) $\sum_{i \in N} \xi_i = \nu(N)$,

where $\nu(\{i\})$ is the value of characteristic function computed for singleton coalition i.

The condition (i) guarantes individual rationality, i.e. every player obtains at least as much as the maximal payoff in case he plays against all the other players. The condition (ii) guarantes Pareto optimality for the imputation and, therefore, group rationality.

Denote imputation set in game Γ_ν by E_ν. *Cooperative optima l ity principle* W_ν in the game Γ_ν is a fixed subset W_ν of imputation set E_ν. If optimality principle W_ν is chosen, then the imputation $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in W_\nu$ is called *optimal* according to given optimality principle W_ν.

Definition 3. *The imputation* $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ *belongs to the core of game* Γ_ν*, if for every coalition* $S \subset N$ *the following inequalities are satisfied:*

$$\sum_{i \in S} \xi_i \geq \nu(S).$$

The core is denoted by C_ν. *The sense of cooperative solution from the core is obvious: if the imputation from the core is chosen, then every coalition of players gets at least as much as he could get playing independently.*

Definition 4 [Shapley, 1953]. *The imputation*

$$\Phi^\nu = (\Phi^\nu_1, \ldots, \Phi^\nu_i, \ldots, \Phi^\nu_n)$$

is called Shapley value, if it is obtained as

$$\Phi^\nu_i = \sum_{S \subset N (i \in S)} \frac{(n-s)!(s-1)!}{n!} [\nu(S) - \nu(S \setminus i)].$$

There exist many other cooperative optimality principles, for example: Neymann–Morgenshtern solution, N-core, nucleus. In all cases they are some subsets of the game imputation set.

2. Time consistency of cooperative solution problem

In previous section we considered static concepts of cooperative solutions. However, management and management science deal with control, and, therefore, with processes (with a conflict evaluation of a large system in time). Control is chosen at initial moment and realized in a given time interval.

2.1. Dynamic stability (time consistency) of optimal control problems

Illustrate the time consistency property of optimal control with a classical example.

Let $M \in R^n$ is a given point which defines in some sense "ideal" state of the system under consideration. Consider the following classical management (control) problem. Let

$$\dot{x}(t) = f(x(t), u(t)), x \in R^m, u \in U \subset R^l, \tag{1}$$

$$x(t_0) = x_0, t \in [t_0, T]$$

be the system of differential equations, where $x(t)$ is a state variable, $u(t)$ control (management) variable which is selected continuously at each time instant t.

The system develops in a given time interval $[t_0, T]$. The aim of the management is to bring the initial point x_0 (initial state of the system) as close as possible to a given point M at the terminal moment t_0. Mathematically this means that the aim of the management is to find such an open-loop control which minimizes the distance $\rho(x(T), M)$ between the terminal point $x(T)$ and the point $M \in R^n$.

Construct the reachability set of the system (1) denoted by $C(x_0, T - t_0)$ from the initial state x_0 at the terminal moment T. $C(x_0, T - t_0)$ is the set of such points $x(T)$ which can arise exactly at the terminal moment T when all possible open-loop controls are used from the initial position x_0 according to the system (1).

Denote our minimization problem by $\Gamma(x_0, T - t_0)$ to underline the dependence of the problem from the initial condition x_0 and the duration of the process $T - t_0$.

For simplicity reasons suppose that the point M does not belong to $C(x_0, T - t_0)$, i.e. $M \cap C(x_0, T - t_0) = \emptyset$.

This means that the point M can not be reached from the initial state x_0 during the time $T - t_0$. The optimality principle in this optimal management problem is to minimize the distance between the point $x(T)$ and the point M.

It is clear that the "optimal motion" or "optimal trajectory" has to bring the initial point x_0 to the point $M^{'}$ $(x(t_0) = t_0, x(T) = M^{'}))$ – the closest point of the reachability set $C(x_0, T - t_0)$ to the point M. Denote by $\bar{x}(t)$ the trajectory connecting x_0 and M' realized under optimal (fixed) open-loop control $\bar{u}(t)$, i.e.

$$\dot{\bar{x}} = f(\bar{x}(t), \bar{u}(t)), \qquad x(t_0) = t_0, \qquad x(T) = M'.$$

Suppose that the process is evaluating along the trajectory $\bar{x}(t)$ as shown in the Figure 1. Consider an intermediate time instant $\tau \in [t_0, T]$ and suppose that at this time instant we want to check if the point $m^{'}$ remains the closest to the point M in the subproblem $\Gamma(\bar{x}(\tau), T - \tau)$ with the initial condition $\bar{x}(\tau)$ on the optimal trajectory and duration $T - \tau$. It is evident that the answer will be "yes". This means that the continuation of the optimal motion along $\bar{x}(t)$ on the time interval $t \geq \tau$ will remain optimal in the subproblem $\Gamma(\bar{x}(\tau), T-\tau)$ (see Fig. 1). This means time consistency or dynamic stability, of the optimal trajectory $\bar{x}(t)$. This was first formulated by R. Bellman (1957) and lies in the bases of dynamic programming. Time consistency nearly always holds in the classical optimal control problems.

At the same time we can see that in this case a stronger condition holds (this was not mentioned by R. Bellman). In the subproblem $\Gamma(\bar{x}(\tau), T-\tau)$ a new optimal trajectory $\tilde{u}(t)$ $t \in [\tau, T)$, $\tilde{u}(t) \neq \bar{u}(t)$ can arise leading from the initial point $\bar{x}(\tau)$ in the subproblem to the point $M^{'}$. It is interesting to mention that the open-loop control of the form

$$\tilde{\tilde{u}} = \begin{cases} \bar{u}(t), t \in [t_0, \tau) \\ \tilde{u}(t), t \in [\tau, T] \end{cases}$$

transfers the point x_0 to the point $M^{'}$ in the problem $\Gamma(x_0, T-t_0)$ and, thus, is optimal in the problem $\Gamma(x_0, T-t_0)$.

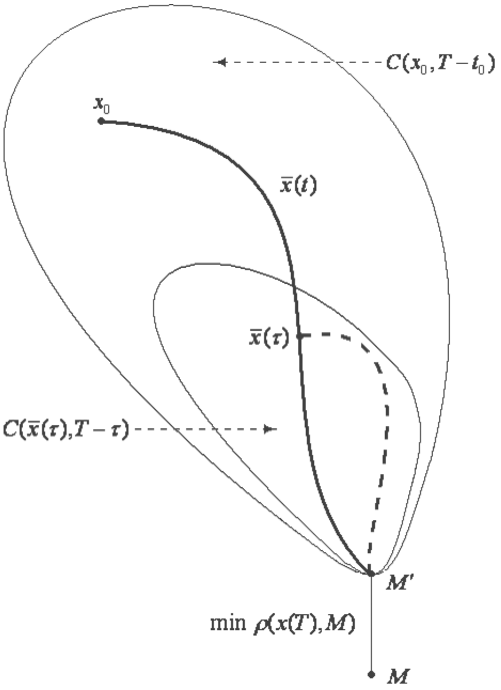

Fig. 1: Dynamic stability of optimal control

So, we get that any optimal prolongation in the subproblem $\Gamma(\bar{x}(\tau), T-\tau)$ together with initially selected optimal motion on the time-interval $[t_0, \tau)$ in $\Gamma(x_0, T-t_0)$ is also optimal in $\Gamma(x_0, T-t_0)$. This property we call “strong dynamic stability”.

The notion of strong dynamic stability was first introduced practically simultaneously and independently by L. Petrosyan (1979) and S. Chistyakov (1981).

2.2. Time consistency (dynamic stability) of Pareto optimal solutions in multicriterial control problems

As in the previous section the management is described by the system of differential equations

$$\dot{x}(t) = f(x(t), u(t)), x \in R^m, u \in U \subset R^l,$$

$$x(t_0) = x_0, t \in [t_0, T].$$

Here as in the previous case $x(t)$ is the state variable and $u(t)$ control variable selected by the manager continuously at each time-instant t from a given set U.

The system develops in the time interval $[t_0, T]$. The difference with the previous problem is in the fact that in this case the quality of the management is evaluated by a number of parameters (in the previous optimal control problem the quality of the management was evaluated only by one single parameter – the distance from a given fixed point M). The aim of the management in this case is to bring the initial point x_0 as close as possible to a finite number of fixed points $M_1, \ldots, M_k$. Mathematically the problem is to minimize the vector criteria

$$[\rho(x(T), M_1), \ldots, \rho(x(T), M_l), \ldots, \rho(x(T), M_k)],$$

where $M_l \in R^n$, $l = 1, 2, \ldots, k$ and $x(T)$ is the terminal state of the management process.

Since we have here a multicriterial optimization problem, as optimality principal we have to consider a Pareto optimal set.

As before, let $C(x_0, T - t_0)$ be the reachability set of the system (1) and denote our optimization problem by $\Gamma(x_0, T - t_0)$ to underline the dependence of the problem from the initial condition x_0 and duration $T - t_0$. Denote by $\hat{M}$ the convex hull of the points $M_1, \ldots, M_k$. For simplicity reasons suppose that

$$\hat{M} \cap C(x_0, T - t_0) = \emptyset.$$

It can be shown that the set of all Pareto optimal trajectories coincides with those with endpoint on the projection of the convex hull $\hat{M}$ on the reachability set.

Denote by $\bar{x}(t)$ a trajectory connecting the initial state x_0 with the some fixed point $M^{'}$ on the projection $\hat{M}$ of the convex hull on the reachability set $C(x_0, T - t_0)$, and let $\bar{u}(t)$ be the corresponding open-loop control.

We shall call $\bar{x}(t)$ optimal trajectory. It is clear that in our problem we may have an infinite number of optimal trajectories with non-comparable outcomes in the sense of the different values for distances to the aim-points, since in general the projection of the set $\hat{M}$ on $C(x_0, T - t_0)$ is closed and may contain infinite number of points.

Consider now an intermediate time instant $\tau \in [t_0, T]$ and ask ourselves if the continuation of the optimal trajectory $M^{'}$ is optimal in subproblem $\Gamma(\bar{x}(\tau), T - \tau)$

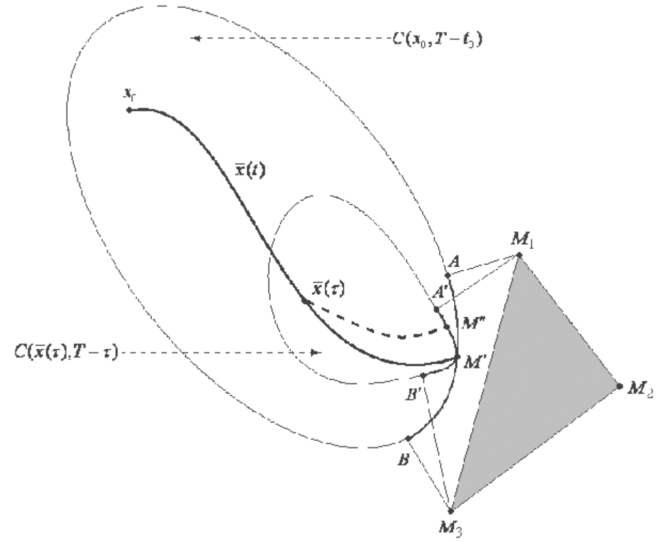

Fig. 2: Strong dynamic in stability for Pareto optimal solution

starting in the state $\bar{x}(\tau)$ on the optimal trajectory with duration $T - \tau$. In other words: will the point $M^{'}$ remain Pareto optimal in the subproblem $\Gamma(\bar{x}(\tau), T - \tau)$.

As in an optimal control problem considered in the previous section the answer will be "yes", thus, the prolongation of the optimal trajectory in the subproblem remains optimal (Pareto optimal) in this subproblem.

At the same time as it is seen from the Figure 2 the Pareto optimal set in problem $\Gamma(\bar{x}(\tau), T - \tau)$ coincides with the arc AB (projection of the set $\hat{M}$ on the reachibility set $C(x_0, T - t_0)$) and differs from the Pareto optimal set in the subproblem which coincides with the arc $A^{'}B^{'}$ (projection of the set $\hat{M}$ on the reachibility set $C(\bar{x}(\tau), T - \tau)$).

But both sets have one point in common. Thus, we see that in the subproblem $\Gamma(\bar{x}(\tau), T - \tau)$ there are new optimal (Pareto optimal) trajectories with endpoints out of the Pareto optimal set of the previous problem $\Gamma(x_0, T - t_0)$.

Consider the following open-loop control:

$$\tilde{\tilde{u}} = \begin{cases} \bar{u}(t), t \in [t_0, \tau) \\ \tilde{u}(t), t \in [\tau, T], \end{cases}$$

where $\bar{u}(t)$ is the segment of the optimal Pareto optimal control in the problem $\Gamma(x_0, T - t_0)$ and $\tilde{u}(t)$ is the Pareto optimal control in the subproblem $\Gamma(\bar{x}(\tau), T - \tau)$ which transfers the initial point $\bar{x}(\tau)$ in the subproblem to the point $M^{''}$.

Since the point $M^{''}$ does not belong to the arc AB the open-loop control is not Pareto optimal in the problem $\Gamma(x_0, T - t_0)$.

And we come to the conclusion that not any Pareto optimal prolongation in the subproblem $\Gamma(\bar{x}(\tau), T-\tau)$ with initial conditions on the Pareto optimal motion in the previous problem (problem $\Gamma(x_0, T-t_0)$) is Pareto optimal in $\Gamma(x_0, T-t_0)$.

This means that Pareto optimal solutions in general are not strongly dynamic stable or strongly time-consistent.

We see that by transition to multicriterial control problems we loose strong time consistency of optimal solutions. This arises difficulties in the practical implementation of optimal solutions in multicriterial control problems, because in some intermediate time instant the manager can change to another Pareto optimal solution (considering this solution for some reason as more attractive) and lose Pareto optimality of the whole process. This implies instability in long-term management and is unacceptable for practical use.

2.3. Time inconsistency of specially selected cooperative solution

The problem of choosing specific Pareto optimal solution is more complicated than in the case considered above. Most of optimality principles (even in non-game theoretical problems), determining the choice of specific Pareto optimal solution from the set of all Pareto optimal solutions are not only strongly dynamically unstable (not strongly time-consistent), but even dynamically unstable (time-inconsistent).

There is a number of approaches to choose a specific Pareto optimal solution from the set of all Pareto optimal solutions. Unfortunately, most complicated and well-defined of them are dynamically unstable (time-inconsistent). Illustrate it with an example. Consider the choice of Pareto optimal solution according to Kalai–Smorodinsky bargaining procedure. Pareto optimal solution chosen in such a way, as we noticed earlier, is called Kalai–Smorodinsky solution, or KS-solution.

Now we suppose that the long-term management process depends on the decisions made by different agents (players). Thus, we shall consider the case when the right side of the differential equations (1) depends upon a number of parameters, each one of them under control of corresponding agent (player) acting in his own interests. So, we have the motion equations

$$\dot{x}(t) = f(x(t), u_1(t), \dots, u_n(t)), \tag{2}$$

$$u_i \in U_i, x(t_0) = x_0, x \in R^m, t \in [t_0, T],$$

where the parameters (control variables) u_i are chosen continuously in time by players.

For simplicity we shall suppose that each of the players $i \in N$ is interested in a payoff which has the form

$$K_i(x_0, T-t_0; u_1, \dots, u_n) = \int\limits_{t_0}^{T} g^i(x(t))dt + q^i(x(T)),$$

where $x(t)$ is the solution of the system (2) corresponding to the choice of controls as functions of current state and time $u(\cdot) = (u_i(\cdot), \dots, u_n(\cdot))$ (strategies, feedback controls) and initial condition $x(t_0) = x_0$. As a result, we have a differential game, which we shall denote by $\Gamma(x_0, T - t_0)$.

Denote by $K(x_0, T - t_0)$ the set of all possible values of vectors

$$[K_1(x_0, T - t_0; u_1, \dots, u_n), \dots, K_i(x_0, T - t_0; u_1, \dots, u_n) \dots$$
$$\dots K_n(x_0, T - t_0; u_1, \dots, u_n)]$$

for all possible n-tuples of strategies $u_1, \dots, u_n$ chosen by players.

Let $\bar{\mathbf{K}}(x_0, T - t_0) \subset \mathbf{K}(x_0, T - t_0)$ be the Pareto frontier of the set $\mathbf{K}(x_0, T - t_0)$. There are different ways for selection of a particular Pareto optimal point from the whole Pareto frontier. In this selection the so-called "status quo" plays an important role. Usually the status-quo point is vector with components K_i^0, where each K_i^0 is equal to the maximal payoff the player i can get in the worst case, when all other players are playing against him (not for themselves). Let

$$K^0(x_0, T - t_0) = [K_1^0(x_0, T - t_0), \dots, K_n^0(x_0, T - t_0)] \in \mathbf{K}(x_0, T - t_0)$$

be the status-quo point. It is clear that this point depends on the initial state of the system x_0 and duration of the process $T - t_0$. Denote by

$$\hat{K}_i(x_0, T - t_0) = \max_{u_1, \dots, u_n} K_i(x_0, T - t_0; u_1, \dots, u_n).$$

The point

$$\hat{K}(x_0, T - t_0) = [\hat{K}_1(x_0, T - t_0), \dots, \hat{K}_i(x_0, T - t_0) \dots \hat{K}_n(x_0, T - t_0)]$$

is called "ideal" point and has the meaning of maximal possible gains of the players. In general, we have

$$\hat{K}(x_0, T - t_0) \notin \mathbf{K}(x_0, T - t_0),$$

otherwise the ideal point will be the "solution" of the problem.

To define the KS-solution draw a line segment connecting the status quo point and the ideal point. Since the ideal point does not belong to the set $\mathbf{K}(x_0, T - t_0)$, exists a point at the intersection of the set $\mathbf{K}(x_0, T - t_0)$, and this line segment is the closest to the ideal point (we suppose that the set $\mathbf{K}(x_0, T - t_0)$ is closed and bounded). This point is called KS-solution. If the set $\mathbf{K}(x_0, T - t_0)$ is convex the KS solution is always Pareto optimal. It is easy to see that even in the simplest cases the KS-solution is not time-consistent (dynamic stable). For the illustration of this property consider the following very trivial example. Suppose that the system (2) has the form

$$\dot{z} = u_1 + u_2, \qquad |u_1| \leq 1, |u_2| \leq 1, \qquad z \in R^2, \qquad u_1, u_2 \in R^2,$$
$$z_0 = (6, 3), \qquad t \in [0, 2], \qquad z = (x, y),$$
$$K_1(z_0, 2; u_1, u_2) = -x(2), \qquad K_2(z_0, 2; u_1, u_2) = -|y(2)|.$$

Show the time inconsistency of the KS-solution. Here the status quo point in the problem $\Gamma(z_0, 2)$ is equal to $K^0(z_0, 2) = (6, 3)$ and corresponds to the initial state of the system $z_0 = (6, 3)$. The ideal point is $\hat{K}(z_0, 2) = (-2, 0)$, since $\max K_1 = -2$, $\max K_2 = 0$. The reachibility set $C^2(6, 3)$ is a circle with the center $(6, 3)$ and radius 4. The optimal trajectory (leading to KS-solution) corresponds to the motion along the line segment from the initial point $z_0 = (6, 3)$ in direction to the point $(z_0, 2) = (2, 0)$ until the intersection with the circumference of the circle $C^2(6, 3)$. This point of intersection defines the KS - solution of the problem $\Gamma(z_0, 2)$ (see Fig.3).

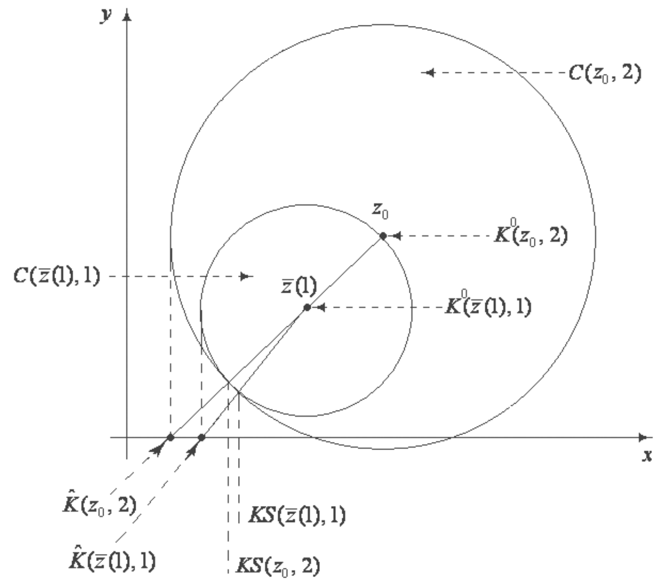

Fig. 3: Time inconsistency of KS-solution

In this figure we see that the KS-solution in a subproblem $\Gamma(\bar{z}(1), 1)$ with the initial conditions on the optimal trajectory is different from the KS-solution of the previously considered problem $\Gamma(z_0, 2)$.

This implies time inconsistency (dynamic instability) of KS-solution.

It is necessary to mention that not only KS-solution is time-inconsistent, but so are all non-trivial bargaining solutions based on the selection of status-quo points. This is also true for Nash bargaining solution.

3. Regularization of cooperative optimality principle

Previous considerations imply that the majority of cooperative solutions are not time-consistent. Therefore, there are serious difficulties for their practical implementation and ultimately it is not possible to get stable solution results. Only classical optimal control solutions and Nash equilibrium with constant discount rate are dynamically stable (time-consistent). Is there a way out of this problem? Yes. We shall explain this in the case of cooperative differential game.

3.1. Definition of cooperative differential game

We begin with the basic formulation of cooperative differential games in characteristic function form and the solution imputations.

Consider a general N-person differential game in which the state dynamics has the form:

$$\dot{x}(s) = f[s, x(s), u_1(s), u_2(s), \ldots, u_n(s)], x(t_0) = x_0. \tag{3}$$

The payoff of player is:

$$\int_{t_0}^{T} g^i[s, x(s), u_1(s), u_2(s), \ldots, u_n(s)]ds + q^i(x(T)), \tag{4}$$

for $i \in N = \{1, 2, \ldots, n\}$, where $x(s) \in X \subset R^m$ denotes the state variables of game, and $u_i \in U^i$ is the control of Player i, for $i \in N$. In particular, the players' payoffs are transferable. A feedback Nash equilibrium solution can be characterized if the players do not play cooperatively.

Now consider the case when the players agree to cooperate. Let $\Gamma_c(x_0, T - t_0)$ denote a cooperative game with the game structure of $\Gamma(x_0, T - t_0)$ in which the players agree to act according to an agreed upon optimality principle. The agreement on how to act cooperatively and allocate cooperative payoff constitutes the solution optimality principle of a cooperative scheme. In particular, the solution optimality principle for a cooperative game $\Gamma_c(x_0, T - t_0)$ includes:

(i) an agreement on a set of cooperative strategies/controls, and

(ii) a mechanism to distribute total payoff among players.

The solution optimality principle will remain in effect along the cooperative state trajectory path $\{x_s^*\}_{s=t_0}^{T}$. Moreover, group rationality requires the players to seek a set of cooperative strategies/controls that yields a Pareto optimal solution. In addition, the allocation principle has to satisfy individual rationality in the sense that neither player would be no worse off than before under cooperation.

To fulfill group rationality in the case of transferable payoffs, the players have to maximize the sum of their payoffs:

$$\sum_{j=1}^{N} \left\{ \int_{t_0}^{T} g^j[s, x(s), u_1(s), u_2(s), \ldots, u_n(s)]ds + q^j(x(T)) \right\}, \tag{5}$$

subject to (3).

A set of optimal controls $u^*(s) = [u_1^*(s), u_2^*(s), \dots, u_n^*(s)]$ is possible to be found using Pontryagin's maximum principle or Bellman equation. Substituting this set of optimal controls into (3) yields the optimal trajectory $\{x_s^*\}_{s=t_0}^T$, where

$$x^*(t) = x_0 + \int\limits_{t_0}^{t} f(s, x^*(s), u^*(s))ds, t \in [t_0, T]. \tag{6}$$

For notational convenience in subsequent exposition, we use $x^*(t)$ and x_t^* interchangeably.

We denote

$$\sum_{j=1}^{N} \left\{ \int\limits_{t_0}^{T} g^j[s, x^*(s), u^*(s)]ds + q^j(x^*(T)) \right\}$$

by $\nu(N; x_0, T - t_0)$. Let $S \subseteq N$ and $S(\nu(S; x_0, T - t_0)$ stands for a characteristic function reflecting the payoff of coalition S. The quantity $\nu(S; x_0, T - t_0)$ yields the maximized payoff to coalition S as a rest of the players form a coalition $N \setminus S$ to play against S. Calling on the superadditivity property of characteristic functions, $\nu(S; x_0, T - t_0) \geq \nu(S^{'}; x_0, T - t_0)$ for $S^{'} \subset S \subseteq N$. Hence, it is advantageous for the players to form a maximal coalition N and obtain a maximum total payoff $\nu(N; x_0, T - t_0)$ that is possible in the game.

One of the integral parts of cooperative game is to explore the possibility of forming coalitions and offer an "agreeable" distribution of the total cooperative payoff among players. In fact, the characteristic function framework displays the possibilities of coalitions in an effective manner and establishes a basis for formulating distribution schemes of the total payoffs that are acceptable to participating players.

We can use $\Gamma_\nu(x_0, T - t_0)$ to denote a *cooperative differential game in characteristic function form.*

Denote with

$$\xi(x_0, T - t_0) = [\xi_1(x_0, T - t_0), \xi_2(x_0, T - t_0), \dots, \xi_n(x_0, T - t_0)]$$

an arbitrary imputation, $C_\nu(x_0, T - t_0)$ – core, $\Phi^\nu(x_0, T - t_0)$ – Shapley value in the game $\Gamma_\nu(x_0, T - t_0)$.

3.2. Imputation in a dynamic context

In dynamic games, the solution imputation along the cooperative trajectory $\{x_s^*\}_{s=t_0}^T$ would be of concern to the players. Now we focus our attention on the dynamic imputation brought about by the solution optimality principle.

Let an optimality principle be chosen in the game $\Gamma_\nu(x_0, T - t_0)$. The solution of this game constructed in the initial state $x(t_0) = x_0$ based on the chosen principle of optimality contains the solution imputation set $W_\nu(x_0, T - t_0) \leq E_\nu(x_0, T - t_0)$ and the conditionally optimal trajectory $\{x_s^*\}_{s=t_0}^T$ which maximizes

$$\sum_{j=1}^{N}\left\{\int_{t_0}^{T} g^j[s,x^*(s),u^*(s)]ds+q^j(x^*(T))\right\}.$$

Assume that $W_\nu(x_0,T-t_0)\neq\emptyset$.

Definition 5. Any trajectory $\{x_s^*\}_{s=t_0}^T$ of the system (3) such that

$$\sum_{j=1}^{N}\left\{\int_{t_0}^{T} g^j[s,x^*(s),u^*(s)]ds+q^j(x^*(T))\right\}=\nu(N;x_0,T-t_0)$$

is called *a conditionally optimal trajectory in the game* $\Gamma_\nu(x_0,T-t_0)$.

Definition 5 suggests that along the conditionally optimal trajectory the players obtain the largest total payoff. For exposition sake, we assume that such a trajectory exits. Now we consider the behavior of the set $W_\nu(x_0,T-t_0)$ along the conditionally optimal trajectory $\{x_s^*\}_{s=t_0}^T$. At time t with state $x^*(t)$, we define the current subgame $\Gamma_\nu(x_t^*,T-t)$ with characteristic function $\nu(N;x_t^*,T-t)$ and the set of imputations $E_\nu(x_t^*,T-t)$.

Consider the family of current games

$$\Gamma_\nu(x_t^*,T-t),\quad t_0\leq t\leq T,$$

and their solutions $W_\nu(x_t^*,T-t)\subset E_\nu(x_t^*,T-t)$, generated by the same principle of optimality that yields the initially solution $W_\nu(x_0,T-t_0)$.

Obviously, the set $W_\nu(x_T^*,0)$ is the solution of current game $\Gamma_\nu(x_T^*,0)$ at the moment T and consists of single imputation

$$q(x^*(T))=[q^1(x^*(T)),q^2(x^*(T)),\ldots,q^n(x^*(T))]=$$
$$=[q^1(x_T^*),q^2(x_T^*),\ldots,q^n(x_T^*)].$$

3.3. Principle of Dynamic Stability

Formulation of optimal behavior for players is a fundamental element in the theory of cooperative games. The players' behaviors satisfying some specific optimality principles constitute a solution of the game. In other words, the solution of a cooperative game is generated by a set of optimality principles (for instance, the Shapley value (1953), the von Neumann–Morgenstern solution (1944) and the Nash bargaining solution (1953)). For dynamic games, an additional stringent condition on their solutions is required: the specific optimality principle must remain optimal at any instant of time throughout the game along the optimal state trajectory chosen at the outset. This condition is known as *dynamic stability or time consistency.* Assume that at the start of the game the players adopt an optimality principle (which includes the consent to maximize the joint payoff and agreed upon payoff distribution principle). When the game proceeds along the “optimal” trajectory, the state of the game changes and the optimality principle may not be feasible or remain optimal to all players. Then, some of the players will have an incentive to deviate from the initially chosen trajectory. If this happens, instability arises. In particular, the dynamic stability of a solution of a cooperative differential game is the property that, when the game proceeds along an “optimal” trajectory, at each instant of time the players are guided by the same optimality principles, and yet do not have any ground for deviation from the previously adopted “optimal” behavior throughout the game.

The question of dynamic stability in differential games has been explored in the past three decades. Haurie (1976) discussed the problem of Stability in extending the Nash bargaining solution to differential games.

Petrosyan (1977) formalized mathematically the notion of dynamic stability in solutions of differential games. Petrosyan and Danilov (1979 and 1982) introduced the notion of “imputation distribution procedure” for cooperative solution. Tolwinski et al. (1986) considered cooperative equilibria in differential games in which memory-dependent strategies and threats are introduced to maintain the agreed-upon control path. Petrosyan and Zenkevich (1996) provided a detailed analysis of dynamic stability in cooperative differential games. In particular, the method of regularization was introduced to construct time-consistent solutions. Yeung and Petrosyan (2001) designed a time-consistent solution in differential games and characterized the conditions that the allocation distribution procedure must satisfy. Petrosyan (2003) used regularization method to construct time-consistent bargaining procedures.

Let there exist solutions $W_\nu(x_t^*, T-t) \neq \emptyset$, $t_0 \leq t \leq T$ along the conditionally optimal trajectory $\{x_s^*\}_{s=t_0}^T$. If this condition is not satisfied, it is impossible for the players to adhere to the chosen principle of optimality, since at the very first instant t $W_\nu(x_t^*, T-t) \neq \emptyset$, $t_0 \leq t \leq T$, the players have no possibility to follow this principle. Assume that at time to when the initial state x_0 is the players agree on the imputation

$$\xi(x_0, T-t_0) = [\xi_1(x_0, T-t_0), \xi_2(x_0, T-t_0), \ldots, \xi_n(x_0, T-t_0)] \in W_\nu(x_0, T-t_0).$$

This means that the players agree on an imputation of the gain in such a way that the share of the i^{th} player over the time interval $[t_0, T]$ is equal to $\xi_i(x_0, T-t_0)$. If according to $\xi(x_0, T-t_0)$ player i is supposed to receive a payoff equaling $\varpi_i[\xi(x_0, T-t_0); x^*(\cdot), t-t_0]$ over the time interval $[t_0, T]$ then over the remaining time interval $[t, T]$ according to the $\xi(x_0, T-t_0)$ player i is supposed to receive:

$$\eta_i[\xi(x_0, T-t_0); x^*(\cdot), T-t] = \xi_i(x_0, T-t_0) - \varpi_i[\xi(x_0, T-t_0); x^*(\cdot), t-t_0]. \quad (7)$$

For the original imputation agreement (that is the imputation $\xi(x_0, T-t_0)$) to remain in force at the instant t, it is essential that the vector

$$\eta[\xi(x_0, T-t_0); x^*(\cdot), T-t] \in W_\nu(x_t^*, T-t), \quad (8)$$

and $\eta[\xi(x_0, T-t_0); x^*(\cdot), T-t]$ is indeed a solution of the current game $\Gamma_\nu(x_t^*, T-t)$. If such a condition is satisfied at each instant of time $t \in [t_0, T]$ along the trajectory $\{x_s^*\}_{s=t_0}^T$, then the imputation $\xi(x_0, T-t_0)$ is dynamical stable.

Dynamic stability or time consistency of the solution imputation $\xi(x_0, T-t_0)$ guarantees that the extension of the solution policy to a situation with a later starting time and along the optimal trajectory remains optimal. Moreover, group and individual rationalities are satisfied throughout the entire game interval.

A payment mechanism leading to the realization of this imputation scheme must be formulated. This will be done in the next section.

3.4. Payoff Distribution Procedure

A payoff distribution procedure (PDP) proposed by Petrosyan (1997) will be formulated so that the agreed upon dynamically stable imputations can be realized. Let the payoff Player i receives over the time interval $[t_0, T]$ be expressed as:

$$\varpi_i[\xi(x_0(\cdot), T-t_0); x^*(\cdot), t-t_0] = \int_{t_0}^{T} B_i(s) ds, \tag{9}$$

where

$$\sum_{j\in N} B_j(s) = \sum_{j\in N} g^i[s, x^*(s), u^*(s)],$$

for $t_0 \le s \le t \le T$.

Therefore,

$$B_i(t) = -\frac{d\eta_i}{dt}, \qquad \text{or } \frac{d\varpi_i}{dt} = B_i(t). \tag{10}$$

This quantity may be interpreted as the instantaneous payoff of the Player i at the moment t. Hence, it is clear the vector $B(t) = [B_1(t), B_2(t), \ldots, B_n(t)]$ prescribes distribution of the total gain among the members of the coalition N. By properly choosing $B(t)$ the players can ensure the desirable outcome that at each instant $t \in [t_0, T]$ there will be no objection against realization of the original agreement (the imputation $\xi(x_0, T-t_0)$) as shown in Figure 4, i.e. the imputation $\xi(x_0, T-t_0)$ is dynamic stable.

Cooperative differential game $\Gamma_\nu(x_0, T-t_0)$ has dynamically stable solution W_ν $(x_0, T-t_0)$, if all imputations $\xi(x_0, T-t_0) \in W_\nu(x_0, T-t_0)$ are dynamically stable. Conditionally optimal trajectory, on which dynamically stable solution of the game $\Gamma_\nu(x_0, T-t_0)$ exists, is *called optimal trajectory.*

We have proved under general conditions that the procedure $B(t)$, $t \in [t_0, T]$ (PDP) leading to dynamic stable cooperative solution exists and realizable [Petrosjan, Zenkevich, 1996].

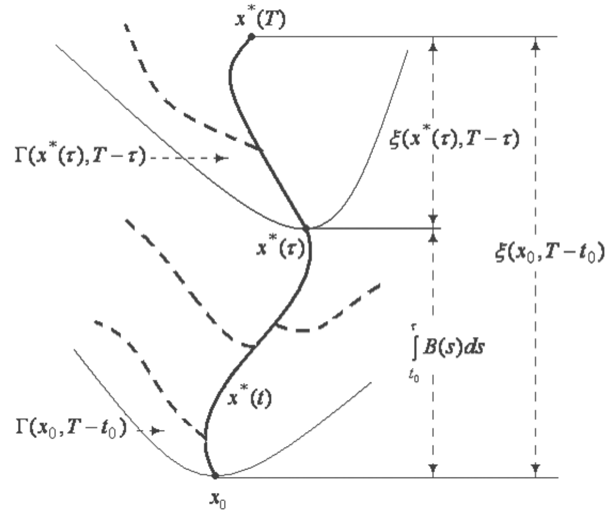

Fig. 4: Dynamically stable cooperative solution

4. A Dynamic Model of Joint Venture

Consider a dynamic joint venture in which there are n firms. The state dynamics of the i^{th} firm is characterized by the set of vector-valued differential equations:

$$\dot{x}_i(s) = f_i^i[s, x_i(s), u_i(s)], x_i(t_0) = x_i, i \in N, \tag{11}$$

where $x_i \in X_i \subset R^m$ denotes the state variables of player i, $u_i \in U^i \subset \text{comp}R^{l_i+}$ is the control vector of firm i. The state of firm i includes its capital stock, level of technology, special skills and productive resources. The objective of firm i is:

$$\int\limits_{t_0}^{T} g^i[s, x_i(s), u_i(s)] \exp\left[-\int\limits_{t_0}^{s} r(y)dy\right] ds + \exp\left[-\int\limits_{t_0}^{T} r(y)dy\right] q^i(x_i(T)),$$

where $\exp\left[-\int\limits_{t_0}^{s} r(y)dy\right]$ is the discount factor, $g^i[s, x_i(s), u_i(s)]$ – the instantaneous profit, and $q^i(x_i(T)$ – the terminal payment. In particular, $g^i[s, x_i(s), u_i(s)]$ and $q^i(x_i(T)$ are positively related to the level of technology x_i.

Consider a joint venture consisting of a subset of companies $K \subseteq N$. There are k firms in the subset K. The participating firms can gain core skills and technology that would be very difficult for them to obtain on their own, and, hence, the state dynamics of firm i in the coalition K becomes

$$\dot{x}_i(s) = f_i^K[s, x_K(s), u_i(s)], x_i(t_0) = x_i, i \in K, \tag{12}$$

where $x_K(s)$ is the concatenation of the vectors $x_j(s)$ for $j \in K$. In particular, $\partial f_i^K[s, x_i, u_K]/\partial u_j \geq 0$ for $j \neq i$. Thus positive effects on the state of firm i could be derived from the technology of other firms within the coalition. Again, without much loss of generalization, the effect of x_j on $f_i^K[s, x_K, u_i]$ remains the same for all possible coalitions K containing firms i and j.

4.1. Coalition Payoffs

At time t_0, the profit to the joint venture becomes:

$$\int_{t_0}^{T} \sum_{j \in K} g^j[s, x_j(s), u_j(s)] \exp\left[-\int_{t_0}^{s} r(y)dy\right] ds + \sum_{j \in K} \exp\left[-\int_{t_0}^{T} r(y)dy\right] q^j(x_j(T)), \tag{13}$$

To compute the profit of the joint venture K we have to consider the optimal control problem $\varpi[K; t_0, x_K^0]$ which maximizes (13) subject to (12).

For notational convenience, we express (12) as:

$$\dot{x}_K(s) = f^K[s, x_K(s), u_K(s)], x_K(t_0) = x_K^0, \tag{14}$$

where u_K is the set of u_j for u_j, $j \in K$; $f^K[t, x_K, u_K]$ is a column vector containing $f_i^K[t, x_K, u_K]$ for $j \in K$.

Using Bellman's technique of dynamic programming the solution of the problem $\varpi[K; t_0, x_K^0]$ can be characterized as follows.

Using the dynamic programming approach, it is possible to describe the solution in the following form. Denote with $\psi_j^{(t_0)K_0}(t, x_K)$ firm's j optimal control (in terms of maximizing the coalition K payoff).

In the case when all the n firms are in the joint venture, that is $K = N$, the optimal control is

$$\psi_N^{(t_0)N_0}(s, x_N(s)) = [\psi_1^{(t_0)K_0}(s, x_N(s)), \psi_2^{(t_0)K_0}(s, x_N(s)), \ldots, \psi_n^{(t_0)K_0}(s, x_N(s))],$$

The dynamics of the optimal state trajectory of the grand coalition can be obtained as:

$$\dot{x}_j(s) = f_j^N[s, x_N(s), \psi_j^{(t_0)N_0}(s, x_N(s))], x_j(t_0) = x_j^0, j \in N,$$

which can also be expressed as

$$\dot{x}_N(s) = f^N[s, x_N(s), \psi_N^{(t_0)N_0}(s, x_N(s))], x_N(t_0) = x_N^0, \tag{15}$$

Let $x_N^*(t) = [x_1^*(t), x_2^*(t), \ldots, x_n^*(t)]$ denote the solution to (15). The optimal trajectories $\{x_t^*\}_{t=t_0}^T$ characterizes the states of the participating firms within the venture period. We use $x_j^{t_0}$ to denote the value of $x_j^*(t)$ at time $t \in [t_0, T]$.

Consider the above joint venture involving n firms. The member firms would maximize their joint profit and share their cooperative profits according to the Shapley value (1953). The problem of profit sharing is inescapable in virtually every joint venture. The Shapley value is one of the most commonly used sharing mechanism in static cooperation games with transferable payoffs. Besides being individually rational and group rational, the Shapley value is also unique. The uniqueness property makes a more desirable cooperative solution relative to other solutions like the Core or the Stable Set. Specifically, the Shapley value gives an imputation rule for sharing the cooperative profit among the members in a coalition as:

$$\Phi_{\nu}^{i} = \sum_{K \subseteq N} \frac{(k-1)!(n-k)!}{n!} [\nu(K) - \nu(K \backslash i)], i \in N, \tag{16}$$

where $K \backslash i$ is the relative complement of i in K, $\nu(K)$ is the profit of coalition K, and $[\nu(K) - \nu(K \backslash i)]$ is the marginal contribution of firm i to the coalition K.

To maximize the joint venture's profits the firms would adopt the control vector $\{\psi_N^{(t_0)N^*}(t, x_N^{t^*})\}_{t=t_0}^{T}$ over the time $[t_0, T]$ interval, and the corresponding optimal state trajectory $\{x_N^*\}_{t=t_0}^{T}$ in (15) would result. At time t_0 with the state $x_n^{t_0}$, the firms agree that firm i's share of profits be:

$$\nu^{(t_0)i}(t_0, x_N^0) = \sum_{K \subseteq N} \frac{(k-1)!(n-k)!}{n!} [W^{(t_0)K}(t_0, x_K^0) - W^{(t_0)K \backslash i}(t_0, x_{K \backslash i}^0)]. \tag{17}$$

However, the Shapley value has to be maintained throughout the venture horizon $[t_0, T]$. In particular, at time $\tau \in [t_0, T]$ with the state being $x_N^{t^*}$ the following imputation principle has to be maintained:

$$\nu^{(\tau)i}(\tau, x_N^{t^*}) = \sum_{K \subseteq N} \frac{(k-1)!(n-k)!}{n!} [W^{(\tau)K}(\tau, x_K^{t^*}) - W^{(\tau)K \backslash i}(\tau, x_{K \backslash i}^{t^*})]. \tag{18}$$

where $i \in N$ and $\tau \in [t_0, T]$.

Note that $\nu^{(\tau)}(\tau, x_N^{t^*}) = [\nu^{(\tau)1}(\tau, x_N^{t^*}), \nu^{(\tau)2}(\tau, x_N^{t^*}), \ldots, \nu^{(\tau)n}(\tau, x_N^{t^*})]$, as specified in (18) satisfies the basic properties of an imputation vector.

Moreover, if condition (18) can be maintained, the solution optimality principle – sharing profits according to the Shapley value – is in effect at any instant of time throughout the game along the optimal state trajectory chosen at the outset. Hence, time consistency is satisfied and no firms would have any incentive to depart the joint venture. Therefore, a dynamic imputation principle leading to (18) is dynamically stable or time-consistent.

Crucial to the analysis is the formulation of a profit distribution mechanism that would lead to the realization of condition (18).

4.2. Transitory Compensation

In this section, a profit distribution mechanism will be developed to compensate transitory changes so that the Shapley value principle could be maintained throughout the venture horizon. First, an imputation distribution procedure (similar to those in Petrosyan and Zaccour (2003) and Yeung and Petrosyan (2004)) must be now formulated so that the imputation scheme in condition (18) can be realized. Let $B_i(t)$ denote the payment received by firm $i \in N$ at time $t \in [t_0, T]$ dictated by $\nu^{(t_0)i}(t_0, x_N^0)$. In particular,

$$\nu^{(t_0)i}(t_0, x_N^0) = \sum_{K \subseteq N} \frac{(k-1)!(n-k)!}{n!} [W^{(t_0)K}(t_0, x_K^0) - W^{(t_0)K \backslash i}(t_0, x_{K \backslash i}^0)] =$$

$$= \int\limits_{t_0}^{T} B_i(s) \exp\left[-\int\limits_{t_0}^{s} r(y)dy\right] ds + q^i(x_i^*(T)) \exp\left[-\int\limits_{t_0}^{T} r(y)dy\right]. \qquad (19)$$

The following formula describes the rule $B_i(\tau)$ for distribution Shapley value in the time, providing time consistency of Shapley value.

$$B_i(\tau) = -\sum_{K \subseteq N} \frac{(k-1)!(n-k)!}{n!} \{[W_t^{(\tau)K}(t, x_K^{\tau^*})|_{t=\tau}] - [W_t^{(\tau)K \backslash i}(t, x_{K \backslash i}^{\tau^*})|_{t=\tau}] +$$

$$+([W_{x_N^{\tau^*}}^{(\tau)K}(t, x_K^{\tau^*})|_{t=\tau}] - [W_{x_N^{\tau^*}}^{(\tau)K \backslash i}(t, x_{K \backslash i}^{\tau^*})|_{t=\tau}])\} \times f^N[\tau, x_N^{\tau^*}, \psi_N^{(\tau)N}(\tau, x_N^{\tau^*})], \qquad (20)$$

or

$$B_i(\tau) = -\sum_{K \subseteq N} \frac{(k-1)!(n-k)!}{n!} \{[W_t^{(\tau)K}(t, x_K^{\tau^*})|_{t=\tau}] - [W_t^{(\tau)K \backslash i}(t, x_{K \backslash i}^{\tau^*})|_{t=\tau}] +$$

$$+ \sum_{j \in K} [W_{x_j^{\tau^*}}^{(\tau)K}(t, x_K^{\tau^*})|_{t=\tau}] \times f_j^N[\tau, x_N^{\tau^*}, \psi_j^{(\tau)N}(\tau, x_N^{\tau^*})] -$$

$$- \sum_{h \in K \backslash i} [W_{x_h^{\tau^*}}^{(\tau)K \backslash i}(t, x_{K \backslash i}^{\tau^*})|_{t=\tau}] \times f_h^N[\tau, x_N^{\tau^*}, \psi_h^{(\tau)N}(\tau, x_N^{\tau^*})]\} =$$

$$= -\sum_{K \subseteq N} \frac{(k-1)!(n-k)!}{n!} \{[W_t^{(\tau)K}(t, x_K^{\tau^*})|_{t=\tau}] - [W_t^{(\tau)K \backslash i}(t, x_{K \backslash i}^{\tau^*})|_{t=\tau}] +$$

$$+ [W_{x_K^{\tau^*}}^{(\tau)K}(t, x_K^{\tau^*})|_{t=\tau}] \times f_K^N[\tau, x_N^{\tau^*}, \psi_K^{(\tau)N}(\tau, x_N^{\tau^*})] -$$

$$- [W_{x_{K \backslash i}^{\tau^*}}^{(\tau)K \backslash i}(t, x_{K \backslash i}^{\tau^*})|_{t=\tau}] \times f_{K \backslash i}^N[\tau, x_N^{\tau^*}, \psi_{K \backslash i}^{(\tau)N}(\tau, x_N^{\tau^*})]\},$$

where

$$f_K^N[\tau, x_N^{\tau^*}, \psi_K^{(\tau)N}(\tau, x_N^{\tau^*})]$$

is a column vector containing

$$f_i^N[\tau, x_N^{\tau^*}, \psi_i^{(\tau)N}(\tau, x_N^{\tau^*})], i \in K.$$

The vector $B(\tau)$ serves as a form equilibrating transitory compensation that guarantees the realization of the Shapley value imputation throughout the game horizon. Note that the instantaneous profit $B_i(\tau)$ offered to Player i at time i is conditional upon the current state $x_N^{\tau^*}$) and current time t. One can elect to express $B_i(\tau)$ as $B_i(\tau, x_N^{\tau^*})$. Hence, an instantaneous payment $B_i(\tau, x_N^{\tau^*})$ to player $i \in N$ yields a dynamically stable solution to the joint venture.

4.3. An Application in Joint Venture

Consider the case when there are 3 companies involved in joint venture. The planning period is $[t_0, T]$. Company i profit is

$$\int_{t_0}^{T} \Big[P_i[x_i(s)]^{\frac{1}{2}} - c_i u_i(s)\Big] \exp[-r(s-t_0)]ds + \exp[-r(T-t_0)]q_i[x_i(T)]^{\frac{1}{2}}, \tag{21}$$

where $i \in N$; P_i, c_i and q_i are positive constants, r is the discount rate, $x_i(t) \in R^+$ is the level of technology of company i at time t, and $u_i(t) \in R^+$ is its physical investment in technological advancement. The term $P_i[x_i(s)]^{\frac{1}{2}}$ reflects the net operating revenue of company i at technology level $x_i(t)$ and $c_i u_i$ is the cost of investment, $q_i[x_i(T)]^{\frac{1}{2}}$ gives the salvage value of company i's technology at time T.

The evolution of the technology level of company i follows the dynamics:

$$\dot{x}(s) = \Big[\alpha_i[u_i(s)x_i(s)]^{\frac{1}{2}} - \delta x_i(s)\Big], \qquad x_i(t_0) = x_i^0 \in X_i, \tag{22}$$

where $\alpha_i[u_i(s)x_i(s)]^{\frac{1}{2}}$ is the addition to the technology brought about by $u_i(s)$ amount of physical investment, and δ is the rate of obsolescence.

Consider the case when all these three firms agree to form a joint venture and share their joint profit according to the dynamic Shapley. Through knowledge diffusion participating firms can gain core skills and technology that would be very difficult for them to obtain on their own. The evolution of the technology level of company i under joint venture becomes:

$$\dot{x}_i(s) = \Big[\alpha_i[u_i(s)x_i(s)]^{\frac{1}{2}} - b_j^{[j,i]}[x_j(s)x_i(s)]^{\frac{1}{2}} + b_k^{[k,j]}[x_k(s)x_i(s)]^{\frac{1}{2}} - \delta x_i(s)\Big],$$

$$x_i(t_0) = x_i^0 \in X_i, \qquad \text{for } i,j,k \in N = \{1,2,3\} \qquad \text{and } i \neq j \neq k, \tag{23}$$

where $b_j^{[j,i]}$ and $b_k^{[k,j]}$ are non-negative constants. In particular, $b_j^{[j,i]}[x_j(s)x_i(s)]^{\frac{1}{2}}$ represents the technology transfer effect under joint venture on firm i brought about by firm j's technology.

The profit of the joint venture is the sum of the participating firms' profits:

$$\int_{t_0}^{T} \sum_{j=1}^{3} \Big[P_j[x_j(s)]^{\frac{1}{2}} - c_j u_j(s)\Big] \exp[-r(s-t_0)]ds + \sum_{j=1}^{3} \exp[-r(T-t_0)]q_j[x_j(T)]^{\frac{1}{2}}. \tag{24}$$

The firms in the joint venture then act cooperatively to maximize (24) subject to (23). Giving up technical calculation, we have

$$f_i^{\{1,2,3\}}[\tau, x_1^{\tau^*}, x_2^{\tau^*}, x_3^{\tau^*}, \psi_i^{(\tau)\{1,2,3\}}(\tau, x_1^{\tau^*}, x_2^{\tau^*}, x_3^{\tau^*})] =$$

$$= \frac{\alpha_i^2}{4c_i} A_i^{\{1,2,3\}}(\tau)(x_i^{\tau^*})^{\frac{1}{2}} + b_j^{[j,i]}[x_j^{\tau^*} x_i^{\tau^*}]^{\frac{1}{2}} + b_k^{[k,j]}[x_k^{\tau^*} x_i^{\tau^*}]^{\frac{1}{2}} - \delta x_i^{\tau^*}, \tag{25}$$

for $i \in 1, 2, 3$.

Denoting $[x_1^{\tau^*}, x_2^{\tau^*}, x_3^{\tau^*}]$ by $x_{\{1,2,3\}}^{\tau^*}$, we can write

$$f_{i,j}^{\{1,2,3\}}[\tau, x_{\{1,2,3\}}^{\tau^*}(\Psi_i^{(\tau)\{1,2,3\}}(\tau, x_{\{1,2,3\}}^{\tau^*}), \Psi_j^{(\tau)\{1,2,3\}}(\tau, x_{\{1,2,3\}}^{\tau^*})] =$$

$$= [f_i^{\{1,2,3\}}[\tau, x_{\{1,2,3\}}^{\tau^*}, \Psi_i^{(\tau)\{1,2,3\}}(\tau, x_{\{1,2,3\}}^{\tau^*})] f_j^{\{1,2,3\}}[\tau, x_{\{1,2,3\}}^{\tau^*},$$

$$\Psi_j^{(\tau)\{1,2,3\}}(\tau, x_{\{1,2,3\}}^{\tau^*})]]$$

for $i, j \in \{1, 2, 3\}$ and $i \neq j$,

$$f_{\{1,2,3\}}^{\{1,2,3\}}[\tau, x_{\{1,2,3\}}^{\tau^*}, \Psi_1^{(\tau)\{1,2,3\}}(\tau, x_{\{1,2,3\}}^{\tau^*}), \Psi_2^{(\tau)\{1,2,3\}}(\tau, x_{\{1,2,3\}}^{\tau^*}),$$

$$\Psi_3^{(\tau)\{1,2,3\}}(\tau, x_{\{1,2,3\}}^{\tau^*})] =$$

$$= \begin{bmatrix} f_1^{\{1,2,3\}}[\tau, x_{\{1,2,3\}}^{\tau^*}, \Psi_1^{(\tau)\{1,2,3\}}(\tau, x_{\{1,2,3\}}^{\tau^*})] \\ f_2^{\{1,2,3\}}[\tau, x_{\{1,2,3\}}^{\tau^*}, \Psi_2^{(\tau)\{1,2,3\}}(\tau, x_{\{1,2,3\}}^{\tau^*})] \\ f_3^{\{1,2,3\}}[\tau, x_{\{1,2,3\}}^{\tau^*}, \Psi_3^{(\tau)\{1,2,3\}}(\tau, x_{\{1,2,3\}}^{\tau^*})] \end{bmatrix}. \tag{26}$$

After analytical transformation we have

$$W_t^{(\tau)\{1,2,3\}}(t, x_{\{1,2,3\}}^{\tau^*})|_{t=\tau} = [\dot{A}_1^{\{1,2,3\}}(\tau)(x_1^{\tau^*})^{1/2} + \dot{A}_2^{\{1,2,3\}}(\tau)(x_2^{\tau^*})^{1/2} +$$

$$+ \dot{A}_3^{\{1,2,3\}}(\tau)(x_3^{\tau^*})^{1/2} + \dot{C}^{\{1,2,3\}}(\tau)] -$$

$$- r[A_1^{\{1,2,3\}}(\tau)(x_1^{\tau^*})^{1/2} + A_2^{\{1,2,3\}}(\tau)(x_2^{\tau^*})^{1/2} +$$

$$+ A_3^{\{1,2,3\}}(\tau)(x_3^{\tau^*})^{1/2} + C^{\{1,2,3\}}(\tau)],$$

$$W_t^{(\tau)\{i,j\}}(t, x_{\{i,j\}}^{\tau^*})\tau|_{t=\tau} =$$

$$= [\dot{A}_i^{\{i,j\}}(\tau)(x_i^{\tau^*})^{1/2} + \dot{A}_j^{\{i,j\}}(\tau)(x_j^{\tau^*})^{1/2} + \dot{C}^{\{i,j\}}(\tau)] -$$

$$- r[A_i^{\{i,j\}}(\tau)(x_i^{\tau^*})^{1/2} + A_j^{\{i,j\}}(\tau)(x_j^{\tau^*})^{1/2} + C^{\{i,j\}}(\tau)],$$

for $i, j \in \{1, 2, 3\}$ and $i \neq j$.

$$W_t^{(\tau)i}(t, x_i^{\tau^*})|_{t=\tau} = [\dot{A}_i^{\{i\}}(\tau)x_i^{\tau^*} + \dot{C}^{\{i\}}(\tau)] - r[A_i^{\{i\}}(\tau)x_i^{\tau^*} + C^{\{i\}}(\tau)],$$

for $i \in \{1, 2, 3\}$.

$$W^{(\tau)K}_{x_i^{\tau^*}}(t, x_K^{\tau^*})|_{t=\tau} = \frac{1}{2} A_i^K(\tau)(x_i^{\tau^*})^{-1/2}, \tag{27}$$

for $i \in K \subseteq \{1, 2, 3\}$.

Note that coefficients A_i, C_j are the solutions of linear differential equation system. The explicit solution is not stated here because of its lengthy expressions.

Using eq. (25) to (27) and (20) we obtain the form for $B_i(\tau)$. A payment $B_i(\tau)$ offered to player $i \in \{1, 2, 3\}$ at time $\tau \in [t_0, T]$ will lead to the realization of the dynamic Shapley value. Hence, a dynamically stable solution to the joint venture will result.

5. Conclusion Venture

Long term cooperative solutions based on interest coordination are considered. It is shown, that basic cooperative optimality principles haven't dynamic stability (time consistency) property. This property requires saving optimality property along the optimal trajectory. We have proposed regularization procedure (PDP), introducing a new control variable. Applying the method of regularization for dynamic cooperation problem, we constructed the control in the form of special payments, paid at each time instant on the optimal trajectory. As a special case the joint venture dynamic model is investigated. For this problem the dynamic stable solution is obtained.

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The Value of Information in Binary, Limited-Liability Principal–Agent Games with Unobservable Outcome

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Abstract. Many applications in accounting, economics, and business, employ binary principal–agent game wherein the work-averse, risk-averse agent takes unobservable actions that determine stochastically the mutually-observable outcome on the basis of which the principal designs the incentives of the agent paid before the principal collects his share. In this paper we study a binary game when the outcome is unobservable and when the agent is protected by limited liability. The principal bases the incentives on the imperfectly audited report of the agent on outcome subject to the constraint that no payment can fall below the agent's limited liability. The most interesting results are that we find that limited liability does not guarantee a higher expected pay-off to the agent since the principal can use additional signal to fine-tune the contract, and that, as a result, there is a demand for a post-outcome signal that is non-informative in Holmstrom's (1979) sense. The binary setting clearly shows the similarities in contract's design of three different contracts: a contract based on mutually-observable outcome (our benchmark contract), a renegotiation-proof contract that induces the agent to report the truth, and a contract based on a report that can be manipulated by the agent. In particular, when limited liability is low (high), the renegotiation-proof contract is similar to a contract based on mutually-observable outcome (managed report).

Keywords: Value of information, earnings management, principal–agent contract.

Introduction

For a truly positive theory of contracts a more reasonable assumption is that buyers and sellers operate in an environment of limited liability [Demougin, Garvie, 1991].

Many applications in accounting, economics, and business employ binary principal–agent game wherein the work-averse, risk-averse agent takes unobservable actions that determine stochastically the mutually-observable outcome on the basis of which the principal designs the incentives of the agent paid before the principal collects his share[1]. Yet, negligible attention has been given to the effect of limited liability on the shape, and the resulting payoffs[2].

In this study, we characterize the effect of the agent's limited liability on the equilibrium one-shot principal–agent contracts with unobservable outcome[3]. As a benchmark, we analyze first the contract when outcome is observed by both principal and agent (MOC). This contract characterizes, as an example, the situation in which the agent is a cost/revenue or profit center, and all costs/revenues are traceable to the agent. Then, we replace the assumption that outcome is observable with a weaker one that the agent alone observes the outcome and the contract is based, instead, on an imperfectly audited report. The agent, therefore, can "manage" the report with some probability. This is the case, for example, for firms whose managers (the agents) release public reports to shareholders (the principals). There is ample evidence that managers receive bonuses based on the accounting numbers (see, e.g., [Healy, 1985]) and that firms "manage earnings" (see, e.g., [Ronen and Sadan, 1981, 2008]). We distinguish between two contracts: in the first, the communication channels between the principal and the agent are blocked after the contract's design stage (NCC); in the second, the principal and agent renegotiate the contract after the agent finished exerting effort and before the agent reports outcome to the principal (RPC). We show that contract's characterization depends on the combination of the limited-liability level (LL) of the agent and available informational signals. In MOC, additional signals are redundant. Hence, the LL spectrum is divided into two sub-regions: low LLs that do not affect the contract and high LLs that allow the agent to enjoy more than his reservation utility payoff – limited-liability rent. In NNC and RPC, post-outcome signal is useful. Hence, the range of LLs is now divided into three sub-regions: low, medium, and high. When LL is low, the payoffs of RPC coincide with the payoffs of MOC. The only difference between the two contracts lies in the role of an imperfectly audited report of outcome: it is redundant in MOC but it

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[1] The appeal of a binary-effort model is that it usually entails little loss of generality. See [Hart and Holmstrom, 1985].

[2] One possible explanation is due to Mirrlees (1974). When the agent's utility is unbounded from below, as in the log family utility functions, the principal can threaten him with an extremely harsh penalty upon observing a low outcome to induce him to exert first-best effort. Limited liability bounds the agent's utility from below. Hence, limited liability that prevents extremely harsh penalties, but does not affect payments of the second-best contract, is assumed implicitly by any study that characterizes the second-best equilibrium. In this regard, our work extends the research into limited liability ranging from minus infinity to an arbitrarily large positive level.

[3] Liability of minus infinity captures the special case in which the agent does not have limited liability.

is essential in RPC to induce the agent to tell the truth. In RPC, the principal elicits the truth from the agent by offering a menu of options and threatening to penalize him when the subsequent audited report reveals that he made the "wrong" choice. When LL is medium, RPC changes to a hybrid of an NCC and MOC. The good-outcome payments are identical to NCC, while the bad-outcome payment is identical to MOC. When LL level is high, NCC and RPC are similar: both pay the same upon a good-outcome realization, but RPC pays the certainty equivalence of NCC upon a bad outcome. The most significant difference between the medium and the high level lies in the principal's flexibility to fine-tune payments to push the agent to his reservation utility level. When the principal can also use pre-decision signal, the limited-liability spectrum is divided into four subregions, because this signal is now used to reduce the cost of NCC when LL is low. We find, however, opposing trends in the demands for the pre- and post-signals. Limited-liability creates demand for a post-signal outcome RPC and suppresses demand for pre-signal outcome in NCC, because the limited-liability of the agent reduces the flexibility of the principal.

Second, when the principal has a post-signal outcome besides the audited report, the limited level is divided into three levels: low, medium, and high. When limited-liability renders the RPC's payoffs infeasible, the structure of the revelation contract (RPC) changes, and now resembles that of NCC. Specifically, both contracts will include a pre- and post-outcome imperfect public signal in compensation's basis, and some of the payments will coincide. The principal fine-tunes payments based on the post-outcome signal to compensate for the loss in flexibility caused by the agent's limited-liability. For some limited-liability levels, the principal, thus, pushes the agent closer to his reservation utility level. Our study is a contribution to three strands of literature: the literature that examines the value of additional signals in principal–agent contract [Christensen and Feltham, 2003]; the earnings management literature [Ronen and Yaari, 2008] and the renegotiation-proof principal–agent contracts; and the literature on limited liability of the agent in principal–agent contracts.

The literature on the value of additional signal has established two main results. A signal is valuable for contracting if it has marginal information on the unobservable actions of the agent – the informativeness criterion of Holmstrom (1979). The other one is that perfect-pre-contract information is not valuable (e.g., [Harris, 1987]). Our study identifies an additional role. To the extent that the limited liability of the agent is a contract friction that preempts low payments to the agent, the principal can use a signal to fine-tune the contract to bypass this friction and push the agent to his reservation utility. So, the ex-post signal is used in RPC despite being non-informative. To sharpen this point, observe that if the unobservability of the agent's effort and conflict of interests between principal and agent render the standard contract between them the second best, limited liability is a third best, which changes the role of additional contracting signals. We also show below a "packing order" of which signals is going to be dropped first as limited liability increases.

For a through review of the contribution of such a study to the earnings management literature, we refer the readers to our companion paper [Ronen and Yaari, 2007], and to part 3 of our book [Ronen and Yaari, 2008]. There we show that if the outcome is not binary, the principal may prefer earnings management to eliciting the truth because the agent's limited liability restricts the penalties on the agent caught in not telling the truth in RPC. In this paper, NCC is always more expensive that RPC. This paper explores renegotiation. To the best of knowledge, in the extant literature our paper is the only one besides Hermalin and Katz (2001) to establish that renegotiation-proof contract is preferable. In their framework, renegotiation is profitable because the principal and agent learn value-relevant information on the agent's effort that can allow them to move to first-best allocation with the benefit of signing enforceable contracts. Other studies show that renegotiation after contracting and before the agent competed exerting effort, is always expensive to the principal. Because the principal is risk neutral and the agent is risk-averse, both have incentives to replace the risky incentive contracts with a fixed salary, which destroys the incentives of the agent to exert effort. Indeed, Aghion et al. showed that the principal may takes steps to commit to not renegotiate the contract. In our paper, this issue does not arise because the principal and agent negotiate after the agent completed exerting effort. Renegotiation is profitable to the principal because the agent alone observes outcome, and renegotiation can elicit the private information of the agent.

In accounting, the value of renegotiation has been recognized recently by Demski and Frimor (1999), Christensen, Demski, and Frimor (2002) in a series of papers in which the principal and the agent can renegotiate a two-period contract, and by Yim (2001). None paid attention to LL. The study most closely related to ours is that of Christensen, Demski, and Frimor (2001), which established the value of earnings management to counteract the unfavorable effect of renegotiation on incentives. Specifically, to induce the agent to exert effort in the first period, the principal can balance week first-period and strong second-period incentives with an audit system that allows conservatism, since such a type of earnings management policy allows the agent to defer remuneration of the fruits of the first period to the second, and, thus, provides him with incentives to exert effort early on. There is a major difference between this study and ours. Because renegotiations take place at different timings with respect to realization of outcomes, they find that it weakens incentives and earnings management is the remedy, while we find that renegotiation strengthens incentives and that had it not been costly, the principal would always use renegotiation to eliminate earnings management. Sappington (1983) first analyzed the effect of the agent's limited liability on the principal–agent contract. As is well known, when a principal contracts with a risk-neutral agent with unlimited wealth, the first-best contract is feasible by selling the enterprise to the agent, thus imposing all risk on the agent [Harris and Raviv, 1979]. Sappington shows that the first-best contract is no longer feasible when the agent has limited liability. This result was exploited by Antle and Eppen (1985), Fellingham and Young (1990), Arya, Glover, and Sivaramakrishnan (1997) and others to derive insights into accounting issues, modeling the problem as a second-best principal–agent game with a risk-neutral agent, whose limited liability protects him from bankruptcy. Innes (1990), Park (1995), and Kim (1997) have analyzed similar situations. Innes finds that a debt contract is optimal when both principal and agent have limited liability. Park and Kim find that a bonus contract can achieve the first-best under some conditions on technology. They explain the apparent inconsistency with Sappington's study as a matter of assumptions: Sappington assumes that the agent exerts effort after he observes a productivity-related signal, while they place the exertion before the observation. Basu (1992) and Sengupta (1997) derive similar results in a somewhat different setting: Basu studies moral hazard in the choice of technologies, and Sengupta extends this model by studying moral hazard in both technology choice and effort. Pitchford (1998) studies a similar framework, allowing the bargaining power to vary with the agent's limited liability (in contrast to the standard model, which assumes that all the bargaining power lies with the principal). He finds that the first-best scheme becomes attainable only when all the bargaining power lies in the hands of the agent. The intuition of this result is that in this case the agent bears all the risk, as required by Harris and Raviv's scheme. Demski, Sappington, and Spiller (1988) extend the multi-agent model of Demski and Sappington (1984) to an environment in which the agents have limited liability on the level of payments. They find that the first-best solution is more likely to be feasible if the limited liability is small relative to the agents' reservation utility[4].

Our paper is different from these papers in that we analyze a second-best scenario because the agent is risk-averse. Hence, we find that the difficulty of inducing the agent to exert effort exacerbates the effect of the agent's limited liability on the costliness of the contract, while the above studies are concerned with the feasibility of the first-best allocation. Second, in contrast to the literature that assumes that limited liability renders some payments infeasible, we give full characterization of how the contracts vary along the range of limited liability. Third, we offer a comparison of contracts that are different in the information available for contracting across a given level of limited liability: in MOC the outcome is observable, in NCC it is not, and in RPC it can be elicited from the agent at some nonnegative cost. The paper proceeds as follows. Section 2 presents the model. Section 3 analyzes MOC. Sections 4 and 5 analyze NCC and RPC, and the effect of the availability of pre-decision signal, respectively. Section 6 summarizes our findings and offers conclusions. Appendices are available upon request from the authors.

1. The Basic Model

We study a binary principal–agent model. The risk-averse, work-averse agent exerts unobservable effort, a, $a \in a_1,a_2$, where $a_1 < a_2$. Jointly with nature, the agent's effort produces a monetary outcome, x, which is either bad – a failure, B, or good – a success, G; i.e., $x \in B, G$, $B < G$[5]. The conditional probability of G given

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[4] Additional papers study the effect of limited liability on the contract when the principal faces an agent of unknown type [Lawarre, Van Audenrode, 1992, 1996] and [Demougin, Garvie, 1991].

[5] The advantage of a binary outcome model is that the earnings management strategy is crystal clear. The agent prefers to report the higher outcome.

effort a_j is denoted by θ_j, $\theta_j = \Pr(G|a_j)$. The greater the agent's effort, the higher the expected outcome, $\theta_1 < \theta_2$.

The principal cannot base the contract on the outcome, x, because it is observed by the agent alone. Instead, he can base the contract on the reported outcome, r, $r \in$ {B,G}, which is audited first by the third party – an outside monitor–based on the input of the agent, d, $d \in \{B, G\}$. In a financial reporting context, the outside monitor is naturally the auditor. In an internal reporting context, the third part could be the controller. The important point here is that the monitor puts a check on the ability of the agent to manage earnings. We assume that the monitor verifies truthful input perfectly, and misrepresentation imperfectly, with probability of π, $1/2 < \pi < 1$[6]. Before reimbursing the agent, the principal learns a public signal, s, which could be either low, ℓ, or high, h, $s \in$ {ℓ,h}. This signal is imperfectly correlated with the outcome, $\Pr(s = h|x =$G$) = \Pr(s = \lambda|x =$B$) = \rho > 1/2$, $\Pr(s = h|x =$B$) = \Pr(s = \lambda|x =$G$) =$ 1-ρ. The time-line of main events is summarized in Table 1. The principal maximizes the expected residual outcome: the expected outcome less the expected compensation costs. He learns the actual outcome and collects the residual outcome (the outcome less the payment to the agent, $x - t$) long after Date 6. But at that point it is too late to use this knowledge to induce the agent to report truthfully.

Table 1: Time-line of main events

Date 1	**Date 2**	**Date 3**	**Date 4**	**Date 5**	**Date 6**
The principal designs the contract, T.	The agent chooses effort, a.	The outcome, x, is realized and observed by the agent alone.	The principal and the agent may renegotiate the contract.	The agent communicates x to the monitor. The principal receives an audited report, r, and a public signal, s.	The agent is paid t, $t \in T$, and quits.

The risk-averse, work-averse agent has a utility function that is separable in monetary income, $W(t)$, and effort, $-V(a)$, where W is a von Neumann–Morgenstern utility function with $W' > 0$ and $W'' < 0$, and the disutility of the agent over effort is an increasing, strictly convex function, $V' > 0$ and $V'' > 0$. He can obtain expected utility of W_0 in alternative employment, $W_0 > L$..

Figure 2 summarizes the information structure and payoffs of the principal and the agent.

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[6] In general, a one-sided audit technology is typical in Management-by-Exception monitoring systems, and when the privately-informed agent must sign the report. Even if the audit technology in the latter case were two-sided stochastic one, the output will be the same as in the one-sided case because the agent would let it slide only when it suited him and refuse to sign the report otherwise.

For the appropriateness of this characterization for external audit, consult Schwartz (1997).

Table 2: The highlights of the information structure and payoffs*

	The Principal	**The Agent**
Private signal	–	Outcome, x
Public signals	Reported outcome, r, Public post-outcome signal, s;	
Decision variables	The contract, T	Effort,a, Input to monitor, d
Payoffs	$E(x-t)$	$E(W(t))-V(a)$

*In the renegotiation game, the agent has also to choose a renegotiation option, before the agent submits the output, d, to the monitor.

We assume that the principal wishes to motivate the agent to exert the higher level of effort; the cost of inducing the agent to exert greater effort is lower than the resulting increase in the expected outcome, $[\theta_2 - \theta_1]$[G − B]. If LL is so high as to render the incentives to exert a_2 prohibitively costly the principal does not contract. We find the equilibrium contracts by solving the principal's program. All contracts maximize the principal's expected residual outcome, subject to the following minimal constraints:

- The contract must guarantee the agent his reservation utility, W_0 :

$$E(W(T)|a_2) - V(a_2) \geq W_0. \qquad (PC).$$

- The agent is induced to choose the higher level of effort:

$$E(W(T)|a_2) - V(a_2) \geq E(W(T)|a_1) - V(a_1). \qquad (MH)$$

- The LL constraint: for each payment t

$$W(t) \geq W(L). \qquad (LC)$$

For parsimony, in all programs, λ, μ, and ϕ denote the Lagrange multipliers of (PC), (MH), and (LC), respectively. We observe that a given program might have additional constraints that are unique to it. Table 3 summarizes the notation.

2. Benchmark: A contract based on Mutually-observable outcome

As a benchmark case, consider the contract based on mutually-observable outcome, M, $M : \{B, G\} \times \{\ell, h\} \to \Re$. Given that the public signal is a noisy signal of perfectly observable outcome, he has no contracting value. This contract is then the solution to the following program:

$$\max_{\{M_x\}} \theta_2[G - M_G] + (1 - \theta_2)[B - M_B]$$

Table 3: Notation

L	=	The limited liability level (LL) of the agent $L \in (-\infty, \bar{L})$.
a	=	The unobservable effort of the agent, $a \in \{a_1, a_2\}, a_1 < a_2$.
x	=	The outcome, $x \in \{B, G\}, B < G$.
Θ_i	=	The probability of $x = G$ conditional on effort a_i, $\Theta_i = Pr(x = = G\|a_i), \Theta_1 < \Theta_2$.
d	=	The agent's communication of outcome to the monitor,$d \in \{B, G\}$.
r	=	The audited report of the outcome, r, $r \in \{B, G\}$.
π	=	The conditional probability that the monitor discovers attempted misrepresentation of the outcome by the agent, $\pi = Pr(r = x\|d \neq \neq x)$, $\frac{1}{2} < \pi < 1$.
s	=	An ex post public, imperfect signal on the outcome, $s \in \{l, h\}$; stands for low, h stands for high.
ρ	=	The conditional probability that the public signal correctly reflects outcome, $\rho = Pr(s = h\|x = G) = Pr(s = l\|x = B)$, $\frac{1}{2} < \rho < 1$.
T	=	The contract that specifies the compensation t to the agent, $t \in T$.
W	=	The utility function of the risk-averse, work-averse agent over compensation.
W_0	=	The reservation utility of the agent.
$V(a)$	=	The agent's disutility over effort.
C, NCC	=	The No-Communication Contract, $C : \{B, G\} \times \{l, h\} \longrightarrow \mathbb{R}$.
M, MOC	=	The Mutually-Observable Outcome contract, $M : \{B, G\} \times \{l, h\} \rightarrow \rightarrow \mathbb{R}$.
$\hat{x}$	=	The post-outcome option in the renegotiation contract $\hat{x} \in \{\hat{B}, \hat{G}\}$.
S, RPC	=	The Renegotiation Contract, $S : \{\hat{B}, \hat{G}\} \times \{B, G\} \times \{l, h\} \longrightarrow \mathbb{R}$.

$$\theta_2 W(M_G) + (1-\theta_2)W(M_B) - V(a_2) \geq W_0, \qquad (PC)$$

$$\theta_2 W(M_G) + (1-\theta_2)W(M_B) - V(a_2) \geq \theta_1 W(M_G) + (1-\theta_1)W(M_B) - V(a_1), \quad (MH)$$

$$M_B \geq L, \quad M_G \geq L. \qquad (LC)$$

The LL of the agent determines whether (PC) is binding. (MH) is binding because if not, the optimal contract where a salary one, which does not provide the agent with incentives to exert effort. Rewriting (PC) and (MH), respectively as: $\theta_2[W(M_G) - W(M_B)] + W(M_B) = W_0 + V(a_2)$, and $W(M_G) - W(M_B) = \nu$, where $\nu = \frac{V(a_2)-V(a_1)}{\theta_2-\theta_1}$, solve for the contract as described in Observation 1.

Observation 1: There is a critical level of LL, L^*, such that:

If $L \leq L^*$, the agent's expected payoff equals his reservation utility, and his incentive contract is:

$$W(M_B) = W_0 + V(a_2) - \theta_2\nu; \quad W(M_G) = W_0 + V(a_2) + (1-\theta_2)\nu.$$

If $L > L^*$, the agent's expected payoff exceeds his reservation utility, and his incentive contract is:

$$W(M_B) = W(L); \quad W(M_G) = W(L) + \nu.$$

Given that MOC is well-known (see, e.g., [Ma, 1991]), and the modification of setting M_B to L when LC is binding, we will not discuss it any further. Observation 1 is useful in the comparison of the contracts studied in the next section.

3. The equilibrium contracts when the outcome is unobservable

3.1. The no-communication contract (NCC)

A principal–agent setting with unobservable outcome is useful for studying games where the agent "manages" the report instead of reporting the truth. Lemma 1 characterizes his earnings management strategy, and Proposition 1 then characterizes the contract.

Lemma 1. *To motivate the agent to exert effort, his expected compensation upon a success, $x = G$, is strictly higher than for a failure, $x = B$. That is, $E_{r,s}\left(W(C(r,s)|x=G)\right) > E_{r,s}\left(W(C(r,s)|x=B)\right)$.*

The agent communicates G truthfully and misrepresents B. That is, $d(x) = G$.

The proof and intuition of this result are based on the incentives feature of the contract. The agent exerts effort because he is paid more for signals that indicate that the outcome is likely to be success, G, than failure, B.

Since, in principle, contracts must be based on mutually observable variables only, the contracting signals are the audited report of the outcome, r, and the public signal,

s[7]. The agent cannot affect the public signal, but he can affect the report. Given that the monitor is imperfect, the agent may succeed is presenting an outcome of B as if it were G. By our assumption on the audit technology, the audited report is truthful upon a success, $Pr(r = r_G|x = G) = 1; Pr(r = r_B|x = G) = 0$, and misrepresented imperfectly upon a failture, $Pr(r = r_G|x = B) = 1 - \pi$; $Pr(r = r_B|x = B) = \pi$.

Lemma 1 implies that when the principal observes a report of B, he believes that the report is truthful. Consequently, he does not condition payoffs on the noisy, non-informative public signal [Holmstrom, 1979]. Denoting NCC by C, C has at most three components: $C = \{C_B, C_{G\ell}, C_{Gh}\}$, which are the solution to the following C-program:

$$\max_{\{C_{rs}\}} E_{x,r,s}(x - C_{rs}|a_2)^8,$$

(PC), (MH) and (LC) obtain[9].

This program yields the following first-order conditions:

$$C_B : \frac{1}{W'(C_B)} = \lambda - \frac{\pi\mu - \varphi_B}{\pi(1-\theta_2)},$$

$$C_{G\ell} : \frac{1}{W'(C_{G\ell})} = \lambda + \frac{(\pi - Q)\mu + \varphi_{G\ell}}{\theta_2(1-\pi) + (1-\pi)(1-\theta_2)\rho} > C_B,$$

$$C_{Gh} : \frac{1}{W'(C_{Gh})} = \lambda + \frac{Q\mu}{\theta_2\rho + (1-\pi)(1-\theta_2)(1-\rho} > C_{G\ell},$$

where: $\pi - Q = 1 - \rho - (1-\pi)\rho$, and $Q = \rho - (1-\pi)(1-\rho)$.

As expected, the first-order conditions show that $C_B < C_{G\ell} < C_{Gh}$. Observe that the effect of limited liability is transitive: if the (LC) constraint is binding for any payment, then it is binding for any payment that is supposed to be lower. Hence, LL can now affect, at most, two payment levels, C_B and $C_{G\ell}$, which are lower than C_{Gh}. It does not affect the highest possible payment directly because, had it done so, the agent would have been paid a fixed payment, L, regardless of the performance measure -an arrangement that provides him with disincentives to exert effort.

Proposition 1. *Identify two levels of LL, L^{C*} and L^{C**}, where $L^{C*} < L^{C**}$, so that LL is low when $L < L^{C*}$, medium – when $L^{C*} \le L \le L^{C**}$, and high – when $L > L^{C**}$.*

Case (a): *When LL is low, NC coincides with the unlimited-liability NC, i.e., $L < C_B < C_{G\ell} < C_{Gh}$, and the agent's expected payoff equals his reservation utility payoff. That is, $E_{x,r,s}(W(C_{rs})) - V(a_2) = W_0$.*

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[7] This represents, for example, publicly traded firms whose managers (the agents) release public reports to shareholders (the principals), who can reward them based on the accounting report, r, and share prices, which provide a noisy signal on the outcome, s.

[8] Since the audit technology is imperfect, we treat the report, r, as a stochastic variable.

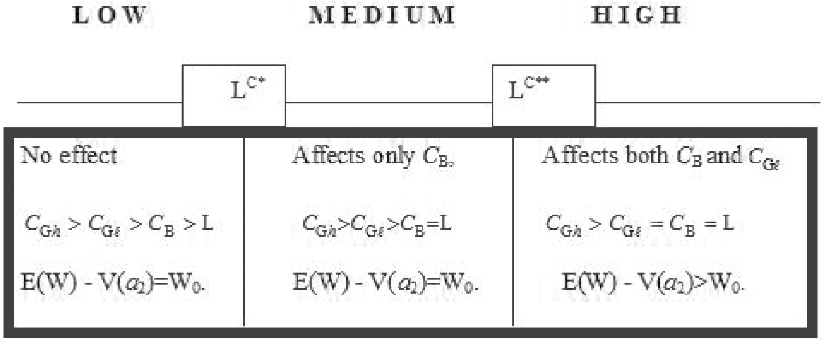
[9] (MH) is: $E_{x,r,s}(W(C_{rs})|a_2) - V(a_2) = \theta_2[\rho W(C_{Gh}) + (1-\rho)W(C_{G\ell})] + (1-\theta_2)$ $\{(1-\pi)[(1-\rho)W(C_{Gh}) + \rho W(C_{G\ell})] + \pi W(C_B)\} - V(a_2) \ge \theta_1[\rho W(C_{Gh}) + (1-\rho)W(C_{G\ell})] +$ $+(1-\theta_1)\{(1-\pi)[(1-\rho)W(C_{Gh}) + \rho W(C_{G\ell})] + \pi W(C_B)\} - V(a_1)$. After rearranging, (MH) is: $QC_{Gh} + (\pi - Q)C_{G\ell} - \pi C_B \ge \nu$, where $Q = \rho - (1-\pi)(1-\rho), \pi - Q = 1 - \rho - (1-\pi)\rho$.

Case (b): *When LL is medium, limited liability affects the only the lowest payment, i.e.,*$L = C_B < C_{G\ell} < C_{Gh}$*, and the agent's expected payoff equals his reservation utility payoff*[10]*. That is,* $E_{x,r,s}(W(C_{rs})) - V(a_2) = W_0$*. The contract is more expensive than in case (a).*

Case c: *When LL is high, limited liability determines all payments except for the highest one,* C_{Gh}*, which is given by:* $C_{Gh} = W^{-1}(W(L) + \frac{\nu}{Q}) > C_{Gh} = C_B = L$*.*

The agent's expected utility exceeds his reservation utility, W_0*,*$E_{x,r,s}(W(C_{rs})) - V(a_2) = W(L) + Pr(r_G, s = h)\frac{\nu}{Q} - V(a_2) > W_0$*. The contract is more expensive than in case (b).*

Table 4: NCC



LOW	MEDIUM	HIGH
No effect	Affects only C_B.	Affects both C_B and $C_{G\ell}$
$C_{Gh} > C_{G\ell} > C_B > L$	$C_{Gh}>C_{G\ell}>C_B=L$	$C_{Gh} > C_{G\ell} = C_B = L$
$E(W) - V(a_2)=W_0$.	$E(W) - V(a_2)=W_0$.	$E(W) - V(a_2)>W_0$.

If L is sufficiently low, $L < L^{C*}$, it has no effect on the contract. When LL is medium, $L^{C*} \leq L \leq L^{C**}$, it determines the lowest payment, C_B. The principal responds by reducing $C_{G\ell}$, leaving the agent's expected utility at his reservation-utility level, W_0. Reducing $C_{G\ell}$ is feasible as long as $C_{G\ell}$ exceeds LL. When LL is high, $L > L^{C**}$, this option is exhausted, both C_B and $C_{G\ell}$ are set at L, and the agent enjoys an LL rent (his expected utility exceeds his reservation-utility level, W_0).

3.2. The Renegotiation-Proof Contract (RPC))

Consider now a contract where the principal and agent can renegotiate the contract on Date 4, after the agent completed exerting effort and before outcome is communicated to the monitor. Renegotiation implies that the principal can replace the gambles according to the original contract by a fixed payment that provides the risk-averse agent the same payoff. At the same time, the cost of the contract is lowered by the risk premium of the agent. Since the players anticipate renegotiation

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[10] The compensation upon a report of G is a lottery based on the public signal, s, as follows:
$C_{Gh} = W^{-1}[v - (\pi - Q)W(C_{G\ell}) + \pi W(L)]/Q > C_{G\ell}$.
$C_{G\ell} = W^{-1}(K_1 - K_2 W(L))$, where:
$K_1 = \frac{Q(W_0+V(a_2))-[\theta_2\rho+(1-\pi)(1-\theta_2)(1-\rho)]\nu}{(1-\pi)(2\rho-1)}$; $K_2 = \frac{\pi\rho}{(1-\pi)(2\rho-1)>0}$.

when they originally sign a contract, they can incorporate the provisions prompted by renegotiation in the contract designed on Date 1, signing, thus, a renegotiation-proof contract ([Fudenberg and Tirole, 1990], Lemma 1). As the rich literature on renegotiation-proof contracts established, renegotiation turns the game into a screening game[11] in which the principal induces the agent to reveal his private knowledge of the outcome by designing two options, one for G and one for B, so that the agent's choice of a specific option reveals what he knows. The principal can monitor the agent's choice by comparing it to the post-outcome report of the monitor. While a profile of choice and a corroborating audited report may hide successful misrepresentation, a contradicting audited report indicates that the agent attempted to misrepresent and the principal can penalize him, as long as the penalty exceeds LL. Denoting the option by $\hat{x}$, $\hat{x} \in \{\hat{B}, \hat{G}\}$, and by S, $S = \{x - S_{\hat{x}_{rs}}\}$, the Renegotiation-Proof Contract (RPC), the principal designs S by solving the following S-program:

$$\max_{\{x - S_{\hat{x}_{rs}}\}} E_{x,r,s}(x - S_{\hat{x}_{rs}}).$$

(PC) obtains.

If

$$x = B, E_{r,s}(W(S_{\hat{x}_{rs}})|B) \geq E_{r,s}(W(S_{\hat{x}_{rs}})|B). \qquad (IC.\hat{B}).$$

If

$$x = G, E_{r,s}(W(S_{\hat{x}_{rs}})|G) \geq E_{r,s}(W(S_{\hat{x}_{rs}})|G) \qquad (IC.\hat{G}).$$

$$E_{x,r,s}(W(S_{\hat{x}_{rs}})|a_2, \hat{x} = x) \geq E_{x,r,s}(W(S_{\hat{x}_{rs}})|a_1).. \qquad (IC.a\hat{x}).$$

The first two new constraints are the incentive-compatibility constraints, $(IC.\hat{B})$ and $(IC.\hat{G})$ stating that once the outcome is realized, the agent (weakly) prefers the option associated with the true outcome. The next constraints involve four independent constraints: one is (MH) stating that the agent refers to exert the higher level of effort given that he self-selects the option honestly. The other three constraints are "double shirking constraints": When the agent chooses the effort, a, he prefers to work harder, $a = a_2$, and select the correct option, $\hat{x} = x$, to shirking an effort, $a = a_1$, and mis-choosing at least one option, just $\hat{B}$, just $\hat{G}$, or both. The last is the (LC) constraint.

Lemma 2. *Neither* $(IC.\hat{B})$ *nor the double-shirking constraints are binding.*

The intuition of Lemma 2 is straightforward. For incentives purposes, the expected payment for the option associated with G is greater than the one associated with B. The agent, then, is willing to choose $\hat{G}$ with no further inducements. The double shirking constraints hold because they are implied by (MH) and the post-outcome renegotiation constraints.

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[11] See, e.g., [Cremer, 1995], [Fudenberg and Tirole, 1990], [Ma, 1991], and [Matthews, 1995].

Lemma 2 implies that RPC can be summarized by a quadruple $(S_{\hat{B}}, S_{\hat{G}_\ell}, S_{\hat{G}_h}; L)$, where $S_{\hat{B}}$,$S_{\hat{G}_\ell}$, $S_{\hat{G}_h}$ are actual payments and L is a penalty upon discovery of misrepresentation.

Denoting the Lagrange multiplier of $(IC.\hat{B})$ by $\eta^{\hat{B}}$ the first-order conditions are as follows:

$$S_{\hat{B}B} : \frac{1}{W'(S_B)} = \lambda - \frac{\mu - \varphi_B + \eta^{\hat{B}}}{1-\theta_2},$$

$$S_{\hat{G}G} : \frac{1}{W'(S_{\hat{G}G\ell})} = \lambda + \frac{(1-\rho)\mu + \varphi_{G\ell}\rho - (1-\pi)\rho\eta^{\hat{B}}}{\theta_2(1-\rho) + (1-\theta_2)(1-\pi)\rho} > (<) S_{\hat{B}B},$$

$$S_{\hat{G}G} : \frac{1}{W'(S_{\hat{G}G\ell})} = \lambda + \frac{\rho\mu - (1-\pi)(1-\rho)\eta^{\hat{B}}}{\theta_2\rho + (1-\theta_2)(1-\pi)(1-\rho)} > S_{\hat{G}G}.$$

The first-order conditions show that LL affects the contract: if it is sufficiently low, $\lambda > g$, $\eta^{\hat{B}} = 0$, for medium levels, $\lambda > g$, $\eta^{\hat{B}} > 0$, and for high levels, $\lambda = g$, $\eta^{\hat{B}} > 0$. The reason follows from the above discussion. If LL is lower than the penalty required to ensure that the agent costlessly chooses when he observes B, the principal can push the agent to his reservation utility level. When LL is not sufficiently low, the principal has to offer the agent inducement to choose, but because he has flexibility over the contract, he can fine-tune it so that the agent still does not earn more than his reservation utility level. When, however, LL is high, the flexibility is exhausted and the agent enjoys a limited-liability rent.

Proposition 2. *Identify two critical levels of LL payments, $L^{R*}, L^{R**}, L^{R*} < L^{R**}$, such that LL is <u>low</u> when $L < L^{R*}$ <u>medium</u> – when $L^{R*} \leq L \leq L^{R**}$, and <u>high</u> – when $L > L^{R**}$. Then,*

Case(a): *When LL is low, $L < L^{R*}$, the RPC is a quadruple ($S_{\hat{B}}$, $S_{\hat{G}}$, $P(\hat{B})$, $P(\hat{B})$ where along the equilibrium play, the payments coincide with unlimited-liability MOC: $S_{\hat{B}} = M_B$, $S_{\hat{G}} = M_G$, with out-of-equilibrium penalties, $P(\hat{B}) = P(\hat{G}) = L$.*

$$L^{R*} = L^{C*}.$$

Case (b): *When LL is medium, $L^{R*} \leq L < L^{R**}$, the RPC is a quintuple (($S_{\hat{G}h}$, $S_{\hat{G}\ell}$, $S_{\hat{B}}$, $P(\hat{B})$, $P(\hat{G})$), where along the equilibrium play, the G-option's payments are the same as in case (b) of the NCC, i.e., the B-option's payment is the same as in MOC, with $P(\hat{B}) = P(\hat{G}) = L$. The agent's expected utility equals his reservation utility, W_0, i.e., $E_{x,r,s}(W(S_{\hat{x}})) - V(a_2) = W_0$. $L^{R**} = L^{C*}$*

Case (c): *When LL is high, $L \geq L^{R**}$, RPC is a quintuple, $S_{\hat{G}h}$, $S_{\hat{G}\ell}$, $S_{\hat{B}}$, $P(\hat{B}), P(\hat{G})$, where along the equilibrium play, the G-option's payments are the same as in case (c) of the NCC, i.e., $S_{\hat{G}h} = C_{Gh}; S_{\hat{G}l} = C_{Gl}$; the B-option's payment is the certainty equivalent of case (c) of the NCC, conditional on $x = B$, i.e., $S_{\hat{B}} = W^{-1}(W(L) + (1-\pi)(1-\rho)\frac{\nu}{Q})$ with $P(\hat{B}) = P(\hat{G}) = L$. The agent obtains the same* *expected utility as in case (c) of the NCC, i.e.,* $E_{x,r,s}(W(S_{\hat{x}})) - V(a_2) = W(L)+$ $+Pr(r = G, s = h)\frac{\nu}{Q} - V(a_2) > W_0$.

Proposition 2 demonstrates the dramatic effect of LL on RPC. Specifically, when LL is:

Low: RPC = MOC.

Medium: RPC = NCC upon success, $x = G$, RC = MOC upon a failure, $x = B$.

High: RPC resembles NCC.

When LL is low, the actual payments are identical to those specified in a contract designed by a principal who observes the outcome. The only difference is that RPC specifies a penalty triggered by an audited report's discovery that the agent misrepresented of G)[12].

When LL is medium or high, the payment upon a success are the same for NCC and RPC and the payment upon a failure is the certainty equivalence of the expected payment of the agent upon a failure in NCC. Since the agent is risk-averse, this reduces the cost of the contract, so that RPC is always Pareto improvement over NCC.

Another interesting result in Proposition 3 is that L^{C*} and L^{C**} of the NCC divide the range of possible LL levels in RPC as well. We find this result perplexing in light of the fact that the NCC and RPC are different regarding the principal's knowledge of the outcome. The similarity is explained by the agent's LL. It prevents the principal from using his ability to elicit the truth from the agent to reduce the information rent of the privately informed agent (who alone observes outcome). Hence, although the public signal is not informative in the sense of Holmstrom (1979) it is still used by the RPC when LL level is medium or higher and the outcome is a success.[13]

Table 5 illustrates Propositions 1 and 2.

Proposition 3. *RPC is less costly than NCC, and although the public signal is not informative, RPC incorporates it in the contract when LL level is at least medium.*

Proposition 3 is a corollary to Proposition 2. RPC is Pareto improvement over NCC because the principal pays the agent less while the agent obtains the same expected utility under both contracts. The public signal is a noisy signal of outcome while the option reveals outcome perfectly. Hence, it is not informative in the sense of Holmstrom (1979)). Still, because of LL it is incorporated in the limited liability contract when it LL matters: LL is medium or high. We conclude this section with numerical examples that highlight the differences between the three cases of the two contracts. The parameters are as follows: The agent's utility function is $W(Z) = W_0 = 9.25$, $V(a_2) = 0.75$, and $V(a_1) = 0$; $\theta_2 = 0.65$, $\theta_1 = 0.55$, $\pi = 0.75$ and $\rho = 0.8$.

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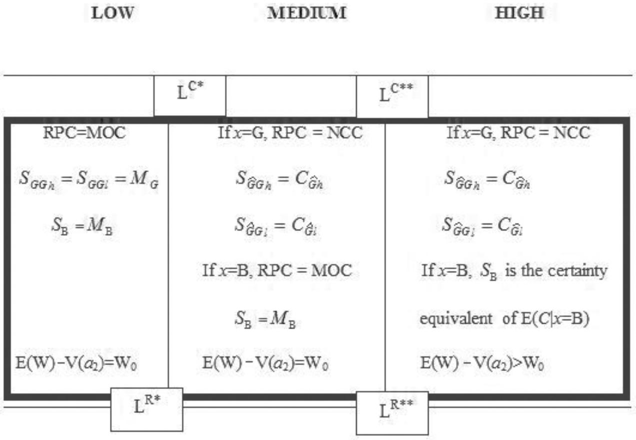
[12] As the numerical examples below show, a penalty payment does not necessarily involve a payment from the agent to the principal. When the penalty is a transfer from the principal, the sentence should read: "Since LL is low, the principal can set the transfer at a level sufficiently low not to need to pay for the elicitation of the agent's private information'.'

[13] That is, $Pr(\hat{x}, s|a) = Pr(\hat{x}, s)Pr(\hat{x}, a)$. See [Holmstrom, 1979].

Table 5: RPC



LOW	MEDIUM	HIGH
RPC=MOC $S_{GGh} = S_{GGl} = M_G$ $S_B = M_B$ $E(W) - V(a_2) = W_0$	If x=G, RPC = NCC $S_{\hat{G}Gh} = C_{\hat{G}h}$ $S_{\hat{G}Gl} = C_{\hat{G}l}$ If x=B, RPC = MOC $S_B = M_B$ $E(W) - V(a_2) = W_0$	If x=G, RPC = NCC $S_{\hat{G}Gh} = C_{\hat{G}h}$ $S_{\hat{G}Gl} = C_{\hat{G}l}$ If x=B, S_B is the certainty equivalent of $E(C\|x=B)$ $E(W) - V(a_2) > W_0$

Table 6: Numerical examples[14]

	NCC	**RPC**
Low LL	117.636	112.80
(L=0)	(9.25)	(9.25)
Medium LL	119.70	117.71
(L=16)	(9.25)	(9.25)
High LL	154.25	152.5875
(L=36)	(10.625)	(10.625)

[14] $C_{Gh} = 176.39$, $C_{Gl} = 100$, $C_B = 10.77$, $S_{\hat{G}G} = 159.39$, $S_{\hat{B}} = 26.2656$, $P(\hat{B}) < 17.5$,
$C_{Gh} = 196$, $C_{Gl} = 50.7656$, $C_B = 16$. $S_{\hat{G}Gh} = 196$, $S_{\hat{G}Gl} = 50.7656$, $S_{\hat{B}} = 26.2656 > P(\hat{B}) = 16$,
$C_{Gh} = 256$, $C_{Gl} = C_B = 36$. $S_{\hat{G}Gh} = 256$, $S_{\hat{G}Gl} = P(\hat{B}) = 36$, $S_{\hat{B}} = 42.25 > P(\hat{B})$,

4. The contracts with pre-decision signals

So far, we are concerned with post-outcome signal only. In this section, we add a pre-decision signal. We assume that prior to choosing effort, the agent alone observes a signal, s, which could be either unfavorable, u, or favorable, f; i.e., $s \in\{u,f\}$. It is common knowledge that the prior probability that the signal is favorable is γ, $\gamma = Prob[s = f]$. The manager alone observes a signal on the probability of outcome, s. It is common knowledge that every manager privately observes this signal. The principal then has the option to design a contract based on the communication, m, of the agent of this signal. We revise the model so that the prior probability that the outcome is x_2 conditional on signal s and effort a by θ^s_a, $\theta^s_a = Pr[x_2|s, a]$. We assume that:

(i) the higher the effort, the higher the expected outcome, $\theta^s_1 < \theta^s_2$, $s = uf$;

(ii) the favorable signal is good news and the unfavorable signal is bad news, $E(x - t(\cdot)|s = f) > E(x - t(\cdot)|s = u)$;

(iii) the productivity of effort is higher when the signal is favorable, $\theta^f_2 - \theta^f_1 > \theta^u_2 - \theta^u_1$;

(iv) for each signal, the owners are better off inducing the higher level of effort, a_2.

Proposition 4.

(a) *Neither RPC nor NCC when LL is medium or high is affected by the availability of the pre-decision signal.*

(b) *NCC uses the pre-decision signal when LL is low*[15]*. It reduces the cost of the NCC to the principal.*

Proposition 4 shows that a pre-decision signal is redundant in RPC and has some use in NCC only if LL is low. Hence, unlike the case with the post-decision signal, we find that limited liability depress the demand for a pre-decision signal.

5. Summary

In many principal–agent relationships, the principal must design a contract that takes into account the limited liability of the agent. We study the effect of his limited liability on the equilibrium contract when the agent alone observes outcome. We find that limited liability:

– increases the cost of the contract even when the agent obtains no more than his reservation utility level – the agent does not enjoy a "limited-liability rent";

– may induce demand for a post-outcome signal that is non-informative in Holmstrom's (1979) sense, to compensate for the loss of flexibility in designing a contract that is caused by the agent's limited liability; and

– suppresses the demand for a pre-decision signal.

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[15] For details of unlimited-liability NCC, consult Ronen and Yaari (2002).

Acknowledgements:

We are grateful to the participants in the First World Congress of the Game Theory Society, Bilbao, Spain, July 2000, where we presented our project, Renegotiation-Proof Principal–Agent Contracts with Unobservable Outcome. Given early criticism that our model was somewhat too rich, this project yielded two papers. One that attempts to explain why a principal will allow the agent to manage earnings rather than report the truth when the assumptions of the Revelation principle do not seem to be violated (The Demand for the truth, published in Review of Accounting Studies), for which we are indebted to the editor, Jerry Feltham, and this one that explores the value of additional signals in a binary principal–agent setting with an unobservable outcome. The second author dedicates this paper to her late father, Gabriel Lewinstein.

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On the Construction of the Characteristic Function in Cooperative Differential Games with Random Duration

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Abstract. The class of cooperative differential games with random duration is studied. The problem of characteristic function construction is researched. The Hamilton–Jacobi–Bellman equation for the problem with random duration is derived. The method of calculating the characteristic function values with the help of given equation is represented as algorithm. The results are illustrated with the examples.

Keywords: Cooperation, differential games, random duration, characteristic function, the Hamilton–Jacobi–Bellman equation, resource extraction.

Introduction

In differential game theory the common way is to consider cooperative differential games with prescribed duration or infinite time horizon. However it seems to be more realistic to expect the end of a game in a random time instant. Therefore, the class of cooperative differential games with random duration was purposed to introduce in [Petrosjan and Shevkoplyas, 2003]. We start by introducing a definition of the game in section 1.

In cooperative differential game players seek to solve the optimal control problem such that maximization of total payoff is subject to a number of constraints, in particular, a differential equation describing the evolution of the state of the game. One of the basic solution techniques for optimal control problem is the Hamilton–Jacobi–Bellman equation [Dockner, 2000]. But we have non-standard dynamical programming problem for games with random duration because of the objective functional form (double integral), that is why till recently the Hamilton–Jacobi–Bellman equation appropriates for using in cooperative differential games with random duration didn't exist. In section 2 we derive the Hamilton–Jacobi–Bellman equation for the problem with random duration in general case for arbitrary probability frequency distribution $f(t) = F'(t)$. In section 3 we discuss methods of the characteristic function construction such classical solution by Neumann and Morgenstern [Vorobyev, 1985] and non-standard Nash equillibrium approach [Petrosjan, 2003]. The main idea of this approach is an assumption that if a subset of players form a coalition then the left-out players stick to their feedback Nash strategies. We represent an algorithm of the characteristic function value calculating based on Petrosjan and Zaccour concept and new Hamilton–Jacobi–Bellman equation derived in section 2.

At last we consider 2 applications of our theory. In section 4 we consider the first example of 3-person cooperative differential game with random duration. In this example open-loop and feedback solutions coincide. In section 5 we research one simple model of common-property nonrenewable resource such as an oil field which had been studied in [Dockner, 2000] with the assumption of infinite time horizon. We consider this problem under condition of a random game duration. Now we use the new Hamilton–Jacobi–Bellman equation to solve the cooperative game of nonrenewable resource under condition of random duration with exponential probability distribution.

1. Definition of the game

Consider n-person differential game $\Gamma(x_0)$ from the initial state x_0 with random duration $T - t_0$. Here the random variable T with distribution function $F(t)$, $t \in \in [t_0, \infty)$,

$$\int_{t_0}^{\infty} dF(t) = 1,$$

is the time instant, when the game $\Gamma(x_0)$ ends. The game starts at the moment t_0 from a position x_0.
Let the motion equations have the form

$$\dot{x} = g(x, u_1, \dots, u_n),\ x \in R^n,\ u_i \in U \subseteq \operatorname{comp} R^l, \qquad (1)$$
$$x(t_0) = x_0.$$

The "instantaneous" payoff at the moment τ, $\tau \in [t_0, \infty)$ is defined as $h_i(x(\tau))$. Then the expected integral payoff of the player i, $i = 1, \dots, n$ is evaluated by the formula

$$K_i(x_0, u_1, \dots, u_n) = \int_{t_0}^{\infty} \int_{t_0}^{t} h_i(x(\tau)) d\tau dF(t), \quad h_i \geq 0, \quad i = 1, \dots, n. \qquad (2)$$

Let $x^*(t)$ and $u^*(t) = (u_1^*(t), \dots, u_n^*(t))$ be the cooperative trajectory and the corresponding n-tuple of open loop controls maximizing the joint expected payoff of players (we suppose that maximum is attained):

$$\max_u \sum_{i=1}^{n} K_i(x_0, u_1, \ldots, u_n) = \sum_{i=1}^{n} K_i(x_0, u_1^*, \ldots, u_n^*) =$$
$$= \sum_{i=1}^{n} \int_{t_0}^{\infty} \int_{t_0}^{t} h_i(x^*(\tau)) d\tau dF(t) = V(I, x_0). \qquad (3)$$

Trajectory $x^*(t)$ and open loop controls $u^*(t) = (u_1^*(t), \ldots, u_n^*(t))$ are called optimal. For simplicity we shall suppose further, that the optimal trajectory is unique. The following will be true for each optimal trajectory.

For the set of subgames $\Gamma(x^*(\vartheta))$ occurring along an optimal trajectory $x^*(\vartheta)$ one can similarly define the expected total integral payoff in cooperative game $\bar{\Gamma}(x^*(\vartheta))$:

$$V(I, x^*(\vartheta)) = \sum_{i=1}^{n} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} h_i(x^*(\tau)) d\tau \, dF_\vartheta(t). \qquad (4)$$

It is clear, that $(1 - F(\vartheta))$ is the probability to start $\Gamma(x^*(\vartheta))$.

Then we have the conditional distribution function $F_\vartheta(t)$ as follows:

$$F_\vartheta(t) = \frac{F(t) - F(\vartheta)}{1 - F(\vartheta)}, \quad t \in [\vartheta, \infty). \qquad (5)$$

In the same way we get the expression for conditional distribution in subgames $\Gamma(x^*(\vartheta + \Delta))$:

$$F_{\vartheta+\Delta}(t) = \frac{F_\vartheta(t) - F_\vartheta(\vartheta + \Delta)}{1 - F_\vartheta(\vartheta + \Delta)} = \frac{F(t) - F(\vartheta + \Delta)}{1 - F(\vartheta + \Delta)}. \qquad (6)$$

Further we assume an existence of a density function $f(t) = F'(t)$. As above we get the formula for conditional density function:

$$f_\vartheta(t) = \frac{f(t)}{1 - F(\vartheta)}, \; t \in [\vartheta, \infty); \; f_{\vartheta+\Delta}(t) = \frac{f(t)}{1 - F(\vartheta + \Delta)}, \; t \in [\vartheta + \Delta, \infty). \qquad (7)$$

From (7) we obtain

$$f_\vartheta(t) = \frac{1 - F(\vartheta + \Delta)}{1 - F(\vartheta)} f_{\vartheta+\Delta}(t). \qquad (8)$$

This is needed for the sequel.

2. The Hamilton–Jacobi–Bellman equation

The Hamilton–Jacobi–Bellman equation lies at the heart of the dynamic programming approach to optimal control problems. Let us remark that the functional (4) doesn't have the standard form for dynamic programming problem. Thus, we need to derive the Hamilton–Jacobi–Bellman equation appropriates for the problem with random duration.

We denote $H(x(t)) = \sum\limits_{i=1}^{n} h_i(x(t))$. In general case we consider $H(x, u)$.

Let $P(x, \vartheta)$ be an optimization problem

$$\max_u \left[\int_{\vartheta}^{\infty} f_{\vartheta}(t) \int_{\vartheta}^{t} H(x(\tau), u(\tau)) d\tau dt \right], \tag{9}$$

subject to $x(t)$ satisfies (1), $x(\vartheta) = x$.

Let $W(x, \vartheta)$ be the optimal value (or Bellman function) of the objective functional of problem $P(x, \vartheta)$ in (9):

$$W(x, \vartheta) = \max_u \left[\int_{\vartheta}^{\infty} f_{\vartheta}(t) \int_{\vartheta}^{t} H(x(\tau), u(\tau)) d\tau dt \right]. \tag{10}$$

We can see that the maximal total payoff in $\Gamma(x_0)$ is

$$V(I, x_0) = W(x_0, t_0).$$

In control theory one usually makes the assumptions that given functions g and H are sufficiently smooth and satisfy certain boundless conditions to ensure that solutions to (10) are uniquelly defined and integral in (10) make sense. Here we don't impose any strong restrictions because we can not easily assume any restrictive properties for the functions g and H, but we make an assumption to objective functional W to be well defined.

Clearly, if we behave optimally from $t + \Delta$ onwards, the total expected payoff is given by formula

$$W(x, \vartheta + \Delta) = \max_u \left[\int_{\vartheta+\Delta}^{\infty} f_{\vartheta+\Delta}(t) \int_{\vartheta+\Delta}^{t} H(x(\tau), u(\tau)) d\tau dt \right]. \tag{11}$$

Using (10), (8), (7) and (11), we get:

$$\begin{aligned} W(x, \vartheta) &= \max_u \int_{\vartheta}^{\vartheta+\Delta} f_{\vartheta}(t) \int_{\vartheta}^{t} H(x(\tau), u(\tau)) d\tau dt + \\ &+ \int_{\vartheta+\Delta}^{\infty} f_{\vartheta}(t) dt \int_{\vartheta}^{\vartheta+\Delta} H(x(\tau), u(\tau)) d\tau + \\ &+ \frac{1 - F(\vartheta + \Delta)}{1 - F(\vartheta)} \int_{\vartheta+\Delta}^{\infty} f_{\vartheta+\Delta}(t) \int_{\vartheta+\Delta}^{t} H(x(\tau), u(\tau)) d\tau \, dt = \\ &= \max \Big(\int_{\vartheta}^{\vartheta+\Delta} f_{\vartheta}(t) \int_{\vartheta}^{t} H(x(\tau), u(\tau)) d\tau \, dt + \\ &+ \frac{1 - F(\vartheta + \Delta)}{1 - F(\vartheta)} \int_{\vartheta}^{\vartheta+\Delta} H(x(\tau), u(\tau)) d\tau + \frac{1 - F(\vartheta + \Delta)}{1 - F(\vartheta)} W(x(\vartheta + \Delta), \vartheta + \Delta) \Big). \end{aligned} \tag{12}$$

Note,

$$\frac{1-F(\vartheta+\Delta)}{1-F(\vartheta)} = 1 + \frac{F(\vartheta)-F(\vartheta+\Delta)}{1-F(\vartheta)}. \tag{13}$$

Now substract $W(x,\vartheta)$ from both sides of (12) and divide the resulting equation by Δ. This yelds

$$\begin{aligned} 0 = \max_u \Bigg(& \frac{1}{\Delta}\int_\vartheta^{\vartheta+\Delta} f_\vartheta(t)\int_\vartheta^t H(x(\tau),u(\tau))d\tau dt + \\ & + \frac{1}{\Delta}\int_\vartheta^{\vartheta+\Delta} H(x(\tau),u(\tau))d\tau + \frac{F(\vartheta)-F(\vartheta+\Delta)}{1-F(\vartheta)}\frac{1}{\Delta}\int_\vartheta^{\vartheta+\Delta} H(x(\tau),u(\tau))d\tau + \\ & + \frac{W(x(\vartheta+\Delta),\vartheta+\Delta)-W(x(\vartheta),\vartheta)}{\Delta} + \frac{F(\vartheta)-F(\vartheta+\Delta)}{1-F(\vartheta)}\frac{1}{\Delta}W(x(\vartheta+\Delta),\vartheta+\Delta)\Bigg). \end{aligned} \tag{14}$$

Let $\Delta \to 0$. From the mean value theorem we know that

$$\lim_{\Delta->0}\frac{1}{\Delta}\int_\vartheta^{\vartheta+\Delta} f_\vartheta(t)\int_\vartheta^t H(x(\tau),u(\tau))d\tau dt = 0. \tag{15}$$

Moreover, we have

$$\begin{aligned} & \lim_{\Delta->0}\frac{1}{\Delta}\int_\vartheta^{\vartheta+\Delta} H(x(\tau),u(\tau))d\tau = H(x(\vartheta),u(\vartheta)); \\ & \lim_{\Delta->0}\frac{1}{\Delta}\frac{F(\vartheta)-F(\vartheta+\Delta)}{1-F(\vartheta)}\int_\vartheta^{\vartheta+\Delta} H(x(\tau),u(\tau))d\tau = 0; \\ & \lim_{\Delta->0}\frac{1}{\Delta}\frac{F(\vartheta)-F(\vartheta+\Delta)}{1-F(\vartheta)} = -\frac{F'(\vartheta)}{1-F(\vartheta)} = -\frac{f(\vartheta)}{1-F(\vartheta)}. \end{aligned} \tag{16}$$

Combining (14), (15) and (16), we obtain

$$\begin{aligned} 0 = \max_u \Bigg(& H(x(\vartheta),u(\vartheta)) + \frac{d}{d\vartheta}W(x,\vartheta) + \\ & + \lim_{\Delta->0}\left[\frac{1}{\Delta}\frac{F(\vartheta)-F(\vartheta+\Delta)}{1-F(\vartheta)}W(x(\vartheta+\Delta),\vartheta+\Delta)\right]\Bigg). \end{aligned}$$

Finally, we have the Hamilton -Jacobi-Bellman equation:

$$\frac{f(\vartheta)}{1-F(\vartheta)}W(x,\vartheta) = \frac{\partial W(x,\vartheta)}{\partial\vartheta} + \max_u\left[H(x(\vartheta),u(\vartheta)) + \frac{\partial W(x,\vartheta)}{\partial x}g(x,u)\right]. \tag{17}$$

Suppose that the final time instant T has the exponential distribution:

$$\begin{aligned} f(t) = \rho e^{-\rho(t-t_0)}, \quad F(t) = 1 - e^{-\rho(t-t_0)} \quad \text{if} \quad t \geq t_0, \\ F(t) = f(t) = 0 \quad \text{if} \quad t < t_0. \end{aligned} \tag{18}$$

Then the Bellman function is as follows

$$W(x, \vartheta) = \max_{u}\left[\int_{\vartheta}^{\infty} \rho e^{-\rho(t-\vartheta)} \int_{\vartheta}^{t} H(x(\tau), u(\tau)) d\tau dt\right], \tag{19}$$

and the Hamilton–Jacobi–Bellman equation (17) has the form:

$$\rho W(x, t) = \frac{\partial W(t, x)}{\partial t} + \max_{u}\left\{H(x(t), u(t)) + \frac{\partial W(x, t)}{\partial x} g(x, u)\right\}. \tag{20}$$

This equation looks like Hamilton–Jacobi–Bellman equation for the problem with prescribed duration with discount factor ρ ([Dockner, 2000]). Let us remark that firstly the Hamilton–Jacobi–Bellman equation for the 2-person game of pursuits with random duration was derived in a paper ([Petrosjan, 1966]).

3. The characteristic function

Classical solution. The common way ([Vorobyev, 1985]) to define the characteristic function in $\Gamma(x_0)$ is as following:

$$V(S, x_0) = \begin{cases} 0, \quad S = \emptyset; \\ \max\limits_{u_S} \min\limits_{u_{N \setminus S}} \sum_{i \in S} K_i(x_0, u), \quad S \subset N, \\ \max\limits_{u} \sum_{i=1}^{n} K_i(x_0, u), \quad S = N. \end{cases} \tag{21}$$

Then $V(S, x_0)$ (21) is superadditive. But this approach doesn't seem to be the best in context of environmental or other problems, because unlikely that if a subset of players form a coalition to tackle an enviromental problem, then the remaining players would form an anti-coalition to harm their efforts. For enviromental problem we can use another method of characteristic function construction ([Petrosjan, 2003]) with assumption that left-out players stick to their feedback Nash strategies. This approach was proposed in ([Petrosjan, 2003]). Then we have the following definition of the characteristic function:

$$V(S, x^*(\vartheta)) = \begin{cases} 0, \quad S = \emptyset; \\ W_i(x^*(\vartheta), \vartheta), \quad i = 1, \ldots, n; \quad \{i\} \in I; \\ W_K(x^*(\vartheta), \vartheta), \quad K \subseteq I, \end{cases} \tag{22}$$

where $W_i(x^*(\vartheta), \vartheta), W_K(x^*(\vartheta), \vartheta)$ are the results of the corresponding Hamilton–Jacobi–Bellman equations. Consider an algorithm for computation of characteristic function values $V(S, x^*(\vartheta))$, $S \subset N$ based on Petrosjan and Zaccour approach with the help the new Hamilton–Jacobi–Bellman equation (17).

(1) Maximize the total expected payoff of the grand coalition I.

$$W_I(x, \vartheta) = \max_{u_i \in U} \frac{1}{1 - F(\vartheta)} \sum_{i=1}^{n} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} h_i(x(\tau)) d\tau dF(t), \tag{23}$$

$$x(\vartheta) = x.$$

Denote $\sum_{i=1}^{n} h_i(\cdot)$ by $H(\cdot)$. Then the Bellman function $W_I(x, \vartheta)$ satisfies the HJB equation (17). Results of optimization are optimal trajectory $x^*(t)$ and optimal strategies $u^* = (u_1^*, \dots, u_n^*)$.

(2) Calculate a feedback Nash equilibrium. Without cooperation each player i seeks to maximize his expected payoff (2). Thus, the player i solves a dynamic programming problem:

$$W_i(x, \vartheta) = \max_{u_i \in U} \frac{1}{1 - F(\vartheta)} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} h_i(x(\tau)) d\tau dF(t), \tag{24}$$
$$x(\vartheta) = x.$$

Denote $h_i(\cdot)$ by $H(\cdot)$. In this notation $W_i(x, \vartheta)$ satisfies the HJB equations (17) for all $i \in I$. Denote by $u^N(\cdot) = \{u_i^N(\cdot), \quad i = 1, \dots, n\}$ any feedback Nash equilibrium of this noncooperative game $\Gamma(x_0)$. Let the corresponding trajectory be $x^N(t)$. We calculate $W_i(x^*(\vartheta), \vartheta)$ under condition that before time instant ϑ players use their optimal strategies u_i^*.

(3) Compute outcomes for all remaining possible coalitions.

$$W_K(x, \vartheta) = \max_{u_i, i \in K} \frac{1}{1 - F(\vartheta)} \sum_{i \in K} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} h_i(x(\tau)) d\tau dF(t), \tag{25}$$
$$u_j = u_j^N \quad \text{for} \quad j \in I \setminus K,$$
$$x(\vartheta) = x.$$

Here we insert for the left-out players $i \in I \setminus K$ their Nash values (see step 2). In the notation $\sum_{i \in K} h_i(\cdot) = H(\cdot)$ the Bellman function $W_K(x, \vartheta)$ satisfies the corresponding HJB equation (17).

(4) Define the characteristic function $V(S, x^*(\vartheta))$, $\forall S \subseteq I$ as

$$V(S, x^*(\vartheta)) = \begin{cases} 0, \quad S = \emptyset; \\ W_i(x^*(\vartheta), \vartheta), \quad i = 1, \dots, n; \quad \{i\} \in I; \\ W_K(x^*(\vartheta), \vartheta), \quad K \subseteq I. \end{cases} \tag{26}$$

Let us remark that the constructed function $V(S, x^*(\vartheta))$ is not superadditive in general.

4. Example I

Now consider an example of 3-person cooperative differential game $\Gamma(z_0)$ with random duration $T - t_0$. The game starts at the moment t_0 from a position z_0.

Suppose the random variable T has a probability density function $\exp^{-t}$. The motion equations have the form

$$\begin{aligned} &\dot{z} = u + v + w, && (27)\\ &z = (x, y), \quad z(t_0) = z_0 = (x_0, y_0);\\ &u = \{u_1; u_2\}, \quad v = \{v_1; v_2\}, \quad w = \{w_1; w_2\},\\ &|u| \leq 1, \quad |v| \leq 1, \quad |w| \leq 1. && (28)\end{aligned}$$

The "instantaneous" payoff at the moment τ, $\tau \in [t_0, \infty)$ is defined as

$$h_i(z(\tau)) = a_i \cdot x(\tau) + b_i \cdot y(\tau) + c_i, \quad a_i, b_i, c_i \geq 0; \qquad (29)$$
$$a_i^2 + b_i^2 + c_i^2 \neq 0, \quad i = 1, 2, 3.$$

The expected integral payoff is evaluated by the formula

$$K_i(z_0, u, v, w) = \int\limits_{t_0}^{\infty} \int\limits_{t_0}^{t} h_i(z, u, v, w) \exp^{-t} d\tau dt, \quad i = 1, 2, 3. \qquad (30)$$

The cooperative form of the game $\Gamma(z_0)$ means that before the beginning of the game the players agree about usage by them such controls u^*, v^*, w^*, that the corresponding trajectory $z^*(t)$ will maximize the joint expected payoff of players, i.e.

$$\begin{aligned}\max_{u,v,w} \sum_{i=1}^{3} K_i(z_0, u, v, w) &= \sum_{i=1}^{3} K_i(z_0, u^*, v^*, w^*) =\\ &= \sum_{i=1}^{3} \int_{t_0}^{\infty} \int_{t_0}^{t} h_i(z^*(\tau)) \exp^{-t} d\tau dt. \qquad (31)\end{aligned}$$

Classical solution

The value Val $G_{S,I\backslash S}$ is defined as:

$$\text{Val } G_{S,I\backslash S} = \begin{cases} \max\limits_{u,v}\min\limits_{w} \left(K_1(.) + K_2(.)\right), \quad S = \{1,2\}, \quad I\backslash S = \{3\};\\ \max\limits_{u,w}\min\limits_{v} \left(K_1(.) + K_3(.)\right), \quad S = \{1,3\}, \quad I\backslash S = \{2\};\\ \max\limits_{v,w}\min\limits_{u} \left(K_2(.) + K_3(.)\right), \quad S = \{2,3\}, \quad I\backslash S = \{1\};\\ \max\limits_{u}\min\limits_{v,w} K_1(.), \quad S = \{1\}, \quad I\backslash S = \{2,3\};\\ \max\limits_{w}\min\limits_{u,w} K_2(.), \quad S = \{2\}, \quad I\backslash S = \{1,3\};\\ \max\limits_{w}\min\limits_{u,v} K_3(.), \quad S = \{3\}, \quad I\backslash S = \{1,2\}. \end{cases} \qquad (32)$$

From (31) we get the formula for the value $V(z_0, I)$ in the game $\Gamma(z_0)$:

$$V(z_0, I) = \max_{u,v,w} \int_{t_0}^{\infty} \int_{t_0}^{t} (a_{123} \cdot x(\tau) + b_{123} \cdot y(\tau) + \\ + c_{123}) \exp^{-t} d\tau dt, \qquad (33)$$

$$\text{where} \qquad a_{123} = a_1 + a_2 + a_3; \quad b_{123} = b_1 + b_2 + b_3; \\ c_{123} = c_1 + c_2 + c_3.$$

Apply maximum principle ([Pontryagin, 1976]) for calculating optimal open loop controls and corresponding trajectory. The functional which is being a subject to maximization is defined by (33). Consider the internal integral in (33). Let us begin to solve the problem:

$$J = \max_{u,v,w} \int_{t_0}^{t_f} (a_{123} \cdot x(\tau) + b_{123} \cdot y(\tau) + c_{123}) d\tau. \qquad (34)$$

Here t_f (t final) is some fixed t from interval $[t_0, \infty)$. Consider the dual problem:

$$J = \min_{u,v,w} \left(- \int_{t_0}^{t_f} (a_{123} \cdot x(\tau) + b_{123} \cdot y(\tau) + c_{123}) d\tau \right). \qquad (35)$$

Hamiltonian for (35) has the form

$$H = \psi_1(u_1 + v_1 + w_1) + \psi_2(u_2 + v_2 + w_2) + (a_{123} \cdot x(.) + b_{123} \cdot y(.) + c_{123}). \qquad (36)$$

Functions ψ_1, ψ_2 satisfy the following differential equations:

$$\frac{dx}{dt} = \frac{\partial H}{\partial \psi_1} = u_1 + v_1 + w_1; \qquad \frac{\partial \psi_1}{\partial t} = -\frac{\partial H}{\partial x} = -a_{123};$$
$$\frac{dy}{dt} = \frac{\partial H}{\partial \psi_2} = u_2 + v_2 + w_2; \qquad \frac{\partial \psi_2}{\partial t} = -\frac{\partial H}{\partial y} = -b_{123}$$

with initial conditions

$$x(t_0) = x_0; \quad y(t_0) = y_0; \qquad \psi_1(t_0) \quad \text{free}; \quad \psi_2(t_0) \quad \text{free};$$
$$x(t_f) \quad \text{free}; \quad y(t_f) \quad \text{free}; \qquad \psi_1(t_f) = 0; \quad \psi_2(t_f) = 0.$$

Assume further $t_0 = 0$.

Optimal controls can be calculated from the condition of maximizing H. As the controls are contained in H linearly, the maximum is reached on boundary. Moreover, taking into account constraints of the admissible controls for the system (28), we have

$$u_2 = \sqrt{1 - u_1^2}; \quad v_2 = \sqrt{1 - v_1^2}; \quad w_2 = \sqrt{1 - w_1^2}. \qquad (37)$$

Substituting expressions (37) into equation (36) for H and also taking a partial derivative, we get

$$u_1^* = \sqrt{\frac{\psi_1^2}{\psi_1^2+\psi_2^2}}; \qquad u_2^* = \sqrt{1-{u_1^*}^2} = \sqrt{\frac{\psi_1^2}{\psi_1^2+\psi_2^2}}. \tag{38}$$

By the similar way we get another optimal controls:

$$v_1^* = w_1^* = \sqrt{\tfrac{\psi_1^2}{\psi_1^2+\psi_2^2}}; \qquad v_2^* = w_2^* = \sqrt{1-{v_1^*}^2} = \sqrt{\tfrac{\psi_1^2}{\psi_1^2+\psi_2^2}}.$$

For $\psi_1(t)$, $\psi_2(t)$ we get the formulas:

$$\begin{aligned} \psi_1(t) &= -a_{123}\cdot t_f + a_{123}\cdot t = a_{123}\cdot(t-t_f); \\ \psi_2(t) &= -b_{123}\cdot t_f + b_{123}\cdot t = b_{123}\cdot(t-t_f). \end{aligned} \tag{39}$$

From (38) and (39) we get optimal controls:

$$u_1^* = v_1^* = w_1^* = \frac{a_{123}}{\sqrt{a_{123}^2+b_{123}^2}}; \qquad u_2^* = v_2^* = w_2^* = \frac{b_{123}}{\sqrt{a_{123}^2+b_{123}^2}}. \tag{40}$$

Note, that the resulting optimal controls are stationary! Thus, the usage of maximum principle for internal integral (34) is well defined. From (27) we get

$$x(t) = (u_1+v_1+w_1)\cdot t + x_0; \qquad y(t) = (u_2+v_2+w_2)\cdot t + y_0,$$

and then for the optimal trajectory:

$$x^*(t) = \frac{3a_{123}}{\sqrt{a_{123}^2+b_{123}^2}}\cdot t + x_0; \qquad y^*(t) = \frac{3b_{123}}{\sqrt{a_{123}^2+b_{123}^2}}\cdot t + y_0. \tag{41}$$

Now we can calculate the value of the functional (33) along the optimal trajectory. Thus, we shall define the value of the characteristic function $V(z_0, I)$ (in a case of all players cooperation). Outgoing from (41), we have

$$\sum_{i=1}^{3} h_i(z^*(t)) = 3\sqrt{a_{123}^2+b_{123}^2}\cdot t + a_{123}\cdot x_0 + b_{123}\cdot y_0 + c_{123}.$$

It is not difficult to show, that the following formula is true:

$$\begin{aligned} V(z_0, I) &= \int_0^{\infty}\int_0^{t}\sum_{i=1}^{3} h_i(z^*(\tau))\exp^{-t} d\tau dt = \\ &= 3\sqrt{a_{123}^2+b_{123}^2} + a_{123}\cdot x_0 + b_{123}\cdot y_0 + c_{123}. \end{aligned} \tag{42}$$

Hence,

$$V(z_0, I) = 3\sqrt{a_{123}^2 + b_{123}^2} + \sum_{i=1}^{3} h_i(z_0). \tag{43}$$

Applying maximum principle for functionals (32), we get the expressions for the characteristic function:

$$\begin{aligned} V(z_0, \{i,j\}) &= \sqrt{a_{ij}^2 + b_{ij}^2} + h_i(z_0) + h_j(z_0), \qquad i,j = 1,2,3; \quad i<j; \\ V(z_0, \{i\}) &= -\sqrt{a_i^2 + b_i^2} + h_i(z_0), \qquad i = 1,2,3. \end{aligned} \tag{44}$$

For simplicity rename the characteristic function $V(z_0, S)$ as $V\{S\}$. It is not difficult to show that the property of convexity ([Vorobyev, 1985]) of the characteristic function

$$V(S_1 \cup S_2) \geq V(S_1) + V(S_2) - V(S_1 \cap S_2), \quad \forall\, S_1, S_2 \subset I \tag{45}$$

is satisfied. It means, that c-core defined for the characteristic function $V(z_0, S)$, $S \subseteq I$ in the game $\Gamma(z_0)$ is not empty and it contains the Shapley Value.

Calculate the Shapley Value in the game $\Gamma(z_0)$. Substituting the values for $V(S)$ from (44), we get

$$\begin{aligned} Sh_1 &= -\frac{1}{3}\sqrt{a_1^2 + b_1^2} + \frac{1}{6}\sqrt{a_2^2 + b_2^2} + \frac{1}{6}\sqrt{a_3^2 + b_3^2} + \frac{1}{6}\sqrt{a_{12}^2 + b_{12}^2} + \\ &+ \frac{1}{6}\sqrt{a_{13}^2 + b_{13}^2} - \frac{1}{3}\sqrt{a_{23}^2 + b_{23}^2} + \sqrt{a_{123}^2 + b_{123}^2} + h_1(z_0), \\ Sh_2 &= -\frac{1}{3}\sqrt{a_2^2 + b_2^2} + \frac{1}{6}\sqrt{a_1^2 + b_1^2} + \frac{1}{6}\sqrt{a_3^2 + b_3^2} + \frac{1}{6}\sqrt{a_{12}^2 + b_{12}^2} + \\ &+ \frac{1}{6}\sqrt{a_{23}^2 + b_{23}^2} - \frac{1}{3}\sqrt{a_{13}^2 + b_{13}^2} + \sqrt{a_{123}^2 + b_{123}^2} + h_2(z_0), \\ Sh_3 &= -\frac{1}{3}\sqrt{a_3^2 + b_3^2} + \frac{1}{6}\sqrt{a_1^2 + b_1^2} + \frac{1}{6}\sqrt{a_2^2 + b_2^2} + \frac{1}{6}\sqrt{a_{13}^2 + b_{13}^2} + \\ &+ \frac{1}{6}\sqrt{a_{23}^2 + b_{23}^2} - \frac{1}{3}\sqrt{a_{12}^2 + b_{12}^2} + \sqrt{a_{123}^2 + b_{123}^2} + h_3(z_0). \end{aligned} \tag{46}$$

In the similar way we get the expressions for the characteristic function in the subgame $\Gamma(z_0)$ starting at the moment ϑ as following:

$$\begin{aligned} V(z^*(\vartheta), \{i,j\}) &= \sqrt{a_{ij}^2 + b_{ij}^2} + h_i(z^*(\vartheta)) + h_j(z^*(\vartheta)), \qquad i,j = 1,2,3; \quad i<j; \\ V(z^*(\vartheta), \{i\}) &= -\sqrt{a_i^2 + b_i^2} + h_i(z^*(\vartheta)), \qquad i = 1,2,3, \end{aligned} \tag{47}$$

and then we get the Shapley Value for the subgame $\Gamma(z^*(\vartheta))$:

$$\begin{aligned} Sh_1^{\vartheta} &= -\frac{1}{3}\sqrt{a_1^2 + b_1^2} + \frac{1}{6}\sqrt{a_2^2 + b_2^2} + \frac{1}{6}\sqrt{a_3^2 + b_3^2} + \frac{1}{6}\sqrt{a_{12}^2 + b_{12}^2} + \\ &+ \frac{1}{6}\sqrt{a_{13}^2 + b_{13}^2} - \frac{1}{3}\sqrt{a_{23}^2 + b_{23}^2} + \sqrt{a_{123}^2 + b_{123}^2} + h_1(z^*(\vartheta)), \end{aligned}$$

$$\begin{aligned} Sh_2^{\vartheta} &= -\frac{1}{3}\sqrt{a_2^2+b_2^2}+\frac{1}{6}\sqrt{a_1^2+b_1^2}+\frac{1}{6}\sqrt{a_3^2+b_3^2}+\frac{1}{6}\sqrt{a_{12}^2+b_{12}^2}+ \\ &+\frac{1}{6}\sqrt{a_{23}^2+b_{23}^2}-\frac{1}{3}\sqrt{a_{13}^2+b_{13}^2}+\sqrt{a_{123}^2+b_{123}^2}+h_2(z^*(\vartheta)), \end{aligned}$$

$$\begin{aligned} Sh_3^{\vartheta} &= -\frac{1}{3}\sqrt{a_3^2+b_3^2}+\frac{1}{6}\sqrt{a_1^2+b_1^2}+\frac{1}{6}\sqrt{a_2^2+b_2^2}+\frac{1}{6}\sqrt{a_{13}^2+b_{13}^2}+ \\ &+\frac{1}{6}\sqrt{a_{23}^2+b_{23}^2}-\frac{1}{3}\sqrt{a_{12}^2+b_{12}^2}+\sqrt{a_{123}^2+b_{123}^2}+h_3(z^*(\vartheta)). \end{aligned}$$

Nash equilibrium approach

To construct the characteristic function we use steps 1–4 of the algorithm. Clearly, the HJB equation (17) has the form:

$$\begin{aligned} W(x,y,t) &= \frac{\partial W(x,y,t)}{\partial t}+ \\ &+ \max_u\Bigg(H(x(t),u(t))+\frac{\partial W(x,y,t)}{\partial x}(u_1+v_1+w_1)+ \\ &+ \frac{\partial W(x,y,t)}{\partial y}(u_2+v_2+w_2)\Bigg). \end{aligned} \tag{48}$$

Step 1. Bellman function:

$$W(x,y,\vartheta,I)=\max_u\int_{\vartheta}^{\infty}\int_{\vartheta}^{t}\Bigg(a_{123}x(\tau)+b_{123}y(\tau)+c_{123}\Bigg)e^{-(t-\vartheta)}d\tau dt.$$

Further denote $W(x,y,\vartheta,I)$ by W. We have the HJB equation:

$$\begin{aligned} W &= \frac{\partial W}{\partial\vartheta}+\max_u\Bigg\{a_{123}x+b_{123}y+c_{123}+\frac{\partial W}{\partial x}(u_1+v_1+w_1)+ \\ &\qquad +\frac{\partial W}{\partial y}(u_2+v_2+w_2)\Bigg\}. \end{aligned} \tag{49}$$

As above, the system (37) is true. Differentiating the right-hand side of (49) with respect to u_1,v_1,w_1 we obtain optimal strategies:

$$u_1=v_1=w_1=\sqrt{\frac{(\frac{\partial W}{\partial x})^2}{(\frac{\partial W}{\partial x})^2+(\frac{\partial W}{\partial y})^2}}; \tag{50}$$

$$u_2=v_2=w_2=\sqrt{\frac{(\frac{\partial W}{\partial y})^2}{(\frac{\partial W}{\partial x})^2+(\frac{\partial W}{\partial y})^2}}.$$

We suppose $W=Ax+By+C$. Then we have $\frac{\partial W}{\partial x}=A, \frac{\partial W}{\partial y}=B$. Further, substituting this to (49) we get

$$A=a_{123};\ B=b_{123};,\ C=3\sqrt{a_{123}^2+b_{123}^2}+c_{123}.$$

Then we obtain

$$W = a_{123}x + b_{123}y + 3\sqrt{a_{123}^2 + b_{123}^2} + c_{123},$$

and the optimal controls

$$u_1^* = v_1^* = w_1^* = \frac{a_{123}}{\sqrt{a_{123}^2 + b_{123}^2}}; \quad u_2^* = v_2^* = w_2^* = \frac{b_{123}}{\sqrt{a_{123}^2 + b_{123}^2}} \qquad (51)$$

Note that open loop controls (40) coincide with feedback solution (51). Hence, it follows that the optimal trajectory has the form

$$x^*(t) = \frac{3a_{123}}{\sqrt{a_{123}^2 + b_{123}^2}} \cdot t + x_0, \quad y^*(t) = \frac{3b_{123}}{\sqrt{a_{123}^2 + b_{123}^2}} \cdot t + y_0, \qquad (52)$$

and the value

$$V(z^*(\vartheta), I) = W(x^*(\vartheta), y^*(\vartheta), \vartheta, I) = a_{123}x^* + b_{123}y^* + 3\sqrt{a_{123}^2 + b_{123}^2} + c_{123} =$$
$$= 3\sqrt{a_{123}^2 + b_{123}^2} + 3\sqrt{a_{123}^2 + b_{123}^2}\vartheta + a_{123}x_0 + b_{123}y_0 + c_{123}z_0.$$

Let us remark, this is the same result as above (see usage maximum principle).

Step 2. Similarly, in noncooperative case we get

$$u_1^N = \frac{a_1}{\sqrt{a_1^2 + b_1^2}}; \qquad u_2^N = \frac{b_1}{\sqrt{a_1^2 + b_1^2}}; \qquad v_1^N = \frac{a_2}{\sqrt{a_2^2 + b_2^2}};$$
$$u_2^N = \frac{b_2}{\sqrt{a_2^2 + b_2^2}}; \qquad w_1^N = \frac{a_3}{\sqrt{a_3^2 + b_3^2}}; \qquad u_2^N = \frac{b_3}{\sqrt{a_3^2 + b_3^2}};$$
$$x^N(t) = (\sum_{i=1}^{3} \frac{a_i}{\sqrt{a_i^2 + b_i^2}})t + x_0; \qquad y^N(t) = (\sum_{i=1}^{3} \frac{b_i}{\sqrt{a_i^2 + b_i^2}})t + y_0;$$
$$V(\{i\}, z^*(\vartheta) = (\sqrt{a_i^2 + b_i^2} + \sum_{j \neq i} \frac{a_i a_j + b_i b_j}{\sqrt{a_j^2 + b_j^2}})\vartheta + \sqrt{a_i^2 + b_i^2} + \sum_{j \neq i} \frac{a_i a_j + b_i b_j}{\sqrt{a_j^2 + b_j^2}} +$$
$$+ a_i x_0 + b_i y_0 + c_i.$$

Step 3. Consider coalition $K = \{1, 2\}$. Then we obtain

$$u_1^{1,2} = \frac{a_{12}}{\sqrt{a_{12}^2 + b_{12}^2}} = v_1^{1,2}; \quad u_2^{1,2} = \frac{b_{12}}{\sqrt{a_{12}^2 + b_{12}^2}} = v_2^{1,2};$$
$$w_1^N = \frac{a_3}{\sqrt{a_3^2 + b_3^2}}, \qquad w_2^N = \frac{b_3}{\sqrt{a_3^2 + b_3^2}}.$$

$$x^{1,2}(t) = \Bigg(\frac{2a_{12}}{\sqrt{a_{12}^2+b_{12}^2}} + \frac{a_3}{\sqrt{a_3^2+b_3^2}}\Bigg)t + x_0,$$

$$y^{1,2}(t) = \Bigg(\frac{2b_{12}}{\sqrt{a_{12}^2+b_{12}^2}} + \frac{b_3}{\sqrt{a_3^2+b_3^2}}\Bigg)t + y_0,$$

$$V(\{1,2\}, z^*(\vartheta) = \Bigg(2\sqrt{a_{12}^2+b_{12}^2} + \frac{a_{12}a_3+b_{12}b_3}{\sqrt{a_3^2+b_3^2}}\Bigg)\vartheta + 2\sqrt{a_{12}^2+b_{12}^2} +$$

$$+ \frac{a_{12}a_3+b_{12}b_3}{\sqrt{a_3^2+b_3^2}} + a_{12}x_0 + b_{12}y_0 + c_{12}.$$

We get the similar results for coalitions $\{1,3\}$ and $\{2,3\}$. Thus, we have constructed the characteristic function $V(S, z^*(\vartheta))$, $S \subseteq I$ with the help of HJB equation (17). However, we can show that $V(S, z^*(\vartheta))$ is not superadditive.

5. A Game Theoretic Model of Nonrenewable Resources with Random Duration

Consider one simple model of common-property nonrenewable resource extraction published by [Dockner, 2000].

Let $x(t)$ and $c_i(t)$ denote respectively the stock of the nonrenewable resource such as an oil field and player i's rate of extraction at time t [Dockner, 2000]. Let the transition equation has the form

$$\dot{x}(t) = -\sum_{i=1}^{n} c_i(t), \quad i = 1, \ldots, n; \tag{53}$$

$$x(t_0) = x_0. \tag{54}$$

The game starts at t_0 from x_0. We suppose that the game ends at the random time instant T with exponential distribution $f(t) = \rho * e^{-\rho(t-t_0)}$, $t \geq t_0$.

The utility function of player i at τ is as follows:

$$h(c_i(\tau)) = Aln(c_i) + B. \tag{55}$$

Here, A is positive and B is a constant which may be positive, negative or zero.

As in general case we define integral expected payoff

$$K_i(x_0, c_1, \ldots, c_n) = \int_{t_0}^{\infty}\int_{t_0}^{t} (A \times ln(c_i(\tau)) + B)\rho \times e^{-\rho(t-t_0)} d\tau dt, \quad i = 1, \ldots, n,$$

and consider total payoff in cooperative form of the game:

$$\max_{c_i} \sum_{i=1}^{n} K_i(x_0, c_1, \ldots, c_n) = \sum_{i=1}^{n} K_i(x_0, c_1^I, \ldots, c_n^I) = \tag{56}$$

$$= \int_{t_0}^{\infty}\int_{t_0}^{t} (A \times \sum_{i=1}^{n} ln(c_i^I(\tau)) + nB)\rho \times e^{-\rho(t-t_0)} d\tau dt.$$

Further we use 4 steps of our algorithm to calculate the characteristic function values.

Step 1. Grand coalition $I = \{1, \dots, n\}$. We have the following Bellman function:

$$W_I(x, \vartheta) = \max_{c_i, i \in I} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} (A \times \sum_{i=1}^{n} ln(c_i(\tau)) + nB)\rho \times e^{-\rho(t-\vartheta)} d\tau dt. \tag{57}$$

Let us define $\sum_{i=1}^{n} h_i(c_i(\cdot)) = H(c(\cdot))$. Then we can use the Hamilton–Jacobi–Bellman equation (17):

$$\rho W_I(x, t) = \frac{\partial W_I(x, t)}{\partial t} + \max_{c} \left(H(c) + \frac{\partial W_I(x, t)}{\partial x} g(x, c) \right). \tag{58}$$

Combining (58) and (57), we obtain

$$\rho W_I(x, t) = \frac{\partial W_I(x, t)}{\partial t} + \max_{c_i} \left(Anln(c_i) + nB + \frac{\partial W_I(x, t)}{\partial x}(-nc_i) \right). \tag{59}$$

Suppose the Bellman function W_I has the form

$$W_I = A_I ln(x) + B_I. \tag{60}$$

Then we get

$$\frac{\partial W_I(x, t)}{\partial x} = \frac{A_I}{x}; \qquad \frac{\partial W_I(x, t)}{\partial t} = \frac{A_I}{x} \dot{x}. \tag{61}$$

Differentiating the right-hand side of (59) with respect to c_i, we obtain optimal strategies

$$c_i^I = \frac{A}{\frac{\partial W_I(x, t)}{\partial x}}. \tag{62}$$

Using (61) and (62), we get

$$\dot{x} = -\frac{nA}{A_I} x. \tag{63}$$

Substituting (69), (62) and (63) in (59), we have an equation for a coefficients:

$$\rho A_I ln(x) + \rho B_I = -2nA + nB + Anln(A) - Anln(A_I) + Anln(x). \tag{64}$$

The result is:

$$A_I = \frac{An}{\rho}; \quad B_I = \frac{Bn}{\rho} - \frac{2An}{\rho} - \frac{Anln(n)}{\rho} + \frac{Anln(\rho)}{\rho}.$$

From (57) and (58) it follows that

$$W_I(x, t) = \frac{An}{\rho} ln(x) + \frac{Bn}{\rho} - \frac{2An}{\rho} - \frac{Anln(n)}{\rho} + \frac{Anln(\rho)}{\rho}. \tag{65}$$

Then we get the optimal strategies $c_i^I = \frac{\rho}{n}x, \qquad i = 1, \ldots, n.$

Finally, we have optimal trajectory and optimal controls

$$x^I(t) = x_o \times e^{-\rho(t-t_0)}; \quad c_i^I(t) = \frac{x_0\rho}{n}e^{-\rho(t-t_0)},$$

and

$$V(I, x^I(\vartheta)) = W_I(x^I, \vartheta) = \frac{An}{\rho}ln(x^I) + \frac{Bn}{\rho} - \frac{2An}{\rho} - \frac{Anln(n)}{\rho} + \frac{Anln(\rho)}{\rho} =$$
$$= \frac{An}{\rho}ln(x_0) - An(\vartheta - t_0) + \frac{Bn}{\rho} - \frac{2An}{\rho} - \frac{Anln(n)}{\rho} + \frac{Anln(\rho)}{\rho}.$$

Let $\vartheta = t_0$. Then

$$V(I, x_0) = W_I(x_0, t_0) = \frac{An}{\rho}ln(x_0) + \frac{Bn}{\rho} - \frac{2An}{\rho} - \frac{Anln(n)}{\rho} + \frac{Anln(\rho)}{\rho}. \tag{66}$$

Step 2. Feedback Nash equilibrium. Bellman function for player i:

$$W_i(x, \vartheta) = \max_{c_i} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} (A \times ln(c_i(\tau)) + B)\rho \times e^{-\rho(t-\vartheta)} d\tau dt. \tag{67}$$

The initial state

$$x(\vartheta) = x^I(\vartheta).$$

Now the HJB equation (58) has the form

$$\rho W_i(x, t) = \frac{\partial W_i(x, t)}{\partial t} + \max_{c_i} \left(Aln(c_i) + B + \frac{\partial W_i(x, t)}{\partial x}(-\sum_{i=1}^{n} c_i) \right). \tag{68}$$

We find W_i in the form

$$W_i = A_N ln(x) + B_N. \tag{69}$$

As before, we get

$$A_N = \frac{A}{\rho}; \tag{70}$$
$$B_N = \frac{B}{\rho} - \frac{2An}{\rho} + \frac{Aln(\rho)}{\rho}.$$

Then we get the Nash feedback strategies and trajectory

$$c_i^N = \rho x, \qquad i = 1, \ldots, n; \tag{71}$$

$$x^N(t) = x^I(\vartheta) \times e^{-n\rho(t-\vartheta)};$$
$$c_i^N(t) = \rho x^I(\vartheta) \times e^{-n\rho(t-\vartheta)};$$

So, we get the value

$$V(\{i\},x^I(\vartheta))=W_i(x^I,\vartheta)=\frac{A}{\rho}ln(x^I(\vartheta))+\frac{B}{\rho}-\frac{2An}{\rho}+\frac{Aln(\rho)}{\rho}.$$

Let $\vartheta=t_0$. Then

$$V(\{i\},x_0)=W_i(x_0,t_0)=\frac{A}{\rho}ln(x_0)+\frac{B}{\rho}-\frac{2An}{\rho}+\frac{Aln(\rho)}{\rho}. \tag{72}$$

Step 3. Coalition $K\subset I$, $|K|=k$, $|I\setminus K|=n-k$. Here k players form a coalition K. Their optimization problem is:

$$W_K(x,\vartheta)=\max_{c_i,i\in K}\int_{\vartheta}^{\infty}\int_{\vartheta}^{t}(A\sum_{i\in K}ln(c_i(\tau))+kB)\rho\times e^{-\rho(t-\vartheta)}d\tau dt. \tag{73}$$

The initial state $x(\vartheta)=x^I(\vartheta)$. Let us recall, that the left-out players $i\in I\setminus K$ will use feedback Nash strategies (71).

In the same way, we get "optimal" for coalition K trajectory, controls

$$x^K(t)=x^I(\vartheta)\times e^{-(n-k+1)\rho(t-\vartheta)};\quad c_i^K(t)=\frac{\rho}{k}x^I(\vartheta)\times e^{-(n-k+1)\rho(t-\vartheta)};$$

and the value of coalition payoff:

$$V(K,x^I(\vartheta))=W_K(x^I,\vartheta)=\frac{Ak}{\rho}ln(x^I(\vartheta))+\frac{kB}{\rho}-\frac{2Ak}{\rho}-\frac{Ak}{\rho}ln(k)+\frac{Akln(\rho)}{\rho}.$$

Let $\vartheta=t_0$. Then

$$V(K,x_0)=W_K(x_0,t_{)})=\frac{Ak}{\rho}ln(x_0)+\frac{kB}{\rho}-\frac{2Ak}{\rho}-\frac{Ak}{\rho}ln(k)+\frac{Akln(\rho)}{\rho}. \tag{74}$$

Thus, we have constructed the characteristic function $V(K,x_0)$,$K\subseteq I$ (see (66),(74)).

Proposition 1. *Suppose the characteristic function $V(K,x_0)$, $K\subseteq I$ is given by (66), (74). Then $V(K,x_0)$ is superadditivity.*

Lemma 1. *Let $s_1\geq 1, s_2\geq 1$. Then*

$$s_1ln(s_1)+s_2ln(s_2)+4s_1s_2\geq(s_1+s_2)ln(s_1+s_2). \tag{75}$$

This lemma can be proved by standard methods. It is easily shown that the left-hand side is faster increasing than the right-hand side. Now proof of the Proposition 1 is by direct calculations.

Finally, we get the Shapley Value in our example:

$$Sh_i(x_0)=\frac{V(I,x_0)}{n}=\frac{A}{\rho}ln(x_0)+\frac{B}{\rho}-\frac{2A}{\rho}-\frac{Aln(n)}{\rho}+\frac{Aln(\rho)}{\rho}.$$

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Contractual Stability and Competitive Equilibrium in a Pure Exchange Economy[1]

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Abstract. In the paper, a game-theoretical analysis of some stable outcomes in pure exchange economies is given. We deal, mostly, with an equilibrium characterization of unblocked allocations generated by recontracting process close to the one introduced by V.L. Makarov [Makarov, 1980]. Rather mild assumptions, providing coincidence of the corresponding contractual core and the set of Walrasian equilibrium allocations, are established, and two examples, demonstrating relevance of the main assumptions, are proposed.

Keywords: M-contract, weak quasi-stable contractual system, weak totally contractual core, competitive equilibrium.

Introduction

This paper contains a game-theoretical analysis of the so-called weak totally contractual allocations, similar to the one introduced by V.L. Makarov (1980) in order to describe stable outcomes of some quite natural recontracting processes in pure exchange economies. Detailed presentation of Makarov's original settings can be found in [Makarov, 1981]. Below, we analyze a slightly strengthened version of contractual blocking, introduced in [Makarov, 1980]. Namely, in what follows, no additional restrictions to the stopping rule of the breaking procedure is posed besides the feasibility of the final contractual system (hence, no minimality condition, applied in

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[1] This work was supported by the Netherlands Organization for Scientific Research in the framework of the Russian-Dutch programme for scientific cooperation (NWO-grant 047.017.017). The author would like to appreciate also financial assistance from Russian Fund of Basic Research (grant 07-06-00363) and Russian Leading Scientific Schools Fund (grant 4999.2006.6).

[Makarov, 1981], [Makarov, 1982], [Vasil'ev, 2006a,b]. At the same time, like in [Makarov, 1980], any such a final contractual system supposed to be an improvement for each member of blocking coalition (not just at least one of the minimal final system, like it appears to be in [Vasil'ev, 2006a, b].

So, being close to the original definition of contractual blocking, the version exploiting in the paper seems to be quite far from the one introduced in [Vasil'ev, 2006b]. In fact, even for the recontracting process, investigated in the latter paper, the multiplicity of outcomes of contractual breaking procedure is quite typical. Remind [Vasil'ev, 2006b], that it may often be the case that when the contracts chosen by some coalition are broken, the rest (including newly concluded by this coalition) do not constitute a feasible system. It is inherent in the model that in such a situation the breaking process proceeds spontaneously, and stops after feasibility of the contract system is recovered. The only requirement we deal with on this step is minimality: the spontaneous process breaks (nullifies) collection of contracts as small, as possible, provided that not nullified ones constitute a feasible contract system. It is clear that by omitting the minimality condition one enlarges the number of outcomes of the breaking procedure considerably. Therefore, to convince the contractual blocking exploiting in the paper is much more stronger than the one applied in [Vasil'ev, 2006b], we only stress once more that the former blocking requires blocking coalition to improve upon the initial contract system with any possible breaking outcome (not just with one of them, like in [Vasil'ev, 2006b]).

Surprisingly, for several rather wide classes of markets, the cores corresponding to the above-mentioned blockings are the same. To clarify this phenomenon, we prove one of the main results of the paper, stating that under rather mild assumptions the weak totally contractual core (consisting, by definition, of the allocations that are stable w.r.t the strengthened blocking) is equal to the set of Walrasian equilibrium allocations. To demonstrate the main assumptions in the core equivalence theorem, mentioned above, are relevant, two examples of pure exchange economies having unblocked allocations with no supporting equilibrium prices are given. The most interesting seems to be the last example with no equilibrium allocations and non-empty weak totally contractual core, exhibiting that the weak totally contractual allocation may be chosen as a compromise solution in case the classical market mechanism doesn't work.

1. Weak Totally Contractual Core

Below, we consider a slightly generalized pure exchange model

$$\mathcal{E} = \langle N, \{X_i, w^i, \alpha_i\}_N, \sigma \rangle, \tag{1}$$

where $N := \{1, \ldots, n\}$ is a set of consumers, and $X_i \subseteq \mathbb{R}^l,\ w^i \in \mathbb{R}^l,\ \alpha_i \subseteq X_i \times X_i$ are their consumption sets, initial endowments, and individual preference relations, respectively. As to σ, which is a non-empty subset of 2^N, called a coalitional structure of $\mathcal{E}$, it defines a collection of admissible coalitions $S \subseteq N$, which may join efforts of their participants in order to improve (block) any current allocation of the total initial endowment $\sum_N w^i$.

Remind also [Vasil'ev, 1984], that in our notations inclusion $(x, y) \in \alpha_i$ means that y is more preferable than x (w.r.t. the preference relation α_i). For any $x \in X_i$, we denote by $\alpha_i(x)$ a collection of all the bundles $y \in X_i$ that are more preferable than x:

$$\alpha_i(x) := \{y \in X_i \mid (x, y) \in \alpha_i\}.$$

As usual, we apply the following shortenings: $x\alpha_i y \Leftrightarrow (x, y) \in \alpha_i$, and

$$\mathcal{P}_i(x) := \{y \in X_i \mid y \in \alpha_i(x),\, x \notin \alpha_i(y)\}$$

(recall that in the standard interpretation inclusion $y \in \mathcal{P}_i(x)$ means that y is strictly more preferable than x).

We present first the main notion of the weak contractual domination (blocking) in $\mathcal{E}$. In order to do that, we modify first some relevant definitions from [Makarov, 1982], aiming to clarify the main features of the elementary interchange structure we deal with. For each $S \in \sigma$ we fix some subset $M_S \subseteq \mathbb{R}^l$ such that $0 \in M_S$, and put $M_\sigma := \{M_S\}_\sigma$.

Definition 1. *A contract (of type M_S) of coalition $S \in \sigma$ is a collection of vectors* $\mathrm{v} = \{\mathrm{v}^{ij}\}_{i,j \in S}$ *satisfying conditions:*

(a) $\mathrm{v}^{ii} = 0$ *for any* $i \in N$,

(b) $\mathrm{v}^{ij} \in M_S$ *for any* $i, j \in S$,

(c) $\mathrm{v}^{ij} = -\mathrm{v}^{ji}$ *for any* $i, j \in S$.

Coalition S entering into a contract v will be denoted by $S(\mathrm{v})$, as well.

Components v^{ij}, appearing in definition of the contract $\mathrm{v} = \{\mathrm{v}^{ij}\}_{i,j \in S}$, indicate amount of the corresponding commodities used in the bilateral exchange between the participants $i, j \in S(\mathrm{v})$. Here nonnegative component $\mathrm{v}^{ij}_k \geq 0$ of the vector v^{ij} denotes the amount of commodity k that agent j is obliged to deliver to agent i, and absolute value of negative component $\mathrm{v}^{ij}_r < 0$ measures the amount of commodity r agent i has to deliver to agent j.

As to the subsets M_S, they define admissible types of elementary exchanges within the contract $\mathrm{v} = \{\mathrm{v}^{ij}\}_{i,j \in S}$, e.g., in case $M_S := \{x \in \mathbb{R}^l \mid p^S \cdot x = 0\}$ the only constraint concerning v^{ij}, $i, j \in S$ is that the bilateral exchanges v^{ij} should be equivalent w.r.t. the (fixed) prices p^S.

We call $\mathrm{v} = \{\mathrm{v}^{ij}\}_{i,j \in S}$ *a proper contract*, if either v is a trivial contract (i.e., $\mathrm{v}^{ij} = 0$ for all $i, j \in S(\mathrm{v})$), or for any $i \in S(\mathrm{v})$ there exist $j, k \in S(\mathrm{v})$ such that v^{ij} (v^{ik}) contains strictly positive (strictly negative) components. Note, that the properness of the contract v makes it possible (in principal) to rescind this contract by each member $i \in S(\mathrm{v})$ simply by not delivering the corresponding commodity to the participant $k = k(i)$.

Any finite set $\mathcal{V} = \{\mathrm{v}_r\}_{\mathcal{R}}$, consisting of proper contracts v_r, is called a (proper) *contract system of type* M_σ (*contractual* M_σ*-system*, or c.s. (of type M_σ), for short). Let us stress at once that the elements of $\mathcal{R}$ are supposed to be the "titles" of any type (not necessary natural numbers), naming the contracts belonging to the set $\mathcal{V}$.

Thus, it is supposed that the members of any coalition $S \in \sigma$ may enter several contracts. Moreover, a c.s. $\mathcal{V}$ may contain several samples of the same contract $\mathrm{v}(S)$ differing just by their titles (or, by their names, to be exact). To simplify the terms, we call a c.s. $\tilde{\mathcal{V}} = \{\tilde{\mathrm{v}}_r\}_{\tilde{\mathcal{R}}}$ *a subsystem of c.s.* $\mathcal{V} = \{\mathrm{v}_r\}_{\mathcal{R}}$ if and only if $\tilde{\mathcal{R}} \subseteq \mathcal{R}$, and $\tilde{\mathrm{v}}_r = \mathrm{v}_r$ for any $r \in \tilde{\mathcal{R}}$. Hence, we consider contract systems $\mathcal{V}$ and $\tilde{\mathcal{V}}$ to be identical if and only if $\mathcal{R} = \tilde{\mathcal{R}}$ and $\tilde{\mathrm{v}}_r = \mathrm{v}_r$ for any $r \in \mathcal{R}$.

Denote by $\Delta^i(\mathcal{V})$ the net outcome of the agent i resulting after a c.s. $\mathcal{V} = \{\mathrm{v}_r\}_{\mathcal{R}}$ is accepted and realized

$$\Delta^i(\mathcal{V}) := \begin{cases} \sum_{r \mid i \in S(\mathrm{v}_r)} \sum_{j \in S(\mathrm{v}_r)} \mathrm{v}_r^{ij}, & i \in \bigcup_{r \in \mathcal{R}} S(\mathrm{v}_r), \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{v}_r^{ij} = -\mathrm{v}_r^{ji}$ for any $i, j \in S(\mathrm{v}_r),\ r \in \mathcal{R}$, we have that $\sum_N \Delta^i(\mathcal{V}) = 0$ and, hence, $\sum_N x^i(\mathcal{V}) = \sum_N w^i$ with

$$x^i(\mathcal{V}) = x^i(\mathcal{V}, \mathcal{E}) := w^i + \Delta^i(\mathcal{V})$$

to be the resulting outcome of the agent i, obtained by means of the entering into c.s. $\mathcal{V}$. In the sequel we call $x(\mathcal{V}) = x(\mathcal{V}, \mathcal{E}) := (x^i(\mathcal{V}, \mathcal{E}))_N$ *a contractual* M_σ*-allocation* of economy $\mathcal{E}$ (c.a. (of type M_σ), for short).

Definition 2. *A contractual system $\mathcal{V}$ is called feasible, if $x^i(\mathcal{V}) \in X_i$ for any $i \in N$.*

By definition, feasibility of a c.s. $\mathcal{V}$ guarantees the contractual M_σ-allocation $x(\mathcal{V})$ to belong to the set

$$X(N) = X_{\mathcal{E}}(N) := \{(x^i)_N \in \prod_N X_i \mid \sum_N x^i = \sum_N w^i\}$$

of balanced allocations of the economy $\mathcal{E}$.

Remind, that due to the properness, any contract $\mathrm{v} \in \mathcal{V}$ may, in principal, be broken by any participant $i \in S(\mathrm{v})$. A formal description of the outcomes of rescinding (breaking) contractual subsystems of a feasible c.s. $\mathcal{V} = \{\mathrm{v}_r\}_{r \in \mathcal{R}}$ takes a bit more space. To start with, fix a subsystem $\mathcal{V}' = \{\mathrm{v}_r\}_{r \in \mathcal{R}'}$ of $\mathcal{V}$, consisting of the contracts to be broken, and put

$$\mathcal{U} := \mathcal{V} \setminus \mathcal{V}' = \{\mathrm{v}_r\}_{r \in \mathcal{T}}$$

with $\mathcal{T} := \mathcal{R} \setminus \mathcal{R}'$. Further, let $\mathcal{F}(\mathcal{V}, \mathcal{V}')$ be a collection of all the feasible contractual systems $\tilde{\mathcal{V}}$ of the economy $\mathcal{E}$, satisfying requirements:

(a) $\tilde{\mathcal{V}} = \{\mathrm{v}_r\}_{r \in \tilde{\mathcal{R}}}$ is a subsystem of $\mathcal{V}$,

(b) $\tilde{\mathcal{R}}$ is a subset of $\mathcal{R} \setminus \mathcal{R}'$.

Introduce, finally, a version of the set $A_*(\mathcal{V}, \mathcal{R}', \mathcal{E})$ of outcomes of the breaking process, investigating in this paper:

$$A_*(\mathcal{V}, \mathcal{R}', \mathcal{E}) := \begin{cases} \mathcal{U}, & \text{if } \mathcal{U} \text{ is a feasible c.s. of } \mathcal{E}, \\ \mathcal{F}(\mathcal{V}, \mathcal{V}'), & \text{otherwise.} \end{cases}$$

Thus, $A_*(\mathcal{V}, \mathcal{R}', \mathcal{E})$ contains all the outcomes of the breaking (cancelation) procedure, consisting of no more than two steps. At the first step of this procedure we nullify all the contracts $\mathrm{v}_r,\ r \in \mathcal{R}'$. In case $\mathcal{U} := \{\mathrm{v}_r\}_{\mathcal{R}\setminus\mathcal{R}'}$ is feasible we put $A_*(\mathcal{V}, \mathcal{R}', \mathcal{E}) := \{\mathcal{U}\}$. Otherwise, at the second, final step, we continue to nullify contracts $\mathrm{v}_\mathrm{r},\ r \in \mathcal{R}''$, with $\mathcal{R}''$ being a subset of $\mathcal{R} \setminus \mathcal{R}'$, in order to provide a feasibility of the subsystem

$$\tilde{\mathcal{U}} = \{\mathrm{v}_r\}_{\mathcal{R}\setminus(\mathcal{R}'\cup\mathcal{R}'')}.$$

So, according to the requirements (a) and (b), any subsystem $\tilde{\mathcal{U}}$ of $\mathcal{U}$ can be chosen as a final outcome of the second step, provided that this subsystem is feasible. Certainly, there may be no outcomes at the second step in case $\mathcal{U}$ has no feasible subsystems at all.

Summarizing, we stress once more, that a breaking procedure at the second step supposed to be a spontaneous one. Hence, any feasible subsystem of $\mathcal{U}$ (if exists) may be realized as an outcome of this procedure. Thus, except when U is a feasible system itself ($A_*(\mathcal{V}, \mathcal{R}', \mathcal{E}) = \{\mathcal{U}\}$), we deal with the multiplicity of outcomes, in general. It is clear even from the first glance that the multiplicity mentioned causes significant obstacles in the study of contractual stability (for more details see Section 2 below).

To describe which way a coalition $S \in \sigma$ can improve a c.a. $x(\mathcal{V})$, generated by a feasible c.s. $\mathcal{V} = \{\mathrm{v}_r\}_\mathcal{R}$ of type M_σ, we suppose first that together with possibility to cancel some contracts v_r with

$$r \in \mathcal{R}^S = \mathcal{R}^S_\mathcal{V} := \{r \in \mathcal{R} \mid S(\mathrm{v}_r) \bigcap S \neq \emptyset\}$$

it is allowed for the coalition S to enter some new contract w. Denote by $\mathcal{E}_\mathrm{w}$ the modification of $\mathcal{E}$ generated by w:

$$\mathcal{E}_\mathrm{w} = \langle N, \{X_i, w^i + \Delta^i(\mathrm{w}), \alpha_i\}_N, \sigma\rangle,$$

where, as before, $\Delta^i(\mathrm{w}) := \Delta^i(\{\mathrm{w}\})$, i.e.,

$$\Delta^i(\mathrm{w}) = \begin{cases} \sum_{j\in\in S(\mathrm{w})} \mathrm{w}^{ij}, & \text{if } i \in S(\mathrm{w}), \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3. *We say that a coalition $S \in \sigma$ with $|S| \geq 2$ w-improves (w-blocks) a feasible contractual M_σ-system $\mathcal{V} = \{\mathrm{v}_r\}_\mathcal{R}$, if there exists a subset $\mathcal{R}' \subseteq \mathcal{R}^S$, and a proper contract* w *of type M_S such that $S(\mathrm{w}) = \mathrm{S}$, and for any $\mathcal{W} \in A_*(\mathcal{V}, \mathcal{R}', \mathcal{E}_\mathrm{w})$ it* *holds:* $x^i(\mathcal{W}, \mathcal{E}_{\mathrm{w}}) \in \alpha_i(x^i(\mathcal{V}, \mathcal{E}))$ *for any* $i \in S$, *with* $x^i(\mathcal{W}, \mathcal{E}_{\mathrm{w}}) \in \mathcal{P}_i(x^i(\mathcal{V}, \mathcal{E}))$ *for at least one* $i \in S$.

As to the case of one-element coalition, $S = \{i\} \in \sigma$, *it is said that* S *w-improves a feasible contractual* M_σ*-system* $\mathcal{V} = \{\mathrm{v}_r\}_{\mathcal{R}}$, *if there exists a subset* $\mathcal{R}' \subseteq \mathcal{R}^S$ *such that for any* $\mathcal{W} \in A_*(\mathcal{V}, \mathcal{R}', \mathcal{E})$ *it holds:* $x^i(\mathcal{W}, \mathcal{E})$ *belongs to* $\mathcal{P}_i(x^i(\mathcal{V}, \mathcal{E}))$.

(Here and below, as before, $\mathcal{P}_i(z) := \{x^i \in \alpha_i(z) \mid (x^i, z) \notin \alpha_i\}$ is the set of those bundles from X_i, which are strictly preferred to z.)

Definition 4. *A feasible contractual* M_σ*-system* $\mathcal{V}$ *is called a weak quasi-stable, if there is no coalition* $S \in \sigma$, *which w-improves* $\mathcal{V}$.

For any $x \in X(N)$ denote by $\mathcal{C}(x) = \mathcal{C}_{M_\sigma}(x)$ the set of all feasible c.s. $\mathcal{V}$ of type M_σ such that $x = x(\mathcal{V})$. To isolate the allocations $x \in X(N)$ with $\mathcal{C}_{M_\sigma}(x) \neq \emptyset$, whose coalitional stability is independent on the concrete representation via some c.s. $\mathcal{V}$ of type M_σ, we introduce the main notion of the paper.

Definition 5. *An allocation* $x \in X(N)$ *is called a weak totally contractual* M_σ*-allocation (w.t.c.a. (of type* M_σ*), for short), if* $\mathcal{C}_{M_\sigma}(x)$ *is non-empty set, and it contains only weak quasi-stable contractual systems (i.e., for every* $\mathcal{V} \in \mathcal{C}_{M_\sigma}(x)$ *it holds:* $\mathcal{V}$ *is a weak quasi-stable allocation).*

A set $D_*^{M_\sigma}(\mathcal{E})$ of all the weak totally contractual M_σ-allocations of $\mathcal{E}$ is called *a weak totally contractual core (of type* M_σ*) of economy* $\mathcal{E}$.

Remark 1. Note, that like in this paper, the main point at issue in [Vasil'ev, 2006b] is the fact that when the contracts with numbers $r \in \mathcal{R}'$ chosen by some coalition $S \in \sigma$ are broken, the rest, with numbers $r \in \mathcal{R} \backslash \mathcal{R}'$ may not constitute a feasible contractual system in $\mathcal{E}_{\mathrm{w}}$. In both papers, it is supposed that the breaking process proceeds spontaneously, and stops after feasibility of the contractual system $\mathcal{U} = \{\mathrm{v}_r\}_{r \in \mathcal{R} \backslash \mathcal{R}'}$ is recovered. One of the distinctive feature of the stopping rule applied in the paper mentioned is that the set of outcomes (denoted there by $A(\mathcal{V}, \mathcal{R}', \mathcal{E}_{\mathrm{w}})$) is supposed to consist of only the maximal feasible subsystems of $\mathcal{U}$ (remind, that here, in this paper, we deal with $A_*(\mathcal{V}, \mathcal{R}', \mathcal{E}_{\mathrm{w}})$ that consists of all the feasible subsystems of $\mathcal{U}$). Consequently, we have

$$A(\mathcal{V}, \mathcal{R}', \mathcal{E}_{\mathrm{w}}) \subseteq A_*(\mathcal{V}, \mathcal{R}', \mathcal{E}_{\mathrm{w}}).$$

Second (and the last) distinction concerns the blocking rule: to improve a feasible contractual M-system $\mathcal{V}$ it is sufficient (according to [Vasil'ev, 2006b] to find at least one maximal feasible subsystem of $\mathcal{U}$ that improves $\mathcal{V}$ (unlike the w-blocking in this paper, which requires any feasible subsystem of $\mathcal{U}$ to be able to improve $\mathcal{V}$).

Summarizing, we have that the sets $D^M(\mathcal{E})$, $M \in \mathcal{M}$, of totally contractual M-allocations (unblocked M-allocation in the sense of blocking from [Vasil'ev, 2006b]) satisfy inclusions

$$D^M(\mathcal{E}) \subseteq \mathcal{D}_*^{\mathcal{M}}(\mathcal{E}), \quad \mathcal{M} \in \mathcal{M}.$$

Surprisingly, for several rather wide classes of pure exchange economies it holds: $D^M(\mathcal{E}) = D_*^M(\mathcal{E})$ (for more details, see Section 4 below).

Note, that any coalitional structure σ admits the (unique) representation as the union of (locally) irreducible coalitional substructures, inscribed into the corresponding components of N. Therefore, in what follows, we suppose σ to be irreducible itself. For the sake of completeness, remind corresponding definition from [Vasil'ev, 1984] (as usual, below we call a partition $\{N_1, N_2\}$ nontrivial, if N_1 and N_2 are non-empty):

- *Irreducibility-assumption*: for any nontrivial partition $\{N_1, N_2\}$ of N there is $S \in \sigma$ such that $S \bigcap N_1$ and $S \bigcap N_2$ are non-empty sets.

Besides, everywhere below it is assumed that all the subsets M_S are identical and equal to some linear subspace $M \subseteq \mathbb{R}^l$ satisfying the following *Sign-assumption*:

- *Sign-assumption*: every nonzero vector $z \in M$ contains both strictly positive and strictly negative components.

Remark 2. Observe, that from the Bipolar Theorem and Minkovski Separation Theorem it follows immediately that a linear subspace $M \neq \mathbb{R}^l$ satisfies the abovementioned Sign-assumption if and only if its polar subspace $M^0 := \{p \in \mathbb{R}^l \mid p \cdot z = 0,\ z \in M\}$ meets the requirement

$$M^0 \bigcap \mathrm{int}\mathbb{R}^l_+ \neq \emptyset. \tag{2}$$

It is clear, that in case relation (2) is satisfied there exists a non-empty finite subset $P_M \subseteq \mathbb{R}^l$ containing at least one strictly positive vector, such that $M = \{z \in \mathbf{R}^l \mid p \cdot z = 0,\ p \in P_M\}$. Hence, the economical meaning of the constraint imposed on the type of the contracts we consider below is as follows: elementary exchanges bundles v^{ij} should be compatible with all the fixed-price vectors p, belonging to some non-empty finite subset $P_M \subseteq \mathbb{R}^l$, containing at least one strictly positive price vector $\bar{p}$.

Under Sign-assumption it can easily be shown also that for any $S \subseteq N$ with $|S| \geq 3$ there exists a proper "zero-contract" w of type M (i.e., a proper contract $\mathrm{w} \neq 0$ of type M such that (i) $S(\mathrm{w}) = S$, and (ii) $\Delta^i(\mathrm{w}) = 0$ for any $i \in S$). It means that the properness of any contract v of type $M \neq \{0\}$ with $|S(\mathrm{v})| \geq 3$ is guaranteed "automatically": $\mathrm{u} := \mathrm{v} + \lambda\mathrm{w}$ is a proper contract of type M with $S(\mathrm{u}) = \mathrm{S}(\mathrm{v})$ and $\Delta^i(\mathrm{u}) = \Delta^i(\mathrm{v})$, $i \in N$, provided that $\mathrm{w} \neq 0$ is a proper "zero-contract" of type M with $S(\mathrm{w}) = \mathrm{S}(\mathrm{v})$, and $\lambda > 0$ is large enough.

Hence, everywhere below we may (and will) deal with some contracts and contract systems of type M, not necessarily satisfying properness-assumption for not-two-person coalitions (at least, in those situations, where only properness "modulo zero-contract" matters).

Denote by $\mathcal{C}_M$ a collection of all c.s. of type M_σ with $M_S = M$ for any $S \in \sigma$. In what follows, the contracts v of type $M_S = M$, as well as the systems $\mathcal{V} \in \mathcal{C}_M$, and allocations $x = x(\mathcal{V})$, will be called *M-contracts, M-systems and M-allocations*, respectively. To present more detailed description of some unblocked (in the sense of Definition 3) M-allocations $x(\mathcal{V})$, we characterize first all feasible allocations, attainable by entering into the contract systems of type M. Put

$$X_{\mathcal{E}}^{M} := \{x \in \prod_{N} X_i \mid \exists \mathcal{V} \in \mathcal{C}_M : x = x(\mathcal{V})\}.$$

Proposition 1 below was established in [Vasil'ev, 2006a]; for its crucial role in the further considerations and for the sake of completeness we reproduce it here together with the proof that seems to be rather instructive.

Proposition 1. *If σ is irreducible coalition structure, then*

$$X_{\mathcal{E}}^{M} = \{x \in X(N) \mid \exists \Delta \in \Delta_M(N) : x = w + \Delta\},$$

where $\Delta_M(N) := \{\Delta = (\Delta^i)_N \in M^N \mid \sum_N \Delta^i = 0\}$, $w := (w^i)_N$.

Proof.

If $x \in X_{\mathcal{E}}^{M}$, then $x = x(\mathcal{V}) = w + \Delta(\mathcal{V})$ for some $\mathcal{V} \in \mathcal{C}_M$, where $\Delta(\mathcal{V}) := (\Delta^i(\mathcal{V}))_N$. By definition of M-contract it follows that $\Delta^i(\mathcal{V}) \in M$, $\sum_N \Delta^i(\mathcal{V}) = 0$ and, consequently, we get the desired: $x = w + \Delta$ for some $\Delta \in \Delta_M(N)$.

Let now $x \in X(N)$ with $x = w + \Delta$ for some $\Delta \in \Delta_M(N)$. To prove that there is $\mathcal{V} \in \mathcal{C}_M$, such that $\Delta = \Delta(\mathcal{V})$, we apply induction on the number $|N|$ of the economical agents of $\mathcal{E}$. It is clear, that in case $|N| = 2$ we have: $\mathcal{V} = \{\mathrm{v}\}$, with $\mathrm{v}^{12} := \Delta^1$, and $\mathrm{v}^{21} := \Delta^2$.

Suppose that our assertion is valid for all economies of type (1) with $|N| \leq m$. Fix some economy $\mathcal{E}$ with $|N| = |N_{\mathcal{E}}| = m+1$. If $N \in \sigma$, then, modulo "zero-contract", in accordance with Remark 2, a c.s. required consists of the only M-contract v, defined by the formula: $\mathrm{v}^{ii+1} := \sum_{k=1}^{i} \Delta^k$, $i = 1, \ldots, m$; $\mathrm{v}^{ij} = 0$, $j - i > 1$ (note, that due to the equalities $\mathrm{v}^{ij} = -\mathrm{v}^{ji}$ to give a complete description of the contract v it is sufficient to indicate all the vectors v^{ij} with $i, j \in S(\mathrm{v})$ satisfying inequalities $i < j$).

Consider the case $N \notin \sigma$. Entering into the trivial contracts for some coalitions $S \in \sigma$, if necessary, we may assume without loss of generality, that σ is minimal (w.r.t. inclusion in 2^N) irreducible coalitional structure. Fix some $S^0 \in \sigma$. It is not hard to prove that there exists a partition $\eta^0 = \{N_1^0, N_2^0\}$ of N such that (i) coalitions $S_k^0 := S^0 \bigcap N_k^0$, $k = 1, 2$, are non-empty; (ii) for any $T \in \sigma \setminus \{S^0\}$ either $T \subseteq N_1^0$, or $T \subseteq N_2^0$; and (iii) coalitional structures $\sigma_k^0 := \{T \in \sigma \mid T \subseteq N_k^0\} \bigcup \{S_k^0\}$, $k = 1, 2$, are irreducible in N_1^0, N_2^0, respectively. Indeed, existence of a partition η^0, satisfying conditions (i) and (ii), follows immediately from the minimality of the irreducible coalitional structure σ (since in case $\sigma \setminus \{T\}$ is irreducible for some $T \in \sigma$ we get a contradiction). As to the irreducibility of σ_1^0, for example, it can easily be shown by consideration of three cases for any nontrivial partition $\{N_1^1, N_1^2\}$ of N_1^0:

1) $S^0 \bigcap N_1^k \neq \emptyset$, $k = 1, 2$;
2) $S^0 \bigcap N_1^1 \neq \emptyset, S^0 \bigcap N_1^2 = \emptyset$;
3) $S^0 \bigcap N_1^1 = \emptyset, S^0 \bigcap N_1^2 \neq \emptyset$.

Specifically, considering partitions $\{N_1^2, N_1^1 \bigcup N_2\}$ and $\{N_1^1, N_1^2 \bigcup N_2\}$ in cases 2) and 3), respectively, we get required coalitions from σ_1^0 on the basis of condition (ii). In fact, since in both these cases coalition S^0 doesn't intersect one of the elements of the corresponding nontrivial partition of N, by (ii) and irreducibility of σ, in each of these cases there exists a coalition $S \in \sigma_1^0$ such that $S \bigcap N_1^k \neq \emptyset$ for any $k = 1, 2$.

Now, by making use of the partition η^0, we divide initial economy $\mathcal{E}$ into two appropriate "smaller" economies:

$$\mathcal{E}_{(k)} := \langle N_{(k)}, X_i^{(k)}, w_{(k)}^i, \alpha_i^{(k)}\}_{N_{(k)}}, \sigma_{(k)} \rangle, \quad k = 1, 2,$$

satisfying inductive hypothesis and "implementing" the allocation $x = w + \Delta$ with $\Delta = (\Delta^1, \ldots, \Delta^n) \in \Delta_M(N)$ by means of some contractual M-system (as a result of a suitable exchange, both within and between these economies). To this end we fix some economic agents $i_k \in S_k^0$, $k = 1, 2$, and put

$$N_{(k)} := N_k^0, \quad k = 1, 2;$$

$$w_{(k)}^i := w^i, \quad i \in N_{(k)},\ k = 1, 2;$$

$$X_i^{(k)} := X_i \quad i \in N_{(k)} \setminus \{i_k\},\ k = 1, 2.$$

As to the consumption sets $X_{i_1}^{(1)}$ and $X_{i_2}^{(2)}$, we put

$$X_{i_1}^{(1)} := \{w^{i_1} - \sum_{i \in N_{(1)} \setminus \{i_1\}} \Delta^i\}$$

and $X_{i_2}^{(2)} := \{w^{i_2} + \sum_{i \in N_{(1)} \cup \{i_2\}} \Delta^i\}$. Finally, we take $\sigma_{(k)}$ to be equal to $\sigma_k^0,\ k = 1, 2$, and put $\alpha_i^{(k)} := \alpha_i$ for any $i \in N_{(k)}$ and $k = 1, 2$. To apply inductive hypothesis, take $\Delta_{(k)} \in \Delta_M(N_{(k)}),\ k = 1, 2,$ with

$$\Delta_{(1)}^{i_1} := -\sum_{i \in N_{(1)} \setminus \{i_1\}} \Delta^i,\ \ \Delta_{(2)}^{i_2} := \sum_{i \in N_{(1)} \cup \{i_2\}} \Delta^i,$$

and $\Delta_{(k)}^i := \Delta^i$ for any $i \in N_{(k)} \setminus \{i_k\},\ \ k = 1, 2$. Further, for each $k = 1, 2$, put $w_{(k)} := (w^i)_{N_{(k)}}$ and $\Delta_{(k)} := (\Delta_{(k)}^i)_{N_{(k)}}$. It is clear, that $x_{(1)} := w_{(1)} + \Delta_{(1)}$ and $x_{(2)} := w_{(2)} + \Delta_{(2)}$ are balanced allocations of the economies $\mathcal{E}_{(1)},\ \mathcal{E}_{(2)}$, respectively. Hence, considerations, given above, imply: economies $\mathcal{E}_{(1)}, \mathcal{E}_{(2)}$ and corresponding allocations $x_{(1)},\ x_{(2)}$ satisfy our inductive hypothesis, and, consequently, there exist M-systems $\mathcal{V}_{(1)}$ and $\mathcal{V}_{(2)}$ such that $x_{(k)} = x(\mathcal{V}_{(k)}, \mathcal{E}_{(k)})$, each $k = 1, 2$.

Now we design a contract $\mathrm{v}_{(S^0)}$, which coalition S^0 should enter in order to organize (together with the M-systems $\mathcal{V}_{(1)}$ and $\mathcal{V}_{(2)}$) an M-system $\mathcal{V}$, satisfying equality $x = x(\mathcal{V}) = x(\mathcal{V}, \mathcal{E})$. To do so, we put

$$\mathrm{v}_{(S^0)}^{i_1 i_2} := \sum_{i \in N_{(1)}} \Delta^i, \quad \mathrm{v}_{(S^0)}^{i_2 i_1} := -\sum_{i \in N_{(1)}} \Delta^i,$$

and $\mathrm{v}^{ij}_{(S^0)} := 0$ for any $i, j \in S^0$ such that $\{i, j\} \neq \{i_1, i_2\}$. It is clear, that the system $\mathcal{V} := \mathcal{V}_{(1)} \bigcup \mathcal{V}_{(2)} \bigcup \{\mathrm{v}_{(S^0)}\}$, with $\mathrm{v}_{(S^0)}$ thus defined (and modified in the spirit of of Remark 2, if necessary), meets our requirement, which proves the proposition.

2. Weak Totally Contractual Set and Equilibrium Allocations

Below, we restrict ourselves to the case, when M is a hypersubspace of $\mathbb{R}^l$ (i.e., we assume, that $\dim M = l - 1$). In this situation it turns out that any w.t.c. allocation is an equilibrium allocation under fairly natural regularity assumptions. Vice versa, under essentially weaker assumptions any (competitive) equilibrium allocation turns out to be a w.t.c. allocation, provided that a subspace M is properly chosen.

Remind (see, e.g., [Arrow and Hahn, 1991], [Hildenbrand and Kirman, 1988]), that $\bar{x} \in X(N)$ is called a *competitive equilibrium allocation* (equilibrium allocation, for short), if there exists a nonzero vector $\bar{p} \in \mathbb{R}^l$ such that

$$\mathcal{P}_i(\bar{x}^i) \bigcap B_i(\bar{p}) = \emptyset, \quad i \in N, \tag{3}$$

where, as usual, $B_i(\bar{p})$ is the budget set of an agent i at prices $\bar{p}$:

$$B_i(\bar{p}) := \{x^i \in X_i \mid \bar{p} \cdot x^i \leq \bar{p} \cdot w^i\}.$$

Denote by $W(\mathcal{E})$ the set of all equilibrium allocations of economy $\mathcal{E}$. Remind, that any vector $\bar{p}$, satisfying (3), is called an equilibrium price vector (equilibrium price, for short), supporting the allocation $\bar{x}$.

Theorem 1. *Let $\bar{x}$ be an equilibrium allocation of the economy $\mathcal{E}$. If there is a strictly positive equilibrium price vector $\bar{p}$, supporting $\bar{x}$, then $\bar{x}$ belongs to $D^M_*(\mathcal{E})$ with $M := \{z \in \mathbb{R}^l \mid \bar{p} \cdot z = 0\}$.*

Proof.

Since $\bar{x} \in W(\mathcal{E})$ implies inclusion $\bar{x} \in X(N) \bigcap \prod_N B_i(\bar{p})$, we have: $\Delta^i := \bar{x}^i - w^i$ belongs to M for any $i \in N$. It is clear, that irreducibility of σ and Proposition 1 imply: $\bar{x} = w + \Delta \in X^M_{\mathcal{E}}$, where, as before, $w = (w^i)_N$, and $\Delta = (\Delta^i)_N$. Suppose, the allocation $\bar{x}$ can be w-improved by some coalition $S \in \sigma$. Then, by Definition 3, there exist at least one contractual system $\mathcal{V} \in \mathcal{C}_M(\bar{x})$ and one participant $i \in S$ such that $x^i(\mathcal{V}) \in \mathcal{P}_i(\bar{x}^i)$ and $x^j(\mathcal{V}) \in \alpha_j(\bar{x}^j)$ for any $j \in S \setminus \{i\}$. By definition of equilibrium allocation, the former inclusion $x^i(\mathcal{V}) \in \mathcal{P}_i(\bar{x}^i)$ implies $\mathcal{P}_i(\bar{x}^i) \bigcap B_i(\bar{p}) = \emptyset$. But then $\bar{p} \cdot x^i(\mathcal{V}) > \bar{p} \cdot w^i$, which contradicts to the inclusion $x^i(\mathcal{V}) \in X^M_i$ yielding $\bar{p} \cdot x^i(\mathcal{V}) = \bar{p} \cdot w^i$.

Thus, there are no coalitions $S \in \sigma$ that can w-improve $\bar{x}$ and, hence, $\bar{x}$ belongs to the weak totally contractual core (of type M) of the economy $\mathcal{E}$.

To prove the reverse inclusion $D^M_*(\mathcal{E}) \subseteq W(\mathcal{E})$ we have to add some compatibility assumptions concerning the coalitional structure σ (as well, as traditional convexity and monotonicity assumptions, imposed on the consumption sets X_i and individual preference relations α_i). Namely, from now on it is supposed that for any $i \in N$ it holds:

(a) X_i is a convex set;

(b) α_i is a reflexive binary relation;

(c) for any $x^i \in \widehat{X}_i := \mathrm{Pr}_{X_i}\, X(N)$ there exists $z \in \mathrm{int}\mathbb{R}^l_+$ such that $x^i + z \in \mathcal{P}_i(x^i)$;

(d) for any $x^i \in X_i$ the set $\mathcal{P}_i(x^i)$ is convex and, besides, $(x^i, y^i] \subseteq \mathcal{P}_i(x^i)$ for any $y^i \in \mathcal{P}_i(x^i)$, where $(x^i, y^i] := \{(1-t)x^i + ty^i \mid t \in (0,1]\}$.

To propose a compatibility assumption concerning σ, remind first, that for any $i \in N$ we denote by σ_i a set of those coalitions $S \in \sigma$ that contain i:

$$\sigma_i := \{S \in \sigma \mid i \in S\}.$$

Second, we introduce a useful strengthening of the notion of σ-divisibility, proposed earlier in ([Vasil'ev, 1984]).

Definition 6. *A coalition $T \subseteq N$ is said to be strongly σ-divisible, if for any $i \in N$ there are two coalitions $R, S \in \sigma_i$ such that*

$$R \cap (T \setminus S) \neq \emptyset. \tag{4}$$

Further, as in [Vasil'ev, 2006b], for any $x \in X^M_{\mathcal{E}}$ put

$$N_x := \{i \in N \mid x^i \in \mathrm{int}_M X_i\},$$

where $\mathrm{int}_M X_i$ is the (relative) interior of $X^M_i := (w^i + M) \cap X_i$ in the affine subspace $w^i + M$.

The main result of this section is as follows.

Theorem 2. *Suppose $\bar{x}$ belongs to $D^M_*(\mathcal{E})$ for some M satisfying Sign-assumption. If $N_{\bar{x}}$ is a strongly σ-divisible subset of N, then $\bar{x}$ is an equilibrium allocation of $\mathcal{E}$.*

Proof.

Let $\bar{x} \in D^M_*(\mathcal{E})$, and i be any agent of the economy $\mathcal{E}$. In order to show that $\bar{x}^i$ is a locally maximal element on X^M_i w.r.t. the binary relation α_i, we fix first some coalitions $R, S \in \sigma_i$ and a participant $k \in R$ such that $k \in N_{\bar{x}}$ and $k \notin S$ (remind, that the existence of such coalitions R, S and a participant k follows directly from the fact that $N_{\bar{x}}$ is a strongly σ-divisible set). Since $\bar{x}^k \in \mathrm{int}_M X_k$, there is some $U \subseteq M$, being a symmetric neighbourhood of zero in M, such that $\bar{x}^k + U \subseteq X^M_k$. Assuming that $\bar{x}^i$ is not a locally maximal element on X^M_i w.r.t. the binary relation α_i, we have that there exists $z \in U$ satisfying inclusion $\bar{x}^i + z \in \mathcal{P}_i(\bar{x}^i)$. To get a contradiction, we construct now a contractual M-system $\bar{\mathcal{V}} \in \mathcal{C}(\bar{x})$, which can be w-improved by coalition S (chosen in the very beginning of the proof). To do so, fix zero M-contract w with $S(\mathrm{w}) = \mathrm{S}$ (and $\mathrm{v}^{ij} = 0$ for any $\{i, j\} \subseteq S$), and consider an arbitrary $\mathcal{V} \in \mathcal{C}(\bar{x})$. Further, denote by $\bar{\mathrm{v}}$ an M-contract satisfying requirements: $S(\bar{\mathrm{v}}) = R$, and

$$\bar{\mathrm{v}}^{jr} = \begin{cases} z, & j = i,\ r = k, \\ 0, & \{j, r\} \neq \{i, k\} \end{cases}$$

(it follows directly from Remark 2 that it always possible to construct a proper "almost zero-contract" v of type M with $|S(\mathrm{v})| \geq 3$ and $\mathrm{v}^{ij} = 0$ for all but one two-element subsets $\{i, j\} \subseteq S$).

Put $\bar{\mathcal{V}} := \mathcal{V} \cup \{\mathrm{v}_1, \mathrm{v}_2\}$ with $\mathrm{v}_1 = \bar{\mathrm{v}}$, $\mathrm{v}_2 = -\bar{\mathrm{v}}$, where, as usual, $(-\mathrm{v})^{ij} = -\mathrm{v}^{ij}$ for any $\{i, j\} \subseteq S(\mathrm{v})$. Further, consider $\mathcal{R}' \subseteq \mathcal{R}^S$ to be equal to $\{2\}$. Note first, that by definition of the contracts v, w and contractual system $\mathcal{V}$ it follows immediately that $A(\bar{\mathcal{V}}, \mathcal{R}', \mathcal{E}_{\mathrm{w}})$ contains only one system, namely, $A(\bar{\mathcal{V}}, \mathcal{R}', \mathcal{E}_{\mathrm{w}}) = \{\mathcal{U}\}$ with $\mathcal{U} := \mathcal{V} \cup \{\mathrm{v}_1\}$ (no multiplicity of outcomes of cancelation process due to a proper chosen breaking system $\mathcal{R}'$). Second, due to the fact that preference relations α_j, $j \in N$, are reflexive, and by definition of z and v_1 we get: $x^j(\mathcal{U}, \mathcal{E}_{\mathrm{w}}) \in \alpha_j(x^j(\bar{\mathcal{V}}, \mathcal{E}))$ for any $j \in S$, with $x^i(\mathcal{U}, \mathcal{E}_{\mathrm{w}}) \in \mathcal{P}_i(x^i(\bar{\mathcal{V}}, \mathcal{E}))$. Hence, according to the Definition 3, we conclude, that coalition S w-improves a feasible contractual M-system $\bar{\mathcal{V}}$ belonging to $\mathcal{C}(\bar{x})$.

Thus, we have obtained the desired contradiction with assumption $\bar{x} \in D_*^M(\mathcal{E})$ and, consequently, finished the proof of the fact that for any $i \in N$, the bundle $\bar{x}^i$ is a local maximum on X_i^M w.r.t. his preference relation α_i.

More precisely, we have established that for any $i \in N$, there exists a symmetric neighbourhood of zero in M, say V, such that $\mathcal{P}_i(\bar{x}^i) \cap (\bar{x}^i + V) \cap X_i = \emptyset$. To continue the proof of the inclusion $\bar{x} \in W(\mathcal{E})$, check first that in fact the intersection $\mathcal{P}_i(\bar{x}^i) \cap X_i^M$ is empty, as well. To verify the latter assertion, just mention, that for any $y^i \in \mathcal{P}_i(\bar{x}^i) \bigcap X_i^M$ (if such y^i exists) the bundle $\bar{z}(t) := ty^i + (1-t)\bar{x}^i$ belongs to the neighbourhood $\bar{x}^i + V$ of $\bar{x}^i$ for any $t \in (0, 1)$ small enough. But this fact (together with the convexity assumption (d)) contradicts to the local maximality of $\bar{x}^i$, which had already been proven above.

Summarizing, we have that for every $i \in N$ the consumption bundle $\bar{x}^i$ is maximal on X_i^M w.r.t. α_i. Proceeding now like in [Vasil'ev, 2006a, Theorem 4.13], pick some $\bar{p} \in M^0 \bigcap \mathrm{int}\mathbb{R}_+^l$ and prove that $\mathcal{P}_i(\bar{x}^i) \bigcap B_i(\bar{p}) = \emptyset$ for any $i \in N$. Let $y^i \in \mathcal{P}_i(\bar{x}^i)$ for some $i \in N$. Since $\bar{p} \cdot x^i = \bar{p} \cdot w^i$ for any $x^i \in X_i^M$ (in particular, $\bar{p} \cdot \bar{x}^i = \bar{p} \cdot w^i$, each $i \in N$), from the maximality of $\bar{x}^i$ on X_i^M it follows that $\bar{p} \cdot y^i \neq \bar{p} \cdot w^i$. Suppose, that $\bar{p} \cdot y^i < \bar{p} \cdot w^i$. Because of the inclusion $\bar{x} \in X(N)$ we have, due to the monotonicity assumption (c): there is $z \in \mathrm{int}\mathbb{R}_+^l$ such that $\bar{y}^i := \bar{x}^i + z \in \mathcal{P}_i(\bar{x}^i)$. It is clear, that $\bar{p} \cdot \bar{y}^i > \bar{p} \cdot w^i$. But then there exists $\bar{t} \in (0, 1)$ such that the bundle $z(\bar{t}) := \bar{t}y^i + (1-\bar{t})\bar{y}^i$ belongs to X_i^M. On the other hand, due to the convexity of $\mathcal{P}_i(\bar{x}^i)$ we have: $z(\bar{t}) \in \mathcal{P}_i(\bar{x}^i)$, which contradicts to the maximality of $\bar{x}^i$ on X_i^M w.r.t. α_i.

The contradiction obtained proves the inequality $\bar{p} \cdot y^i > \bar{p} \cdot w^i$, which implies the relation $y^i \notin B_i(\bar{p})$. Due to the arbitrariness of element y^i taken from $\mathcal{P}_i(\bar{x}^i)$, we have the desired: $\mathcal{P}_i(\bar{x}^i) \cap B_i(\bar{p}) = \emptyset$.

Note, that the cardinality of $N_{\bar{x}}$ may be very small. It is clear, however, that σ-divisibility condition takes the simplest form in case $N_{\bar{x}} = N$.

To simplify formulations, from now on we assume that hypersubspace M under consideration always satisfies Sign-assumption.

Corollary 1. *If $|\sigma_i| \geq 2$ for any $i \in N$, then $D^M_{**}(\mathcal{E}) \subseteq W(\mathcal{E})$, where*

$$D^M_{**}(\mathcal{E}) := D^M_*(\mathcal{E}) \bigcap X^M_0, \quad X^M_0 := \prod_N \mathrm{int}_M X_i.$$

Proof.

Let $\bar{x} \in D^M_{**}(\mathcal{E})$ be given. To prove that $\bar{x}$ belongs to $W(\mathcal{E})$, we mention first that due to the inclusion $\bar{x} \in X^M_0$ we have $N_{\bar{x}} = N$. To show that $\bar{x}$ satisfies all the requirements of Theorem 2, it is sufficient to prove that under assumptions of our corollary the grand coalition N is strongly σ-divisible. Fix an arbitrary $i \in N$. By applying assumption $|\sigma_i| \geq 2$, select two different coalitions R and S, belonging to σ_i. Due to $R \neq S$ we have that either $R \setminus S \neq \emptyset$, or $S \setminus R \neq \emptyset$ (otherwise we get equality $R = S$ contradicting the choice of R, S). It is clear, that in both cases the relation (4) is satisfied (with $T = N$ and renaming the coalitions selected, if necessary). Hence, $N_{\bar{x}}$ is a strongly σ-divisible subset of N and, consequently, by Theorem 2 we get inclusion required: $\bar{x} \in W(\mathcal{E})$. The results obtained allow us to find some rather large classes of exchange models, admitting an equilibrium characterization for any w.t.c. allocation $\bar{x}$, without direct analysis of the structure of $N_{\bar{x}}$. To give some examples, introduce first additional characteristics of the exchange model $\mathcal{E}$. Denote by $\mathcal{M}$ the set of all hypersubspaces $M \subseteq \mathbb{R}^l$ satisfying Sign-assumption, and for any $M \in \mathcal{M}$ put

$$S^M_{\mathcal{E}} := \{i \in N \mid \alpha_i \text{ is complete, transitive, } w^i \in X_i, \text{ and } \alpha_i(w^i) \bigcap X^M_i \subseteq \mathrm{int}_M X_i\}.$$

Definition 7. *An exchange model $\mathcal{E}$ is called C_M-regular, if $S^M_{\mathcal{E}} \neq \emptyset$.*

Definition 8. *An exchange model $\mathcal{E}$ is called $C_{\mathcal{M}}$-regular, if it is C_M-regular for any $M \in \mathcal{M}$.*

Remark 3. Note, that by definition of the sets $S^M_{\mathcal{E}}$ we have that for any $M \in \mathcal{M}$ it holds: $w^i \in \mathrm{int}_M X_i$ for each $i \in S^M_{\mathcal{E}}$. Consequently, for any $C_{\mathcal{M}}$-regular economy $\mathcal{E}$ we have: $w^i \in \mathrm{int}_M X_i$ for any $M \in \mathcal{M}$ and $i \in S^M_{\mathcal{E}}$.

By applying, mutatis mutandis, argumentation, used in the proof of Theorem 2, we obtain the following results.

Proposition 2. *Let $\mathcal{E}$ be a C_M-regular pure exchange model with w^i belonging to X_i for any $i \in N \setminus S^M_{\mathcal{E}}$. If $S^M_{\mathcal{E}}$ is a strongly σ-divisible subset of N, and one-element coalition $\{i\}$ belongs to σ for any $i \in S^M_{\mathcal{E}}$, then $D^M_*(\mathcal{E}) \subseteq W(\mathcal{E})$.*

Proof.

Observe first, that by definition of the set $S_{\mathcal{E}}^M$ and due to the assumption $w^i \in X_i,\ i \in N \setminus S_{\mathcal{E}}^M$, of our proposition, we have that $w^i \in X_i$ for any $i \in N$ (initial endowments of the agents belong to their consumption sets). Let $\bar{x}$ be an arbitrary element of $D_*^M(\mathcal{E})$. It is not very hard to verify that $S_{\mathcal{E}}^M$ is contained in $N_{\bar{x}}$. To start with the proof of this assertion, note that due to the completeness of α_i and inclusion $\{i\} \in \sigma$, fulfilled for any $i \in S_{\mathcal{E}}^M$, we get: $\bar{x}^i \in \alpha_i(w^i)$ for any $i \in S_{\mathcal{E}}$. Indeed, suppose $\bar{x}^i \notin \alpha_i(w^i)$ for some $i \in S_{\mathcal{E}}^M$. Then, by completeness of α_i we get $w^i \in \mathcal{P}_i(\bar{x}^i)$. Applying the latter inclusion, we prove that the coalition $\{i\}$ can w-improve the allocation $\bar{x}$. Doing so, fix some $\mathcal{V} \in \mathcal{C}(\bar{x})$ and put $S = \{i\}$, $\mathrm{w} = 0$ and $\mathcal{R}' = \mathcal{R}^S$. Note, that due to the inclusions $w^i \in X_i,\ i \in N$, we have that $A(\mathcal{V}, \mathcal{R}', \mathcal{E}_{\mathrm{w}}) \neq \emptyset$ (to argue, just mention, that in case the contractual system $\mathcal{U} := \{\mathrm{v_r}\}_{\mathrm{r} \in \mathcal{R} \setminus \mathcal{R}^{\mathrm{S}}}$ happens not to be feasible, we can just break all the rest, nullifying all the contracts from $\mathcal{V} \setminus \mathcal{U}$; the resulting empty system gives us initial endowment allocation $w = (w^1, \ldots, w^n)$, which belongs to $X(N)$ due to the inclusions $w^i \in X_i$ mentioned). Moreover, it is easy to check that $x^i(\mathcal{V}', \mathcal{E}_{\mathrm{w}}) = w^i$ for any $\mathcal{V}' \in A(\mathcal{V}, \mathcal{R}', \mathcal{E}_{\mathrm{w}})$ and, hence, according to the Definition 3 and our supposition $w^i \in \mathcal{P}_i(\bar{x}^i)$ we have: coalition $S = \{i\}$ w-improves the allocation $\bar{x}$. But the latter contradicts to the assumption $\bar{x} \in D_*^M(\mathcal{E})$. Due to the contradiction obtained, we get: $\bar{x}^i$ belongs to $\alpha_i(w^i)$ for any $i \in S_{\mathcal{E}}^M$. Since, evidently, $\bar{x}^i$ belongs to X_i^M for any $i \in N$, by inclusions $\bar{x}^i \in \alpha_i(w^i)$ and $\alpha_i(w^i) \cap X_i^M \subseteq \mathrm{int}_M X_i,\ i \in S_{\mathcal{E}}^M$, we get: $\bar{x}^i \in \mathrm{int}_M X_i$ for any $i \in S_{\mathcal{E}}^M$. Thus, the inclusion

$$S_{\mathcal{E}}^M \subseteq N_{\bar{x}}, \tag{5}$$

mentioned in the beginning of our proof, is established.

It is clear, that any superset of a strongly σ-divisible set is a strongly σ-divisible set, as well. Hence, by inclusion (5) we have that $N_{\bar{x}}$ is a strongly σ-divisible set. Consequently, by applying Theorem 2 we get required: $\bar{x} \in W(\mathcal{E})$.

Below, we demonstrate one of the quite clear implications of Proposition 2.

Corollary 2. *If $\mathcal{E}$ is C_M-regular exchange economy with $w^i \in X_i$ for any $i \in N \setminus S_{\mathcal{E}}^M$, one-element coalition $\{i\}$ belongs to σ for any $i \in N$, and, moreover, for any $i \in N$ there exists a coalition $R \in \sigma_i$ such that*

$$(R \setminus \{i\}) \cap S_{\mathcal{E}}^M \neq \emptyset, \tag{6}$$

then every weak totally contractual M-allocation is an equilibrium allocation.

Proof.

The main lines of the proof are the same, as in the proof of Proposition 2. First, we establish that $\bar{x}^i$ belongs to $\alpha_i(w^i)$ for any $\bar{x} \in D_*^M(\mathcal{E})$ and $i \in S_{\mathcal{E}}^M$. Here we apply argumentation from the proof of Proposition 2, stating that assumption $w^i \in \mathcal{P}_i(\bar{x}^i)$ with $i \in S_{\mathcal{E}}^M$ implies that the one-element coalition $\{i\}$ w-improves allocation $\bar{x}$. Second, as a consequence of inclusions $\bar{x}^i \in \alpha_i(w^i),\ i \in S_{\mathcal{E}}^M$, we get the insertion $S_{\mathcal{E}}^M \subseteq N_{\bar{x}}$. Finally, by taking for any $i \in N$ coalitions $S = \{i\}$ and R to be equal to that appearing in (6), we have that $S^M_{\mathcal{E}}$ is a strongly σ-divisible subset of N. Referring to Proposition 2, we complete the proof.

To present a "global" version of the results, obtained in term of the C_M-regularity, we introduce one more important notion. Put

$$D_*(\mathcal{E}) := \bigcup_{M \in \mathcal{M}} D^M_*(\mathcal{E}), \tag{7}$$

$$S_{\mathcal{E}} := \bigcup_{M \in \mathcal{M}} S^M_{\mathcal{E}}. \tag{8}$$

Definition 9. *We call $D_*(\mathcal{E})$, defined by the formula (7), a weak totally contractual core of an economy $\mathcal{E}$.*

Theorem 3. *If $\mathcal{E}$ is $C_{\mathcal{M}}$-regular, $S^M_{\mathcal{E}}$ is strongly σ-divisible for every $M \in \mathcal{M}$, $\{i\} \in \sigma$ for every $i \in S_{\mathcal{E}}$ (with $S_{\mathcal{E}}$ to be defined by (8)), and $w^i \in X_i$ for any $i \in N \setminus S_{\mathcal{E}}$, then the weak totally contractual core $D_*(\mathcal{E})$ of economy $\mathcal{E}$ is contained in the set $W(\mathcal{E})$ of competitive equilibria of $\mathcal{E}$.*

In order to provide an easier verifiable regularity condition than that, given in Definition 8, we introduce another characteristic of the exchange model under consideration:

$$T_{\mathcal{E}} := \{i \in N \mid \alpha_i \text{ is complete, transitive, } w^i \in X_i, \text{ and } \alpha_i(w^i) \bigcap \check{X}_i \subseteq \mathrm{int} X_i\},$$

where $\check{X}_i := X_i \setminus (w^i + \mathrm{int}\mathbb{R}^l_+)$, $i \in N$.

Definition 10. *An exchange model $\mathcal{E}$ is called C-regular, if $T_{\mathcal{E}} \neq \emptyset$.*

It is clear, that a slight modification of the proof of Proposition 2 gives the following analogs of Theorem 3.

Theorem 4. *If $T_{\mathcal{E}}$ is strongly σ-divisible, $w^i \in X_i$ for any $i \in N \setminus T_{\mathcal{E}}$, and $\{i\} \in \sigma$ for every $i \in T_{\mathcal{E}}$, then $D_*(\mathcal{E}) \subseteq W(\mathcal{E})$.*

Proof.

Observe, that in the theorem under consideration C-regularity of $\mathcal{E}$ follows from the fact that it is assumed $T_{\mathcal{E}}$ to be a strongly σ-divisible subset of N and, hence, $T_{\mathcal{E}} \neq \emptyset$. Since, in some sense, the role of $T_{\mathcal{E}}$ in our theorem is quite similar to that, played by $S^M_{\mathcal{E}}$ in Proposition 2, we just adapt here the main lines of the proof of this proposition.

Let $\bar{x} \in D^M_*(\mathcal{E})$ for some $M \in \mathcal{M}$. Due to the assumption that $T_{\mathcal{E}}$ is strongly σ-divisible, the only thing we need to directly apply Theorem 2 and get the inclusion required $\bar{x} \in W(\mathcal{E})$, is to prove the insertion $T_{\mathcal{E}} \subseteq N_{\bar{x}}$. To this end, according to the assumptions $\alpha_i(w^i) \cap \check{X}_i \subseteq \mathrm{int}X_i$, $i \in T_{\mathcal{E}}$, given, it would be enough to establish inclusion: $\bar{x}^i \in \check{X}_i$, and $\bar{x}^i \in \sigma_i(w^i)$ for any $i \in T_{\mathcal{E}}$. Note, that the first inclusion, $\bar{x}^i \in \check{X}_i$, follows directly from the relations $\bar{x}^i - w^i \in M$ and $M \in \mathcal{M}$, fulfilled for any $i \in N$ (remind, that by Sign-assumption, for any non-zero $x \in M$ with

$M \in \mathcal{M}$ we have that $x_k < 0$ for some k). As to the second one, we apply the same argumentation, as in the proof of Proposition 2, which shows that supposition $w^i \in \mathcal{P}_i(\bar{x}^i)$ for some $i \in T_{\mathcal{E}}$ implies that coalition $\{i\}$ can w-improve the allocation $\bar{x}$. Applying the contradiction obtained (remind, it was supposed that $\bar{x}$ belongs to $D^M_{\mathcal{E}}$) and completeness of the preference relations α_i, $i \in T_{\mathcal{E}}$, we get required: $\bar{x}^i \in \alpha_i(w^i)$ for any $i \in T_{\mathcal{E}}$.

According to the remarks, given above, the proof of our theorem is completed.

Corollary 3. *If $\mathcal{E}$ is C-regular with $\{i\} \in \sigma$, $i \in T_{\mathcal{E}}$, $w^i \in X_i$ for any $i \in N \setminus T_{\mathcal{E}}$, and, besides, for any $i \in N$ there exists a coalition $R \in \sigma_i$ such that*

$$(R \setminus \{i\}) \cap T_{\mathcal{E}} \neq \emptyset,$$

then $D_*(\mathcal{E}) \subseteq W(\mathcal{E})$.

We omit the proof of Corollary 3 because it almost literally reproduces the proof of Corollary 2 (with $S_{\mathcal{E}}$ replaced by $T_{\mathcal{E}}$).

As usual, we say that α_i is locally monotonic, if for any $x^i \in \mathrm{Pr}_{X_i} X(N)$ there exists $\delta(x^i) > 0$ such that $x^i + z \in \mathcal{P}_i(x^i)$, whenever $z \in \mathbb{R}^l_+$ and $0 < \|z\| < \delta(x^i)$. Observe, that α_i is always locally monotonic under the following standard assumptions: α_i is strongly monotonic, and $X_i = \mathbb{R}^l_+$.

Taking into account that local monotonicity guarantees equilibrium prices to be strictly positive, and summarizing the results, which has been proven above, we obtain the following core equivalence result.

Theorem 5. *Suppose a pure exchange economy $\mathcal{E}$ satisfies the assumptions of either one of Theorems 3, 4, or Corollary 3. If α_i is locally monotonic for at least one agent of the economy $\mathcal{E}$, then $D_*(\mathcal{E}) = W(\mathcal{E})$.*

3. Conclusion

Two remarks should be given in the conclusion. First, let us mention once more that w-blocking introduced in the paper is not that much stronger than blocking considered in [Vasil'ev, 2006b]. Although we have that directly by definition it follows that coalition S can block a feasible contractual M-system $\mathcal{V}$ whenever $\mathcal{V}$ can be w-blocked by S (i.e., w-blocking implies blocking, which means in our terms that the former is stronger than the latter), there are several rather wide classes of pure exchange economies with totally contractual set $D(\mathcal{E})$ and weak totally contractual core $D_*(\mathcal{E})$ to be equal. In fact, due to the inclusion $W_{\mathcal{M}}(\mathcal{E}) \subseteq D(\mathcal{E})$[2] established in [Vasil'ev, 2006b], one can easily see that any condition, which provides insertion $D_*(\mathcal{E}) \subseteq W(\mathcal{E})$, guarantees the coincidence of $D(\mathcal{E})$ and $D_*(\mathcal{E})$. In particular, due to Theorem 5 we have the following assertion.

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[2] Here $W_{\mathcal{M}}(\mathcal{E})$ consists of those allocations from $W(\mathcal{E})$ that can be supported by the strictly positive prices.

Proposition 3. *Let $\mathcal{E}$ satisfies the assumptions of either one of Theorems 3, 4, or Corollary 3. If α_i is locally monotonic for at least one agent of the economy $\mathcal{E}$, then $D(\mathcal{E}) = D_*(\mathcal{E})$.*

Almost the same argumentation, based on Corollary 2 and Proposition 2, yields the "individual" version of Proposition 3 (the only alteration we need in the corresponding proof is the replacement of $W(\mathcal{E})$, $D(\mathcal{E})$, and $D_*(\mathcal{E})$ by $W^M(\mathcal{E})$, $D^M(\mathcal{E})$, and $D^M_*(\mathcal{E})$ with $W^M(\mathcal{E}) := W(\mathcal{E}) \bigcap X^M_{\mathcal{E}},\ M \in \mathcal{M}$).

Proposition 4. *If $\mathcal{E}$ is C_M-regular exchange economy with $w^i \in X_i$ for any $i \in N \setminus S^M_{\mathcal{E}}$, one-element coalition $\{i\}$ belongs to σ for any $i \in N$, and, moreover, for any $i \in N$ there exists a coalition $R \in \sigma_i$ such that*

$$(R \setminus \{i\}) \cap S^M_{\mathcal{E}} \neq \emptyset,$$

then $D^M(\mathcal{E}) = D^M_*(\mathcal{E})$.

Proposition 5. *Let $\mathcal{E}$ be a C_M-regular pure exchange model with w^i belonging to X_i for any $i \in N \setminus S^M_{\mathcal{E}}$. If $S^M_{\mathcal{E}}$ is a strongly σ-divisible subset of N, and one-element coalition $\{i\}$ belongs to σ for any $i \in S^M_{\mathcal{E}}$, then $D^M(\mathcal{E}) = D^M_*(\mathcal{E})$.*

Second (and the final) remark concerns the importance of the strong σ-divisibility in Theorem 5. To demonstrate the main assumption in the core equivalence theorem is relevant, two examples of pure exchange economies having unblocked (in the sense of Definition 3) allocations with no supporting equilibrium prices are given. The most interesting seems to be the second example with no equilibrium allocations and non-empty weak totally contractual core, exhibiting that the weak totally contractual allocation may be chosen as a compromise solution in case the classical market mechanism doesn't work.

Example. $[W(\mathcal{E}_1) \neq \emptyset, D_*(\mathcal{E}_1) \setminus W(\mathcal{E}_1) \neq \emptyset]$.

Let $\mathcal{E}_1$ be pure exchange economy defined by the following parameters:

$$N = \{1,2,3,4,5\}, \sigma = \big\{\{1,2\},\{2,3\},\{3,4,5\}\big\};$$

$$X_i = \mathbb{R}^2_+,\ i \in N, \quad w^1 = (3,0),\ w^2 = (6,0),\ w^3 = (6,1),\ w^4 = (9,1),\ w^5 = (6,3);$$

$$u_i(x_1,x_2) = \begin{cases} x_1 + 4x_2, & i = 1,2, \\ 2x_1 + 3x_2, & i = 3,4,5. \end{cases}$$

Put

$$M = \{x \in \mathbb{R}^2 \mid x_1 + 3x_2 = 0\}$$

and consider the balanced allocation $\bar{x} = (\bar{x}^i)_{i \in N} \in X(N)$ of the economy $\mathcal{E}_1$, defined as follows:

$$\bar{x}^1 = (0,1),\ \bar{x}^2 = (0,2),\ \bar{x}^3 = (9,0),\ \bar{x}^4 = (12,0),\ \bar{x}^5 = (9,2).$$

It is clear, that $\bar{x}^i \in X^M_i := (w^i + M) \cap X_i$ for any $i \in N$, and $N_{\bar{x}} = \{5\}$. Note, that directly from the definition 5 it follows that any strongly σ-divisible coalition contains at least two participants, since in case $T = \{i\}$ we have $T \setminus S = \emptyset$ for any $S \in \sigma_i$. Hence, due to $N_{\bar{x}} = \{5\}$ we get that in our case $N_{\bar{x}}$ is not a strongly σ-divisible subset of N. Further, by analyzing restrictions of utility functions u_i to the corresponding intervals X_i^M, one can easily check that for any $i \in N$ the bundle $\bar{x}^i$ is a maximal element in X_i^M w.r.t. the individual preference relation, generated by u_i. Consequently, there is no coalition $S \in \sigma$, which can w-improve some contractual system from $\mathcal{C}(\bar{x})$ (remind, that according to the definition of w-improvement, all the contracts we deal with are M-contracts, and, hence, any output of the blocking procedure belongs to $\prod_{i \in N} X_i^M$). So, the allocation $\bar{x}$ belongs to the weak totally contractual core $D_*(\mathcal{E}_1)$.

At the same time, one can easily check, that the set $W(\mathcal{E}_1)$ of competitive equilibria of the economy $\mathcal{E}_1$ consists of the only allocation $\widehat{x} = (\widehat{x}^i)_{i \in N} \in X(N)$, given by the data:

$$\widehat{x}^1 = (0, 5/3),\ \widehat{x}^2 = (0, 10/3),\ \widehat{x}^3 = (39/5, 0),\ \widehat{x}^4 = (54/5, 0),\ \widehat{x}^5 = (57/5, 0),$$

with the (normed) supporting equilibrium price vector $\widehat{p}$, given by the equality: $\widehat{p} = (5/14, 9/14)$.

Hence, we have got the required: $W(\mathcal{E}_1) \neq \emptyset$ and $D_*(\mathcal{E}_1) \setminus W(\mathcal{E}_1) \neq \emptyset$.

Example. $[W(\mathcal{E}_2) = \emptyset,\ D_*(\mathcal{E}_2) \neq \emptyset]$.

Consider one more pure exchange economy $\mathcal{E}_2$, given by the data:

$$N = \{1, 2, 3\}, \quad \sigma = \big\{\{1, 2\}, \{2, 3\}\big\};$$

$$X_i = \mathbb{R}^2_+,\ i \in N; \quad w^1 = (0, 1),\ w^2 = (6, 1),\ w^3 = (6, 3);$$

$$u_i(x_1, x_2) = \begin{cases} x_1 + 4x_2, & i = 1, \\ \quad x_1, & i = 2, 3. \end{cases}$$

Put, as in the previous example,

$$M = \{x \in \mathbb{R}^2 \mid x_1 + 3x_2 = 0\},$$

and pick the balanced allocation $\bar{x} = (\bar{x}^i)_{i \in N}, \in M_{\mathcal{E}_2}(N)$ of the economy $\mathcal{E}_2$, defined as,follows:

$$\bar{x}^1 = (0, 1),\ \bar{x}^2 = (9, 0),\ \bar{x}^3 = (3, 4).$$

Note, that as in the first example, we have that $N_{\bar{x}} = \{3\}$ is not a strongly σ-divisible set in $\mathcal{E}_2$. Further, suppose, $\bar{x} \notin D_*^M(\mathcal{E}_2)$. Doing, like in the previous example, one can easily check that the bundles $\bar{x}^1$ and $\bar{x}^2$ are maximal elements in X_1^M and X_2^M, respectively (w.r.t. the individual preferences, defined by the corresponding utility functions u_1 and u_2). Consequently, the only coalition $S \in \sigma$, which might be w-improving for some contractual system from $\mathcal{C}(\bar{x})$, is $S = \{2, 3\}$ (remind once more, that according to the definition of w-improvement, all the contracts we deal with are M-contracts). Suppose, that $S = \{2, 3\}$ really w-improves some $\mathcal{V} \in \mathcal{C}(\bar{x})$. By definition of w-improvement and construction of $\bar{x}$ and utility functions u_2, u_3, it means that there exists $x = (x^i)_{i \in N} \in X^M_{\mathcal{E}_2}$ such that $x^1_1 \geq 0$, $x^2_1 \geq 9$, and $x^3_1 > 3$. But these inequalities apparently contradict to the relations

$$\sum_{i \in N} x^i_1 = \sum_{i \in N} w^i_1 = 12,$$

following from the obvious inclusion $X^M_{\mathcal{E}_2} \subseteq X(N)$.

So, our economy $\mathcal{E}_2$ possesses at least one weak totally contractual M-allocation. As to the competitive equilibrium, from the famous "irreducibility" criterion for the linear exchange economies, proposed by D. Gale (1976), it follows immediately that $W(\mathcal{E}_2) = \emptyset$ (the latter fact can be verified directly, as well, just by applying the definition of equilibrium and taking into account that coalition $S = \{2, 3\}$ is too "1self-sufficient"). Hence, we get required: $W(\mathcal{E}_2) = \emptyset$ and $D_*(\mathcal{E}_2) \neq \emptyset$.

Acknowlegments

The author express his gratitude to prof. S. Weber for useful discussion on the subject.

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Multilingualism

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Abstract. *Multilingualism* or *linguistic diversity* in a heterogeneous society provides extraordinary challenges and room for policies which may have important economic implications in shaping the flows of interregional or international trade, investment and migrations. Given the often uncompromising nature of linguistic conflicts, linguistic policies and, especially, the choice of official languages should take into account the preferences of those groups of individuals whose cultural, societal, historical values and sensibilities could be affected. In evaluating linguistic policies an important role is played by the dynamic nature of language environments driven by individual choices of learning other languages.

Keywords: Communicative benefits, linguistic disenfranchisement, linguistic standardization, multilinguialism, official languages.

Introduction

Multilingualism, or linguistic diversity, is an important societal phenomenon that can generate gains or losses resulting from the economic interactions between individuals, regions or countries. The effects of multilingualism have recently come to the forefront of public policy debates. Linguistic issues and, in particular, the treatment of minority languages are almost unparalleled in terms of their explosiveness and emotional appeal, much more so than any other question of resource allocation or responsibility sharing within a polity. As noted by Bretton, 'language may be the most explosive issue universally and over time. This mainly because language alone, unlike all other concerns associated with nationalism and ethnocentrism is so closely tied to the individual self. Fear of being deprived of communicating skills seems to raise political passion to fever pitch'.

Language policies in multilingual societies are beset by the trade-off between standardization and disenfranchisement. Linguistic *standardization* comprises any set of policies that promote the dominant use of a unique or several languages while limiting the usage of languages spoken by other population groups. Indeed, linguistic standardization may deliver important benefits in terms of greater ease of communication, reducing costs of translation, increased trade, improved economic performance and administrative efficiency. However, excessive standardization may exacerbate the alienation of large minorities and widen the existing chasm between linguistic communities [Laponce, 2003].

A restriction of basic linguistic rights may create *disenfranchisement* of groups of individuals and cause citizens to lose their ability to communicate in the language of their choice. Standardization, which is often represented by a selection of official languages and allocation of linguistic rights, may alienate those groups of individuals whose cultural, societal and historical values and sensibilities are not represented by the official languages [Laitin, 1989]. As Pool (1991) points out, non-official languages may suffer from their "minority status" and limit employment and advancement possibilities of their native speakers.

Since in many cases it is not feasible to include all the languages in the set of official ones, a multilingual society must design some language standardization policies (for example, the "three-language formula" in India; [Baldridge, 1996] and the implementation of certain standardization measures [De Swaan, 2001], [Grin, 2004]).

However, the explosive and uncompromising nature of linguistic conflicts, the reluctance of linguistic majorities to concede rights to minorities make the choice of official languages a challenging and daunting task.

Thus, the choice of the set of official languages has to take into account the sensitivity of a society towards possible disenfranchisement of large groups of its citizens [Ginsburgh, Ortuño-Ortín and Weber, 2005] and has to rely on a delicate resolution of the interplay between administrative and cost efficiency, on the one hand, and the rights and desires of various linguistic groups, on the other [Van Parijs, 2005].

1. The model

To illustrate the individual and aggregate cost and benefits of standardization and disenfranchisement, we consider a society M and the set of languages L spoken in this society. We assume that every citizen i is endowed with a unique native language $n(i) \in L$ and a set of languages $L(i) \subset L$ that, to simplify, (s)he commands with identical ease. A linguistic profile of each individual i is the pair $(n(i), L(i))$, and society's linguistic profile is given by $P = (n(i), L(i))_{i \in M}$. A linguistic policy is represented by a set of official languages $K \subset L$ that is chosen for administrative, educational, and official communication functions in the society [Pool, 1991, 1996], and the extensive list of references therein [Ginsburgh, Ortuño-Ortín and Weber, 2005].

The choice of the set K represents a linguistic *standardization policy.* If the set of official languages K is non-empty and smaller than L, those members of the society whose native language is not included in K will be *disenfranchised* and some of their linguistic rights will be denied.

In order to evaluate the costs of disenfranchisement, we assume that every citizen i has utility function u_i defined over all subsets of L. We will denote $u_i(K)$ for $i \in M$ and $K \subset L$, where citizens with the same linguistic profiles have identical utility functions. It is important to stress that the functions u_i are defined over the set of languages as a whole, rather than being dissected into preferences over single languages. Though citizens may have preferences over single languages, their evaluation of the set of official languages could be crucially affected by inclusion or exclusion of their native language. The aggregate utility (welfare) function for the entire society is given by $W(u, P, K)$, where u is the vector of u_i's.

Our description indicates the special role played by the native languages of citizens in M, which can be viewed as the union of linguistic clusters M_l, where, for each $l \in L$, M_l consists of citizens whose native language is l. Assuming additivity of the aggregate utility, we have $W(u, P, K) = \sum_{l \in L} \sum_{i \in M_l} u_i(K)$. As a simple example, consider the *dichotomous* function based on the citizens' native languages [Ginsburgh and Weber, 2005], for which the value of $u_i(K)$ is 1 if i's native language, $n(i)$, is included in K, and zero if it is not. The latter group contains individuals who are *disenfranchised* by the imposed standardized measures. The value taken by the function W is the number of citizens whose native language belongs to the set K,

$$W^1(u, P, K) = \sum_{\{i \in N | n(i) \in K\}} .$$

One generalization of the dichotomous approach is to take into account the entire language profile of every citizen rather than her native language only. Then, the value of her/his utility function is 1 if at least one of the languages spoken by her/his is included in K and zero otherwise. Here, the notion of disenfranchisement is limited to those who speak no official language:

$$W^2(u, P, K) = \sum_{\{i \in N | L(i) \bigcap K \neq \emptyset\}} 1.$$

In evaluating citizens' preferences over subsets of languages one may take into account the similarity or the proximity between languages (see, for example, [Dyen, Kruskal and Black, 1992], for a matrix of distances between 95 Indo-European languages). Let $\delta(l, l')$ be the linguistic distance between two languages l and l'. Denote the linguistic distance between any two subsets T, T' of L as the minimal distance between a language from T and a language from T':

$$\delta(T, T') = \min_{l \in T, l' \in T'} \delta(l, l').$$

Then, the "linguistic welfare" of the society is function of the distances between citizens' native languages and the set of official languages K:

$$W^3(u, P, K) = w(\delta(n(1), K), \delta(n(2), K), \ldots, \delta(n(M), K)),$$

where $w : \Re_+^M \to \Re$ is decreasing in each of its M arguments. Again, a modified utility function could be defined over the distances between the sets $L(i)$ and K instead:

$$W^4(u, P, K) = w(\delta(L(1), K), \delta(L(2), K), \ldots, \delta(L(M), K)).$$

Note that enlarging the set of official language is welfare improving in all four specifications above. Thus, if the only goal of the society is to maximize aggregate utility, it should set $K = L$. However, there are also other considerations to take into account. Difficulties of communication, costs incurred by translation and interpretation, possible errors causing delays and sometimes paralysing multilateral discussions and negotiations impose a non-negligible burden on societies with a large number of official languages (in 2007, the European Union had to manage 23 official languages at a cost over \$1.5 billion).

Denote then by $C(K)$ the cost of maintaining the set K of official languages. Obviously, C is increasing, but its specific form depends on the intensity of the linguistic regime. There could be various requirements, including a "full" regime that every official document needs to exist in all official languages.

There is, thus, a trade-off between language standardization (and disenfranchisement of some citizens) and the translation, interpretation and communication costs generated by every additional official language. Formally, the society's objective is to find a set of languages K that maximizes the difference between aggregate utility and costs:

$$\max_{K \subset L} W(u, P, K) - C(K).$$

A solution to this problem is discussed by Grin, who argues that there must be an optimum, since "it is reasonable to assume that the benefits of diversity increase at a decreasing rate, while its costs increase at an increasing rate", and is addressed in [Ginsburgh, Ortuño-Ortín and Weber, 2005].

Language profiles considered so far as they are assumed given. In fact, they can be remarkably dynamic and change over time as individuals may decide to learn other languages. The reasons that induce citizens to do so can be analysed by examining the benefits and the costs that such learning generates. Benefits are often linked with the increased earning potential, especially in the case of immigrants who acquire the native language of the country in which they live (see, for example, [MacManus, Gould and Welsch, 1978], [Grenier, 1985], [Lang, 1986], [Chiswick, 1998] and references in [Grin and Vaillancourt, 1997]).

We consider the Selten and Pool "communicative benefits" approach that frees itself from the restriction that "earnings [are] a mechanism and firms a milieu of the incentive to learn languages". For every language l consider the set M_l of its native speakers, whose number is denoted by m_l.

Assume for simplicity that $L = j, k$ and that all citizens speak only their native language, so that the linguistic profile $L(i)$ consists of $n(i)$ for every $i \in M$. Citizens may learn the other language. Denote by $m_{j,k}, (m_{k,j})$ the number of citizens in $M_j(M_k)$ who do so. A citizen $i \in M_j$ who learns language k incurs a cost $C(\delta(j, k))$, where C is an increasing function of linguistic distance.

Let $u_j(m_j, \cdot)$ be the utility of $i \in M_j$, where the second argument indicates the number of individuals i can communicate with. We assume that the utility functions are increasing and, moreover, identical for all individuals with the same native language. If i learns k, it costs her/him $C_{j,k}$, but (s)he will be able to communicate with all citizens in M_k. Her/his gross benefit will be given by $u_j(m_j, m_k)$. If i does not learn k, (s)he will be able to communicate with those in M_k who learn language j, and her/his gross (and net) benefit will be $u_j(m_j, m_{k,j})$. This formulation leads to the following equilibrium condition that makes individuals in M_k indifferent between learning the other language and deciding not to do so: $u_j(m_j, m_k) - C_{j,k} = u_j(m_j, m_{k,j})$. This equation allows us to determine the number of citizens in group M_k who learn j, and in a similar manner the number of those in group M_j who learn k (see [Selten and Pool, 1991], [Church and King, 1993], [Shy, 2001], [Gabszewicz, Ginsburgh and Weber, 2005], [Ginsburgh, Ortuño-Ortín and Weber, 2006]. By imposing some additional conditions, such as continuity, concavity and super-modularity of the utility functions one can derive some comparative statics results. In particular, one can show that the number of learners of the foreign language j in country k is positively correlated with the number of j-speakers in other countries and negatively correlated with the population size of their own country k [Lazear, 1999], [Ginsburgh, Ortuño-Ortín and Weber, 2006]. These results also show that public policies may be useful in stimulating learning (for a cost-benefit analysis of linguistic policies in Quebec, see, for example, [Breton and Mieskowski, 1975], [Vaillancourt, 1987], see also [Fidrmuc and Ginsburgh, 2006] for policy suggestions in the EU).

2. Conclusion

In short, the questions raised by multilingualism offer serious challenges and the main reason is that linguistic policies are concerned not only with difficult trade-offs and resource allocation issues, but enter also the area of public policies that touch so closely personal values, beliefs and traditions.

See also culture and economics; social welfare function.

Acknowlegments

We should like to thank Yuval Weber for his help in preparing this manuscript.

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One More Uniqueness of the Shapley Value [3]

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Abstract. The class of TU games whose maximal per capita characteristic function values are attained on the grand coalition. Three axiomatic characterizations of the Shapley value – Shapley's original axiomatization, Sobolev's axiomatization with the aid of consistency, and Young's axiomatization by means of marginality – are used for the corresponding axiomatization of the Shapley value on the class of totally cooperative games. It is shown that only two last axiomatizations characterize the Shapley value uniquely, and Shapley's axiomatization leads to linear combinations of the Shapley value and the egalitarian value.

Keywords: maximal per capita values, Shapley value, axiomatic characterization.

Introduction

Axioms describing properties of cooperative game solutions can be divided into two classes: "within games" and "between games" axioms. The axioms from the first class formulate properties of a solution/value for single games from some class of games (such are the symmetry, the efficiency and the dummy axioms). They can be considered for every class of games, because the axioms are formulated for each game from the class separately. The "between games" axioms connect between themselves the solutions of different games. For example, all independence axioms belong to this class. Hence, the classes of games axiomatized with the help of between games axioms cannot be arbitrary. For example, if we define invariance or covariance of a solution w.r.t. some transformations of games, then all transformed games, together with the initial game, should belong to the class.

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[3] This work was supported by the NWO grant 047.017.017 and by Russian Foundation for Basic Research, grant 05-01-89005.

In the original axiomatization of the Shapley value with the aid of efficiency, dummy, symmetry, and additivity the first three axioms are “within games” ones, and the last one – a “between games” axiom.

A player faced with the prospect of playing a cooperative game can easily answer whether he agrees or not with some within games axioms, but only for this game. He may not have an opinion about these axioms with regard to other games. He may not even imagine to be a player of this or that game (e.g. a rich man does not represent himself as a poor one and vice versa), and in this case he cannot accept or reject a within games axiom on the whole class of games under consideration.

It is still more reasonable to consider a player not being sure to accept or not the “between games” axioms.

On the other hand, if a value satisfies a certain set of “within games” axioms on some class of games, then it trivially satisfies those axioms on every subclass of games. As for ”between games” axioms, then this fact is only true when all games being compared belong to the class. For example, consistency axioms require that the reduced games belong to the same class as the original game.

Therefore, the smaller the subclass of games under axiomatization, the more convincing the corresponding values turn out to be for games from this class, but the less likely these axioms will determine a unique value on it. The ultimate goal of the values’ axiomatizations should be the finding of minimal classes of games admitting the axiomatic characterization of a value with the help of certain system of axioms.

In this paper we consider the famous Shapley value and its axiomatizations on a subclass of cooperative games with transferable utilities (TU).

The first axiomatization of the Shapley value was obtained by Shapley (1953) for the class of superadditive games, but his proof can also be used to characterize the Shapley value on the space of all games with finite player sets. Weber (1988) obtained the characterization of the Shapley value with the same axioms for the class of monotonic games. Dubey (1975) characterized the Shapley value on the class of monotonic simple games by means of Shapley’s axioms where additivity was replaced by modularity (transfer): the sum of solutions of two simple games equals to the sum of solutions of two simple games being the minimum and the maximim of the initial games.

Neyman (1989) found the minimal class of games characterized by Shapley’s axioms. It is the additive class spanned by a single game, i.e. this class consists of the initial game and all linear combinations of the restricted on coalitions games. Neyman obtained the corresponding characterizations of the Shapley value on these classes of games. This result is a unique result characterizing the Shapley value by means of Shapley’s axioms on the minimal class of games.

The consistency axioms establish connections between the solution vector of a cooperative game and those of its reduced game. The last one is obtained from the initial game by removing one or more players and by giving them the payoffs according to a specific principle (e.g. a proposed payoff vector). Consistency of a value means that the restriction of the value payoff vector of the initial game to any coalition is the value payoff vector of the corresponding reduced game. Therefore, to characterize a value by means of consistency, it is necessary to define a universal set of players $\mathcal{N}$ and then to consider collections of all classes of TU games with the player sets $N \subset \mathcal{N}$.

Since reduced games are defined not uniquely, there are several definitions of value's consistency. In particular, the Shapley value has two axiomatizations with the help of consistency: in the sense of definitions by Sobolev [Sobolev, 1975] and by Hart–Mas-Colell [Hart and Mas-Colell, 1989]. The first definition requires the unboundedness of the universal set of players. The second one considers only games with at most n players. In both definitions consistency and standardness of a value for two-person games uniquely characterize the Shapley value on the corresponding classes of TU games.

One more axiomatization of the Shapley value for games with a fixed player set was obtained by Young (1985). His axiomatization of the Shapley value on the space of all TU games with a fixed player set consists of three axioms: efficiency, symmetry, and marginality. The last axiom states the dependence of players' payoffs only on their marginal contributions to the characteristic function of the game. Khmelnitskaya (2003) applied these axioms to the characterization of the Shapley value on the class of TU constant sum games.

In the paper a subclass of TU games with high per capita values of the grand coalition is considered. For games from this class the maximal per capita characteristic function value is attained on the grand coalition, so in some sense the players are interested in the complete cooperation. In fact, on this class Dutta's egalitarian solution [Dutta, 1990] is unique as well as for the convex games, and coincides with the egalitarian value assigning equal shares of the total payoff to all players. It turns out that the mentioned systems of axioms (except for the Hart–Mas-Colell axiomatization) characterize the Shapley value on the class under consideration either uniquely, or, together with the Shapley value, they describe the set of values being linear combinations of the Shapley and the egalitarian values.

The paper is organized as follows. In Section 1 we give a list of properties of cooperative games solutions (values) used in axiomatizations of the Shapley value. Three main axiomatizations of the Shapley value on the classes of all TU games with a fixed and variable player sets are given.

The main results are contained in Section 2. Here the class of games with high per capita grand coalition values is considered and three axiomatizations of the Shapley value cited in Section 1 are applied to the characterization of the same value on the class of games with high per capita grand coalition values. It turns out that Shapley's original axiomatization leads to the set of values being linear combinations of the Shapley and the egalitarian values. Consistency à la Hart–Mas-Colell cannot be applied to the class under consideration because subgames of games from this class (which take part in the definition of the reduced games) may not belong to it. Other two axiomatizations characterize the Shapley value uniquely on this class.

1. Properties of TU values and the known axiomatizations of the Shapley value

Let N be an arbitrary finite set. We denote by $\mathcal{G}_N$ a class of TU games with the player set N. Let $\Gamma = \langle, N, v\rangle \in \mathcal{G}_N$ denote an arbitrary game, $x \in X(\Gamma)$ denote an arbitrary payoff vector, where

$$X(\Gamma) = \{y \in \mathrm{I\!R}^N \mid \sum_{i \in N} y_i \leq v(N)\},$$

the set of payoff vectors of the game Γ, and

$$Y(\Gamma) = \{y \in \mathrm{I\!R}^n \mid \sum_{i \in N} y_i = v(N)\}$$

be the set of *efficient* (Pareto optimal) payoff vectors.

For any $x \in \mathrm{I\!R}^N$, $S \subset N$ denote by x_S the projection of x on the space $\mathrm{I\!R}^S$, and by $x(S)$ – the sum $\sum_{i \in S} x_i$, with a convention $x(\emptyset) = 0$.

A *solution* to a class $\mathcal{G}_N$ is a mapping σ, assigning to each game $\Gamma \in \mathcal{G}_N$ a subset $\sigma(\Gamma) \subset X(\Gamma)$ of its payoff vectors.

If $|\sigma(N, v) = 1|$ for a solution σ and for each game $\langle N, v\rangle \in \mathcal{G}_N$, then the solution σ is called a *value*.

Give now some properties of values, which will be applied in the following characterizations of the Shapley value on some classes of TU games (Section 4).

Recall that the player $i \in N$ in the game $\langle N, v\rangle$ is a *dummy*, if $v(S \cup \{i\}) = = v(S) + v(\{i\})$ for all $S \subset N, i \notin S$.

A value Φ to a class $\mathcal{G}_N$

- is *efficient*, if $\sum_{i \in N} \Phi_i(N, v) = v(N)$ for every TU-game $\langle N, v\rangle$;
- is *anonymous,* if $\Phi_{\pi(i)}(N, \pi v) = \Phi_i(N, v)$ for all games $\langle N, v\rangle$, all $i \in N$ and every permutation π of N. Here the game $\langle N, \pi v\rangle$ is defined by $(\pi v)(\pi S) := v(S)$ for all $S \subseteq N$;
- is *symmetric,* if for each $\langle N, v\rangle \in \mathcal{G}_N$, such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for some $i, j \in N$, and all $S \subset N, i, j \notin S$

$$\Phi_i(N, v) = \Phi_j(N, v).$$

It is known that anonymity implies symmetry for values.

- is *(weakly) translation covariant*, if $\Phi(N, v + a) = \Phi(N, v) + a$ for all games $\langle N, v\rangle$, and $a = (a_i)_{i \in N} \in \mathrm{I\!R}^N$ $(a_i = a_j, i, j \in N)$. Here the game $\langle N, v + a\rangle$ is defined by $(v + a)(S) := v(S) + \sum_{j \in S} a_j$ for all $S \subseteq N$;
- is *scale covariant*, if $\Phi(N, \alpha v) = \alpha \Phi(N, v)$ for all $\alpha > 0$;

- is *covariant*, if Φ is both translation and scale covariant;
- is *additive*, if $\Phi(N,v)+\Phi(N,w)=\Phi(N,v+w)$, where for every $S\subset N$ $(v+w)(S)=v(S)+w(S)$.
- is *linear* if and for all $\alpha,\beta\in\mathrm{I\!R}$ and for all games $\langle N,v\rangle$, $\langle N,w\rangle\in\mathcal{G}_N$ the game $\langle N,\alpha v+\beta w\rangle\in\mathcal{G}_N$, and

  $$\Phi(N,\alpha v+\beta w)=\alpha\cdot\Phi(N,v)+\beta\cdot\Phi(N,w).$$

  Here the game $\langle\alpha v+\beta w\rangle$ is defined by $(\alpha v+\beta w)(S):=\alpha\cdot v(S)+\beta\cdot w(S)$ for all $S\subseteq N$.
- satisfies the *dummy* axioms, if for every $\langle N,v\rangle\in\mathcal{G}_N$ where $i\in N$ is a dummy player, it holds $\Phi_i(N,v)=v(\{i\})$.
- is *marginalist*, or satisfies the *marginality* axiom, if from $\langle N,v\rangle,\langle N,w\rangle\in\mathcal{G}_N$, $v(S\cup\{i\})-v(S)=w(S\cup\{i\})-w(S)$ for some $i\in N$ and all coalitions $S\not\ni i$, it follows

  $$\Phi_i(N,v)=\Phi_i(N,w).$$

The following axioms connect between themselves the values of games with different player sets. Hence, to formulate them, we should consider the collections of classes of TU games $\mathcal{G}=\bigcup_{N\subset\mathcal{N}}\mathcal{G}_N$ for some infinite universal set $\mathcal{N}$ such that

$$N\subset\mathcal{N}\implies N'\subset\mathcal{N}\text{ for all }N'\subset N. \tag{1}$$

- A value Φ on a class $\mathcal{G}$ of TU games is *consistent* or satisfies the *reduced game property* if from $\langle N,v\rangle\in\mathcal{G},x=\Phi(N,v)$ it follows that for every $N'\subset N$ the *reduced game* $\langle N',v^x\rangle$ or $\langle N',v^\Phi\rangle$ belong to the class $\mathcal{G}$ as well, and

  $$\Phi'_N(N,v)=\Phi(N',v^x)\quad(=\Phi(N',v^\Phi)).$$

In this definition different notations for reduced games are given. This follows from non-uniqueness of the definitions of the reduced games. For the axiomatization of the Shapley value the following definitions of the reduced games are applied:

- A *linear reduced game* of the game $\langle N,v\rangle\in\mathcal{G}$ on the player set $N\setminus\{i\}$ and w.r.t. a payoff vector x is the game $\langle N\setminus\{i\},v^x_{N\setminus\{i\}}\rangle$, whose characteristic function $v^x_{N\setminus\{i\}}$ is defined by

$$v^x_{N\setminus\{i\}}(S)=\begin{cases}v(N)-x, & \text{if } S=N\setminus\{i\},\\ 0, & \text{if } S=\emptyset,\\ w_{n,s}v(S)+(1-w_{n,s})(v(S\cup\{i\})-x) & \\ & \text{otherwise.}\end{cases} \tag{2}$$

Here the numbers $w_{n,s} \in [0,1]$ are weights, depending on the numbers of players in the initial game and in the corresponding coalition.

The reduced games on arbitrary player sets $N' \subset N$ are defined by a consecutive elimination of players from the game. To obtain the same result for different permutations of players leaving the game, the weights w_{ns} should satisfy the equalities

$$w_{n-1,s}(1-w_{n,s}) = w_{n,s+1}(1-w_{n-1,s}) \text{ for all } N, s \leq n-2$$

[Yanovskaya and Driessen, 2002].

The following definition due to Hart–Mas-Colell [Hart and Mas–Colell, 1990] define the reduced games w.r.t. some value F to the class $\bigcup_{N' \subset N} \mathcal{G}_{N'}$ for arbitrary finite N.

- A *reduced game in the sense of Hart–Mas-Colell* of the game $\langle N, v\rangle$ on the player set $N' \subset N$ is the game $\langle N', v^F\rangle$, where the characteristic function v^F is defined as follows:

$$v^F(S) = \begin{cases} v(N) - \sum_{i \in N \setminus N'} F_i(N, v), & \text{for } S = N', \\ v(S \cup (N \setminus N')) - \sum_{i \in N \setminus N'} F_i(S \cup (N \setminus N'), v), \\ \hfill \text{for other } S, \end{cases} \tag{3}$$

where $\langle S \cup N \setminus N', v\rangle$ is the subgame of $\langle N, v\rangle$.

Shapley's amazing result consisted in the fact that four axioms[4] from the list above characterize a value uniquely.

Theorem [Shapley, 1953]. *For each finite set N there exists a unique value on the class $\mathcal{G}_N$ satisfying the efficiency, symmetry, dummy, and additivity axioms: it is the Shapley value given as follows:*

$$\Phi_i(N, v) = \sum_{S : S \ni i} \frac{(n-s)!(s-1)!}{n!} \Big(v(S \cup \{i\}) - v(S) \Big) \text{ for all } i \in N. \tag{4}$$

The most appealing property following from (4) is that the Shapley value is a marginalist value. Young (1985) obtained another characterization of the Shapley value with the help of marginality axiom.

Theorem [Young, 1985]. *For each finite N there exists a unique value on the class $\mathcal{G}_N$ satisfying efficiency, symmetry, and marginality. It is the Shapley value.*

Consider now axiomatizations of the Shapley value on some classes with variable player sets by some consistency axioms. The first such a characterization was given by A. Sobolev (1975) who used the linear consistency axiom with the weights

$$w_{n,s} = \frac{n-s-1}{n-1}. \tag{5}$$

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[4] In the original paper (Shapley, 1953) Shapley used a unique carrier axiom instead of both the efficiency and the dummy axioms.

Theorem [Sobolev, 1975]. *The Shapley value is the unique value on the class $\mathcal{G}$, satisfying efficiency, symmetry, and linear consistency with the weight defined in (5).*

The analogous result, but for the classes of games with at most n players was obtained by Hart and Mas-Colell for their definition of consistency.

Theorem [Hart, Mas-Colell, 1989]. *For each finite set N the Shapley value is the unique value on the class $\bigcup_{N'\subset N}\mathcal{G}_{N'}$ satisfying efficiency, symmetry, and consistency in the sense of the definition (3) of the reduced games.*

In the two last Theorems axioms efficiency and symmetry can be replaced by property of *standardness* for two-person games. Recall that a value Φ is *standard* on a class of two-person games, if for every game $\langle\{i,j\},v\rangle$ from the class and every player $k\in\{i,j\}$

$$\Phi_k(\{i,j\},v)=v(\{k\})+\frac{1}{2}(v(\{i,j\})-v(\{i\})-v(\{j\}),i,j=1,2.$$

The standard value is efficient, linear and symmetric. The class of all efficient, linear, and symmetric values for two-person games consists of λ-standard values, $\lambda\in\mathbb{R}$. A value Φ^λ is λ-*standard* for a class of two-person games if for every game $\langle\{i,j\},v\rangle$ and every player $k\in\{i,j\}$

$$\Phi^\lambda_k(\{i,j\},v)=\lambda\cdot v(\{k\})+\frac{1}{2}\Big(v(\{i,j)\}-\lambda\cdot v(\{i\})-\lambda\cdot v(\{j\}\Big).$$

It is clear that λ-standard values are linear combinations of the egalitarian value and the standard value.

Up to the present there are many other axiomatizations of the Shapley value (e.a. Myerson (1977), Chun (1989), Brink (2001)), but in this paper we will apply only the systems of axioms used in the cited theorems to characterize the Shapley value on a subclass of games.

3. Games with high per capita grand coalition values

3.1. Definition and properties

Consider the class $\mathcal{G}^h_N$ of TU games with the player set N, such that the maximum in coalitions of the per capita characteristic function values $\frac{v(S)}{s}$ is attained on the grand coalition N:

$$\langle N,v\rangle\in\mathcal{G}^h_N \iff \frac{v(N)}{n}\geq\frac{v(S)}{s}\text{ for all } S\subset N, \tag{6}$$

where $n=|N|, s=|S|$ – are the numbers of players in the coalitions.

We call $\mathcal{G}^h_N$ the class of games with *high per capita grand coalition values.* In this section we show the uniqueness or nonuniqueness of the Shapley value on the classes $\mathcal{G}^h_N$ and $\mathcal{G}^h=\bigcup_{N\subset\mathcal{N}}\mathcal{G}^h_N$ in the axiomatizations given in Section 2.

The class $\mathcal{G}_N^h$ has some attractive properties:

Proposition 1. *The class $\mathcal{G}_N^h$ is balanced.*

Proof.

Let $\langle N, v\rangle \in \mathcal{G}_N^h$ be an arbitrary game. Then the egalitarian payoff vector $\left(\frac{v(N)}{n}, \dots, \frac{v(N)}{n}\right)$, evidently, belongs to the core of $\langle N, v\rangle$.

Proposition 2. *If $i \in N$ is a dummy in a game $\langle N, v\rangle \in \mathcal{G}_N^h$, then*

$$v(\{i\}) = \frac{v(N)}{n}. \tag{7}$$

Proof.

Equality (7) follows from the following system of inequalities and equality:

$$\begin{aligned} &v(\{i\}) + v(N \setminus \{i\}) = v(N), \\ &v(\{i\}) \leq \frac{v(N)}{n}, \\ &\frac{v(N \setminus \{i\})}{n-1} \leq \frac{v(N)}{n}. \end{aligned}$$

Moreover, the egalitarian value on the class $\mathcal{G}_N^h$ is the unique value of constrained egalitarianism [Dutta, 1990]. The last value was defined on the class of convex games, because for each game from this class there is a unique coalitions of maximal size on which the maximum of per capita values $\frac{v(S)}{s}$ is attained. This property provides the single-valuedness of the constrained egalitarian solution.

Games from the class $\mathcal{G}_N^h$ may not be convex. However, if we apply Dutta's algorithm to any game from $\mathcal{G}_N^h$ we obtain the unique payoff vector $\left(\frac{v(N)}{n}, \dots, \frac{v(N)}{n}\right)$.

Let $\mathcal{N}$ be an infinite universal set of players satisfying the condition (1).

Denote $\mathcal{G}^1 = \bigcup_{N \subset \mathcal{N}} \mathcal{G}_N^h$. For the classes $\mathcal{G}_N^h$ with finite N, and $\mathcal{G}^h$ we give the analogs of Theorems given in Section 2 characterizing the Shapley value for the classes of all TU games with fixed player sets $\mathcal{G}_N$ for each finite N, and for the class $\mathcal{G}$.

3.2. Shapley's axiomatization

We begin with Shapley's axiomatization [Shapley, 1953]. Note that, besides the Shapley value itself, the egalitarian solution also satisfies the Shapley axioms on the class $\mathcal{G}_N^1$ because of (7).

First, let us find the general form of efficient, linear, and symmetric values on the class $\mathcal{G}_N^h$.

Lemma 1. *If a value Psi on the class $\mathcal{G}_N^h$ is efficient, linear, and symmetric, then*

$$\Psi_i(N, v) = \frac{v(N)}{n} + \sum_{\substack{S: S \ni i \\ S \subsetneq N}} \frac{a_s}{s} v(S) - \sum_{S: S \not\ni i} \frac{a_s}{n-s} v(S) \text{ for all } \langle N, v\rangle \in \mathcal{G}_N^1. \tag{8}$$

Proof.

Note that (8) gives the general form of efficient, linear, and symmetric values on the class of all TU games $\mathcal{G}_N$ (see, e.g., [Ruiz et al., 1998]).

Therefore, the value Ψ in (8) is efficient, linear, and symmetric on the class $\mathcal{G}_N^1$ as well.

Let now Ψ be an arbitrary efficient, linear, and symmetric value on the class $\mathcal{G}_N^1$. For any number A denote by v_A the following characteristic function:

$$v_A(S) = \begin{cases} A, & \text{if } S = N, \\ 0, & \text{for other } S. \end{cases} \tag{9}$$

Then $\langle N, v_A\rangle \in \mathcal{G}_N^1$ for $A > 0$, and for each game $\langle N, v\rangle \in \mathcal{G}_N$ there is $A > 0$ such that $\langle N, v + v_A\rangle \in \mathcal{G}_N^1$. Denote $w_A = v + v_A$ (or $w_A(v)$, if we would like to fix the game v). Define the value Φ on the class $\mathcal{G}_N$ as follows:

$$\Phi(N, v) = \Psi(w_A) - \Psi(v_A). \tag{10}$$

Equality (10) is well-defined, since Φ does not depend on A by additivity of Ψ. In fact, if $v = w_A - v_A = w_B - v_B$, then

$$w_A + v_B = w_B + v_A \implies \Psi(N, w_A) + \Psi(N, v_B) = \Psi(N, w_B) + \Psi(N, v_A).$$

Thus, the value Φ is determined on the whole class $\mathcal{G}_N$. It is clear that it is efficient and symmetric.

Let us show additivity:

Consider two games $\langle N, v_1\rangle, \langle N, v_2\rangle \in \mathcal{G}_N$. Let A_1, A_2 be arbitrary numbers such that $\langle N, w_{A_1}(v_1)\rangle, \langle N, w_{A_2}(v_2)\rangle \in \mathcal{G}_N^1$. Denote $A = A_1 + A_2$. Then $\langle N, (v_1+v_2)+v_A\rangle \in \mathcal{G}_N^1$. By definition of Φ

$$\begin{aligned} \Phi(N, v_1 + v_2) &= \Psi(N, w_A(v_1 + v_2)) - \Psi(N, v_A), \\ w_A(v_1 + v_2) &= v_1 + v_2 + v_A = v_1 + v_2 + v_{A_1} + v_{A_2}, \end{aligned}$$

from where by additivity of Ψ we have $\Phi(N, v_1 + v_2) =$

$$= \Psi(N, v_1 + v_{A_1}) + \Psi(N, v_2 + v_{A_2}) - \Psi(N, v_{A_1}) - \Psi(N, v_{A_2}) = \Phi(N, v_1) + \Phi(N, v_2).$$

Positive homogeneity of Φ also follows from that of Ψ. Therefore, the value Φ has the form (8). By symmetry $\Psi(N, v_A) = (A/n, \ldots, A/n)$, and, hence, $\Psi(N, w_A)$ can be also represented by (8). An arbitrary characteristic function w such that $\langle N, w\rangle \in \mathcal{G}_N^1$, can be represented as $w = v + v_A, A \geq 0, v \in \mathcal{G}_N$, i.e. $w = w_A(v)$, therefore, the value Ψ on the class $\mathcal{G}_N^1$ has the form (8).

Theorem 1. *Values on the class $\mathcal{G}_N^h$ satisfying efficiency, linearity, symmetry, and dummy form an one-parametric family of values, being linear combinations of the Shapley value and the egalitarian value.*

Proof.

Let us show the formula for coefficients in (8):

$$a_s = \alpha \frac{(s-1)!(n-s)!}{n!} \text{ for all } \alpha \in \mathbb{R}. \tag{11}$$

It is clear that the value Ψ over the class $\mathcal{G}_N^h$, determined in (8) with coefficients (11) verifies the conditions of the Theorem.

Now let Ψ be an arbitrary value on the class $\mathcal{G}_N^h$, verifying the conditions of the Theorem. Then by Lemma 1 it is defined by (8) for some α_s. Let $\langle N, v\rangle \in \mathcal{G}_N^h$ be an arbitrary game with the dummy player i. Then $\Phi_i(N,v) = v(\{i\}) = \frac{v(N)}{n}$, and (8) implies

$$\sum_{\substack{S:S\ni i\\ S\subsetneq N}} \frac{a_s}{s}(v(S\setminus\{i\}) + v(\{i\})) - \sum_{S:S\not\ni i} \frac{a_s}{n-s} v(S) = 0. \tag{12}$$

for all $v(S), S \not\ni i$. The numbers $v(S), S\subset N$ satisfy the following equalities: $v(\{i\}) = = \frac{v(N)}{n} = \frac{v(N\setminus\{i\})}{n-1}$, $v(S)+v(\{i\}) = v(S\cup\{i\})$. Put in (12) $v(N\setminus\{i\}) = (n-1)v(\{i\})$. Then in this equality the values $v(S), S\not\ni i, S\neq N\setminus\{i\}$ and $v(i)$ may be arbitrary, only satisfying inequalities $v(S)\leq sv(\{i\})$, and (12) becomes an identity w.r.t. such $v(S)$. Therefore, the coefficients in $v(S), S\not\ni i, S\neq N\setminus\{i\}$, and in $bv(\{i\})$ may be nullified, $a_s = \frac{s!(n-s)!}{n!}$ up to an arbitrary multiplier α. Putting these values a_s in (8) we obtain

$$\Psi_i(N,v) = \frac{v(N)}{n} + \alpha\left(\sum_{\substack{S:S\ni i\\ S\subsetneq N}} \frac{(n-s)!(s-1)!}{n!} v(S) - \sum_{S:S\not\ni i} \frac{s!(n-s-1)!}{n!} v(S)\right) =$$

$$= (1-\alpha)\frac{v(N)}{n} + \alpha \sum_{S:S\ni i} \frac{(s-1)!(n-s)!}{n!}(v(S) - v(S\setminus\{i\})) = (1-\alpha)\frac{v(N)}{n} +$$

$$+\alpha Sh_i(N,v).$$

3.3. Axiomatization with the help of consistency

Definition (3) of reduced games w.r.t. some value Φ can not be applied for axiomatizations of values on the class $\mathcal{G}_N^h$, because subgames of a game from $\mathcal{G}_N^h$ w.r.t. the Shapley value may not belong to this class. For example, consider the three-person game $\langle N, v\rangle \in \mathcal{G}_N^h$, $N = \{1,2,3\}$ such that

$$\begin{aligned} v(N) &= 30, \quad v(\{1\}) = 0,\ v(\{2\}) = 5,\ v(\{3\}) = 0,\\ v(1,2) &= 10, \quad v(2,3) = 9,\ v(1,3) = 20. \end{aligned}$$

The Shapley value $Sh(N,v) = (12, 9, 9)$.

Consider the reduced game $\langle\{1,2\}, v^{Sh}\rangle$ in the sense of Hart–Mas-Colell. We obtain

$$v^{Sh}(1,2) = 21, \quad v^{Sh}(\{1\}) = v(1,3) - Sh_3(1,3) = 20 - 5 = 15,$$

Since $\frac{21}{2} < 15$, the game $\langle\{1,2\}, v^{Sh}\rangle \notin \mathcal{G}^h_N$.

Thus, the Hart–Mas-Colell theorem [Hart and Mas-Colell, 1989] cannot be extended to classes of games whose subgames do not belong to these classes.

Let us consider linear consistency determined by the reduced games given by (2) where the weights $w_{n,s}$ are defined in (5). To deal with linear consistency we should admit an infinite universal set $\mathcal{N}$ of players and to consider the class $\mathcal{G}^h = \bigcup_{N\subset\mathcal{N}} \mathcal{G}^h_N$, where the union is taken up on all finite subsets of $\mathcal{N}$.

Denote by Φ^α the value for the class $\mathcal{G}^h_N$, being the linear combination of the Shapley value and the egalitarian value:

$$\Phi^\alpha(N,v) = \alpha\Phi(N,v) + (1-\alpha)EG(N,v) \text{ for all games } \langle N,v\rangle \in \mathcal{G}^h_N,$$

and let $x^\alpha = \Phi^\alpha(N,v)$ for some arbitrary game $\langle N,v\rangle \in \mathcal{G}^h_N$. Consider the reduced game of $\langle N,v\rangle$ on the player set $N\setminus\{i\}$ for some $i\in N$ w.r.t. the payoff x^α. The reduced game will be denoted by $\langle N\setminus\{i\}, v^\alpha\rangle$. Then by (2)

$$v^\alpha(S) = \begin{cases} v(N) - \Phi^\alpha_i(N,v), & \text{for } S = N\setminus\{i\}, \\ \dfrac{n-s-1}{n-1}v(S) + \dfrac{s}{n-1}(v(S\cup\{i\}) - \Phi^\alpha_i(N,v)), & \text{for } S \subsetneqq N\setminus\{i\}. \end{cases}$$

It turns out that all values Φ^α permit the axiomatization with the help of linear consistency with the coefficients $w_{n,s} = \frac{n-s-1}{n-1}$.

Theorem 2. *The values Φ^α are the unique values on the class $\mathcal{G}^h_N$ satisfying linear consistency with the coefficients $w_{n,s} = \frac{n-s-1}{n-1}$ and being the α-standard values on the class of two-person games $\mathcal{G}^h_N, |N| = 2$.*

Proof.

First, let us show that the linear reduced games $\langle N\setminus\{i\}, v^x\rangle$, $i\in N$ of the games from the class $\mathcal{G}^h_N$ on the players set $N\setminus\{i\}$ and w.r.t. arbitrary payoff vector x, belong to the class $\mathcal{G}^h_{N\setminus\{i\}}$.

By the definition of the linear reduced games for any $S\subset N\setminus\{i\}$ he have

$$v^x(S) = \begin{cases} v(N) - x_i, & \text{if } S = N\setminus\{i\}, \\ \dfrac{n-s-1}{n-1}v(S) + \dfrac{s}{n-1}\Big(v(S\cup\{i\}) - x_i\Big), & \text{if } S \subsetneqq N\setminus\{i\}. \end{cases} \tag{13}$$

Equalities (13) imply that the difference

$$\frac{v^x(N\setminus\{i\})}{n-1} - \frac{v^x(S)}{s}$$

does not depend on x, and from $\langle N, v\rangle \in \mathcal{G}^h_N$ it follows that the reduced game $\langle N \setminus \{i\}, v^x\rangle \in \mathcal{G}^h_{N\setminus\{i\}}$ for any x.

Therefore, the Shapley value is linear consistent (with the coefficients $w_{n,s} = \frac{n-s-1}{n-1}, s = 1, 2, \leq n-2, n = 3, 4, \ldots$) on the class $\mathcal{G}^h$. It is clear that the egalitarian value is also linear consistent on this class, and their linear combinations are also linear consistent on the same class.

Now let Ψ be an arbitrary value that is linear consistent on the class $\mathcal{G}^h$ and such that $\Psi = \Phi^\alpha$ on the class of two-person games from $\mathcal{G}^h$. The α-standardness of the value Φ^α and linear consistency of Ψ imply that there is at most one value satisfying the conditions of the Theorem [Yanovskaya and Driessen, 2002][5] As it has been already proved, it is the value Φ^α.

If in the conditions of the Theorem we put $\alpha = 1$, then we obtain the next characterization of the Shapley value:

Corollary 1. *The Shapley value is a unique value on the class* $\mathcal{G}^h$ *that satisfies linear consistency with the coefficients* $w_{n,s} = \frac{n-s-1}{n-1}$ *and is standard on the class* $\mathcal{G}^{\langle}_{\mathcal{N}}, |\mathcal{N}| = \in$ *of two-person games.*

3.4. Axiomatization with the help of marginality

3.4.1 Some properties of marginalist values

Given a TU game $\langle N, v\rangle$ we denote the differences $a^i_S = v(S) - v(S \setminus \{i\})$, $i \notin$ $\notin S \subset N$. Every such a difference is called the *marginal contributions* of the player i to the coalition S. We will denote the vectors of marginal contributions of the player i by $a^i(v)$, or simply by a^i when it is clear what a game $\langle N, v\rangle$ is meant.

In this section we consider marginalist values Φ for TU games.

It is convenient to consider such values in the dividend form. Let $\langle N, v\rangle$ be a TU game in the characteristic function form. Then the Möbius transform

$$\begin{aligned} \Delta_S = & \sum_{T\subset S} (-1)^{s-t} v(T) \\ v(S) = & \sum_{T\subset S} \Delta_S \end{aligned} \tag{14}$$

permits to define the game $\langle N, v\rangle$ in the *dividend form* $\langle N, \Delta\rangle$, where the Harsanyi dividends $\Delta_S, S \subset N$ are interpreted as values that the coalition $S \subset N$ should pay for the entry in a larger coalition. Note that $\Delta_{\{i\}} = v(\{i\})$ for all $i \in N$.

As it has been already determined in the previous section a *marginalist value* for a class $\mathcal{G}_N$ of TU game is a mapping Φ associating with each game $\langle N, v\rangle \in \mathcal{G}_N$ a vector $\Phi \in \mathbb{R}^N$ whose coordinates Φ_i, $i \in N$ depend only on the differences $a^i_S =$ $= v(S) - v(S \setminus \{i\})$, $i \notin S$.

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[5] In the cited paper the class of all TU game was considered, however, the proof of uniqueness of the value was fulfilled separately for the arbitrary game, including, certainly, games from the class $\mathcal{G}^h$.

From (14) it follows that if a game $\langle N, v\rangle$ is given in the dividend form $\langle N, \Delta\rangle$, then a marginalist value is defined as a mapping $\Phi : \mathbb{R}^{2^{n-1}} \to \mathbb{R}^n$ such that for each game $\langle N, \Delta\rangle$ the coordinates of the vector $\Phi(N, \Delta)$ depend only on the values Δ_S, $S \ni i$, we denote them by Δ^i_S.

First, give some properties of the marginalist values .

Proposition 3. *Let Φ be an efficient and marginalist value on some class $\mathcal{G}_N$. Then for every game $\langle N, v\rangle \in \mathcal{G}_N$ and player $i \in N$*

$$\Phi_i(N, v) = v(i) + F_i(\{\Delta_S\}_{S\ni i, S\neq i})$$

for some function F_i. Here Δ_S are Harsanyi's dividends for the function v.

Proof.

By marginality of Φ $\Phi_i(N, v) = \Phi_i(\{\Delta_S\}_{S\ni i})$, and by efficiency $\sum_{i\in N} \Phi_i(N, v) = = v(N)$, i.e.

$$\sum_{i\in N} \Phi_i(\{\Delta_S\}_{S\ni i}) = \sum_{S\subset N} \Delta_S.$$

In the left-hand side of this equality only the component Φ_i depends on Δ_i. Put other components in the right-hand side. Then we obtain

$$\Phi_i(N, v) = \Delta_i + \sum_{S\subset N, S\neq i} \Delta_S - \sum_{j\in N, j\neq i} \Phi_j(N, v).$$

Denote by F_i two sums in the right-hand side:

$$F_i = \sum_{S\subset N, S\neq i} \Delta_S - \sum_{j\neq i} \Phi_j(N, v).$$

By marginality of Φ F_i depends only on $\Delta_S, S \ni i$, but it does not depend on Δ_i, i.e. $F_i = F_i(\{\Delta_S\}_{S\ni i, S\neq i})$.

Corollary 2. *If $\mathcal{G}_N$ is a class of games closed under identical translations $v(S) \to v(S) + \beta \cdot s$ for every number β, then $\Phi_i(N, v + \beta) = \Phi_i(N, v) + \beta$ for all $i \in N$.*

Proof.

Let a game $\langle N, v\rangle$ and, hence, the games $\langle N, v + \beta\rangle \in \mathcal{G}_N$. The dividends of $\langle N, v\rangle$ and $\langle N, v + \beta\rangle$ differ only on one-element coalitions. Therefore, by Proposition 1

$$\begin{aligned} \Phi_i(N, v) &= v(i) + F_i(\{\Delta_S\}_{S\ni i, S\neq i}) = \\ = \Phi_i(N, v + \beta) &= v(i) + \beta + F_i(\{\Delta_S\}_{S\ni i, S\neq i}). \end{aligned}$$

These equalities prove the Corollary.

The result of Corollary 2 can be strengthen. The proofs of two following Corollaries coincide with that of Corollary 2.

Corollary 3. *Let for some vector* $b = (b_i)_{i \in N}$ $\langle N, v\rangle, \langle N, v + b\rangle \in \mathcal{G}_N$, *where* $(v+$ $+b)(S) = v(S) + \sum_{i \in S} b_i$ *(though the class* $\mathcal{G}_N$ *may be not closed under covariant transformations). Then if* Φ *is efficient and marginalist value for the class* $\mathcal{G}_N$, *then* $\Phi_i(N, v + b) = \Phi_i(N, v) + \beta_i$ *for all* $i \in N$.

Corollary 4. *Let* $\langle N, v\rangle \in \mathcal{G}_N$, *and there is another game* $\langle N, v'\rangle \in \mathcal{G}_N$ *such that* $\Delta'_i = \Delta_i + \beta, \Delta'_S = \Delta_S$ *for other* $S \ni i$. *Then for every efficient and marginalist value* Φ

$$\Phi_i(N, v) + \beta = \Phi_i(N, v').$$

3.4.2 The class $\mathcal{G}_N^{h1}$

We begin to consider marginalist values with a subclass $\mathcal{G}_N^{h1} \subset \mathcal{G}_N^{h}$. To define this class it is more convenient to consider games in the dividend's form. Let $\langle N, v\rangle$ be an arbitrary game, $\langle N, \Delta\rangle$ be the same game in the dividend's form. Let for coalitions Q, S, $Q \subset S$ the following inequalities hold:

$$\sum_{T: Q \subset T \subset N} \Delta_T \geq \sum_{Q \subset T \subset S} \Delta_T. \tag{15}$$

The class $\mathcal{G}_N^{h1}$ is defined as follows: a game $\langle N, v\rangle \in \mathcal{G}_N^{h1}$ if and only if it belongs to the class $\mathcal{G}_N^{h}$ and inequalities (15) hold for all Q, S, $Q \subset S$ (including $Q = \emptyset$).

It is clear that $\mathcal{G}_N^{h1} \supset \mathcal{G}_N^{h} \cap \mathcal{G}_N^{tp}$, where $\mathcal{G}_N^{tp}$ is the class of *totally positive* TU games with the player sets N. In fact, if $\Delta_S \geq 0$ for all $S \subset N$, then inequalities (15) trivially hold. Moreover, this class is *monotonic* in Δ_N in the sense that

$$\langle N, \Delta\rangle \in \mathcal{G}_N^{h1}, \implies \langle N, \Delta'\rangle \in \mathcal{G}_N^{h1},$$

where

$$\begin{aligned} \Delta'_S &= \Delta_S \text{ for all } S \subsetneqq N, \\ \Delta'_N &> \Delta_N. \end{aligned}$$

Evidently, the class $\mathcal{G}_N^{h1}$, as well as $\mathcal{G}_N^{h}$, is invariant w.r.t. identical translation mappings. Therefore, for the class $\mathcal{G}_N^{h1}$ Corollary 2 holds, and in the sequel we may assume without loss of generality that, given a game $\langle N, v\rangle \in \mathcal{G}_N^{h1}$, its marginal contributions $a^i(v) \geq 0$ for all $i \in N$.

First, we prove the uniqueness of the Shapley value for the class $\mathcal{G}_N^{h1}$.

Recall that equalities

$$a_S^i = \sum_{T: i \in T \subset S \cup i} \Delta_T, \quad S \not\ni i \tag{16}$$

put one-to-one correspondence between marginal contributions of a player i and dividends of coalitions containing this player. So, a marginalist value for any player is completely determined by the dividends of coalitions containing him.

Lemma 2. *Let $a^i = \{a^i_S\}, S \subset N,\ i \notin S$ be a vector of marginal contributions of the player i, for whom the dividends in (16) satisfy (15) for $Q \ni i$. Then there exists a game $\langle N, v^i\rangle \in \mathcal{G}^{h1}_N$ such that $a^i = a^i(v^i)$ and $k(\bar{v}^i) = k_i(v^i)$, where $k(v^i)$ is the number of non-zero dividends of coalitions $S, |S| > 1$ in the game $\langle N, v^i\rangle$, and $k_i(v^i)$ is the number of non-zero dividends of coalitions $S, i \in S, S \neq \{i\}$.*

Proof.

Let us find the dividends $\Delta^i_S, S \subset N, i \notin S$, which, together with the dividends $\Delta^i_S, i \in S$, defined by (16), will determine the required game $\langle N, v^i\rangle$. Define these dividends as follows:

$$\Delta^i_S = \begin{cases} 0, & \text{if } S \not\ni i, S \neq \{j\}, \\ a^i_{N\setminus\{i\}} = \sum_{T:i\in T\subset N} \Delta_T, & \text{of } S = \{j\}, j \neq i. \end{cases} \tag{17}$$

Then the game $\langle N, v^i\rangle$ has been completely determined. Evidently, its vector of marginal contributions of the player i coincides with $\{a^i_S\}_{S\subset N\setminus\{i\}}$, and $v(N) = na^i_{N\setminus\{i\}}$, $v(N\setminus\{i\}) = (n-1)a^i_{N\setminus\{i\}}$.

Let us show that $\langle N, v^i\rangle \in \mathcal{G}^{h1}_N$. First, check that it belongs to the class $\mathcal{G}^h_N$. By the definition of the game $\langle N, v^i\rangle$ we have $\frac{v(N)}{n} = a^i_{N\setminus\{i\}}$, and

$$\begin{aligned} \frac{v(S)}{s} &= \frac{a^i_S + (s-1)a^i_{N\setminus\{i\}}}{s} \leq \frac{v(N)}{n}, \text{ for } S \ni i, \\ \frac{v(S)}{s} &= \frac{sa^i_{N\setminus\{i\}}}{s} = a^i_{N\setminus\{i\}} = \frac{v(N)}{n}. \end{aligned} \tag{18}$$

Inequalities (18) hold because of (15) for $Q = \{i\}$ and of equalities $a^i_{S\setminus\{i\}} = = \sum_{T:i\in T\subset S} \Delta^i_S$. Now let us prove inequalities (15) for Δ^i_S, $\forall\, S \subset N$. Consider the following cases:

1) If $Q \ni i$, then the inequalities hold by the conditions of the Theorem.

2) $Q \not\ni i, |Q| > 1, S \ni i$. Then

$$\sum_{T:Q\subset T\subset S} \Delta^i_T = \sum_{T:Q\cup\{i\}\subset T\subset S} \Delta^i_T, \tag{19}$$

and the corresponding inequality is equivalent to (15) for $Q \cup i \subset S$ $\Delta^i_T, T \ni i$.

3) $Q = \{j\}, S \ni i$.

$$\sum_{T:j\in T\subset N} \Delta^i_T = \Delta^i_j + \sum_{T:i,j\in T\subset N} \Delta^i_T \geq \Delta^i_j + \sum_{T:i,j\in T\subset S} \Delta^i_T = \sum_{T:j\in T\subset S} .$$

4) $i \notin Q, i \notin S, |Q| > 1$. Then by the definition of the game $\langle N, v^i\rangle$ (17)

$$\sum_{T:Q\subset T\subset S} \Delta^i_T = 0.$$

Therefore, we should show that

$$\sum_{T:Q\subset T\subset N} \Delta_T^i = \sum_{T:Q\cup\{i\}\subset T\subset N} \Delta_T^i \geq 0 \text{ for every } Q \not\ni i. \tag{20}$$

The condition of the Lemma implies

$$\sum_{T:Q\cup\{i\}\subset T\subset N} \Delta_T^i \geq \sum_{T:Q\cup\{i\}\subset T\subset S} \Delta_T^i \tag{21}$$

for every $S \ni i$. Put a coalition $S = N \setminus \{j\}, j \notin Q$, consider inequality (21) for it, and delete identical components in both sides of the inequality. Then we obtain

$$\sum_{T:Q\cup\{i,j\}\subset T\subset N} \Delta_T^i \geq 0. \tag{22}$$

These inequalities hold for all $Q \not\ni i$ including $Q = \emptyset$, and for all $j \notin Q$, hence, they imply the required inequalities (20).

5) $Q = \{j\}, S \ni i$. In this case

$$\sum_{T:j\in T\subset S} \Delta_T^i \qquad = \Delta_j^i + \textstyle\sum_{T:i,j\in T\subset S} \Delta_T^i, \tag{23}$$

$$\sum_{T:j\in T\subset N} \Delta_T^i \quad = \Delta_j^i + \textstyle\sum_{T:i,j\in T\subset N} \Delta_T^i \geq \Delta_j^i + \textstyle\sum_{T:i,j\in T\subset S} \Delta_T^i, \tag{24}$$

where inequality (24) follows from the conditions of the Lemma.

Thus, the game $\langle N, v^i\rangle \in \mathcal{G}_N^{h1}$.

The number of non-zero dividends of non one-elements coalitions in the game $\langle N, v^i\rangle$ equals to $k_i(v^i) = k(v^i)$.

This Lemma implies that the individual values $\Phi_i, i \in N$ for the class $\mathcal{G}_N^{h1}$ have been determined for each vector of marginal contributions a_S^i, satisfying inequalities (15). The player i does not know whether the corresponding game belongs to the class $\mathcal{G}_N^{h1}$ or not. If the game does not belong to this class, the value Φ has been defined as well, but it may turn out inefficient.

Theorem 3. *On the class $\mathcal{G}_N^{h1}$ the Shapley value is the unique efficient, symmetric, and marginalist value.*

Proof.

Evidently, the Shapley value verifies all the axioms in the class $\mathcal{G}_N^{h1}$. Let us check the uniqueness. Let Φ be an arbitrary efficient, symmetric, and marginalist value on the class $\mathcal{G}_N^{h1}$. Then for each game $\langle N, v\rangle \in \mathcal{G}_N^{h1}$ and player $i \in N$ the value $\Phi_i(N, v)$ depends only on the vector of its marginal contributions a^i, or, that is the same, on the dividends $\Delta_S, S \ni i$ for which inequalities (15) hold.

The proof is fulfilled by Young's induction [Young, 1875], but in the number of non-zero dividends of non one-element coalitions. This number $k(v) = |\{\Delta_S \neq 0, |S| > 1\}|$ is invariant w.r.t. identical positive translations of the characteristic functions $v \to v + b, b > 0$ and the value Φ is covariant w.r.t. such translations. Thus, without loss of generality, we can consider only games from $\mathcal{G}_N^{h1}$ with non-negative $\Delta_i = v(i) \geq 0$ for all $i \in N$.

Let us prove the Theorem for $k(v) = 1$.

First, consider the case $\Delta_N \neq 0, \Delta_S = 0$ for all $S \subsetneq N, |S| > 1$. Then $\Delta_N > 0$. In this case the values $\Phi_i(N, v)$ for all $i \in N$ depend only on Δ_i, Δ_N, and, by symmetry of Φ, $\Phi_i(N, v) = f(\Delta_i, \Delta_N)$ for some function f, which is the same for all players. Since the value Φ on the class $\mathcal{G}_N^{h1}$ is covariant w.r.t. identical positive translations, for an arbitrary number $b > 0$ the following equality holds:

$$\Phi_i(N, v + b) = \Phi_i(N, v) + b \implies f(\Delta_i + b, \Delta_N) = f(\Delta_i, \Delta_N) + b, \tag{25}$$

where $(v + b)(S) = v(S) + bs$.

From (25) it follows that for any fixed $y > 0$ $f(x, y) = f(0, y) + x$, i.e.

$$\Phi_i(N, v) = \Delta_i + \Phi_i(N, v^0), \tag{26}$$

where $v^0(N) = \Delta_N, v^0(S) = 0$ for other S. Since the game $\langle N, v^0 \rangle$ is symmetric, $\Phi_i(N, v^0) = \frac{\Delta_N}{n}$ for all $i \in N$. Therefore, equalities (26) imply $\Phi(N, v) = Sh(N, v)$.

Let now $\Delta_S \neq 0, S \neq N$, and $\Delta_T = 0$ for all $T \neq S, |T| > 1$. Then all players $j \in N \setminus S$ are dummies, and $\Phi_j(N, v) = \varphi(\Delta_j)$ for some function φ. Similarly to the case above, covariance of Φ w.r.t. identical positive translations implies $\varphi(x + b) = = \varphi(x) + b$, i.e. $\varphi(x) = \varphi(0) + x$ for all $x > 0$.

If $\Delta_S = 0$ for all S, then the game $\langle N, v \rangle$ is zero, and $\Phi_i(N, v) = 0$ for all $i \in N$. Therefore, $\varphi(0) = 0$, and

$$\Phi_j(N, v) = \Delta_j \text{ for all } j \in N \setminus S. \tag{27}$$

For other players $i \in S$ we have the first case: $\Phi_i(N, v) = \phi(\Delta_i, \Delta_S)$ for all $i \in S$ and some function ϕ. Similar to that case we obtain

$$\Phi_i(N, v) = \Delta_i + \frac{\Delta_S}{s} \text{ for all } i \in S. \tag{28}$$

Equalities (27) and (28) imply $\Phi(N, v) = Sh(N, v)$.

Assume that the Theorem is true for all games $\langle N, v \rangle \in \mathcal{G}_N^{h1}$ with $k(v) < m$. Let $\langle N, v \rangle \in \mathcal{G}_N^{h1}$ be an arbitrary game with $k(v) = m$. Denote $R = \bigcap_{S:|S|>1} \Delta_S \neq 0$. Then $R \neq N$ (otherwise we would come to the first case $\Delta_N \neq 0, \Delta_S = 0$ for all $S \subsetneq N, |S| > 1$). For $i \notin R$ consider the game $\langle N, v^i \rangle$, determined in Lemma 4. The Lemma implies that $\langle N, v^i \rangle \in \mathcal{G}_N^{h1}$ and

$$\Phi_i(N, v^i) = \Phi_i(N, v). \tag{29}$$

The number of non-zero dividends of non one-element coalitions in the game $\langle N, v^i \rangle$ is less than m. Hence, by the inductive hypothesis, $\Phi(N, v^i) = Sh(N, v^i)$, and this equality together with (29) gives $\Phi_i(N, v) = Sh_i(N, v)$ for all $i \in N \setminus R$. Let $R \neq \emptyset$, then

$$\Phi_j(N, v) = f(\Delta_j, \{\Delta_S\}_{\substack{\Delta_S \neq 0, \\ S \supset R, |S|>1}})$$

for all $j \in R$ and some function f, which is the same for all players from R. By the definition of the set R for different players the arguments of F differ one from another only by the first coordinate, whose domain is set of non-negative numbers $\mathbb{R}_+$ (though in total the values $\Delta_j, j \in R$ cannot be arbitrary, because we should stay in the class $\mathcal{G}_N^{h1}$.) Covariance of Φ w.r.t. identical positive translations leads to the equality

$$f(x + b, \{\Delta_S\}_{S:\Delta_S \neq 0}) = f(x, \{\Delta_S\}_{S:\Delta_S \neq 0}) + b$$

for all $x > 0 \;\; b > 0$, ..

$$f(x, \{\Delta_S\}_{S:\Delta_S \neq 0}) = f(0, \{\Delta_S\}_{S:\Delta_S \neq 0}) + x. \tag{30}$$

By efficiency of Φ

$$\sum_{i \in R} \Phi_i(N, v) = \sum_{i \in R} Sh_i(N, v). \tag{31}$$

Equalities (30) and (31) imply

$$f(0, \{\Delta_S\}_{S:\Delta_S \neq 0}) = \frac{1}{r} \left(\sum_{i \in R} (Sh_i(N, v) - \Delta_i) \right),$$

i.e.

$$f(0, \{\Delta_S\}_{S:\Delta_S \neq 0}) = Sh_i(N, v^0), \tag{32}$$

where $v^0(S) = v(S) - \sum_{i \in S} \Delta_i$. At last, (32) and (27) imply

$$\Phi_i(N, v) = f(\Delta_i, \{\Delta_S\}_{S:\Delta_S \neq 0}) = Sh_i(N, v^0) + \Delta_i = Sh_i(N, v) \text{ for all } i \in R.$$

Note that Lemma 2 implies that the solution Φ satisfying the conditions of Theorem 3 is determined for the games from $\mathcal{G}_N^h \setminus \mathcal{G}_N^{h1}$ as well, only for such games it may be not efficient. Return to the whole class $\mathcal{G}_N^h$.

3.4.3 Class $\mathcal{G}_N^h$

We begin with the individual domains of marginal, efficient, and symmetric values for the class $\mathcal{G}_N^h$. Consider vectors of marginal contributions $a = \{a_S\}_{S \subset N \setminus \{i\}}$ satisfying the inequalities

$$0 \leq a_{N \setminus \{i\}} \geq \frac{a_S}{s} \text{ for all } S \subset N \setminus \{i\}. \tag{33}$$

Note that an arbitrary vector $a^i = \{a^i_S\}_{S \subset N \setminus \{i\}}$ of marginal contributions can be transformed by an identical translation mapping to the vector $a^i + b = \{a^i_S + b\}_{S \subset N \setminus \{i\}}$ satisfying inequalities (33). Therefore, by Lemma 1 (33) are not restrictive when we consider marginal and efficient values for the class $\mathcal{G}^h_N$.

Lemma 3. *Given a vector of marginal contributions* $a = \{a_S\}, S \subset N, S \not\ni i$ *satisfying inequalities (33), there exists a game* $\langle N, \bar{v}^i \rangle \in \mathcal{G}^h_N$ *such that* $a^i(\bar{v}^i) = a^i$.

Proof.

Define the game $\langle N, \bar{v}^i \rangle$ as follows:

$$\bar{v}^i(S) = \begin{cases} a_S, & \text{if } S \ni i, S \neq N, \\ 0, & \text{if } S \not\ni i, S \neq N \setminus \{i\}, \\ (n-1)a_{N \setminus \{i\}}, & \text{if } S = N \setminus \{i\}, \\ na_{N \setminus \{i\}}, & \text{if } S = N. \end{cases}$$

Then in the game $\langle N, \bar{v}^i \rangle$ $a^i_S(v) = a_S$. Let us show that $\langle N, \bar{v}^i \rangle \in \mathcal{G}^h_N$. We have

$$\frac{\bar{v}^i(N)}{n} = \frac{\bar{v}^i(N \setminus \{i\})}{n-1} = a_{N \setminus \{i\}}.$$

For other S either $v(S) = 0$, or $v(S) = a_S$, and by the condition of the Theorem $\frac{v(S)}{s} \leq a_{N \setminus \{i\}}$. Give the last uniqueness Theorem:

Theorem 4. *On the class* $\mathcal{G}^h_N$ *the Shapley value is the unique value that is efficient, anonymous, and marginalist.*

Proof.

Since (33) is not essential, the domain of marginalist and efficient values for each player $i \in N$ is the whole space $\mathbb{R}^{2^N - 1}$. Let Φ be an arbitrary marginalist, efficient, and symmetric value for the class $\mathcal{G}^1_N$. Then by Theorem 3 $\Phi(N, v) = Sh(N, v)$ for any game $\langle N, v \rangle \in \mathcal{G}^{h1}_N$. Assume that $\Phi \neq Sh$. it means that there exists a game $\langle N, v' \rangle \in \mathcal{G}^h_N$ such that $\Phi(N, v') \neq Sh(N, v')$.

For arbitrary $A > 0$ define the values Φ^A for the class $\mathcal{G}^h_N$ as follows:

$$\Phi^A_i(N, v) = \Phi_i(N, v^A) - \frac{A}{n} \text{ for all } i \in N,$$

where

$$(v^A)(S) = \begin{cases} v(S). & \text{if } S \neq N, \\ v(N) + A, & \text{if } S = N. \end{cases}$$

It is clear that for any $A > 0$ $\langle N, v^A \rangle \in \mathcal{G}^h_N$, and there exists A_0 such that for all $A \geq A_0$ $\langle N, v^A \rangle \in \mathcal{G}^2_N$.

The values Φ^A are marginalist, efficient, and symmetric. Let A be a number such that $\langle N, v' + A \rangle \in \mathcal{G}^{h1}_N$. Then for this game $\Phi^A(N, v' + A) \neq Sh(N, v' + A)$ that contradicts Theorem 3.

Acknowledgments
The author is grateful to Anna Khmelnitskaya for valuable remarks and discussions.

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Managing Catastrophe-bound Industrial Pollution with Game-theoretic Algorithm: the St Petersburg Initiative [1]

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Abstract. After several decades of rapid technological advancement and economic growth, alarming levels of pollutions and environmental degradation are emerging all over the world. Reports are portraying the situation as an industrial civilization on the verge of suicide, destroying its environmental conditions of existence with people being held as prisoners on a runaway catastrophe-bound train. Though cooperation in environmental control holds out the best promise of effective action, limited success has been observed. Existing multinational joint initiatives like the Kyoto Protocol can hardly be expected to offer a long-term solution because (i) the plans are limited to a confined set of controls like gas emissions and permits which is unlikely be able to offer an effective mean to reverse the accelerating trend of environmental deterioration, and (ii) there is no guarantee that participants will always be better off and, hence, be committed within the entire duration of the agreement.To create a cooperative solution a comprehensive set of environmental policy instruments including taxes, subsidies, technology choices, pollution abatement activities, pollution legislations and green technology R&D has to be taken into consideration.

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[1] This research was supported by Hong Kong Baptist University FRG Grant FRG/05-06/II-22, FRG/06-07/II-39, Strategic Development Fund 03-17-224 and European Commission TOCSIN Project RTD REG/I.5 (2006) D/553242.

The implementation of such a scheme would inevitably bring about different implications in cost and benefit to each of the participating nations. To construct a cooperative solution that every party would commit to from beginning to end, the proposed arrangement must guarantee that every participant will be better-off and the originally agreed upon arrangement remain effective at any time within the cooperative period for any feasible state brought about by prior optimal behavior. This is a "classic" game-theoretic problem. This paper applies the latest discoveries in cooperative game theory and mathematics by researchers at the Center of Game theory in St. Petersburg State University to suggest solutions and means to solve this deadlock problem of global environmental management. Such an approach would yield an effective policy menu to tackle one of the gravest problems facing the global market economy.

Keywords: Differential games, cooperative solution, subgame consistency, Industrial pollution.

Introduction

After decades of rapid technological advancement and economic growth, alarming levels of pollutions and environmental degradation are emerging globally. Due to the geographical diffusion of pollutants, unilateral response of one nation or region is often ineffective. Reports are portraying the situation as an industrial civilization on the verge of suicide, destroying its environmental conditions of existence with people being held as prisoners on a runaway catastrophe-bound train. Though global cooperation in environmental control holds out the best promise of effective action, limited success has been observed. This is the result of many hurdles, ranging from commitment and sharing of costs to disparities in future developments. It is hard to be convinced that multinational joint initiatives like the Kyoto Protocol or pollution permit trading can offer a long-term solution because (i) the plans are limited to a confined set of controls like gas emissions and permits which is unlikely be able to offer an effective mean to reverse the accelerating trend of environmental deterioration, and (ii) there is no guarantee that participants will always be better off and, hence, be committed within the entire duration of the agreement.

To construct a theoretical framework capturing the essence of a transboundary industrial pollution paradigm a differential game approach is adopted. Differential games provide an effective tool to study pollution control problems and to analyze the interactions between the participants' strategic behaviors and dynamic evolution of pollution. Applications of noncooperative differential games in environmental studies can be found in [Yeung, 1992], [Dockner and Long, 1993], [Tahvonen, 1994], [Stimming, 1999], [Feenstra et al 2001] and [Dockner and Leitmann, 2001]. Cooperative differential games in environmental control are presented by [Dockner and Long, 1993], [Jorgensen and Zaccour, 2001], [Fredj et al., (2004)], [Breton et al., 2005 and 2006], and [Petrosyan and Zaccour, 2003]. To incorporate the widely observed uncertainty phenomenon in pollution accumulation a stochastic differential game framework is adopted. Cooperative stochastic differential games with resource and environmental management contents include [Haurie et al., 1994], [Yeung and Petrosyan, 2004] and [Yeung, 2007].

To formulate the foundation for an effective policy menu to tackle one of the gravest problems facing the global market economy this analysis proposes a new cooperative initiative involving a comprehensive set of environmental policy instruments including taxes, subsidies, technology choices, pollution abatement activities, pollution legislations and green technology R&D. The implementation of such a scheme would inevitably bring about different implications in cost and benefit to each of the participating nations. To construct a cooperative solution that every party would commit to from beginning to end, the proposed arrangement must guarantee that every participant will be better-off and the originally agreed upon arrangement remain effective at any time within the cooperative period for any feasible state brought about by prior optimal behavior. This condition is known as *a subgame consistency* (see [Yeung and Petrosyan, 2004]) which guarantees that no participants would possess incentives to deviate from the previously adopted optimal cooperative behavior.To resolve this “classic” game-theoretic problem we further develop and apply the latest discoveries in cooperative game theory and mathematics by researchers at the Center of Game theory in St Petersburg State University (see [Yeung and Petrosyan, 2004, 2005, 2006a and 2006b], and [Yeung et.al., 2006], [Petrosyan and Yeung, 2007], and [Yeung, 2007]) to suggest solutions and means to solve this deadlock problem of global environmental management.

Last but not least, the analysis provides a payment distribution mechanism which governs each nation’s share of benefits and costs under the cooperative scheme so that the proposed subgame consistent solution can be realized. The analysis is also expected to open up a forum for policy research on global cooperative initiatives in environmental management.

The paper is organized as follows. Section 1 provides an analytical framework to study transboundary industrial pollution management. Noncooperative outcomes are characterized in Section 2. Cooperative arrangements, group optimal actions, and individually rational and subgame-consistent imputations are examined in Section 3. A payment distribution mechanism bringing about the proposed subgame-consistent solution is derived in Section 4. Policy Implications are discussed in Section 5 and concluding remarks are given in Section 6.

1. An Analytical Framework

In this section we present an analytical framework to study transboundary industrial pollution management.

1.1. The Industrial Sector

Consider a global economy which is comprised of n nations. At time instant s the demand system of the outputs of the nations is

$$P_i(s) = f^i[q_1(s), q_2(s), \cdots, q_n(s), s], \qquad i \in N \equiv \{1, 2, \cdots n\}, \tag{1}$$

where $P_i(s)$ is the price vector of the output vector of nation i and $q_j(s)$ is the output of nation j. The demand system (1) shows that the world economy is a form of generalized differentiated products oligopoly.
Industrial profits of nation i at time s can be expressed as:

$$f^i[q_1(s), q_2(s), \cdots, q_n(s), s]q_i(s) - c^i[q_i(s), v_i(s)], \qquad \text{for } i \in N, \tag{2}$$

where $v_i(s)$ is the set of environmental policy instruments of government i and $c^i[q_i(s), v_i(s)]$ is the cost of producing $q_i(s)$ under policy $v_i(s)$.

As mentioned before $v_i(s)$ is nation i's comprehensive set of policy instruments including taxes, subsidies, technology choices, pollution abatement activities, pollution legislations and green technology R&D. Profit maximization by the industrial sectors yields:

$$\begin{aligned} &f^i[q_1(s), q_2(s), \cdots, q_n(s), s] + f^i_{q_i}[q_1(s), q_2(s), \cdots q_n(s), s]q_i(s) - \\ &\quad - c^i_{q_i}[q_i(s), v_i(s)] = 0, \qquad \text{for } i \in N. \end{aligned} \tag{3}$$

Condition (3) is a system of implicit functions in $q(s) = [q_1(s), q_2(s), \cdots q_n(s)]$ with government policies $v(s) = [v_1(s), v_2(s), \cdots v_n(s)]$ being regarded as parameters. The existence of a market equilibrium reflects the satisfaction of the Implicit Function Theorem in (3), and nation i's instantaneous market equilibrium output can be expressed as:

$$q_i^*(s) = \hat{q}^i[v_1(s), v_2(s), \cdots, v_n(s), s] \equiv \hat{q}^i[v(s), s], \text{ for } i \in N. \tag{4}$$

One can readily observe from (4) that each nation's output decision depends on government environmental policies.

1.2. Accumulation Dynamics of Pollutants

Industrial production emits pollutants into the environment, and the amount of pollution created by different nations' outputs may be different. The pollutant will then add to the stock of existing pollution. Each government adopts its own pollution abatement policy to reduce the pollution stock. Let $x(s) \subset R^m$ denote the level of pollution at time s, the dynamics of pollution stock is governed by the stochastic differential equation:

$$\begin{aligned} dx(s) &= \Big[\sum_{j=1}^n a_j[q_j(s), v_j(s)] - \sum_{j=1}^n b_j[u_j(s), x(s)] - \delta[x(s]x(s))\Big]ds + \\ &+ \sigma[x(s)]dz(s), \\ &\quad x(t_0) = x_{t_0}, \end{aligned} \tag{5}$$

where
σ is a noise parameter and $z(s)$ is a Wiener process,
$a_j[q_j(s), v_j(s)]$ is the amount of pollution created by $q_j(s)$ amount of output produced under policy $v_i(s)$,
$u_j(s)$ is the pollution abatement effort of nation j,
$b_j[u_j(s), x(s)]$ is the amount of pollution removed by $u_j(s)$ unit of abatement effort of nation j, and $\delta[x(s)]$ is the natural rate of decay of the pollutants.

Moreover, $\delta(x)$ is negatively related to x reflecting the phenomenon that the natural rate of decay declines as the level of pollution stock rises.

The stochastic nature of (5) reflects the uncertainty in the evolution of the pollution stock.

1.3. The Governments' Objectives

The governments have to promote business interests and at the same time handle the financing of the costs brought about by pollution. In particular, each government maximizes the gains in the industrial sector plus tax revenue minus expenditures on pollution abatement and damages from pollution. The instantaneous objective of government i at time s can be expressed as:

$$\begin{aligned} & f^i[q_1(s), q_2(s), \cdots, q_n(s), s]q_i(s) - c^i[q_i(s), v_i(s)] - c_i^P[v_i(s)] \\ & - c_i^a[u_i(s)] - h_i[x(s)], \qquad i \in N, \end{aligned} \tag{6}$$

where $c_i^P[v_i(s)]$ is the cost of implementing the vector policy instrument $v_i(s)$, $c_i^a[u_i(s)]$ is the cost of employing u_i amount of pollution abatement effort, and $h_i[x(s)]$ is the value of damage to country i from $x(s)$ amount of pollution.

The governments' planning horizon is $[t_0, T]$. It is possible that T may be very large. The discount rate is r. At time T, the terminal appraisal of pollution damage is $g^i[x(T)]$ where $\partial g^i/\partial x < 0$. Each one of the n governments seeks to maximize the integral of its instantaneous objective (6) over the planning horizon subject to pollution dynamics (5) with controls on the level of abatement effort and output tax.

Substituting $q_i(s)$, for $i \in N$, from (4) into (5) and (6) one obtains a stochastic differential game in which government $i \in N$ seeks to:

$$\begin{aligned} \max_{v_i(s), u_i(s)} E_{t_0}\{ & \int_{t_0}^{T} \Big[f^i\{\hat{q}^1 v(s), s], \hat{q}^2[v(s), s], \cdots, \hat{q}^n[v(s), s], s\}\hat{q}^i[v(s), s] - \\ & - c^i\{\hat{q}^i[v(s), s], v_i(s)\} - c_i^P[v_i(s)] - c_i^a[u_i(s)] - h_i[x(s)\Big] e^{-r(s-t_0)} ds + \\ & + g^i[x(T)] e^{-r(T-t_0)}\} \end{aligned} \tag{7}$$

subject to

$$\begin{aligned} dx(s) = \Big[& \sum_{j=1}^{n} a_j\{\hat{q}^j[v(s), s], v_j(s)\} - \sum_{j=1}^{n} b_j[u_j(s), x(s)] - \delta[x(s)]x(s)\Big] ds + \\ & + \sigma[x(s)] dz(s), \qquad x(t_0) = x_{t_0}. \end{aligned} \tag{8}$$

2. Noncooperative Outcomes Characterization

Since the payoffs of nations are measured in monetary terms, the game (7)-(8) is a transferable payoff game.

Under a noncooperative framework, a feedback Nash equilibrium solution (if it exists) can be characterized as (see [Basar and Olsder, 1995] and [Yeung and Petrosyan, 2006b]):

Definition 1. *A set of feedback strategies* $\{u_i^*(t) = \mu_i(t,x),\ v_i^*(t) = \phi_i(t,x),$ *for* $i \in N\}$ *provides a Nash equilibrium solution to the game (7)–(8) if there exist suitably smooth functions :* $V^{(t_0)i}(t,x) : [t_0,T] \times R \to R$, $i \in N$, *satisfying the following partial differential equations:*

$$-V_t^{(t_0)i}(t,x) - \frac{1}{2}\sum_{k,j}\sigma^{kj}(x)V_{x_k x_j}^{(t_0)i}(t,x) =$$

$$= \max_{v_i,u_i}\Big\{\Big[f^i\{\hat{q}^1[v_i,\phi_{\neq i}(t,x),t],\hat{q}^2[v_i,\phi_{\neq i}(t,x),t],\cdots,$$

$$\hat{q}^n[v_i,\phi_{\neq i}(t,x),t]\}\times\hat{q}^i[v_i,\phi_{\neq i}(t,x),t] - c\{\hat{q}^i[v_i,\phi_{\neq i}(t,x),t],v_i\} -$$

$$-c_i^P[v_i] - c_i^a[u_i] - h_i(x)\Big]e^{-r(t-t_0)} + V_x^{(t_0)i}\Big[\sum_{j=1}^n a_j\{\hat{q}^j[v_i,\phi_{\neq i}(t,x),t],v_j\} -$$

$$-b_i(u_i,x) - \sum_{j=1,j\neq i}^n b_j[\mu_j(t,x),x] - \delta(x)x\Big]\Big\},$$

$$V^{(t_0)i}(T,x) = g^i[x]e^{-r(T-t_0)}, \tag{9}$$

where

$$\phi_{\neq i}(t,x) = [\phi^1(t,x),\phi^2(t,x),\cdots,\phi^{i-1}(t,x),\phi^{i+1}(t,x),\cdots,\phi^n(t,x)]. \tag{10}$$

In a prevailing Nash equilibrium the function $V^{(t_0)i}(t,x)$ is then the integral:

$$E_{t_0}\Big\{\int_t^T\Big[f^i\{\hat{q}^1[\phi(s,x(s)),s],\hat{q}^2[\phi(s,x(s)),s],\cdots,\hat{q}^n[\phi(s,x(s)),s],s\}\times$$

$$\times\hat{q}^i[\phi(s,x(s)),s] - c^i\{\hat{q}^i[\phi(s,x(s)),s],\phi_i(s,x(s))\} - c_i^P[\phi_i(s,x(s))] -$$

$$-c_i^a[\mu_i(s,x(s))] - h_i[x(s)]\Big]e^{-r(s-t_0)}ds +$$

$$+g^i[x(T)]e^{-r(T-t_0)}\Big|x(t) = x,\Big\} \tag{11}$$

for $i \in N$.

The game equilibrium dynamics then becomes:

$$dx(s) = \Big[\sum_{j=1}^n a_j\{\hat{q}^i[\phi(s,x(s)),s],\phi_j(s,x(s))\} - \sum_{j=1}^n b_j[\mu_j(s,x(s)),x(s)] -$$

$$-\delta[x(s)]x(s)\Big]ds + \sigma[x(s)]dz(s), \qquad x(t_0) = x_{t_0}. \tag{12}$$

Remark 1. One can readily verify that $V^{(\tau)i}(t, x_t) = V^{(t_0)i}(t, x_t)e^{r(\tau - t_0)}$, for $\tau \in$ $\in [t_0, T]$, is the value function to player i at time $t \in [\tau, T]$ when the state $x(t) = x_t$ in the game (7)–(8) which starts at time τ.

3. Cooperative Arrangement

Now consider the case when all the nations want to cooperate and agree to act so that an international optimum could be achieved. For the cooperative scheme to be upheld throughout the game horizon both group rationality and individual rationality are required to be satisfied at any time.

Group optimality ensures that all potential gains from cooperation are captured. Failure to fulfill group optimality leads to condition where the participants prefer to deviate from the agreed upon solution plan in order to extract the unexploited gains. Individual rationality is required to hold so that the payoff allocated to a nation under cooperation will be no less than its noncooperative payoff. Failure to guarantee individual rationality leads to condition where the concerned participants would reject the agreed upon solution plan and play noncooperatively.

Finally, as mentioned in Introduction, to ensure that the cooperative solution is dynamically consistent, the agreement must be subgame-consistent. In the absence of a punishment scheme, the cooperative plan will dissolve if any of the nations deviates from the agreed-upon plan.

3.1. Group Optimality and Cooperative State Trajectory

Consider the cooperative stochastic differential games with payoff structure (7) and dynamics (8). To secure group optimality the participating nations seek to maximize their joint expected payoff by solving the following stochastic control problem:

$$\max_{v_1, v_2, \cdots, v_n; u_1, u_2, \cdots, u_n} E_{t_0}\{\int_{t_0}^{T} \Big[\sum_{i=1}^{n} f^i\{\hat{q}^1[v(s), s], \hat{q}^2[v(s), s], \cdots,$$
$$\hat{q}^n[v(s), s], s]\hat{q}^i[v(s), s] - c^i\{\hat{q}[v(s), s], v_i(s)]\} - c_i^P[v_i(s)] - c_i^a[u_i(s)] -$$
$$- h_i[x(s)]\Big] e^{-r(t-t_0)} ds + \sum_{i=1}^{n} g^i[x(T)]e^{-r(T-t_0)}\}. \tag{13}$$

subject to (8).

Invoking Fleming's (1969) technique in stochastic control a set of controls

$$\{[v_i^*(t), u_i^*(t)] = [\psi_i(t, x), \varpi_i(t, x)],\ i \in N\}$$

constitutes an optimal solution to the stochastic control problem (13) and (8) if there exists continuously differentiable function:

$$W^{(t_0)}(t, x) : [t_0, T] \times R \to R,$$

$i \in N$, satisfying the following partial differential equations:

$$-W_t^{(t_0)}(t,x) - \frac{1}{2}\sum_{k,j}\sigma^{kj}(x)W_{x_k x_j}^{(t_0)}(t,x) =$$

$$= \max_{v_1,v_2,\cdots,v_n;u_1,u_2,\cdots,u_n}\Big\{\sum_{i=1}^{n} f^i[\hat{q}^1(v,t),\hat{q}^2(v,t),\cdots,\hat{q}^n(v,t),t]\hat{q}^i(v,t)-$$
$$-c^i[\hat{q}^i(v,t),v_i] - c_i^P(v_i) - c_i^a(v_i) - h_i(x)]e^{-r(t-t_0)}+$$
$$+W_x(t,x)\Big[\sum_{j=1}^{n} a_j[\hat{q}^j(v,t),v_j] - \sum_{j=1}^{n} b_j(u_j,x) - \delta(x)x\Big]\Big\},$$

and

$$W^{(t_0)}(T,x) = \sum_{i=1}^{n} g^i(x)e^{-r(T-t_0)}. \tag{14}$$

Hence, the players will adopt the cooperative control $\{[\psi_i(t,x),\varpi_i(t,x)]$, for $i \in N$ and $t \in [t_0,T]\}$. The value function $W^{(t_0)}(t,x)$ is then the integral:

$$E_{t_0}\Big\{\int_t^T \Big[\sum_{i=1}^{n} f^i\{\hat{q}^1[\psi(s,x(s)),s],\hat{q}^2[\psi(s,x(s)),s],\cdots$$
$$\cdots,\hat{q}^n[\psi(s,x(s)),s],s\}\hat{q}^i[\psi(s,x(s)),s] - c^i\{\hat{q}^i[\psi(s,x(s)),s],\psi_i(s,x(s))\} -$$
$$-c_i^P[\psi_i(s,x(s))] - c_i^a[\varpi_i(s,x(s))] - h_i[x(s)]\Big]e^{-r(s-t_0)}ds +$$
$$+\sum_{i=1}^{n} g^i[x(T)]e^{-r(T-t_0)}\Big|x(t) = x\Big\} \text{ for } i \in N. \tag{15}$$

The optimal trajectory under cooperation becomes

$$dx(s) = [\sum_{j=1}^{n} a_j\{\hat{q}^j[\psi(s,x(s)),s],\psi_j(s,x(s))\} - \sum_{j=1}^{n} b_j[\varpi_j(s,x(s)),x(s)] -$$
$$-\delta[x(s)]x(s)]ds + \sigma[x(s)]dz(s), \qquad x(t_0) = x_{t_0}. \tag{16}$$

The solution to (16) can be expressed as:

$$x^*(t) = x_0 + \int_{t_0}^{t}\{\sum_{j=1}^{n} a_j\{\hat{q}^j[\psi(s,x^*(s)),s],\psi_j(s,x^*(s))\} -$$
$$-\sum_{j=1}^{n} b_j[\varpi_j(s,x^*(s)),x^*(s)] - \delta[x^*(s)]x^*(s)]\}ds + \int_{t_0}^{t}\sigma x^*(s)dz(s). \tag{17}$$

We use X_t^* to denote the set of realizable values of $x^*(t)$ at time t generated by (17). The term x_t^* is used to denote an element in the set X_t^*.

The cooperative control for the game $\Gamma_c(x_0, T-t_0)$ over the time interval $[t_0, T]$ can be expressed more precisely as:

$$\psi_i(t, x^*(t)) \text{ and } \varpi_i(t, x^*(t)) \text{ for } t \in [t_0, T] \text{ and } i \in N. \tag{18}$$

Note that for group optimality to be achievable, the cooperative controls (18) must be exercised throughout time interval $[t_0, T]$.

Remark 2. One can readily verify that $W^{(\tau)}(t, x_t^*) = W^{(t_0)}(t, x_t^*)e^{r(\tau - t_0)}$, for $\tau \in [t_0, T]$, is the value function at time $t \in [\tau, T]$ of the stochastic control problem (8) and (13) which starts at time τ with $x(t) = x_t^* \in X_t^*$.

3.2. Individually Rational and Subgame-consistent Imputation

An agreed upon optimality principle must be sought to allocate the cooperative payoff. In a dynamic framework individual rationality has to be maintained at every instant of time within the cooperative duration $[t_0, T]$ given any feasible state generated by the cooperative trajectory (21). For $\tau \in [t_0, T]$, let $\xi^{(\tau)i}(\tau, x_\tau^*)$ denote the solution imputation (payoff under cooperation) over the period $[\tau, T]$ to player $i \in N$ given that the state is $x_\tau^* \in X_\tau^*$. Individual rationality along the cooperative trajectory requires:

$$\xi^{(\tau)i}(\tau, x_\tau^*) \geq V^{(\tau)i}(\tau, x_\tau^*), \text{ for } i \in N, x_\tau^* \in X_\tau^* \text{ and } \tau \in [t_0, T]. \tag{19}$$

Since nations are asymmetric and the number of nations may be large, a reasonable solution optimality principle for gain distribution is to share the expected gain from cooperation proportional to the nations' relative sizes of expected noncooperative payoffs. As mentioned before, a very stringent condition – subgame consistency – is required for a credible cooperative solution under a dynamic stochastic framework. In particular, the solution optimality principle must be maintained in any subgame which starts at a later time with any feasible state brought about by prior optimal behaviors so that no player has incentives to deviate from the previously adopted optimal behavior throughout the game.

In order to satisfy the property of subgame consistency, the optimality principle of sharing the expected gain proportional to the nations' relative sizes of expected noncooperative payoffs has to remain in effect throughout the cooperation period. Hence, the solution imputation scheme $\{\xi^{(\tau)i}(\tau, x_\tau^*); \text{ for } i \in N\}$ has to satisfy:

Condition 1.

$$\xi^{(\tau)i}(\tau, x_\tau^*) = V^{(\tau)i}(\tau, x_\tau^*) + \frac{V^{(\tau)i}(\tau, x_\tau^*)}{\sum_{j=1}^n V^{(\tau)j}(\tau, x_\tau^*)}[W^{(\tau)}(\tau, x_\tau^*) -$$

$$-\sum_{j=1}^n V^{(\tau)j}(\tau, x_\tau^*)] = \frac{V^{(\tau)i}(\tau, x_\tau^*)}{\sum_{j=1}^n V^{(\tau)i}(\tau, x_\tau^*)} W^{(\tau)}(\tau, x_t^*) \tag{20}$$

for $i \in N$, $x_\tau^* \in X_\tau^*$ and $\tau \in [t_0, T]$.

One can easily verify that the imputation scheme in Condition 1 satisfies both group optimality and individual rationality. Crucial to the analysis is the formulation of a payment distribution mechanism that would lead to the realization of Condition 1. This will be done in the next Section.

4. Payment Distribution Mechanism

Following Yeung and Petrosyan (2004 and 2006b), we formulate a payment distribution scheme over time so that the agreed upon imputation Condition 1 can be realized for any time instant $\tau \in [t_0, T]$ with the state being $x^*_\tau \in X^*_\tau$. Let the vectors

$$B(s, x^*_s) = [B_1(s, x^*_s), B_2(s, x^*_s), \cdots, B_n(s, x^*_s)]$$

denote the instantaneous payment to the n nations at time instant s when the state is $x^*_s \in X^*_s$. A terminal value of $g^i[x^*_T]$ is realized by nation i at time T.

To satisfy Condition 1 it is required that

$$\begin{aligned} &\xi^{(\tau)i}(\tau, x^*_\tau) = \frac{V^{(\tau)i}(\tau, x^*_\tau)}{\sum_{j=1}^n V^{(\tau)j}(\tau, x^*_\tau)} W^{(\tau)}(t, x^*_t) = \\ &\quad = E_\tau\{\int_\tau^T B_i(s, x^*(s) e^{-r(s-\tau)} ds + g^i[x^*_T] e^{-r(T-\tau)} \Big| x(\tau) = x^*_\tau\}, \\ &\text{for } i \in N, x^*_\tau \in X^*_\tau \text{ and } \tau \in [t_0, T]. \end{aligned} \tag{21}$$

To facilitate further exposition, we use the term $\xi^{(\tau)i}(t, x^*_\tau)$ which equals to:

$$\begin{aligned} &E_\tau\{\int_t^T B_i(s, x^*(s)) e^{-r(s-\tau)} ds + g^i[x^*_T] e^{-r(T-\tau)} \Big| x(t) = x^*_t\} = \\ &= \frac{V^{(\tau)i}(t, x^*_t)}{\sum_{j=1}^n V^{(\tau)i}(t, x^*_t)} W^{(\tau)}(t, x^*_t) = \frac{V^{(t)i}(t, x^*_t)}{\sum_{j=1}^n V^{(t)j}(t, x^*_t)} W^{(t)}(t, x^*_t) e^{-r(t-\tau)} \\ &= \xi^{(t)i}(t, x^*_\tau) e^{-r(t-\tau)}, \qquad \text{for } x^*_\tau \in X^*_\tau \text{ and } t \in [\tau, T]. \end{aligned} \tag{22}$$

to denote the expected present value (with initial time set at τ) of nation i's cooperative payoff over the time interval $[t, T]$.

Theorem 1. *A distribution scheme with a terminal payment $-g^i[x^*_T - \overline{x}^i]$ at time T and an instantaneous payment at time $\tau \in [t_0, T]$:*

$$\begin{aligned} B_i(\tau, x^*_\tau) &= -\Big[\xi^{(\tau)i}_t(t, x^*_t)\Big|_{t=\tau}\Big] - \frac{1}{2}\sum_{k,j} \sigma^{kj}(x^*_t)\Big[\xi^{(t_0)i}_{x^*_k x^*_j}(t, x^*_t)\Big|_{t=\tau}\Big] - \\ &\quad -\Big[\xi^{(\tau)i}_{x^*_t}(t, x^*_t)\Big|_{t=\tau}\Big] \times \Big[\sum_{j=1}^n a_j\{\hat{q}^j[\psi(\tau, x^*_\tau), \tau], \psi_j(\tau, x^*_\tau)\} - \\ &\quad -\sum_{j=1}^n b_j[\varpi_j(\tau, x^*_\tau), x^*_\tau] - \delta(x^*_\tau) x^*_\tau\Big], \qquad \text{for } i \in N \ \text{ yields Condition 1.} \end{aligned} \tag{23}$$

Proof.

Since $\xi^{(\tau)i}(t, x_t^*)$ is continuously differentiable in t and x_t^* , using (22) and Remarks 1 and 2 one can obtain:

$$\begin{aligned}
&E_\tau\{\int_\tau^{\tau+\Delta t} B_i(s, x^*(s))e^{-r(s-\tau)}ds\Big|x(\tau) = x_\tau^*\} = \\
&\quad = E_\tau\{\xi^{(\tau)i}(\tau, x_\tau^*) - e^{-r\Delta t}\xi^{(\tau+\Delta t)i}(\tau + \Delta t, x_{\tau+\Delta t}^*)\Big|x(\tau) = x_\tau^*\} = \\
&\quad = E_\tau\{\xi^{(\tau)i}(\tau, x_\tau^*) - \xi^{(\tau)i}(\tau + \Delta t, x_{\tau+\Delta t}^*)\Big|x(\tau) = x_\tau^*\}, \\
&\text{for } i \in N \text{ and } \tau \in [t_0, T],
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
&\Delta x_\tau = \Big[\sum_{j=1}^n a_j\{\hat{q}^j[\psi(\tau, x_\tau^*), \tau], \psi_j(\tau, x_\tau^*)\} - \sum_{j=1}^n b_j[\varpi_j(\tau, x_\tau^*), x_\tau^* - \delta(x_\tau^*)x_\tau^*\Big]\Delta t + \\
&+\sigma(x_\tau^*)\Delta z_\tau + o(\Delta t),
\end{aligned}$$

$\Delta z_\tau = z(\tau + \Delta t) - z(\tau)$, and $E_\tau[o(\Delta t)]/\Delta t \to 0$ as $\Delta t \to 0$.

With $\Delta t \to 0$, condition (24) can be expressed as:

$$\begin{aligned}
&E_\tau\{B_i(\tau, x_\tau^*)\Delta t + o(\Delta t)\} = E_\tau\{-\Big[\xi_t^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\Big]\Delta t - \\
&-\Big[\xi_{x_t^*}^{(\tau)i}(t, x_\tau^*)\Big|_{t=\tau}\Big] \times \Big[\sum_{j=1}^n a_j\{\hat{q}^j[\psi(\tau, x_\tau^*), \tau], \psi_j(\tau, x_\tau^*)\} - \\
&-\sum_{j=1}^n b_j[\varpi_j(\tau, x_\tau^*), x_\tau^*] - \delta(x_\tau^*)x_\tau^*\Big]\Delta t - \frac{1}{2}\sum_{k,j}\sigma^{kj}(x_t^*)\Big[\xi_{x_k^* x_j^*}^{(t_0)i}(t, x_t^*)\Big|_{t=\tau}\Big]\Delta t - \\
&-\Big[\xi_{x_t^*}^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\Big]\sigma(x)\Delta z_\tau - o(\Delta t)\},
\end{aligned} \tag{25}$$

Taking expectation and dividing (25) throughout by Δt, with $\Delta t \to 0$, yields (23). Hence, Theorem 21 follows.

Finally, explicit illustrative examples of the theoretical framework can be found in Yeung (2007) and Yeung and Petrosyan (2008).

5. Policy Implications

Facing with increasing demand for a sustainable solution the international community has responded to the deteriorating problem of global pollution. Over a decade ago, most countries joined an international treaty – the United Nations Framework Convention on Climate Change (UNFCCC) – to consider what can be done to reduce global warming and to cope with whatever temperature increases are inevitable.

Recently, a number of nations have approved an addition to the treaty: the Kyoto Protocol, which has more powerful and legally binding measures. In brief, the Kyoto Protocol is an international agreement, which builds on the United Nations Framework Convention on Climate Change, and sets legally binding targets and timetables for cutting the greenhouse-gas emissions of industrialized countries. Conditions for entry into effect are that some UNFCCC parties cut greenhouse-gas emissions of at least 5% from 1990 levels in the commitment period 2008-2012. As for December 2006, 169 countries and other governmental entities ratified the agreement. Notable exceptions include the United States. Other countries, like India and China, which have ratified the protocol, are not required to reduce carbon emissions under the present agreement despite their relatively large industrial production activities.

As mentioned before placing a constraint just on certain types of pollution emissions cannot offer a long-term solution, because the plans are limited to a confined set of controls like gas emissions and permits which is unlikely be able to offer an effective mean to reverse the accelerating trend of environmental deterioration, and there is no guarantee that participants will always be better off and, hence, be committed within the entire duration of the agreement. Guided by the analysis shown above, a grand coalition of all nations should be formed to pursue a comprehensive cooperative scheme of industrial pollution abatement. In particular, the entire set of policy instruments available – including environmental taxes and charges, adoption of environment-friendly production technology, subsidy to the replacement of polluting techniques, joint research and development in clean technology, restoration and preservation of the natural ecosystem, and legislations to outlaw environmentally unacceptable practices – will be used achieve an optimal cooperative outcome. A payment distribution mechanism has to be formulated so that cooperative gains will be shared according to the proportions of the nations relative sizes of expected noncooperative payoffs throughout the planning horizon. In sum, appropriate policy coordination will lead to the enhancement of economic performance and the realization of a cleaner environment.

This analysis opens up a novel policy forum for the international community. A particularly relevant instance would be the formation of a United Nations Agency to coordinate international cooperative actions on pollution and climate change. The Agency is proposed to be comprised of three divisions. An executive branch would be established to coordinate adoption and development of clean technology, pollution abatement activities, use of materials, waste disposal, mode of resource extraction and cooperation in environmental R&D. A financial branch (or FUND) would be set up to handle pollution charges, clean technology subsidies and allocate payoff distributions so that the agreed upon optimality principle will be realized throughout the cooperative period. Lastly, a legislative body would be in place to enact regulations on the use of dirty technologies, toxic disposal, pollutant emissions, activities damaging the environment and violation of the cooperative agreement.

Finally, a large scale scheme is in order for research in mechanism design theory initiated by Hurwicz (1973) and refined and applied Myerson (1989) and Maskin

(1999). In particular, mechanism designs for conventional markets in the face of impacts from a comprehensive set of environmental policy instruments including taxes, subsidies, technology choices, pollution abatement activities, pollution legislations and green technology R&D have to be considered. In addition, mechanism designs for inter-government transfers, institution formation, like-market and beyond-conventional market arrangements have also to be investigated.

6. Conclusions

After several decades of rapid technological advancement and economic growth, alarming levels of pollutions and environmental degradation are emerging all over the world. Though cooperation in environmental control holds out the best promise of effective action, limited success has been observed. Existing multinational joint initiatives like the Kyoto Protocol or pollution permit trading can hardly be expected to offer a long-term solution because there is no guarantee that participants will always be better off within the entire duration of the agreement.

A practicable cooperative scheme which guarantees that every participant will be better-off and the agreed upon arrangement will remain optimal along the cooperative path is characterized. Such an approach would yield an effective policy menu to tackle one of the gravest problems facing the global market economy. Finally this analysis opens up a novel policy forum for the international community.

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The Environment Protecting Dynamics. An Evolutionary Game Theory approach

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Abstract. The action of large human conglomerates is behind many of the environmental catastrophes in the last decades. However, these same conglomerates are the main actors in some important environment-protection battles. People might lead to the construction of social organizations, and achieve a kind of accumulation of a protecting stock (laws, protocols, social conscience, etc). Frequently, those social movements grow up and become strong, but they do not remain forever, they often follow a kind of cycle. We study emergence and dynamics of social organizations through the repetition of a game in an evolutionary context. The history of the conflict goes through stages. The reached stochastic stable equilibrium in each stage determines which organization is formed, and which is the accumulated stock in next stage, leading a new game, and new equilibria.

Keywords: Prisoner Dilemma, Public Good, Organization that Forces cooperation, Organization Game, Strict Nash Equilibria, Learning Dynamics, Myopic People, Sample of Information, Mistakes, Markov Process, Perturbation, Stochastically Stable Equilibria, Accumulation, Learning-Accumulation Dynamics, Depreciation.

Introduction

We study dynamics behavior of large populations facing any kind of environmental problem. There are destructive trends to face those problems inside most of such communities. The selfish individual interests might be too aggressive against the environment. However, in the real world, many of those populations face the undesired effects of their negative trends and try to achieve cooperation. In order to do that, they create organizations (such as community assemblies, trade-unions, ecologist organizations) that force their members to cooperate, and avoid others' destructive actions. Why do those organizations emerge?

All around the world, numerous communities organize themselves to resist the anti-environment tendencies of modern world. They can be found in Asia, Africa, Latin America, as well as in developed countries. Those organizations are not imposed by some forces outside the community. On the contrary, they emerge as free decisions of uncoordinated individuals. Some of those communities remain organized for centuries, for example, in underdeveloped countries. Nature and environmental protection are a part of their cultures. Of course, there are people who choose to free ride. But, it will surely emerge from a nucleus in charge of protecting the environment and other communal interests. This nucleus might change from time to time, and its size is increasing and decreasing in the long run. In contrast, other communities or social groups act as movements. That is to say, an organization of part of their population fights to reach obligatory protective conditions (laws, protocols, conscience). When those conditions are achieved, less and less people want to organize themselves and cooperate. However, laws and social forces wear down through time, and after some periods of apathy, people realize that laws have become only paper, and it is necessary to do something again. That is to say, there is a cycle as follows: people form an organization that grows in time, it disappears when goals are reached, and reemerges when those profits decreases through time. It is similar to trade-unions dynamics. Workers build strong organizations to fight for better wages and other benefits. However, when those conditions are established in contracts as an accumulated public wealth, people loose interest in collective actions, and the organization becomes a bureaucratic apparatus which is only useful to the leaders, until it is necessary to renew the organization.

We consider laws, protocols and conscience as a public good stock that can be accumulated, and it also can be depreciated (laws and social forces wear down on time). That good is public, that is, it is non-excludable. In this paper we study the dynamics of accumulation and depreciation of that special stock, and the responsibility of it on the kind of pattern that the organization achieves. Patterns where the community remains organized, with high probability, maybe changing the organization size and its composition or patterns with organization–disorganization cycles.

Conflicts similar to those have been studied in the literature, using a very simple Game theoretical model that, nevertheless, captures the core of the problem. That model is known as n-personal prisoner's dilemma. [Hardin, 1964] who generalized the model to n players called it: The Tragedy of the Commons. Authors who have been modeled that kind of conflicts through prisoner's dilemmas are, among many others, [Weissing and Ostrom, 1991], [Okada and Sakakibara, 1991], [Maruta and Okada, 2001], [Glance and Huberman, 1993, 1997]. The conditions under which people build organizations to overcome a prisoner's dilemma have been studied by, among others, [Ostrom, 1990, 1998], [Weissing and Ostrom, 1991], [Okada, Sakakibara, 1991], [Maruta and Okada, 2001].

"Collective action" is the name applied in the literature to the process by which social organizations form and operate. [Olson, 1965], who pointed out the essential question related to the emergence of organizations to overcome social dilemmas, suggested that a social organization emerges to allow people to enjoy some common goods. But those goods are public goods, and nobody can be excluded from enjoying them. He asked why people decide to make some effort participating in those organizations. [Ostrom, 1990, 1998] studied the ways in which communities involved in common pool resource dilemmas (of the prisoner's kind) overcame those dilemmas. She considered a repetition process, where individuals have bound rationality. Repetition of the conflict might lead people to design norms in order to overcome the dilemma, when there is good communication and trust among people. Then, some level of cooperation is reached. Repetition of the common pool resource dilemma, in her paper, is a game similar to the repeated prisoner's dilemma that [Axelrod, 1984] analyzed.

In this paper, we follow an evolutionary approach [à la Kandori, Mailath, Rob, 1993], [Kandori, Mailath, 1995] and [Young, 1993, 1998]. Why an evolutionary approach should be meaningful in this setting? Because we want to study great populations that are involved in similar social conflicts. And, we want to find the social patterns that emerge in the long run as a result of the behavior of an enormous quantity of individuals that are repeatedly involved in that kind of conflict. We have to build a realistic model on the way about real people learn with many limitations from experience, and adjust their actions in order to improve their utility facing others' behavior. Then we have to study the emergent social pattern in that process in this context. The Evolutionary Games Theory has been applied to the study of the setting up of the conventions in a society, the making of its ethical concepts and its social institutions; using a similar approach we have explained. See, for example, [Binmore, 1995, 1998, 2003], [Young, 1993, 1998]. Some authors, such as [Glance and Huberman, 1997], have studied the evolution of cooperative behaviors in the prisoner's dilemma and other dilemmas. [Maruta and Okada, 2001] follow Young's approach (adaptive play) in the non-accumulation case. Accumulation and depreciation provoke a process that is the history of development of environment and nature protecting laws. It is a process that goes through stages. In each of those stages, public stock (laws, etc.) does not change. Then a large population is involved in an organization game with fixed stock. That is, members of a community consider if they should voluntarily form an organization that forces themselves to cooperate. We study the emergent organization dynamics, when the conflict repeats in the long run relative to a stage. Encounters of that game occur between small groups of the members of that population drawn at random. The process of repetition of that organization game is a perturbed Markov process; it is not a repeated game. The bounded or limited rationality implies that individuals are able just to gather small amounts of information in a stochastic way and to improve their payoffs in response, without necessarily achieving the optimum. They also make mistakes in a stochastic way. Finally, the equilibria of the process are stochastic, and they describe which social behavior patterns occur with the highest probability, for a given public stock k. There are two cases that depend on the size of the public stock. In the first one, stochastically stable equilibria that are reached in the stage correspond to some of the strict Nash equilibria, and it means that emerge a minimal organization, according to k. In another case, none organization is formed, in each stochastically stable equilibrium. Then, accumulation in a stage is determined by the stochastically stable equilibrium that is achieved in that stage. Accumulation and depreciation provoke community to get to another stage, that is, the community will be involved in an organization game where the stock has changed by the accumulation achieved in a lower stage, and by depreciation. Which pattern does that accumulation dynamics follow?.

The paper is organized as follows: in section 1, we describe the prisoner's dilemma that shows the destructive trends existing inside the community. We also study the organization game, through which people can overcome those negative trends. It is similar to Okada and Maruta (2001). In the last part of that section, we explain the process of protective laws and conscience development that goes trough stages where people might organize in order to obtain stronger laws, and improve their concern. We establish, in section 3, the dynamics occurring in each stage as a perturbed Markov process that expresses a learning dynamics where community people achieve a stochastically stable equilibrium. That equilibrium might mean an organized activity that allows people to achieve better protective environment conditions or it might mean community people's apathy and passivity. In section 4, we study which patterns of organization-accumulation could be established through the stages that happens in the long run. Finally, we present some conclusions, and some open questions for future works.

1. The Models

1.1. The Prisoner Dilemma

In order to develop the ideas in a specific way, let us think in a community N, $N = \{1, 2, \ldots, n\}$, that is involved in a conflict about its water resources. This sort of conflicts are very common in many countries Let H be a large company that wants to establish an industrial or tourist project that might concentrate all the regional water resources in its hands. If carried out, this project might also pollute the groundwater reserves. However, the individuals, who supported H, would obtain some income from the company. Let us assume that the community members are players. Each one of those players has two possible strategies, either cooperation (C), which means to oppose company's project, or non-cooperation (NC), supporting it and taking advantage of it. *Cooperation* provokes protective forces in two senses: a) Past cooperation has accumulated a kind of stock (laws, social consciences, etc.) that gives individual rights to each people over water resources. Because the laws and social force that protect the regional water resources guarantee each person's from the community right to obtain some quantity of water, we consider that laws and forces as a stock that prevent community members' rights over water. This stock have a size that measure the quality of rights. Let k be the accumulated stock that has been accumulated until a stage t, we will say that each member of the community obtains an extra payoff $f_i(k)$ due to stock k. b) Current cooperation provokes that laws and consciences are more effective today and reach more accumulation for tomorrow. That is, individuals that choose C, in a period, create a protective tools that reinforce the laws, and each one obtains an extra payoff. Then, if s persons choose C we will say that each member of the community receives $f_s(s)$ utility from those s persons. Those protective forces benefit everybody, because laws and social force are public goods, that is, they are not excludable. We will assume f_i and f_s are strictly increasing functions, $f_s(0) = 0$ and $f_i(0) = 0$. On the other hand, we will denote by γ the utility each individual expects to obtain from H. If an individual decided not to support the project and act against it, he would obtain γ' smaller than γ. We will say that $d_C = \gamma - \gamma'$ is the cost of cooperation.

The previous suppositions can be expressed in a strategic game, which has N as a set of players. Each of them has $D_j = \{\text{Cooperation}\,(C),\ \text{Defection}\ (NC)\}$ as a set of pure strategies. Let σ be $(\sigma^1, \sigma^2, \ldots, \sigma^n)$, a profile of pure strategies in $D = D_1 \times D_2 \times \ldots \times D_n$, then the payoff got by any j is denoted by the function φ_j that is defined in D as

$$\varphi_j(\sigma) = \begin{cases} \gamma - d_C + f_i(k) + f_s(s_\sigma + 1) & \text{if } \sigma^j = C, \\ \gamma + f_i(k) + f_s(s_\sigma) & \text{if } \sigma^j = NC, \end{cases} \tag{1}$$

$s_\sigma = \#\left\{ i \in N \,\middle|\, i \neq j \text{ and } \sigma^i = C \right\}$.

Let us assume $f_s(s+1) - f_s(s) > d_C$, for each s. It would be better to cooperate, for each $j \in N$, without anyone forcing them to do it, independently on the decisions of the rest of the people. The fact that everybody cooperates would be, then, the only Nash equilibrium of the game. In that case, each individual looking only for own welfare would be part of the common welfare construction, confirming Adam Smith's conjecture.

We are here more interested, however, in the case where $f_s(s+1) - f_s(s) < d_C$, for each $0 \leq s \leq n$. Therefore, the strategy Defection (NC) is dominant and the only Nash equilibrium is $(NC, NC, \ldots, NC)$. In this situation, if $f_s(n) > d_C$, which we also assume, we would be facing a similar tragedy to the Hardin one. In order to refer to this game we will use the notation $G = (N, \{D_j\}, \varphi)$. In G, no one would choose an ecologist attitude without anything forcing them to do it, although there is a natural number l^*, $n \geq l^* > 1$, such that if l^* members of the community cooperated, the utility of each people in N would be larger than $\gamma + f_i(k)$. However, inside the community rises a new game which each of its members will think about in pursue of getting the best advantages of cooperation, even in an obligatory manner. That new game is the organization game.

1.2. Model I: The Organization Game

Organization Game that we consider is similar to Maruta–Okada one [2001]. But, we assume it happens in a stage, when community has accumulated a stock k.

Let us denote that game by $\widehat{G_k}$. Then,

$$\widehat{G_k} = \left(N, \left\{\widehat{D}_j = \left\{ \text{ Participation } (P), \text{ No Participation } (NP)\right\}\right\}, \widehat{\varphi_k}\right). \quad (2)$$

The organization could be, for instance, an assembly open to all members of the community who would like to participate in it. The objective of such organization would be to reach cooperation of their members in order to permit the existence of better conditions for the water resources. Members of the organization must cooperate. People not pertaining to the organization are free riders. We consider that the enforcement system has a cost $d_0(k)$.

Our assumptions in order to establish the payoff function are: 1) If no group in the community forms an assembly, everyone earns $\gamma + f_i(k)$. 2) s_k^* is the size of the minimum group S that can constitute an effective organization, in the sense that, if that group cooperated, each member of the community would obtain more than $\gamma + f_i(k)$. Let us denote as $f_s(s) - d_o(k)$, the earn that each person obtained if s persons choose P, and the accumulated stock is k. It is an increasing function relative to s, and decreasing relative to k. $d_o(k)$ is the organization cost, and it is an increasing function on k (it depends on $f_i(k)$). If the number of people that decided P is less than s_k^*, the assembly will not form. That is, s_k^* is the minimum integer number such that $f_s(s_k^*) > d_C + d_o(k)$, s_k^* is an increasing function of k. 3) If $\sigma^j = P$, but $s_\sigma < s_k^* - 1$, j earns the payoff of a free rider.

Then the j payoff function $\widehat{\varphi}$ is

$$\widehat{\varphi}_j(\widehat{\sigma}) = \begin{cases} \gamma - d_C + f_i(k) + f_s(s_{\widehat{\sigma}} + 1) - d_o(k) & \text{if } s_{\widehat{\sigma}} \geq s_k^* - 1 \text{ and } \widehat{\sigma}^j = P \\ \gamma + f_i(k) & \text{if } s_{\widehat{\sigma}} < s_k^* - 1 \text{ and } \widehat{\sigma}^j = P \\ \gamma + f_i(k) + f_s(s_{\widehat{\sigma}}) - d_o(k) & \text{if } s_{\widehat{\sigma}} \geq s_k^* \text{ and } \widehat{\sigma}^j = NP \\ \gamma + f_i(k) & \text{if } s_{\widehat{\sigma}} < s_k^* \text{ and } \widehat{\sigma}^j = NP \end{cases} \quad (3)$$

$s_\sigma = \#\left\{i \in N \;\middle|\; i \neq j \text{ and } \sigma^i = P\right\}$

The game has two important properties that are enunciated in the two following propositions.

Proposition 1. *If the community is "large" ($n \geq s_k^*$), a profile of pure strategies σ^* is a strict Nash equilibrium of the game $\widehat{G_k}$ if and only if the number of players that chose P in σ^* is s_k^*. If $n < s_k^*$, there is not strict Nash equilibrium of the game $\widehat{G_k}$.*

Proof.

Let us assume k is the accumulated stock. Let σ be a profile such that the number of people who chose P is smaller than $s_k^* - 1$. No organization appears in that profile, and everybody earns $\gamma + f_i(k)$. If any player j changed its strategy, and the others did not, it is still not possible to form an organization, and j would continue earning $\gamma + f_i(k)$. σ is Nash equilibrium, it does not matter if s_k^* is smaller than n, as is assumed, or if it is larger than n or equal to it. These equilibria are not strict.

Let be a profile such that exactly s_k^* people choose P, whoever they are. An organization is formed if s_k^*. Let j be a player who chose P. Then, j earned $\gamma - d_C + f_i\left(k\right) + f_s\left(s_k^*\right) - d_o\left(k\right)$, but if it changes its strategy, and the others do not, the organization cannot be formed, and j would earn $\gamma + f_i\left(k\right)$, which is smaller than $\gamma - d_C + f_i\left(k\right) + f_s\left(s_k^*\right) - d_o\left(k\right)$. Let i be a player who chose NP in σ. i earned $\gamma + f_i\left(k\right) + f_s\left(s_k^*\right) - d_o\left(k\right)$; if it changes its strategy, and the others do not, it would earn $\gamma - d_C + f_i\left(k\right) + f_s\left(s_k^* + 1\right) - d_o\left(k\right)$, which is smaller than $\gamma + f_i\left(k\right) + f_s\left(s_k^*\right) - d_o\left(k\right)$. Then σ is strict Nash equilibrium.

Let us examine the other profiles. Let σ be a profile of pure strategies such that the number of people who chose P is $s_k^* - 1$. No organization appears in that profile and each player earns $\gamma + f_i\left(k\right)$. If j chose NP in σ, and it changes its strategy, and the others do not, there would be s_k^* players that would choose P, and an organization would appear. Then j would earn $\gamma - d_C + f_i\left(k\right) + f_s\left(s_k^*\right) - d_o\left(k\right)$ which is larger than $\gamma + f_i\left(k\right)$. That is to say, σ is not Nash equilibrium.

Let σ be a profile where the number of people who chose P is larger than s_k^*. Let j be a player who chose P. An organization, whose size is $s_\sigma + 1$ would appear, then j would have earned $\gamma - d_C + f_i\left(k\right) + f_s\left(s_{\widehat{\sigma}} + 1\right) - d_o\left(k\right)$, but if it changes its strategy, and the others do not, an organization, whose size was s_σ, would appear. Then, j would earn $\gamma + f_i\left(k\right) + f_s\left(s_{\widehat{\sigma}}\right) - d_o\left(k\right)$, which is larger than $\gamma - d_C + f_i\left(k\right) + f_s\left(s_{\widehat{\sigma}} + 1\right) - d_o\left(k\right)$. Then it is not Nash equilibrium. There is not another Nash equilibrium in pure strategies in the organization game.

Proposition 2. *If the community is "large" ($n \geq s_k^*$), the Organization Game $\widehat{G_k}$ is weakly acyclic for each k.*

Proof.

Let us assume k is the accumulated stock. Let $\sigma\left(1\right)$ be a profile of pure strategies such that w, the number of people who chose P, is larger than s_k^*. If a player chose P in $\sigma\left(1\right)$, it will be called j_1. The payoff of j_1 in $\sigma\left(1\right)$ is $\gamma - d_C + f_i\left(k\right) + f_s\left(s_{\sigma(1)} + 1\right) - d_o\left(k\right)$. The best reply of j_1 to $\sigma\left(1\right)$ is NP. Let $\sigma\left(2\right)$ be the profile where j_1 chooses NP and each one of the rest of the players chooses the same strategy that it chose in $\sigma\left(1\right)$. If the number of people who chose P in $\sigma\left(2\right)$ is s_k^*, $\sigma\left(2\right)$ is a strict Nash equilibrium. If that number is larger than s_k^*, it could proceed by induction to build $\{\sigma\left(1\right), \sigma\left(2\right), \ldots, \sigma\left(r\right)\}$, such that s_k^* persons choose P in $\sigma\left(r\right)$, and, for each $i = 2, \ldots, r$, there is a player j_i who changed its strategy from the profile $\sigma\left(i-1\right)$, where it chose P, to the profile $\sigma\left(i\right)$ and NP is a j_i best reply to $\sigma\left(i-1\right)$. All of the rest of players did not change their strategies from $\sigma\left(i-1\right)$ to $\sigma\left(i\right)$. $\sigma\left(r\right)$ is a strict Nash equilibrium that is reached in a finite number of steps.

Let $\sigma\left(1\right)$ be a profile of pure strategies such that w, the number of people who chose P, is smaller than s_k^*. If a player chose NP in $\sigma\left(1\right)$, it will be called j_1. The payoff of that player in $\sigma\left(1\right)$ is $\gamma + f_i\left(k\right)$. If it changed its strategy, and $w + 1$ is smaller than s_k^*, it would again earn $\gamma + f_i\left(k\right)$. If it changed its strategy, and $w + 1$ is equal to s_k^*, it would earn $\gamma - d_C + f_i\left(k\right) + f_s\left(s_k^*\right) - d_o\left(k\right)$. In both cases, P is the best reply of j_1 to $\sigma\left(1\right)$. Let $\sigma\left(2\right)$ be the profile where j_1 chooses P and each one of the rest of the players chooses the same strategy that it chose in $\sigma\left(1\right)$. If the number of people who chose P in $\sigma(2)$ is s_k^*, $\sigma(2)$ is strict Nash equilibrium. If that number is smaller than s_k^* the building of the search sequence could again proceed by induction. In a finite number of steps a strict Nash equilibrium would reach.

We establish a strategic game, and we consider that the solution is reached when the game is repeated in an evolutionary context. In the organization game, each j in N should choose if it takes part in an organization S or if rather not.

More important than Nash equilibria of the game are the equilibria in the long run, when the conflict is repeated period after period by people with limited rationality as it happens in the real world.

1.3. Model I: The Learning-Accumulation Process

We will study organization and accumulation dynamics. In those process, time is considered in two senses. In the first sense, the history of the process is the history of the development, stagnation and backwards in laws, protocols and people conscience. That is, the history of the public good stock accumulation and depreciation in the long run. As we said in this paper, the establishment of laws or decrees related to the environment protection is considered as a sort of accumulation of an environment-protecting stock. That long run process goes through stages. The public good that protects the environment is accumulated at the end of each stage. That is, people cooperation provokes, at the end of each stage, a growing of that social capital stock, and it survives in the following stage, although it may be depreciated. In each of those stages, the repetition of the organization game, and the learning dynamics (an evolutionary process) occurs once and once through short periods of time. During a stage the accumulated stock does not change. People learn from past periods experience in a limited way. They have, for example, an incomplete information about other people attitude, and they make mistakes with a small probability. They change their behavior according to that. Then, a stochastic stable equilibrium is reached. That equilibrium determines how many people participate in the organization and which is the accumulated stock in the next stage. Then, at the end of each stage t, the "stock" might be accumulated or not. Besides, the stock that had been accumulated until the beginning of the stage t is depreciated, and the following stage begins with a new accumulated stock, which is the depreciated old stock plus the accumulated stock in t.

It is possible that the process, in the long run, leads to an organization that achieves a "stock", and then, keeps it forever, after repeated accumulation and depreciation, but this fact has very small probability. It is more probable that the process follows a kind of cycle. It could be an organization-disorganization cycle or a cycle where organization is changing its size. In both cases, the dynamics turns around a given stock.

In the two following subsections, we will study what is happened in a stage when the community has reached a stock k, and which dynamics could be provoked in the long run.

2. Learning Process

It is necessary to model the organization process using an adequate learning dynamics. It is a gradual process that would follow the members of a community involved in the considered water conflict. That is, many conflicts similar to the organization game take place daily among different groups of members of the community in which the decision to act individually or in an organized way was involved. Day by day such conflicts arose in one or another neighborhood; in the fields, at the market, in schools, or in other settings. Water has many different effects on the community and people have to adopt a position on the issue, often changing their way of thinking and acting according to their own and others' experiences over time. Of course, each person did not have full information about other people's behavior or attitude. Nobody gathered information systematically; instead, it was received at random. According to the information obtained people take the best decisions they can, although sometimes they make mistakes. That is, the emergence and evolution of an organization are spontaneous orders provoked by the repeated actions of the members of a great population.

2.1. Model I: The Repetition of the Conflict by a Great Population

Let us assume, in a stage t, there is an accumulated stock k and the game $\widehat{G_k}$ is repeated in that stage. As we said, people who play that repeated game are not always the same. The process happens among the members of a great population N (for example, people from the community). Let m be the size of N. Meetings of the organization game (right over water disputes) in a set of n people from N (different ones each time) would happen daily, where the decision of acting individually or in an organized way was involved. Sometimes the conflict would arise between people who are neighbors, or among co-workers, etc. Water disputes are present in many different ways and people have to adopt elections about it. People would change their way of thinking and would act according to their own and others' experiences. That is, we assume there is a large population N that contains many water conflicts expressed, in a period, by the game $\widehat{G_k}$. There are many conflicts of this kind, in the long stage, period after period. We also assume that people from N learn by experience, but they are myopic, and make mistakes in each period, and we study the stochastically stable equilibria of that process. That is the approach of [Kandori et al., 1993, 1995] and [Young, 1993, 1998].

We consider that the population N is partitioned in subpopulations $N_1, N_2, ..., N_n$, one for each player. For each j, N_j is large. We assume that partition in order to non-symmetric profiles in pure strategies were a part of the states of the dynamic process. Besides, in this way, it would be easier to introduce later the assumption that players might not have the same participation trends. A group of n persons, one of each N_i, could be the players of $\widehat{G_k}$.

In each period τ, only one encounter happens. j knows about others' attitude in the period $\tau - 1$ (that is, some strategies people chose in the last period). j uses that experience in order to choose its strategy for next period. Each one makes mistakes with small probability. We will study the behavior of people in the stage t.

Let us assume that everybody in N chose a strategy, in a period τ, and we denoted z_i^j as the strategy that chose member i from N_j (P or NP), the strategic structure of society would be z, where

$$z = \left(z^1 = \left(z_1^1, z_2^1, ..., z_{\theta_1}^1\right), ..., z^n = \left(z_1^n, z_2^n, ..., z_{\theta_n}^n\right)\right), \sum_{i=1}^{n} \theta_i = \theta\,. \tag{4}$$

Let Z be the set of strategic structures of society.

People learn by their own and others' experience. However, they are not very rational, they are myopic and have many limitations. According to their experiences, each j obtains a sample of size r from z, that is a vector

$$z\left(r\right) = \left(z^{1(r)} = \left(z_{i_1}^1, z_{i_2}^1, ..., z_{i_r}^1\right), ..., z^{n(r)} = \left(z_{j_1}^n, z_{j_2}^n, ..., z_{j_r}^n\right)\right) \tag{5}$$

Then, j chooses his best reply to the sample $z\left(r\right)$. That is, he considers $\widetilde{z\left(r\right)}$ the profile of mixed strategies that is determined by $z\left(r\right)$ and chooses the best reply according to $\widehat{G_k}$.

$\widetilde{z\left(r\right)} = \frac{1}{r}\left(\left(x^1, r - x^1\right), ..., \left(x^n, r - x^n\right)\right)$, where x^j is the number of P that appears in $z^{j(r)}$.

Each $z\left(r\right)$ might be elected as a sample with positive probability.

We build h as a correspondence from Z to Z such that:

If $z' \in h\left(z\right)$, there are at most n coordinates of z', one of each $j = 1, 2, \ldots, n$, that are different to the correspondent in z. Besides, if $z_i'^j$ is different to z_i^j the strategy $z_i'^j$ is the best reply to one $\widetilde{z\left(r\right)}$ for player j according to $\widehat{G_k}$.

Definition 1. *The learning dynamics of the process is a correspondence h from Z to Z such that if z' is in $h(z)$, there are m samples of z with size r, one for each player, such that for m persons of N, each one from one of the N_i, $z^{j'}$ is the j's best reply to his sample. Other coordinates of z do not change.*

If the state of the process is z, there might not be a succession of states $\{z_1 = z, z_1, ..., z_{s-1}, z_s = z'\}$ such that $z_{i+1} \in h\left(z_i\right)$, for $i = 1, .., s - 1$. That is, it might not be possible to reach some state z' through a finite number of periods, where the learning dynamics h is working. However, as we said, we consider that each member from N makes mistakes with a small probability. Then we will obtain a perturbation of the dynamics h, which allows to reach any z' from each z, through a finite number of periods.

Definition 2. *A player j's mistake in the state z is a strategy that is not the best reply to any sample $z\left(r\right)$.*

It is clear that it is possible to reach any state z' from z through a finite number of periods, when a perturbation of learning dynamics h is working. That is, there is a succession of states $\{z_1 = z, z_1, ..., z_{s-1}, z_s = z'\}$ such that, for $i = 1, .., s-1$, z_{i+1} is obtained due to deviations of at most m players from one of the strategic structures in $h(z_i)$.

Let us represent the learning dynamics and its perturbations by Markov matrixes.

Definition 3. *A Markov matrix Q, which order is $|Z|$, represents the learning dynamics h if, for each couple (z, z'),*

$$Q_{zz'} = \begin{cases} \lambda_{zz'} & \text{if } z' \in h(z), \\ 0 & \text{if } z' \notin h(z), \end{cases} \tag{6}$$

where $\lambda_{zz'}$ is the probability of choosing a combination of samples of z that determines z'.

If we consider that each member from N makes mistakes with a small probability η, and the mistakes of each person are independent of the mistakes of the other's.

Definition 4. *For each $\eta \in (0, a]$, a Markov matrix $Q(\eta)$ represents the perturbed learning dynamics Q with mistakes rate η if*

$$Q_{zz'}(\eta) = (1-\eta)^n Q_{zz'} + \sum_{y \in h(z),\ y \neq z'} \sum_{J_{yz'} \neq \theta} \alpha_{yz'}^{J_{yz'}} \eta^{\left|J_{yz'}\right|} (1-\eta)^{n-\left|J_{yz'}\right|} Q_{zy},$$

where $\alpha_{xx'}^{J}$ is the probability to go from x to x' when only the members of J make mistakes, and $J_{xx'}$ is such that $\alpha_{xx'}^{J_{xx'}}$ is positive.

Proposition 3. *For each $\eta \in (0, a]$, $Q(\eta)$ is regular. That is,*

a) *For each couple (z, z'), $Q_{zz'}(\eta)$ tends to $Q_{zz'}$ as η tends to 0.*

b) There is $\widehat{q}(\eta)$ a unique distribution vector such that $\widehat{q}(\eta) Q(\eta) = \widehat{q}(\eta)$.

c) For each couple (z, z') such that $Q_{zz'}(\eta) > 0$, there is a non-negative integer number $\nu_{zz'}$ such that the limit of $\frac{Q_{zz'}(\eta)}{\eta^{\nu_{zz'}}}$, as η tends to 0, exists and is positive.

Proof.

a) $Q_{zz'}(\eta)$ is a polynomial in η, such that $Q_{zz'}$ is the coefficient of η^0, then it is obvious that $Q_{zz'}(\eta)$ tends to $Q_{zz'}$ as η tends to 0.

b) Consider two states z and z' $(z \neq z')$. Then, for each (z, z'), there is a succession $\{z_1 = z, z_1, ..., z_{s-1}, z_s = z'\}$ such that $Q_{z_i z_{i+1}}(\eta)$ is positive, that is $Q(\eta)$ is irreducible. Then there is $\widehat{q}(\eta)$ a unique distribution vector such that $\widehat{q}(\eta) Q(\eta) = \widehat{q}(\eta)$.

c) If (z, z') is such that $Q_{zz'}(\eta) > 0$, let $\nu_{zz'}$ be the minimum exponent of η in $Q_{zz'}(\eta)$ such that corresponds to a positive coefficient, then $\frac{Q_{zz'}(\eta)}{\eta^{\nu_{zz'}}}$ is a polynomial, where the coefficient of η^0 is positive. Then, the limit of $\frac{Q_{zz'}(\eta)}{\eta^{\nu_{zz'}}}$, as η tends to 0, exists and it is positive.

That proposition said that the studied learning process perturbed by mistakes is regular as Young defined in his paper of conventions [1993]. In that paper, Young works with the concept of stochastic potential of each state of a perturbed Markov process P^{ε}. The stochastic potential of the state z is the minimum of the costs of all the z-trees in the graphic $\left(V, \overrightarrow{A}\right)$, where $V = Z$, $\overrightarrow{A} = \{(z, z') \,|P^{\varepsilon}_{zz'} > 0\}$, and the cost function is defined as $c(z, z')$ equals to the minimum number of mistakes to go from z to z'. Then the cost of a z-tree is the addition of all the $c(z, z')$ that correspond to (z, z') in the z-tree. Young's theorem (1993) about the stochastically stable equilibria of a regular perturbed Markov process.

Theorem 1 [Young, 1993]. *Let P^{ε} be a regular perturbed Markov process, and μ^{ε} – the unique stationary distribution of P^{ε} for each $\varepsilon > 0$. Then $\lim_{\varepsilon\to 0} \mu^{\varepsilon} = \mu^0$ exists, and μ^0 is a stationary distribution of P^0. The stochastically stable states are precisely those which have minimum stochastic potential.*

Which strategic structures of society are the stochastically stable equilibria of the organization game's learning process considered?

Young (1993, 1998) studied the special case of a weakly acyclic game G under an adaptive play process, with samples incomplete enough. He found that the stochastically stable equilibria, in that case, are some of the states that correspond to strict Nash equilibria, those which have minimum stochastic potential. If L_{Γ} denotes the maximum of the trajectories that join a profile of pure strategies to a strict Nash equilibrium in the best-reply graphic of G, the theorem is expressed as follows.

Theorem 2 [Young, 1993]. *Let G be a weakly acyclic n-person game, and let P be the adaptive process the unperturbed adaptive play with sizes of memory and sample α and r, respectively. If $r < \frac{\alpha}{L_{\Gamma}+2}$, the process converges with probability one to a convention from any initial state. The stochastically stable states are those that correspond to strict Nash equilibria with minimum stochastic potential.*

The learning dynamics we are considering is not an adaptive play, the states are strategic structures of society instead of histories of size m. However, it has the same properties that Young's adaptive play has according to weakly acyclic games. Then, it is possible to establish the same result when $\widehat{G_k}$. is weakly acyclic, that is, when $n \geq s^*_k$ (the community is large enough). If $r < \frac{\min|N_j|}{L_{\Gamma}+2}$ it is possible to find an integer positive number M and proving that from each state z the process reaches, with positive probability, a state which corresponds to a strict Nash equilibrium in at most M periods. Then if the community is large enough the stochastically stable equilibria correspond to strict Nash equilibria. That is, an organization is formed. Instead, if the population is small $\widehat{G_k}$ is not weakly acyclic, and none organization would form.

Theorem 3. a) *If $m \geq s^*_k$ and $r < \frac{\min|N_j|}{L_{\Gamma}+2}$ a social strategic structure is a stochastic stable equilibrium if and only if it is such that all the members of exactly s^*_k subpopulations choose P and the rest of the community people choose NP.* b) *If $m < s^*_k$ none organization is formed in a stochastically stable equilibrium.*

Proof.

Demonstration of a) follows the steps of Young's one (1993) due to the fact that $\widehat{G_k}$ is weakly acyclic, when $n \geq s_k^*$.

i) If the process is in the state z, in period t, it is possible to consider there is a positive probability that, in the period $t+i-1$, for $i = 1, 2, \ldots, r$, the players are the members $r+i$ of all of the N_j subpopulation. There is a positive probability that all of them chose as sample the first r persons in each of the vectors of z, in each of the considered periods. Let μ^1 be that sample. We considered that each player changed his strategy for a best reply to μ^1, for next period. Let σ be the profile of those best replies. Let us assume that member $r+i$ of N_j chooses s_j, the same player $j's$ best reply to μ^1, during periods $i = 1, 2, \ldots, r$. Then it has been generated a state $z(1)$ that can be reached, with positive probability, where decisions of the first r members of each N_j correspond to μ^1 and decisions of the $r+1, \ldots 2r$ correspond to the profile σ. Let μ^2 be the vector which has size r and all of its coordinates are σ.

ii) Let us consider a sequence $\left\{\sigma^0 = \sigma, \sigma^1, \sigma^2, \ldots, \sigma^l = \sigma^*\right\}$ of the best reply graphic that joins σ with the strict Nash equilibrium σ^*. For $i = 1, 2, \ldots, r$, let the members $2r+i$ of each N_j be the players in period $2r+i$. If j_1 is the unique player that changed his strategy from σ to σ^1 in the best reply graphic. People who represent j_1 in each of the considered r periods choose as sample μ^2. Other players choose μ^1. Then, it has been generated a state $z(2)$ that can be reached, with positive probability, where decisions of the first r members of each N_j correspond to μ^1, decisions of the members $r+1$, $r+2$, ..., $2r$ correspond to μ^2, and decisions of the $2r+1, ...3r$ correspond to the profile σ^1. Let μ^3 be the vector which has size r and all of its coordinates are σ^1.

iii) It can be constructed by induction the sequence $\{z, z\left(1\right), \ldots, z(l+2)\}$ such that decisions of the first r members of each N_j correspond to μ^1, for $i = 3, .., l+2$, decisions of the members $ir+1$, $ir+2$, ..., $(i+1)r$ correspond to μ^i, which is the vector that has size r and all of its coordinates are σ^{i-2}. That sample is μ^{l+2}, let us denote it as μ^*.

iv) When $z(l+2)$ that contains a sample μ^* is reached, we consider that each person who represents one of the players in the following periods chooses μ^* as sample, then period by period all the coordinates of the reached states are changing and becoming σ^*. The process without mistakes reaches $z^* = \left(\left(\sigma_1^*, ..., \sigma_1^*\right), ..., \left(\sigma_m^*, ..., \sigma_m^*\right)\right)$ in at most $max\ |N_j| + r\left(L_\Gamma + 1\right)$ periods.

On the other hand, all the strict Nash equilibria correspond to similar strategic structure of society. In those structures, all the members of exactly s_k^* subpopulations choose P and the rest of the community people choose NP. Then, all of those strategic structures are stochastically stable equilibria.

Demonstration of b): If $m < s_k^*$, the process without mistakes is connected, then all of the states are stochastically stable equilibria. Each of them means that none organization is formed.

The stochastically stable equilibria tell us that in each stage, if the population is large enough, people form an organization and achieve a larger "accumulated stock", and if the population is small, there is not accumulation in the stage. In order to complete the dynamics of the stock, it is necessary to consider that the stock depreciates. The stochastically stable equilibria and depreciation determine which game will be played at the next stage.

3. Accumulation in the long run

We have studied that people, after period t, might accumulate a new stock that continues acting in next period. We assume that a new stock does not depreciate, but the stock that existed at the beginning of t depreciates before the period $t+1$ began. The conflict in the stage t is the game G_k if k is the stock that has been accumulated until t. At the end of t, if $s_k^* \leq m$, the stock is accumulated by the cooperation of s_k^* persons. If $s_k^* > m$, there is no accumulation. But k has suffered a depreciation.

Depreciation: The stock depreciates, because laws and social forces wear down on time. Let us assume that the accumulated stock is depreciated by a rate ρ, $\rho \in (0,1)$. If at the beginning of stage t there were k unities of stock, and Δk unities are accumulated during t, in $t+1$ there will be $k\prime$ equal to $(1-\rho)\,k + \Delta k$ unities of stock.

If, in period t, k is the accumulated stock, and there is a rate of depreciation equal to ρ, the stocks that will be accumulated with probability almost one in $t+1$ are:

$$k' = \begin{cases} \;k\,(1-\rho) & \text{if } \; s_k^* > m, \\ \;k\,(1-\rho) + h - h' & \text{if } \; s_k^* \leq m, \, h \text{ is the stock accumulated by } \; s_k^* \\ \text{persons, and } h' \text{ is the loss in stock due to organization cost.} \end{cases} \tag{7}$$

In the long run, the community might remain organized, although the number of people that participate might be increasing and decreasing. This is the case, when it is reached a stock k, such that $s_k^* \leq m$, but Δk is smaller or equal to ρk. On the other hand, when it is possible to reach a stock k such that $s_k^* > m$, there is not any group inside the community that can allow people earn more than $\gamma + f_i\,(k)$, and periods of disorganization happens until depreciation leads to a stock k' such that $s_k^* \leq m$. That is, an organization-disorganization cycle would be provoked.

Example. Let us consider that $f_i\,(k) = \beta k$, $f_{ac}\,(s) = \beta s$, $d_0\,(k) = \beta\,(\gamma + \beta k)$. β is a unitary (marginal) "productivity" of the public good. We consider the corresponding game G. If $\beta > 1$, G is not a prisoner dilemma, the individual selfish interests lead people to cooperate. They always cooperate without an enforcing organization. On the contrary if $\beta < 1$, G is a prisoner dilemma. Let us study the behavior of the sizes of the stock and of the organization that emerges in the stochastically stable equilibria according to a stock that is more or less productive ($\beta = 0.5$) and according to a stock that is not productive ($\beta = 0.75$). The remainder parameters (γ,ρ,d_c,n) do not change.

Then, $s_k^* = \left[\frac{d_c}{\beta} + \gamma + \beta k\right]$, and

$$k' = \begin{cases} k\ (1-\rho) & \text{if } \ s_k^* > m, \\ k\ (1-\rho) + s_k^* - (\gamma + \beta k) & \text{if } \ s_k^* \leq m. \end{cases}$$

Let us assume that $\gamma = 1$, $\rho = 0.1$, $d_c = 0.6$, $n = 10$.

$\beta = 0.5$

stock	organization size	stock	organization size
$k_0 = 0$	$s_0^* = 3$	$k_{19} = 13.330$	$s_{13.330}^* = 9$
$k_1 = 2$	$s_2^* = 4$	$k_{20} = 13.332$	$s_{13.332}^* = 9$
$k_2 = 3.8$	$s_{3.8}^* = 5$	$k_{21} = 13.333$	$s_{13.333}^* = 9$
$k_3 = 5.52$	$s_{5.52}^* = 5$	$k_{22} = 13.334$	$s_{13.334}^* = 9$
...	...	$k_{23} = 13.334$	$s_{13.334}^* = 9$

(8)

$\beta = 0.075$

stock	organization size	stock	organization size
$k_0 = 0$	$s_0^* = 10$	$k_{18} = 14.38$	0
$k_1 = 9$	$s_9^* = 10$	$k_{19} = 12.942$	$s_{12.942}^* = 10$
$k_2 = 16.425$	0	...	...
$k_3 = 14.783$	0	$k_{45} = 19.622$	0
$k_4 = 13.305$	$s_{13.305}^* = 10$	$k_{46} = 17.660$	0
...	...	$k_{47} = 15.824$	0
$k_{14} = 13$	$s_{13}^* = 10$	$k_{48} = 14.305$	0
$k_{15} = 19.725$	0	$k_{49} = 12.330$	$s_{12.330}^* = 10$
$k_{16} = 17.753$	0	$k_{50} = 19.622$	0
$k_{17} = 15.978$	0	cycle: 5 stages	cycle: 5 stages

(9)

4. Conclusions

The presented model only intends to study the qualitative behavior of communities with environmental problems. Nonetheless, in spite of its limitations, we can study some essential aspects in the trend of those communities to organize themselves, and in the relationship between those trends and the accumulation of the community wealth. We stress as well that the model dynamics reproduces qualitative properties that are seen in the real dynamics of the emerging organizations inside different social sectors.

Let us try to review the qualitative conclusions obtained with this dynamics model.

1. It is obvious that the large human conglomerations are some of the main actors responsible for the modern environmental catastrophe. Other important actors are the big corporations and the national states, particularly the world powers, as they try to satisfy their huge economical and political interests. Our model justifies the opinion that these human conglomerates are also main actors in facing the catastrophe. They may act both forcing themselves to change their own environment-damaging activities, and stopping those of the big corporations and state powers. If

this is true, the Evolutionary Game approach is the proper one to discover what behavior patterns will emerge in the long run. Many policies are inspired by the idea that prohibitive laws, punishments and several other projects can by themselves abolish the negative behavior patterns. If these patterns are Long Run Equilibria of the process where the communities are involved that idea may be chimerical and those laws and projects may not achieve their objectives. On the other hand, the environment-protecting laws, decrees and punishments can be very useful if they strengthen the virtuous Long Run Equilibria of the process in which the community is involved. They may even transform the game conditions in order to obtain virtuous equilibria.

2. The qualitative analysis carried out in this work let us obtain the following conclusions: a) If the direct payoff the members of the communities receive from the environment is very small relative to the cost necessary to protect it, they will aggressively act against the environment either directly or indirectly abandoning it. b) If, on the other hand, the benefit is large enough (the larger the better), it will emerge from inside the community a nucleus in charge of protecting the environment and other communal interests, either continuously or cyclically, watching over it and stopping negative activities from both other members of the community and actors with a bigger damaging capacity. This nucleus will also maintain the unity of the members of the community among themselves, as well as their link with their place of origin, reverting some of trends to emigrate.

3. We think that this approach can also be useful to study other sorts of social problems, such as the rising of crime in a society, migration, etc. In our opinion, they also have their origin in the destruction of communal and social links that provoke different social dilemmas of the kind of the prisoner's one. In order to enrich the model, it would be interesting to build a unique dynamics that model learning and accumulation at the same time, in such a way that it would be necessary only to consider only the equilibria of that process. In the paper, it is necessary that the community achieves an equilibrium of the learning dynamics, in each stage, in order to build the accumulation dynamics. In order to enrich the learning process, it could be interesting to consider that in the communities (specially in the big ones), the individuals may have a much bigger probability to contact their neighbors, establishing an Organization Game with characteristics of a Spatial Game as those of Young (1998). The other interesting topic to study is the evolution of free riders due to their opportunities to accumulate wealth and power. They may have a set of strategies different to the rest. This is true also for those representing the enforcement system that may get out of the community control. That is, they may become new players looking for the use of the power they have got, in their own benefit.

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Financial Instruments in a Problem of Stochastic Characteristic Function and Imputation Construction

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Introduction

The paper is dedicated to the problem of characteristic function and time-consistent imputation procedure construction in multistage game with stochastically variable parameter [Yeung, Petrosyan, 2006]. It is known, that there is a problem of time-consistency even in the deterministic case, so the results obtained in this investigation could be used in stochastic solution developing. A multistage game with stochastically variable parameter was introduced. The main subject of the work was the problem of stochastic dynamics described by Ito's process with constant parameters μ and δ:

$$dx = \mu x dt + \delta x dz.$$

It was assumed, that multistage game was constructed on a graph. To solve the problem of uncertainty a discretization for continuous stochastic process was considered.

Finally, multistage games were divided into two classes: games with comparable payoffs and games with incomparable payoffs. For each one a certain approach was constructed to solve the problems of optimal trajectory determining and imputation procedure construction.

1. Ito's process

Ordinary Brownian motion is a scalar stochastic process where for the initial value X_0 at time $t = 0$, random variable X at $t > 0$ is normally distributed with mean $(X_0 + \mu t)$ and variance $\sigma^2 t$.

Denote $\Delta X = X(t+\Delta t) - X(t)$, $X(0) = X_0$. Let

$$\Delta X = \mu \Delta t + \sigma \Delta w, \tag{1}$$

where, $\Delta w = \varepsilon \sqrt{\Delta t}$, $\varepsilon \sim N(0,1)$, so we consider ordinary Brownian motion as a limit when $\Delta t \to 0$: $dX = \mu dt + \sigma dw$, where dw is normally distributed with zero mean and variance dt. Ito's lemma [Ito, 1951] asserts that if x follows the Ito's process

$$dx/x = \mu dt + \sigma dw \tag{2}$$

then $X = \ln x$ follows ordinary Brownian motion

$$dX = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dw. \tag{3}$$

So, the discretization of the process (2) is the following:

$$x(t+\Delta t) = x(t)e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma w}, \quad x(0) = x_0.$$

2. Multistage game definition

Consider multistage game $\Gamma(x_0, T)$ with perfect information starting from x_0 and finite duration $T < \infty$. Numerate every player with number: $N = \{1, 2, \ldots, n\}$. Consider a finite set Z and the partition of the set on $n+1$ sets: $Z_1, \ Z_2, ..., Z_n, Z_{n+1}$, $\bigcup\limits_{i=1}^{n+1} Z_i = Z$, $Z_k \bigcap Z_l = \emptyset$, $k \neq l$. Let F be a point-to-set mapping, where $F(z) = Z_{k+1}$, $z \in Z_k$, and for $z \in Z_{n+1} \quad F(z) = \emptyset$. Suppose that game $\Gamma(x_0, T)$ is played on a finite tree graph $Gr = (Z, F)$, where Z is a set of nodes. The set $Z_i \in Z$ we call *order set of player i*. In each node we define functions $h_i(z, x)$ – player's i payoff. Multistage game $\Gamma(x_0, T)$ is played ain the following way. Let $z_0 \in Z_1$, then in the node z_0 acts player 1 and chooses the next node $z_1 \in F(z_0)$. Then in the node z_1 acts player 2 and chooses the next node $z_2 \in F(z_1)$, and so on. The game ends up, when such node z_l is reached that $F(z_l) = \emptyset$.

Define player's i payoff in the game $\Gamma(x_0, T)$:

$$K_i(z_1, ..., z_{l+1}, x_1, ..., x_{l+1}) = \sum_{k=1}^{l+1} h_i(z_k, x(k)),$$

where $(z_1, ..., z_{l+1})$ is a path on a graph $Gr = (Z, F)$.

Let $\bar{x}$ be a certain realization of stochastic process.

Definition 1 [Yeung, Petrosyan, 2006]. *A real function V we will call a characteristic function in a game of n players, defined on coalitions $S \subset N$, if for any nonintersecting coalitions R, S $(R \subset N, S \subset N)$ the following inequality is fulfilled:*

$$V(R, T, z_0, \bar{x}) + V(S, T, z_0, \bar{x}) \leq V(R \cup S, T, z_0, \bar{x}), V(\emptyset, T, z_0, \bar{x}) = 0.$$

Define value of characteristic function for N players as a sum of their payoffs:

$$\max_{z_1,\ldots,z_{l+1}} \sum_{i=1}^{n} \sum_{k=1}^{l+1} h_i(z_k, x(k)) = \sum_{i=1}^{n} \sum_{k=1}^{l+1} h_i(\bar{z}_k, x(k)) = V(N, T, z_0, \bar{x}).$$

The characteristic function $V(S, T, z_0, \bar{x})$, $S \subset N$, is defined as the value of the zero-sum game $G_{S,N\setminus S}(z_0, x, T)$, constructed over the structure of game $G(z_0, x, T)$ and played between coalition S as the first (maximizing) player and coalition $N\setminus S$ as the second (minimizing) player. We suppose, that such values exist for each $z \in Z$ and $S \subset N$. For game $\Gamma(x_0, T)$ consider the corresponding cooperative game $G(z_0, x, T) = \langle N, T, V(S, T, z_0, x)\rangle$.

3. Problem statement

We can consider problems usually met in cooperative multistage games. The first issue is if it possible to create a coalition and appropriate imputation. The second one is about optimal trajectory – which one should be realized to earn maximum payoff for the coalition? And the third issue is time consistency of the solution. We need to solve all these problems in the stochastic game $G(z_0, \bar{x}, T) = \langle N, V(S, T, z_0, \bar{x})\rangle$. Similar issues are considered in [Yeung, Petrosyan, 2006]. Here the approach is constructed for monetary environment using financial instruments.

4. Comparable payoffs

Definition 2 [Hardy, Littlewood, Polya, 1934].

Functions $f(a) = f(a_1, a_2, ..., a_n)$ and $g(a) = (a_1, a_2, ..., a_n)$ are comparable, if the inequality $f \leq g$ is fulfilled for all positive a, or $f \geq g$ for all positive a.

Example 1. Consider Ito's process (2). It is known [Dixit, 1993] that all states $x(t)$ of the process are positive for all $t > 0$. So, it is clear, that functions $f(x) = 3x + 1, g(x) = x$ are comparable: $f \geq g$.

Definition 3. *Vector* $\xi(T, z_0, \bar{x}) = [\xi_1(T, z_0, \bar{x}), \xi_2(T, z_0, \bar{x}), \ldots, \xi_n(T, z_0, \bar{x})]$, *satisfying conditions* $\xi_i(T, z_0, \bar{x}) \geq V(i; T, z_0, \bar{x})$, *for* $i \in N$, *and*

$$\sum_{j \in N} \xi_j(T, z_0, \bar{x}) = V(N; T, z_0, \bar{x})$$

is called an imputation in the game $G(z_0, \bar{x}, T) = \langle N, V(S, T, z_0, \bar{x})\rangle$.

In the case of comparable payoffs, it is possible to use the same approach to define cooperative trajectory as in a deterministic one.

Consider an example of the game, when on each node players' payoffs are comparable with corresponding payoffs of a player on other nodes.

Suppose that the decision is to be made by two players on the graph tree (fig. 1). The first player acts first. Suppose that variable x evaluates according to Ito's process: $dx = \mu x dt + \delta x dz$.

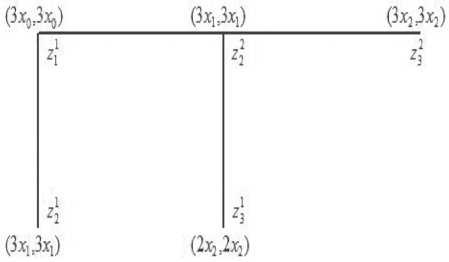

Fig. 1: A game with comparable payoffs

We obtain the following values for characteristic function:

$$V(\{1,2\},2,z_1^1,x_0,x_1,x_2) = 6(x_0+x_1+x_2), \quad V(\{1,2\},1,z_2^2,x_1,x_2) = 6(x_1+x_2),$$

$$V(\{1\},2,z_1^1,x_0,x_1,x_2) = 3x_0+3x_1+2x_2, \quad V(\{1\},1,z_2^2,x_1,x_2) = 3x_1+2x_2,$$

$$V(\{2\},2,z_1^1,x_0,x_1,x_2) = 3x_0+3x_1, \quad V(\{2\},1,z_2^2,x_1,x_2) = 3x_1+3x_2.$$

Note, that in the example all values of the characteristic function are expressions of stochastic variable, nevertheless all the values are comparable. It brings a unique property that despite of stochasticity of the process, characteristic function possesses super-additivity. It allows us to divide multistage stochastic games into two classes: games with comparable payoffs and games with incomparable payoffs. Consider stochastic imputation distribution procedure (IDP) for Shapley value:

$$Sh_1(2,z_1^1,x_0,x_1,x_2) = 3x_0+3x_1+4x_2, \quad Sh_1(1,z_2^2,x_1,x_2) = 3x_1+\frac{5}{2}x_2,$$

$$Sh_2(2,z_1^1,x_0,x_1,x_2) = 3x_0+3x_1+2x_2, \quad Sh_2(1,z_2^2,x_1,x_2) = 3x_1+\frac{7}{2}x_2.$$

Obviously, it is impossible to obtain numeric Shapley value at the beginning of the game because of stochasticity of the process. In spite of it the condition of individual rationality is fulfilled, making players to cooperate.

Consider stochastic imputation distribution procedure:

$$\beta_1(2,z_1^1,x_0,x_1,x_2) = 3x_0+\frac{3}{2}x_2, \quad \beta_1(1,z_2^2,x_1,x_2) = 3x_1+\frac{5}{2}x_2,$$

$$\beta_2(2,z_1^1,x_0,x_1,x_2) = 3x_0-\frac{1}{2}x_2, \quad \beta_2 = 3x_1+\frac{7}{2}x_2.$$

It is known, that states of stochastic Ito's process are positive (see [Dixit, 1993]). It means, that $\beta_2(z_1^1,x_0,x_1,x_2)$ could be negative. In other words, positive probability exists to earn negative value by the second player. This problem could be solved by constructing time-consistent imputation distribution procedure (TCIDP) on the base of originally chosen solution:

$$\bar{\beta}_1(2, z_1^1, x_0, x_1, x_2) = \frac{(3x_0 + 3x_1 + 4x_2)3(x_0 + x_1)}{6(x_0 + x_1 + x_2)}, \tag{4}$$

$$\bar{\beta}_2(2, z_1^1, x_0, x_1, x_2) = \frac{(3x_0 + 3x_1 + 2x_2)3(x_0 + x_1)}{6(x_0 + x_1 + x_2)},$$

$$\bar{\beta}_1(1, z_2^2, x_1, x_2) = \frac{(3x_1 + \frac{5}{2}x_2)3x_2}{6(x_1 + x_2)},$$

$$\bar{\beta}_2(1, z_2^2, x_1, x_2) = \frac{(3x_1 + \frac{7}{2}x_2)3x_2}{6(x_1 + x_2)}.$$

Note at this regulated scheme all payoffs are positive and this solution possesses the property of time-consistency.

The second property of TCIDP is group rationality. Indeed, on each step overall payoff earned is distributed between players:

$$\bar{\beta}_1(2, z_1^1, x_0, x_1, x_2) + \bar{\beta}_2(2, z_1^1, x_0, x_1, x_2) = 3(x_0 + x_1),$$

$$\bar{\beta}_1(1, z_2^2, x_1, x_2) + \bar{\beta}_2(1, z_2^2, x_1, x_2) = 3x_2,$$

$$\bar{\beta}_1(2, z_1^1, x_0, x_1, x_2) + \bar{\beta}_2(2, z_1^1, x_0, x_1, x_2) + \bar{\beta}_1(1, z_2^2, x_1, x_2) + \bar{\beta}_2(1, z_2^2, x_1, x_2) =$$

$$= V(\{1, 2\}, 2, z_1^1, x_0, x_1, x_2).$$

It seems, that it is impossible to distribute payoff earned on each step between participants, because, for example, being on the first step participants can not use scheme (4): future values x_2 are not known. In spite of it, payment is possible. It could be realized using special form of promissory notes, according to scheme (4). At the end of the game all players know the whole realization of stochastic process $\bar{x}_0,\ \bar{x}_1,\ \bar{x}_2$, and they are able to distribute overall payoff between all players according to rules inserted to promissory notes.

5. Incomparable payoffs

At this section we suppose general case, when payoffs are not comparable in sense of Definition 2.

In the case of incomparable payoff it is impossible to determine optimal trajectory by comparing payoffs. Methods, where decision is to be made on base of mathematical expectation are widely developed. Unfortunately, in this case real value of the process is equal to expected one with zero probability. Therefore, imputation and regularization of the solution are to be improved.

The problem could be solved using financial option pricing technique (we suppose here payoffs are transferable). Namely, we offer to buy financial options in certain amount to guarantee payment at the value of mathematical expectation. Of course, all players are to pay to obtain such option.

Consider multistage stochastic game $G(z_0, \bar{x}, T) = \langle N, V(S, T, z_0, \bar{x})\rangle$ on a tree graph. Let v be the number of node at an order set. We suppose, that payoffs $h_i(z^v, x)$ are incomparable with each other. As mentioned above, it is possible to make decision about optimal trajectory using mathematical expectation $E_{x_0} h_i(z^v, x)$. In monetary environment we are able to use the following technique:

1. Denote mathematical expectation in each node of the graph:
$$E_{x_0} h_i(z_k^v, x(k)) = \bar{h}_i(z_k^v, k).$$
2. To solve all implicit equations of the form
$$\bar{x}_i(z_k^v) = \max\{0, (x|h_i(z_k^v, x(k)) - \bar{h}_i(z_k^v, k) = 0)\}$$
to obtain strike prices of the options in each node of the graph.
3. Required number of options $\frac{\bar{h}_i(z_k^v, x(k))}{\bar{x}_i(z_k^v)}$ it is easy to determine from identity $\bar{h}_i(z_k^v, k) \equiv \bar{x}_i(z_k^v)\frac{\bar{h}_i(z_k^v, k)}{\bar{x}_i(z_k^v)}$.
4. For each player i and every value $\bar{x}_i(z_k^v)$ it is possible to determine an option fee $P(z_k^v)$, i.e. by Black–Scholes formula:
$$P(z_k^v, k) = -x_0 \left[1 - F(z)\right] + e^{-ru\Delta t}\bar{x}_i(z_k^v)\left[1 - F\left(z - \sigma\sqrt{t}\right)\right],$$
where
$$z = \frac{\ln\left(\frac{x_0}{\bar{x}_i(z_k^v)}\right) + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}.$$
So at the beginning of the game every player has to pay a certain costs to guarantee earning of expected payoff in future. Note the game with perfect information is considered. Therefore, every player can compute payoff and option price.
5. Define deterministic analog of the stochastic game, where every player's payoff on each node is $\bar{h}_i(z_k^v, k) - P_i(z_k^v)\frac{\bar{h}_i(z_k^v, k)}{\bar{x}_i(z_k^v)}$.
6. Compute characteristic function in the deterministic game using classical approaches, i.e. maximin.

Example. Consider multistage game on the following graph (fig. 2).
where $dx = 0.1xdt + 0.1xdz$, $r = 0.1$, $x_0 = 0.5$.

Here two stage game is represented. Suppose that $n = 2$. On each vertex there is a vector presenting payoffs of the first and the second players. On each of two steps

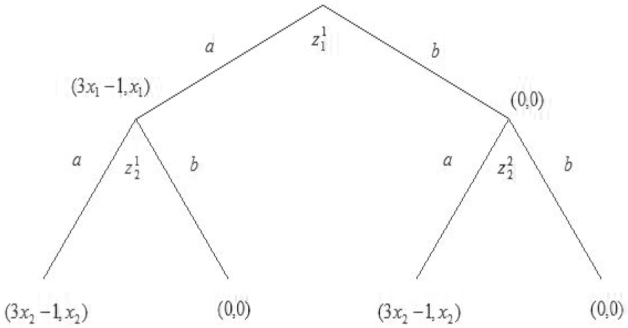

Fig. 2: Incomparable payoff

every player is able to invest and earn stochastic payoff or to give up investment and earn zero payoff. The functions $f(x) = 3x_1 - 1$ and $g(x) \equiv 0$ are incomparable, therefore, it is a game with incomparable payoffs.

The game is played in the following way. Let the node z_0 belongs to order set of player i, i.e. player 1 is to move first. He chooses one of two possible alternatives:

$a1$ – action, earning him payoff in amount of $(3x_1 - 1)$ and x_1 for the second player,

$b1$ – inaction. Obviously corresponding payoffs are null.

On the second stage player 2 acts. We consider a game with perfect information, therefore, player 2 knows the choice of the first player. Now it is a turn for the second player to decide whether to provide stochastic payoff for players or to end up with zeros.

Consider deterministic equivalent construction and characteristic function computation. Compute mathematical expectation of payoffs at each node. For the function $f(x) = x$, where $dx/x = \mu dt + \sigma dw$, mathematical expectation is $Ex(t) = x_0 e^{\mu t}$. Then it is possible to compute $\bar{h}_i(z_k^v, k)$ at each node. In other words, we have obtained deterministic equivalent of the stochastic game (fig. 3).

For deterministic game obtained it is possible to use an ordinary approach from cooperative games to construct characteristic function. The main problem remained is to provide an actual payoff to be equal to the expected one. To solve it we consider financial options technique. We have not given any interpretation of the payoff earned on each step. Now we suppose that it is profit as a function of stochastic parameter. For instance, we can consider producers selling on a foreign market and treat x as a currency exchange rate.

When deterministic equivalent of the game is obtained, it is possible to compute the strike price of the put option (in our example it is an option to exchange currency obtained into domestic currency with certain rate).

Strike prices we found from the equation

$$\bar{x}_i(z_k^v) = \max\{0, (x | h_i(z_k^v, x(k)) - \bar{h}_i(z_k^v, k) = 0)\}$$

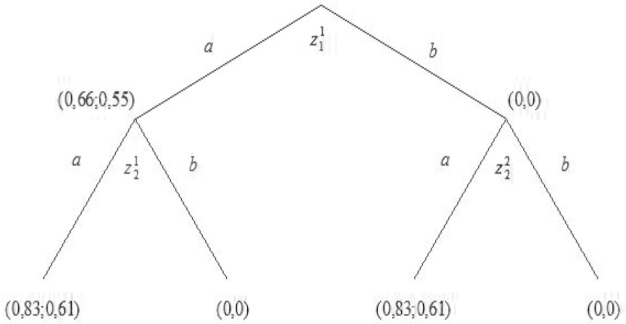

Fig. 3: Deterministic game

:

$$\bar{x}_1^{11} = \bar{x}_2^{11} \approx 0.55, \quad \bar{x}_1^{12} = 0, \quad \bar{x}_2^{12} = 0$$

$$\bar{x}_1^{21} = \bar{x}_2^{21} = \bar{x}_1^{23} = \bar{x}_2^{23} \approx 0.61, \quad \bar{x}_1^{22} = \bar{x}_2^{22} = \bar{x}_1^{24} = \bar{x}_2^{24} = 0$$

Then we compute option fees and amount of options needed:
$P_1(z_1^1) = 0.02, 1.2$ options; $P_2(z_1^1) = 0.02$, 1 option;
$P_1(z_2^1) = 0.03, 1.36$ options; $P_2(z_2^1) = 0.03$, 1 option;
$P_1(z_3^1) = 0.03, 1.36$ options; $P_2(z_3^1) = 0.03$, 1 option.

Taking into account discount factor and option fees we obtain the following values of characteristic function:

$$V(\{1,2\}, 2, z_1^1, x_0) = 2.3,\ V(\{1,2\}, 1, z_2^1, x_1) = 1.24,\ V(\{1,2\}, 0, z_3^1, x_2) = 1.14,$$

$$V(\{1\}, 2, z_1^1, x_0) = 0.58, \quad V(\{1\}, 1, z_2^1, x_1) = 0, \quad V(\{1\}, 0, z_3^1, x_2) = 0,$$

$$V(\{2\}, 2, z_1^1, x_0) = 0.47, \quad V(\{2\}, 1, z_2^1, x_1) = 0.47, \quad V(\{2\}, 0, z_3^1, x_2) = 0.47.$$

Therefore, optimal trajectory is $\{a, a\}$. Shapley value is

$$Sh_1(1, z_1^1, x_0) = 1.205, \quad Sh_1(0, z_2^1, x_1) = 0.385,$$

$$Sh_2(1, z_1^1, x_0) = 1.095, \quad Sh_2(0, z_2^1, x_1) = 1.095.$$

Then we can regularize the solution obtained using TCIDP [Yeung, Petrosyan, 2004].

Conclusion

Two classes of games have been represented on a tree graph: games with comparable payoffs and games with incomparable payoffs. It is clear, two approaches proposed could be used in more complicated classes of games, for example, multistage stochastic game, where on each step a game in normal form is introduced, or games with incomplete information.

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